

Word-Valued Sources: an Ergodic Theorem, an AEP and the Conservation of Entropy

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Abstract

A word-valued source $\mathbf{Y} = Y_1, Y_2, \dots$ is discrete random process that is formed by sequentially encoding the symbols of a random process $\mathbf{X} = X_1, X_2, \dots$ with codewords from a codebook \mathcal{C} . These processes appear frequently in information theory (in particular, in the analysis of source-coding algorithms), so it is of interest to give conditions on \mathbf{X} and \mathcal{C} for which \mathbf{Y} will satisfy an ergodic theorem and possess an Asymptotic Equipartition Property (AEP). In this correspondence, we prove the following: (1) if \mathbf{X} is asymptotically mean stationary, then \mathbf{Y} will satisfy a pointwise ergodic theorem and possess an AEP; and, (2) if the codebook \mathcal{C} is prefix-free, then the entropy rate of \mathbf{Y} is equal to the entropy rate of \mathbf{X} normalized by the average codeword length.

Index Terms

Word-Valued Source, Pointwise Ergodic Theorem, Asymptotic Equipartition Property, Asymptotically Mean Stationary.

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I. INTRODUCTION

The following notion of a word-valued source appears frequently in source-coding theory [1–4]. Suppose that \mathcal{A} and \mathcal{B} are discrete-finite alphabets and $\mathbf{X} = X_1, X_2, \dots$ is an \mathcal{A} -valued random process. Let \mathcal{C} be a codebook whose codewords take symbols from \mathcal{B} and have different lengths, and let $f : \mathcal{A} \rightarrow \mathcal{C}$ be a mapping. The word-valued source generated by \mathbf{X} and f is the \mathcal{B} -valued random process $\mathbf{Y} = f(X_1), f(X_2), \dots$, which is formed by sequentially encoding the symbols of \mathbf{X} with f and concatenating (placing end-to-end) the resulting codewords.

It is of fundamental interest to give broad conditions on \mathbf{X} , f and \mathcal{C} for which \mathbf{Y} is guaranteed to possess an Asymptotic Equipartition Property (AEP). A common approach to this type of problem is to determine when the random processes of interest are stationary, after which the classic Shannon-McMillan-Breiman Theorem [5, Thm. 15.7.1] may be used to achieve an AEP. However, this approach is not particularly useful for word-valued sources: for most choices of f and \mathcal{C} , \mathbf{Y} will not be stationary – even when \mathbf{X} is stationary. Thus, the primary focuss of this paper is to give broad conditions for an AEP without direct recourse to stationarity and the Shannon-McMillan-Breiman Theorem.

Nishiara and Morita [1, Thms. 1 & 2] derived an AEP as well as a conservation of entropy law for \mathbf{Y} when \mathbf{X} is independent and identically distributed (i.i.d.), f is a bijection and \mathcal{C} is prefix-free. (A codebook is said to be prefix-free if no codeword is a prefix of another codeword [5, Chap. 5].) These results were later extended from the i.i.d. case to the more general stationary and ergodic case by Goto *et al.* in [2, Thm. 2]. We further generalize the results of [1, 2] to the setting where \mathbf{X} is Asymptotically Mean Stationary (AMS), f is a bijection and \mathcal{C} is prefix-free. (This AMS condition is a weaker version of the stationary condition that permits short-term non-stationary properties [6].) As we will see, the resulting AEP and entropy-conservation law do not retain the simplicity of those results reported in [1, 2] for stationary and ergodic \mathbf{X} ; namely, both extensions are ineluctably linked to an ergodic-decomposition theorem.

In contrast to the aforementioned results for prefix-free codebooks, very little is known about word-valued sources generated by codebooks without the prefix-free property. In [1], Nishiara and Morita derived an upper bound for the sample-entropy rate of \mathbf{Y} when \mathbf{X} is an i.i.d. process and \mathcal{C} is not prefix-free. This upper bound was later supplemented with a non-matching lower bound by Ishida *et al.* in [4]. These bounds, however, fell short of proving an AEP. We prove an ergodic theorem as well as an AEP for \mathbf{Y} when \mathbf{X} is AMS and \mathcal{C} is arbitrary; and, in doing so, we resolve the open problem reported in [1, 2, 4].

Our results will follow from a new lemma (Lemma 8) for AMS random processes. This lemma is an extension of a result by Gray and Saadat [7, Cor. 2.1], and it demonstrates that the AMS property is invariant to variable-length time shifts: an AMS random process will remain AMS when it is viewed under different time scales. This invariance property will, in turn, allow us to show that \mathbf{Y} is AMS whenever \mathbf{X} is AMS – no matter which f and \mathcal{C} is used. Finally, Gray and Kieffer’s AEP for AMS processes [8, Cor. 4] will provide the desired AEP for \mathbf{Y} .

An outline of the paper is as follows. We introduce some notation and definitions in Section II. We present an ergodic theorem (Theorem 1-A) in Section III, and in Section IV we restate this ergodic theorem using the language of AMS random processes (Theorem 1-B). We present an AEP (Theorem 2) in Section V. Finally, Theorems 1-B and 2 are proved in Sections VI and VII respectively.

II. DYNAMICAL SYSTEMS & WORD-VALUED SOURCES

The notion of “time” is problematic for the development of word-valued sources. In particular, each symbol X_i , $i = 1, 2, \dots$, will produce multiple symbols (a codeword) $f(X_i)$; thus, \mathbf{X} and \mathbf{Y} are naturally defined by different time scales. We simplify notation for these different time scales by using various shift transformations to model the passage of time. A brief review of these transformations and the resulting dynamical systems is given in this section – a complete treatment can be found in [6] and [9]. After this review, we formally define word-valued sources.

A. A Dynamical Systems Model for \mathbf{X}

Let us first introduce some notation. Suppose that \mathcal{A} is a discrete-finite alphabet. For any natural number n (i.e. $n \in \{1, 2, \dots\}$), let

$$\mathcal{A}^n = \underbrace{\mathcal{A} \times \mathcal{A} \times \dots \times \mathcal{A}}_n$$

denote the n -fold Cartesian product of \mathcal{A} , and let¹ $a^n = a_1, a_2, \dots, a_n$ denote an arbitrary n -tuple from \mathcal{A}^n . (These notation conventions will apply to the Cartesian product of every discrete-finite alphabet used in this paper.)

Now suppose that $\mathbf{X} = X_1, X_2, \dots$ is an \mathcal{A} -valued random process that is characterised by a sequence of joint probability distributions

$$p^{(n)}(a^n) = \Pr(X_1 = a_1, X_2 = a_2, \dots, X_n = a_n), \quad n = 1, 2, \dots, \quad (1)$$

¹When $n = 1$, we shall omit the superscript for brevity, e.g., $a^1 = a$ and $\mathcal{A}^1 = \mathcal{A}$.

for which the consistency condition

$$p^{(n)}(a_1, a_2, \dots, a_n) = \sum_{\tilde{a} \in \mathcal{A}} p^{(n+1)}(a_1, a_2, \dots, a_n, \tilde{a}) \quad , \quad n = 1, 2, \dots \quad , \quad (2)$$

is satisfied. Instead of characterising \mathbf{X} with the sequence of joint distributions given in (1), we may use a dynamical system without loss of generality. A brief review of this fact is as follows.

Let $\mathcal{X} = \mathcal{A} \times \mathcal{A} \times \dots$ denote the set of all sequences with elements from \mathcal{A} , and let $\mathbf{x} = x_1, x_2, \dots$ denote an arbitrary member of \mathcal{X} . Now let

$$[a^n] = \{\mathbf{x} \in \mathcal{X} : x_1 = a_1, x_2 = a_2, \dots, x_n = a_n\}$$

denote the cylinder set determined by an n -tuple $a^n \in \mathcal{A}^n$, and define $\mathcal{F}(\mathcal{X})$ to be the σ -field of subsets of \mathcal{X} that is generated by the collection of all cylinder sets. Let $T_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X}$ be the left-shift transform that is defined by $T_{\mathcal{X}}(\mathbf{x}) = x_2, x_3, \dots$. For integers $n \geq 0$, let²

$$\begin{aligned} T_{\mathcal{X}}^n(\mathbf{x}) &= \underbrace{T_{\mathcal{X}}(T_{\mathcal{X}}(\dots T_{\mathcal{X}}(\mathbf{x}) \dots))}_n \\ &= x_{n+1}, x_{n+2}, \dots \end{aligned}$$

denote the n -fold composition of $T_{\mathcal{X}}$, and let

$$T_{\mathcal{X}}^{-n}A = \{\mathbf{x} \in \mathcal{X} : T_{\mathcal{X}}^n(\mathbf{x}) \in A\}$$

denote the preimage of an arbitrary set $A \in \mathcal{F}(\mathcal{X})$ under $T_{\mathcal{X}}^n$. Finally, consider the partition $\mathcal{Q} = \{[a] : a \in \mathcal{A}\}$ of \mathcal{X} , and define the function $X_{\mathcal{Q}} : \mathcal{X} \rightarrow \mathcal{A}$ by setting $X_{\mathcal{Q}}(\mathbf{x}) = a$ if $\mathbf{x} \in [a]$. I.e. $X_{\mathcal{Q}}(\mathbf{x})$ returns the value of the first symbol, x_1 , from \mathbf{x} .

