Bit Commitment from Non-Signaling Correlations

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Abstract—Central cryptographic functionalities such as encryption, authentication, or secure two-party computation cannot be realized in an information-theoretically secure way from scratch. This serves as a motivation to study what (possibly weak) primitives they *can* be based on. We consider as such starting points general two-party input-output systems that do not allow for message transmission, and show that they can be used for realizing unconditionally secure bit commitment as soon as they are non-trivial, i.e., cannot be securely realized from distributed randomness only.

Index Terms—Unconditional security, bit commitment, non-locality.

I. INTRODUCTION

Modern cryptography deals — besides the classical tasks of encryption and authentication — with secure cooperation between two (or more) parties willing to collaborate but distrusting each other. Examples of important functionalities of such secure *two-party computation* are *bit commitment* and *oblivious transfer*. In this note, we concentrate on bit commitment, a primitive which, for instance, allows for fair coin flipping [1] and has central applications in interactive proof systems.

A bit commitment scheme is a protocol between two parties, Alice and Bob, that consists of two stages. First, they execute *Commit* where Alice chooses a bit b as input. Later, they execute *Open* where Alice reveals the bit b to Bob. The security properties of bit commitment are the following. Security for Alice ensures that the *Commit* protocol does not give any information about the bit b to Bob. Security for Bob, on the other hand, means that after the execution of *Commit*, bcannot be changed anymore by Alice. Ideally, one would like these security properties to hold even against an adversary with unlimited computing power.

It is well known that unconditionally secure bit commitment cannot be implemented from (noiseless) classical communication only — and the same is true even for (noiseless) quantum communication [2], [3]. Therefore, it is interesting to study unconditionally secure reductions of bit commitment to weaker primitives, e.g., to physical assumptions. It is known that bit commitment can be realized from communication over noisy channels [4], [5] or from pieces of correlated randomness [6], [7], [8].

Measurements on entangled quantum states can produce socalled *non-local correlations*, i.e., correlations that cannot be simulated with shared classical information. These correlations can be modeled as *bipartite input-output systems* that are characterized by a conditional distribution $P_{XY|UV}$, where Uand V stand for the inputs and X and Y for the outputs of the system, respectively. We only consider correlations that are *non-signaling*, i.e., which do not allow for message transmission from one side to the other. When using a nonsignaling system, a party receives its output immediately after giving its input, independently of whether the other party has given its input already. This prevents the parties from signaling by delaying their inputs. An example of such a system is the *non-local box* (*NL box* for short) proposed by Popescu and Rohrlich [9], where the inputs and outputs are binary, each output is a uniform bit, independent of the pair of inputs, but $X \oplus Y = U \land V$ always holds.

As bit commitment cannot be implemented from quantum communication, the question has been studied whether bit commitment can be realized when the two parties share trusted non-local correlations as a resource. It has been proven in [10] that unconditionally secure bit commitment can be implemented from NL boxes. This result shows that unconditionally secure computation can be realized from non-signaling systems in principle. In particular, it implies that the problems that arise from the fact that any non-signaling system allows the parties to delay their inputs can be circumvented. However, the correlations of an NL box cannot be realized by measurements on a quantum state [11]. In the present article we show that any non-signaling system providing binary outputs can either be simulated securely with shared randomness, or allows for information-theoretically secure bit commitment (Theorem 3); our condition is thus tight. This implies in particular that even local non-signaling correlations can be used to implement unconditionally secure bit commitment if they are provided as a trusted resource to the two parties.

II. PRELIMINARIES

A. Bit Commitment

A bit commitment scheme is a pair of protocols Commit and Open executed by two parties Alice and Bob. First, Alice and Bob execute Commit where Alice has a bit b as input. Bob either accepts or rejects the execution of Commit. Later, they execute Open where Bob has output (accept, b') or reject. The two protocols must have the following (ideal) properties:

- Correctness: If both parties follow the protocol, then Bob always accepts with b' = b.
- Hiding: If Alice is honest, then committing to *b* does not reveal any information about *b* to Bob.¹
- Binding: If Bob is honest and accepts after the execution of Commit, then there exists only one value b' (which is

¹Bob's views for b = 0 and b = 1 are indistinguishable.

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equal to *b*, if Alice is honest) that Bob accepts as output after the execution of Open.

In the following we call a bit commitment scheme secure, if it fulfills the above ideal requirements except with an error that can be made negligible (as a function of some security parameter n).

B. Notation

Let $W: \mathcal{X} \to \mathcal{Y}$ be a *stochastic matrix* with rows indexed by elements of \mathcal{X} and columns indexed by elements of \mathcal{Y} . We denote the entries of W by $W(y|x) = W_x(y)$ and the row vector indexed by x by W_x . $W_x(\cdot)$ defines a probability distribution on \mathcal{Y} for every $x \in \mathcal{X}$, i.e., for all x it holds that $W(y|x) \ge 0$ for all y and $\sum_{y} W(y|x) = 1$. We denote by $\operatorname{conv}(W)$ the convex hull of the set $\{W_x | x \in \mathcal{X}\}$, i.e., the convex hull of the row vectors of W. We call W_{x_0} an extreme point of this set if the convex hull of the set $(\{W_x|x \in \mathcal{X}\} \setminus \{W_{x_0}\})$ is strictly smaller. We denote the set of extreme points by extr(conv(W)). We call W_{z_0} nonextreme if it is not an extreme point of conv(W). We denote by $x^n = (x_1, \ldots, x_n)$ a sequence of elements in \mathcal{X} or a vector in \mathcal{X}^n . If $I := \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$ then x^I denotes the sub-sequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ of x^n . We denote by $h(\cdot)$ the binary entropy function.

We call a function $f(n) \ge 0$ negligible if for any c > 0, there exists n_c such that for all $n > n_c$, $f(n) < 1/n^c$. We call f(n) overwhelming if 1 - f(n) is negligible.

