# Capacity Regions and Sum-Rate Capacities of Vector Gaussian Interference Channels 

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#### Abstract

The capacity regions of vector, or multiple-input multiple-output, Gaussian interference channels are established for very strong interference and aligned strong interference. Furthermore, the sum-rate capacities are established for Z interference, noisy interference, and mixed (aligned weak/intermediate and aligned strong) interference. These results generalize known results for scalar Gaussian interference channels.


## I. Introduction

The interference channel (IC) models the situation in which transmitters communicate with their respective receivers while generating interference to all other receivers. This channel model was mentioned in [1, Section 14] and its capacity region is still generally unknown.

In [2] Carleial showed that interference does not reduce capacity when it is very strong. This result follows because the interference can be decoded and subtracted at each receiver before decoding the desired message. Later Han and Kobayashi [3] and Sato [4] showed that the capacity region of the strong interference channel is the same as the capacity region of a compound multiple access channel. In these cases, the interference is fully decoded at both receivers.

[^0]

Fig. 1. The two-user MIMO IC.

When the interference is not strong, the capacity region is unknown. The best inner bound is by Han and Kobayashi [3], which was later simplified by Chong et al. in [5] and [6]. Etkin et al. and Telatar and Tse showed that Han and Kobayashi's inner bound is within one bit of the capacity region for scalar Gaussian ICs [7] and [8]. Various outer bounds have been developed in [7]-[12].

Special ICs such as the degraded IC and the Z interference channel (ZIC) were studied in [13] and [14]. Costa proved that the capacity regions of degraded ICs and ZICs are the same for the scalar Gaussian case [14]. The sum-rate capacity for the ZIC was established in [13] and [15]. A recent result in [10]-[12] showed that if a two-user Gaussian scalar IC has noisy interference, then treating interference as noise can achieve the sum-rate capacity. This result has been extended to multi-user Gaussian ICs in [16] and [12]. The sum-rate capacity for mixed interference, i.e., one receiver has strong interference and the other has weak/intermediate interference, was derived in [11] and [17].

In this paper, we study the capacity of the two-user Gaussian vector IC or multiple-input multiple-output (MIMO) IC. As shown in Fig. 1, the received signals are defined as

$$
\begin{align*}
& \boldsymbol{y}_{1}=\mathbf{H}_{1} \boldsymbol{x}_{1}+\mathbf{H}_{2} \boldsymbol{x}_{2}+\boldsymbol{z}_{1} \quad \text { and } \\
& \boldsymbol{y}_{2}=\mathbf{H}_{3} \boldsymbol{x}_{1}+\mathbf{H}_{4} \boldsymbol{x}_{4}+\boldsymbol{z}_{2}, \tag{1}
\end{align*}
$$

where $\boldsymbol{x}_{i}, i=1,2$, is the transmitted (column) vector signal of user $i$ which is subject to the average covariance matrix constraint

$$
\begin{equation*}
\sum_{j=1}^{n} E\left[\boldsymbol{x}_{i j} \boldsymbol{x}_{i j}^{\dagger}\right] \preceq n \mathbf{S}_{i}, \tag{2}
\end{equation*}
$$

where $\boldsymbol{x}_{i 1}, \boldsymbol{x}_{i 2}, \ldots, \boldsymbol{x}_{i n}$, is the transmitted vector sequence of user $i$, and $\mathbf{S}_{i}$ is a fixed positive semidefinite matrix. Inequality $\mathbf{A} \preceq \mathbf{B}$ means that $\mathbf{A}-\mathbf{B}$ is Hermitian positive semi-definite. The noise $z_{i}$ is a circularly symmetric complex Gaussian random vector with zero mean and identity covariance matrix;
and $\mathbf{H}_{k}, j=1, \ldots, 4$, are the complex channel matrices known at both the transmitters and receivers. Transmitter $i$ has $t_{i}$ antennas and receiver $i$ has $r_{i}$ antennas.

For the MIMO IC, Telatar and Tse [8] showed that Han and Kobayashi's region is within one bit per receive antenna of the capacity region. Some outer bounds for the capacity region were discussed in [18] and some lower bounds for the sum-rate capacity based on Han and Kobayashi's region were given in [19]. Recent work in [20] and [21] extended the existing capacity results from scalar ICs to MIMO ICs under average power constraints. Specifically, [20] and [21] derived the capacity region for aligned strong interference, and the sum-rate capacity for Z interference, noisy interference and mixed interference under average power constraints. It should be noted that some of the results in [20] and [21] require the channel matrices to be square and invertible, and the noisy-interference sum-rate capacity is obtained by requiring all possible covariance matrices of $x_{1}$ and $x_{2}$ to satisfy a condition. A partially strengthened noisy interference condition for MIMO ICs was later presented in [22] which required only that the optimizing covariance matrices of $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$ satisfy the condition of [20] and [21], as long as these optimizing covariance matrices have full rank (see [22, Remarks 2 and 3 and Theorem 1]). A special case of the MIMO IC, the so-called parallel Gaussian IC where the $\mathbf{H}_{i}$ 's are all square and diagonal matrices, was studied in [23] and [24], and it was shown that under suitable conditions for channel matrices and the power constraints, separate coding among antennas (or the transmit vector entries) and treating interference as noise achieves the sum-rate capacity. In addition, the optimal covariance matrices can be singular for this special case. Using the result of [25] that beamforming is optimal for the single-user detection rate region of the multiple-input single-output (MISO) IC, [22] derived noisy-interference sumrate capacities for symmetric MISO ICs, i.e., the $\mathbf{H}_{j}, j=1, \cdots, 4$, are all row vectors with $\mathbf{H}_{1}=\mathbf{H}_{4}$, $\mathbf{H}_{2}=\mathbf{H}_{3}$ and the two users have identical power constraints.

In this paper, we use the covariance matrix constraint (2) and derive the sum-rate capacity of the MIMO IC with noisy interference, mixed aligned interference, as well as one-sided interference. The capacity regions of the MIMO IC with very strong interference and aligned strong interference are also obtained. For all the results, $\mathbf{S}_{i}, i=1,2$, can be any positive semi-definite matrix, and the channel matrices $\mathbf{H}_{j}$, $j=1, \cdots, 4$, can be singular or non-square unless otherwise specified.

The rest of the paper is organized as follows: we present our main results and numerical examples in Sections II and III, and the proofs of the main results are given in Section IV.

Before proceeding, we introduce some notation that will be used in the paper.

- Italic letters (e.g. $X$ ) denote scalars; and bold letters $\boldsymbol{x}$ and $\mathbf{X}$ denote column vectors and matrices, respectively.
- I denotes the identity matrix and $\mathbf{0}$ denotes the all-zero matrix.
- $|\mathbf{X}|, \mathbf{X}^{\dagger}$ and $\mathbf{X}^{-1}$ denote respectively the determinant, conjugate transpose, and inverse of the matrix $\mathbf{X}$, and $\|x\|$ denotes the Euclidean vector norm of $x$.
- $\operatorname{radius}(\mathbf{X})$ is the numerical radius [26, p.g. 321] of the square matrix $\mathbf{X}$, and is defined as

$$
\operatorname{radius}(\mathbf{X})=\max _{\boldsymbol{\alpha}^{\dagger} \boldsymbol{\alpha} \leq 1} a b s\left(\boldsymbol{\alpha}^{\dagger} \mathbf{X} \boldsymbol{\alpha}\right)
$$

where $\boldsymbol{\alpha}$ is a complex vector, and $a b s(\cdot)$ denotes the absolute value.

- $\boldsymbol{x}^{n}=\left[x_{1}^{\dagger}, \boldsymbol{x}_{2}^{\dagger}, \ldots, \boldsymbol{x}_{n}^{\dagger}\right]^{\dagger}$ is a long vector which consists of a sequence of vectors $\boldsymbol{x}_{i}, i=1, \ldots, n$.
- $\boldsymbol{x} \sim \mathcal{C N}(\mathbf{0}, \boldsymbol{\Sigma})$ means that the random vector $\boldsymbol{x}$ has the circularly symmetric complex Gaussian distribution with zero mean and covariance matrix $\boldsymbol{\Sigma}$.
- $E[\cdot]$ denotes expectation; $\operatorname{Cov}(\cdot)$ denotes covariance matrix; $I(\cdot ; \cdot)$ denotes mutual information; $h(\cdot)$ denotes differential entropy with the logarithm base $e$, and $\log (\cdot)=\log _{e}(\cdot)$.


## II. Main Results

In this section, we give the capacity regions for MIMO ICs under very strong interference and aligned strong interference, and the sum-rate capacities for MIMO ICs under Z interference, noisy interference and mixed interference.

For economy of notation, we introduce a set of matrices

$$
\begin{equation*}
\mathcal{B}_{i}=\left\{\mathbf{B} \mid \text { all columns of } \mathbf{B}^{\dagger} \text { are in the null space of } \mathbf{S}_{i}\right\}, \quad i=1,2 \tag{3}
\end{equation*}
$$

i.e., each column of $\mathbf{B}^{\dagger}$ is either a zero vector, or an eigenvector of the covariance matrix constraint $\mathbf{S}_{i}$ associated with the zero eigenvalue (if $\mathbf{S}_{i}$ has one). This condition is equivalent to the condition $\mathbf{S}_{i} \mathbf{B}^{\dagger}=\mathbf{0}$.

## A. Capacity region of MIMO IC under very strong interference

We begin with the result for the MIMO ZIC (MIMO IC with one-sided interference) with very strong interference.

Theorem 1: For the MIMO IC defined in (1) with $\mathbf{H}_{3}=0$, if

$$
\begin{equation*}
\log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right|-\log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\right| \geq \log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right| \tag{4}
\end{equation*}
$$

then the capacity region of the MIMO IC is

$$
\begin{equation*}
\left\{\left(R_{1}, R_{2}\right): \quad 0 \leq R_{1} \leq \log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\right|, \quad 0 \leq R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|\right\} \tag{5}
\end{equation*}
$$

where $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$ are the respective covariance matrix constraints defined in (2).
We say that a MIMO ZIC has very strong interference if (4) is satisfied. In this case the interference does not reduce the capacity region. Theorem 1 can be easily extended to obtain the capacity region for a two-sided MIMO IC under very strong interference.

Theorem 2: For the MIMO IC defined in (1) and $\mathbf{H}_{2} \neq \mathbf{0}$ and $\mathbf{H}_{3} \neq \mathbf{0}$, if

$$
\begin{align*}
& \log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right|-\log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\right| \geq \log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|  \tag{6}\\
& \log \left|\mathbf{I}+\mathbf{H}_{3} \mathbf{S}_{1} \mathbf{H}_{3}^{\dagger}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|-\log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right| \geq \log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\right|, \tag{7}
\end{align*}
$$

then the capacity region of the MIMO IC is

$$
\begin{equation*}
\left\{\left(R_{1}, R_{2}\right): \quad 0 \leq R_{1} \leq \log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\right|, \quad 0 \leq R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|\right\} \tag{8}
\end{equation*}
$$

where $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$ are the respective covariance matrix constraints defined in (2).
Inequalities (6) and (7) are the very strong interference conditions for a two-sided MIMO IC, which means that when both users transmit at the maximum rate, both receivers can first decode the interference by treating the desired signal as noise, i.e., we have

$$
\begin{aligned}
& I\left(\boldsymbol{x}_{2}^{*} ; \boldsymbol{y}_{1}^{*}\right) \geq I\left(\boldsymbol{x}_{2}^{*} ; \boldsymbol{y}_{2}^{*} \mid \boldsymbol{x}_{1}^{*}\right) \quad \text { and } \\
& I\left(\boldsymbol{x}_{1}^{*} ; \boldsymbol{y}_{2}^{*}\right) \geq I\left(\boldsymbol{x}_{1}^{*} ; \boldsymbol{y}_{1}^{*} \mid \boldsymbol{x}_{2}^{*}\right),
\end{aligned}
$$

where $\boldsymbol{x}_{i}^{*} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{S}_{i}\right)$ and $\boldsymbol{y}_{i}^{*}$ is defined in (1) with $\boldsymbol{x}_{i}$ replaced by $\boldsymbol{x}_{i}^{*}, i=1,2$. As with the scalar Gaussian IC where the notion of very strong interference depends on both the channel coefficients and power constraints, for the MIMO IC our definition of very strong interference involves both the channel matrices and the covariance matrix constraints. Let $\mathbf{H}_{1}=\mathbf{H}_{4}=1, \mathbf{H}_{2}=\sqrt{a}, \mathbf{H}_{3}=\sqrt{b}, \mathbf{S}_{1}=P_{1}$ and $\mathbf{S}_{2}=P_{2}$, then (6) and (7) become $a \geq 1+P_{1}$ and $b \geq 1+P_{2}$, respectively. Therefore, Theorem 2 generalizes the capacity region for scalar Gaussian ICs under very strong interference [2].

We remark that an alternative definition of MIMO with very strong interference is to use the power constraints instead of the the covariance matrix constraints. The conditions as well the corresponding capacity region have the same expression as that of Theorem 2 except that $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$ are now replaced with the waterfilling covariance matrices for the two intended links in the absence of interference. This alternative definition gives a capacity region that is a superset of that defined using the covariance matrix constraints with the trace of the covariance matrices being equal to the power constraints. This alternative definition also includes the scalar Gaussian ICs under very strong interference as its special case.

## B. Capacity region of MIMO IC under aligned strong interference

We begin with the result for the MIMO ZIC under aligned strong interference.
Theorem 3: For the MIMO IC defined in (1) with $\mathbf{H}_{3}=0$, if there exist matrices $\mathbf{A}$ and $\mathbf{B}$ such that

$$
\begin{equation*}
\mathbf{H}_{4}=\mathbf{A H}_{2}+\mathbf{B}, \tag{9}
\end{equation*}
$$

where $\mathbf{A}^{\dagger} \mathbf{A} \preceq \mathbf{I}$ and $\mathbf{B} \in \mathcal{B}_{2}$, then the capacity region of the MIMO IC is

$$
\left\{\begin{array}{c}
0 \leq R_{1} \leq \log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\right|  \tag{10}\\
0 \leq R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right| \\
R_{1}+R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right|
\end{array}\right\}
$$

where $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$ are the respective covariance matrix constraints defined in (2).
Theorem 3 gives the capacity region of a MIMO ZIC under aligned strong interference. If $\mathbf{S}_{2}$ is singular, then (9) means that all the columns of $\mathbf{H}_{4}^{\dagger}-\mathbf{H}_{2}^{\dagger} \mathbf{A}^{\dagger}$ are either zero vectors or the eigenvectors of $\mathbf{S}_{2}$ associated with eigenvalue 0 . If $\mathbf{S}_{2}$ is nonsingular, then $\mathbf{H}_{4}=\mathbf{A H}$, i.e., $\mathbf{H}_{4}$ is a linear transformation of $\mathbf{H}_{2}$. Therefore, users 1 and 2 see $\boldsymbol{x}_{2}$ in the forms of $\mathbf{H}_{2} \boldsymbol{x}_{2}$ and $\mathbf{A H}_{2} \boldsymbol{x}_{2}$, respectively. If $\mathbf{A}^{\dagger} \mathbf{A} \preceq \mathbf{I}$, then user 1 can decode $x_{2}$ if user 2 can.

The following is a special case of Theorem 3 where we can choose A explicitly as

$$
\begin{equation*}
\mathbf{A}=\left(\mathbf{H}_{4}-\mathbf{B}\right)\left(\mathbf{H}_{2}^{\dagger} \mathbf{H}_{2}\right)^{-1} \mathbf{H}_{2}^{\dagger} \tag{11}
\end{equation*}
$$

Proposition 1: For the MIMO IC defined in (1) with $\mathbf{H}_{3}=0$, if $\mathbf{H}_{2}$ is left-invertible, i.e., has full column rank, and there exists $\mathbf{B} \in \mathcal{B}_{2}$ such that

$$
\begin{equation*}
\mathbf{H}_{2}^{\dagger} \mathbf{H}_{2} \succeq\left(\mathbf{H}_{4}-\mathbf{B}\right)^{\dagger}\left(\mathbf{H}_{4}-\mathbf{B}\right), \tag{12}
\end{equation*}
$$

then the capacity region of the MIMO IC is given by (10).
By choosing $\mathbf{B}_{i}=\mathbf{0}$, (12) becomes $\mathbf{H}_{2}^{\dagger} \mathbf{H}_{2} \succeq \mathbf{H}_{4}^{\dagger} \mathbf{H}_{4}$, which is related only to $\mathbf{H}_{2}$ and $\mathbf{H}_{4}$ and directly mimics that of the scalar Gaussian IC.

Using Theorem 3, we obtain the capacity region for the two-sided MIMO IC under aligned strong interference.

