# Noisy-interference Sum-rate Capacity of Parallel Gaussian Interference Channels 

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#### Abstract

The sum-rate capacity of the parallel Gaussian interference channel is shown to be achieved by independent transmission across sub-channels and treating interference as noise in each sub-channel if the channel coefficients and power constraints satisfy a certain condition. The condition requires the interference to be weak, a situation commonly encountered in, e.g., digital subscriber line transmission. The optimal power allocation is characterized by using the concavity of sum-rate capacity as a function of the power constraints.


## I. Introduction

Parallel Gaussian interference channels (PGICs) model the situation in which several transceiver pairs communicate through a number of independent sub-channels, with each sub-channel being a Gaussian interference channel. Fig. 1 illustrates a two-user PGIC where a pair of users, each subject to a total power constraint, has access to a set of $m$ Gaussian interference channels (GIC). Existing systems that can be accurately modelled as PGICs include both wired systems such as digital subscriber lines (DSL) and wireless systems employing orthogonal frequency division multiple access (OFDMA). Both of these systems have been and will be major players in broadband systems.

While there have been extensions of information theory for the classical single-channel GIC to the PGIC (e.g, [1]), most existing research, especially for DSL systems, often relies on the following two assumptions [2]-[4]:

[^0]

Fig. 1. Illustration of a PGIC system where the two transceiver pairs have access to $m$ independent parallel channels.

- Transmissions in sub-channels are independent of each other.
- Each receiver treats interference as noise.

These assumptions greatly simplify an otherwise intractable problem. The independent transmission assumption ensures that the total sum rate is expressed as a sum of all sub-channels' sum rates. The assumption of using single-user detection permits a simple closed-form expression for the rate pair of each sub-channel ${ }^{1}$. Our main goal in this paper is to provide a sound theoretical basis for such assumptions, i.e., to understand under what conditions such a transceiver structure leads to optimal throughput performance. It is certainly not obvious that this structure could ever be optimal, but we show that it is optimal for systems with weak interference, a situation encountered in many deployed systems such as DSL.

Our approach in characterizing the sum-rate capacity of a two-user PGIC leverages recent breakthroughs in determining the sum-rate capacity of the GIC under noisy interference [6]-[8]. We determine conditions on the channel gains and power constraints such that there is no loss in terms of sum rate when we impose the above two assumptions. This is accomplished in two steps. First, under the independent transmission assumption, we find conditions such that the maximum sum rate can be achieved by treating interference as noise in each sub-channel. The key to establishing these conditions is the concavity of sum-rate capacity in power constraints for a GIC (cf. Lemma 2). Second, we show that with the same power constraints and channel gains obtained in the first step, independent transmission and single-user detection in each sub-channel achieves the sum-rate capacity of the PGIC. The proof utilizes a genie-aided approach that generalizes that of [6].

This paper is organized as follows. In Section II we introduce the system model and review recent results. In Section III, we consider the maximum sum rate of a special transmission scheme, i.e., independent transmission and single-user detection for each sub-channel. We obtain conditions on the

[^1]power constraints and channel coefficients under which the above strategy maximizes the total sum rate. We prove in Section IV that the maximum sum rate we obtain is the sum-rate capacity. Numerical results are given in Section V. Section VI concludes the paper.

## II. System Model and Preliminaries

The received signals of the $i$ th sub-channel $i=1, \cdots, m$ are defined as

$$
\begin{align*}
& Y_{1 i}=\sqrt{c_{i}} X_{1 i}+\sqrt{a_{i}} X_{2 i}+Z_{1 i},  \tag{1}\\
& Y_{2 i}=\sqrt{d_{i}} X_{2 i}+\sqrt{b_{i}} X_{1 i}+Z_{2 i},
\end{align*}
$$

where $0 \leq a_{i}<d_{i}, 0 \leq b_{i}<c_{i} ; Z_{1 i}$ and $Z_{2 i}$ are unit variance Gaussian noise, the total block power constraints are $P$ and $Q$ for users 1 and 2 respectively:

$$
\sum_{i=1}^{m}\left[\frac{1}{n} \sum_{j=1}^{n} E\left(X_{1 i_{-} j}^{2}\right)\right] \leq P
$$

and

$$
\sum_{i=1}^{m}\left[\frac{1}{n} \sum_{j=1}^{n} E\left(X_{2 i \_j}^{2}\right)\right] \leq Q
$$

where $n$ is the block length, and $X_{1 i_{j} j}$ and $X_{2 i_{-} j}, j=1, \ldots, n$, are the user/channel input sequences for the $i$ th sub-channel. We remark that this model is a special case of the multiple-input multiple-output (MIMO) GIC [9]. We denote the sum-rate capacity of the $i$ th sub-channel as $C_{i}\left(P_{i}, Q_{i}\right)$, where $P_{i}$ and $Q_{i}$ are the respective powers allocated to the two users in this sub-channel.

To find the sum-rate capacity of the PGIC, we need to solve three problems: the first problem is whether the sub-channels can be treated separately like the parallel Gaussian multiple-access channel [10] and parallel Gaussian broadcast channel [11]-[13], i.e., whether the sum-rate capacity of the PGIC is in the form of $\sum_{i=1}^{m} C_{i}\left(P_{i}, Q_{i}\right)$. Such a strategy is suboptimal for PGICs in general [14], [15]. The second problem is the optimal distribution of the input signals. It has been shown respectively in [16]-[18] and [6]-[8] that Gaussian inputs are sum-rate optimal for a single-channel GIC under strong or noisy interference. However, whether this is still the case for PGICs is not known. The third problem is to find the optimal power allocation among sub-channels. Existing works on this problem treat the sub-channels separately, they use Gaussian inputs, and they use single-user detection at the receivers [2]-[4].

Before proceeding, we introduce the following notation.

- Bold fonts $\boldsymbol{x}$ and $\mathbf{X}$ denote vectors and matrices respectively.
- I denotes the identity matrix and $\mathbf{0}$ denotes the zero matrix.
- $|\mathbf{X}|, \mathbf{X}^{T}, \mathbf{X}^{-1}$, denote the respective determinant, transpose and inverse of the matrix $\mathbf{X}$.
- $\boldsymbol{x}^{n}=\left[\boldsymbol{x}_{1}^{T}, \boldsymbol{x}_{2}^{T}, \ldots, \boldsymbol{x}_{n}^{T}\right]^{T}$ is a long vector which consists of the vectors $\boldsymbol{x}_{i}, i=1, \ldots, n$.
- $\boldsymbol{x} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ means that the random vector $\boldsymbol{x}$ is Gaussian distributed with zero mean and covariance matrix $\Sigma$.
- $E(\cdot)$ denotes expectation; $\operatorname{Var}(\cdot)$ denotes variance; $\operatorname{Cov}(\cdot)$ denotes covariance matrix; $I(\cdot ; \cdot)$ denotes mutual information; $h(\cdot)$ denotes differential entropy with the logarithm base $e$ and $\log (\cdot)=\log _{e}(\cdot)$.


## A. Noisy-interference sum-rate capacity

The noisy-interference sum-rate capacity for single-channel GICs [6]-[8] is summarized as follows.
Lemma 1: The sum-rate capacity of the $i$ th sub-channel with $a_{i}<c_{i}, b_{i}<d_{i}$ and power allocation $p$, $q$ is

$$
\begin{equation*}
C_{i}(p, q)=\frac{1}{2} \log \left(1+\frac{c_{i} p}{1+a_{i} q}\right)+\frac{1}{2} \log \left(1+\frac{d_{i} q}{1+b_{i} p}\right) \tag{2}
\end{equation*}
$$

provided $(p, q) \in \mathcal{A}_{i}$ :

$$
\mathcal{A}_{i}=\left\{(\tilde{p}, \tilde{q}) \left\lvert\, \begin{array}{c}
\sqrt{a_{i} c_{i}}\left(1+b_{i} \tilde{p}\right)+\sqrt{b_{i} d_{i}}\left(1+a_{i} \tilde{q}\right) \leq \sqrt{c_{i} d_{i}}  \tag{3}\\
\tilde{p} \geq 0, \quad \tilde{q} \geq 0
\end{array}\right.\right\}
$$

In the case of a symmetric GIC, i.e., $a_{i}=b_{i}, c_{i}=d_{i}$ and $p=q$, the noisy interference condition reduces to

$$
\begin{equation*}
\frac{a_{i}}{c_{i}} \leq \frac{1}{4}, \quad p=q \leq \frac{\sqrt{a_{i} c_{i}}-2 a_{i}}{2 a_{i}^{2}} \tag{4}
\end{equation*}
$$

In the case of a ZIC where $a_{i}=0$, the noisy interference condition reduces to

$$
\begin{equation*}
b_{i}<1, \quad p \geq 0, \quad q \geq 0 \tag{5}
\end{equation*}
$$

The main difficulty in maximizing $\sum_{i=1}^{m} C_{i}\left(P_{i}, Q_{i}\right)$ is that $C_{i}\left(P_{i}, Q_{i}\right)$ is generally unknown if $\left(P_{i}, Q_{i}\right) \notin$ $\mathcal{A}_{i}$. To solve this problem we use the following results.

## B. Concavity of sum-rate capacity

The key to our study of the PGIC is the concavity of the sum-rate capacity as a function of the power constraint. We establish a slightly more general result by using a modified frequency division multiplexing (FDM) argument [19].

Lemma 2: Let $C_{\mu}(p, q)$ denote the weighted sum rate capacity of a GIC with powers $p$ and $q$ :

$$
C_{\mu}(p, q)=\max _{R_{1}, R_{2} \text { achievable }}\left\{R_{1}+\mu R_{2}\right\}
$$

where $\mu \geq 0$ is a constant. Then $C_{\mu}(p, q)$ is concave in the powers $(p, q)$, i.e., for any $0 \leq \lambda \leq 1$ we have

$$
\begin{equation*}
C_{\mu}(p, q) \geq \lambda C_{\mu}\left(p^{\prime}, q^{\prime}\right)+(1-\lambda) C_{\mu}\left(p^{\prime \prime}, q^{\prime \prime}\right), \tag{6}
\end{equation*}
$$

where $p^{\prime}, p^{\prime \prime}, q^{\prime}$, and $q^{\prime \prime}$ are chosen to satisfy

$$
\begin{equation*}
\lambda p^{\prime}+(1-\lambda) p^{\prime \prime}=p, \quad \lambda q^{\prime}+(1-\lambda) q^{\prime \prime}=q . \tag{7}
\end{equation*}
$$

Proof: Consider a potentially suboptimal strategy that divides the total frequency band into two subbands: one with a fraction $\lambda$ and the other with a fraction $1-\lambda$ of the total bandwidth. Powers are allocated into these two sub-bands as $\left(\lambda p^{\prime}, \lambda q^{\prime}\right)$ and $\left((1-\lambda) p^{\prime \prime},(1-\lambda) q^{\prime \prime}\right)$, where $p^{\prime}, q^{\prime}, p^{\prime \prime}, q^{\prime \prime}$ are such that (7) is satisfied. The information transmitted in these two sub-bands is independent and the decoding is also independent. Then the maximum weighted sum rate for the first sub-band is reduced by a factor $\lambda$ and becomes $\lambda C_{\mu}\left(p^{\prime}, q^{\prime}\right)$. Similarly, the maximum weighted sum rate at the second sub-band is $(1-\lambda) C_{\mu}\left(p^{\prime \prime}, q^{\prime \prime}\right)$. Therefore, the right-hand side of (6) is an achievable weighted sum rate.

Lemma 2 provides a fundamental result for weighted sum-rate capacities. It applies not only to two-user GICs but also to many-user GICs, Gaussian multiaccess channels, and Gaussian broadcast channels.

## C. Subgradient and subdifferential

To apply the concavity of sum-rate capacity, we need to use several properties of subgradients and subdifferentials (see [20]).

Definition 1: If $f: \mathcal{R}^{n} \rightarrow \mathcal{R}$ is a real-valued concave function defined on a convex set $\mathcal{S} \subset \mathcal{R}^{n}$, a vector $\boldsymbol{y}$ is a subgradient at point $\boldsymbol{x}_{0}$ if

$$
\begin{equation*}
f(\boldsymbol{x})-f\left(\boldsymbol{x}_{0}\right) \leq \boldsymbol{y}^{T}\left(\boldsymbol{x}-\boldsymbol{x}_{0}\right), \quad \forall x \in \mathcal{S} . \tag{8}
\end{equation*}
$$

Definition 2: For the concave function $f$ defined in Definition 1, the collection $\partial f\left(\boldsymbol{x}_{0}\right)$ of all subgradients at point $x_{0}$ is the subdifferential at this point.

If the function $f$ is differentiable at $x_{0}$, then the subgradient and subdifferential both coincide with the gradient. We introduce a lemma related to subdifferentials which we use to prove our main result.