C. Non-Signaling Boxes

A non-signaling box is defined by a stochastic matrix

$$W: \mathcal{U} \times \mathcal{V} \to \mathcal{X} \times \mathcal{Y}$$

as follows: Alice gives an input $u \in \mathcal{U}$ and Bob gives an input $v \in \mathcal{V}$. Alice gets output $x \in \mathcal{X}$ and Bob $y \in \mathcal{Y}$ with probability W(xy|uv). Furthermore, the following non-signaling conditions must hold

$$\sum_{y} W(xy|uv) = \sum_{y} W(xy|uv') \quad \forall u, v, v', x,$$
$$\sum_{x} W(xy|uv) = \sum_{x} W(xy|u'v) \quad \forall u, u', v, y,$$

i.e., the distribution of Alice's output is independent of Bob's input (and vice-versa). A party receives its output immediately after giving its input, independently of whether the other party has given its input already. Note that this is possible, since the box is non-signaling. Furthermore, after a box is used once, it is destroyed. The set of non-signaling boxes can be divided into two types: local and non-local. A box is local if and only if it can be simulated by non-communicating parties with only shared randomness as a resource. This means that there exist probabilities p_i and stochastic matrices V_A^i, V_B^i such that

$$W(xy|uv) = \sum_{i=1}^{n} p_i V_A^i(x|u) V_B^i(y|v) \quad \forall u, v, x, y.$$
(1)

A box is called *independent* if there exist stochastic matrices V_A, V_B such that

$$W(xy|uv) = V_A(x|u)V_B(y|v) \quad \forall u, v, x, y,$$

i.e., such a box can be simulated without any shared resources at all. In the following we only consider boxes with binary outputs, i.e., $\mathcal{X} = \mathcal{Y} = \{0, 1\}$. We define

$$\begin{split} W^A(x|u) &:= \sum_y W(xy|uv) \quad \forall u, v, x, \\ W^B(y|v) &:= \sum_x W(xy|uv) \quad \forall u, v, y. \end{split}$$

We call a box with binary outputs *perfectly correlated* for an input pair $(u, v) \in \mathcal{U} \times \mathcal{V}$ if

$$W(01|uv) = W(10|uv) = 0$$

and perfectly anti-correlated if

$$W(00|uv) = W(11|uv) = 0.$$

An input u for Alice is called *redundant* if there exists $\tilde{u} \neq u$ such that

$$W(xy|uv) = W(xy|\tilde{u}v) \ \forall x, y, v.$$

D. Chernoff/Hoeffding Bounds

We will use the following bounds attributed to Chernoff [12] and Hoeffding [13].

Lemma 1. Let $X_1, X_2, ..., X_n$ be independent random variables with $\Pr[X_i = 1] = p_i$ and $\Pr[X_i = 0] = 1 - p_i$. Let $X = \sum_{i=1}^n X_i$ and $\mu = E[X]$. Then for any $0 < \delta < 1$ it holds that

$$\Pr[X > (1+\delta)\mu] \le \exp(-\delta^2 \mu/3) ,$$

$$\Pr[X < (1-\delta)\mu] \le \exp(-\delta^2 \mu/2) .$$

Lemma 2. Let $X_1, X_2, ..., X_n$ be independent random variables with $\Pr[X_i = 1] = p_i$ and $\Pr[X_i = 0] = 1 - p_i$. Let $X = \sum_{i=1}^n X_i$ and $\mu = E[X]$. Then for any $0 < \delta < 1$ it holds that

$$\Pr[X > \mu + \delta] \le \exp(-2\delta^2/n) ,$$

$$\Pr[X < \mu - \delta] \le \exp(-2\delta^2/n) .$$

E. Information Theory

We will use the smoothed versions of the min-entropy [14]. For an event \mathcal{E} , let $P_{X\mathcal{E}|Y=y}(x)$ be the probability that X = xand the event \mathcal{E} occurs, conditioned on Y = y. We define

$$\mathrm{H}^{\epsilon}_{\infty}(X|Y) := \max_{\mathcal{E}: \mathrm{Pr}(\mathcal{E}) \ge 1-\epsilon} \min_{y} \min_{x} (-\log P_{X\mathcal{E}|Y=y}(x)).$$

We will make use of the following lemma from [14].

Lemma 3. Let P_{XYZ} be a probability distribution. For any $\epsilon, \epsilon' > 0$,

$$\mathrm{H}_{\infty}^{\epsilon+\epsilon'}(X|YZ) \geq \mathrm{H}_{\infty}^{\epsilon}(XY|Z) - \log(|\mathcal{Y}|) - \log(1/\epsilon') \; .$$

The following lemma from [15] gives a lower bound for the smooth entropy of n-fold product distributions:

Lemma 4. Let $P_{X^nY^n} := P_{X_1Y_1} \dots P_{X_nY_n}$ be a probability distribution over $\mathcal{X}^n \times \mathcal{Y}^n$ and let $\epsilon > 0$. Then

$$\mathrm{H}_{\infty}^{\epsilon}(X^{n}|Y^{n}) \geq \mathrm{H}(X^{n}|Y^{n}) - 4\sqrt{n\log(1/\epsilon)}\log(|\mathcal{X}|) \;.$$

F. Randomness Extraction

A function $f : \mathcal{X} \times \mathcal{S} \to \mathcal{Y}$ is called a *two-universal hash* function [16] if for all $x_0 \neq x_1$ we have

$$\Pr[f(x_0, S) = f(x_1, S)] \le \frac{1}{|\mathcal{Y}|}$$

if S is uniform over S. The following lemma from [17], [18] shows that two-universal hash functions are strong extractors, i.e., the concatenation of the seed and the output of the extractor is close to uniform.

Lemma 5 (Leftover hash lemma). Let $f : \mathcal{X} \times S \to \mathcal{Y}$ be a two-universal hash function with m > 0. Let X be a random variable over \mathcal{X} and let $\epsilon > 0$. If

$$H_{\infty}(X) - 2\log(1/\epsilon) \ge m,$$

then $\frac{1}{2}||(f(S,X),S) - (U,S)||_1 \le \epsilon$ for S and U independent and uniform over S and Y.

G. Typical Sequences

In this section we will state and prove some basic results on typical sequences. More details on this topic can be found in the book by Csiszár and Körner [19].

Definition 1. Let P be a probability distribution on X and $\epsilon > 0$. Then the set of ϵ -typical sequences is defined as:

$$\mathcal{T}_{P,\epsilon}^{n} := \{ x^{n} \in \mathcal{X}^{n} : \forall x \in \mathcal{X} | N(x|x^{n}) - P(x)n| \le \epsilon n \\ and P(x) = 0 \Rightarrow N(x|x^{n}) = 0 \},$$

where $N(x|x^n)$ denotes the number of letters x in x^n .

Definition 2. For a stochastic matrix $W : \mathcal{X} \to \mathcal{Z}$ we define the set of W-typical sequences under the condition $x^n \in \mathcal{X}^n$ with constant ϵ as

$$\mathcal{T}^n_{W,\epsilon}(x^n) = \{ z^n : \forall x, z | N(xz|x^n z^n) - W_x(z)N(x|x^n) | \le \epsilon n \\ and \ W_x(z) = 0 \Rightarrow N(xz|x^n z^n) = 0 \}.$$

The following two well-known lemmas follow directly from Lemma 1.