Theorem 4: For the MIMO IC defined in (1), if there exist matrices $\mathbf{A}_{1}, \mathbf{A}_{2}, \mathbf{B}_{1}$ and $\mathbf{B}_{2}$ such that

$$
\begin{align*}
& \mathbf{H}_{1}=\mathbf{A}_{1} \mathbf{H}_{3}+\mathbf{B}_{1} \quad \text { and }  \tag{13}\\
& \mathbf{H}_{4}=\mathbf{A}_{2} \mathbf{H}_{2}+\mathbf{B}_{2}, \tag{14}
\end{align*}
$$

where $\mathbf{A}_{i}^{\dagger} \mathbf{A}_{i} \preceq \mathbf{I}$ and $\mathbf{B}_{i} \in \mathcal{B}_{i}, i=1,2$, then the capacity region of the MIMO IC is

$$
\left\{\begin{array}{c}
0 \leq R_{1} \leq \log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\right|  \tag{15}\\
0 \leq R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right| \\
R_{1}+R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right| \\
R_{1}+R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{3} \mathbf{S}_{1} \mathbf{H}_{3}^{\dagger}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|
\end{array}\right\},
$$

where $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$ are the respective covariance matrix constraints defined in (2).
Similarly to Proposition 1, we have the following proposition.
Proposition 2: For the MIMO IC defined in (1), and where the channel matrices $\mathbf{H}_{2}$ and $\mathbf{H}_{3}$ are both left-invertible, if there exist $\mathbf{B}_{i} \in \mathcal{B}_{i}, i=1,2$, such that

$$
\begin{align*}
& \mathbf{H}_{2}^{\dagger} \mathbf{H}_{2} \succeq\left(\mathbf{H}_{4}-\mathbf{B}_{2}\right)^{\dagger}\left(\mathbf{H}_{4}-\mathbf{B}_{2}\right) \quad \text { and }  \tag{16}\\
& \mathbf{H}_{3}^{\dagger} \mathbf{H}_{3} \succeq\left(\mathbf{H}_{1}-\mathbf{B}_{1}\right)^{\dagger}\left(\mathbf{H}_{1}-\mathbf{B}_{1}\right), \tag{17}
\end{align*}
$$

then the capacity region of the MIMO IC is given by (15).
Obviously, Proposition 2 generalizes the capacity region of the scalar Gaussian ICs under strong interference. Furthermore, Proposition 2 also generalizes the result of [18] for single-input multipleoutput (SIMO) ICs under strong interference. In this case, $\mathbf{H}_{2}$ and $\mathbf{H}_{3}$ are both non-zero column vectors, and hence are left-invertible. Therefore, (16) and (17) become $\mathbf{H}_{2}^{\dagger} \mathbf{H}_{2} \succeq \mathbf{H}_{4}^{\dagger} \mathbf{H}_{4}$ and $\mathbf{H}_{3}^{\dagger} \mathbf{H}_{3} \succeq \mathbf{H}_{1}^{\dagger} \mathbf{H}_{1}$ which are the same as $\left\|\mathbf{H}_{2}\right\| \geq\left\|\mathbf{H}_{4}\right\|$ and $\left\|\mathbf{H}_{3}\right\| \geq\left\|\mathbf{H}_{1}\right\|$.

Let $\mathbf{B}_{1}=\mathbf{B}_{2}=\mathbf{0}$ and assume that there exist $\mathbf{A}_{1}$ and $\mathbf{A}_{2}$ satisfying (13) and (14). We can verify Theorem 4 in a way similar to that done in [3] and [4] for scalar Gaussian ICs under strong interference. Assuming the rate pair $\left(R_{1}, R_{2}\right)$ is achievable, then $x_{1}$ and $x_{2}$ can be reliably recovered at user 1 and user 2 , respectively. After subtracting $\boldsymbol{x}_{1}$ from $\boldsymbol{y}_{1}$, user 1 obtains

$$
\begin{equation*}
\boldsymbol{y}_{1}^{\prime}=\mathbf{H}_{2} \boldsymbol{x}_{2}+\boldsymbol{z}_{1} . \tag{18}
\end{equation*}
$$

We can pre-multiply $\boldsymbol{y}_{1}^{\prime}$ by $\mathbf{A}_{2}$ and get

$$
\begin{align*}
\boldsymbol{y}_{1}^{\prime \prime} & =\mathbf{A}_{2} \mathbf{H}_{2} \boldsymbol{x}_{2}+\mathbf{A}_{2} \boldsymbol{z}_{1} \\
& =\mathbf{H}_{4} \boldsymbol{x}_{2}+\mathbf{A}_{2} \boldsymbol{z}_{1} . \tag{19}
\end{align*}
$$

Since $\boldsymbol{x}_{1}$ is recovered at user 1 , we can add $\mathbf{H}_{3} \boldsymbol{x}_{1}$ to (19). Thus user 1 can eventually compute

$$
\begin{equation*}
\boldsymbol{y}_{1}^{\prime \prime \prime}=\mathbf{H}_{3} \boldsymbol{x}_{1}+\mathbf{H}_{4} \boldsymbol{x}_{2}+\mathbf{A}_{2} \boldsymbol{z}_{1} . \tag{20}
\end{equation*}
$$

If $\mathbf{A}_{2}^{\dagger} \mathbf{A}_{2} \preceq \mathbf{I}$, by Lemma 6 we have $\mathbf{A}_{2} \mathbf{A}_{2}^{\dagger} \preceq \mathbf{I}$ and the received signal at user 2 can be written as

$$
\begin{align*}
\boldsymbol{y}_{2} & =\mathbf{H}_{3} \boldsymbol{x}_{1}+\mathbf{H}_{4} x_{2}+z_{2} \\
& =\boldsymbol{y}_{1}^{\prime \prime \prime}+\boldsymbol{w}, \tag{21}
\end{align*}
$$

where $\boldsymbol{w} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{I}-\mathbf{A}_{2} \mathbf{A}_{2}^{\dagger}\right)$, and $\boldsymbol{w}$ is independent of all other random vectors. Since $\boldsymbol{x}_{2}$ can be recovered from $\boldsymbol{y}_{2}, \boldsymbol{x}_{2}$ can also be recovered from $\boldsymbol{y}_{1}^{\prime \prime \prime}$. Thus, user 1 can decode both $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$. Similarly, user 2 can also decode both $x_{1}$ and $x_{2}$. Therefore, the MIMO IC is now a compound MIMO multipleaccess channel, whose capacity region coincides with (15) [27]. The above development imposes no structure on $\boldsymbol{x}_{i}, i=1,2$. Therefore, as long as the input signal $\boldsymbol{x}_{i}$ (which can be non-Gaussian with arbitrary covariance matrix) can be decoded by its desired receiver, it can also be decoded by the other receiver. This result applies to MIMO ICs under a variety of power constraints, for example, peak power constraints, average power constraints and per-antenna power constraints. We state this formally in the following proposition.

Proposition 3: For the MIMO IC defined in (1) with expected per-symbol power constraints, or expected block power constraints, or per-antenna expected block power constraints, if there exist matrices $\mathbf{A}_{i}, i=1,2$, such that $\mathbf{A}_{i}^{\dagger} \mathbf{A}_{i} \preceq \mathbf{I}$ and

$$
\begin{align*}
& \mathbf{H}_{1}=\mathbf{A}_{1} \mathbf{H}_{3} \quad \text { and }  \tag{22}\\
& \mathbf{H}_{4}=\mathbf{A}_{2} \mathbf{H}_{2}, \tag{23}
\end{align*}
$$

then the capacity region of the MIMO IC is

$$
\underset{\left(\widehat{\mathbf{S}}_{1}, \widehat{\mathbf{S}}_{2}\right) \in \mathcal{P}}{\bigcup}\left\{\begin{array}{c}
0 \leq R_{1} \leq \log \left|\mathbf{I}+\mathbf{H}_{1} \widehat{\mathbf{S}}_{1} \mathbf{H}_{1}^{\dagger}\right|  \tag{24}\\
0 \leq R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{4} \widehat{\mathbf{S}}_{2} \mathbf{H}_{4}^{\dagger}\right| \\
R_{1}+R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{1} \widehat{\mathbf{S}}_{1} \mathbf{H}_{1}^{\dagger}+\mathbf{H}_{2} \widehat{\mathbf{S}}_{2} \mathbf{H}_{2}^{\dagger}\right| \\
R_{1}+R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{3} \widehat{\mathbf{S}}_{1} \mathbf{H}_{3}^{\dagger}+\mathbf{H}_{4} \widehat{\mathbf{S}}_{2} \mathbf{H}_{4}^{\dagger}\right|
\end{array}\right\},
$$

where $\mathcal{P}$ denotes the specified power constraints.
For this result, we say that there is an expected per-symbol power constraint, an expected block power constraint, and a per-antenna expected block power constraint, respectively, if the following conditions must be satisfied:

$$
\begin{align*}
& \operatorname{tr}\left(E\left[x_{i j} \boldsymbol{x}_{i j}^{\dagger}\right]\right) \leq P_{i}, \quad j=1, \cdots, n  \tag{25}\\
& \sum_{j=1}^{n} \operatorname{tr}\left(E\left[x_{i j} \boldsymbol{x}_{i j}^{\dagger}\right]\right) \leq n P_{i} \quad \text { or } \tag{26}
\end{align*}
$$

$$
\begin{equation*}
\sum_{j=1}^{n}\left(E\left[x_{i j} x_{i j}^{\dagger}\right]\right)_{k} \leq n P_{i k}, \tag{27}
\end{equation*}
$$

where $(\cdot)_{k}$ denotes the $k$ th diagonal element of a square matrix, and $P_{i k}$ is the power constraint for the $k$ th antenna of user $i$.

Theorem 4 has relaxed conditions on the channel matrices as compared to Proposition 3. The extra term $\mathbf{B}_{i}$ in Theorem 4 results from the covariance matrix constraint $\mathbf{S}_{i}$. Suppose (13) and (14) hold and the input signal of user $i$ is $\boldsymbol{x}_{i}^{*} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{S}_{i}\right)$. From Theorem 4, $\boldsymbol{x}_{i}^{*}$ achieves the capacity. Applying the same procedure in (18)-(20) to $\boldsymbol{y}_{1}$, we obtain the counterpart of (19)

$$
\begin{align*}
\bar{y}^{\prime \prime} & =\mathbf{A}_{2} \mathbf{H}_{2} x_{2}^{*}+\mathbf{A}_{2} \boldsymbol{z}_{1} \\
& =\left(\mathbf{A}_{2} \mathbf{H}_{2}+\mathbf{B}_{2}\right) \boldsymbol{x}_{2}^{*}+\mathbf{A}_{2} \boldsymbol{z}_{1} \\
& =\mathbf{H}_{4} \boldsymbol{x}_{2}^{*}+\mathbf{A}_{2} \boldsymbol{z}_{1}, \tag{28}
\end{align*}
$$

where the second equality holds since

$$
\begin{equation*}
\operatorname{Cov}\left(\mathbf{B}_{2} x_{2}^{*}\right)=\mathbf{B}_{2} \mathbf{S}_{2} \mathbf{B}_{2}^{\dagger}=\mathbf{0} \tag{29}
\end{equation*}
$$

and hence $\mathbf{B}_{2} \boldsymbol{x}_{2}^{*}=\mathbf{0}$. Therefore, $\boldsymbol{y}_{2}$ can also be written as (21).
The difference between Proposition 3 and Theorem 4 is that (22) and (23) ensure that $\boldsymbol{x}_{i}$ can be reliably decoded at both receivers as long as it can be decoded at the desired receiver, while (13) and (14) ensure that the capacity-achieving $x_{i}^{*}$ can be reliably decoded at both receivers.

## C. Sum-rate capacity of MIMO IC under noisy interference

In [10], we say that an IC has noisy interference when treating interference as noise achieves the sum-rate capacity. In this section, we present the sum-rate capacity results for MIMO ICs that have noisy interference.

Theorem 5: For the MIMO IC defined in (1) with $\mathbf{H}_{3}=\mathbf{0}$, if there exist matrices $\mathbf{A}$ and $\mathbf{B}$ that satisfy

$$
\begin{equation*}
\mathbf{H}_{2}=\mathbf{A}^{\dagger} \mathbf{H}_{4}+\mathbf{B}, \tag{30}
\end{equation*}
$$

where $\mathbf{A}^{\dagger} \mathbf{A} \preceq \mathbf{I}$ and $\mathbf{B} \in \mathcal{B}_{2}$, then the sum-rate capacity of the MIMO IC is

$$
\begin{equation*}
\log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right)^{-1}\right|+\log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right| \tag{31}
\end{equation*}
$$

where $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$ are the respective covariance matrix constraints defined in (2).
Similarly to Proposition 1, we obtain the following result.

Proposition 4: For the MIMO IC defined in (1) with $\mathbf{H}_{3}=\mathbf{0}$, if $\mathbf{H}_{4}$ is left-invertible and there exists $\mathbf{B} \in \mathcal{B}_{2}$ such that

$$
\begin{equation*}
\mathbf{H}_{4}^{\dagger} \mathbf{H}_{4} \succeq\left(\mathbf{H}_{2}-\mathbf{B}\right)^{\dagger}\left(\mathbf{H}_{2}-\mathbf{B}\right), \tag{32}
\end{equation*}
$$

then the sum-rate capacity of the MIMO IC is given by (31).
Theorem 5 gives the noisy-interference sum-rate capacity of a MIMO ZIC. Specifically, when (30) is satisfied, the sum-rate capacity can be achieved by treating interference as noise. Consider a scalar Gaussian IC where $\mathbf{H}_{1}=\mathbf{H}_{4}=1, \mathbf{H}_{2}=\sqrt{a}$ and $\mathbf{H}_{3}=0$. Equation (30) is $0 \leq a \leq 1$. Therefore, Theorem 5 includes the scalar Gaussian ZIC noisy-interference sum-rate capacity as a special case ${ }^{1}$. For a SIMO IC where $\mathbf{H}_{1}, \mathbf{H}_{3}$ and $\mathbf{H}_{4}$ are all nonzero column vectors, Proposition 4 shows that if $\left\|\mathbf{H}_{2}\right\| \leq\left\|\mathbf{H}_{4}\right\|$, the sum-rate capacity is achieved by treating interference as noise.

Similarly to Proposition 3, if we choose $\mathbf{B}=\mathbf{0}$ in (30), then Theorem 5 can be extended for different power constraints. We state this formally in the following proposition.

Proposition 5: For the MIMO IC defined in (1) with expected per-symbol power constraints, or expected block power constraints, or per-antenna expected block power constraints, if $\mathbf{H}_{3}=\mathbf{0}$ and there exists a matrix $\mathbf{A}$ such that $\mathbf{A}^{\dagger} \mathbf{A} \preceq \mathbf{I}$ and

$$
\begin{equation*}
\mathbf{H}_{2}=\mathbf{A}^{\dagger} \mathbf{H}_{4}, \tag{33}
\end{equation*}
$$

then the sum-rate capacity is

$$
\begin{equation*}
\max _{\left(\widehat{\mathbf{S}}_{1}, \widehat{\mathbf{S}}_{2}\right) \in \mathcal{P}}\left(\log \left|\mathbf{I}+\mathbf{H}_{1} \widehat{\mathbf{S}}_{1} \mathbf{H}_{1}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{2} \widehat{\mathbf{S}}_{2} \mathbf{H}_{2}^{\dagger}\right)^{-1}\right|+\log \left|\mathbf{I}+\mathbf{H}_{4} \widehat{\mathbf{S}}_{2} \mathbf{H}_{4}^{\dagger}\right|\right) \tag{34}
\end{equation*}
$$

where $\mathcal{P}$ denotes the specified power constraints.
Next, we give the noisy-interference sum-rate capacity of a two-sided MIMO IC. Note that this result does not require $\mathbf{S}_{1}$ or $\mathbf{S}_{2}$ to have full rank (see [22] and Example 4 below).