Lemma 3: Let $f_{i}(\boldsymbol{x}), i=1, \ldots, m$, be finite, concave, real-valued functions on $\mathcal{S} \subset \mathcal{R}^{n}$ and let $\boldsymbol{x}_{i}^{*} \in \mathcal{S}, i=1, \cdots, m$. If there is a vector $\boldsymbol{y}$ such that $\boldsymbol{y} \in \partial f_{i}\left(\boldsymbol{x}_{i}^{*}\right), i=1, \ldots m$, and $\sum_{i=1}^{m} \boldsymbol{x}_{i}^{*}=\boldsymbol{u}$, then
$\boldsymbol{x}^{*}=\left[\boldsymbol{x}_{1}^{* T}, \cdots, \boldsymbol{x}_{m}^{* T}\right]$ is a solution for the following optimization problem:

$$
\begin{align*}
\max & \sum_{i=1}^{m} f_{i}\left(\boldsymbol{x}_{i}\right) \\
\text { subject to } & \sum_{i=1}^{m} \boldsymbol{x}_{i}=\boldsymbol{u}, \quad \boldsymbol{x}_{i} \in \mathcal{S} . \tag{9}
\end{align*}
$$

Proof: Since $\boldsymbol{y} \in \partial f_{i}\left(\boldsymbol{x}_{i}^{*}\right), i=1, \ldots m$, we have

$$
\begin{equation*}
f_{i}(\boldsymbol{x}) \leq f_{i}\left(\boldsymbol{x}_{i}^{*}\right)+\boldsymbol{y}^{T}\left(\boldsymbol{x}-\boldsymbol{x}_{i}^{*}\right), \quad \forall \boldsymbol{x} \in \mathcal{S} \tag{10}
\end{equation*}
$$

Let $\hat{\boldsymbol{x}}_{i}, i=1, \ldots, m$, be any vectors satisfying $\hat{\boldsymbol{x}}_{i} \in \mathcal{S}$ and $\sum_{i=1}^{m} \hat{\boldsymbol{x}}_{i}=\boldsymbol{u}$, then using (10) we have

$$
\begin{equation*}
f_{i}\left(\hat{\boldsymbol{x}}_{i}\right) \leq f_{i}\left(\boldsymbol{x}_{i}^{*}\right)+\boldsymbol{y}^{T}\left(\hat{\boldsymbol{x}}_{i}-\boldsymbol{x}_{i}^{*}\right) \tag{11}
\end{equation*}
$$

Therefore

$$
\begin{align*}
\sum_{i=1}^{m} f_{i}\left(\hat{\boldsymbol{x}}_{i}\right) & \leq \sum_{i=1}^{m} f_{i}\left(\boldsymbol{x}_{i}^{*}\right)+\boldsymbol{y}^{T}\left(\sum_{i=1}^{m} \hat{\boldsymbol{x}}_{i}-\sum_{i=1}^{m} \boldsymbol{x}_{i}^{*}\right) \\
& =\sum_{i=1}^{m} f_{i}\left(\boldsymbol{x}^{*}\right) \tag{12}
\end{align*}
$$

where the last equality is from $\sum_{i=1}^{m} \hat{\boldsymbol{x}}_{i}=\sum_{i=1}^{m} \boldsymbol{x}_{i}^{*}=\boldsymbol{u}$.
In the Appendix, we compute the subdifferential $\partial C_{i}(p, q)$ when $(p, q) \in \mathcal{A}_{i}$. We are also interested in the set of pairs

$$
\begin{equation*}
\mathcal{B}_{i}=\bigcup_{(p, q) \in \mathcal{A}_{i}} \partial C_{i}(p, q) \tag{13}
\end{equation*}
$$

The mapping from $\mathcal{A}_{i}$ to $\mathcal{B}_{i}$ is illustrated in Fig. 2. As seen from (3), $\mathcal{A}_{i}$ is a triangle region with the corner points

$$
\begin{aligned}
& O(0,0) \\
& S\left(0, \frac{\sqrt{c_{i} d_{i}}-\sqrt{a_{i} c_{i}}-\sqrt{b_{i} d_{i}}}{a_{i} \sqrt{b_{i} d_{i}}}\right) \triangleq\left(0, q_{s}\right)
\end{aligned}
$$

and

$$
T\left(\frac{\sqrt{c_{i} d_{i}}-\sqrt{a_{i} c_{i}}-\sqrt{b_{i} d_{i}}}{b_{i} \sqrt{a_{i} c_{i}}}, 0\right) \triangleq\left(p_{t}, 0\right)
$$

The corresponding points in $\mathcal{B}_{i}$ are respectively

$$
\begin{aligned}
& O^{\prime}\left(\frac{c_{i}}{2}, \frac{d_{i}}{2}\right) \\
& S^{\prime}\left(\frac{c_{i}}{2\left(1+a_{i} q_{s}\right)}-\frac{b_{i} d_{i} q_{s}}{2\left(1+d_{i} q_{s}\right)}, \frac{d_{i}}{2\left(1+d_{i} q_{s}\right)}\right)
\end{aligned}
$$

and

$$
T^{\prime}\left(\frac{c_{i}}{2\left(1+c_{i} p_{t}\right)}, \frac{d_{i}}{2\left(1+b_{i} p_{t}\right)}-\frac{a_{i} c_{i} p_{t}}{2\left(1+c_{i} p_{t}\right)}\right) .
$$




Fig. 2. The mapping from $\mathcal{A}_{i}$ to $\mathcal{B}_{i}$.

Let $\mathcal{A}_{i}^{(1)}$ be the inner points of $\mathcal{A}_{i}$ and the line segment $\overline{S T}$, and let $\mathcal{B}_{i}^{(1)}$ be the inner points of the closed area defined by $O^{\prime} S^{\prime} T^{\prime}$ and the curve $\widehat{S^{\prime} T^{\prime}}$. As shown in the Appendix, $\mathcal{A}_{i}^{(1)}$ maps to $\mathcal{B}_{i}^{(1)}$ and this mapping is one-to-one. Let $\mathcal{A}_{i}^{(2)}$ be the line segment $\overline{O T}$ and $\mathcal{B}_{i}^{(2)}$ be the curve $\widehat{O^{\prime} T^{\prime}}$ and the points above it (labeled as region II in Fig. 2). $\mathcal{A}_{i}^{(2)}$ maps to $\mathcal{B}_{i}^{(2)}$ and this mapping is one-to-many. Specifically, let $\left(P_{i}, 0\right)$ be a point on $\overline{O T}$. The partial derivatives of $C_{i}(p, q)$ with respect to $p$ and $q$ at this point are denoted as $K_{p}$ and $K_{q}$, respectively, where $K_{p}$ is a two-sided partial derivative and $K_{q}$ is a one-sided partial derivative. Point $\left(K_{p}, K_{q}\right)$ is on the curve $\widehat{O^{\prime} T^{\prime}}$ of $\mathcal{B}_{i}^{(2)}$. The subdifferential of $C_{i}(p, q)$ at point $\left(P_{i}, 0\right)$ is a ray in $\mathcal{B}_{i}^{(2)}$ defined as $k_{p}=K_{p}, k_{q} \geq K_{q}$.

Similarly to the above, let $\mathcal{A}_{i}^{(3)}$ be the line segment $\overline{O S}$ and $\mathcal{B}_{i}^{(3)}$ be the curve $\widehat{O^{\prime} S^{\prime}}$ and the points to the right (labeled as region III in Fig. 2). $\mathcal{A}_{i}^{(3)}$ maps to $\mathcal{B}_{i}^{(3)}$ and this mapping is also one-to-many. Let $\mathcal{A}_{i}^{(4)}$ be the origin and let $\mathcal{B}_{i}^{(4)}$ be the collection of points ( $k_{p}, k_{q}$ ) satisfying $k_{p} \geq K_{p}$ and $k_{q} \geq K_{q}$ (labeled as region IV in Fig. 2), where $K_{p}$ and $K_{q}$ are the two one-sided partial derivatives at the origin. $\mathcal{A}_{i}^{(4)}$ maps to $\mathcal{B}_{i}^{(4)}$.

## D. Concave-like property of conditional entropy

The following Lemma is proved in [9] based on the fact that a Gaussian distribution maximizes conditional entropy under a covariance matrix constraint [21].

Lemma 4: [9, Lemma 2] Let $\boldsymbol{x}_{i}^{n}=\left[\boldsymbol{x}_{i, 1}^{T}, \ldots, \boldsymbol{x}_{i, n}^{T}\right]^{T}, i=1, \ldots, k$, be $k$ long random vectors each of which consists of $n$ vectors. Suppose the $\boldsymbol{x}_{i, j}, i=1, \cdots, k$ all have the same length $L_{j}, j=1, \cdots, n$. Let $\boldsymbol{y}^{n}=\left[\boldsymbol{y}_{1}^{T}, \ldots, \boldsymbol{y}_{n}^{T}\right]^{T}$, where $\boldsymbol{y}_{j}$ has length $L_{j}$, be a long Gaussian random vector with covariance matrix

$$
\begin{equation*}
\operatorname{Cov}\left(\boldsymbol{y}^{n}\right)=\sum_{i=1}^{k} \lambda_{i} \operatorname{Cov}\left(\boldsymbol{x}_{i}^{n}\right) \tag{14}
\end{equation*}
$$

where $\sum_{i=1}^{k} \lambda_{i}=1, \lambda_{i} \geq 0$. Let $\mathcal{S}$ be a subset of $\{1,2, \ldots, n\}$ and $\mathcal{T}$ be a subset of $\mathcal{S}$ 's complement. Then we have

$$
\begin{equation*}
\sum_{i=1}^{k} \lambda_{i} h\left(\boldsymbol{x}_{i, \mathcal{S}} \mid \boldsymbol{x}_{i, \mathcal{T}}\right) \leq h\left(\boldsymbol{y}_{\mathcal{S}} \mid \boldsymbol{y}_{\mathcal{T}}\right) \tag{15}
\end{equation*}
$$

When $\boldsymbol{x}_{k}, k=1, \cdots, n$ are all Gaussian distributed, Lemma 4 shows that $h\left(\boldsymbol{x}_{\mathcal{S}} \mid \boldsymbol{x}_{\overline{\mathcal{S}}}\right)$ is concave over the covariance matrices.

## III. A LOWER bound For the sum-rate capacity

If the sum-rate capacity of a PGIC can be achieved by (1) transmitting independent symbol streams in each sub-channel and (2) treating interference as noise in each sub-channel, we say this PGIC has noisy interference. Before proceeding to the main theorem of noisy-interference sum-rate capacity, we first consider the following optimization problem:

$$
\begin{align*}
\max & \sum_{i=1}^{m} C_{i}\left(P_{i}, Q_{i}\right) \\
\text { subject to } & \sum_{i=1}^{m} P_{i}=P, \quad \sum_{i=1}^{m} Q_{i}=Q \\
& P_{i} \geq 0, \quad Q_{i} \geq 0, \quad i=1, \ldots, m \tag{16}
\end{align*}
$$

Problem (16) is to find the maximum of the sum of the sum-rate capacities of individual sub-channels and the corresponding power allocation. In general, the optimal solution of (16) is not the sum-rate capacity of the PGIC, since it presumes that the signals transmitted in each sub-channel are independent and no joint decoding across sub-channels is allowed. However, solving problem (16) is important to derive the
sum-rate capacity of a PGIC. We are interested in the case where the optimal power allocations $P_{i}^{*}, Q_{i}^{*}$ satisfy the following noisy interference conditions

$$
\begin{equation*}
\sqrt{a_{i} c_{i}}\left(1+b_{i} P_{i}^{*}\right)+\sqrt{b_{i} d_{i}}\left(1+a_{i} Q_{i}^{*}\right) \leq \sqrt{c_{i} d_{i}}, \quad i=1, \ldots, m . \tag{17}
\end{equation*}
$$

In such a case, it turns out that the sum of the sum-rate capacities in (16) is maximized when each sub-channel experiences noisy interference.

For the rest of this section, we first consider the general PGIC and derive the optimal solution of problem (16) based on Lemmas 2 and 3. We further find conditions on the total power $P$ and $Q$ such that the optimal solution of (16) satisfies (17). Then we focus on symmetric PGICs and provide some insights on this solution.

## A. General parallel Gaussian interference channel

Theorem 1: For a PGIC defined in (1), if $\sqrt{a_{i} c_{i}}+\sqrt{b_{i} d_{i}}<\sqrt{c_{i} d_{i}}$ and the power constraint $(P, Q)$ is in the following set

$$
\bigcup_{\left[k_{p}^{*}, k_{q}^{*}\right]^{T} \in \bigcap_{i=1}^{m} \mathcal{B}_{i}}\left\{(P, Q) \left\lvert\, \begin{array}{l}
P=\sum_{i=1}^{m} P_{i}^{*}, \quad Q=\sum_{i=1}^{m} Q_{i}^{*},  \tag{18}\\
{\left[k_{p}^{*}, k_{q}^{*}\right]^{T} \in \partial C_{i}\left(P_{i}^{*}, Q_{i}^{*}\right) \quad i=1, \ldots, m .}
\end{array}\right.\right\},
$$

then the optimal solution of (16) satisfies (17).
Proof: The proof is straightforward from Lemma 3. For any $\left[k_{p}^{*}, k_{q}^{*}\right]^{T} \in \bigcap_{i=1}^{m} \mathcal{B}_{i}$, there exist $P_{i}^{*}, Q_{i}^{*}$ such that $\left[k_{p}^{*}, k_{q}^{*}\right]^{T} \in \partial C_{i}\left(P_{i}^{*}, Q_{i}^{*}\right), i=1, \ldots, m$. Thus if $P=\sum_{i=1}^{m} P_{i}^{*}, Q=\sum_{i=1}^{m} Q_{i}^{*}$, then from Lemma $3, P_{i}^{*}, Q_{i}^{*}$ are optimal for the optimization problem (16). Since $\left(P_{i}^{*}, Q_{i}^{*}\right) \in \mathcal{A}_{i}$, then from Lemma $1,\left(P_{i}^{*}, Q_{i}^{*}\right)$ satisfies (17).

Theorem 1 provides conditions on the power and channel coefficients such that treating interference as noise (or single-user detection) maximizes the sum rate of a PGIC under the assumption of independent transmission among sub-channels. The conditions of Theorem 1 ensures that the power constraints $P$ and $Q$ are associated with a subgradient $\left[k_{p}^{*}, k_{q}^{*}\right]^{T}$ shared by $C_{i}\left(P_{i}^{*}, Q_{i}^{*}\right)$ for all $i=1, \cdots, m$. Therefore, at the points of the optimal power allocations $\left(P_{i}^{*}, Q_{i}^{*}\right)$, all the functions $C_{i}\left(p_{i}, q_{i}\right)$ have parallel supporting hyperplanes. We will discuss this in more details in Remark 1 below.

In general, the closed-form expression (18) of the power region for $P$ and $Q$ is very complex. However, for some special cases like symmetric PGICs, we can obtain simpler closed-form solutions.

## B. Symmetric parallel Gaussian interference channel

In this section, we consider PGICs with symmetric parameters, namely $a_{i}=b_{i}, c_{i}=d_{i}$ and $P=Q$. Without loss of generality we assume $c_{1} \geq c_{2} \cdots \geq c_{m}$. Define

$$
\begin{align*}
w_{i} & =\frac{4 a_{i}^{2}}{\left(\sqrt{c_{i}}-\sqrt{a_{i}}\right)^{2}},  \tag{19}\\
\hat{w} & =\max _{i}\left\{w_{i}\right\}, \tag{20}
\end{align*}
$$

and let $r$ to be an index between 1 and $m$ such that

$$
\begin{equation*}
c_{r+1}<\hat{w}<c_{r} \tag{21}
\end{equation*}
$$

where we let $c_{m+1}=0$ for convention. Then we have the following theorem.
Theorem 2: For a symmetric PGIC, if $\frac{a_{i}}{c_{i}}<\frac{1}{4}, i=1, \ldots, m$, and

$$
\begin{equation*}
0<P \leq \bar{P} \tag{22}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{P}=\sum_{i=1}^{r} \frac{\sqrt{c_{i}^{2}+\frac{4 a_{i} c_{i}}{\hat{w}}\left(a_{i}+c_{i}\right)}-\left(2 a_{i}+c_{i}\right)}{2 a_{i}\left(a_{i}+c_{i}\right)} \tag{23}
\end{equation*}
$$

then the optimal solution of (16) satisfies (17). Furthermore, only the first $r$ sub-channels are active.
Proof: By symmetry, we simplify the proof by considering the following optimization problem:

$$
\begin{align*}
\max & \sum_{i=1}^{m} C_{i}\left(P_{i}, P_{i}\right) \\
\text { subject to } & \sum_{i=1}^{m} P_{i}=P  \tag{24}\\
& P_{i} \geq 0, \quad i=1, \ldots, m
\end{align*}
$$

That is, we require that the power allocated to both users be $P_{i}$ for the $i$ th sub-channel. Obviously the maximum of (24) is no greater than the maximum of (16) because of the extra constraint $P_{i}=Q_{i}$. To prove Theorem 2, it suffices to show that under the condition $0<P \leq \bar{P}: 1$ ) the optimal $P_{i}^{*}$ for (24) satisfy the noisy interference condition; 2) the optimization problems (16) and (24) are equivalent.