Lemma 6. $P^n(\mathcal{T}_{P,\epsilon}^n) \ge 1 - 2|\mathcal{X}|\exp(-n\epsilon^2/3)$ Lemma 7. $W_{x^n}^n(\mathcal{T}_{W,\epsilon}^n(x^n)) \ge 1 - 2|\mathcal{X}||\mathcal{Z}|\exp(-n\epsilon^2/3)$

Using the results above we will prove a lemma that we will use in the security proofs in this paper. The lemma is similar to Lemma 14 in [5]. Let $W : \mathcal{X} \to \mathcal{Z}$ be a (discrete memoryless) channel, let $a \in \mathcal{X}$ be an input such that the output distribution of a is not a convex combination of the other output distributions and let $x^n, \tilde{x}^n \in \mathcal{X}^n$ be sequences such that $|\{k : x_k \neq a \text{ and } \tilde{x}_k = a\}| \geq \kappa n$. Then the lemma states that the output of the channel, given x^n as input, will not be W-typical conditioned on \tilde{x}^n with overwhelming probability if $\exp(-\kappa^2 n)$ is negligible.

Lemma 8. Let $W : \mathcal{X} \to \mathcal{Z}$ be a stochastic matrix and $a \in \mathcal{X}$ such that for all probability distributions P over \mathcal{X} such that P(a) = 0 and

$$\left\| W_a - \sum_x P(x) W_x \right\|_1 \ge \delta$$

Let $x^n, \tilde{x}^n \in \mathcal{X}^n$ with $d_H(x^{I_a}, \tilde{x}^{I_a}) \ge \kappa n$ where $I_a := \{k : \tilde{x}_k = a\}$. If $n_a := |I_a| \ge \lambda n$, then

$$W_{x^n}^n(\mathcal{T}_{W,\epsilon}^n(\tilde{x}^n)) \le 2\exp(-n\epsilon^2/3)$$

where $\epsilon := \frac{1}{2|\mathcal{Z}|} \lambda \delta \kappa$.

Proof: Let $D := \{k \in I_a : x_k \neq \tilde{x}_k\}$. Then it follows that

$$\left\|\frac{1}{n_a}\sum_{k\in I_a} W_{x_k} - W_a\right\|_1 = \frac{|D|}{n_a} \left\|W_a - \frac{1}{|D|}\sum_{k\in D} W_{x_k}\right\|_1$$
$$\geq \frac{|D|}{n_a}\delta \geq \kappa\delta . \tag{2}$$

This implies that there exists $b \in \mathcal{Z}$ such that

$$\left|\frac{1}{n_a}\sum_{k\in I_a} W_{x_k}(b) - W_a(b)\right| \ge \frac{1}{|\mathcal{Z}|}\kappa\delta.$$

Let $w^n \in \mathcal{T}^n_{W,\epsilon}(\tilde{x}^n)$. Then it holds that

$$\begin{split} \left| N(b|w^{I_a}) - \sum_{k \in I_a} W_{x_k}(b) \right| &= \left| N(ab|\tilde{x}^n w^n) - \sum_{k \in I_a} W_{x_k}(b) \right| \\ &\geq \left| \sum_{k \in I_a} W_{z_k}(b) - n_a W_a(b) \right| - \left| n_a W_a(b) - N(ab|\tilde{x}^n w^n) \right| \\ &\geq \frac{1}{|\mathcal{Z}|} \kappa \delta n_a - \epsilon n \\ &\geq \frac{1}{2|\mathcal{Z}|} \kappa \delta \lambda n_a \;. \end{split}$$

We define independent binary random variables X_k , $k \in I_a$, with distributions $P_{X_k}(1) := W_{x_k}(b)$. Let $X = \sum_{k \in I_a} X_i$ and $\mu := E[X] = \sum_{k \in I_a} W_{x_k}(b)$. Let $t := \frac{1}{2|Z|} \kappa \delta \lambda n_a \mu^{-1}$ (assuming $\mu \neq 0$). Using the Chernoff bound it follows that

$$W_{x^n}^n(\mathcal{T}_{W,\epsilon}^n(\tilde{x}^n)) \leq \Pr\left[|X-\mu| \geq \frac{1}{2|\mathcal{Z}|} \kappa \delta \lambda n_a\right]$$

= $\Pr[|X-\mu| \geq t\mu]$
 $\leq 2 \exp(-\epsilon^2 n/3).$

III. IMPOSSIBILITY

The following theorem proves that a certain class of nonsignaling boxes can be securely implemented from shared randomness alone and does, therefore, not allow for unconditinally secure bit commitment (otherwise bit commitment could be implemented form noiseless communication only, which is well known to be impossible).

Theorem 1. Let a local non-signaling box with binary output be defined by $W : U \times V \rightarrow \{0, 1\}^2$ such that

$$W(xy|uv) = pV_A^0(x|u)V_B^0(y|v) + (1-p)V_A^1(x|u)V_B^1(y|v)$$

and there exists $u_0 \in \mathcal{U}, v_0 \in \mathcal{V}$ and $b_0, b_1 \in \{0, 1\}$ with:

$$\begin{split} V^0_A(0|u_0) &= V^1_A(1|u_0) = b_0 \\ V^0_A(1|u_0) &= V^1_A(0|u_0) = 1 - b_0 \\ V^0_B(0|v_0) &= V^1_B(1|v_0) = b_1 \\ V^1_B(1|v_0) &= V^0_B(0|v_0) = 1 - b_1 \;, \end{split}$$

then there is no reduction of information-theoretically secure bit commitment to the box W (with noiseless communication only).

Proof: We prove the statement by showing that one can securely implement such a box from noiseless communication and shared randomness alone. The implementation directly follows the definition of the box: Let λ be the shared random bit. Alice on input u outputs 0 with probability $V_A^{\lambda}(0|u)$ and 1 with probability $V_A^{\lambda}(1|u) = 1 - V_A^{\lambda}(0|u)$. Bob on input v outputs $b \in \{0, 1\}$ with probability $V_B^{\lambda}(b|v)$. This perfectly implements the behavior of the box. Furthermore, this implementation is secure, since Alice and Bob can get the same information (i.e. the shared randomness λ) if they only have black-box access to W, if they always input u_0 and v_0 , respectively.

IV. TWO PROTOCOLS

We will now give two slightly different protocols, which work for two different kinds of non-signaling boxes.

A. Protocol I

Informally, the first protocol works as follows: in the Commit protocol an honest Alice gives a fixed input to all her boxes, while Bob chooses his inputs randomly. Alice applies privacy amplification to the outputs of the boxes and uses the resulting key K to hide the bit B she wants to commit to. Alice then sends $K \oplus B$ and the randomness used for privacy amplification to Bob. In the Open protocol Alice sends her outputs from the boxes. Alice's input is chosen such that there is a statistical test that allows Bob to detect if Alice has changed more than $O(\sqrt{n})$ output values while Bob has only limited information about the output of the boxes before the opening phase. A dishonest Alice might still be able to change $O(\sqrt{n})$ output values. To ensure that this is not possible, we use a linear code and let Alice send parity check bits of the output to Bob in the Commit protocol. If the minimal distance of the code is large enough, no two strings with the same parity check bits lie in a hamming sphere with radius proportional to \sqrt{n} .