Proposition 1 ([6, 9]): If \mathbf{X} is an \mathcal{A} -valued random process that is characterised by a distribution (1) for which the consistency condition (2) holds, then there exists a unique probability measure μ on $(\mathcal{X}, \mathcal{F}(\mathcal{X}))$ such that $p^{(n)}(a^n) = \mu([a^n])$ for every tuple $a^n \in \mathcal{A}^n$ and every $n = 1, 2, \dots$. In particular, the distribution of the sequence of \mathcal{A} -valued random variables $X_{\mathcal{Q}} \circ T_{\mathcal{X}}^n$, $n = 0, 1, \dots$, defined on $(\mathcal{X}, \mathcal{F}(\mathcal{X}), \mu)$ matches that of \mathbf{X} :

$$\mu\left(\left\{\mathbf{x} \in \mathcal{X} : X_{\mathcal{Q}}(\mathbf{x}) = a_1, X_{\mathcal{Q}}(T_{\mathcal{X}}(\mathbf{x})) = a_2, \dots, X_{\mathcal{Q}}(T_{\mathcal{X}}^{n-1}(\mathbf{x})) = a_n\right\}\right) = \mu\left(\bigcap_{i=1}^n T_{\mathcal{X}}^{-i+1}[a_i]\right) = \mu([a^n]) \quad .$$

The probability measure μ is called the Kolmogorov measure of the process \mathbf{X} .

Proposition 1 shows that the quadruple $(\mathcal{X}, \mathcal{F}(\mathcal{X}), \mu, T_{\mathcal{X}})$ may be used in place of \mathbf{X} without loss of generality. We shall use $(\mathcal{X}, \mathcal{F}(\mathcal{X}), \mu, T_{\mathcal{X}})$ and \mathbf{X} interchangeably.

²If $n = 0$, define $T_{\mathcal{X}}^0(\mathbf{x}) = \mathbf{x}$.

B. A Dynamical System Model for \mathbf{Y}

Suppose that \mathcal{B} is a discrete-finite alphabet, N is a natural number, and

$$\mathcal{B}^* = \bigcup_{i=1}^N \mathcal{B}^i$$

is the set of all \mathcal{B} -valued tuples $b^i = b_1, b_2, \dots, b_i$ whose length i is greater than or equal to 1 and no more than N . Let $f : \mathcal{A} \rightarrow \mathcal{B}^*$ be a mapping and $\mathcal{C} = \text{Range}(f)$. Finally, let c denote an arbitrary member of \mathcal{C} and $|c|$ its length. We call f a *word function*, \mathcal{C} a *codebook*³, and c a *codeword*.

Definition 1 (Word-Valued Source): Suppose that \mathbf{X} is an \mathcal{A} -valued random process and f is a word function. The word-valued source \mathbf{Y} generated by \mathbf{X} and f is defined to be the \mathcal{B} -valued random process that is formed by:

- (i) sequentially coding the symbols X_i , $i = 1, 2, \dots$, with f , and
- (ii) concatenating the resulting sequence of codewords: $\mathbf{Y} = f(X_1), f(X_2), f(X_3), \dots$

For arbitrary f , the particular realisation of \mathbf{X} may not be uniquely determined by observing \mathbf{Y} . The following definition describes a class of word functions where \mathbf{X} can be uniquely recovered from \mathbf{Y} .

Definition 2 (Prefix-Free Word Function): A word function f is said to be prefix free if:

- (i) $f : \mathcal{A} \rightarrow \mathcal{C}$ is a bijection, and
- (ii) there does not exist two codewords c and c' in \mathcal{C} such that $c_i = c'_i$ for $i = 1, 2, \dots, \min\{|c|, |c'|\}$.

The distribution of the word-valued source \mathbf{Y} ,

$$q^{(n)}(b^n) = \Pr(Y_1 = b_1, Y_2 = b_2, \dots, Y_n = b_n) \ , \ n = 1, 2, \dots \ ,$$

may be calculated by combining the distribution of \mathbf{X} with f . With a slight abuse of notation, let $f^{-1}b^n$ denote the set of n -tuples a^n where the first n symbols of the n concatenated codewords $f(a_1), f(a_2), \dots, f(a_n)$ are equal to b^n ; that is,

$$f^{-1}b^n = \left\{ a^n \in \mathcal{A}^n : \phi_n(f(a_1), f(a_2), \dots, f(a_n)) = b^n \right\} \ ,$$

where $\phi_n : \bigcup_{n \leq m \leq nN} \mathcal{B}^m \rightarrow \mathcal{B}^n$ is the projection defined by $\phi_n(b_1, b_2, \dots, b_n, b_{n+1}, \dots, b_m) = b_1, b_2, \dots, b_n$. Using this notation, we have that

$$q^{(n)}(b^n) = \begin{cases} \sum_{a^n \in f^{-1}b^n} p^{(n)}(a^n), & \text{if } f^{-1}b^n \neq \emptyset \text{ and} \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

³By construction, we have that the length $|c|$ of each codeword $c \in \mathcal{C}$ is bound by $1 \leq |c| \leq N$. In practice, however, the restriction to codewords with finite length may not be suitable for all applications [1].

where \emptyset denotes the empty set.

Describing \mathbf{Y} directly with (3) is rather cumbersome, and it is more convenient to use a dynamical system that is formed by coding $(\mathcal{X}, \mathcal{F}(\mathcal{X}), \mu, T_{\mathcal{X}})$ with a sequence-to-sequence coder. To this end, let $\mathcal{Y} = \mathcal{B} \times \mathcal{B} \times \dots$ denote the collection of all sequences with elements from \mathcal{B} , let $\mathbf{b} = b_1, b_2, \dots$ denote an arbitrary member of \mathcal{Y} , and let $\mathcal{F}(\mathcal{Y})$ be the σ -field of subsets of \mathcal{Y} generated by cylinder sets. Now consider the sequence-to-sequence coder (measurable mapping) $F : \mathcal{X} \rightarrow \mathcal{Y}$ that is formed by setting $F(\mathbf{x}) = f(x_1), f(x_2), \dots$. When F acts on the abstract probability space $(\mathcal{X}, \mathcal{F}(\mathcal{X}), \mu)$, it induces a probability measure η on $(\mathcal{Y}, \mathcal{F}(\mathcal{Y}))$ [10, Ex. 9.4.3] [9, Pg. 80]. In particular, η and μ are related by

$$\eta(A) = \mu(F^{-1}A), \quad A \in \mathcal{F}(\mathcal{Y}), \quad (4)$$

where $F^{-1}A = \{\mathbf{x} \in \mathcal{X} : F(\mathbf{x}) \in A\}$ denotes the preimage of a set $A \in \mathcal{F}(\mathcal{Y})$ under F . Finally, when $(\mathcal{Y}, \mathcal{F}(\mathcal{Y}), \eta)$ is combined with the left-shift transform $T_{\mathcal{Y}}(\mathbf{y}) = y_2, y_3, \dots$ and the partition $\{[b] : b \in \mathcal{B}\}$ of \mathcal{Y} , the result is a dynamical system model $(\mathcal{Y}, \mathcal{F}(\mathcal{Y}), \eta, T_{\mathcal{Y}})$ for \mathbf{Y} . In particular, for each $n = 1, 2, \dots$ and $b^n \in \mathcal{B}^n$, we have that $\eta([b^n]) = \mu(F^{-1}[b^n]) = q^{(n)}(b^n)$.

Throughout the remainder of this paper, we shall use the following notation: $(\mathcal{X}, \mathcal{F}(\mathcal{X}), \mu, T_{\mathcal{X}})$ and \mathbf{X} will denote an arbitrary \mathcal{A} -valued random process; $f : \mathcal{A} \rightarrow \mathcal{C}$ will denote a word function; $F : \mathcal{X} \rightarrow \mathcal{Y}$ will denote the sequence-to-sequence coder generated by f ; and, $(\mathcal{Y}, \mathcal{F}(\mathcal{Y}), \eta, T_{\mathcal{Y}})$ and \mathbf{Y} will denote the word-valued source generated by coding $(\mathcal{X}, \mathcal{F}(\mathcal{X}), \mu, T_{\mathcal{X}})$ with F , where μ and η are related via (4). In addition, we will use $(\mathcal{W}, \mathcal{F}(\mathcal{W}), \rho, T)$ to represent an arbitrary dynamical system. Here it should always be understood that \mathcal{W} is the sequence space corresponding to some discrete-finite alphabet (an element of which will be written $\mathbf{w} = w_1, w_2, \dots$); $\mathcal{F}(\mathcal{W})$ is the σ -field generated by cylinder sets; ρ is a probability measure on $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$; and, $T : \mathcal{W} \rightarrow \mathcal{W}$ is an arbitrary measurable mapping. When we are explicitly interested in the special case where T is the left-shift transform, we shall use the notation $T_{\mathcal{W}}(\mathbf{w}) = w_2, w_3, \dots$.