Let $P_{i}=Q_{i}$, we obtain from Lemma 1

$$
\begin{equation*}
\mathcal{A}_{i}^{\prime}=\left\{p \left\lvert\, 0 \leq p \leq \frac{\sqrt{a_{i} c_{i}}-2 a_{i}}{2 a_{i}^{2}}\right.\right\} . \tag{25}
\end{equation*}
$$

The subdifferential is computed in the Appendix and is given by (see (89))

$$
\partial C_{i}\left(P_{i}\right)= \begin{cases}\left\{k \mid c_{i} \leq k \leq \hat{c}\right\}, & P_{i}=0,  \tag{26}\\ \left\{k \left\lvert\, k=\frac{c_{i}}{\left(1+a_{i} P_{i}\right)\left(1+a_{i} P_{i}+c_{i} P_{i}\right)}\right.\right\}, & P_{i} \in \mathcal{A}_{i}^{\prime}, \quad P_{i} \neq 0,\end{cases}
$$

where $\hat{c}=\max \left\{c_{i}\right\}$. Therefore

$$
\begin{equation*}
\mathcal{B}_{i}^{\prime}=\bigcup_{P_{i} \in \mathcal{A}_{i}^{\prime}} \partial C_{i}\left(P_{i}\right)=\left\{k \mid w_{i} \leq k \leq \hat{c}\right\} \tag{27}
\end{equation*}
$$

and

$$
\begin{equation*}
\bigcap_{i=1}^{m} \mathcal{B}_{i}^{\prime}=\{k \mid \hat{w} \leq k \leq \hat{c}\} \tag{28}
\end{equation*}
$$

For any $k^{*} \in \bigcap_{i=1}^{m} \mathcal{B}_{i}^{\prime}$, equation (26) determines a one-to-one mapping from $k^{*}$ to $P_{i}^{*} \in \mathcal{A}_{i}^{\prime}$, namely

$$
P_{i}^{*}\left(k^{*}\right)= \begin{cases}0, & k^{*} \geq c_{i}  \tag{29}\\ \frac{\sqrt{c_{i}^{2}+\frac{4 a_{i} c_{i}}{k^{*}}\left(a_{i}+c_{i}\right)}-\left(2 a_{i}+c_{i}\right)}{2 a_{i}\left(a_{i}+c_{i}\right)}, & w_{i} \leq k^{*}<c_{i}\end{cases}
$$

So consider the region (18) which is here

$$
\begin{equation*}
\bigcup_{k^{*} \in[\hat{w}, \hat{c}]}\left\{P \mid P=\sum_{i=1}^{m} P_{i}^{*}\left(k^{*}\right)\right\} \tag{30}
\end{equation*}
$$

From (29), $\sum_{i=1}^{m} P_{i}^{*}\left(k^{*}\right)$ is decreasing in $k^{*}$, therefore

$$
\begin{align*}
P & \geq P\left(k^{*}=\hat{c}\right)=0  \tag{31}\\
P & \leq P\left(k^{*}=\hat{w}\right)=\sum_{i=1}^{r} P_{i}^{*}\left(k^{*}=\hat{w}\right) \\
& =\sum_{i=1}^{r} \frac{\sqrt{c_{i}^{2}+\frac{4 a_{i} c_{i}}{\hat{w}}\left(a_{i}+c_{i}\right)}-\left(2 a_{i}+c_{i}\right)}{2 a_{i}\left(a_{i}+c_{i}\right)} \triangleq \bar{P} \tag{32}
\end{align*}
$$

where the first equality of (32) is from (21). Since $P\left(k^{*}\right)=\sum_{i=1}^{m} P_{i}^{*}\left(k^{*}\right)$ is continuous over $k^{*}$, for any $P \in[0, \bar{P}]$ there exists a $k^{*}$, and the corresponding $P_{i}^{*}, i=1, \ldots, m$, that solve the optimization problem (24).

We complete the proof by showing that the optimal $P_{i}^{*}$ for (24) is also optimal for (16) for a symmetric PGIC. Assume that for a given $P$ and the optimal $P_{i}^{*}$ of (24), the corresponding subgradient (which is identical for all $i$ ) is $k^{*}$. Then by symmetry, the subderivative of $C_{i}\left(P_{i}, Q_{i}\right)$ in (16) is $\left[\frac{k^{*}}{2}, \frac{k^{*}}{2}\right]^{T}$ by choosing $P_{i}=Q_{i}=P_{i}^{*}$. Therefore the subderivatives are identical for all the $C_{i}\left(P_{i}, Q_{i}\right)$ at $P_{i}=Q_{i}=P_{i}^{*}$. From Lemma 3, $P_{i}=Q_{i}=P_{i}^{*}$ is an optimal choice for (16). Since $P_{i}^{*}$ satisfies the noisy-interference condition in (25), $P_{i}=Q_{i}=P_{i}^{*}$ also satisfies the noisy-interference condition in (3).

Remark 1: From the proof of Theorem 2, all the $C_{i}\left(P_{i}, Q_{i}\right)$ have parallel supporting hyperplanes at the optimal point $P_{i}=Q_{i}=P_{i}^{*}$. This gives rise to a geometric interpretation, as illustrated in Fig. 3. For clarity, we use the simplified optimization problem (24). $C_{i}(p)$ is the sum-rate capacity for the $i$ th


Fig. 3. An illustration of sum-rate capacity achieving power allocation for a symmetric parallel Gaussian interference channel.
sub-channel, points $A_{1}$ and $A_{2}$ correspond to the power allocations $P_{1}$ and $P_{2}$, respectively, and the two supporting hyperplanes pass through $A_{1}$ and $A_{2}$. The power allocation satisfies $P=P_{1}+P_{2}, P_{i} \in \mathcal{A}_{i}^{\prime}, i=$ 1,2 , and $k=\left.\frac{\partial C_{1}(p)}{\partial p}\right|_{p=P_{1}}=\left.\frac{\partial C_{2}(p)}{\partial p}\right|_{p=P_{2}}$ (we assume the subgradient is equal to the gradient in this case, and hence the supporting hyperplane is the tangent hyperplane), and the corresponding sum rate is $C_{s 1}^{*}+C_{s 2}^{*}$. Consider the power allocation $P_{1}-\delta$ and $P_{2}+\delta$ and the corresponding sum-rate capacities for the two sub-channels $C_{s 1}^{*}-\triangle R_{1}$ and $C_{s 2}^{*}+\triangle R_{2}$, respectively. By concavity, $\triangle R_{1}>k \delta$ and $\triangle R_{2}<k \delta$. Therefore the new sum-rate is $\left(C_{s 1}^{*}-\triangle R_{1}\right)+\left(C_{s 2}^{*}+\triangle R_{2}\right)<\left(C_{s 1}^{*}-k \delta\right)+\left(C_{s 2}^{*}+k \delta\right)=C_{s 1}^{*}+C_{s 2}^{*}$. Remark 2: When $\min _{i}\left\{\frac{\sqrt{a_{i} c_{i}}-2 a_{i}}{2 a_{i}^{2}}\right\}<P \leq \bar{P}$, there exist power allocations such that some subchannels do not have noisy interference. As such the sum-rate capacities of those sub-channels are unknown. Surprisingly, in this case we do not need to derive upper bounds for those unknown sum-rate capacities. Instead, the concavity of the sum-rate capacity (as a function of the power) and the existing noisy-interference sum-rate capacity results ensure the validity of Theorem 2.

Remark 3: The parallel supporting hyperplanes condition for the optimal power allocation is applicable to a broad class of parallel channels in which 1) transmissions across subchannels are independent, 2)
the capacity of each subchannel is concave in its power constraint. For example, this condition applies to parallel multi-access and broadcast channels. In particular, applying the condition to single user parallel Gaussian channels, it is easy to verify that the parallel supporting hyperplanes condition reduces to the classic waterfilling interpretation.

Remark 4: Intuitively, since each sub-channel is a symmetric Gaussian IC with noisy interference, the power allocated to the two users in each sub-channel ought to be identical. While Theorem 2 does not explicitly address the power allocation scheme, we see from the proof of the theorem that this is indeed the case.

Remark 5: If (22) is satisfied, the optimal power allocation $P_{i}^{*}$ is unique, and there exists a $k^{*} \in[\hat{w}, \hat{c}]$ such that $P_{i}^{*}$ and $k^{*}$ satisfy (29). To see this, observe that in the proof of Theorem 2, $\sum_{i=1}^{m} P_{i}^{*}\left(k^{*}\right)$ is continuous and monotonically decreasing in $k^{*}$ when $k^{*} \in[\hat{w}, \hat{c}]$, and $P$ varies from 0 to $\bar{P}$ when $k^{*}$ varies from $\hat{c}$ to $\hat{w}$. Thus, if $0 \leq P \leq \bar{P}$, there exists a corresponding unique $k^{*}$ in $[\hat{w}, \hat{c}]$ that solves problem (24). Since the mapping from $k^{*}$ to $P_{i}^{*}$ in (29) is a one-to-one mapping, $P_{i}^{*}$ is also unique.

Remark 6: As shown in (29), whether a sub-channel is active or not depends only on the direct channel gain $c_{i}$. The amount of power allocated to a sub-channel depends on both the direct channel gain $c_{i}$ and the interference channel gain $a_{i}$. When the total power constraint $P$ increases from 0 to $\bar{P}$, the corresponding $k^{*}$ decreases from $\hat{c}$ to $\hat{w}$. As such, from (29) the sub-channels with larger $c_{i}$ become active earlier than those with smaller $c_{i}$.

## IV. Noisy interference sum-rate capacity

The following theorem gives the noisy-interference sum-rate capacity of a PGIC.
Theorem 3: For the PGIC defined in (1), if

$$
\begin{equation*}
\sqrt{a_{i} c_{i}}+\sqrt{b_{i} d_{i}}<\sqrt{c_{i} d_{i}} \tag{33}
\end{equation*}
$$

for all $i=1, \cdots, m$, and the power constraint pair $(P, Q)$ is in the set (18), the sum-rate capacity is the maximum of problem (16), and the sum-rate capacity is achieved by independent transmission across sub-channels and treating interference as noise for each sub-channel.

The following theorem is a special case of Theorem 1 for symmetric PGICs.
Theorem 4: For a symmetric PGIC, if $\frac{a_{i}}{c_{i}}<\frac{1}{4}$ for all $i=1, \cdots, m$, and the power constraint $P$ satisfies (22), then the sum-rate capacity is the maximum of problem (24) and is achieved by independent transmission across sub-channels and treating interference as noise in each sub-channel.

For the PGIC, there may exist some sub-channels with one-sided interference or no interference, i.e., $b_{j}=0$, or $a_{k}=0$ or $a_{r}=b_{r}=0$ for some integers $j, k$ and $r$ between 1 and $m$. We prove Theorem 3 for all such cases.

Proof: Let $i, j, k, r$ be integers and $1 \leq i \leq m_{1}, m_{1}+1 \leq j \leq m_{2}, m_{2}+1 \leq k \leq m_{3}$ and $m_{3}+1 \leq r \leq m$ throughout this proof. We denote $\underline{i}, \underline{j}, \underline{k}, \underline{r}$ as index sets and $\underline{i}=\left\{1, \cdots, m_{1}\right\}$, $\underline{j}=\left\{m_{1}+1, \cdots, m_{2}\right\}, \underline{k}=\left\{m_{2}+1, \cdots, m_{3}\right\}$, and $\underline{r}=\left\{m_{3}+1, \cdots, m\right\}$. Without loss of generality we can assume that sub-channels with index $i$ are all two-sided GICs with $a_{i} \neq 0, b_{i} \neq 0$; the sub-channels with index $j$ are all GICs with $a_{j} \neq 0, b_{j}=0$; the sub-channels with index $k$ are all GICs with $a_{k}=0$, $b_{k} \neq 0$; and the sub-channels with index $r$ are all GICs with $a_{r}=b_{r}=0$. Let $\mathbf{A}=\operatorname{diag}\left(\sqrt{a_{1}}, \cdots, \sqrt{a_{m}}\right)$, $\mathbf{B}=\operatorname{diag}\left(\sqrt{b_{1}}, \cdots, \sqrt{b_{m}}\right), \mathbf{C}=\operatorname{diag}\left(\sqrt{c_{1}}, \cdots, \sqrt{c_{m}}\right), \mathbf{D}=\operatorname{diag}\left(\sqrt{d_{1}}, \cdots, \sqrt{d_{m}}\right)$. Then we can rewrite (1) in the following form

$$
\begin{align*}
& \boldsymbol{y}_{1}=\mathbf{C} \boldsymbol{x}_{1}+\mathbf{A} \boldsymbol{x}_{2}+\boldsymbol{z}_{1},  \tag{34}\\
& \boldsymbol{y}_{2}=\mathbf{B} \boldsymbol{x}_{1}+\mathbf{D} \boldsymbol{x}_{2}+\boldsymbol{z}_{2},
\end{align*}
$$

where $z_{1} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ and $z_{2} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.
We further define

$$
\begin{aligned}
& \mathbf{A}_{\underline{i}}=\operatorname{diag}\left(\sqrt{a_{1}}, \cdots, \sqrt{a_{m_{1}}}\right)=\operatorname{diag}\left(\sqrt{a_{\underline{i}}}\right), \\
& \mathbf{A}_{\underline{j}}=\operatorname{diag}\left(\sqrt{a_{m_{1}+1}}, \cdots, \sqrt{a_{m_{2}}}\right)=\operatorname{diag}\left(\sqrt{a_{\underline{j}}}\right), \\
& \mathbf{A}_{\underline{k}}=\operatorname{diag}\left(\sqrt{a_{m_{2}+1}}, \cdots, \sqrt{a_{m_{3}}}\right)=\operatorname{diag}\left(\sqrt{a_{\underline{k}}}\right), \\
& \mathbf{A}_{\underline{r}}=\operatorname{diag}\left(\sqrt{a_{m_{3}}+1}, \cdots, \sqrt{a_{m}}\right)=\operatorname{diag}\left(\sqrt{a_{\underline{r}}}\right),
\end{aligned}
$$

and similarly for $\mathbf{B}, \mathbf{C}$ and $\mathbf{D}$. Denote the transmitted vector of user 1 as $\boldsymbol{x}_{1}=\left[X_{1}, \cdots, X_{m}\right]$, where each entry is the transmitted signal at the corresponding sub-channel. Similarly, we let $\boldsymbol{x}_{1 \underline{i}}=\left[X_{1}, \cdots, X_{m_{1}}\right]=$ $\left[X_{1 \underline{\underline{1}}}\right], \boldsymbol{x}_{1 \underline{j}}=\left[X_{m_{1}+1}, \cdots, X_{m_{2}}\right]=\left[X_{1 \underline{j}}\right], \boldsymbol{x}_{1 \underline{k}}=\left[X_{m_{2}+1}, \cdots, X_{m_{3}}\right]=\left[X_{1 \underline{k}}\right]$ and $\boldsymbol{x}_{1 \underline{\underline{r}}}=\left[X_{m_{3}+1}, \cdots, X_{m}\right]=$ [ $X_{1 \underline{r}}$ ]. The input vectors for the second user are similarly defined.