Let Alice and Bob share n identical non-signaling boxes given by $W : \mathcal{U} \times \mathcal{V} \to \{0, 1\}^2$. In our protocol, we will require Bob to choose his input uniformly from \mathcal{V} . For an honest Bob and a potentially malicious Alice, we can define a stochastic matrix $\hat{W} : \{0, 1\} \times \mathcal{U} \to \{0, 1\} \times \mathcal{V}$ describing the probability of Bob's input and output values v and y, conditioned on Alice's input u and output x as

$$\hat{W}(yv|xu) := \frac{1}{|\mathcal{V}|} \frac{W(xy|uv)}{W^A(x|u)} ,$$

if $W^A(x|u) \neq 0$, and undefined otherwise. Furthermore, we will require an honest Alice to always input a fixed value u_a to the box. For an honest Alice, and a potentially malicious Bob that chooses his input $v \in \{0, 1\}$ freely, we can define random variables X_v, Y_v depending on Bob's input that describe the output of Alice and Bob, respectively, i.e. with a joint distribution

$$P_{X_v Y_v}(x, y) := W(xy|u_a v)$$

The protocol below is secure if there exists a value $a = (x_a, u_a)$ such that the following condition is fulfilled:

Condition 1. (1) There exists $\delta > 0$ such that for all probability distributions P over $\{0,1\}^2$ with P(a) = 0 it holds that

$$\left\|\hat{W}_a - \sum_x P(x)\hat{W}_x\right\|_1 \ge \delta \ .$$

(2) There exists $\gamma > 0$ such that for all $v \in \mathcal{V}$ it holds that

$$\mathrm{H}(X_v|Y_v) \ge \gamma,$$

i.e., the Shannon entropy of Alice's output given Bob's output is non-zero for all possible inputs of Bob.

We label the inputs of Alice as $\{0, \ldots, |\mathcal{U}| - 1\}$. Furthermore, we define the distribution of Alice's output x if her input is u_a as $P(x) := W^A(x|u_a)$ for all $x \in \{0, 1\}$. Let $\lambda := \frac{1}{2}\min\{P(x), x \in \{0, 1\}\}$. Let k be the security parameter. Let $\epsilon := \frac{1}{4}\lambda\delta k/n$. Let d > 2k and let H be the parity check matrix of a linear [n, Rn, d]-code with $R > (1 - \gamma)$. Since we do not have to decode, this could be a random linear code chosen by Bob. Let $l := \gamma n - n(1 - R) - 4\sqrt{nk} - 3k$. We choose $k := n^{2/3}$, which implies that $k, \sqrt{nk} \in O(n^{5/6})$ and $k, k^2/n, n\epsilon^2 \in \Omega(n^{1/3})$. It follows that $l \in (\gamma + R - 1)n - O(n^{5/6})$. If n is big enough, we have l > 0. Let ext : $\{0, 1\}^* \times \{0, 1\}^n \to \{0, 1\}^l$ be a two-universal hash function. We define $syn(x^n) := H^Tx^n$.

$\operatorname{Commit}(b^l)$:

- Bob chooses $v^n \in_R \{0,1\}^n$
- Alice and Bob input uⁿ_a and vⁿ component-wise to the boxes. Alice gets xⁿ ∈ {0,1}ⁿ and Bob yⁿ ∈ {0,1}ⁿ.
- Alice chooses r ∈_R {0,1}* and sends (syn(xⁿ), r, b^l ⊕ ext(r, xⁿ)) to Bob.

Open():

- Alice sends Bob x^n and b^l .
- Bob checks:

$$- syn(x^n)$$
 is correct

- $b \oplus \operatorname{ext}(r, x^n)$ is correct
- $((y_1, v_1), ..., (y_n, v_n)) \in \mathcal{T}^n_{\hat{W}, \epsilon}((x_1, u_a), ..., (x_n, u_a))$ $- x^n \in \mathcal{T}^n_{P, \epsilon}$
- If all the checks pass successfully, Bob accepts and outputs b^l , otherwise he rejects.

B. Security

Let $u^n := (u_1, \ldots, u_n)$ be Alice's inputs to the boxes, let $x^n := (x_1, \ldots, x_n)$ be her outputs from the boxes and let $\tilde{x}^n := (\tilde{x}_1, \ldots, \tilde{x}_n)$ be the values Alice sends to Bob in the opening phase. We define $z^n :=$ $((x_1, u_1), \ldots, (x_n, u_n))$ and $\tilde{z}^n := ((\tilde{x}_1, u_a), \ldots, (\tilde{x}_n, u_a))$. Let $r^n := ((y_1, v_1), \ldots, (y_n, v_n))$ be Bob's inputs and outputs.

Lemma 9. The protocols Commit and Open satisfy the correctness condition.

Proof: Bob always accepts Commit. If Alice follows the protocol, then $syn(x^n)$ and $b^l \oplus ext(r_1, u^n)$ are correct. From

Lemma 7 it follows that

$$\Pr[r^n \in \mathcal{T}^n_{\hat{W},\epsilon}(z^n)] = \hat{W}_{z^n}(\mathcal{T}_{\hat{W},\epsilon}(z^n))$$

$$\geq 1 - 16|\mathcal{V}|\exp(-n\epsilon^2/3) ,$$

and from Lemma 6 it follows that

$$\Pr[x^n \in \mathcal{T}_{P,\epsilon}^n] = P_{x^n}^n(\mathcal{T}_{P,\epsilon})$$

$$\geq 1 - 4\exp(-n\epsilon^2/3)$$

Thus, Bob accepts Open with overwhelming probability and outputs b^l , the value Alice was committed to.

Lemma 10. The protocol Commit satisfies the privacy condition with an error negligible in n.

Proof: Let us assume that Alice is honest. Alice inputs u_a into the boxes as required by the protocol, while Bob can choose its input $v^n = (v_1, \ldots, v_n)$ freely. We then define the random variables $X^n = X_{v_1} \times \ldots \times X_{v_n}$ and $Y^n = Y_{v_1} \times \ldots \times Y_{v_n}$. Let $\epsilon_1 := 2^{-k}$. According to Lemma 4 it holds that

$$\mathrm{H}_{\mathrm{min}}^{\epsilon_1}(X^n|Y^n) \ge \mathrm{H}(X^n|Y^n) - 4\sqrt{nk}.$$

Using Lemma 3 with get that

$$\begin{split} \mathrm{H}_{\min}^{2\epsilon_{1}}(X^{n}|syn(X^{n})V^{n}) &\geq \mathrm{H}_{\min}^{\epsilon_{1}}(X^{n}|Y^{n}) - n(1-R) \\ &\quad -\log(1/\epsilon_{1}) \\ &\geq \gamma n - n(1-R) - 4\sqrt{nk} - k \\ &\quad = l + 2k \;. \end{split}$$

According to Lemma 5 Bob has no information about $ext(r_1, x^n)$ except with probability $2\epsilon_1 + \epsilon_1$.