Theorem 6: For the MIMO IC defined in (1), if there exist matrices $\mathbf{A}_{i}, \mathbf{B}_{i} \in \mathcal{B}_{i}$, and Hermitian positive definite matrices $\boldsymbol{\Sigma}_{i}, i=1,2$, such that

$$
\begin{align*}
& \mathbf{A}_{1}^{\dagger} \mathbf{A}_{1} \preceq \boldsymbol{\Sigma}_{1} \preceq \mathbf{I}-\mathbf{A}_{2} \boldsymbol{\Sigma}_{2}^{-1} \mathbf{A}_{2}^{\dagger},  \tag{35}\\
& \mathbf{A}_{2}^{\dagger} \mathbf{A}_{2} \preceq \boldsymbol{\Sigma}_{2} \preceq \mathbf{I}-\mathbf{A}_{1} \boldsymbol{\Sigma}_{1}^{-1} \mathbf{A}_{1}^{\dagger},  \tag{36}\\
& \mathbf{H}_{3}=\mathbf{A}_{1}^{\dagger}\left(\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}+\mathbf{I}\right)^{-1} \mathbf{H}_{1}+\mathbf{B}_{1} \quad \text { and } \tag{37}
\end{align*}
$$

[^1]\[

$$
\begin{equation*}
\mathbf{H}_{2}=\mathbf{A}_{2}^{\dagger}\left(\mathbf{H}_{3} \mathbf{S}_{1} \mathbf{H}_{3}^{\dagger}+\mathbf{I}\right)^{-1} \mathbf{H}_{4}+\mathbf{B}_{2}, \tag{38}
\end{equation*}
$$

\]

then the sum-rate capacity of the MIMO IC is

$$
\begin{equation*}
\log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right)^{-1}\right|+\log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{3} \mathbf{S}_{1} \mathbf{H}_{3}^{\dagger}\right)^{-1}\right| \tag{39}
\end{equation*}
$$

where $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$ are the respective covariance matrix constraints defined in (2).
Theorem 6 gives sufficient conditions for the MIMO IC under which treating interference as noise achieves the sum-rate capacity. In the case where both $\mathbf{H}_{1}$ and $\mathbf{H}_{4}$ are left-invertible, the following conditions are sufficient for (37) and (38):

$$
\begin{align*}
& \mathbf{A}_{1}=\left(\mathbf{I}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right) \mathbf{H}_{1}\left(\mathbf{H}_{1}^{\dagger} \mathbf{H}_{1}\right)^{-1}\left(\mathbf{H}_{3}^{\dagger}-\mathbf{B}_{1}^{\dagger}\right) \quad \text { and }  \tag{40}\\
& \mathbf{A}_{2}=\left(\mathbf{I}+\mathbf{H}_{3} \mathbf{S}_{1} \mathbf{H}_{3}^{\dagger}\right) \mathbf{H}_{4}\left(\mathbf{H}_{4}^{\dagger} \mathbf{H}_{4}\right)^{-1}\left(\mathbf{H}_{2}^{\dagger}-\mathbf{B}_{2}^{\dagger}\right) . \tag{41}
\end{align*}
$$

That is, such matrices $\mathbf{A}_{1}$ and $\mathbf{A}_{2}$ exist when $\mathbf{H}_{1}$ and $\mathbf{H}_{4}$ are left-invertible. It remains to find matrices $\mathbf{B}_{1} \in \mathcal{B}_{1}$ and $\mathbf{B}_{2} \in \mathcal{B}_{2}$ such the matrix inequalities (35) and (36) have solutions. We state this formally in the following proposition.

Proposition 6: For the MIMO IC defined in (1), if $\mathbf{H}_{1}$ and $\mathbf{H}_{4}$ are left-invertible, and there exist symmetric positive definite matrices $\boldsymbol{\Sigma}_{1}$ and $\boldsymbol{\Sigma}_{2}$ that satisfy (35) and (36) with $\mathbf{A}_{1}$ and $\mathbf{A}_{2}$ defined in (40) and (41) for some $\mathbf{B}_{1} \in \mathcal{B}_{1}$ and $\mathbf{B}_{2} \in \mathcal{B}_{2}$, then the sum-rate capacity is given by (39).

Although Theorem 6 gives the noisy interference conditions for a MIMO IC, finding explicit solution of the matrix inequalities (35) and (36) can be very complex. Therefore, using Theorem 6 to check whether a MIMO IC has noisy interference is not practical. We thus derive the following proposition that is a special case of Theorem 6 but is more amenable to computation.

Proposition 7: For the MIMO IC defined in (1), the sum-rate capacity is given by (39) if

$$
\begin{equation*}
\operatorname{radius}\left(\boldsymbol{\Phi}_{i}\right) \leq \frac{1}{2}, \quad i=1,2, \tag{42}
\end{equation*}
$$

where

$$
\begin{align*}
& \mathbf{\Phi}_{1}=\left(\mathbf{I}-\mathbf{A}_{1}^{\dagger} \mathbf{A}_{1}-\mathbf{A}_{2} \mathbf{A}_{2}^{\dagger}\right)^{-\frac{1}{2}} \mathbf{A}_{1}^{\dagger} \mathbf{A}_{2}^{\dagger}\left(\mathbf{I}-\mathbf{A}_{1}^{\dagger} \mathbf{A}_{1}-\mathbf{A}_{2} \mathbf{A}_{2}^{\dagger}\right)^{-\frac{1}{2}}  \tag{43}\\
& \mathbf{\Phi}_{2}=\left(\mathbf{I}-\mathbf{A}_{1} \mathbf{A}_{1}^{\dagger}-\mathbf{A}_{2}^{\dagger} \mathbf{A}_{2}\right)^{-\frac{1}{2}} \mathbf{A}_{2}^{\dagger} \mathbf{A}_{1}^{\dagger}\left(\mathbf{I}-\mathbf{A}_{1} \mathbf{A}_{1}^{\dagger}-\mathbf{A}_{2}^{\dagger} \mathbf{A}_{2}\right)^{-\frac{1}{2}} \tag{44}
\end{align*}
$$

and $\mathbf{A}_{1}$ and $\mathbf{A}_{2}$ are chosen to satisfy (37) and (38) respectively, and $\mathbf{B}_{i} \in \mathcal{B}_{i}, i=1,2$.
In the scalar case, if we have $\mathbf{H}_{1}=\mathbf{H}_{4}=1, \mathbf{H}_{2}=\sqrt{a}, \mathbf{H}_{3}=\sqrt{b}, \mathbf{S}_{1}=P_{1}$ and $\mathbf{S}_{2}=P_{2}$, from (42) we directly have

$$
\sqrt{a}\left(1+b P_{1}\right)+\sqrt{b}\left(1+a P_{2}\right) \leq 1 .
$$

The above condition can also be obtained from Theorem 6 after some mathematical manipulations. Therefore Theorem 6 and Proposition 7 generalize the noisy-interference sum-rate capacity of the scalar Gaussian IC [10]-[12] to the MIMO IC.

Similarly to Proposition 6, we obtain the following proposition.
Proposition 8: For the MIMO IC defined in (1), if both $\mathbf{H}_{1}$ and $\mathbf{H}_{4}$ are left-invertible, and there exist matrices $\mathbf{B}_{1} \in \mathcal{B}_{1}$ and $\mathbf{B}_{2} \in \mathcal{B}_{2}$ such that the $\mathbf{A}_{1}$ and $\mathbf{A}_{2}$ defined in (40) and (41) satisfy (42), then the sum-rate capacity is (39).

## D. Sum-rate capacity of MIMO IC under mixed aligned interference

Theorem 7: For the MIMO IC defined in (1), if there exist matrices $\mathbf{A}_{1}, \mathbf{A}_{2}, \mathbf{B}_{1}$ and $\mathbf{B}_{2}$ that satisfy

$$
\begin{align*}
& \mathbf{H}_{1}=\mathbf{A}_{1} \mathbf{H}_{3}+\mathbf{B}_{1} \quad \text { and }  \tag{45}\\
& \mathbf{H}_{2}=\mathbf{A}_{2}^{\dagger} \mathbf{H}_{4}+\mathbf{B}_{2} \tag{46}
\end{align*}
$$

where $\mathbf{A}_{i}^{\dagger} \mathbf{A}_{i} \preceq \mathbf{I}$ and $\mathbf{B}_{i} \in \mathcal{B}_{i}, i=1,2$, then the sum-rate capacity of the MIMO IC is

$$
\min \left\{\begin{array}{c}
\log \left|\mathbf{I}+\mathbf{H}_{3} \mathbf{S}_{1} \mathbf{H}_{3}^{\dagger}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|  \tag{47}\\
\log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right)^{-1}\right|+\log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|
\end{array}\right\}
$$

where $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$ are the respective covariance matrix constraints defined in (2).
Proposition 9: For the MIMO IC defined in (1) where $\mathbf{H}_{3}$ and $\mathbf{H}_{4}$ are left-invertible, if there exist $\mathbf{B}_{i} \in \mathcal{B}_{i}, i=1,2$, such that

$$
\begin{align*}
& \mathbf{H}_{4}^{\dagger} \mathbf{H}_{4} \succ\left(\mathbf{H}_{2}-\mathbf{B}_{2}\right)^{\dagger}\left(\mathbf{H}_{2}-\mathbf{B}_{2}\right) \quad \text { and }  \tag{48}\\
& \mathbf{H}_{3}^{\dagger} \mathbf{H}_{3} \succeq\left(\mathbf{H}_{1}-\mathbf{B}_{1}\right)^{\dagger}\left(\mathbf{H}_{1}-\mathbf{B}_{1}\right) \tag{49}
\end{align*}
$$

then the sum-rate capacity is given by (47).
Theorem 7 gives the sum-rate capacity of the MIMO IC under mixed aligned interference, i.e., one user sees aligned weak/intermediate interference and the other user sees aligned strong interference. The sum-rate capacity is achieved by treating interference as noise at the receiver that sees aligned weak/intermediate interference, and fully decoding the interference at the receiver that sees aligned strong interference. Proposition 9 includes the sum-rate capacity of scalar Gaussian ICs with mixed interference as a special case. If we choose $\mathbf{B}_{1}=\mathbf{0}$ and $\mathbf{B}_{2}=\mathbf{0}$, the constraints (48) and (49) reduce to $\mathbf{H}_{4}^{\dagger} \mathbf{H}_{4} \succ$ $\mathbf{H}_{2}^{\dagger} \mathbf{H}_{2}$ and $\mathbf{H}_{3}^{\dagger} \mathbf{H}_{3} \succeq \mathbf{H}_{1}^{\dagger} \mathbf{H}_{1}$. The MIMO ICs that satisfy these two simplified conditions have mixed interference and this result applies to channels with other power constraints.

Similar to Propositions 3 and 5, we obtain the sum-rate capacity for MIMO ICs with aligned mixed interference under different power constraints.

Proposition 10: For the MIMO IC defined in (1) with expected per-symbol power constraints, or expected block power constraints, or per-antenna expected block power constraints, if there exist matrices $\mathbf{A}_{i}, i=1,2$, such that $\mathbf{A}_{i}^{\dagger} \mathbf{A}_{i} \preceq \mathbf{I}$ and

$$
\begin{align*}
& \mathbf{H}_{1}=\mathbf{A}_{1} \mathbf{H}_{3} \quad \text { and }  \tag{50}\\
& \mathbf{H}_{2}=\mathbf{A}_{2}^{\dagger} \mathbf{H}_{4} \tag{51}
\end{align*}
$$

then the sum-rate capacity is

$$
\begin{equation*}
\max _{\left(\widehat{\mathbf{S}}_{1}, \widehat{\mathbf{S}}_{2}\right) \in \mathcal{P}} \min \left\{\log \left|\mathbf{I}+\mathbf{H}_{3} \widehat{\mathbf{S}}_{1} \mathbf{H}_{3}^{\dagger}+\mathbf{H}_{4} \widehat{\mathbf{S}}_{2} \mathbf{H}_{4}^{\dagger}\right| \mid, ~\left(\log \left|\mathbf{I}+\mathbf{H}_{1} \widehat{\mathbf{S}}_{1} \mathbf{H}_{1}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{2} \widehat{\mathbf{S}}_{2} \mathbf{H}_{2}^{\dagger}\right)^{-1}\right|+\log \left|\mathbf{I}+\mathbf{H}_{4} \widehat{\mathbf{S}}_{2} \mathbf{H}_{4}^{\dagger}\right|\right\},\right. \tag{52}
\end{equation*}
$$

where $\mathcal{P}$ denotes the specified power constraints.

## E. Generalizations

The results in the previous sections are for MIMO ICs whose capacities are achieved by $\boldsymbol{x}_{i} \sim$ $\mathcal{C N}\left(\mathbf{0}, \mathbf{S}_{i}\right), i=1,2$, where $\mathbf{S}_{i}$ is the covariance matrix constraint for user $i$ defined in (2). The methods introduced can also be applied to more general cases in which the capacity is achieved by $\boldsymbol{x}_{i}^{\prime} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{S}_{i}^{\prime}\right)$ where $\mathbf{S}_{i}^{\prime} \preceq \mathbf{S}_{i}$. For example, consider the following generalization of Theorem 5 that gives the sum-rate capacity of a class of MIMO ZICs under noisy interference. This extension applies to all the corresponding theorems for other kinds of interference.

Theorem 8: For the MIMO IC defined in (1), if $\mathbf{H}_{3}=\mathbf{0}$ and the optimal $\mathbf{A}^{*}, \mathbf{S}_{1}^{*}$ and $\mathbf{S}_{2}^{*}$ for the following optimization problem

$$
\begin{array}{ll}
\min _{\mathbf{A}} \max _{\widehat{\mathbf{S}}_{1}, \widehat{\mathbf{S}}_{2}} & C\left(\mathbf{A}, \widehat{\mathbf{S}}_{1}, \widehat{\mathbf{S}}_{2}\right) \\
\text { subject to } & \mathbf{A} \mathbf{A}^{\dagger} \preceq \mathbf{I}, \quad \mathbf{0} \preceq \widehat{\mathbf{S}}_{1} \preceq \mathbf{S}_{1}, \quad \mathbf{0} \preceq \widehat{\mathbf{S}}_{2} \preceq \mathbf{S}_{2}, \tag{53}
\end{array}
$$

satisfy

$$
\begin{equation*}
\mathbf{H}_{2}=\mathbf{A}^{* \dagger} \mathbf{H}_{4}+\mathbf{B} \tag{54}
\end{equation*}
$$

where

$$
\begin{align*}
C\left(\mathbf{A}, \widehat{\mathbf{S}}_{1}, \widehat{\mathbf{S}}_{2}\right)= & \log \left|\mathbf{H}_{1} \widehat{\mathbf{S}}_{1} \mathbf{H}_{1}^{\dagger}+\mathbf{H}_{2} \widehat{\mathbf{S}}_{2} \mathbf{H}_{2}^{\dagger}+\mathbf{I}\right|-\log \left|\mathbf{I}-\mathbf{A} \mathbf{A}^{\dagger}\right| \\
& +\log \left|\mathbf{H}_{4} \widehat{\mathbf{S}}_{2} \mathbf{H}_{4}^{\dagger}+\mathbf{I}-\left(\mathbf{H}_{4} \widehat{\mathbf{S}}_{2} \mathbf{H}_{2}^{\dagger}+\mathbf{A}\right)\left(\mathbf{H}_{2} \widehat{\mathbf{S}}_{2} \mathbf{H}_{2}^{\dagger}+\mathbf{I}\right)^{-1}\left(\mathbf{H}_{2} \widehat{\mathbf{S}}_{2} \mathbf{H}_{4}^{\dagger}+\mathbf{A}^{\dagger}\right)\right|, \tag{55}
\end{align*}
$$

and

$$
\begin{equation*}
\mathbf{B} \in\left\{\tilde{\mathbf{B}} \mid \text { all columns of } \tilde{\mathbf{B}}^{\dagger} \text { are in the null space of } \mathbf{S}_{2}^{*}\right\} \tag{56}
\end{equation*}
$$

then the sum-rate capacity for the MIMO IC is

$$
\begin{equation*}
\log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1}^{*} \mathbf{H}_{1}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{2} \mathbf{S}_{2}^{*} \mathbf{H}_{2}^{\dagger}\right)^{-1}\right|+\log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2}^{*} \mathbf{H}_{4}^{\dagger}\right| . \tag{57}
\end{equation*}
$$

The solution of problem (53) is an upper bound on the sum-rate capacity of this MIMO ZIC. The bound is tight when (54) is satisfied. Theorem 8 includes Theorem 5 as a special case in which $\mathbf{S}_{1}$ and $\mathrm{S}_{2}$ are optimal for problem (53).