Since the power constraint $[P, Q]^{T}$ is in the set (18), so there exists a subgradient $\left[k_{p}^{*}, k_{q}^{*}\right]^{T} \in \bigcap_{l=1}^{m} \mathcal{B}_{l}$ and the corresponding $\left[P_{l}^{*}, Q_{l}^{*}\right]^{T} \in \mathcal{A}_{l}$ such that

$$
\begin{equation*}
\left[k_{p}^{*}, k_{q}^{*}\right]^{T} \in \partial C_{l}\left(P_{l}^{*}, Q_{l}^{*}\right), \quad l=1, \cdots, m . \tag{35}
\end{equation*}
$$

From Theorem 1, the $P_{l}^{*}, Q_{l}^{*}$ optimize problem (16).

Assuming the channel is used $n$ times, the transmitted vector sequences are denoted as $\boldsymbol{x}_{1}^{n}=\left[\boldsymbol{x}_{11}^{T}, \cdots, \boldsymbol{x}_{1 n}^{T}\right]^{T}$ and $\boldsymbol{x}_{2}^{n}=\left[\boldsymbol{x}_{21}^{T}, \cdots, \boldsymbol{x}_{2 n}^{T}\right]^{T}$ which satisfy the average power constraints

$$
\begin{aligned}
& \sum_{l=1}^{n} \operatorname{tr}\left[E\left(\boldsymbol{x}_{1 l} \boldsymbol{x}_{1 l}^{T}\right)\right] \leq n P \\
& \sum_{l=1}^{n} \operatorname{tr}\left[E\left(\boldsymbol{x}_{2 l} \boldsymbol{x}_{2 l}^{T}\right)\right] \leq n Q
\end{aligned}
$$

Define zero-mean Gaussian vectors $\widehat{\boldsymbol{x}}_{1}^{*}=\left[\widehat{X}_{11}^{*}, \cdots, \widehat{X}_{1 m}^{*}\right]$ and $\widehat{\boldsymbol{x}}_{2}^{*}=\left[\widehat{X}_{21}^{*}, \cdots, \widehat{X}_{2 m}^{*}\right]$ with the covariance matrices

$$
\begin{align*}
& \operatorname{Cov}\left(\widehat{\boldsymbol{x}}_{1}^{*}\right)=\frac{1}{n} \sum_{l=1}^{n} \operatorname{Cov}\left(\boldsymbol{x}_{1 l}\right),  \tag{36}\\
& \operatorname{Cov}\left(\widehat{\boldsymbol{x}}_{2}^{*}\right)=\frac{1}{n} \sum_{l=1}^{n} \operatorname{Cov}\left(\boldsymbol{x}_{2 l}\right) . \tag{37}
\end{align*}
$$

Obviously, $\widehat{\boldsymbol{x}}_{1}^{*}$ and $\widehat{\boldsymbol{x}}_{2}^{*}$ satisfy the power constraints. We define

$$
\begin{align*}
P_{l} & =\operatorname{Var}\left(\widehat{X}_{1 l}^{*}\right)  \tag{38}\\
Q_{l} & =\operatorname{Var}\left(\widehat{X}_{2 l}^{*}\right) \tag{39}
\end{align*}
$$

Vectors $\widehat{\boldsymbol{y}}_{1}^{*}$ and $\widehat{\boldsymbol{y}}_{2}^{*}$ are defined by (34) with $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$ being replaced by $\widehat{\boldsymbol{x}}_{1}^{*}$ and $\widehat{\boldsymbol{x}}_{2}^{*}$ respectively. Similar to $\boldsymbol{x}_{1}, \widehat{\boldsymbol{x}}_{1}^{*}$ is also partitioned as $\widehat{\boldsymbol{x}}_{1 \underline{i}}^{*}, \widehat{\boldsymbol{x}}_{1 \underline{j}}^{*}, \widehat{\boldsymbol{x}}_{1 \underline{k}}^{*}$ and $\widehat{\boldsymbol{x}}_{1 \underline{\underline{r}}}^{*}$.

Define Gaussian random vectors $\boldsymbol{n}_{1 \underline{i}}, \boldsymbol{n}_{1 \underline{j}}, \boldsymbol{n}_{2 \underline{i}}$ and $\boldsymbol{n}_{2 \underline{k}}$ independent of $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$, and let

$$
\begin{align*}
& {\left[\begin{array}{c}
\boldsymbol{z}_{1 \underline{i}} \\
\boldsymbol{z}_{1 \underline{j}} \\
\boldsymbol{z}_{1 \underline{k}} \\
\boldsymbol{n}_{1 \underline{i}} \\
\boldsymbol{n}_{1 \underline{j}}
\end{array}\right] \sim \mathcal{N}\left(\mathbf{0},\left[\begin{array}{ccccc}
\mathbf{I}_{\underline{i}} & \mathbf{0} & \mathbf{0} & \rho_{1 \underline{i}} \sigma_{1 \underline{i}} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}_{\underline{j}} & \mathbf{0} & \mathbf{0} & \rho_{1 \underline{j}} \sigma_{1 \underline{j}} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}_{\underline{k}} & \mathbf{0} & \mathbf{0} \\
\rho_{1 \underline{i}} \sigma_{1 \underline{i}} & \mathbf{0} & \mathbf{0} & \sigma_{1 \underline{i}}^{2} & \mathbf{0} \\
\mathbf{0} & \rho_{1 \underline{j}} \sigma_{1 \underline{j}} & \mathbf{0} & \mathbf{0} & \sigma_{1 \underline{j}}^{2}
\end{array}\right]\right),}  \tag{40}\\
& {\left[\begin{array}{c}
\boldsymbol{z}_{2 \underline{i}} \\
\boldsymbol{z}_{2 \underline{j}} \\
\boldsymbol{z}_{2 \underline{k}} \\
\boldsymbol{n}_{2 \underline{i}} \\
\boldsymbol{n}_{2 \underline{k}}
\end{array}\right] \sim \mathcal{N}\left(\mathbf{0},\left[\begin{array}{ccccc}
\mathbf{I}_{\underline{i}} & \mathbf{0} & \mathbf{0} & \rho_{2 \underline{i}} \sigma_{2 \underline{i}} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}_{\underline{j}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}_{\underline{k}} & \mathbf{0} & \rho_{2 \underline{k}} \sigma_{2 \underline{k}} \\
\rho_{2 \underline{i}} \sigma_{2 \underline{i}} \mathbf{0} & \mathbf{0} & \sigma_{2 \underline{i}}^{2} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \rho_{2 \underline{k}} \sigma_{2 \underline{k}} & \mathbf{0} & \sigma_{2 \underline{k}}^{2}
\end{array}\right]\right),} \tag{41}
\end{align*}
$$

where $\rho_{\underline{i}}$ and $\sigma_{\underline{i}}$ are diagonal matrices with the diagonal entries being $\rho_{i}$ and $\sigma_{i}, i \in \underline{i}$, respectively. Furthermore, we let

$$
\begin{align*}
& \begin{aligned}
\sigma_{1 i}^{2}= & \frac{1}{2 b_{i}}\left(\frac{b_{i}}{c_{i}}\left(a_{i} Q_{i}^{*}+1\right)^{2}-\frac{a_{i}}{d_{i}}\left(b_{i} P_{i}^{*}+1\right)^{2}+1\right. \\
& \left.\quad \pm \sqrt{\left[\frac{b_{i}}{c_{i}}\left(a_{i} Q_{i}^{*}+1\right)^{2}-\frac{a_{i}}{d_{i}}\left(b_{i} P_{i}^{*}+1\right)^{2}+1\right]^{2}-\frac{4 b_{i}}{c_{i}}\left(a_{i} Q_{i}^{*}+1\right)^{2}}\right), \\
\sigma_{2 i}^{2}= & \frac{1}{2 a_{i}}\left(\frac{a_{i}}{d_{i}}\left(b_{i} P_{i}^{*}+1\right)^{2}-\frac{b_{i}}{c_{i}}\left(a_{i} Q_{i}^{*}+1\right)^{2}+1\right. \\
& \left.\quad \pm \sqrt{\left[\frac{a_{i}}{d_{i}}\left(b_{i} P_{i}^{*}+1\right)^{2}-\frac{b_{i}}{c_{i}}\left(a_{i} Q_{i}^{*}+1\right)^{2}+1\right]^{2}-\frac{4 a_{i}}{d_{i}}\left(b_{i} P_{i}^{*}+1\right)^{2}}\right), \\
\rho_{1 i}= & \sqrt{1-a_{i} \sigma_{2 i}^{2}}, \\
\rho_{2 i}= & \sqrt{1-b_{i} \sigma_{1 i}^{2}}, \\
\sigma_{1 j}^{2}= & \frac{\left(1+a_{j} Q_{j}^{*}\right)^{2}}{c_{j} \rho_{1 j}^{2}} \\
\rho_{1 j}= & \sqrt{1-\frac{a_{j}}{d_{j}}}, \\
\sigma_{2 k}^{2}= & \frac{\left(1+b_{k} P_{k}^{*}\right)^{2}}{d_{k} \rho_{2 k}^{2}}, \\
\rho_{2 k}= & \sqrt{1-\frac{b_{k}}{c_{k}}} .
\end{aligned} .
\end{align*}
$$

We emphasize that the $P_{l}^{*}$ and $Q_{l}^{*}$ in (42)-(49) are the optimal powers for the problem (16) and can be considered as constants in what follows. It has been shown in [6, equations (51),(52)] that (42)-(45) are feasible (i.e., there exist at least one choice of $\left\{\sigma_{1 i}^{2}, \sigma_{2 i}^{2}, \rho_{1 i}, \rho_{2 i}\right\}$ such that the covariance matrices are symmetric and semi-positive definite, and thus the defined Gaussian random vectors exist) for the definition in (40) and (41) if and only if $\left[P_{i}^{*}, Q_{i}^{*}\right]^{T} \in \mathcal{A}_{i}$. Obviously, (46)-(49) are feasible for the definitions in (40) and (41) if and only if $a_{j} \leq d_{j}$ and $b_{k} \leq c_{k}$. Moreover, (42)-(45), (46) and (48) satisfy

$$
\begin{align*}
& \sqrt{c_{l}} \rho_{1 l} \sigma_{1 l}=1+a_{l} Q_{l}^{*},  \tag{50}\\
& \sqrt{d_{l}} \rho_{2 l} \sigma_{2 l}=1+b_{l} P_{l}^{*}, \tag{51}
\end{align*}
$$

for all $l=1, \cdots, m$.

Let $\epsilon>0$ and $\epsilon \rightarrow 0$ as $n \rightarrow \infty$. From Fano's inequality, any achievable rate $R_{1}$ and $R_{2}$ for the PGIC must satisfy

$$
\begin{align*}
& n\left(R_{1}+R_{2}\right)-n \epsilon \\
& \leq I\left(\boldsymbol{x}_{1}^{n} ; \boldsymbol{y}_{1}^{n}\right)+I\left(\boldsymbol{x}_{2}^{n} ; \boldsymbol{y}_{2}^{n}\right) \\
& \leq I\left(\boldsymbol{x}_{1}^{n} ; \boldsymbol{y}_{1}^{n}, \boldsymbol{x}_{1 \underline{i}}^{n}+\boldsymbol{n}_{1 \underline{i}}^{n}, \boldsymbol{x}_{1 \underline{j}}^{n}+\boldsymbol{n}_{1 \underline{j}}^{n}\right)+I\left(\boldsymbol{x}_{2}^{n} ; \boldsymbol{y}_{2}^{n}, \boldsymbol{x}_{2 \underline{i}}^{n}+\boldsymbol{n}_{2 \underline{2}}^{n}, \boldsymbol{x}_{2 \underline{k}}^{n}+\boldsymbol{n}_{2 \underline{k}}^{n}\right) \\
& =h\left(\boldsymbol{y}_{1}^{n}, \boldsymbol{x}_{1 \underline{\underline{i}}}^{n}+\boldsymbol{n}_{1 \underline{i}}^{n}, \boldsymbol{x}_{1 \underline{j}}^{n}+\boldsymbol{n}_{1 \underline{j}}^{n}\right)-h\left(\boldsymbol{y}_{1}^{n}, \boldsymbol{x}_{1 \underline{i}}^{n}+\boldsymbol{n}_{1 \underline{i}}^{n}, \boldsymbol{x}_{1 \underline{\underline{j}}}^{n}+\boldsymbol{n}_{1 \underline{j}}^{n} \mid \boldsymbol{x}_{1}^{n}\right)+h\left(\boldsymbol{y}_{2}^{n}, \boldsymbol{x}_{2 \underline{i}}^{n}+\boldsymbol{n}_{2 \underline{i}}^{n}, \boldsymbol{x}_{2 \underline{k}}^{n}+\boldsymbol{n}_{2 \underline{k}}^{n}\right) \\
& \quad-h\left(\boldsymbol{y}_{2}^{n}, \boldsymbol{x}_{2 \underline{i}}^{n}+\boldsymbol{n}_{2 \underline{i}}^{n}, \boldsymbol{x}_{2 \underline{k}}^{n}+\boldsymbol{n}_{2 \underline{k}}^{n} \mid \boldsymbol{x}_{2}^{n}\right) . \tag{52}
\end{align*}
$$

In (52), we provide side information $x_{1 \underline{\underline{i}}}^{n}+\boldsymbol{n}_{1 \underline{\underline{i}}}^{n}$ and $x_{1 \underline{j}}^{n}+\boldsymbol{n}_{1 \underline{j}}^{n}$ to receiver one, and $x_{2 \underline{\underline{i}}}^{n}+\boldsymbol{n}_{2 \underline{\underline{i}}}^{n}$ and $\boldsymbol{x}_{2 \underline{\underline{k}}}^{n}+\boldsymbol{n}_{2 \underline{k}}^{n}$ to receiver two, respectively. For the first $m_{1}$ sub-channels which have two-sided interference, both receivers have side information. For the sub-channels which have one-sided interference, only the receivers suffering from interference have the corresponding side information. For the sub-channels without interference, no side information is given.