Lemma 11. If $d_H(x^n, \tilde{x}^n) \ge k$, then the probability that Bob accepts \tilde{x}^n is negligible in n.

Proof: From $d_H(x^n, \tilde{x}^n) \ge k$ follows $d_H(z^n, \tilde{z}^n) \ge k$. Let $n_a := N(u_a|u^n)$, $I_a := \{k : \tilde{z}_k = (x_a, u_a)\}$ and $p := W^A(x_a|u_a)$. For all $w^n \in \mathcal{T}^n_{P,\epsilon}$, we have

$$|N(x_a|w^n) - np| \le \epsilon n = \frac{1}{4}\lambda \delta k/n \cdot n \le \frac{1}{8}kp$$

since $\lambda \leq p/2$ and $\delta \leq 1$. We distinguish two cases:

(1) $n_a \leq (n - k/2)$: The expectation of $N((x_a, u_a)|z^n)$ is smaller than or equal to $(n - \frac{k}{2})p$. Since $k^2/n \in \Omega(n^{1/3})$, it follows from Lemma 1 that with overwhelming probability

$$N((x_a, u_a)|z^n) \le \left(n - \frac{k}{2}\right)p + \frac{k}{8}p$$
$$= \left(n - \frac{3}{8}k\right)p.$$

But since Bob only accepts if $\tilde{x}^n \in \mathcal{T}_{P,\epsilon}^n$, we have

$$d_H(z^{I_a}, \tilde{z}^{I_a}) \ge \left(n - \frac{1}{4}k\right)p - \left(n - \frac{3}{8}k\right)p$$
$$= \frac{1}{4}kp$$

and the claim follows from Lemma 8.

(2) $n_a > (n - k/2)$: Then the expectation of $N((1 - x_a, u_a)|z^n)$ is greater than or equal to $(n - \frac{k}{2})(1 - p)$. As

 $k^2/n \in \Omega(n^{1/3})$ Lemma 1 implies that with overwhelming probability

$$N((1 - x_a, u_a)|z^n) \ge \left(n - \frac{k}{2}\right)(1 - p) - \frac{k}{8}(1 - p)$$
$$= n(1 - p) - \frac{5}{8}k(1 - p).$$

But since Bob only accepts if $\tilde{x}^n \in \mathcal{T}_{P,\epsilon}^n$, we have

$$d_H(z^{I_a}, \tilde{z}^{I_a}) \ge \left(n - \frac{5}{8}k\right)(1-p) - \left(n - \frac{1}{4}k\right)(1-p) \\ = \frac{1}{4}k(1-p)$$

and the claim follows from Lemma 8.

Lemma 12. The protocol satisfies the binding condition with an error negligible in n.

Proof: Any two strings $s^n \neq \tilde{s}^n$ with $syn(s^n) = syn(\tilde{s}^n)$ have distance at least d. So at least one of the two strings has distance at least k from Alice's output x^n . The probability that Bob accepts this string in the opening phase is negligible according to lemma 11.

C. Protocol II

Protocol I is not hiding if for every fixed input of Alice a dishonest Bob can choose an input such that he has perfect information about Alice's output. This is the case for example with the above mentioned NL box. But, as shown in [10], this box allows for bit commitment. Therefore, we present a second protocol that allows to securely implement bit commitment for such boxes. The protocol, which is similar to a protocol proposed without a security proof in [20] already, works as follows: Alice gives random inputs to all her boxes. Then she applies privacy amplification to the string of inputs and uses the resulting key to hide the bit she is committed to. In the opening phase Alice sends all her inputs/outputs to Bob. Bob performs statistical tests on the input/output of Alice that allow him to detect if Alice has changed more than \sqrt{n} values. We use again parity check bits of a linear code to make sure that a dishonest Alice cannot change \sqrt{n} values except with negligible probability.

Alice and Bob share n identical non-signaling boxes given by $W: \mathcal{U} \times \mathcal{V} \to \{0, 1\}^2$. We define the corresponding matrix \hat{W} as in Section IV-A. In the following we always assume that $W^A(x|u) \neq 0$ for all $x \in \{0, 1\}, u \in \mathcal{U}$. For the following protocol to be secure we require W to fulfill the following condition:

Condition 2. There exist $u_0, u_1 \in \mathcal{U}, u_0 \neq u_1$, such that the set $D := \{\hat{W}_{0u_0}, \hat{W}_{1u_1}, \hat{W}_{0u_0}, \hat{W}_{1u_1}\}$ contains at most one non-extreme point of $\operatorname{conv}(\hat{W})$, i.e., there is $c_0 \in \{0u_0, 1u_1, 0u_0, 1u_1\}$ such that for all $c \in \{0u_0, 1u_1, 0u_0, 1u_1\} \setminus \{c_0\}$ it holds that for all probability distributions P with P(c) = 0

$$\left\|\hat{W}_c - \sum_z P(z)\hat{W}_z\right\|_1 \ge \delta$$

We label the inputs of Alice as $\{0, \ldots, |\mathcal{U}| - 1\}$ and assume that $u_0 = 0$ and $u_1 = 1$. In the protocol, we will require

Alice to choose her input uniformly from $\{0, 1\}$, and Bob to choose his input uniformly from \mathcal{V} . If both are honest, the joint distribution of the inputs and outputs of Alice and Bob is

$$P(x, y, u, v) := \begin{cases} \frac{1}{2|\mathcal{V}|} W(xy|uv), & \text{if } u \in \{0, 1\}\\ 0, & \text{else.} \end{cases}$$

If Alice is honest, the joint distribution of her input and output is

$$Q(x, u) := \begin{cases} \frac{1}{2} W^A(x|u), & \text{if } u \in \{0, 1\} \\ 0, & \text{else.} \end{cases}$$

Let $\lambda := \frac{1}{4}\min\{Q(x,u), (x,u) \in \{0,1\}^2\}$. Let $p_0 := \min\{W^A(x|u), (x,u) \in \{0,1\}^2\}$. Note that we assumed $p_0 > 0$ and that obviously we also have $p_0 \leq \frac{1}{2}$. Let k_1 be the security parameter, $k_2 := k_1(4p_0+1)/2p_0^2, \epsilon := \frac{1}{4}\lambda\delta k_1/n, d \geq k_1 + 2k_2 + 1, l > 0$ and let H be the parity check matrix of a [n, Rn, d]-linear code with $Rn \geq n/2 + \frac{3}{2}k_1 + l/2$. We choose $k_1 := n^{2/3}$ and $l := n - 2n(1-R) - 3k_1$. This implies $k_1, k_1^2/n, n\epsilon^2 \in \Omega(n^{1/3})$ and $l \in (2R-1)n - O(n^{2/3})$. If n is big enough, then l > 0. Let ext : $\{0,1\}^* \times \{0,1\}^n \to \{0,1\}^l$ be a two-universal hash function.