III. A POINTWISE ERGODIC THEOREM

Theorem 1-A:

(i) *If the limit*

$$\langle g \rangle(\mathbf{x}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(T_{\mathcal{X}}^i(\mathbf{x})) \quad (5)$$

exists almost surely with respect to μ (a.s. $[\mu]$) for every bounded-measurable $g : \mathcal{X} \rightarrow (-\infty, \infty)$, then the limit

$$\langle \tilde{g} \rangle(\mathbf{y}) = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{j=0}^{m-1} \tilde{g}(T_{\mathcal{Y}}^j(\mathbf{y})) \quad (6)$$

exists a.s. $[\eta]$ for every bounded-measurable $\tilde{g} : \mathcal{Y} \rightarrow (-\infty, \infty)$. If f is prefix-free, then the reverse implication also holds.

- (ii) If the limit (5) exists and takes a constant value a.s. $[\mu]$ for every bounded-measurable $g : \mathcal{X} \rightarrow (-\infty, \infty)$, then the limit (6) exists and takes a constant value a.s. $[\eta]$ for every bounded-measurable $\tilde{g} : \mathcal{Y} \rightarrow (-\infty, \infty)$.

IV. ASYMPTOTICALLY MEAN STATIONARY RANDOM PROCESSES

Theorem 1-A may be restated in a more compact form using the language of asymptotically mean stationary random processes. For this purpose, let us recall the following definitions from Gray [6].

Consider a dynamical system $(\mathcal{W}, \mathcal{F}(\mathcal{W}), \rho, T)$, where $T : \mathcal{W} \rightarrow \mathcal{W}$ is an arbitrary measurable mapping. The system is said to be *stationary* if $\rho(A) = \rho(T^{-1}A)$ for every $A \in \mathcal{F}(\mathcal{W})$. A set $A \in \mathcal{F}(\mathcal{W})$ is said to be *T -invariant* if $A = T^{-1}A$. The system is said to be *ergodic* if $\rho(A) = 0$ or 1 for every T -invariant set A . Finally, the system is said to be *Asymptotically Mean Stationary* (AMS) if the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \rho(T^{-i}A)$$

exists for every $A \in \mathcal{F}(\mathcal{W})$, in which case the set function

$$\bar{\rho}(A) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \rho(T^{-i}A), \quad A \in \mathcal{F}(\mathcal{W}),$$

is a stationary probability measure on $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$; that is, the system $(\mathcal{W}, \mathcal{F}(\mathcal{W}), \bar{\rho}, T)$ is stationary. The measure $\bar{\rho}$ is called the *stationary mean* of ρ .

For brevity, we will say that the measure ρ is *T -stationary* / *T -ergodic* / *T -AMS* if the corresponding dynamical system is stationary / ergodic / AMS respectively. The next lemma gives necessary and sufficient conditions for a system to be ergodic and AMS.

Lemma 1:

- (i) The system $(\mathcal{W}, \mathcal{F}(\mathcal{W}), \rho, T)$ is AMS if and only if the limit

$$\langle g \rangle(\mathbf{w}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(T^i(\mathbf{w})) \quad (7)$$

exists a.s. $[\rho]$ for every bounded-measurable $g : \mathcal{W} \rightarrow (-\infty, \infty)$.

- (ii) The system $(\mathcal{W}, \mathcal{F}(\mathcal{W}), \rho, T)$ is ergodic if and only if the limit (7) takes a constant finite value a.s. $[\rho]$ for every bounded-measurable $g : \mathcal{W} \rightarrow (-\infty, \infty)$.

The AMS component of Lemma 1 was proved by Gray and Kieffer [8, Thm. 1], and the ergodic component follows from the definition of ergodicity [6, Sec. 6.7]. Using Lemma 1, we may restate Theorem 1-A as follows. A proof of this result can be found in Section VI.

Theorem 1-B:

- (i) If μ is $T_{\mathcal{X}}$ -AMS, then η is $T_{\mathcal{Y}}$ -AMS.
- (ii) If f is prefix-free, then η is $T_{\mathcal{Y}}$ -AMS if and only if μ is $T_{\mathcal{X}}$ -AMS.
- (iii) If μ is $T_{\mathcal{X}}$ -ergodic, then η is $T_{\mathcal{Y}}$ -ergodic.

V. AN ASYMPTOTIC EQUIPARTITION PROPERTY

In this section, we extend the AEP of [1, 2, 4] to the setting where μ is $T_{\mathcal{X}}$ -AMS and f is arbitrary. Two fundamental features of this extension will be the ergodic-decomposition theorem and the AEP for AMS random processes. We briefly review each of these ideas in Subsections V-A and V-B before stating our main results in Subsection V-C.

A. The Ergodic Decomposition Theorem

Suppose that $\mathbf{W} = W_1, W_2, \dots$ is a discrete-finite alphabet random process and $(\mathcal{W}, \mathcal{F}(\mathcal{W}), \rho, T_{\mathcal{W}})$ is the corresponding dynamical system in the sense of Proposition 1, where $T_{\mathcal{W}}(\mathbf{w}) = w_2, w_3, \dots$ is the left-shift transformation. For each set $A \in \mathcal{F}(\mathcal{W})$, let $\mathbf{1}_A$ denote its indicator function:

$$\mathbf{1}_A(\mathbf{w}) = \begin{cases} 1, & \text{if } \mathbf{w} \in A \\ 0, & \text{otherwise.} \end{cases}$$

When the limit exists, let

$$\langle \mathbf{1}_A \rangle(\mathbf{w}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_A(T_{\mathcal{W}}^i(\mathbf{w}))$$

denote the relative frequency of the set A in the sequence \mathbf{w} . Finally, for each bounded-measurable function $g : \mathcal{W} \rightarrow (-\infty, \infty)$, let $\mathbb{E}[\rho, g]$ denote its expected value:

$$\mathbb{E}[\rho, g] = \int g(\mathbf{w}) d\rho(\mathbf{w}) .$$

The pair $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$ belongs to a family of measurable spaces called standard spaces [6, Chap. 2]. A distinctive property of these spaces is that they possess a countable generating field [6, Cor. 2.2.1]. Let \mathcal{S} be a countable generating field for $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$. Now let $G(\mathcal{S})$ denote the collection of sequences \mathbf{w}

from \mathcal{W} such that the limit $\langle \mathbf{1}_A \rangle(\mathbf{w})$ exists for every generating set $A \in \mathcal{S}$. It can be shown that, for each $\mathbf{w} \in G(\mathcal{S})$, the set function $P_{\mathbf{w}}$ obtained by setting $P_{\mathbf{w}}(A) = \langle \mathbf{1}_A \rangle(\mathbf{w})$ induces a unique $T_{\mathcal{W}}$ -stationary probability measure $p_{\mathbf{w}}$ on $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$. Let E denote the set of sequences \mathbf{w} from $G(\mathcal{S})$ where the induced $T_{\mathcal{W}}$ -stationary probability measure $p_{\mathbf{w}}$ is also $T_{\mathcal{W}}$ -ergodic:

$$E = \{ \mathbf{w} \in \mathcal{W} : \mathbf{w} \in G(\mathcal{S}) \text{ and } p_{\mathbf{w}} \text{ is } T_{\mathcal{W}}\text{-ergodic} \} .$$

The set E is called the set of *ergodic sequences*. Finally, let p^* be an arbitrary $T_{\mathcal{W}}$ -stationary and $T_{\mathcal{W}}$ -ergodic probability measure on $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$, and for each sequence $\mathbf{w} \in \mathcal{W}$ define

$$\bar{p}_{\mathbf{w}} = \begin{cases} p_{\mathbf{w}}, & \text{if } \mathbf{w} \in E \\ p^*, & \text{otherwise.} \end{cases}$$

The collection of probability measures $\{\bar{p}_{\mathbf{w}} : \mathbf{w} \in \mathcal{W}\}$ is called the *ergodic decomposition* of $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$.

Lemma 2 (AMS Ergodic Decomposition Theorem [6, 9]): Let $\{\bar{p}_{\mathbf{w}} : \mathbf{w} \in \mathcal{W}\}$ be the ergodic decomposition of $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$ and E the set of ergodic sequences. Then,

- (i) the set E is $T_{\mathcal{W}}$ -invariant: $E = T_{\mathcal{W}}^{-1}E$,
- (ii) $\bar{p}_{\mathbf{w}}(A) = \bar{p}_{T_{\mathcal{W}}(\mathbf{w})}(A)$ for every set $A \in \mathcal{F}(\mathcal{W})$ and every sequence $\mathbf{w} \in \mathcal{W}$,
- (iii) for any pair \mathbf{w} and \mathbf{w}' , the probability measures $\bar{p}_{\mathbf{w}}$ and $\bar{p}_{\mathbf{w}'}$ are either identical or mutually singular.

Additionally, if ρ is T -AMS with stationary mean $\bar{\rho}$, then

- (iv) $\rho(E) = \bar{\rho}(E) = 1$,
- (v) for each set $A \in \mathcal{F}(\mathcal{W})$

$$\bar{\rho}(A) = \int \bar{p}_{\mathbf{w}}(A) d\rho(\mathbf{w}) ,$$

- (vi) the limit

$$\langle g \rangle(\mathbf{w}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(T_{\mathcal{W}}^i(\mathbf{w})) = \mathbb{E}[\bar{p}_{\mathbf{w}}, g]$$

holds a.s. $[\rho]$ for each bounded-measurable function $g : \mathcal{W} \rightarrow (-\infty, \infty)$.