For the first term of (52), we have

$$
\begin{align*}
& h\left(\boldsymbol{y}_{1}^{n}, \boldsymbol{x}_{1 \underline{\underline{l}}}^{n}+\boldsymbol{n}_{1 \underline{1}}^{n}, \boldsymbol{x}_{1 \underline{j}}^{n}+\boldsymbol{n}_{1 \underline{j}}^{n}\right) \\
& =h\left(\boldsymbol{y}_{1 \underline{i}}^{n}, \boldsymbol{y}_{1 \underline{j}}^{n}, \boldsymbol{y}_{1 \underline{k}}^{n}, \boldsymbol{y}_{1 \underline{\underline{r}}}^{n}, \boldsymbol{x}_{1 \underline{i}}^{n}+\boldsymbol{n}_{1 \underline{i}}^{n}, \boldsymbol{x}_{1 \underline{j}}^{n}+\boldsymbol{n}_{1 \underline{j}}^{n}\right) \\
& \leq h\left(\boldsymbol{y}_{1 \underline{\underline{k}}}^{n}, \boldsymbol{x}_{1 \underline{\underline{l}}}^{n}+\boldsymbol{n}_{1 \underline{i}}^{n}\right)+h\left(\boldsymbol{y}_{1 \underline{\underline{l}}}^{n} \mid \boldsymbol{x}_{1 \underline{i}}^{n}+\boldsymbol{n}_{1 \underline{\underline{i}}}^{n}\right)+h\left(\boldsymbol{y}_{1 \underline{j}}^{n}, \boldsymbol{x}_{1 \underline{j}}^{n}+\boldsymbol{n}_{1 \underline{j}}^{n}\right)+h\left(\boldsymbol{y}_{1 \underline{\underline{q}}}^{n}\right) \\
& \leq h\left(\boldsymbol{y}_{1 \underline{\underline{L}}}^{n}, \boldsymbol{x}_{1 \underline{\underline{i}}}^{n}+\boldsymbol{n}_{1 \underline{\underline{L}}}^{n}\right)+n h\left(\widehat{\boldsymbol{y}}_{1 \underline{\underline{L}}}^{*} \mid \widehat{\boldsymbol{x}}_{1 \underline{i}}^{*}+\boldsymbol{n}_{1 \underline{i}}\right)+n h\left(\widehat{\boldsymbol{y}}_{1 \underline{j}}^{*}, \widehat{\boldsymbol{x}}_{1 \underline{j}}^{*}+\boldsymbol{n}_{1 \underline{j}}\right)+n h\left(\widehat{\boldsymbol{y}}_{1 \underline{\underline{q}}}^{*}\right) \\
& \leq h\left(\boldsymbol{y}_{1 \underline{k}}^{n}, \boldsymbol{x}_{1 \underline{i}}^{n}+\boldsymbol{n}_{1 \underline{\underline{l}}}^{n}\right)+n \sum_{i} h\left(\widehat{Y}_{1 i}^{*} \mid \widehat{X}_{1 i}^{*}+N_{1 i}\right)+n \sum_{j} h\left(\widehat{Y}_{1 j}^{*}, \widehat{X}_{1 j}^{*}+N_{1 j}\right)+n \sum_{r} h\left(\widehat{Y}_{1 r}^{*}\right) \tag{53}
\end{align*}
$$

where the first inequality follows by the chain rule and the fact that conditioning does not increase entropy, and the second inequality is from Lemma 4.

For the fourth term of (52), we have

$$
\begin{aligned}
& -h\left(\boldsymbol{y}_{2}^{n}, \boldsymbol{x}_{2 \underline{i}}^{n}+\boldsymbol{n}_{2 \underline{i}}^{n}, \boldsymbol{x}_{2 \underline{k}}^{n}+\boldsymbol{n}_{2 \underline{k}}^{n} \mid \boldsymbol{x}_{2}^{n}\right) \\
& =-h\left(\boldsymbol{y}_{2 \underline{u}}^{n}, \boldsymbol{y}_{2 \underline{j}}^{n}, \boldsymbol{y}_{2 \underline{k}}^{n}, \boldsymbol{y}_{2 \underline{r}}^{n}, \boldsymbol{x}_{2 \underline{\underline{L}}}^{n}+\boldsymbol{n}_{2 \underline{i}}^{n}, \boldsymbol{x}_{2 \underline{k}}^{n}+\boldsymbol{n}_{2 \underline{k}}^{n} \mid \boldsymbol{x}_{2 \underline{i}}^{n}, \boldsymbol{x}_{2 \underline{j}}^{n}, \boldsymbol{x}_{2 \underline{k}}^{n}, \boldsymbol{x}_{2 \underline{\underline{r}}}^{n}\right) \\
& =-h\left(\mathbf{B}_{\underline{i}} x_{1 \underline{1}}^{n}+z_{2 \underline{2}}^{n}, z_{2 \underline{j}}^{n}, \mathbf{B}_{\underline{k}} x_{1 \underline{k}}^{n}+z_{2 \underline{k}}^{n}, z_{2 \underline{\underline{2}}}^{n}, \boldsymbol{n}_{2 \underline{i}}^{n}, \boldsymbol{n}_{2 \underline{k}}^{n}\right) \\
& =-h\left(\mathbf{B}_{\underline{\underline{z}}} x_{1 \underline{\underline{i}}}^{n}+z_{2 \underline{i}}^{n}, \mathbf{B}_{\underline{k}} x_{1 \underline{k}}^{n}+z_{2 \underline{k}}^{n}, \boldsymbol{n}_{2 \underline{i}}^{n}, \boldsymbol{n}_{2 \underline{k}}^{n}\right)-h\left(\boldsymbol{z}_{2 \underline{j}}^{n}\right)-h\left(z_{2 \underline{r}}^{n}\right) \\
& =-h\left(\mathbf{B}_{\underline{i}} x_{1 \underline{i}}^{n}+z_{2 \underline{i}}^{n}, \mathbf{B}_{\underline{k}} x_{1 \underline{k}}^{n}+z_{2 \underline{k}}^{n} \mid n_{2 \underline{i}}^{n}, n_{2 \underline{k}}^{n}\right)-h\left(n_{2 \underline{j}}^{n}\right)-h\left(n_{2 \underline{k}}^{n}\right)-h\left(z_{2 \underline{j}}^{n}\right)-h\left(z_{2 \underline{\underline{r}}}^{n}\right)
\end{aligned}
$$

$$
\begin{align*}
= & -h\left(\mathbf{B}_{\underline{i}} x_{1 \underline{\underline{i}}}^{n}+z_{2 \underline{i}}^{n}, \mathbf{B}_{\underline{k}} x_{1 \underline{k}}^{n}+z_{2 \underline{k}}^{n} \mid \boldsymbol{n}_{2 \underline{i}}^{n}, \boldsymbol{n}_{2 \underline{k}}^{n}\right)-n \sum_{i} h\left(N_{2 i}\right)-n \sum_{k} h\left(N_{2 k}\right)-n \sum_{j} h\left(Z_{2 j}\right) \\
& -n \sum_{r} h\left(Z_{2 r}\right) . \tag{54}
\end{align*}
$$

where the third equality holds since $\boldsymbol{z}_{2 \underline{j}}^{n}$ and $\boldsymbol{z}_{2 \underline{r}}^{n}$ are independent of all other variables.
Combine the first terms of (53) and (54), we have

$$
\begin{align*}
& h\left(\boldsymbol{y}_{1 \underline{k}}^{n}, x_{1 \underline{\underline{i}}}^{n}+n_{1 \underline{i}}^{n}\right)-h\left(\mathbf{B}_{\underline{i}} x_{1 \underline{\underline{l}}}^{n}+z_{2 \underline{i}}^{n}, \mathbf{B}_{\underline{k}} x_{1 \underline{\underline{k}}}^{n}+z_{2 \underline{k}}^{n} \mid n_{2 \underline{i}}^{n}, \boldsymbol{n}_{2 \underline{k}}^{n}\right) \\
& \stackrel{(a)}{=} h\left(\boldsymbol{x}_{1 \underline{\underline{i}}}^{n}+n_{1 \underline{\underline{i}}}^{n}, \mathbf{C}_{\underline{k}} x_{1 \underline{\underline{k}}}^{n}+z_{1 \underline{\underline{k}}}^{n}\right)-h\left(\mathbf{B}_{\underline{i}} x_{1 \underline{\underline{i}}}^{n}+\boldsymbol{w}_{2 \underline{\underline{i}}}^{n}, \mathbf{B}_{\underline{k}} x_{1 \underline{k}}^{n}+\boldsymbol{w}_{2 \underline{k}}^{n}\right) \\
& =h\left(x_{1 \underline{i}}^{n}+\boldsymbol{n}_{1 \underline{i}}^{n}, \boldsymbol{x}_{1 \underline{k}}^{n}+\mathbf{C}_{\underline{k}}^{-1} \boldsymbol{z}_{1 \underline{k}}^{n}\right)-h\left(\boldsymbol{x}_{1 \underline{\underline{n}}}^{n}+\mathbf{B}_{\underline{i}}^{-1} \boldsymbol{w}_{2 \underline{i}}^{n}, \boldsymbol{x}_{1 \underline{k}}^{n}+\mathbf{B}_{\underline{k}}^{-1} \boldsymbol{w}_{2 \underline{k}}^{n}\right)+n \log \frac{\left|\mathbf{C}_{\underline{k}}\right|}{\left|\mathbf{B}_{\underline{i}}\right| \cdot\left|\mathbf{B}_{\underline{k}}\right|} \\
& \stackrel{(b)}{=} n \log \frac{\left|\mathbf{C}_{\underline{k}}\right|}{\left|\mathbf{B}_{\underline{i}}\right| \cdot\left|\mathbf{B}_{\underline{k}}\right|} \\
& \stackrel{(c)}{=} n \sum_{i} h\left(\widehat{X}_{1 i}^{*}+N_{1 i}\right)+n \sum_{k} h\left(\widehat{X}_{1 k}^{*}+\frac{1}{\sqrt{c_{k}}} Z_{1 k}\right)-n \sum_{i} h\left(\widehat{X}_{1 i}^{*}+\frac{1}{\sqrt{b_{i}}} W_{2 i}\right) \\
& -n \sum_{k} h\left(\hat{X}_{1 k}^{*}+\frac{1}{\sqrt{b_{k}}} W_{2 k}\right)+n \sum_{k} \log \sqrt{c_{k}}-n \sum_{i} \log \sqrt{b_{i}}-n \sum_{k} \log \sqrt{b_{k}} \\
& =n \sum_{i} h\left(\widehat{X}_{1 i}^{*}+N_{1 i}\right)+n \sum_{k} h\left(\sqrt{c_{k}} \widehat{X}_{1 k}^{*}+Z_{1 k}\right)-n \sum_{i} h\left(\sqrt{b_{i}} \widehat{X}_{1 i}^{*}+Z_{2 i} \mid N_{2 i}\right) \\
& -n \sum_{k} h\left(\sqrt{b_{k}} \widehat{X}_{1 k}^{*}+Z_{2 k} \mid N_{2 k}\right), \tag{55}
\end{align*}
$$

where in (a) we let $\boldsymbol{w}_{2 \underline{i}}$ and $\boldsymbol{w}_{2 \underline{k}}$ be independent Gaussian vectors and

$$
\begin{align*}
& \operatorname{Cov}\left(\boldsymbol{w}_{2 \underline{i}}\right)=\operatorname{Cov}\left(\boldsymbol{z}_{2 \underline{2}} \mid \boldsymbol{n}_{2 \underline{i}}\right)=\mathbf{I}_{\underline{i}}-\operatorname{diag}\left(\rho_{2 \underline{\underline{i}}}^{2}\right) \\
& \operatorname{Cov}\left(\boldsymbol{w}_{2 \underline{k}}\right)=\operatorname{Cov}\left(\boldsymbol{z}_{2 \underline{k}} \mid \boldsymbol{n}_{2 \underline{k}}\right)=\mathbf{I}_{\underline{k}}-\operatorname{diag}\left(\rho_{2 \underline{k}}^{2}\right) . \tag{56}
\end{align*}
$$

The stacked vectors $\boldsymbol{w}_{2 \underline{i}}^{n}$ and $\boldsymbol{w}_{2 \underline{k}}^{n}$ each have independent and identical distribution (i.i.d) entries. Equality (b) holds because of (45) and (49) which imply

$$
\begin{aligned}
\operatorname{Cov}\left(n_{1 \underline{i}}\right) & =\operatorname{Cov}\left(\mathbf{B}_{\underline{i}}^{-1} \boldsymbol{w}_{2 \underline{i}}\right), \\
\operatorname{Cov}\left(\mathbf{C}_{\underline{k}}^{-1} \boldsymbol{z}_{1 \underline{k}}\right) & =\operatorname{Cov}\left(\mathbf{B}_{\underline{k}}^{-1} \boldsymbol{w}_{2 \underline{k}}\right),
\end{aligned}
$$

and

$$
h\left(x_{1 \underline{i}}^{n}+\boldsymbol{n}_{1 \underline{1}}^{n}, x_{1 \underline{k}}^{n}+\mathbf{C}_{\underline{k}}^{-1} z_{1 \underline{k}}^{n}\right)-h\left(\boldsymbol{x}_{1 \underline{i}}^{n}+\mathbf{B}_{\underline{i}}^{-1} \boldsymbol{w}_{2 \underline{i}}^{n}, x_{1 \underline{k}}^{n}+\mathbf{B}_{\underline{k}}^{-1} \boldsymbol{w}_{2 \underline{k}}^{n}\right)=0
$$

regardless of the distribution of $\boldsymbol{x}_{1 \underline{\underline{i}}}^{n}$ and $\boldsymbol{x}_{1 \underline{k}}^{n}$.