 $\operatorname{Commit}(b^l)$:

- Alice chooses $u^n \in_R \{0,1\}^n$, Bob chooses $v^n \in_R \mathcal{V}$.
- Alice and Bob input u^n and v^n component-wise to the boxes. Alice gets $x^n \in \{0,1\}^n$ and Bob $y^n \in \{0,1\}^n$.
- Alice chooses $r_2 \in_R \{0,1\}^*$ and sends $(syn(u^n), syn(x^n), r_2, b^l \oplus ext(r_2, x^n))$ to Bob.

Open():

• Alice sends Bob u^n, x^n and b^l .

Bob checks:

- $syn(u^n)$ and $syn(x^n)$ are correct
- $b^l \oplus \operatorname{ext}(r_2, u^n)$ is correct
- $((y_1, v_1), ..., (y_n, v_n)) \in \mathcal{T}^n_{\hat{W}, \epsilon}((x_1, u_1), ..., (x_n, u_n))$ $- ((x_1, u_1), ..., (x_n, u_n)) \in \mathcal{T}^n_{Q, \epsilon}$
- If all the checks pass successfully, Bob accepts and outputs b^l , otherwise he rejects.

D. Security

Let $z^n := ((x_1, u_1), \ldots, (x_n, u_n))$ be Alice's input and output, $\tilde{z}^n := ((\tilde{x}_1, \tilde{u}_1), \ldots, (\tilde{x}_n, \tilde{u}_n))$ the values Alice sends to Bob in the opening phase and $r^n := ((y_1, v_1), \ldots, (y_n, v_n))$ Bob's inputs and outputs. For all $c \in (\{0, 1\} \times \mathcal{U})$ we define the sets $I_c := \{i : \tilde{z}_i = c\}$.

Lemma 13. The protocols Commit and Open satisfy the correctness condition.

Proof: Bob always accepts Commit. If Alice follows the protocol, then $syn(u^n)$, $syn(x^n)$ and $b^l \oplus ext(r_2, u^n)$ are correct. From Lemma 7 it follows that

$$\Pr[r^n \in \mathcal{T}^n_{\hat{W},\epsilon}(z^n))] = \hat{W}_{z^n}(\mathcal{T}_{\hat{W},\epsilon}(z^n))$$

$$\geq 1 - 8|\mathcal{U}||\mathcal{V}|\exp(-n\epsilon^2/2)$$

and from Lemma 6 it follows that

$$\Pr[z^n \in \mathcal{T}_{Q,\epsilon}^n] = Q^n(\mathcal{T}_{Q,\epsilon}(z^n))$$

$$\geq 1 - 4|\mathcal{U}|\exp(-n\epsilon^2/2).$$

Thus, Bob accepts Open with overwhelming probability and outputs b^l , the value Alice was committed to.

Lemma 14. The protocol Commit satisfies the privacy condition with an error negligible in n.

Proof: Let us assume that Alice is honest. Since the box is non-signaling, Bob's values Y^n and V^n are independent of U^n . Since Alice chooses U^n uniformly from $\{0,1\}^n$, we have

$$\operatorname{H}_{\infty}(U^n) = n$$
 .

All the information Bob gets about U^n is $syn(U^n)$ and $syn(X^n)$. Let $\epsilon_1 := 2^{-k_1}$. Using Lemma 3 we get

$$\begin{aligned} \mathrm{H}_{\infty}^{\epsilon_1}(U^n|syn(U^n)syn(X^n)) &\geq n - 2n(1-R) - k_1 \\ &\geq l + 2k_1 . \end{aligned}$$

If follows from Lemma 5 that extracting l bits makes the key uniform with an error of at most $2\epsilon_1 = 2 \cdot 2^{-k_1}$. The statement follows.

The proof of the binding condition is slightly more involved. Because our boxes are non-signaling, Alice has the possibility of delaying her input to the box until the opening phase. Hence, a general strategy for her is to give input to some of the boxes in the commit phase, and to delay the input to some of the boxes until the opening phase. And she may send incorrect values about her input/output to/from the boxes to Bob in the opening phase. Note that we can ignore the case where she does not give any input to some boxes, as she might as well just give input but ignore the output.

Lemma 15. If $d_H(z^n, \tilde{z}^n) \ge k_1$, then the probability that Bob accepts \tilde{z}^n is negligible.

Proof: For all $w^n \in \mathcal{T}^n_{Q,\epsilon}$ it holds that

$$N(xu|w^n) - nQ(x,u)| \le \epsilon n = \frac{1}{4}\lambda \delta k_1/n \cdot n \le \frac{1}{64}k_1 ,$$

since $\lambda \leq \min_{x,u} Q(x,u)/4 \leq 1/16$ and $\delta \leq 1$.

We distinguish the following two cases:

(1) There exists $u' \in \{0, 1\}$ such that $N(u'|u^n) \le n/2 - k_1/8$: For all $x \in \{0, 1\}$ the expectation of $N(xu'|z^n)$ is equal to $(n/2 - \frac{k_1}{8})W^A(x|u')$. Since $k_1^2/n \in \Omega(n^{1/3})$ it follows from Lemma 1 that with overwhelming probability

$$N(xu'|z^n) \le \left(\frac{n}{2} - \frac{1}{8}k_1\right) W^A(x|u') + \frac{1}{16}k_1 W^A(x|u')$$
$$= \left(\frac{n}{2} - \frac{1}{16}k_1\right) W^A(x|u') .$$

But since Bob only accepts if $\tilde{z}^n \in \mathcal{T}^n_{Q,\epsilon}$, we have $d_H(z^{I_{0u'}}, \tilde{z}^{I_{0u'}}) \geq \frac{1}{32}k_1$ and $d_H(z^{I_{1u'}}, \tilde{z}^{I_{1u'}}) \geq \frac{1}{32}k_1$, and the claim follows from Lemma 8.

(2) For all $u \in \{0, 1\}$ we have $|n/2 - N(u|u^n)| \le k_1/8$: Since $\epsilon^2 n \in \Omega(n^{1/3})$ it follows from Lemma 7 that with overwhelming probability we have $z^n \in \mathcal{T}^n_{W^A,\epsilon}(u^n)$. Assume $z^n \in \mathcal{T}^n_{W^A,\epsilon}(u^n)$. There exists a value $(x', u') \in \{0, 1\}^2$ such that $d_H(z^{I_{x'u'}}, \tilde{z}^{I_{x'u'}}) \geq \frac{1}{4}k_1$. Therefore

$$N(x'u'|z^n) + d_H(z^{I_{x'u'}}, \tilde{z}^{I_{x'u'}})$$

$$\geq nW^A(x'|u')/2 - k_1W^A(x'|u')/8 - \epsilon n + k_1/4$$

$$\geq nW^A(x'|u')/2 + \frac{7}{64}k_1.$$

If there exists $(x'', u'') \neq (x', u') \in \{0, 1\}^2$ such that $d_H(z^{I_{x''u''}}, \tilde{z}^{I_{x''u''}}) \geq \frac{1}{32}k_1$, then the claim follows from Lemma 8. Otherwise $\tilde{z}^n \notin \mathcal{T}^n_{Q,\epsilon}$.