B. An AEP for AMS Random Processes

As before, suppose that $\mathbf{W} = W_1, W_2, \dots$ is a discrete-finite alphabet random process and $(\mathcal{W}, \mathcal{F}(\mathcal{W}), \rho, T_{\mathcal{W}})$ is the corresponding dynamical system. For each sequence $\mathbf{w} \in \mathcal{W}$, the probability $\rho([w^n])$ is non-increasing in n . If ρ is $T_{\mathcal{W}}$ -AMS, then Gray and Kieffer's AEP [8] asserts that this decrease is exponential in n on a set of probability one; in particular, the (asymptotic) rate of decent is given by the

entropy rate of the underlying $T_{\mathcal{W}}$ -stationary and $T_{\mathcal{W}}$ -ergodic probability measure $\bar{\rho}_{\mathbf{w}}$ from the ergodic decomposition theorem. A formal statement of this idea is given in the next lemma. However, before this lemma is given, we briefly review the concepts of joint entropy, entropy rate and sample-entropy rate.

The *joint entropy* $H(W^n)$ of the first n -random variables W^n from \mathbf{W} is defined as [5]

$$H(W^n) = \sum_{w^n} \Pr[W^n = w^n] \log \frac{1}{\Pr[W^n = w^n]} .$$

With respect to the Kolmogorov measure ρ , we define the joint entropy of the first n random variables to be

$$H_n(\rho) = \sum_{w^n} \rho([w^n]) \log \frac{1}{\rho([w^n])} .$$

From Proposition 1, these functionals are consistent in that $H(W^n) = H_n(\rho)$. When the limit exists, the *entropy rate* of \mathbf{W} is defined as $\bar{H}(\mathbf{W}) = \lim_{n \rightarrow \infty} (1/n)H(W^n)$ [5, Chap. 4]. Similarly, we define the entropy rate of \mathbf{W} with respect to ρ to be $\bar{H}(\rho) = \lim_{n \rightarrow \infty} (1/n)H_n(\rho)$ when the limit exists. Finally, we define the *sample-entropy rate* of a sequence $\mathbf{w} \in \mathcal{W}$ with respect to ρ as

$$h(\rho, \mathbf{w}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{\rho([w^n])} ,$$

when the limit exists.

Lemma 3 (Asymptotic Equipartition Property [10]): Let $\{\bar{\rho}_{\mathbf{w}} : \mathbf{w} \in \mathcal{W}\}$ be the ergodic decomposition of $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$. If ρ is $T_{\mathcal{W}}$ -AMS with stationary mean $\bar{\rho}$, then there exists a set $\Omega \in \mathcal{F}(\mathcal{W})$ with probability $\rho(\Omega) = 1$ such that the sample-entropy rate $h(\rho, \mathbf{w})$ of any sequence $\mathbf{w} \in \Omega$ exists and is given by

$$h(\rho, \mathbf{w}) = \varphi(\mathbf{w}) , \tag{8}$$

where φ is the $T_{\mathcal{W}}$ -invariant function that is defined by $\varphi(\mathbf{w}) = \bar{H}(\bar{\rho}_{\mathbf{w}})$. Furthermore, the entropy rate of ρ exists and is given by

$$\bar{H}(\rho) = \bar{H}(\bar{\rho}) = \mathbb{E}[\rho, \varphi] .$$

Finally, if ρ is $T_{\mathcal{W}}$ -ergodic, then $h(\rho, \mathbf{w}) = \bar{H}(\rho) = \bar{H}(\bar{\rho})$ for every $\mathbf{w} \in \Omega$.

C. An AEP for Word Valued Sources

We now return to the problem of establishing an AEP for \mathbf{Y} . From Theorem 1-B and Lemma 3, it is clear that \mathbf{Y} satisfies an AEP whenever μ is $T_{\mathcal{X}}$ -AMS. It turns out, however, that not only does the limit $h(\eta, \mathbf{y})$ exist almost surely, but its value may also be bound from above by the entropy rate of \mathbf{X} normalized by the expected codeword length. We formalize this idea in the following theorem.

Theorem 2: Let $\{\bar{\mu}_{\mathbf{x}} : \mathbf{x} \in \mathcal{X}\}$ be the ergodic decomposition of $(\mathcal{X}, \mathcal{F}(\mathcal{X}))$. If μ is $T_{\mathcal{X}}$ -AMS, then η is $T_{\mathcal{Y}}$ -AMS and there exists a set $\Omega_x \in \mathcal{F}(\mathcal{X})$ with probability $\mu(\Omega_x) = 1$ such that, for every sequence $\mathbf{x} \in \Omega_x$, the sample-entropy rate $h(\eta, F(\mathbf{x}))$ of the word-valued sequence $F(\mathbf{x}) = f(x_1), f(x_2), \dots$ exists and is bound from above by

$$h(\eta, F(\mathbf{x})) \leq \frac{\overline{H}(\bar{\mu}_{\mathbf{x}})}{\mathbb{E}[\bar{\mu}_{\mathbf{x}}, l]}, \quad (9)$$

where $l : \mathcal{X} \rightarrow \{1, 2, \dots, N\}$ is given by $l(\mathbf{x}) = |f(x_1)|$. In addition, if f is prefix free, then the inequality in (9) becomes an equality.

A proof of Theorem 2 follows in Section VII. The next corollary demonstrates that if \mathbf{X} is AMS, then the entropy in each symbol of \mathbf{X} is conserved with respect to each stationary and ergodic sub-source from the ergodic-decomposition theorem. This behaviour is consistent with the entropy-conservation laws of variable-to-fixed length source codes [11, 12].

Corollary 2.1: If μ is $T_{\mathcal{X}}$ -AMS, then the entropy rate of η exists and is bound from above by

$$\overline{H}(\eta) \leq \int \frac{\overline{H}(\bar{\mu}_{\mathbf{x}})}{\mathbb{E}[\bar{\mu}_{\mathbf{x}}, l]} d\mu(\mathbf{x}). \quad (10)$$

In addition, if f is prefix-free, then the inequality in (10) becomes an equality.

Finally, the next corollary resolves the open problem reported in [1, 2, 4]: if \mathbf{X} is stationary and ergodic, then an AEP holds for \mathbf{Y} .

Corollary 2.2: If μ is $T_{\mathcal{X}}$ -stationary and $T_{\mathcal{X}}$ -ergodic, then η is $T_{\mathcal{Y}}$ -ergodic and

$$h(\eta, \mathbf{y}) \leq \frac{\overline{H}(\mu)}{\mathbb{E}[\mu, l]} \text{ a.s. } [\eta]. \quad (11)$$

In addition, if f is prefix-free, then the inequality in (11) becomes an equality.

VI. PROOF OF THEOREM 1

The proof of Theorem 1-B (and Theorem 1-A) will use Lemmas 4 through 9, which are given respectively in Subsections VI-A through VI-E. The forward and reverse implications of Theorem 1-B are proved in Subsections VI-F and VI-G respectively.

A. Subsequences, Weighted Sequences & Density

Suppose that $\zeta = \zeta_0, \zeta_1, \zeta_2, \dots$ is a strictly increasing subsequence in the non-negative integers $\mathbb{Z}^* = \{0, 1, 2, \dots\}$. Let $\xi = \xi_0, \xi_1, \xi_2, \dots$ be the *weight sequence* obtained from ζ by setting

$$\xi_n = \begin{cases} 1, & \text{if } n = \zeta_k \text{ for some } k = 0, 1, \dots \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

When the limit exists, the density d_ζ of ζ in \mathbb{Z}^* is defined as

$$d_\zeta = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi_i . \quad (13)$$

The next lemma follows directly from these definitions, e.g., see [13, Prop. 1.7].

Lemma 4: Suppose that ζ is a strictly increasing subsequence in \mathbb{Z}^ with density $d_\zeta > 0$ and weight sequence ξ . For any sequence $\mathbf{r} = r_0, r_1, \dots$ of real numbers, we have that*

$$d_\zeta \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^{k-1} r_{\zeta_j} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi_i r_i ;$$

that is, the existence of either limit implies the existence of the other.

B. Invariant Sets & Asymptotic Mean Stationarity

The next lemma gives some equivalence conditions for AMS dynamical systems.

Lemma 5 (Cor. 6.3.4, [6]; Thm. 2.2, [14]): For a dynamical system $(\mathcal{W}, \mathcal{F}(\mathcal{W}), \rho, T)$, the following statements are equivalent:

- (i) ρ is T -AMS.
- (ii) *There exists a T -stationary probability measure $\tilde{\rho}$ on $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$ such that $\tilde{\rho}$ asymptotically dominates ρ ; that is, $\tilde{\rho}(A) = 0$ implies $\lim_{n \rightarrow \infty} \rho(T^{-n}A) = 0$.*
- (iii) *The limit $\lim_{n \rightarrow \infty} (1/n) \sum_{i=0}^{n-1} g(T^i \mathbf{w})$ exists a.s. $[\rho]$ for every bounded-measurable $g : \mathcal{W} \rightarrow (-\infty, \infty)$. (See also Lemma 1.)*
- (iv) *There exists a T -stationary probability measure $\tilde{\rho}$ on $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$ such that $A = T^{-1}A$ and $\tilde{\rho}(A) = 0$ together imply that $\rho(A) = 0$.*

C. Stationary, Ergodic & AMS Sequence Coders

In Section II, we defined the word-valued source $(\mathcal{Y}, \mathcal{F}(\mathcal{Y}), \eta, T_{\mathcal{Y}})$ using a sequence coder $F : \mathcal{X} \rightarrow \mathcal{Y}$. In the proof of Theorem 1-B, it will be necessary to determine when such a sequence coder will transfer stationary / ergodic / AMS properties from the input to the output. For this purpose, we now review the notions of stationary, ergodic and AMS sequence coders.