## III. Numerical Results

Example 1: Consider a MIMO IC with

$$
\mathbf{H}_{1}=\mathbf{H}_{4}=\mathbf{I}, \quad \mathbf{H}_{2}=\left[\begin{array}{ll}
2.0 & 1.5 \\
0.8 & 1.0
\end{array}\right], \quad \mathbf{H}_{3}=\left[\begin{array}{cc}
1.2 & 2.0 \\
0 & 0.8
\end{array}\right] \quad \text { and } \quad \mathbf{S}_{1}=\mathbf{S}_{2}=\mathbf{I} .
$$

Conditions (6) and (7) are satisfied. Therefore this MIMO IC has very strong interference and the capacity region is

$$
\left\{\left(R_{1}, R_{2}\right): \quad 0 \leq R_{1} \leq 1.3863, \quad 0 \leq R_{2} \leq 1.3863\right\}
$$

However, consider the aligned strong interference conditions (13) and (14) for this channel. We have $\mathbf{A}_{1}=\mathbf{H}_{3}^{-1}, \mathbf{A}_{2}=\mathbf{H}_{2}^{-1}$ and $\mathbf{B}_{1}=\mathbf{B}_{2}=\mathbf{0}$, where $\mathbf{A}_{1}^{\dagger} \mathbf{A}_{1} \npreceq \mathbf{I}$ and $\mathbf{A}_{2}^{\dagger} \mathbf{A}_{2} \npreceq \mathbf{I}$. Therefore, the above channel has very strong interference but not aligned strong interference.

Example 2: Consider a MIMO IC with

$$
\begin{array}{ll}
\mathbf{H}_{1} & =\left[\begin{array}{cccc}
1.8 & 0.8 & -0.6 & 1.4 \\
1.2 & -1.9 & 0.5 & -0.7
\end{array}\right], \quad \mathbf{H}_{2}=\left[\begin{array}{ccc}
0.8 & 1.0 & -0.5 \\
0.6 \\
1.0 & -1.2 & 0.4 \\
1.2
\end{array}\right] \\
\mathbf{H}_{3}=\left[\begin{array}{cccc}
1.0 & 1.0 & 0.5 & 0.5 \\
0.4 & 0.2 & 1 & 0.6
\end{array}\right], \quad \mathbf{H}_{4}=\left[\begin{array}{ccc}
0.68 & 0.36 & -0.22 \\
1.04 & -0.66 & 0.17 \\
1.14
\end{array}\right] \\
\mathbf{S}_{1}=\left[\begin{array}{cccc}
0.9 & 0.4 & 1.0 & 0.1 \\
0.4 & 0.4 & 0 & -0.4 \\
1.0 & 0 & 2.0 & 1.0 \\
0.1 & -0.4 & 1.0 & 0.9
\end{array}\right] \quad \text { and } \mathbf{S}_{2}=\mathbf{I} .
\end{array}
$$

Conditions (13)-(14) are both satisfied by choosing

$$
\mathbf{A}_{1}=\left[\begin{array}{cc}
0.8 & 0 \\
0 & 0.5
\end{array}\right], \quad \mathbf{A}_{2}=\left[\begin{array}{cc}
0.6 & 0.2 \\
0.3 & 0.8
\end{array}\right], \quad \mathbf{B}_{1}=\left[\begin{array}{cccc}
1 & 0 & -1 & 1 \\
1 & -2 & 0 & -1
\end{array}\right] \quad \text { and } \quad \mathbf{B}_{2}=\mathbf{0}
$$

By Theorem 4, this MIMO IC is under aligned strong interference and the capacity region is

$$
\left\{\left(R_{1}, R_{2}\right): \quad 0 \leq R_{1} \leq 1.6770, \quad 0 \leq R_{2} \leq 1.8636, \quad 0 \leq R_{1}+R_{2} \leq 3.2812\right\}
$$

Example 3: Consider a MIMO ZIC where

$$
\begin{aligned}
& \mathbf{H}_{1}=\mathbf{I}, \quad \mathbf{H}_{2}=\left[\begin{array}{ccc}
1.3 & 1.1 & 1.4 \\
1.5 & -0.5 & 3.0 \\
0.9 & -0.36 & 1.5
\end{array}\right], \quad \mathbf{H}_{3}=\mathbf{0}, \quad \mathbf{H}_{4}=\left[\begin{array}{ccc}
1.0 & 2.0 & 0.5 \\
1.0 & 1.0 & 2 \\
0.5 & 0.4 & 0.5
\end{array}\right], \\
& \mathbf{S}_{1}=\mathbf{I} \text { and } \mathbf{S}_{2}=\left[\begin{array}{ccc}
1.8 & 1.0 & -0.4 \\
1.0 & 5.0 & 2.0 \\
-0.4 & 2.0 & 1.2
\end{array}\right] .
\end{aligned}
$$

Condition (30) is satisfied by choosing

$$
\mathbf{A}=\left[\begin{array}{ccc}
0.8 & 0 & 0 \\
0 & 0.5 & 0 \\
0 & 0 & 0.6
\end{array}\right] \text { and } \mathbf{B}=\left[\begin{array}{ccc}
0.5 & -0.5 & 1.0 \\
1.0 & -1.0 & 2.0 \\
0.6 & -0.6 & 1.2
\end{array}\right]
$$

By Theorem 5 or Proposition 4, the above MIMO ZIC is under noisy interference and the sum-rate capacity $C=5.6622$ is obtained from (31).

Example 4: Consider a MISO IC with

$$
\left[\begin{array}{l}
\mathbf{H}_{1} \\
\mathbf{H}_{2} \\
\mathbf{H}_{3} \\
\mathbf{H}_{4}
\end{array}\right]=\left[\begin{array}{ccc}
6.0 & 4.0 & 5.0 \\
0.5 & 0.8 & 1.0 \\
0.4 & 0.6 & 0.1 \\
3.0 & -2.0 & 6.0
\end{array}\right], \quad \mathbf{S}_{1}=\left[\begin{array}{ccc}
0.9 & 0.5 & -0.2 \\
0.5 & 2.5 & 1 \\
-0.2 & 1 & 0.6
\end{array}\right] \quad \text { and } \mathbf{S}_{2}=\left[\begin{array}{ccc}
2.2 & -0.2 & -0.6 \\
-0.2 & 0.2 & -0.4 \\
-0.6 & -0.4 & 1.3
\end{array}\right] .
$$

Condition (42) is satisfied by choosing

$$
\mathbf{A}_{1}=0.1578, \quad \mathbf{A}_{2}=0.2394, \quad \mathbf{B}_{1}=[-0.2,0.2,-0.4] \quad \text { and } \quad \mathbf{B}_{2}=[0.2,1.0,0.4] .
$$

By Proposition 7, this MISO IC is under noisy interference and the sum-rate capacity $C=7.7171$ is achieved by treating interference as noise. In this case $\operatorname{rank}\left(\mathbf{S}_{1}\right)=\operatorname{rank}\left(\mathbf{S}_{2}\right)=2$. However, if we use average power constraints $P_{1}=\operatorname{tr}\left(\mathbf{S}_{1}\right)=4.0$ and $P_{2}=\operatorname{tr}\left(\mathbf{S}_{2}\right)=3.7$ instead of the covariance
matrix constraints $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$, then using the optimality of beamforming for single-user detection of MISO ICs [25], we can achieve a sum rate of $R_{1}+R_{2}=9.9162$ by treating interference as noise and choosing $\mathbf{S}_{i}=\gamma_{i} \boldsymbol{\gamma}_{i}^{\dagger}$, rank $\left(\mathbf{S}_{i}\right)=1, i=1,2$, where $\gamma_{1}=[1.2133,-0.0181,1.5899]^{\dagger}$ and $\gamma_{2}=$ $[0.5673,-1.4460,1.1345]^{\dagger}$.

Example 5: Consider a MIMO IC under average power constraints $P_{1}=8$ and $P_{2}=1$ with

$$
\begin{array}{ll}
\mathbf{H}_{1}=\operatorname{diag}[1.0392,1.5937,1.2689], & \mathbf{H}_{2}=\operatorname{diag}[0.7746,0.2646,0.3162], \\
\mathbf{H}_{3}=\operatorname{diag}[0.3000,0.6083,0.3162]
\end{array} \quad \text { and } \quad \mathbf{H}_{4}=\operatorname{diag}[1.5330,1.2124,1.3784] .
$$

Since all the channel matrices are diagonal, this MIMO IC can be considered as a parallel IC. From [24, Theorem 3], this MIMO IC is under noisy interference and the sum-rate capacity $C=6.1066$ can be achieved by independent coding across antennas and treating interference as noise. The optimal input signals are Gaussian with covariance matrices

$$
\overline{\mathbf{S}}_{1} \triangleq \operatorname{diag}[2.0922,3.3021,2.6057] \quad \text { and } \quad \overline{\mathbf{S}}_{2} \triangleq \operatorname{diag}[0.4472,0,0.5528],
$$

where $\operatorname{tr}\left(\overline{\mathbf{S}}_{1}\right)=P_{1}$ and $\operatorname{tr}\left(\overline{\mathbf{S}}_{2}\right)=P_{2}$. The input covariance matrix of the second user is singular and the second antenna is inactive.

If the average power constraints $P_{1}$ and $P_{2}$ are replaced by covariance constraints:

$$
\mathbf{S}_{1}=\left[\begin{array}{ccc}
2.0922 & 0.5000 & 1.0000 \\
0.5000 & 3.3021 & 0 \\
1.0000 & 0 & 2.6057
\end{array}\right] \text { and } \mathbf{S}_{2}=\left[\begin{array}{ccc}
0.4472 & 0 & 0.1500 \\
0 & 0 & 0 \\
0.1500 & 0 & 0.5528
\end{array}\right]
$$

where $\operatorname{tr}\left(\mathbf{S}_{1}\right)=P_{1}$ and $\operatorname{tr}\left(\mathbf{S}_{2}\right)=P_{2}$ but $\mathbf{S}_{1} \nsucceq \overline{\mathbf{S}}_{1}$ and $\mathbf{S}_{2} \nsucceq \overline{\mathbf{S}}_{2}$. Conditions (40) and (41) are satisfied by choosing

$$
\mathbf{A}_{1}=\left[\begin{array}{ccc}
0.3661 & 0 & 0.0092 \\
0 & 0.3817 & 0 \\
0.0106 & 0 & 0.2630
\end{array}\right], \quad \mathbf{A}_{2}=\left[\begin{array}{ccc}
0.6004 & 0.0199 & 0.0218 \\
0.0461 & 0.4848 & 0 \\
0.0479 & 0 & 0.2892
\end{array}\right], \quad \text { and } \quad \mathbf{B}_{1}=\mathbf{B}_{2}=\mathbf{0}
$$

It can be obtained from (42) that $\operatorname{radius}\left(\boldsymbol{\Phi}_{1}\right)=0.4614$ and radius $\left(\boldsymbol{\Phi}_{2}\right)=0.1822$. Therefore, from Proposition 7 this MIMO IC is under noisy interference and the sum-rate capacity $C=5.9541$ is achieved by treating interference as noise.

## IV. Proofs of the Main Results

We first introduce some lemmas which will be used to prove our main results.

## A. Preliminaries

The following lemma is based on the fact that a Gaussian distribution maximizes conditional entropy under a covariance matrix constraint [28].

Lemma 1: Let $\boldsymbol{x}_{i}^{n}=\left[x_{i, 1}^{\dagger}, \ldots, \boldsymbol{x}_{i, n}^{\dagger}\right]^{\dagger}, i=1, \ldots, k$, be $k$ long random vectors each of which consists of $n$ vectors. Suppose the $\boldsymbol{x}_{i, j}, i=1, \cdots, k$ all have the same length $L_{j}, j=1, \cdots, n$. Let $\boldsymbol{y}^{n}=$ $\left[\boldsymbol{y}_{1}^{\dagger}, \ldots, \boldsymbol{y}_{n}^{\dagger}\right]^{\dagger}$, where $\boldsymbol{y}_{j}$ has length $L_{j}$, be a long Gaussian random vector with covariance matrix

$$
\begin{equation*}
\operatorname{Cov}\left(\boldsymbol{y}^{n}\right)=\sum_{i=1}^{k} \lambda_{i} \operatorname{Cov}\left(\boldsymbol{x}_{i}^{n}\right), \tag{58}
\end{equation*}
$$

where $\sum_{i=1}^{k} \lambda_{i}=1, \lambda_{i} \geq 0$ and $\left|\operatorname{Cov}\left(\boldsymbol{x}_{i}^{n}\right)\right|>0$. Let $\mathcal{S}$ be a subset of $\{1,2, \ldots, n\}$ and $\mathcal{T}$ be a subset of $\mathcal{S}$ 's complement. Then we have

$$
\begin{equation*}
\sum_{i=1}^{k} \lambda_{i} h\left(\boldsymbol{x}_{i, \mathcal{S}} \mid \boldsymbol{x}_{i, \mathcal{T}}\right) \leq h\left(\boldsymbol{y}_{\mathcal{S}} \mid \boldsymbol{y}_{\mathcal{T}}\right) \tag{59}
\end{equation*}
$$

## Proof: See Appendix A.

When the $\boldsymbol{x}_{k}, k=1, \cdots, n$ are all Gaussian distributed, Lemma 1 shows that $h\left(\boldsymbol{x}_{\mathcal{S}} \mid \boldsymbol{x}_{\mathcal{T}}\right)$ is concave over the covariance matrices.

Lemma 2 includes some special cases of Lemma 1.
Lemma 2: Let $\boldsymbol{x}^{k}=\left\{\boldsymbol{x}_{1}, \cdots, \boldsymbol{x}_{k}\right\}$ and $\boldsymbol{y}^{k}=\left\{\boldsymbol{y}_{1}, \cdots, \boldsymbol{y}_{k}\right\}$ be two sequences of random vectors, and let $\widehat{\boldsymbol{x}}^{*}, \widehat{\boldsymbol{y}}^{*}, \overline{\boldsymbol{x}}^{*}$ and $\overline{\boldsymbol{y}}^{*}$ be Gaussian vectors with covariance matrices satisfying

$$
\operatorname{Cov}\left[\begin{array}{l}
\widehat{\boldsymbol{x}}^{*}  \tag{60}\\
\widehat{\boldsymbol{y}}^{*}
\end{array}\right]=\frac{1}{k} \sum_{i=1}^{k} \operatorname{Cov}\left[\begin{array}{l}
\boldsymbol{x}_{i} \\
\boldsymbol{y}_{i}
\end{array}\right] \preceq \operatorname{Cov}\left[\begin{array}{l}
\overline{\boldsymbol{x}}^{*} \\
\overline{\boldsymbol{y}}^{*}
\end{array}\right] .
$$

Then we have

$$
\begin{align*}
& h\left(\boldsymbol{x}^{k}\right) \leq k \cdot h\left(\widehat{\boldsymbol{x}}^{*}\right) \leq k \cdot h\left(\overline{\boldsymbol{x}}^{*}\right) \quad \text { and }  \tag{61}\\
& h\left(\boldsymbol{y}^{k} \mid \boldsymbol{x}^{k}\right) \leq k \cdot h\left(\widehat{\boldsymbol{y}}^{*} \mid \widehat{\boldsymbol{x}}^{*}\right) \leq k \cdot h\left(\overline{\boldsymbol{y}}^{*} \mid \overline{\boldsymbol{x}}^{*}\right) \tag{62}
\end{align*}
$$

Proof: See Appendix B.
Lemma 3: Let $\boldsymbol{x}^{n}=\left\{\boldsymbol{x}_{1}, \cdots, \boldsymbol{x}_{n}\right\}$ be a sequence of $n$ random vectors and let $\overline{\boldsymbol{x}}^{*}$ and $\widehat{\boldsymbol{x}}^{*}$ be Gaussian random vectors with covariance matrices

$$
\begin{equation*}
\operatorname{Cov}\left(\widehat{\boldsymbol{x}}^{*}\right)=\frac{1}{n} \sum_{i=1}^{n} \operatorname{Cov}\left(\boldsymbol{x}_{i}\right) \preceq \operatorname{Cov}\left(\overline{\boldsymbol{x}}^{*}\right) . \tag{63}
\end{equation*}
$$

Let $z$ and $\tilde{\boldsymbol{z}}$ be two independent Gaussian random vectors and $\boldsymbol{z}^{n}$ and $\tilde{\boldsymbol{z}}^{n}$ be two sequences of random vectors each independent and identically distributed (i.i.d.) as $z$ and $\tilde{z}$, respectively. We have

$$
\begin{align*}
h\left(\boldsymbol{x}^{n}+\boldsymbol{z}^{n}\right)-h\left(\boldsymbol{x}^{n}+\boldsymbol{z}^{n}+\tilde{\boldsymbol{z}}^{n}\right) & \leq n h\left(\widehat{\boldsymbol{x}}^{*}+\boldsymbol{z}\right)-n h\left(\widehat{\boldsymbol{x}}^{*}+\boldsymbol{z}+\tilde{\boldsymbol{z}}\right)  \tag{64}\\
& \leq n h\left(\overline{\boldsymbol{x}}^{*}+\boldsymbol{z}\right)-n h\left(\overline{\boldsymbol{x}}^{*}+\boldsymbol{z}+\tilde{\boldsymbol{z}}\right) \tag{65}
\end{align*}
$$

Proof: See Appendix C.
Lemma 4: [29, page 107] [30] Let $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ be joint Gaussian vectors. If $\operatorname{Cov}(\boldsymbol{y})$ is invertible, then $\boldsymbol{x} \rightarrow \boldsymbol{y} \rightarrow \boldsymbol{z}$ forms a Markov chain if and only if

$$
\operatorname{Cov}(x, z)=\operatorname{Cov}(x, y) \operatorname{Cov}(\boldsymbol{y})^{-1} \operatorname{Cov}(\boldsymbol{y}, \boldsymbol{z})
$$

Using Lemma 4 we obtain the following lemma.
Lemma 5: Let $\boldsymbol{x}, \boldsymbol{u}$ and $\boldsymbol{v}$ be jointly Gaussian vectors, such that $\boldsymbol{x}$ is independent of $\boldsymbol{u}$ and $\boldsymbol{v}$. Denote $\operatorname{Cov}(\boldsymbol{x})=\mathbf{S}_{x}, \operatorname{Cov}(\boldsymbol{u})=\mathbf{S}_{u}$ and $\operatorname{Cov}(\boldsymbol{u}, \boldsymbol{v})=\mathbf{S}_{u v}$. If $\mathbf{S}_{u}$ is invertible, then $\boldsymbol{x} \rightarrow \mathbf{H} \boldsymbol{x}+\boldsymbol{u} \rightarrow \mathbf{G} \boldsymbol{x}+\boldsymbol{v}$ forms a Markov chain if and only if

$$
\begin{equation*}
\mathbf{S}_{x} \mathbf{G}^{\dagger}=\mathbf{S}_{x} \mathbf{H}^{\dagger} \mathbf{S}_{u}^{-1} \mathbf{S}_{u v} \tag{66}
\end{equation*}
$$

Proof: See Appendix D.
Lemma 6: $\left[\begin{array}{ll}\mathbf{I} & \mathbf{A} \\ \mathbf{A}^{\dagger} & \mathbf{B}\end{array}\right] \succeq \mathbf{0}$ if and only if $\mathbf{B} \succeq \mathbf{A}^{\dagger} \mathbf{A}$. If $\mathbf{B} \succ \mathbf{0}$, then $\mathbf{B} \succeq \mathbf{A}^{\dagger} \mathbf{A}$ if and only if $\mathbf{I} \succeq \mathbf{A B}^{-1} \mathbf{A}^{\dagger}$.