Equality (c) holds also because of (45) and (49), which imply

$$
\begin{aligned}
h\left(\widehat{X}_{1 i}^{*}+N_{1 i}\right)-h\left(X_{1 i}^{n}+\frac{1}{\sqrt{b_{i}}} W_{2 i}\right) & =0, \\
h\left(\widehat{X}_{1 k}^{*}+\frac{1}{\sqrt{c_{k}}} Z_{1 k}\right)-h\left(\widehat{X}_{1 k}^{*}+\frac{1}{\sqrt{b_{k}}} W_{2 k}\right) & =0,
\end{aligned}
$$

regardless of the distributions of $\widehat{X}_{1 i}$ and $\widehat{X}_{1 k}$.
Combining (53) and (54) and using (55), we have

$$
\begin{align*}
& h\left(\boldsymbol{y}_{1}^{n}, \boldsymbol{x}_{1 \underline{i}}^{n}+\boldsymbol{n}_{1 \underline{i}}^{n}, \boldsymbol{x}_{1 \underline{j}}^{n}+\boldsymbol{n}_{1 \underline{j}}^{n}\right)-h\left(\boldsymbol{y}_{2}^{n}, \boldsymbol{x}_{2 \underline{\underline{i}}}^{n}+\boldsymbol{n}_{2 \underline{i}}^{n}, \boldsymbol{x}_{2 \underline{k}}^{n}+\boldsymbol{n}_{2 \underline{k}}^{n} \mid \boldsymbol{x}_{2}^{n}\right) \\
& \leq n \sum_{i} h\left(\widehat{X}_{1 i}^{*}+N_{1 i}\right)+n \sum_{k} h\left(\sqrt{c_{k}} \widehat{X}_{1 k}^{*}+Z_{1 k}\right)+n \sum_{i} h\left(\widehat{Y}_{1 i}^{*} \mid \widehat{X}_{1 i}^{*}+N_{1 i}\right)+n \sum_{r} h\left(\widehat{Y}_{1 r}^{*}\right) \\
& \quad+n \sum_{j} h\left(\widehat{Y}_{1 j}^{*}, \widehat{X}_{1 j}^{*}+N_{1 j}\right)-n \sum_{i} h\left(\sqrt{b_{i}} \widehat{X}_{1 i}^{*}+Z_{2 i} \mid N_{2 i}\right)-n \sum_{k} h\left(\sqrt{b_{k}} \widehat{X}_{1 k}^{*}+Z_{2 k} \mid N_{2 k}\right) \\
& \quad-n \sum_{i} h\left(N_{2 i}\right)-n \sum_{k} h\left(N_{2 k}\right)-n \sum_{j} h\left(Z_{2 j}\right)-n \sum_{r} h\left(Z_{2 r}\right) . \tag{57}
\end{align*}
$$

Similarly, because of (44) and (47) we have

$$
\begin{align*}
& h\left(\boldsymbol{y}_{2}^{n}, \boldsymbol{x}_{2 \underline{i}}^{n}+\boldsymbol{n}_{2 \underline{i}}^{n}, \boldsymbol{x}_{2 \underline{k}}^{n}+\boldsymbol{n}_{2 \underline{k}}^{n}\right)-h\left(\boldsymbol{y}_{1}^{n}, \boldsymbol{x}_{1 \underline{i}}^{n}+\boldsymbol{n}_{1 \underline{i}}^{n}, \boldsymbol{x}_{1 \underline{j}}^{n}+\boldsymbol{n}_{1 \underline{j}}^{n} \mid \boldsymbol{x}_{1}^{n}\right) \\
& \leq n \sum_{i} h\left(\widehat{X}_{2 i}^{*}+N_{2 i}\right)+n \sum_{j} h\left(\sqrt{d_{j}} \widehat{X}_{2 j}^{*}+Z_{2 j}\right)+n \sum_{i}\left(Y_{2 i}^{*} \mid \widehat{X}_{2 i}^{*}+N_{2 i}\right)+n \sum_{r} h\left(\widehat{Y}_{2 r}^{*}\right) \\
& \quad+n \sum_{k} h\left(\widehat{Y}_{2 k}^{*}, \widehat{X}_{2 k}^{*}+N_{2 k}\right)-n \sum_{i} h\left(\sqrt{a_{i}} \widehat{X}_{2 i}^{*}+Z_{1 i} \mid N_{1 i}\right)-n \sum_{j} h\left(\sqrt{a_{j}} \widehat{X}_{2 j}^{*}+Z_{1 j} \mid N_{1 j}\right) \\
& \quad-n \sum_{i} h\left(N_{1 i}\right)-n \sum_{j} h\left(N_{1 j}\right)-n \sum_{k} h\left(Z_{1 k}\right)-n \sum_{r} h\left(Z_{1 r}\right) . \tag{58}
\end{align*}
$$

Substituting (57) and (58) into (52), we have

$$
\begin{aligned}
& R_{1}+R_{2}-\epsilon \\
& \leq \sum_{i}\left[h\left(\widehat{X}_{1 i}^{*}+N_{1 i}\right)+h\left(\widehat{Y}_{1 i}^{*} \mid \widehat{X}_{1 i}^{*}+N_{1 i}\right)-h\left(N_{1 i}\right)-h\left(\sqrt{a_{i}} \widehat{X}_{2 i}^{*}+Z_{1 i} \mid N_{1 i}\right)\right. \\
& \left.\quad h\left(\widehat{X}_{2 i}^{*}+N_{2 i}\right)+h\left(\widehat{Y}_{2 i}^{*} \mid \widehat{X}_{2 i}^{*}+N_{2 i}\right)-h\left(N_{2 i}\right)-h\left(\sqrt{b_{i}} \widehat{X}_{1 i}^{*}+Z_{2 i} \mid N_{2 i}\right)\right] \\
& \quad+\sum_{j}\left[h\left(\widehat{Y}_{1 j}^{*}, \widehat{X}_{1 j}^{*}+N_{1 j}\right)-h\left(N_{1 j}\right)-h\left(\sqrt{a_{j}} \widehat{X}_{2 j}^{*}+Z_{1 j} \mid N_{1 j}\right)+h\left(\sqrt{d_{j}} \widehat{X}_{2 j}^{*}+Z_{2 j}\right)-h\left(Z_{2 j}\right)\right] \\
& \quad+\sum_{k}\left[h\left(\widehat{Y}_{2 k}^{*}, \widehat{X}_{2 k}^{*}+N_{2 k}\right)-h\left(N_{2 k}\right)-h\left(\sqrt{b_{k}} \widehat{X}_{1 k}^{*}+Z_{2 k} \mid N_{2 k}\right)+h\left(\sqrt{c_{k}} \widehat{X}_{1 k}^{*}+Z_{1 k}\right)-h\left(Z_{1 k}\right)\right] \\
& \quad+\sum_{r}\left[h\left(\widehat{Y}_{1 r}^{*}\right)-h\left(Z_{1 r}\right)\right]+\sum_{r}\left[h\left(\widehat{Y}_{2 r}^{*}\right)-h\left(Z_{2 r}\right)\right] \\
& =\sum_{i} f_{i}\left(P_{i}, Q_{i}\right)+\sum_{j} f_{j}\left(P_{j}, Q_{j}\right)+\sum_{k} f_{k}\left(P_{k}, Q_{k}\right)+\sum_{r} f_{r}\left(P_{r}, Q_{r}\right)
\end{aligned}
$$

$=\sum_{l=1}^{m} f_{l}\left(P_{l}, Q_{l}\right)$,
where

$$
\begin{align*}
f_{i}\left(P_{i}, Q_{i}\right)= & \frac{1}{2} \log \left[\frac{\left(1+a_{i} Q_{i}\right) P_{i}}{1+a_{i} Q_{i}-\rho_{1 i}^{2}}\left(\frac{1}{\sigma_{1 i}}-\frac{\sqrt{c_{i}} \rho_{1 i}}{1+a_{i} Q_{i}}\right)^{2}+1+\frac{c_{i} P_{i}}{1+a_{i} Q_{i}}\right] \\
& +\frac{1}{2} \log \left[\frac{\left(1+b_{i} P_{i}\right) Q_{i}}{1+b_{i} P_{i}-\rho_{2 i}^{2}}\left(\frac{1}{\sigma_{2 i}}-\frac{\sqrt{d_{i}} \rho_{2 i}}{1+b_{i} P_{i}}\right)^{2}+1+\frac{d_{i} Q_{i}}{1+b_{i} P_{i}}\right]  \tag{60}\\
f_{j}\left(P_{j}, Q_{j}\right)= & \frac{1}{2} \log \left[\frac{\left(1+a_{j} Q_{j}\right) P_{j}}{1+a_{j} Q_{j}-\rho_{1 j}^{2}}\left(\frac{1}{\sigma_{1 j}}-\frac{\sqrt{c_{j}} \rho_{1 j}}{1+a_{j} Q_{j}}\right)^{2}+1+\frac{c_{j} P_{j}}{1+a_{j} Q_{j}}\right]+\frac{1}{2} \log \left(1+d_{j} Q_{j}\right),  \tag{61}\\
f_{k}\left(P_{k}, Q_{k}\right)= & \frac{1}{2} \log \left[\frac{\left(1+b_{k} P_{k}\right) Q_{k}}{1+b_{k} P_{k}-\rho_{2 k}^{2}}\left(\frac{1}{\sigma_{2 k}}-\frac{\sqrt{d_{k}} \rho_{2 k}}{1+b_{k} P_{k}}\right)^{2}+1+\frac{d_{k} Q_{k}}{1+b_{k} P_{k}}\right]+\frac{1}{2} \log \left(1+c_{k} P_{k}\right),  \tag{62}\\
f_{r}\left(P_{r}, Q_{r}\right)= & \frac{1}{2} \log \left(1+P_{r}\right)+\frac{1}{2} \log \left(1+Q_{r}\right) . \tag{63}
\end{align*}
$$

Next we will show that the $f_{l}\left(P_{l}, Q_{l}\right), l=1, \cdots, m$ are all concave and non-decreasing functions of $\left(P_{l}, Q_{l}\right)$. From (59) and the fact that

$$
\begin{aligned}
& h\left(\widehat{X}_{1 i}^{*}+N_{1 i}\right)-h\left(\sqrt{b_{i}} \widehat{X}_{1 i}^{*}+Z_{2 i} \mid N_{2 i}\right)=-\log \sqrt{b_{i}}, \\
& h\left(\widehat{X}_{2 i}^{*}+N_{2 i}\right)-h\left(\sqrt{a_{i}} \widehat{X}_{2 i}^{*}+Z_{1 i} \mid N_{1 i}\right)=-\log \sqrt{a_{i}},
\end{aligned}
$$

we have

$$
\begin{equation*}
f_{i}\left(P_{i}, Q_{i}\right)=h\left(\widehat{Y}_{1 i}^{*} \mid \widehat{X}_{1 i}^{*}+N_{1 i}\right)-h\left(N_{1 i}\right)+h\left(\widehat{Y}_{2 i}^{*} \mid \widehat{X}_{2 i}^{*}+N_{2 i}\right)-h\left(N_{2 i}\right)-\log \sqrt{a_{i} b_{i}} . \tag{64}
\end{equation*}
$$

Define Gaussian variables $\widehat{X}_{1 i_{-} t}^{*}, \widehat{X}_{2 i_{-} t}^{*}$ and $\widehat{Y}_{1 i_{-} t}^{*}$ independent of $N_{i}$, and $\widehat{Y}_{1 i_{-} t}^{*}=\sqrt{c_{i}} \widehat{X}_{1 i_{-} t}^{*}+\sqrt{a_{i}} \widehat{X}_{2 i_{-} t}^{*}+Z_{i}$, $t=1, \cdots, s$ where $s$ is an integer. Let $\operatorname{Var}\left(\widehat{X}_{1 i_{-} t}^{*}\right)=P_{i_{-} t}, \operatorname{Var}\left(\widehat{X}_{2 i_{-} t}^{*}\right)=Q_{i_{-} t}$ and

$$
\begin{align*}
& \sum_{t=1}^{s} \lambda_{t} \operatorname{Var}\left(\widehat{X}_{1 \imath_{-} t}^{*}\right)=P_{i}=\operatorname{Var}\left(\widehat{X}_{1 i}^{*}\right),  \tag{65}\\
& \sum_{t=1}^{s} \lambda_{t} \operatorname{Var}\left(\widehat{X}_{2 i \_t}^{*}\right)=Q_{i}=\operatorname{Var}\left(\widehat{X}_{2 i}^{*}\right), \tag{66}
\end{align*}
$$

where $\left\{\lambda_{t}\right\}$ is a non-negative sequence with $\sum_{t=1}^{s} \lambda_{t}=1$. Then we have

$$
\sum_{t=1}^{s} \lambda_{l} \operatorname{Cov}\left(\left[\begin{array}{c}
\widehat{Y}_{1 i_{-} t}^{*}  \tag{67}\\
\widehat{X}_{1 i_{-} t}^{*}+N_{i}
\end{array}\right]\right)=\operatorname{Cov}\left(\left[\begin{array}{c}
\widehat{Y}_{1 i}^{*} \\
\widehat{X}_{1 i}^{*}+N_{i}
\end{array}\right]\right),
$$

From Lemma 4 we have

$$
\begin{equation*}
h\left(\widehat{Y}_{1 i}^{*} \mid \widehat{X}_{1 i}^{*}+N_{1 i}\right) \geq \sum_{l=1}^{k} \lambda_{t} h\left(\widehat{Y}_{1 i_{-} t}^{*} \mid \widehat{X}_{1 i_{-} t}^{*}+N_{1 i}\right) . \tag{68}
\end{equation*}
$$

Therefore, $h\left(\widehat{Y}_{1 i}^{*} \mid \widehat{X}_{1 i}^{*}+N_{1 i}\right)$ is a concave function of $\left(P_{i}, Q_{i}\right)$. For the same reason $h\left(\widehat{Y}_{2 i}^{*} \mid \widehat{X}_{2 i}^{*}+N_{2 i}\right)$ is also a concave function of $\left(P_{i}, Q_{i}\right)$. Therefore $f_{i}\left(P_{i}, Q_{i}\right)$ is a concave function of $\left(P_{i}, Q_{i}\right)$. Similar steps show that $f_{j}, f_{k}$, and $f_{r}$ are concave in $\left(P_{l}, Q_{l}\right)$.