Suppose that $(\mathcal{W}, \mathcal{F}(\mathcal{W}), \rho_\alpha, T_\alpha)$ and $(\mathcal{U}, \mathcal{F}(\mathcal{U}), \rho_\beta, T_\beta)$ are dynamical systems, where \mathcal{W} and \mathcal{U} are sequence spaces corresponding to some discrete-finite alphabets; $\mathcal{F}(\mathcal{W})$ and $\mathcal{F}(\mathcal{U})$ are σ -fields generated by cylinder sets; $T_\alpha : \mathcal{W} \rightarrow \mathcal{W}$ and $T_\beta : \mathcal{U} \rightarrow \mathcal{U}$ are arbitrary measurable maps; $G : \mathcal{W} \rightarrow \mathcal{U}$ is a sequence coder; ρ_α is a probability measure on $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$; and, ρ_β is induced by G

$$\rho_\beta(A) = \rho_\alpha(G^{-1}A) , \quad A \in \mathcal{F}(\mathcal{U}) .$$

The sequence coder G also induces a probability measure $\rho_{\alpha\beta}$ on the product space⁴ $(\mathcal{W} \times \mathcal{U}, \mathcal{F}(\mathcal{W}) \times \mathcal{F}(\mathcal{U}))$ via

$$\rho_{\alpha\beta}(A \times B) = \rho_{\alpha}(A \cap G^{-1}B), \quad A \in \mathcal{F}(\mathcal{W}), \quad B \in \mathcal{F}(\mathcal{U}).$$

The two shifts T_{α} and T_{β} together define a product shift $T_{\alpha\beta} : \mathcal{W} \times \mathcal{U} \rightarrow \mathcal{W} \times \mathcal{U}$ via $T_{\alpha\beta}(\mathbf{w}, \mathbf{u}) = (T_{\alpha}(\mathbf{w}), T_{\beta}(\mathbf{u}))$. The combination of $\rho_{\alpha\beta}$ and $T_{\alpha\beta}$ yields a dynamical system $(\mathcal{W} \times \mathcal{U}, \mathcal{F}(\mathcal{W}) \times \mathcal{F}(\mathcal{U}), \rho_{\alpha\beta}, T_{\alpha\beta})$.

The sequence coder G is said to be (T_{α}, T_{β}) -stationary / (T_{α}, T_{β}) -ergodic / (T_{α}, T_{β}) -AMS if, for any T_{α} -stationary / T_{α} -ergodic / T_{α} -AMS probability measure ρ_{α} , the induced measure $\rho_{\alpha\beta}$ is $T_{\alpha\beta}$ -stationary / $T_{\alpha\beta}$ -ergodic / $T_{\alpha\beta}$ -AMS.

Lemma 6 (Ex. 9.4.3, [10]): A sequence coder G is (T_{α}, T_{β}) -stationary if and only if $G(T_{\alpha}(\mathbf{w})) = T_{\beta}(G(\mathbf{w}))$.

Lemma 7 (Lems. 9.3.2 & 9.4.1, [10]): If G is (T_{α}, T_{β}) -stationary, then G is also (T_{α}, T_{β}) -ergodic and (T_{α}, T_{β}) -AMS.

We note in passing that the sequence coder F generated by the word function f is not $(T_{\mathcal{X}}, T_{\mathcal{Y}})$ -stationary. Thus, Theorem 1-B does not follow directly from Lemma 7. The additional result needed to prove Theorem 1-B is given in the next section.

D. AMS Processes & Variable Length Shifts

Suppose that \mathbf{W} is a discrete-finite alphabet random process and $(\mathcal{W}, \mathcal{F}(\mathcal{W}), \rho, T_{\mathcal{W}})$ is the corresponding dynamical system, where $T_{\mathcal{W}}(\mathbf{w}) = w_2, w_3, \dots$ is the left-shift transform. Now, suppose that N is a natural number and \mathbf{W} is parsed into a sequence of non-overlapping blocks of length N to form the block-valued process $\mathbf{W}^N = \{(W_{nN+1}, W_{nN+2}, \dots, W_{(n+1)N}); n = 0, 1, \dots\}$. I.e. \mathbf{W}^N is simply \mathbf{W} viewed in blocks of length N . The appropriate shift transform for \mathbf{W}^N is the N -block shift $T_{\mathcal{W}^N} : \mathcal{W} \rightarrow \mathcal{W}$ of Gray and Kieffer [8] (see also Gray and Saadat [7]), which is defined by

$$T_{\mathcal{W}^N}(\mathbf{w}) = T_{\mathcal{W}}^N(\mathbf{w}) = w_{N+1}, w_{N+2}, \dots$$

The following proposition shows that the AMS property transcends block-time scales.

⁴We use $\mathcal{F}(\mathcal{W}) \times \mathcal{F}(\mathcal{U})$ to denote the product σ -field induced by rectangles of the form $A \times B$, $A \in \mathcal{F}(\mathcal{W})$, $B \in \mathcal{F}(\mathcal{U})$ [15, Pg. 97].

Proposition 2 (Cor. 2.1, [7]): If ρ is $T_{\mathcal{W}^N}$ -AMS for any natural number N , then ρ is $T_{\mathcal{W}^M}$ -AMS for every natural number M .

Proposition 2 does not have analogues for stationary and / or ergodic random processes; it is a unique property of AMS random processes. We now extend this proposition to include the more general notion of “variable-length” parsing, which will be necessary for our study of word-valued sources.

Suppose now that \mathbf{W} is parsed into a sequence of non-overlapping blocks, where the length of each block is determined by a simple function $\gamma : \mathcal{W} \rightarrow \{1, 2, \dots, N\}$. The appropriate transform for this variable-length parsing is the variable-length shift of Gray and Kieffer [8, Ex. 6].

Definition 3 (Variable-Length Shift): Suppose that $\gamma : \mathcal{W} \rightarrow \{1, 2, \dots, N\}$ is a simple measurable function and that there exists a natural number M such that $\gamma(\mathbf{w}) = \gamma(\mathbf{w}')$ for every pair of sequences $\mathbf{w}, \mathbf{w}' \in \mathcal{W}$ with $w_i = w'_i$ for every $i = 1, 2, \dots, M$. The variable-length shift $T_{\mathcal{W}^\gamma} : \mathcal{W} \rightarrow \mathcal{W}$ generated by γ is defined by [8]

$$T_{\mathcal{W}^\gamma}(\mathbf{w}) = T_{\mathcal{W}^\gamma}^{\gamma(\mathbf{w})}(\mathbf{w}) = w_{\gamma(\mathbf{w})+1}, w_{\gamma(\mathbf{w})+2}, \dots$$

Our extension of Proposition 2 is given in the next lemma. This lemma will be the centrepiece of our proof of Theorem 1-B.

Lemma 8: If ρ is $T_{\mathcal{W}^\gamma}$ -AMS for any variable-length shift $T_{\mathcal{W}^\gamma} : \mathcal{W} \rightarrow \mathcal{W}$, then ρ is $T_{\mathcal{W}^\lambda}$ -AMS for every variable-length shift $T_{\mathcal{W}^\lambda} : \mathcal{W} \rightarrow \mathcal{W}$.

We note that Gray’s proof of Proposition 2 [6, Sec. 7.3] elegantly combines convergent subsequences with the notion of asymptotic dominance. It is not clear if this argument can be extended to prove the more general Lemma 8. Instead, we take a more laborious approach and prove the lemma by showing an ergodic theorem and applying Lemma 5 (iii).

Proof: We first show that if ρ is $T_{\mathcal{W}^\gamma}$ -AMS, then ρ must also be $T_{\mathcal{W}}$ -AMS. We then show that if ρ is $T_{\mathcal{W}}$ -AMS, then ρ must also be $T_{\mathcal{W}^\lambda}$ -AMS.