## Proof: See Appendix E.

Lemma 7: If $\mathbf{B}$ is left-invertible (or $\mathbf{B}^{\dagger} \mathbf{B}$ is invertible) and $\mathbf{A}=\mathbf{B}\left(\mathbf{B}^{\dagger} \mathbf{B}\right)^{-1} \mathbf{C}^{\dagger}$, then $\mathbf{A}^{\dagger} \mathbf{A} \preceq \mathbf{I}$ or $\mathbf{A} \mathbf{A}^{\dagger} \preceq \mathbf{I}$ if and only if $\mathbf{B}^{\dagger} \mathbf{B} \succeq \mathbf{C}^{\dagger} \mathbf{C}$.

Proof: See Appendix F.
Lemma 8: [31, Theorem 5.2] Suppose $\mathbf{W}$ is nonsingular and $\mathbf{M}$ is positive definite. Then the matrix equation

$$
\mathbf{X}+\mathbf{W}^{\dagger} \mathbf{X}^{-1} \mathbf{W}=\mathbf{M}
$$

has a positive definite solution $\mathbf{X}$ if and only if

$$
\text { radius }\left(\mathbf{M}^{-\frac{1}{2}} \mathbf{W} \mathbf{M}^{-\frac{1}{2}}\right) \leq \frac{1}{2}
$$

Using Lemma 8, we obtain necessary and sufficient conditions for a pair of matrix equations to have positive definite solutions.

Lemma 9: Suppose $\mathbf{A}_{1}$ and $\mathbf{A}_{2}$ are fixed, and $\mathbf{I}$ is the identity matrix, the following matrix equations

$$
\begin{align*}
& \boldsymbol{\Sigma}_{1}=\mathbf{I}-\mathbf{A}_{2} \boldsymbol{\Sigma}_{2}^{-1} \mathbf{A}_{2}^{\dagger} \quad \text { and }  \tag{67}\\
& \boldsymbol{\Sigma}_{2}=\mathbf{I}-\mathbf{A}_{1} \boldsymbol{\Sigma}_{1}^{-1} \mathbf{A}_{1}^{\dagger}, \tag{68}
\end{align*}
$$

have positive definite solutions $\boldsymbol{\Sigma}_{1} \succ \mathbf{A}_{1}^{\dagger} \mathbf{A}_{1}$ and $\boldsymbol{\Sigma}_{2} \succ \mathbf{A}_{2}^{\dagger} \mathbf{A}_{2}$ if and only if

$$
\begin{equation*}
\operatorname{radius}\left(\boldsymbol{\Phi}_{i}\right) \leq \frac{1}{2}, \quad i=1,2, \tag{69}
\end{equation*}
$$

where

$$
\begin{align*}
& \mathbf{\Phi}_{1}=\left(\mathbf{I}-\mathbf{A}_{1}^{\dagger} \mathbf{A}_{1}-\mathbf{A}_{2} \mathbf{A}_{2}^{\dagger}\right)^{-\frac{1}{2}} \mathbf{A}_{1}^{\dagger} \mathbf{A}_{2}^{\dagger}\left(\mathbf{I}-\mathbf{A}_{1}^{\dagger} \mathbf{A}_{1}-\mathbf{A}_{2} \mathbf{A}_{2}^{\dagger}\right)^{-\frac{1}{2}} \text { and }  \tag{70}\\
& \mathbf{\Phi}_{2}=\left(\mathbf{I}-\mathbf{A}_{1} \mathbf{A}_{1}^{\dagger}-\mathbf{A}_{2}^{\dagger} \mathbf{A}_{2}\right)^{-\frac{1}{2}} \mathbf{A}_{2}^{\dagger} \mathbf{A}_{1}^{\dagger}\left(\mathbf{I}-\mathbf{A}_{1} \mathbf{A}_{1}^{\dagger}-\mathbf{A}_{2}^{\dagger} \mathbf{A}_{2}\right)^{-\frac{1}{2}} \tag{71}
\end{align*}
$$

Proof: See Appendix G.

## B. Proof of Theorem 1

The converse follows by giving receiver 1 the message not destined for it and applying the maximumentropy theory to show that Gaussian input distributions are optimal. To prove achievability, let $\boldsymbol{x}_{1} \sim$ $\mathcal{C N}\left(\mathbf{0}, \mathbf{S}_{1}\right)$ and $\boldsymbol{x}_{2} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{S}_{2}\right)$, and let user 1 transmit at rate $R_{1}=\log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\right|$, and user 2 transmit at rate $R_{2}=\log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|$. Inequality (4) guarantees that user 1 can first decode $\boldsymbol{x}_{2}$ by treating $x_{1}$ as noise. After the interference is subtracted, user 1 sees a single-user Gaussian MIMO channel. Therefore, the rate region (5) is achievable.

## C. Proof of Theorem 2

Similarly to the proof of Theorem 1, the converse follows by giving each receiver the message not destined for it and applying the maximum-entropy theory to show that Gaussian input distributions are optimal. To prove the achievability, let $\boldsymbol{x}_{1} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{S}_{1}\right)$ and $\boldsymbol{x}_{2} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{S}_{2}\right)$, and let user 1 transmit at rate $R_{1}=\log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\right|$, and user 2 transmit at rate $R_{2}=\log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|$. Inequalities (6) and (7) guarantee that each user can first fully decode the interference by treating the desired signals as noise. After the interference is subtracted, each user sees a single-user Gaussian MIMO channel. Therefore, the rate region (8) is achievable.

## D. Proof of Theorem 3 and Proposition 1

Suppose the channel is used $n$ times. The transmitted and received vector sequences are denoted by $\boldsymbol{x}_{i}^{n}$ and $\boldsymbol{y}_{i}^{n}$ for user $i, i=1,2$, and $\boldsymbol{x}_{i}^{n}$ satisfies (2).

Since $\mathbf{A}^{\dagger} \mathbf{A} \preceq \mathbf{I}$, from Lemma 6 , there exists a Gaussian random vector $\boldsymbol{n}$ whose joint distribution with $z_{2}$ is

$$
\left[\begin{array}{l}
z_{2}  \tag{72}\\
n
\end{array}\right] \sim \mathcal{C N}\left(\mathbf{0},\left[\begin{array}{cc}
\mathbf{I} & \mathbf{A} \\
\mathbf{A}^{\dagger} & \mathbf{I}
\end{array}\right]\right)
$$

Moreover, from (9), $\boldsymbol{n}$ is of the same dimension as $\boldsymbol{z}_{1}$ hence has the same marginal distribution as $\boldsymbol{z}_{1}$.
Let $\epsilon>0$ and $\epsilon \rightarrow 0$ as $n \rightarrow+\infty$, From Fano's inequality, any achievable rates must satisfy

$$
\begin{align*}
& n\left(R_{1}+R_{2}\right)-n \epsilon \\
& \leq I\left(\boldsymbol{x}_{1}^{n} ; \boldsymbol{y}_{1}^{n}\right)+I\left(\boldsymbol{x}_{2}^{n} ; \boldsymbol{y}_{2}^{n}\right) \\
& \leq I\left(\boldsymbol{x}_{1}^{n} ; \boldsymbol{y}_{1}^{n}\right)+I\left(\boldsymbol{x}_{2}^{n} ; \boldsymbol{y}_{2}^{n}, \mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}^{n}\right) \\
&= h\left(\mathbf{H}_{1} \boldsymbol{x}_{1}^{n}+\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{1}^{n}\right)-h\left(\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{1}^{n}\right)+h\left(\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}^{n}\right)-h\left(\boldsymbol{n}^{n}\right) \\
&+h\left(\mathbf{H}_{4} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{2}^{n} \mid \mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}^{n}\right)-h\left(\boldsymbol{z}_{2}^{n} \mid \boldsymbol{n}^{n}\right) \\
& \stackrel{(a)}{=} I\left(\boldsymbol{x}_{1}^{n}, \boldsymbol{x}_{2}^{n} ; \mathbf{H}_{1} \boldsymbol{x}_{1}^{n}+\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{1}^{n}\right)+h\left(\mathbf{H}_{4} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{2}^{n} \mid \mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}^{n}\right)-h\left(\boldsymbol{z}_{2}^{n} \mid \boldsymbol{n}^{n}\right) \\
& \stackrel{(b)}{\leq} I\left(\boldsymbol{x}_{1}^{n}, \boldsymbol{x}_{2}^{n} ; \mathbf{H}_{1} \boldsymbol{x}_{1}^{n}+\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{1}^{n}\right)+n h\left(\mathbf{H}_{4} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{2} \mid \mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n}\right)-n h\left(\boldsymbol{z}_{2} \mid \boldsymbol{n}\right) \\
& \stackrel{(c)}{=} I\left(\boldsymbol{x}_{1}^{n}, \boldsymbol{x}_{2}^{n} ; \mathbf{H}_{1} \boldsymbol{x}_{1}^{n}+\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{1}^{n}\right)+n h\left(\mathbf{H}_{4} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{2} \mid \mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n}, \overline{\boldsymbol{x}}_{2}^{*}\right)-n h\left(\boldsymbol{z}_{2} \mid \boldsymbol{n}\right) \\
&= I\left(\boldsymbol{x}_{1}^{n}, \boldsymbol{x}_{2}^{n} ; \mathbf{H}_{1} \boldsymbol{x}_{1}^{n}+\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{1}^{n}\right) \\
& \leq n \log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right|, \tag{73}
\end{align*}
$$

where $\boldsymbol{z}_{i}^{n}=\left[\boldsymbol{z}_{i, 1}^{\dagger}, \boldsymbol{z}_{i, 2}^{\dagger}, \ldots, \boldsymbol{z}_{i, n}^{\dagger}\right]^{\dagger}$ and $\boldsymbol{n}^{n}=\left[\boldsymbol{n}_{1}^{\dagger}, \boldsymbol{n}_{2}^{\dagger}, \ldots, \boldsymbol{n}_{n}^{\dagger}\right]^{\dagger}, i=1,2$, and $\left[\boldsymbol{z}_{2, j}^{\dagger}, \boldsymbol{n}_{j}^{\dagger}\right]^{\dagger}, j=1, \ldots, n$, are i.i.d. as (72).

Equality (a) is from the fact that $n$ and $z_{1}$ have the same marginal distribution. Inequality (b) is by Lemma 2, and we let $\overline{\boldsymbol{x}}_{i}^{*} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{S}_{i}\right), i=1,2 . \overline{\boldsymbol{x}}_{1}^{*}$ is independent of $\overline{\boldsymbol{x}}_{2}^{*}$ and $\overline{\boldsymbol{y}}_{i}^{*}$ is defined in (1) with $x_{i}$ replaced by $\bar{x}_{i}^{*}$. Equality (c) is from (9) which means

$$
\mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}=\mathbf{S}_{2}\left(\mathbf{H}_{2}^{\dagger} \mathbf{A}^{\dagger}+\mathbf{B}^{\dagger}\right)=\mathbf{S}_{2} \mathbf{H}_{2}^{\dagger} \mathbf{A}^{\dagger} .
$$

By Lemma 5, $\overline{\boldsymbol{x}}_{2}^{*} \rightarrow \mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n} \rightarrow \mathbf{H}_{4} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{2}$ forms a Markov chain.
Therefore, (10) is an outer bound for the capacity region. On the other hand, (10) is also achievable by requiring user 1 to decode messages from both users. Therefore, Theorem 3 is proved.

If $\mathbf{H}_{2}$ is left-invertible, we can choose

$$
\begin{equation*}
\mathbf{A}^{\dagger}=\mathbf{H}_{2}\left(\mathbf{H}_{2}^{\dagger} \mathbf{H}_{2}\right)^{-1}\left(\mathbf{H}_{4}^{\dagger}-\mathbf{B}^{\dagger}\right) \tag{74}
\end{equation*}
$$

so that (9) is satisfied. By Lemma $7, \mathbf{A}^{\dagger} \mathbf{A} \preceq \mathbf{I}$ is equivalent to (12). Thus Proposition 1 is proved.

## E. Proof of Theorem 4 and Proposition 2

Theorem 4 can be proved by using Theorem 3 twice. To prove a converse, we first remove the interference link from transmitter 1 to receiver 2 and obtain a MIMO ZIC with $\mathbf{H}_{3}=\mathbf{0}$. The capacity region of the original MIMO IC is a subset of the capacity region of this MIMO ZIC because we are effectively giving user 1's message to receiver 2 . Theorem 3 gives the capacity region of this MIMO ZIC with (14). Similarly, we remove the interference link from transmitter 2 to receiver 1 and obtain a MIMO ZIC with $\mathbf{H}_{2}=\mathbf{0}$. Theorem 3 gives the capacity region of this MIMO ZIC with (13):

$$
\left\{\begin{array}{c}
R_{1} \leq \log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\right|  \tag{75}\\
R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right| \\
R_{1}+R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{3} \mathbf{S}_{1} \mathbf{H}_{3}^{\dagger}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|
\end{array}\right\} .
$$

Thus, the capacity region of the original MIMO IC is included in the intersection of (10) and (75) which is (15). On the other hand (15) is achievable by requiring both receivers to decode messages from both transmitters, and therefore (15) is the capacity region.

Proposition 2 is similarly proved as Proposition 1.