To show that $f_{i}\left(P_{i}, Q_{i}\right)$ is a non-decreasing function of $P_{i}, Q_{i}$, we let $\bar{X}_{1 i}, \bar{X}_{2 i}, U_{1}$ and $U_{2}$ be four independent Gaussian variables and

$$
\begin{aligned}
& \widehat{X}_{1 i}^{*}=\bar{X}_{1 i}+U_{1}, \\
& \widehat{X}_{2 i}^{*}=\bar{X}_{2 i}+U_{2} .
\end{aligned}
$$

Then

$$
\begin{aligned}
& h\left(\widehat{Y}_{1 i}^{*} \mid \widehat{X}_{1 i}^{*}+N_{1 i}\right) \\
& =h\left(\sqrt{c_{i}} \widehat{X}_{1 i}^{*}+\sqrt{a_{i}} \widehat{X}_{2 i}^{*}+Z_{1 i} \mid \widehat{X}_{1 i}^{*}+N_{1 i}\right) \\
& \leq h\left(\sqrt{c_{i}} \widehat{X}_{1 i}^{*}+\sqrt{a_{i}} \widehat{X}_{2 i}^{*}+Z_{1 i} \mid \widehat{X}_{1 i}^{*}+N_{1 i}, U_{1}, U_{2}\right) \\
& =h\left(\sqrt{c_{i}} \bar{X}_{1 i}+\sqrt{a_{i}} \bar{X}_{2 i}+Z_{1 i} \mid \bar{X}_{1 i}+N_{1 i}\right) .
\end{aligned}
$$

Therefore, $h\left(\widehat{Y}_{1 i}^{*} \mid \widehat{X}_{1 i}^{*}+N_{1 i}\right)$ is a non-decreasing function of $\left(P_{i}, Q_{i}\right)$. For the same reason, $h\left(\widehat{Y}_{2 i}^{*} \mid \widehat{X}_{2 i}^{*}+N_{2 i}\right)$ is also a non-decreasing function of $\left(P_{i}, Q_{i}\right)$. Therefore, $f_{i}$ is a non-decreasing function of $\left(P_{i}, Q_{i}\right)$.

From (50) and (51) we have

$$
\begin{align*}
\sum_{l=1}^{m} f_{l}\left(P_{l}^{*}, Q_{l}^{*}\right) & =\frac{1}{2} \sum_{l=1}^{m}\left[\log \left(1+\frac{c_{l} P_{l}^{*}}{1+a_{l} Q_{l}^{*}}\right)+\log \left(1+\frac{d_{l} Q_{l}^{*}}{1+b_{l} P_{l}^{*}}\right)\right] \\
& =\sum_{l=1}^{m} C_{l}\left(P_{l}^{*}, Q_{l}^{*}\right) . \tag{69}
\end{align*}
$$

Next we will show that $\sum_{l=1}^{m} f\left(P_{l}^{*}, Q_{l}^{*}\right) \geq \sum_{l=1}^{m} f\left(P_{l}, Q_{l}\right)$ for any $P_{l}, Q_{l}$ that satisfy $\sum_{l=1}^{m} P_{l}=P$, $\sum_{l=1}^{m} Q_{l}=Q$.

Using (50) and (51), we have

$$
\begin{align*}
& \left.\frac{\partial f_{l}(p, q)}{\partial p}\right|_{\substack{p=P_{l}^{*} \\
q=Q_{l}^{*}}}=\left.\left.\frac{\partial C_{l}(p, q)}{\partial p}\right|_{\substack{p=P_{l}^{*} \\
q=Q_{l}^{*}}}\right|_{\substack{p=P_{l}^{*} \\
q=Q_{l}^{*}}}=\left.\frac{\partial C_{l}(p, q)}{\partial q}\right|_{\substack{p=P_{l}^{*} \\
q=Q_{l}^{*}}}, \tag{70}
\end{align*}
$$

for all $l=1, \cdots, m$. Therefore, $f_{l}$ and $C_{l}$ have the same partial derivatives at point $\left(P_{l}^{*}, Q_{l}^{*}\right)$. From the Appendix, the subgradient of a function is determined by the derivatives, therefore, $f_{l}$ and $C_{l}$ have the same subgradient at point $\left(P_{l}^{*}, Q_{l}^{*}\right)$ for each $l$. From (35), we have

$$
\begin{equation*}
\left[k_{p}^{*}, k_{q}^{*}\right] \in \partial f_{l}\left(P_{l}^{*}, Q_{l}^{*}\right), \quad l=1, \cdots, m . \tag{72}
\end{equation*}
$$

Therefore, from Lemma 3 we have

$$
\begin{equation*}
\sum_{l=1}^{m} f_{l}\left(P_{l}^{*}, Q_{l}^{*}\right) \geq \sum_{l=1}^{m} f_{l}\left(P_{l}, Q_{l}\right) \tag{73}
\end{equation*}
$$

if $\sum_{l=1}^{m} P_{l}=P$ and $\sum_{l=1}^{m} Q_{l}=Q$.
If $\sum_{l=1}^{m} P_{l} \leq P$ and $\sum_{l=1}^{m} Q_{l} \leq Q$, there exist two non-negative sequences $\left\{s_{1}, \cdots, s_{m}\right\}$ and $\left\{t_{1}, \cdots, t_{m}\right\}$ such that $\sum_{l=1}^{m}\left(P_{l}+s_{l}\right)=P$ and $\sum_{l=1}^{m}\left(Q_{l}+t_{l}\right)=Q$. Therefore we have from Lemma 3

$$
\begin{equation*}
\sum_{l=1}^{m} f_{l}\left(P_{l}^{*}, Q_{l}^{*}\right) \geq \sum_{i=1}^{m} f_{l}\left(P_{l}+s_{l}, Q_{l}+t_{l}\right) \tag{74}
\end{equation*}
$$

Since $f_{l}\left(P_{l}, Q_{l}\right), l=1, \cdots, m$ are all non-decreasing functions of $P_{l}$ and $Q_{l}$, we have

$$
\begin{equation*}
\sum_{i=1}^{m} f_{l}\left(P_{l}+s_{l}, Q_{l}+t_{l}\right) \geq \sum_{i=1}^{m} f_{l}\left(P_{l}, Q_{l}\right) \tag{75}
\end{equation*}
$$

Combining (74) and (75) we have

$$
\begin{equation*}
\sum_{l=1}^{m} f_{l}\left(P_{l}^{*}, Q_{l}^{*}\right) \geq \sum_{i=1}^{m} f_{l}\left(P_{l}, Q_{l}\right) \tag{76}
\end{equation*}
$$

for any $\sum_{l=1}^{m} P_{l} \leq P$ and $\sum_{l=1}^{m} Q_{l} \leq Q$. Therefore we have from (59) and (69) that

$$
\begin{equation*}
R_{1}+R_{2}-\epsilon \leq \frac{1}{2} \sum_{l=1}^{m}\left[\log \left(1+\frac{c_{l} P_{l}^{*}}{1+a_{l} Q_{l}^{*}}\right)+\log \left(1+\frac{d_{l} Q_{l}^{*}}{1+b_{l} P_{l}^{*}}\right)\right] \tag{77}
\end{equation*}
$$

The above sum rate is achievable by independent transmission across sub-channels and single-user detection in each sub-channel. Therefore (77) is the sum-rate capacity of the PGIC if the power constraints satisfy (18).

Remark 7: The main idea of the proof can be summarized as follows. We first assume an arbitrary power allocation $\left(P_{l}, Q_{l}\right), l=1, \cdots, m$. Then we show that the sum rate for this power allocation is upper bounded by $\sum_{l=1}^{m} f_{l}\left(P_{l}, Q_{l}\right)$. This upper bound decomposes the sum rate bound into the sum of the individual sub-channel's sum-rate capacity upper bounds. By Lemma 3, the maximum of $\sum_{l=1}^{m} f_{l}\left(P_{l}, Q_{l}\right)$ is $\sum_{l=1}^{m} C_{l}\left(P_{l}^{*}, Q_{l}^{*}\right)$ which is an achievable sum rate for a special power allocation. To ease the proof, the upper bound $f_{l}$ can not be arbitrarily chosen. Compared to the sum-rate capacity $C_{l}$ for sub-channel $l, f_{l}$ has the following properties:

- $f_{l}$ is concave over the powers;
- $f_{l}$ is tight at the optimal point $\left(P_{l}^{*}, Q_{l}^{*}\right)$;
- $f_{l}$ and $C_{l}$ have the same subdifferentials at the optimal point $\left(P_{l}^{*}, Q_{l}^{*}\right)$.

Therefore we choose the noise vectors $\boldsymbol{n}_{1}$ and $\boldsymbol{n}_{2}$ such that the above conditions are satisfied. Fig. 4 illustrates such an upper bound.


Fig. 4. An illustration of the upper bound suitable for the proof of Theorem 1.

## V. Numerical examples

Figs. 5 and 6 are examples for a symmetric PGIC with directly link gain $c_{i}=d_{i}=1$. Fig. 5 shows the ratio of maximum noisy interference power constraint $\bar{P}$ for a two-channel symmetric PGIC with $a$ varying from 0 to 0.25 , and the sum of the maximum noisy interference power constraints for $S_{1}+S_{2}$ where $S_{i}=\frac{\sqrt{a_{i}}-2 a_{i}}{2 a_{i}^{2}}, i=1,2$. Thus, for $a_{1}=a_{2}$ (i.e., these two sub-channels are identical), the ratio is 1 and $S_{1}=S_{2}=\frac{\bar{P}}{2}$, and when $a_{1}$ and $a_{2}$ are far apart, then $\bar{P} \ll S_{1}+S_{2}$. Thus, for $a_{1}=a_{2}$ we achieve capacity despite the fact that we do not know the capacity if all power is placed in one sub-channel. Again, this is where Lemma 2 is useful.

Fig. 6 shows the maximum noisy interference power constraint $\bar{P}$ for a two-channel symmetric PGIC with $a_{1}$ varying in $\left[0, \frac{1}{4}\right]$ and $a_{2}=\frac{1}{8}$. When $a_{1}=\frac{1}{4}$, the first sub-channel no longer has noisy interference, therefore the maximum noisy interference power constraint is $\bar{P}=0$. Fig. 6 also shows that $\bar{P}$ decreases with $a_{1}$. The discontinuity at $a_{1}=\frac{1}{8}$ is because the second sub-channel becomes the worse channel.

Fig. 7 shows the regions of $\mathcal{B}_{1}$ and $\mathcal{B}_{2}$ for a PGIC with two sub-channels. In this case $\mathcal{B}_{1} \subset \mathcal{B}_{2}$. In Fig. 8, $O_{1} C M N D$ is the noisy-interference power region for this PGIC. Points $A, E$ coincide in

Fig. 8 since in Fig. $7 A, E \in \mathcal{B}_{1}^{(2)} \cap \mathcal{B}_{2}^{(4)}$ and thus the power allocation is $Q_{1}^{*}=P_{2}^{*}=Q_{2}^{*}=0$ and $P_{1}^{*}>0$. Similarly, points $C, T$ coincide in Fig. 8 since $C, T \in \mathcal{B}_{1}^{(2)} \bigcap \mathcal{B}_{2}^{(2)}$ and thus $Q_{1}^{*}=Q_{2}^{*}=0$ and $P_{1}^{*}>0, P_{2}^{*}>0$. Similar arguments apply to points $B, F$ and $D, S$. By considering the relationship between the power regions and the respective subdifferentials we can determine the activity of the two users in each sub-channel. This activity is summarized in Tab. I, where 0 indicates inactive (zero power) and + indicates active (positive power) for the user in the corresponding sub-channel. In the following we illustrate the regions in Figs. 7 and 8.

We first remind the reader that $\mathcal{B}_{i}^{(1)}$ corresponds to the case where both users are active; $\mathcal{B}_{i}^{(2)}$ corresponds to the case where only user 1 is active; $\mathcal{B}_{i}^{(3)}$ corresponds to the case where only user 2 is active; and $\mathcal{B}_{i}^{(4)}$ corresponds to the case where both users are inactive in sub-channel $i$. Consider the following regions.

- The region $O_{2} M N$ in Tab. I denotes regions in both Figs. 7 and 8. In Fig. 7, it is the intersection of $\mathcal{B}_{1}^{(1)}$ and $\mathcal{B}_{2}^{(1)}$. Thus, to achieve the sum-rate capacity both users are active in both sub-channels. The corresponding power region $O_{2} M N$ is shown in Fig. 8.
- Region $O_{1} E O_{2} F$ is the intersection of $\mathcal{B}_{1}^{(1)}$ and $\mathcal{B}_{2}^{(4)}$. So both users are active only in sub-channel 1. In this case, both of the power constraints $P$ and $Q$ are small, so that only the better subchannel which produces larger sum-rate capacity than the other is allocated power. Therefore, this two-channel PGIC behaves as a GIC.
- Region $O_{1} A E$ is the intersection of $\mathcal{B}_{1}^{(2)}$ and $\mathcal{B}_{2}^{(4)}$. So user 1 is active in sub-channel 1 and user 2 is inactive in both sub-channels. The power region $O_{1} A E$ is shown in Fig. 8. The overall power for user 2 is $Q=0$.
- Similar to the above case, region $A E T C$ of Fig. 7 is the intersection of $\mathcal{B}_{1}^{(2)}$ and $\mathcal{B}_{2}^{(2)}$. The optimal power allocation makes user 1 active in both sub-channels while user 2 is inactive in both subchannels. In Fig. 8, the overall power for user 2 is also $Q=0$. Actually, regions $O_{1} A E$ and $A E T C$ are examples of single-user parallel Gaussian channels whose optimal power allocation is the waterfilling scheme. In the former case, the power constraint is so small that only the sub-channel with larger direct link channel gain (sub-channel 1) is allocated power. In the latter case, the power constraint increases to a critical level (point $A$ in Fig. 8) so that both sub-channels are allocated power.
- Region $E T M O_{2}$ is the intersection of $\mathcal{B}_{1}^{(1)}$ and $\mathcal{B}_{2}^{(2)}$. User 1 is active in both sub-channels while user 2 is active only in sub-channel 1 which has larger direct channel gain for user 2 . In this case,
the two-channel PGIC with two-sided interference works like a PGIC with one sub-channel having two-sided interference and the other having one-sided interference.
- Regions $O_{1} B F, B F S D$ and $F O_{2} N S$ are counterparts of regions $O_{1} A E, A E T C$ and $E T M O_{2}$, respectively, by swapping the roles of the two users.

Also plotted in Fig. 8 are the noisy-interference power regions for the individual sub-channels, where sub-channel 2 has a larger noisy-interference power region than that of sub-channel 1 . In this case, the overall noisy-interference power region is larger than that of either of these two sub-channels.


Fig. 5. Ratio of the total power constraint and the sum of the individual sub-channel power constraints for different channel gains.

## VI. CONCLUSION

Based on the concavity of the sum-rate capacity in the power constraints, we have shown that the noisy-interference sum-rate capacity of a PGIC can be achieved by independent transmission across subchannels and treating interference as noise in each sub-channel. The optimal power allocations have


Fig. 6. The maximum noisy interference power constraint $\bar{P}$ for $a_{1}$ with $a_{2}=\frac{1}{8}$.