Next, we will prove a technical lemma:

Lemma 16. For any n it holds that, if $k \leq np$,

$$\sum_{i=0}^{k} \binom{n}{i} p^{i} (1-p)^{n-i} \le 2^{-2np^{2}+4pk}.$$

Proof: Let X_1, X_2, \ldots, X_n be random variables with $Pr[X_i = 1] = p$ and $Pr[X_i = 0] = (1 - p)$. Let $X = \sum_{i=1}^n X_i$. Then using Lemma 2 and setting t := np - k

$$\sum_{i=0}^{k} \binom{n}{i} p^{i} (1-p)^{n-i} = \Pr[X \le k] \le \exp(-2t^{2}/n)$$
$$\le 2^{-2(np-k)^{2}/n}$$
$$\le 2^{-2np^{2}+4pk}.$$

Lemma 17. If Alice does not input any values to at least k_2 boxes before sending $syn(x^n)$ to Bob, then Bob does accept the opening of the protocol with negligible probability.

Proof: Alice does not give any input to at least k_2 boxes before sending a syndrome s_0 to Bob. Later she gives her inputs to the remaining k_2 boxes and gets a random output x_i for each box. We know that any two strings $s^n \neq \tilde{s}^n$ with $syn(s^n) = syn(\tilde{s}^n)$ have distance at least $d > 2k_2$. We can bound the probability that the output string has distance at most k_1 to a string with syndrome s_0 by

$$\sum_{i=0}^{k_1} \binom{k_2}{i} p_0^i (1-p_0)^{k_2-i}$$

Note that since $4p_0 + 1 > 1$ and $2p_0 \le 1$, we have $p_0k_2 \ge k_1$. So we can apply Lemma 16 and get an upper bound on this probability of

$$2^{-2k_2p_0^2+4k_1p_0} = 2^{-2\frac{k_1(4p_0+1)}{2p_0^2}p_0^2+4k_1p_0} = 2^{-k_1}.$$

The statement now follows from Lemma 15.

Lemma 18. If Alice changes only k_1 values and delays only k_2 inputs, then the protocol is binding.

Proof: Any two input strings s^n and \tilde{s}^n with $s_0 = syn(s^n) = syn(\tilde{s}^n)$ have distance at least d. If we ignore all the positions where Alice did not input anything to the box, s^n and \tilde{s}^n still have distance at least $d - k_2 > 2k_1$.

V. TIGHTNESS OF OUR RESULTS

In this section we show that every non-signaling box with binary outputs that cannot be securely implemented from shared randomness allows to realize bit commitment with one of the above protocols.

Lemma 19. Let $W : \mathcal{U} \times \mathcal{V} \rightarrow \{0,1\}^2$ be a non-signaling box with $|\mathcal{U}| \geq 2$. If there exists (x_0, u_0) such that either $W^A(x_0|u_0) = 0$ or $\hat{W}_{x_0u_0} = \hat{W}_{x_1u_1}$ for some $(x_1, u_1) \neq$ (x_0, u_0) with $W^A(x_0|u_0) \leq W^A(x_1|u_1)$, then bit commitment can be implemented from W if and only if bit commitment can be implemented from the reduced box \tilde{W} that is obtained by removing input u_0 from W. Furthermore, W is local if and only if \tilde{W} is local.

Proof: We prove the statement by showing that Alice having access to \tilde{W} can simulate the behavior of W on input u_0 using local randomness: We first consider the case where $\hat{W}_{x_0u_0} = \hat{W}_{x_1u_1}$ with $u_1 \neq u_0$ and $W^A(x_1|u_1) \neq 0$. We define $p := W^A(x_0|u_0)/W^A(x_1|u_1)$. Then it holds that

$$W(x_0y|u_0v) = pW(x_1y|u_1v)$$

for all $y \in \{0,1\}, v \in \mathcal{V}$. It follows from the non-signaling conditions that

$$W((1-x_0)y|u_0v) = (1-p)W(x_1y|u_1v) + W((1-x_0)y|u_1v)$$

for all $y \in \{0, 1\}, v \in \mathcal{V}$. We assume $x_0 = x_1 = 0$. Then we can simulate W using \tilde{W} in the following way: Alice gives input u_1 to \tilde{W} and gets output x. If x = 1, then Alice outputs 1. If x = 0, then Alice outputs 0 with probability p and 1 with probability 1 - p. If $W^A(x_0|u_0) = 0$ or $\hat{W}_{0u_0} = \hat{W}_{1u_0}$, then Alice on input u_0 outputs 0 with probability $W^A(0|u_0)$ and 1 with probability $W^A(1|u_0)$.

Theorem 2. A non-signaling box $W : \{0,1\}^2 \rightarrow \{0,1\}^2$ that fulfills neither Condition 1 nor Condition 2 does not allow for information-theoretically secure bit commitment (with noiseless communication only) and is local.

Proof: We first consider the case where there exists (x_0, u_0) such that $W^A(x_0|u_0) = 0$ or $\hat{W}_{x_0u_0} = \hat{W}_{x_1u_1}$ for some $(x_1, u_1) \neq (x_0, u_0)$. We assume $W^A(x_0|u_0) \leq$ $W^A(x_1|u_1)$ and examine the box W that is obtained by removing input u_0 . \hat{W} is obviously local. If $\hat{W}(0|u_1) =$ $\hat{W}(1|u_1)$, the box is independent and doesn't allow for bit commitment. If there is a perfectly correlated or anti-correlated input pair, the box doesn't allow for bit commitment according to Theorem 1. Otherwise bit commitment can be reduced to this box using Protocol I. From Lemma 19 it follows that we can implement bit commitment from W if and only if bit commitment can be implemented from W. Thus, the claim follows for all boxes with $W^A(x_0|u_0) = 0$ or $\hat{W}_{x_0u_0} = \hat{W}_{x_1u_1}$ for some $(x_1, u_1) \neq (x_0, u_0)$. In the following we assume $W^{A}(x_{0}|u_{0}) \neq 0$ for all $x_{0}, u_{0} \in \{0, 1\}$ and $\hat{W}_{z} \neq \hat{W}_{z'}$ for all $z, z' \in \{0, 1\}^2$ with $z \neq z'$.

(1) $|\operatorname{extr}(\operatorname{conv}(W))| \ge 3$: Then the box fulfills Condition 2 and we can securely implement bit commitment using Protocol II.