## F. Proof of Theorem 5 and Propositions 4 and 5

Since $\mathbf{A}^{\dagger} \mathbf{A} \preceq \mathbf{I}$, from Lemma 6 there exists a Gaussian random vector $\boldsymbol{n}$ whose joint distribution with $z_{2}$ is

$$
\left[\begin{array}{l}
z_{2}  \tag{76}\\
n
\end{array}\right] \sim \mathcal{C N}\left(\mathbf{0},\left[\begin{array}{cc}
\mathbf{I} & \mathbf{A} \\
\mathbf{A}^{\dagger} & \mathbf{I}
\end{array}\right]\right) .
$$

Moreover, (30) and (76) mean that $n$ and $z_{1}$ have the same dimension and distribution.
From Fano's inequality, any achievable rates must satisfy

$$
\begin{aligned}
& n\left(R_{1}+R_{2}\right)-n \epsilon \\
& \leq I\left(\boldsymbol{x}_{1}^{n} ; \boldsymbol{y}_{1}^{n}\right)+I\left(\boldsymbol{x}_{2}^{n} ; \boldsymbol{y}_{2}^{n}\right) \\
& \leq I\left(\boldsymbol{x}_{1}^{n} ; \boldsymbol{y}_{1}^{n}\right)+I\left(\boldsymbol{x}_{2}^{n} ; \boldsymbol{y}_{2}^{n}, \mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}^{n}\right)
\end{aligned}
$$

$$
\begin{align*}
&= h\left(\mathbf{H}_{1} \boldsymbol{x}_{1}^{n}+\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{1}^{n}\right)-h\left(\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{1}^{n}\right)+h\left(\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}^{n}\right)-h\left(\boldsymbol{n}^{n}\right) \\
&+h\left(\mathbf{H}_{4} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{2}^{n} \mid \mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}^{n}\right)-h\left(\boldsymbol{z}_{2}^{n} \mid \boldsymbol{n}^{n}\right) \\
& \stackrel{(a)}{=} h\left(\mathbf{H}_{1} \boldsymbol{x}_{1}^{n}+\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{1}^{n}\right)-h\left(\boldsymbol{n}^{n}\right)+h\left(\mathbf{H}_{4} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{2}^{n} \mid \mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}^{n}\right)-h\left(\boldsymbol{z}_{2}^{n} \mid \boldsymbol{n}^{n}\right) \\
& \stackrel{(b)}{\leq} n h\left(\mathbf{H}_{1} \overline{\boldsymbol{x}}_{1}^{*}+\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{1}\right)-n h(\boldsymbol{n})+n h\left(\mathbf{H}_{4} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{2} \mid \mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n}\right)-n h\left(\boldsymbol{z}_{2} \mid \boldsymbol{n}\right)  \tag{77}\\
&= n h\left(\mathbf{H}_{1} \overline{\boldsymbol{x}}_{1}^{*}+\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{1}\right)-n h(\boldsymbol{n})+n h\left(\mathbf{H}_{4} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{2}\right)+n h\left(\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n} \mid \mathbf{H}_{4} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{2}\right) \\
&-n h\left(\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n}\right)-n h\left(\boldsymbol{z}_{2} \mid \boldsymbol{n}\right) \\
& \stackrel{(c)}{=} n h\left(\mathbf{H}_{1} \overline{\boldsymbol{x}}_{1}^{*}+\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{1}\right)-n h(\boldsymbol{n})+n h\left(\mathbf{H}_{4} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{2}\right)+n h\left(\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n} \mid \mathbf{H}_{4} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{2}, \overline{\boldsymbol{x}}_{2}^{*}\right) \\
&-n h\left(\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n}\right)-n h\left(\boldsymbol{z}_{2} \mid \boldsymbol{n}\right) \\
& \stackrel{(d)}{=} n h\left(\mathbf{H}_{1} \overline{\boldsymbol{x}}_{1}^{*}+\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{1}\right)-n h(\boldsymbol{n})+n h\left(\mathbf{H}_{4} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{2}\right)+n h\left(\boldsymbol{n} \mid \boldsymbol{z}_{2}\right)-n h\left(\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{1}\right)-n h\left(\boldsymbol{z}_{2} \mid \boldsymbol{n}\right) \\
&= n h\left(\mathbf{H}_{1} \overline{\boldsymbol{x}}_{1}^{*}+\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{1}\right)-n h\left(\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{1}\right)+n h\left(\mathbf{H}_{4} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{2}\right)-n h\left(\boldsymbol{z}_{2}\right) \\
&= n \log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right)\right|+n \log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|, \tag{78}
\end{align*}
$$

where $\boldsymbol{n}^{n}=\left[\boldsymbol{n}_{1}^{\dagger}, \boldsymbol{n}_{2}^{\dagger}, \ldots, \boldsymbol{n}_{n}^{\dagger}\right]^{\dagger}$, and the $\boldsymbol{n}_{i}$ are i.i.d. Gaussian vectors distributed as (76).
Equalities (a) and (d) are both from the fact that $n$ and $z_{1}$ have the same marginal distribution. Inequality (b) is from Lemma 2, and we let $\overline{\boldsymbol{x}}_{i}^{*} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{S}_{i}\right), i=1,2 . \overline{\boldsymbol{x}}_{1}^{*}$ is independent of $\overline{\boldsymbol{x}}_{2}^{*}$ and $\overline{\boldsymbol{y}}_{i}^{*}$ is defined in (1) with $\boldsymbol{x}_{i}$ replaced by $\overline{\boldsymbol{x}}_{i}^{*}$. Equality (c) is from (30) which means

$$
\mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}=\mathbf{S}_{2} \mathbf{H}_{4}^{\dagger} \mathbf{A}
$$

By Lemma 5, $\overline{\boldsymbol{x}}_{2}^{*} \rightarrow \mathbf{H}_{4} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{2} \rightarrow \mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n}$ forms a Markov chain.
Since (31) is achievable, the sum-rate capacity is (31) if (30) holds. Therefore, Theorem 5 is proved.
When $\mathbf{H}_{4}$ is left-invertible, we can choose

$$
\begin{equation*}
\mathbf{A}=\mathbf{H}_{4}\left(\mathbf{H}_{4}^{\dagger} \mathbf{H}_{4}\right)^{-1}\left(\mathbf{H}_{2}^{\dagger}-\mathbf{B}^{\dagger}\right) \tag{79}
\end{equation*}
$$

Then (30) is satisfied. By Lemma $7, \mathbf{A}^{\dagger} \mathbf{A} \preceq \mathbf{I}$ is equivalent to (32), therefore Proposition 4 is proved.
Proposition 5 is proved in a similar way as Theorem 5. Let $\widehat{\boldsymbol{x}}_{i} \sim \mathcal{C N}\left(\mathbf{0}, \widehat{\mathbf{S}}_{i}\right), i=1,2$, where

$$
\begin{equation*}
\widehat{\mathbf{S}}_{i}=\frac{1}{n} \sum_{j=1}^{n} \operatorname{Cov}\left(\boldsymbol{x}_{i j}\right) . \tag{80}
\end{equation*}
$$

From Fano's inequality, we have

$$
\begin{aligned}
& n\left(R_{1}+R_{2}\right)-n \epsilon \\
& \leq I\left(\boldsymbol{x}_{1}^{n} ; \boldsymbol{y}_{1}^{n}\right)+I\left(\boldsymbol{x}_{2}^{n} ; \boldsymbol{y}_{2}^{n}, \mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}^{n}\right)
\end{aligned}
$$

$$
\begin{align*}
& =h\left(\mathbf{H}_{1} \boldsymbol{x}_{1}^{n}+\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{1}^{n}\right)-h\left(\boldsymbol{n}^{n}\right)+h\left(\mathbf{H}_{4} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{2}^{n} \mid \mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}^{n}\right)-h\left(\boldsymbol{z}_{2}^{n} \mid \boldsymbol{n}^{n}\right) \\
& \stackrel{(a)}{\leq} n h\left(\mathbf{H}_{1} \widehat{\boldsymbol{x}}_{1}+\mathbf{H}_{2} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{z}_{1}\right)-n h(\boldsymbol{n})+n h\left(\mathbf{H}_{4} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{z}_{2} \mid \mathbf{H}_{2} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{n}\right)-n h\left(\boldsymbol{z}_{2} \mid \boldsymbol{n}\right) \\
& \stackrel{(b)}{=} n h\left(\mathbf{H}_{1} \widehat{\boldsymbol{x}}_{1}+\mathbf{H}_{2} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{z}_{1}\right)-n h(\boldsymbol{n})+n h\left(\mathbf{H}_{4} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{z}_{2}\right)+n h\left(\mathbf{H}_{2} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{n} \mid \mathbf{H}_{4} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{z}_{2}, \widehat{\boldsymbol{x}}_{2}\right) \\
& \\
& \quad-n h\left(\mathbf{H}_{2} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{n}\right)-n h\left(\boldsymbol{z}_{2} \mid \boldsymbol{n}\right) \\
& =n h\left(\mathbf{H}_{1} \widehat{\boldsymbol{x}}_{1}+\mathbf{H}_{2} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{z}_{1}\right)-n h\left(\mathbf{H}_{2} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{n}\right)+n h\left(\mathbf{H}_{4} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{z}_{2}\right)-n h\left(\boldsymbol{z}_{2}\right)  \tag{81}\\
& =\log \left|\mathbf{I}+\mathbf{H}_{1} \widehat{\mathbf{S}}_{1} \mathbf{H}_{1}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{2} \widehat{\mathbf{S}}_{2} \mathbf{H}_{2}^{\dagger}\right)^{-1}\right|+\log \left|\mathbf{I}+\mathbf{H}_{4} \widehat{\mathbf{S}}_{2} \mathbf{H}_{4}^{\dagger}\right|
\end{align*}
$$

where (a) is from Lemma 2; and (b) is from (33) which means $\widehat{\mathbf{S}}_{2} \mathbf{H}_{2}^{\dagger}=\widehat{\mathbf{S}}_{2} \mathbf{H}_{4}^{\dagger} \mathbf{A}$ and thus by Lemma 5, $\widehat{\boldsymbol{x}}_{2} \rightarrow \mathbf{H}_{4} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{z}_{2} \rightarrow \mathbf{H}_{2} \widehat{\boldsymbol{x}}_{2}+\boldsymbol{n}$ forms a Markov chain.

## G. Proof of Theorem 6 and Proposition 6

Since there exist $\boldsymbol{\Sigma}_{1}$ and $\boldsymbol{\Sigma}_{2}$ which satisfy (35) and (36), by Lemma 6, there exist two random vectors $\boldsymbol{n}_{1}$ and $\boldsymbol{n}_{2}$ whose joint distributions with $z_{1}$ and $z_{2}$ are

$$
\left[\begin{array}{l}
\boldsymbol{z}_{i}  \tag{82}\\
\boldsymbol{n}_{i}
\end{array}\right] \sim \mathcal{C N}\left(\mathbf{0},\left[\begin{array}{cc}
\mathbf{I} & \mathbf{A}_{i} \\
\mathbf{A}_{i}^{\dagger} & \boldsymbol{\Sigma}_{i}
\end{array}\right]\right), \quad i=1,2
$$

Furthermore, from (35) and (36) we have

$$
\begin{align*}
& \operatorname{Cov}\left(\boldsymbol{n}_{1}\right) \preceq \operatorname{Cov}\left(\boldsymbol{z}_{2} \mid \boldsymbol{n}_{2}\right) \quad \text { and }  \tag{83}\\
& \operatorname{Cov}\left(\boldsymbol{n}_{2}\right) \preceq \operatorname{Cov}\left(\boldsymbol{z}_{1} \mid \boldsymbol{n}_{1}\right) . \tag{84}
\end{align*}
$$

From Fano's inequality, any achievable sum rate $R_{1}+R_{2}$ must satisfy

$$
\begin{align*}
& n\left(R_{1}+R_{2}\right)-n \epsilon \\
& \leq I\left(\boldsymbol{x}_{1}^{n} ; \boldsymbol{y}_{1}^{n}\right)+I\left(\boldsymbol{x}_{2}^{n} ; \boldsymbol{y}_{2}^{n}\right) \\
& \leq I\left(\boldsymbol{x}_{1}^{n} ; \boldsymbol{y}_{1}^{n}, \mathbf{H}_{3} \boldsymbol{x}_{1}^{n}+\boldsymbol{n}_{1}^{n}\right)+I\left(\boldsymbol{x}_{2}^{n} ; \boldsymbol{y}_{2}^{n}, \mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}_{2}^{n}\right) \\
& =h\left(\mathbf{H}_{3} \boldsymbol{x}_{1}^{n}+\boldsymbol{n}_{1}^{n}\right)-h\left(\boldsymbol{n}_{1}^{n}\right)+h\left(\boldsymbol{y}_{1}^{n} \mid \mathbf{H}_{3} \boldsymbol{x}_{1}^{n}+\boldsymbol{n}_{1}^{n}\right)-h\left(\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{1}^{n} \mid \boldsymbol{n}_{1}^{n}\right)+h\left(\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}_{2}^{n}\right)-h\left(\boldsymbol{n}_{2}^{n}\right) \\
& \quad+h\left(\boldsymbol{y}_{2}^{n} \mid \mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}_{2}^{n}\right)-h\left(\mathbf{H}_{3} \boldsymbol{x}_{1}^{n}+\boldsymbol{z}_{2}^{n} \mid \boldsymbol{n}_{2}^{n}\right), \tag{85}
\end{align*}
$$

where $\boldsymbol{n}_{i}^{n}=\left[\boldsymbol{n}_{i, 1}^{\dagger}, \boldsymbol{n}_{i, 2}^{\dagger}, \ldots, \boldsymbol{n}_{i, n}^{\dagger}\right]^{\dagger}$, and the $\boldsymbol{n}_{i, j}$ are i.i.d. Gaussian vectors distributed as (82). Since $\boldsymbol{n}_{1, j}$ is independent of $\boldsymbol{n}_{1, k}$, and $\boldsymbol{z}_{2, j}$ is independent of $\boldsymbol{n}_{2, k}$, for any $j \neq k$, from (83) we have

$$
\begin{equation*}
\operatorname{Cov}\left(\boldsymbol{n}_{1}^{n}\right) \preceq \operatorname{Cov}\left(z_{2}^{n} \mid \boldsymbol{n}_{2}^{n}\right) . \tag{86}
\end{equation*}
$$

By Lemma 3 we have

$$
\begin{equation*}
h\left(\mathbf{H}_{3} \boldsymbol{x}_{1}^{n}+\boldsymbol{n}_{1}^{n}\right)-h\left(\mathbf{H}_{3} \boldsymbol{x}_{1}^{n}+\boldsymbol{z}_{2}^{n} \mid \boldsymbol{n}_{2}^{n}\right) \leq n h\left(\mathbf{H}_{3} \overline{\boldsymbol{x}}_{1}^{*}+\boldsymbol{n}_{1}\right)-n h\left(\mathbf{H}_{3} \overline{\boldsymbol{x}}_{1}^{*}+\boldsymbol{z}_{2} \mid \boldsymbol{n}_{2}\right), \tag{87}
\end{equation*}
$$

where $\overline{\boldsymbol{x}}_{1}^{*} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{S}_{1}\right)$. Similarly, we have

$$
\begin{equation*}
h\left(\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}_{2}^{n}\right)-h\left(\mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{z}_{1}^{n} \mid \boldsymbol{n}_{1}^{n}\right) \leq n h\left(\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n}_{2}\right)-n h\left(\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{1} \mid \boldsymbol{n}_{1}\right), \tag{88}
\end{equation*}
$$

where $\overline{\boldsymbol{x}}_{2}^{*} \sim \mathcal{C N}\left(\mathbf{0}, \mathbf{S}_{2}\right)$.
By Lemma 2 we have

$$
\begin{align*}
& h\left(\boldsymbol{y}_{1}^{n} \mid \mathbf{H}_{3} \boldsymbol{x}_{1}^{n}+\boldsymbol{n}_{1}^{n}\right) \leq n h\left(\overline{\boldsymbol{y}}_{1}^{*} \mid \mathbf{H}_{3} \overline{\boldsymbol{x}}_{1}^{*}+\boldsymbol{n}_{1}\right) \quad \text { and }  \tag{89}\\
& h\left(\boldsymbol{y}_{2}^{n} \mid \mathbf{H}_{2} \boldsymbol{x}_{2}^{n}+\boldsymbol{n}_{2}^{n}\right) \leq n h\left(\overline{\boldsymbol{y}}_{2}^{*} \mid \mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n}_{2}\right), \tag{90}
\end{align*}
$$

where $\overline{\boldsymbol{y}}_{i}^{*}$ is defined in (1) with $\boldsymbol{x}_{j}, j=1,2$, replaced by $\overline{\boldsymbol{x}}_{j}^{*}$.
On substituting (87)-(90) into (85) we have

$$
\begin{align*}
& R_{1}+R_{2}-\epsilon \\
& \leq h\left(\mathbf{H}_{3} \overline{\boldsymbol{x}}_{1}^{*}+\boldsymbol{n}_{1}\right)-h\left(\boldsymbol{n}_{1}\right)+h\left(\overline{\boldsymbol{y}}_{1}^{*} \mid \mathbf{H}_{3} \overline{\boldsymbol{x}}_{1}^{*}+\boldsymbol{n}_{1}\right)-h\left(\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{z}_{1} \mid \boldsymbol{n}_{1}\right) \\
& \quad+h\left(\mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n}_{2}\right)-h\left(\boldsymbol{n}_{2}\right)+h\left(\overline{\boldsymbol{y}}_{2}^{*} \mid \mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n}_{2}\right)-h\left(\mathbf{H}_{3} \overline{\boldsymbol{x}}_{1}^{*}+\boldsymbol{z}_{2} \mid \boldsymbol{n}_{2}\right) \\
& =I\left(\overline{\boldsymbol{x}}_{1}^{*} ; \overline{\boldsymbol{y}}_{1}^{*}, \mathbf{H}_{3} \overline{\boldsymbol{x}}_{1}^{*}+\boldsymbol{n}_{1}\right)+I\left(\overline{\boldsymbol{x}}_{2}^{*} ; \overline{\boldsymbol{y}}_{2}^{*}, \mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n}_{2}\right) \\
& \stackrel{(a)}{=} I\left(\overline{\boldsymbol{x}}_{1}^{*} ; \overline{\boldsymbol{y}}_{1}^{*}\right)+I\left(\overline{\boldsymbol{x}}_{2}^{*} ; \overline{\boldsymbol{y}}_{2}^{*}\right), \\
& =\log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right)^{-1}\right|+\log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{3} \mathbf{S}_{1} \mathbf{H}_{3}^{\dagger}\right)^{-1}\right|, \tag{91}
\end{align*}
$$

where (a) is from (37), (38) and Lemma 5 since $\overline{\boldsymbol{x}}_{1}^{*} \rightarrow \overline{\boldsymbol{y}}_{1}^{*} \rightarrow \mathbf{H}_{3} \overline{\boldsymbol{x}}_{1}^{*}+\boldsymbol{n}_{1}$ and $\overline{\boldsymbol{x}}_{2}^{*} \rightarrow \overline{\boldsymbol{y}}_{2}^{*} \rightarrow \mathbf{H}_{2} \overline{\boldsymbol{x}}_{2}^{*}+\boldsymbol{n}_{2}$ form two Markov chains.