Assume that ρ is $T_{\mathcal{W}^\gamma}$ -AMS. From Lemma 5 (iv), there exists a $T_{\mathcal{W}^\gamma}$ -stationary probability measure $\bar{\rho}^\gamma$ on $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$ such that $T_{\mathcal{W}^\gamma}^{-1}A = A$ and $\bar{\rho}^\gamma(A) = 0$ together imply that $\rho(A) = 0$. Using the procedure given by Gray and Kieffer in [8, Ex. 6], it can be shown that $\bar{\rho}^\gamma$ is also $T_{\mathcal{W}}$ -AMS. A second application of Lemma 5 (iv) shows that there exists a $T_{\mathcal{W}}$ -stationary probability measure $\bar{\rho}$ on $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$ such that $T_{\mathcal{W}}^{-1}A = A$ and $\bar{\rho}(A) = 0$ together imply that $\bar{\rho}^\gamma(A) = 0$. Note also that if a set A is $T_{\mathcal{W}}$ -invariant, then it is also $T_{\mathcal{W}^\gamma}$ -invariant: $A = T_{\mathcal{W}}^{-1}A \Rightarrow A = T_{\mathcal{W}^\gamma}^{-1}A$. On combining these facts, we have the following: if $A = T_{\mathcal{W}}^{-1}A$ and $\bar{\rho}(A) = 0$, then it must be true that $\bar{\rho}^\gamma(A) = 0$, $A = T_{\mathcal{W}^\gamma}^{-1}A$ and $\rho(A) = 0$. Thus,

we have demonstrated the existence of a $T_{\mathcal{W}}$ -stationary probability measure $\bar{\rho}$ on $(\mathcal{W}, \mathcal{F}(\mathcal{W}))$ such that $T_{\mathcal{W}}^{-1}A = A$ and $\bar{\rho}(A) = 0$ together imply that $\rho(A) = 0$. A third application of Lemma 5 (iv) shows that ρ must indeed be $T_{\mathcal{W}}$ -AMS.

We now show: if ρ is $T_{\mathcal{W}}$ -AMS, then ρ must also be $T_{\mathcal{W}\lambda}$ -AMS. To do this, it will be useful to identify the orbit⁵ of $T_{\mathcal{W}\lambda}$ on each sequence $\mathbf{w} \in \mathcal{W}$ with a time subsequence $\zeta = \zeta_0, \zeta_1, \dots$. Namely, for each $n = 0, 1, \dots$ set ζ_n to be

$$\zeta_n = \begin{cases} 0, & \text{if } n = 0 \\ \sum_{i=0}^{n-1} \lambda(T_{\mathcal{W}\lambda}^i(\mathbf{w})), & \text{if } n \geq 1, \end{cases} \quad (14)$$

so, by construction, we have that

$$T_{\mathcal{W}\lambda}^n(\mathbf{w}) = w_{\zeta_n+1}, w_{\zeta_n+2}, \dots = T_{\mathcal{W}}^{\zeta_n}(\mathbf{w}). \quad (15)$$

Let $\xi = \xi_0, \xi_1, \dots$ be the weight sequence that corresponds to ζ , as given by (12). Since the length of each shift is at most N , the density d_ζ of ζ in \mathbb{Z}^* , as given by (13), can be no smaller than $1/N$ (when the limit exists).

Let \mathcal{U} denote the collection of all sequences with elements from $\{1, 2, \dots, N\}$, let $\mathcal{F}(\mathcal{U})$ be the σ -field on \mathcal{U} generated by cylinder sets, and let $T_{\mathcal{U}}(\mathbf{u}) = u_2, u_3, \dots$ be the left-shift transform. Let $\Lambda : \mathcal{W} \rightarrow \mathcal{U}$ be the mapping defined by

$$\Lambda(\mathbf{w}) = \lambda(\mathbf{w}), \lambda(T_{\mathcal{W}}(\mathbf{w})), \lambda(T_{\mathcal{W}}^2(\mathbf{w})), \dots$$

From Lemma 6, this mapping is $(T_{\mathcal{W}}, T_{\mathcal{U}})$ -stationary since $T_{\mathcal{U}}(\Lambda(\mathbf{w})) = \Lambda(T_{\mathcal{W}}(\mathbf{w}))$. Finally, from Lemma 7 the induced measure $\rho_{wu}(A \times B) = \rho(A \cap \Lambda^{-1}B)$ on $(\mathcal{W} \times \mathcal{U}, \mathcal{F}(\mathcal{W}) \times \mathcal{F}(\mathcal{U}))$ is $T_{\mathcal{W}\mathcal{U}}$ -AMS, where $T_{\mathcal{W}\mathcal{U}}(\mathbf{w}, \mathbf{u}) = (T_{\mathcal{W}}(\mathbf{w}), T_{\mathcal{U}}(\mathbf{u}))$.

Let \mathcal{Z} denote the collection of all sequences with elements from $\{0, 1\}$, let $\mathcal{F}(\mathcal{Z})$ be the σ -field generated by cylinder sets, and let $T_{\mathcal{Z}}(\mathbf{z}) = z_2, z_3, \dots$ be the left-shift transform. We now construct a finite-state coder $G : \mathcal{W} \times \mathcal{U} \rightarrow \mathcal{Z}$, which identifies the orbit of the variable-length shift $T_{\mathcal{W}\lambda}$. Define $\mathcal{G} = \{0, 1, \dots, N-1\}$ to be the internal state space of the coder, and define the state update function g_s and the output function g_o by

$$g_s(w, u, s) = \begin{cases} u - 1, & \text{if } s = 0 \\ s - 1, & \text{otherwise.} \end{cases}$$

$$g_o(w, u, s) = \begin{cases} 1, & \text{if } s = 0 \\ 0, & \text{otherwise.} \end{cases}$$

⁵The orbit of $T_{\mathcal{W}\lambda}$ on \mathbf{w} is the sequence of points $\mathbf{w}, T_{\mathcal{W}\lambda}(\mathbf{w}), T_{\mathcal{W}\lambda}^2(\mathbf{w}), \dots$ from \mathcal{W} .

Set $s_1 = 0$ and calculate the first output $z_1 = g_o(w_1, u_1, 0) = 1$. Update the state $s_2 = g_s(w_1, u_1, 0) = u_1 - 1$ and determine the next output $z_2 = g_o(w_2, u_2, u_1 - 1)$. Continue in this fashion to obtain the finite state coder $G : \mathcal{W} \times \mathcal{U} \rightarrow \mathcal{Z}$. As with sequence coders, the finite-state coder G is measurable and it induces a probability measure

$$\rho_{wuz}(A \times B \times C) = \rho_{wu}((A \times B) \cap G^{-1}C)$$

on $(\mathcal{W} \times \mathcal{U} \times \mathcal{Z}, \mathcal{F}(\mathcal{W}) \times \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{Z}))$. Moreover, this finite state coder is an example of a one-sided Markov channel [16], so it follows from⁶ [16, Thm. 6] that ρ_{wuz} is $T_{\mathcal{W}\mathcal{U}\mathcal{Z}}$ -AMS, where $T_{\mathcal{W}\mathcal{U}\mathcal{Z}}(\mathbf{w}, \mathbf{u}, \mathbf{z}) = (T_{\mathcal{W}}(\mathbf{w}), T_{\mathcal{U}}(\mathbf{u}), T_{\mathcal{Z}}(\mathbf{z}))$.

Consider the set

$$\Upsilon = \{(\mathbf{w}, \mathbf{u}, \mathbf{z}) : \mathbf{w} \in \mathcal{W}, \mathbf{u} = \Lambda(\mathbf{w}), \mathbf{z} = G(\mathbf{w}, \Lambda(\mathbf{w}))\}$$

It can be shown that Υ is measurable and $\rho_{wuz}(\Upsilon) = 1$. Suppose $(\mathbf{w}, \mathbf{u}, \mathbf{z}) \in \Upsilon$, ζ is the time subsequence from (14), and ξ is the weight sequence corresponding to ζ . If $\mathbf{1}_\lambda : \mathcal{W} \times \mathcal{U} \times \mathcal{Z} \rightarrow \{0, 1\}$ is the indicator function defined by

$$\mathbf{1}_\lambda(\mathbf{w}, \mathbf{u}, \mathbf{z}) = \begin{cases} 1, & \text{if } z_1 = 1 \\ 0, & \text{otherwise,} \end{cases}$$

then, by construction, we have that

$$\xi_i = \mathbf{1}_\lambda(T_{\mathcal{W}\mathcal{U}\mathcal{Z}}^i(\mathbf{w}, \mathbf{u}, \mathbf{z})) \quad (16)$$

for all $i = 0, 1, 2, \dots$. Moreover, the density of ζ is given by (if the limit exists)

$$\begin{aligned} d_\zeta &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi_i \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_\lambda(T_{\mathcal{W}\mathcal{U}\mathcal{Z}}^i(\mathbf{w}, \mathbf{u}, \mathbf{z})) \\ &= \langle \mathbf{1}_\lambda \rangle(\mathbf{w}, \mathbf{u}, \mathbf{z}) . \end{aligned} \quad (17)$$

Finally, since the length of each codeword is no more than L , it must be true that $d_\zeta \geq 1/L$ (when this limit exists.)

Since ρ_{wuz} is $T_{\mathcal{W}\mathcal{U}\mathcal{Z}}$ -AMS, it follows from Lemma 5 (iii) that there exists a subset Ω with probability $\rho_{wuz}(\Omega) = 1$ such that, for each $(\mathbf{w}, \mathbf{u}, \mathbf{z}) \in \Omega$, the limit

$$\langle g \rangle(\mathbf{w}, \mathbf{u}, \mathbf{z}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(T_{\mathcal{W}\mathcal{U}\mathcal{Z}}^i(\mathbf{w}, \mathbf{u}, \mathbf{z}))$$

⁶Example (b) from [16] demonstrates that a finite-state coder is a special case of a one-sided Markov channel.

exists for every bounded-measurable g . Since $\mathbf{1}_\lambda$ is bounded and measurable, this ergodic theorem guarantees the density (17) exists for every $(\mathbf{w}, \mathbf{u}, \mathbf{z}) \in \Omega \cap \Upsilon$.