TABLE I
Power constraints and the activeness of users

|  | $O_{1} A(E)$ | $A(E)(T) C$ | $E T M O_{2}$ | $O_{1} E O_{2} F$ |
| :---: | :---: | :---: | :---: | :---: |
|  | $\mathcal{B}_{1}^{(2)} \cap \mathcal{B}_{2}^{(4)}$ | $\mathcal{B}_{1}^{(2)} \cap \mathcal{B}_{2}^{(2)}$ | $\mathcal{B}_{1}^{(1)} \cap \mathcal{B}_{2}^{(2)}$ | $\mathcal{B}_{1}^{(1)} \cap \mathcal{B}_{2}^{(4)}$ |
| $\left(P_{1}^{*}, Q_{1}^{*}\right)$ | $(+, 0)$ | $(+, 0)$ | $(+,+)$ | $(+,+)$ |
| $\left(P_{2}^{*}, Q_{2}^{*}\right)$ | $(0,0)$ | $(+, 0)$ | $(+, 0)$ | $(0,0)$ |
|  | $O_{2} M N$ | $O_{1} B(F)$ | $B(F)(S) D$ | $F O_{2} N S$ |
|  | $\mathcal{B}_{1}^{(1)} \cap \mathcal{B}_{2}^{(1)}$ | $\mathcal{B}_{1}^{(3)} \cap \mathcal{B}_{2}^{(4)}$ | $\mathcal{B}_{1}^{(3)} \cap \mathcal{B}_{2}^{(3)}$ | $\mathcal{B}_{1}^{(1)} \cap \mathcal{B}_{2}^{(3)}$ |
| $\left(P_{1}^{*}, Q_{1}^{*}\right)$ | $(+,+)$ | $(0,+)$ | $(0,+)$ | $(+,+)$ |
| $\left(P_{2}^{*}, Q_{2}^{*}\right)$ | $(+,+)$ | $(0,0)$ | $(0,+)$ | $(0,+)$ |



Fig. 7. $\mathcal{B}_{1}$ and $\mathcal{B}_{2}$ for the parallel Gaussian interference channel with $a_{1}=b_{1}=0.6, c_{1}=d_{1}=4, a_{2}=b_{2}=0.24, c_{2}=$ $d_{2}=1.2$.
the property that the sub-channel sum-rate capacity curves have parallel supporting hyperplanes at these powers. The methods introduced in this paper can also be used to obtain the optimal power allocation and capacity regions of parallel Gaussian multiple access and broadcast channels [22].

## APPENDIX: SUBDIFFERENTIAL OF $C_{i}(p, q)$

The subdifferential $\partial C_{i}(p, q)$ depends on the location of the point $(p, q)$. If the sub-channel is a twosided GIC, from Lemma $1, \mathcal{A}_{i}$ is defined in (3). We derive the subdifferentials as outlined in (78)-(88) below and present the evaluations in (89) below.

- If $(p, q)$ is an interior point of $\mathcal{A}_{i}, C_{i}(p, q)$ is differentiable. From the concavity of $C_{i}(p, q), \nabla C_{i}(p, q)$


Fig. 8. Noisy-interference power region for the parallel Gaussian interference channel with $a_{1}=b_{1}=0.6, c_{1}=d_{1}=4, a_{2}=$ $b_{2}=0.24, c_{2}=d_{2}=1.2$.
satisfies (8) and the subdifferential $\partial C_{i}(p, q)$ consists of a unique vector $\left[k_{p}, k_{q}\right]^{T}=\nabla C_{i}(p, q)$ where

$$
\begin{align*}
& k_{p}=\frac{\partial C_{i}(p, q)}{\partial p}  \tag{78}\\
& k_{q}=\frac{\partial C_{i}(p, q)}{\partial q} \tag{79}
\end{align*}
$$

- If $\sqrt{a_{i} c_{i}}\left(1+b_{i} p\right)+\sqrt{b_{i} d_{i}}\left(1+a_{i} q\right)=\sqrt{c_{i} d_{i}}, p>0, q>0$, we can compute only one-sided partial derivatives since $C_{i}(p, q)$ is unknown on the other side. The subdifferential includes a vector $\left[k_{p}, k_{q}\right]^{T}$ where

$$
\begin{align*}
& k_{p}=\lim _{\delta \uparrow 0} \frac{C_{i}(p+\delta, q)-C_{i}(p, q)}{\delta},  \tag{80}\\
& k_{q}=\lim _{\delta \uparrow 0} \frac{C_{i}(p, q+\delta)-C_{i}(p, q)}{\delta}, \tag{81}
\end{align*}
$$

where $\lim _{\delta \uparrow 0} \triangleq \lim _{\delta \leq 0, \delta \rightarrow 0}$ and similarly $\lim _{\delta \downarrow 0} \triangleq \lim _{\delta \geq 0, \delta \rightarrow 0}$

- If $q=0, p>0$, we have only a one-sided partial derivative in $q$. From the concavity of $C_{i}(p, q)$, all the $\left[k_{p}, k_{q}\right]^{T}$ vectors satisfying following conditions are subgradients

$$
\begin{align*}
& k_{p}= \begin{cases}\frac{\partial C_{i}(p, 0)}{\partial p}, & 0<p<\frac{\sqrt{c_{i} d_{i}}-\sqrt{a_{i} c_{i}}-\sqrt{b_{i} d_{i}}}{b_{i} \sqrt{a_{i} c_{i}}}, \\
\lim _{\delta \uparrow 0} \frac{C_{i}(p+\delta, 0)-C_{i}(p, 0)}{\delta}, & p=\frac{\sqrt{c_{i} d_{i}}-\sqrt{a_{i} c_{i}}-\sqrt{b_{i} d_{i}}}{b_{i} \sqrt{a_{i} c_{i}}}\end{cases}  \tag{82}\\
& k_{q}=\lim _{\delta \downarrow 0} \frac{C_{i}(p, \delta)-C_{i}(p, 0)}{\delta} . \tag{83}
\end{align*}
$$

On the other hand, for the same point $(p, q)$ and the corresponding $\left[k_{p}, k_{q}\right]^{T}$ defined above, by choosing $k_{q}^{\prime}>\lim _{\delta \downarrow 0} \frac{C_{i}(p, \delta)-C_{i}(p, 0)}{\delta}$, the vector $\left[k_{p}, k_{q}\right]^{T}$ satisfies (8) for all points $[p, q]^{T} \in \mathcal{A}_{i}$, and thus is also a subgradient. Therefore we can replace (83) with

$$
\begin{equation*}
k_{q} \geq \lim _{\delta \downarrow 0} \frac{C_{i}(p, \delta)-C_{i}(p, 0)}{\delta} \tag{84}
\end{equation*}
$$

- Similarly, if $p=0, q>0$, the subdifferential is the set of $\left[k_{p}, k_{q}\right]^{T}$ with

$$
\begin{align*}
& k_{p} \geq \lim _{\delta \downarrow 0} \frac{C_{i}(\delta, q)-C_{i}(0, q)}{\delta},  \tag{85}\\
& k_{q}= \begin{cases}\frac{\partial C_{i}(0, q)}{\partial q}, & 0<q<\frac{\sqrt{c_{i} d_{i}}-\sqrt{a_{i} c_{i}}-\sqrt{b_{i} d_{i}}}{a_{i} \sqrt{b_{i} d_{i}}} \\
\lim _{\delta \uparrow 0} \frac{C_{i}(p+\delta, 0)-C_{i}(p, 0)}{\delta}, & q=\frac{\sqrt{c_{i} d_{i}}-\sqrt{a_{i} c_{i}}-\sqrt{b_{i} d_{i}}}{a_{i} \sqrt{b_{i} d_{i}}}\end{cases} \tag{86}
\end{align*}
$$

- If $p=q=0$ the subdifferential is the set of $\left[k_{p}, k_{q}\right]^{T}$ with

$$
\begin{align*}
& k_{p} \geq \lim _{\delta \downarrow 0} \frac{C_{i}(\delta, 0)-C_{i}(0,0)}{\delta}  \tag{87}\\
& k_{q} \geq \lim _{\delta \downarrow 0} \frac{C_{i}(0, \delta)-C_{i}(0,0)}{\delta} \tag{88}
\end{align*}
$$

For completeness, we summarize $\partial C_{i}(p, q)$ as follows

$$
\partial C_{i}(p, q)
$$

$$
=\left\{\begin{array}{l|l}
\left\{\begin{array}{ll}
\left.\left\{k_{p}, k_{q}\right) \left\lvert\, \begin{array}{ll}
k_{p}=\frac{1}{2}\left(\frac{c_{i}}{1+c_{i} p+a_{i} q}+\frac{b_{i}}{1+b_{i} p+d_{i} q}-\frac{b_{i}}{1+b_{i} p}\right) \\
k_{q}=\frac{1}{2}\left(\frac{a_{i}}{1+c_{i} p+a_{i} q}+\frac{d_{i}}{1+b_{i} p+d_{i} q}-\frac{a_{i}}{1+a_{i} q}\right)
\end{array}\right.\right\}, & (p, q) \in \mathcal{A}_{i}^{(1)} \\
k_{p}=\frac{c_{i}}{2\left(1+c_{i} p\right)} & (p, q) \in \mathcal{A}_{i}^{(2)} \\
\frac{d_{i}}{2\left(1+b_{i} p\right)}-\frac{a_{i} c_{i} p}{2\left(1+c_{i} p\right)} \leq k_{q} \leq \frac{\hat{d}}{2}
\end{array}\right\}, & (p, q) \in \mathcal{A}_{i}^{(3)}  \tag{89}\\
\left\{\left(k_{p}, k_{q}\right) \left\lvert\, \begin{array}{ll}
\frac{c_{i}}{2\left(1+a_{i} q\right)}-\frac{b_{i} d_{i} q}{2\left(1+d_{i} q\right)} \leq k_{p} \leq \frac{\hat{c}}{2} \\
k_{q}=\frac{d_{i}}{2\left(1+d_{i} q\right)}
\end{array}\right.\right\}, & (p, q) \in \mathcal{A}_{i}^{(4)} .
\end{array}\right.
$$

where

$$
\begin{align*}
& \mathcal{A}_{i}^{(1)}=\left\{(p, q) \mid \sqrt{a_{i}}\left(1+b_{i} p\right)+\sqrt{b_{i}}\left(1+a_{i} q\right) \leq \sqrt{c_{i} d_{i}}, p>0, q>0\right\},  \tag{90}\\
& \mathcal{A}_{i}^{(2)}=\left\{(p, q) \left\lvert\, 0<p \leq \frac{\sqrt{c_{i} d_{i}}-\sqrt{a_{i} c_{i}}-\sqrt{b_{i} d_{i}}}{b_{i} \sqrt{a_{i} c_{i}}}\right., q=0\right\},  \tag{91}\\
& \mathcal{A}_{i}^{(3)}=\left\{(p, q) \mid p=0,0<q \leq \frac{\sqrt{c_{i} d_{i}}-\sqrt{a_{i} c_{i}}-\sqrt{b_{i} d_{i}}}{a_{i} \sqrt{b_{i} d_{i}}}\right\},  \tag{92}\\
& \mathcal{A}_{i}^{(4)}=\{(p, q) \mid p=q=0\}, \tag{93}
\end{align*}
$$

and

$$
\begin{align*}
& \hat{c}=\max _{i=1, \cdots, m}\left\{c_{i}\right\},  \tag{94}\\
& \hat{d}=\max _{i=1, \cdots, m}\left\{d_{i}\right\} . \tag{95}
\end{align*}
$$

In (89) $\mathcal{A}_{i}=\bigcup_{l=1}^{4} \mathcal{A}_{i}^{(l)}$. For the subgradient $\left[k_{p}, k_{q}\right]^{T}$, when $p=0$ or $q=0, k_{p}$ or $k_{q}$ varies from some constants to infinity. In (89) we upper bound $k_{p}$ and $k_{q}$ with $\frac{\hat{c}}{2}$ and $\frac{\hat{d}}{2}$ respectively for convenience and without loss of generality. The main reason is that we are interested only in $\bigcup_{i=1}^{m} \mathcal{B}_{i}$. To relate the mapping of $\mathcal{A}_{i}^{(j)}$ to different regions of $\mathcal{B}_{i}$, we further define

$$
\begin{equation*}
\mathcal{B}_{i}^{(j)}=\bigcup_{[p, q]^{T} \in \mathcal{A}_{i}^{(j)}} \partial C_{i}(p, q), \quad j=1, \cdots, 4 . \tag{96}
\end{equation*}
$$

If the sub-channel has one-sided interference $b_{i}=0,0<a_{i}<1$ or no interference $a_{i}=b_{i}=0$, then from Lemma 1 we have $\mathcal{A}_{i}=\{p \geq 0, q \geq 0\}$. We similarly obtain $\partial C_{i}(p, q)$ as follows:

$$
\begin{align*}
& \partial C_{i}(p, q) \\
& = \begin{cases}\left\{\left(k_{p}, k_{q}\right) \left\lvert\, \begin{array}{ll}
k_{p}=\frac{c_{i}}{2\left(1+c_{i} p+a_{i} q\right)} \\
k_{q}=\frac{1}{2}\left(\frac{a_{i}}{1+c_{i} p+a_{i} q}+\frac{d_{i}}{1+d_{i} q}-\frac{a_{i}}{1+a_{i} q}\right)
\end{array}\right.\right\}, & p>0, q>0 \\
\left\{\left(k_{p}, k_{q}\right) \left\lvert\, \begin{array}{ll}
k_{p}=\frac{c_{i}}{2\left(1+c_{i} p\right)} \\
\frac{d_{i}}{2}-\frac{a_{i} c_{i} p}{2\left(1+c_{i} p\right)} \leq k_{q} \leq \frac{\hat{d}}{2}
\end{array}\right.\right\}, & p>0, q=0 \\
\left\{\left(k_{p}, k_{q}\right) \left\lvert\, \begin{array}{ll}
\frac{c_{i}}{2\left(1+a_{i} q\right)} \leq k_{p} \leq \frac{\hat{c}}{2} \\
k_{q}=\frac{d_{i}}{2\left(1+d_{i} q\right)}
\end{array}\right.\right\}, & p=0, q>0\end{cases}  \tag{97}\\
& \left\{\left(k_{p}, k_{q}\right) \left\lvert\, \begin{array}{ll}
\frac{c_{i}}{2} \leq k_{p} \leq \frac{\hat{c}}{2}, \quad \frac{d_{i}}{2} \leq k_{q} \leq \frac{\hat{d}}{2}
\end{array}\right.\right\},
\end{align*}
$$

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[^1]:    ${ }^{1}$ Even under this simplified assumption, finding the optimal power allocation is an NP hard problem [5]