On the other hand (91) is achievable by treating interference as noise, and therefore (91) is the sum-rate capacity.

Proposition 6 is straightforward from Theorem 6.

## H. Proof of Proposition 7 and Proposition 8

Since matrices $\mathbf{A}_{1}$ and $\mathbf{A}_{2}$ satisfy (42), by Lemma 9 there exist two Hermitian positive definite matrices $\Sigma_{1}$ and $\boldsymbol{\Sigma}_{2}$ that satisfy

$$
\begin{align*}
& \mathbf{A}_{1}^{\dagger} \mathbf{A}_{1} \preceq \boldsymbol{\Sigma}_{1}=\mathbf{I}-\mathbf{A}_{2} \boldsymbol{\Sigma}_{2}^{-1} \mathbf{A}_{2}^{\dagger} \text { and }  \tag{92}\\
& \mathbf{A}_{2}^{\dagger} \mathbf{A}_{2} \preceq \boldsymbol{\Sigma}_{2}=\mathbf{I}-\mathbf{A}_{1} \boldsymbol{\Sigma}_{1}^{-1} \mathbf{A}_{1}^{\dagger} . \tag{93}
\end{align*}
$$

Thus, we see (35) and (36) are satisfied. Since (37) and (38) are satisfied by hypothesis, Proposition 7 follows by Theorem 6 .

Proposition 8 is straightforward from Proposition 7.

## I. Proof of Theorem 7 and Propositions 9 and 10

The achievability part is straightforward by letting user 2 first decode the message from user 1 and then decode its own message, and by letting user 1 treat signals from user 2 as noise. Then user 1 and user 2 have the respective rates

$$
\begin{aligned}
& R_{1}=\min \left\{\begin{array}{l}
\log \mid \mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right)^{-1} \\
\log \left|\mathbf{I}+\mathbf{H}_{3} \mathbf{S}_{1} \mathbf{H}_{3}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right)^{-1}\right|
\end{array}\right\} \text { and } \\
& R_{2}=\log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|
\end{aligned}
$$

Therefore, the sum rate (47) is achievable.
To prove the converse, we first let $\mathbf{H}_{2}=\mathbf{0}$. By using (45) and Theorem 3, the sum rate satisfies

$$
R_{1}+R_{2} \leq \min \left\{\begin{array}{c}
\log \left|\mathbf{I}+\mathbf{H}_{3} \mathbf{S}_{1} \mathbf{H}_{3}^{\dagger}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|  \tag{94}\\
\log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\right|+\log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|
\end{array}\right\}
$$

Alternatively, we let $\mathbf{H}_{3}=\mathbf{0}$. By using (46) and Theorem 5, the sum rate also satisfies

$$
\begin{equation*}
R_{1}+R_{2} \leq \log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right)^{-1}\right|+\log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right| \tag{95}
\end{equation*}
$$

Combining (94) and (95), we have

$$
R_{1}+R_{2} \leq \min \left\{\log \left|\mathbf{I}+\mathbf{H}_{3} \mathbf{S}_{1} \mathbf{H}_{3}^{\dagger}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|, \begin{array}{c}
\log +\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\left(\mathbf{I}+\mathbf{H}_{2} \mathbf{S}_{2} \mathbf{H}_{2}^{\dagger}\right)^{-1}|+\log | \mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger} \mid  \tag{96}\\
\log \left|\mathbf{I}+\mathbf{H}_{1} \mathbf{S}_{1} \mathbf{H}_{1}^{\dagger}\right|+\log \left|\mathbf{I}+\mathbf{H}_{4} \mathbf{S}_{2} \mathbf{H}_{4}^{\dagger}\right|
\end{array}\right\}
$$

We complete the proof by pointing out that the last line of (96) is redundant because of the second line.
Proposition 9 is similarly proved by Propositions 1 and 4. Proposition 10 is similarly proved by Propositions 3 and 5.

## J. Proof of Theorem 8

The proof of Theorem 8 follows that of Theorem 5. The bound in problem (53) is derived from (77) by assuming $\overline{\boldsymbol{x}}_{i}^{*} \sim \mathcal{C N}\left(\mathbf{0}, \widehat{\mathbf{S}}_{i}\right), i=1,2$. Following similar steps as in (78), one can verify that the sum-rate capacity is (57) if (54) is satisfied.

## ApPENDIX

## A. Proof of Lemma 1

Let $\boldsymbol{x}_{i, S}^{*}$ be a Gaussian vector with covariance matrix $\operatorname{Cov}\left(\boldsymbol{x}_{i, S}\right)$. We have

$$
\begin{align*}
\sum_{i=1}^{k} \lambda_{i} h\left(\boldsymbol{x}_{i, \mathcal{S}} \mid \boldsymbol{x}_{i, \mathcal{T}}\right) & \stackrel{(a)}{\leq} \sum_{i=1}^{k} \lambda_{i} h\left(\boldsymbol{x}_{i, \mathcal{S}}^{*} \mid \boldsymbol{x}_{i, \mathcal{T}}^{*}\right) \\
& =\sum_{i=1}^{k} \lambda_{i}\left[h\left(\boldsymbol{x}_{i, \mathcal{S} \cup \mathcal{T}}^{*}\right)-h\left(\boldsymbol{x}_{i, \mathcal{T}}^{*}\right)\right] \\
& =\sum_{i=1}^{k} \lambda_{i} \log \left(\frac{\left|\operatorname{Cov}\left(\boldsymbol{x}_{i, \mathcal{S} \cup \mathcal{T}}^{*}\right)\right|}{\left|\operatorname{Cov}\left(x_{i, \mathcal{T}}^{*}\right)\right|} \cdot(\pi e)^{\sum_{j \in \mathcal{S}} L_{j}}\right) \\
& \leq \sum_{i=1}^{(b)} \log \left(\frac{\left|\operatorname{Cov}\left(\boldsymbol{y}_{\mathcal{S} \cup \mathcal{T}}\right)\right|}{\left|\operatorname{Cov}\left(\boldsymbol{y}_{\mathcal{T}}\right)\right|} \cdot(\pi e)^{\sum_{j \in \mathcal{S}} L_{j}}\right) \\
& =h\left(\boldsymbol{y}_{\mathcal{S}} \mid \boldsymbol{y}_{\mathcal{T}}\right), \tag{97}
\end{align*}
$$

where inequality (a) is from [28, Lemma 2], and inequality (b) is from [32, Theorem 17.10.1].

## B. Proof of Lemma 2

The first inequalities of (61) and (62) are straightforward from Lemma 1. It suffices to prove the second inequality of (62). Since (60) holds, we can define two random vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ that are joint Gaussian, independent of $\widehat{\boldsymbol{x}}^{*}$ and $\widehat{\boldsymbol{y}}^{*}$, and satisfy

$$
\left[\begin{array}{l}
\bar{x}^{*}  \tag{98}\\
\bar{y}^{*}
\end{array}\right]=\left[\begin{array}{l}
\widehat{x}^{*} \\
\widehat{y}^{*}
\end{array}\right]+\left[\begin{array}{l}
u \\
v
\end{array}\right] .
$$

Therefore,

$$
\begin{equation*}
h\left(\overline{\boldsymbol{y}}^{*} \mid \overline{\boldsymbol{x}}^{*}\right) \geq h\left(\overline{\boldsymbol{y}}^{*} \mid \overline{\boldsymbol{x}}^{*}, \boldsymbol{u}, \boldsymbol{v}\right)=h\left(\widehat{\boldsymbol{y}}^{*} \mid \widehat{\boldsymbol{x}}^{*}\right) . \tag{99}
\end{equation*}
$$

C. Proof of Lemma 3

$$
\begin{aligned}
& h\left(\boldsymbol{x}^{n}+\boldsymbol{z}^{n}\right)-h\left(\boldsymbol{x}^{n}+\boldsymbol{z}^{n}+\tilde{\boldsymbol{z}}^{n}\right) \\
& =-I\left(\tilde{\boldsymbol{z}}^{n} ; \boldsymbol{x}^{n}+\boldsymbol{z}^{n}+\tilde{\boldsymbol{z}}^{n}\right) \\
& \stackrel{(a)}{\leq}-I\left(\tilde{\boldsymbol{z}}^{n} ; \boldsymbol{x}^{* n}+\boldsymbol{z}^{n}+\tilde{\boldsymbol{z}}^{n}\right) \\
& =-h\left(\tilde{\boldsymbol{z}}^{n}\right)+h\left(\tilde{\boldsymbol{z}}^{n} \mid \boldsymbol{x}^{* n}+\boldsymbol{z}^{n}+\tilde{\boldsymbol{z}}^{n}\right)
\end{aligned}
$$

$$
\begin{align*}
& \stackrel{(b)}{\leq}-n h(\tilde{\boldsymbol{z}})+n h\left(\tilde{\boldsymbol{z}} \mid \widehat{\boldsymbol{x}}^{*}+\boldsymbol{z}+\tilde{\boldsymbol{z}}\right) \\
& =n h\left(\widehat{\boldsymbol{x}}^{*}+\boldsymbol{z}\right)-n h\left(\widehat{\boldsymbol{x}}^{*}+\boldsymbol{z}+\tilde{\boldsymbol{z}}\right), \tag{100}
\end{align*}
$$

where (a) is from [33, Lemma II.2], and $x^{* n}$ is a Gaussian vector sequence that has the same covariance matrix as $\boldsymbol{x}^{n}$. Inequality (b) is from Lemma 2. Alternatively, we can use Lemma 2 to bound (100) as

$$
\begin{align*}
& n h\left(\widehat{\boldsymbol{x}}^{*}+\boldsymbol{z}\right)-n h\left(\widehat{\boldsymbol{x}}^{*}+\boldsymbol{z}+\tilde{\boldsymbol{z}}\right) \\
& =-n h(\tilde{\boldsymbol{z}})+n h\left(\tilde{\boldsymbol{z}} \mid \widehat{\boldsymbol{x}}^{*}+\boldsymbol{z}+\tilde{\boldsymbol{z}}\right) \\
& \leq-n h(\tilde{\boldsymbol{z}})+n h\left(\tilde{\boldsymbol{z}} \mid \overline{\boldsymbol{x}}^{*}+\boldsymbol{z}+\tilde{\boldsymbol{z}}\right) \\
& =n h\left(\overline{\boldsymbol{x}}^{*}+\boldsymbol{z}\right)-n h\left(\overline{\boldsymbol{x}}^{*}+\boldsymbol{z}+\tilde{\boldsymbol{z}}\right) . \tag{101}
\end{align*}
$$

## D. Proof of Lemma 5

Let the eigenvalue decomposition of $\mathbf{S}_{u}$ be

$$
\begin{equation*}
\mathbf{S}_{u}=\mathbf{Q} \boldsymbol{\Lambda} \mathbf{Q}^{\dagger}, \tag{102}
\end{equation*}
$$

where $\mathbf{Q}$ is a unitary matrix and $\boldsymbol{\Lambda}$ is a diagonal matrix with strictly positive diagonal elements. Since

$$
\begin{equation*}
\operatorname{Cov}(\boldsymbol{x}, \boldsymbol{y}) \operatorname{Cov}(\boldsymbol{y})^{-1} \operatorname{Cov}(\boldsymbol{y}, \boldsymbol{z})=\operatorname{Cov}(\boldsymbol{x}, \mathbf{A} \boldsymbol{y}) \operatorname{Cov}(\mathbf{A} \boldsymbol{y})^{-1} \operatorname{Cov}(\mathbf{A} \boldsymbol{y}, \boldsymbol{z}) \tag{103}
\end{equation*}
$$

for any invertible matrix $\mathbf{A}$, we choose $\mathbf{A}=\boldsymbol{\Lambda}^{-\frac{1}{2}} \mathbf{Q}$ and then $\boldsymbol{x} \rightarrow \mathbf{H} \boldsymbol{x}+\boldsymbol{u} \rightarrow \mathbf{G} \boldsymbol{x}+\boldsymbol{v}$ forms a Markov chain if and only if $\boldsymbol{x} \rightarrow \tilde{\mathbf{H}} \boldsymbol{x}+\tilde{\boldsymbol{u}} \rightarrow \mathbf{G} \boldsymbol{x}+\boldsymbol{v}$ forms a Markov chain, where

$$
\begin{align*}
\tilde{\mathbf{H}} & =\boldsymbol{\Lambda}^{-\frac{1}{2}} \mathbf{Q} \mathbf{H} \quad \text { and }  \tag{104}\\
\tilde{\boldsymbol{u}} & =\boldsymbol{\Lambda}^{-\frac{1}{2}} \mathbf{Q} \boldsymbol{u}
\end{align*}
$$

and we have

$$
\begin{align*}
\operatorname{Cov}(\tilde{\boldsymbol{u}}) & =\mathbf{I} \quad \text { and } \\
\operatorname{Cov}(\tilde{\boldsymbol{u}}, \boldsymbol{v}) & =\boldsymbol{\Lambda}^{-\frac{1}{2}} \mathbf{Q S} \mathbf{S}_{u v} \triangleq \tilde{\mathbf{S}}_{u v} . \tag{105}
\end{align*}
$$

By Lemma 4, $\boldsymbol{x} \rightarrow \tilde{\mathbf{H}} \boldsymbol{x}+\tilde{\boldsymbol{u}} \rightarrow \mathbf{G} \boldsymbol{x}+\boldsymbol{v}$ forms a Markov chain if and only if

$$
\begin{aligned}
\mathbf{S}_{x} \mathbf{G}^{\dagger} & =\mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger}\left(\mathbf{I}+\tilde{\mathbf{H}} \mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger}\right)^{-1}\left(\tilde{\mathbf{H}} \mathbf{S}_{x} \mathbf{G}^{\dagger}+\tilde{\mathbf{S}}_{u v}\right) \\
& =\mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger}\left(\mathbf{I}+\tilde{\mathbf{H}} \mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger}\right)^{-1} \tilde{\mathbf{H}} \mathbf{S}_{x} \mathbf{G}^{\dagger}+\mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger}\left(\mathbf{I}+\tilde{\mathbf{H}} \mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger}\right)^{-1} \tilde{\mathbf{S}}_{u v} \\
& \stackrel{(a)}{=} \mathbf{S}_{x}\left[\mathbf{I}-\left(\mathbf{I}+\tilde{\mathbf{H}}^{\dagger} \tilde{\mathbf{H}} \mathbf{S}_{x}\right)^{-1}\right] \mathbf{G}^{\dagger}+\mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger}\left(\mathbf{I}+\tilde{\mathbf{H}} \mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger}\right)^{-1} \tilde{\mathbf{S}}_{u v}
\end{aligned}
$$

$$
\begin{align*}
& =\mathbf{S}_{x} \mathbf{G}^{\dagger}-\mathbf{S}_{x}\left(\mathbf{I}+\tilde{\mathbf{H}}^{\dagger} \tilde{\mathbf{H}} \mathbf{S}_{x}\right)^{-1} \mathbf{G}^{\dagger}+\mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger}\left(\mathbf{I}+\tilde{\mathbf{H}} \mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger}\right)^{-1} \tilde{\mathbf{S}}_{u v} \\
& \stackrel{(b)}{=} \mathbf{S}_{x} \mathbf{G}^{\dagger}-\left(\mathbf{I}+\mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger} \tilde{\mathbf{H}}\right)^{-1}\left(\mathbf{S}_{x} \mathbf{G}^{\dagger}-\mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger} \tilde{\mathbf{S}}_{u v}\right) \\
& \stackrel{(c)}{=} \mathbf{S}_{x} \mathbf{G}^{\dagger}-\left(\mathbf{I}+\mathbf{S}_{x} \tilde{\mathbf{H}}^{\dagger} \tilde{\mathbf{H}}\right)^{-1}\left(\mathbf{S}_{x} \mathbf{G}^{\dagger}-\mathbf{S}_{x} \mathbf{H}^{\dagger} \mathbf{S}_{u}^{-1} \mathbf{S}_{u v}\right) \tag{106}
\end{align*}
$$

where (a) is from the matrix inverse identity [34, page 151]

$$
\mathbf{A}(\mathbf{I}+\mathbf{B A})^{-1} \mathbf{B}=\mathbf{I}-(\mathbf{I}+\mathbf{A B})^{-1}
$$

Equality (b) is from the matrix inverse identity [34, page 151]

$$
\mathbf{A}(\mathbf{I}+\mathbf{B A})^{-1}=(\mathbf{I}+\mathbf{A B})^{-1} \mathbf{A}
$$

Equality (c) is from (102), (104) and (105). We complete the proof by pointing out that (106) is equivalent to (66).