Let $T_{\mathcal{W}\mathcal{U}\mathcal{Z}^\lambda}$ denote the variable-length shift on the product space $\mathcal{W} \times \mathcal{U} \times \mathcal{V}$ defined by

$$T_{\mathcal{W}\mathcal{U}\mathcal{Z}^\lambda}(\mathbf{w}, \mathbf{u}, \mathbf{z}) = T_{\mathcal{W}\mathcal{U}\mathcal{Z}}^{\lambda(\mathbf{w})}(\mathbf{w}, \mathbf{u}, \mathbf{z}) .$$

From (14), we have that $T_{\mathcal{W}\mathcal{U}\mathcal{Z}^\lambda}^n(\mathbf{w}, \mathbf{u}, \mathbf{z}) = T_{\mathcal{W}\mathcal{U}\mathcal{Z}}^{\zeta_n}(\mathbf{w}, \mathbf{u}, \mathbf{z})$ for all $n = 0, 1, 2, \dots$

If $g : \mathcal{W} \times \mathcal{U} \times \mathcal{Z} \rightarrow (-\infty, \infty)$ is bounded-measurable, then $\mathbf{1}_\lambda \times g$ is bounded and measurable, and for each $(\mathbf{w}, \mathbf{u}, \mathbf{z}) \in \Omega \cap \Upsilon$ the following limits will exist:

$$\begin{aligned} \langle \mathbf{1}_\lambda \times g \rangle &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_\lambda(T_{\mathcal{W}\mathcal{U}\mathcal{Z}}^i(\mathbf{w}, \mathbf{u}, \mathbf{z})) g(T_{\mathcal{W}\mathcal{U}\mathcal{Z}}^i(\mathbf{w}, \mathbf{u}, \mathbf{z})) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi_i g(T_{\mathcal{W}\mathcal{U}\mathcal{Z}}^i(\mathbf{w}, \mathbf{u}, \mathbf{z})) \end{aligned} \quad (18)$$

$$= d_\zeta \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{j=0}^{m-1} g(T_{\mathcal{W}\mathcal{U}\mathcal{Z}}^{\zeta_j}(\mathbf{w}, \mathbf{u}, \mathbf{z})) \quad (19)$$

$$= d_\zeta \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{j=0}^{m-1} g(T_{\mathcal{W}\mathcal{U}\mathcal{Z}^\lambda}^j(\mathbf{w}, \mathbf{u}, \mathbf{z})) , \quad (20)$$

where (18) follows from (16), (19) follows from Lemma 4, and (20) follows from (14). This chain of equalities guarantees the limit in (20) exists for every $(\mathbf{w}, \mathbf{u}, \mathbf{z}) \in \Omega \cap \Upsilon$. Since g is an arbitrary bounded measurable function, it follows from Lemma 5 (iii) that ρ_{wuz} is $T_{\mathcal{W}\mathcal{U}\mathcal{Z}^\lambda}$ -AMS. Finally, since ρ is a marginal of ρ_{wuz} , it follows that ρ is $T_{\mathcal{W}\mathcal{Z}^\lambda}$ -AMS. ■

E. Ergodic Processes & Variable Length Shifts

In Lemma 8, it was shown that an AMS random process remains AMS under all variable-length time shifts. The next lemma proves a weaker result for ergodic processes. Again, suppose that \mathbf{W} is a discrete-finite alphabet random process and $(\mathcal{W}, \mathcal{F}(\mathcal{W}), \rho, T_{\mathcal{W}})$ is the corresponding dynamical system.

Lemma 9: If ρ is $T_{\mathcal{W}^\gamma}$ -ergodic for some variable-length shift $T_{\mathcal{W}^\gamma} : \mathcal{W} \rightarrow \mathcal{W}$, then ρ is also $T_{\mathcal{W}}$ -ergodic.

Proof: If ρ is $T_{\mathcal{W}^\gamma}$ -ergodic and A is an $T_{\mathcal{W}^\gamma}$ -invariant set, then $\rho(A) = 0$ or 1. Since $A = T_{\mathcal{W}}^{-1}A$ implies that $A = T_{\mathcal{W}^\gamma}^{-1}A$, it follows that $\rho(A) = 0$ or 1 for every $T_{\mathcal{W}}$ -invariant set A . ■

F. Proof of Theorem 1-B (Forward Claim)

We now prove the forward claim of Theorem 1-B: if μ is $T_{\mathcal{X}}$ -AMS (and $T_{\mathcal{X}}$ -ergodic), then η is $T_{\mathcal{Y}}$ -AMS (and $T_{\mathcal{Y}}$ -ergodic). Let \mathcal{Z} denote the set of all sequences with elements from $\{1, 2, \dots, N\}$, let $\mathcal{F}(\mathcal{Z})$ denote the σ -field generated by cylinder sets, and let $T_{\mathcal{Z}}(\mathbf{z}) = z_2, z_3, \dots$ denote the left-shift transform. Using the word function f , define the mapping

$$\tilde{f}(x) = (f(x)_1, |f(x)|), (f(x)_2, |f(x)| - 1), \dots, (f(x)_{|f(x)|}, 1),$$

where $f(x)_j$, $1 \leq j \leq |f(x)|$, denotes the j^{th} symbol of the codeword $f(x)$. By construction, $\tilde{f}(x)$ couples the codeword $f(x)$ with a sequence of indices $|f(x)_1|, |f(x)_1| - 1, \dots, 1$, which mark the distance from the current symbol to the end of the codeword. Using \tilde{f} , define the sequence coder $\tilde{F} : \mathcal{X} \rightarrow \mathcal{Y} \times \mathcal{Z}$ via $\tilde{F}(\mathbf{x}) = \tilde{f}(x_1), \tilde{f}(x_2), \dots$. As before, this sequence coder induces a probability measure $\eta_{yz}(A \times B) = \mu(\tilde{F}^{-1}(A \times B))$ on $(\mathcal{Y} \times \mathcal{Z}, \mathcal{F}(\mathcal{Y}) \times \mathcal{F}(\mathcal{Z}))$. Let $T_{\mathcal{Y}\mathcal{Z}}(\mathbf{y}, \mathbf{z}) = (T_{\mathcal{Y}}(\mathbf{y}), T_{\mathcal{Z}}(\mathbf{z}))$, and let $T_{\mathcal{Y}\mathcal{Z}\gamma}$ be the variable-length shift defined by setting $\gamma(\mathbf{y}, \mathbf{z}) = z_1$. Since

$$\tilde{F}(T_{\mathcal{X}}(\mathbf{x})) = T_{\mathcal{Y}\mathcal{Z}\gamma}(\tilde{F}(\mathbf{x})).$$

it follows from Lemma 6 that \tilde{F} is a $(T_{\mathcal{X}}, T_{\mathcal{Y}\mathcal{Z}\gamma})$ -stationary sequence coder. Since μ is $T_{\mathcal{X}}$ -AMS (and $T_{\mathcal{X}}$ -ergodic), we have from Lemma 7 that η_{yz} is $T_{\mathcal{Y}\mathcal{Z}\gamma}$ -AMS (and $T_{\mathcal{Y}\mathcal{Z}\gamma}$ -ergodic). Finally, from Lemmas 8 and 9, we can see that η_{yz} must also be $T_{\mathcal{Y}\mathcal{Z}}$ -AMS (and $T_{\mathcal{Y}\mathcal{Z}}$ -ergodic); therefore, η must be $T_{\mathcal{Y}}$ -AMS (and $T_{\mathcal{Y}}$ -ergodic).

G. Proof of Theorem 1-B (Reverse Claim)

We now prove the reverse claim of Theorem 1-B: if η is $T_{\mathcal{Y}}$ -AMS and f is prefix-free, then μ is $T_{\mathcal{X}}$ -AMS. Define the variable-length shift $T_{\mathcal{Y}\gamma} : \mathcal{Y} \rightarrow \mathcal{Y}$ by setting

$$\gamma(\mathbf{y}) = \begin{cases} |c|, & \text{if there exists a unique } c \in \mathcal{C} \text{ such that } y_i = c_i \\ & \text{for all } i = 1, 2, \dots, |c|. \\ 1, & \text{otherwise.} \end{cases}$$

From Lemma 7, it follows that η is $T_{\mathcal{Y}\gamma}$ -AMS.

Define

$$\Omega = \{\mathbf{y} \in \mathcal{Y} : \text{there exists } \mathbf{x} \in \mathcal{X} \text{ such that } \mathbf{y} = F(\mathbf{x})\},$$

where it can be shown that $\Omega \in \mathcal{F}(\mathcal{Y})$ and $\eta(\Omega) = 1$.

Let $g : \mathcal{C} \rightarrow \mathcal{A}$ denote the inverse of f . If \mathbf{y} is in Ω , then there exists a unique sequence of codewords c_1, c_2, \dots from \mathcal{C} such that $\mathbf{y} = c_1, c_2, \dots$. Therefore, using g , we may define the sequence-coder $G : \Omega \rightarrow \mathcal{X}$ by setting $G(\mathbf{y}) = F^{-1}(c_1, c_2, \dots) = g(c_1), g(c_2), \dots$.