## E. Proof of Lemma 6

Let $\boldsymbol{x}$ be a vector with dimension equal to the number of rows of $\mathbf{A}$, and $\boldsymbol{y}$ be a vector with dimension equal to the number of columns of $\mathbf{A}$. We have $\mathbf{B} \succeq \mathbf{A}^{\dagger} \mathbf{A}$ so that $\boldsymbol{y}^{\dagger} \mathbf{B} \boldsymbol{y} \geq \boldsymbol{y}^{\dagger} \mathbf{A}^{\dagger} \mathbf{A} \boldsymbol{y}$ and

$$
\begin{aligned}
{\left[\begin{array}{l}
x \\
y
\end{array}\right]^{\dagger}\left[\begin{array}{cc}
\mathbf{I} & \mathbf{A} \\
\mathbf{A}^{\dagger} & \mathbf{B}
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\boldsymbol{x}^{\dagger} \boldsymbol{x}+\boldsymbol{y}^{\dagger} \mathbf{A}^{\dagger} \boldsymbol{x}+\boldsymbol{x}^{\dagger} \mathbf{A} \boldsymbol{y}+\boldsymbol{y}^{\dagger} \mathbf{B} \boldsymbol{y} \\
& \geq \boldsymbol{x}^{\dagger} \boldsymbol{x}+\boldsymbol{y}^{\dagger} \mathbf{A}^{\dagger} \boldsymbol{x}+\boldsymbol{x}^{\dagger} \mathbf{A} \boldsymbol{y}+\boldsymbol{y}^{\dagger} \mathbf{A}^{\dagger} \mathbf{A} \boldsymbol{y} \\
& =(\mathbf{A} \boldsymbol{y}+\boldsymbol{x})^{\dagger}(\mathbf{A} \boldsymbol{y}+\boldsymbol{x}) \\
& \geq 0
\end{aligned}
$$

Therefore, sufficiency is proved. On the other hand, if $\left[\begin{array}{cc}\mathbf{I} & \mathbf{A} \\ \mathbf{A}^{\dagger} & \mathbf{B}\end{array}\right] \succeq \mathbf{0}$, we have

$$
\left[\begin{array}{l}
x  \tag{107}\\
y
\end{array}\right]^{\dagger}\left[\begin{array}{cc}
\mathbf{I} & \mathbf{A} \\
\mathbf{A}^{\dagger} & \mathbf{B}
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\boldsymbol{x}^{\dagger} \boldsymbol{x}+\boldsymbol{y}^{\dagger} \mathbf{A}^{\dagger} x+\boldsymbol{x}^{\dagger} \mathbf{A} \boldsymbol{y}+\boldsymbol{y}^{\dagger} \mathbf{B} \boldsymbol{y} \geq 0
$$

We choose $\boldsymbol{x}=-\mathbf{A} \boldsymbol{y}$ and substitute it into (107), then we have

$$
\begin{equation*}
\boldsymbol{y}^{\dagger}\left(\mathbf{B}-\mathbf{A}^{\dagger} \mathbf{A}\right) \boldsymbol{y} \geq 0 \tag{108}
\end{equation*}
$$

Therefore, $\mathbf{B} \succeq \mathbf{A}^{\dagger} \mathbf{A}$.
If $\mathbf{B} \succ \mathbf{0}$, then $\mathbf{B} \succeq \mathbf{A}^{\dagger} \mathbf{A}$ is equivalent to

$$
0 \leq \boldsymbol{y}^{\dagger}\left(\mathbf{B}-\mathbf{A}^{\dagger} \mathbf{A}\right) \boldsymbol{y}
$$

$$
\begin{align*}
& =\boldsymbol{y}^{\dagger} \mathbf{B}^{\frac{1}{2}}\left(\mathbf{I}-\mathbf{B}^{-\frac{1}{2}} \mathbf{A}^{\dagger} \mathbf{A} \mathbf{B}^{-\frac{1}{2}}\right) \mathbf{B}^{\frac{1}{2}} \boldsymbol{y} \\
& =\tilde{\boldsymbol{y}}^{\dagger}\left(\mathbf{I}-\mathbf{B}^{-\frac{1}{2}} \mathbf{A}^{\dagger} \mathbf{A} \mathbf{B}^{-\frac{1}{2}}\right) \tilde{\boldsymbol{y}} \tag{109}
\end{align*}
$$

where

$$
\begin{aligned}
\mathbf{B}^{\frac{1}{2}} & =\mathbf{U} \boldsymbol{\Lambda}^{\frac{1}{2}} \mathbf{U}^{\dagger} \quad \text { and } \\
\tilde{\boldsymbol{y}} & =\mathbf{B}^{\frac{1}{2}} \boldsymbol{y}
\end{aligned}
$$

and

$$
\mathbf{B}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\dagger}
$$

is the eigenvalue decomposition of $\mathbf{B}$ with $\mathbf{U}$ being a unitary matrix and $\boldsymbol{\Lambda}$ being a diagonal matrix with strictly positive diagonal elements. Since $\tilde{\boldsymbol{y}}$ can be any vector, (109) means

$$
\mathbf{I} \succeq \mathbf{B}^{-\frac{1}{2}} \mathbf{A}^{\dagger} \mathbf{A} \mathbf{B}^{-\frac{1}{2}}
$$

Suppose that the singular value decomposition of $\mathbf{B}^{-\frac{1}{2}} \mathbf{A}^{\dagger}$ is

$$
\mathbf{B}^{-\frac{1}{2}} \mathbf{A}^{\dagger}=\mathbf{P}\left[\begin{array}{cc}
\boldsymbol{\Sigma} & 0 \\
0 & 0
\end{array}\right] \mathbf{Q}^{\dagger},
$$

where both $\mathbf{P}$ and $\mathbf{Q}$ are unitary matrices and $\boldsymbol{\Sigma}$ is a diagonal matrix with strictly positive diagonal elements. Then we have

$$
\begin{aligned}
& \mathbf{B}^{-\frac{1}{2}} \mathbf{A}^{\dagger} \mathbf{A} \mathbf{B}^{-\frac{1}{2}}=\mathbf{P}\left[\begin{array}{ll}
\boldsymbol{\Sigma} & \mathbf{0} \\
\mathbf{0} & \mathbf{0}
\end{array}\right] \mathbf{P}^{\dagger} \text { and } \\
& \mathbf{A B}^{-\frac{1}{2}} \mathbf{B}^{-\frac{1}{2}} \mathbf{A}^{\dagger}=\mathbf{Q}\left[\begin{array}{ll}
\boldsymbol{\Sigma} & \mathbf{0} \\
\mathbf{0} & \mathbf{0}
\end{array}\right] \mathbf{Q}^{\dagger} .
\end{aligned}
$$

Therefore, $\mathbf{I} \succeq \mathbf{B}^{-\frac{1}{2}} \mathbf{A}^{\dagger} \mathbf{A} \mathbf{B}^{-\frac{1}{2}}$ if and only if $\mathbf{I} \succeq \boldsymbol{\Sigma}$ which is also the necessary and sufficient condition for $\mathbf{I} \succeq \mathbf{A B}^{-\frac{1}{2}} \mathbf{B}^{-\frac{1}{2}} \mathbf{A}^{\dagger}=\mathbf{A B}^{-1} \mathbf{A}^{\dagger}$.
F. Proof of Lemma 7

Let $\mathbf{A}=\mathbf{B}\left(\mathbf{B}^{\dagger} \mathbf{B}\right)^{-1} \mathbf{C}^{\dagger}$ and suppose that the singular value decomposition of $\mathbf{B}$ is

$$
\mathbf{B}=\mathbf{U}\left[\begin{array}{l}
\boldsymbol{\Sigma}  \tag{110}\\
\mathbf{0}
\end{array}\right] \mathbf{V}^{\dagger}
$$

where both $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices, and $\boldsymbol{\Sigma}$ is a diagonal matrix with strictly positive diagonal elements. Suppose further that

$$
\begin{align*}
\mathbf{I} & \succeq \mathbf{A}^{\dagger} \mathbf{A} \\
& =\mathbf{C}\left(\mathbf{B}^{\dagger} \mathbf{B}\right)^{-1} \mathbf{C}^{\dagger} \\
& =\mathbf{C} \mathbf{V} \boldsymbol{\Sigma}^{-2} \mathbf{V}^{\dagger} \mathbf{C}^{\dagger} \tag{111}
\end{align*}
$$

Lemma 6 implies that $\mathbf{X}^{\dagger} \mathbf{X} \preceq \mathbf{I}$ if and only if $\mathbf{X} \mathbf{X}^{\dagger} \preceq \mathbf{I}$, therefore (111) is equivalent to

$$
\begin{equation*}
\mathbf{I} \succeq \boldsymbol{\Sigma}^{-1} \mathbf{V}^{\dagger} \mathbf{C}^{\dagger} \mathbf{C V} \boldsymbol{\Sigma}^{-1}, \tag{112}
\end{equation*}
$$

i.e., for any vector $x$ we have

$$
\begin{align*}
0 & \leq \boldsymbol{x}^{\dagger}\left(\mathbf{I}-\boldsymbol{\Sigma}^{-1} \mathbf{V}^{\dagger} \mathbf{C}^{\dagger} \mathbf{C V} \boldsymbol{\Sigma}^{-1}\right) \boldsymbol{x} \\
& =\boldsymbol{x}^{\dagger} \boldsymbol{\Sigma}^{-1} \mathbf{V}^{\dagger}\left(\mathbf{V} \boldsymbol{\Sigma}^{2} \mathbf{V}^{\dagger}-\mathbf{C}^{\dagger} \mathbf{C}\right) \mathbf{V} \boldsymbol{\Sigma}^{-1} \boldsymbol{x} \\
& =\boldsymbol{y}^{\dagger}\left(\mathbf{B}^{\dagger} \mathbf{B}-\mathbf{C}^{\dagger} \mathbf{C}\right) \boldsymbol{y} \tag{113}
\end{align*}
$$

where the last line is from (110), and we define $\boldsymbol{y}=\mathbf{V} \boldsymbol{\Sigma}^{-1} \boldsymbol{x}$. Since $\boldsymbol{x}$ can be any vector and $\boldsymbol{\Sigma}^{-1} \mathbf{V}^{\dagger}$ is invertible, $\boldsymbol{y}$ can also be any vector. Therefore, (113) proves Lemma 7.

## G. Proof of Lemma 9

From (68) and the Woodbury matrix identity [35]:

$$
\left(\mathbf{E}+\mathbf{C B C}^{\dagger}\right)^{-1}=\mathbf{E}^{-1}-\mathbf{E}^{-1} \mathbf{C}\left(\mathbf{B}^{-1}+\mathbf{C}^{\dagger} \mathbf{E}^{-1} \mathbf{C}\right)^{-1} \mathbf{C}^{\dagger} \mathbf{E}^{-1}
$$

we have

$$
\begin{equation*}
\boldsymbol{\Sigma}_{2}^{-1}=\mathbf{I}-\mathbf{A}_{1}\left(-\boldsymbol{\Sigma}_{1}+\mathbf{A}_{1}^{\dagger} \mathbf{A}_{1}\right)^{-1} \mathbf{A}_{1}^{\dagger} . \tag{114}
\end{equation*}
$$

Substituting (114) into (67) we have

$$
\begin{equation*}
\boldsymbol{\Sigma}_{1}=\mathbf{I}-\mathbf{A}_{2} \mathbf{A}_{2}^{\dagger}+\mathbf{A}_{2} \mathbf{A}_{1}\left(\mathbf{A}_{1}^{\dagger} \mathbf{A}_{1}-\boldsymbol{\Sigma}_{1}\right)^{-1} \mathbf{A}_{1}^{\dagger} \mathbf{A}_{2}^{\dagger} \tag{115}
\end{equation*}
$$

Define

$$
\begin{align*}
\mathbf{X}_{1} & =\boldsymbol{\Sigma}_{1}-\mathbf{A}_{1}^{\dagger} \mathbf{A}_{1},  \tag{116}\\
\mathbf{M}_{1} & =\mathbf{I}-\mathbf{A}_{1}^{\dagger} \mathbf{A}_{1}-\mathbf{A}_{2} \mathbf{A}_{2}^{\dagger}  \tag{117}\\
\mathbf{M}_{2} & =\mathbf{I}-\mathbf{A}_{1} \mathbf{A}_{1}^{\dagger}-\mathbf{A}_{2}^{\dagger} \mathbf{A}_{2}, \tag{118}
\end{align*}
$$

$$
\begin{align*}
& \mathbf{W}_{1}=\mathbf{A}_{1}^{\dagger} \mathbf{A}_{2}^{\dagger} \quad \text { and }  \tag{119}\\
& \mathbf{W}_{2}=\mathbf{A}_{2}^{\dagger} \mathbf{A}_{1}^{\dagger} \tag{120}
\end{align*}
$$

On substituting (116)-(119) into (115), we have the following matrix equation:

$$
\begin{equation*}
\mathbf{X}_{1}+\mathbf{W}_{1}^{\dagger} \mathbf{X}_{1}^{-1} \mathbf{W}_{1}=\mathbf{M}_{1} . \tag{121}
\end{equation*}
$$

Equation (121) is a special case of a discrete algebraic Ricatti equation [31]. From Lemma 8, with $\mathbf{M}_{1}$ Hermitian and positive definite, (121) has a positive definite solution $\mathbf{X}_{1}$ (i.e., $\boldsymbol{\Sigma}_{1} \succ \mathbf{A}_{1}^{\dagger} \mathbf{A}_{1}$ ) if and only if

$$
\operatorname{radius}\left(\mathbf{M}_{1}^{-\frac{1}{2}} \mathbf{W}_{1} \mathbf{M}_{1}^{-\frac{1}{2}}\right)=\operatorname{radius}\left(\boldsymbol{\Phi}_{1}\right) \leq \frac{1}{2} .
$$

Similarly, applying the Woodbury matrix identity to invert $\boldsymbol{\Sigma}_{2}$ in (67) and substituting the result into (68), we obtain

$$
\begin{equation*}
\mathbf{X}_{2}+\mathbf{W}_{2}^{\dagger} \mathbf{X}_{2}^{-1} \mathbf{W}_{2}=\mathbf{M}_{2} \tag{122}
\end{equation*}
$$

where

$$
\mathbf{X}_{2}=\boldsymbol{\Sigma}_{2}-\mathbf{A}_{2}^{\dagger} \mathbf{A}_{2} .
$$

Matrix equation (122) has a positive definite solution $\mathbf{X}_{2}$ (i.e., $\boldsymbol{\Sigma}_{2} \succeq \mathbf{A}_{2}^{\dagger} \mathbf{A}_{2}$ ) if and only if

$$
\operatorname{radius}\left(\mathbf{M}_{2}^{-\frac{1}{2}} \mathbf{W}_{2} \mathbf{M}_{2}^{-\frac{1}{2}}\right)=\operatorname{radius}\left(\boldsymbol{\Phi}_{2}\right) \leq \frac{1}{2}
$$

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[^1]:    ${ }^{1}$ The case with $a<1$ is often referred to as ZIC with weak interference in the literature. We use the term noisy-interference simply because of the fact that treating interference as noise achieves the sum-rate capacity.