For each $\mathbf{y} \in \Omega$ we have that $G(T_{\mathcal{Y}^\gamma}(\mathbf{y})) = T_{\mathcal{X}}(G(\mathbf{y}))$, so it follows from Lemma 6 that G is a $(T_{\mathcal{Y}^\gamma}, T_{\mathcal{X}})$ -stationary sequence coder. From Lemma 6, the induced probability measure $\tilde{\mu}(A) = \eta(G^{-1}A)$ on $(\mathcal{X}, \mathcal{F}(\mathcal{X}))$ is $T_{\mathcal{X}}$ -AMS. Since $\tilde{\mu}(A) = \eta(G^{-1}A) = \mu(F^{-1}G^{-1}A) = \mu(A)$ for each $A \in \mathcal{F}(\mathcal{X})$, it follows that μ is $T_{\mathcal{X}}$ -AMS. \blacksquare

VII. PROOF OF THEOREM 2 & COROLLARIES

A. Proof of Theorem 2

Let $\{\bar{\mu}_{\mathbf{x}} : \mathbf{x} \in \mathcal{X}\}$ and $\{\bar{\eta}_{\mathbf{y}} : \mathbf{y} \in \mathcal{Y}\}$ be the ergodic decompositions of $(\mathcal{X}, \mathcal{F}(\mathcal{X}))$ and $(\mathcal{Y}, \mathcal{F}(\mathcal{Y}))$ respectively. For each $n = 1, 2, \dots$, let $\phi_n : \mathcal{Y} \rightarrow \mathcal{B}^n$ be the projection $\phi_n(\mathbf{y}) = y_1, y_2, \dots, y_n$. From Lemma 3, there exists a subset $\Omega_{x,1} \in \mathcal{F}(\mathcal{X})$ with probability $\mu(\Omega_{x,1}) = 1$ such that the sample-entropy rate of each sequence $\mathbf{x} \in \Omega_{x,1}$ exists and is given by $h(\mu, \mathbf{x}) = \varphi_x(\mathbf{x})$, where $\varphi_x(\mathbf{x}) = \overline{H}(\bar{\mu}_{\mathbf{x}})$. Similarly, there exists a subset $\Omega_y \in \mathcal{F}(\mathcal{Y})$ with probability $\eta(\Omega_y) = 1$ such that the sample-entropy rate of each sequence $\mathbf{y} \in \Omega_y$ exists and is given by $h(\eta, \mathbf{y}) = \varphi_y(\mathbf{y})$, where $\varphi_y(\mathbf{y}) = \overline{H}(\bar{\eta}_{\mathbf{y}})$. Finally, from Lemma 2 there exists a subset $\Omega_{x,2} \in \mathcal{F}(\mathcal{X})$ with probability $\mu(\Omega_{x,2}) = 1$ such that for each sequence $\mathbf{x} \in \Omega_{x,2}$ the time-averaged codeword-length exists and is given by

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n |f(x_i)| = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} l(T_{\mathcal{X}}^i(\mathbf{x})) = \mathbb{E}[\bar{\mu}_{\mathbf{x}}, l] .$$

For each $\mathbf{x} \in \mathcal{X}$, define the time subsequence $\zeta = \zeta_0, \zeta_1, \dots$ by setting

$$\zeta_n = \begin{cases} 0, & \text{if } n = 0 \\ \sum_{i=1}^n |f(x_i)|, & \text{if } n \geq 1 . \end{cases}$$

For each $n = 1, 2, \dots$, we have that $F^{-1}[\phi_{\zeta_n}(F(\mathbf{x}))] \supseteq [x^n]$, with set equality if f is prefix free. This implies

$$\frac{1}{n} \log_2 \frac{1}{\mu([x^n])} \geq \frac{\zeta_n}{n} \frac{1}{\zeta_n} \log_2 \frac{1}{\eta([\phi_{\zeta_n}(F(\mathbf{x}))])} , \quad (21)$$

with equality if f is prefix free. Furthermore,

$$\frac{1}{\zeta_n} \log_2 \frac{1}{\eta([\phi_{\zeta_n}(F(\mathbf{x}))])} , \quad n = 1, 2, \dots , \quad (22)$$

is a subsequence of

$$\frac{1}{n} \log_2 \frac{1}{\eta([\phi_n(F(\mathbf{x}))])} , \quad n = 1, 2, \dots ; \quad (23)$$

thus, if $\mathbf{x} \in F^{-1}\Omega_y$, then (22) and (23) both converge to $\varphi_y(F(\mathbf{x}))$ as $n \rightarrow \infty$. To complete the proof, note that Theorem 2 follows from (21) since $\lim_{n \rightarrow \infty} \zeta_n/n = \mathbb{E}[\bar{\mu}_{\mathbf{x}}, l]$, $\lim_{n \rightarrow \infty} -(1/n) \log_2 \mu([x^n]) = \bar{H}(\bar{\mu}_{\mathbf{x}})$ and $\lim_{n \rightarrow \infty} -(1/n) \log_2 \eta([\phi_{\zeta_n}(F(\mathbf{x}))])$ exists for every $\mathbf{x} \in \Omega_{x,1} \cap \Omega_{x,2} \cap F^{-1}\Omega_y$. ■

B. Proof of Corollary 2.1

Let $\{\bar{\mu}_{\mathbf{x}} : \mathbf{x} \in \mathcal{X}\}$ and $\{\bar{\eta}_{\mathbf{y}} : \mathbf{y} \in \mathcal{Y}\}$ be the ergodic decompositions of $(\mathcal{X}, \mathcal{F}(\mathcal{X}))$ and $(\mathcal{Y}, \mathcal{F}(\mathcal{Y}))$ respectively. As usual, define $\varphi_x(\mathbf{x}) = \bar{H}(\bar{\mu}_{\mathbf{x}})$ and $\varphi_y(\mathbf{y}) = \bar{H}(\bar{\eta}_{\mathbf{y}})$. Now define $\tilde{\varphi}_x(\mathbf{x}) = \varphi_y(F(\mathbf{x}))$ and

$$g(\mathbf{x}) = \frac{\varphi_{\mathbf{x}}(\mathbf{x})}{\mathbb{E}[\bar{\mu}_{\mathbf{x}}, l]} .$$

Suppose μ is $T_{\mathcal{X}}$ -AMS. From Theorem 2, we have that η is $T_{\mathcal{Y}}$ -AMS and $\tilde{\varphi}_x(\mathbf{x}) \leq g(\mathbf{x})$ on a set Ω_x of probability $\mu(\Omega_x) = 1$ (with equality if f is prefix-free). Therefore,

$$\int \tilde{\varphi}_x(\mathbf{x}) d\mu(\mathbf{x}) \leq \int g(\mathbf{x}) d\mu(\mathbf{x}) . \quad (24)$$

Note, the R.H.S. of (24) is equal to the R.H.S. of (10). By the change of variables formula [6, Lem. 4.4.7] and Lemma 3, we have

$$\int \tilde{\varphi}_x(\mathbf{x}) d\mu(\mathbf{x}) = \int \varphi_y(\mathbf{y}) d\eta(\mathbf{y}) = \bar{H}(\eta) . \quad (25)$$

which is the desired result. ■

C. Proof of Corollary 2.2

Suppose that μ is $T_{\mathcal{X}}$ -stationary and $T_{\mathcal{X}}$ -ergodic. From Theorem 1-B, η is $T_{\mathcal{Y}}$ -ergodic. From Lemma 3, there exists a subset $\Omega_y \in \mathcal{F}(\mathcal{Y})$ with probability $\eta(\Omega_y) = 1$ such that the sample-entropy rate of each sequence $\mathbf{y} \in \Omega_y$ takes the same constant value $h(\eta, \mathbf{y}) = \bar{H}(\eta)$. From Theorem 2, there exists a subset $\Omega_x \in \mathcal{F}(\mathcal{X})$ with probability $\mu(\Omega_x) = 1$ such that the sample-entropy rate of each coded sequence $F(\mathbf{x})$, $\mathbf{x} \in \Omega_x$, exists and is bound from above by

$$h(\eta, F(\mathbf{x})) \leq \frac{\bar{H}(\bar{\mu}_{\mathbf{x}})}{\mathbb{E}[\bar{\mu}_{\mathbf{x}}, l]} . \quad (26)$$

Since $F^{-1}\Omega_y \cap \Omega_x \neq \emptyset$, there exists $\mathbf{x} \in \Omega_x$ and $\mathbf{y} \in \Omega_y$ such that $\mathbf{y} = F(\mathbf{x})$ and

$$h(\eta, \mathbf{y}) \leq \frac{\bar{H}(\bar{\mu}_{\mathbf{x}})}{\mathbb{E}[\bar{\mu}_{\mathbf{x}}, l]} = \frac{\bar{H}(\mu)}{\mathbb{E}[\mu, l]} \quad (27)$$

where the R.H.S. equality in (27) follows from the fact that μ is $T_{\mathcal{X}}$ -stationary and $T_{\mathcal{X}}$ -ergodic. The result follows since $h(\eta, \mathbf{y})$ exists and takes the constant value $\bar{H}(\eta)$ on Ω_y . Finally, note that for prefix-free codes (26) and therefore (27) are equalities. ■

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