(2)
$$|\operatorname{extr}(\operatorname{conv}(W))| = 2$$
: We first consider the case

 $\hat{W}_{1u}, \hat{W}_{0u} \in D$. Without loss of generality, we can assume u = 0. Then there exist $0 < \lambda_0, \mu_0 < 1$ such that

$$\begin{split} \ddot{W}_{01} &= \lambda_0 \ddot{W}_{00} + (1 - \lambda_0) \ddot{W}_{10}, \\ \\ \dot{W}_{11} &= \mu_0 \dot{W}_{00} + (1 - \mu_0) \dot{W}_{10}. \end{split}$$

We define $\lambda_1 := 1 - \lambda_0$ and $\mu_1 := 1 - \mu_0$. Then it follows from the non-signaling conditions that for all $(y, v) \in \{0, 1\} \times \mathcal{V}$

$$\begin{split} \frac{W(0y|1v)}{W^A(0|1)} &= \frac{\lambda_0 W(0y|0v)}{W^A(0|0)} + \frac{\lambda_1 W(1y|0v)}{W^A(1|0)},\\ \frac{W(1y|1v)}{W^A(1|1)} &= \frac{\mu_0 W(0y|0v)}{W^A(0|0)} + \frac{\mu_1 W(1y|0v)}{W^A(1|0)}. \end{split}$$

We define

$$a_x := \frac{\lambda_x W^A(0|1)}{W^A(x|0)}, \ x \in \{0,1\},$$

$$b_x := \frac{\mu_x W^A(1|1)}{W^A(x|0)}, \ x \in \{0,1\}.$$

Then it follows from the non-signaling conditions that for all $(y,v) \in \{0,1\} \times \mathcal{V}$ it holds that W(0y|0v) + W(1y|0v) is equal to

$$(a_0 + b_0)W(0y|0v) + (a_1 + b_1)W(1y|0v)$$

As we have have excluded the case $\hat{W}_{10} = \hat{W}_{00}$, it follows that $a_0 + b_0 = a_1 + b_1 = 1$. Then the box is local as follows from $W(xy|uv) = W^A(0|0)V^0_A(x|u)V^0_B(y|v) + W^A(1|0)V^1_A(x|u)V^1_B(y|v)$ with

(x, u)	$V_A^0(x u)$	$V_A^1(x u)$
(0, 0)	1	0
(0, 1)	a_0	a_1
(1, 0)	0	1
(1,1)	b_0	b_1

and

$$V_B^0(y|v) := W(0y|0v)/W^A(0|0),$$

$$V_B^1(y|v) := W(1y|0v)/W^A(1|0)$$

for all $y, v \in \{0, 1\}$. If one of the inputs (0, 0) or (0, 1) is perfectly correlated or anti-correlated, then we cannot reduce bit commitment to this box (Theorem 1). Otherwise we can securely implement bit commitment from this box using Protocol I.

Next, we consider the case $\hat{W}_{x0}, \hat{W}_{x'1} \in D, x, x' \in \{0, 1\}$. We assume x = x' = 0. Then it holds that

$$\hat{W}_{10} = \lambda_{00}\hat{W}_{00} + \lambda_{01}\hat{W}_{01},$$
$$\hat{W}_{11} = \mu_{00}\hat{W}_{00} + \mu_{10}\hat{W}_{10}.$$

If there is $u \in \{0, 1\}$ such that for all $v \in \{0, 1\}$ the box is neither perfectly correlated nor perfectly anti-correlated for input (u, v), then the box fulfills Condition 1. Otherwise, there must be $v_0, v_1 \in \{0, 1\}$ such that the box is perfectly correlated or anti-correlated for both $(0, v_0)$ and $(1, v_1)$. Then it follows that $\lambda_{00} = 0$ and $\mu_{10} = 0$, which is a contradiction to our assumptions.

The case $|\operatorname{extr}(\operatorname{conv}(\hat{W}))| \leq 1$ we have already excluded.

In order to prove that we can reduce bit commitment to any box with binary outputs (and general input alphabets \mathcal{U} and \mathcal{V}) that cannot be securely implemented from shared randomness we need to give an alternative condition for the security of Protocol II.

Condition 3. There exist $u_0, u_1 \in U$, $u_0 \neq u_1$ and $x_0, x_1 \in \{0, 1\}$ such that the following two conditions hold:

(1) $W_{x_0u_0}, W_{x_1u_1}$ are extreme points of $\operatorname{conv}(\hat{W})$, i.e., for all $c \in \{(x_0, u_0), (x_1, u_1)\}$ it holds that for all probability distributions P s.t. P(c) = 0

$$\left\|\hat{W}_c - \sum_z P(z)\hat{W}_z\right\|_1 \ge \delta.$$

(2) Let $c, c' \in \{(1 - x_0, u_0), (1 - x_1, u_1)\}$ with $c \neq c'$. Then for all probability distributions P such that P(c') > 0 and P(c) = 0 it holds that

$$\left\|\hat{W}_c - \sum_z P(z)\hat{W}_z\right\|_1 \ge \delta.$$

To prove Protocol II secure for all boxes that fulfill Condition 3, we replace Lemma 15 with the following lemma. We assume that $(x_0, u_0) = (0, 0)$ and $(x_1, u_1) = (0, 1)$.

Lemma 20. If $d_H(z^n, \tilde{z}^n) \ge k_1$, then the probability that Bob accepts \tilde{z}^n is negligible in n.

Proof: For all $w^n \in \mathcal{T}^n_{Q,\epsilon}$ it holds that $|N(xu|w^n) - \frac{n}{2}W^A(x|u)| \le \epsilon n \le \frac{1}{32}k_1$. We distinguish the following two cases:

(1) If there exists $u' \in \{0,1\}$ such that $N(u'|u^n) \leq n/2 - k_1/8$, then the statement follows from the proof of Lemma 15. (2) If $|n/2 - N(u|u^n)| \leq k_1/8$ for all $u \in \{0,1\}$, then it follows from Lemma 7 that with overwhelming probability $z^n \in \mathcal{T}_{W^A,\epsilon}^n(u^n)$. Assume $z^n \in \mathcal{T}_{W^A,\epsilon}^n(u^n)$. If $d_H(z^{I_{01}}, \tilde{z}^{I_{01}}) \geq \frac{1}{8}k_1$, then the claim follows from Lemma 8 and Condition 3. If $|\{i \in I_{10} : z_i = 11\}| \geq \frac{1}{8}k_1$, then the claim follows from Lemma 8 as follows: Let $D := \{k \in I_{10} : z_k \neq \tilde{z}_k\}$. We use Condition 3 and replace (2) with

$$\begin{aligned} \left\| \frac{1}{|I_{10}|} \sum_{k \in I_{10}} W_{z_k} - W_{10} \right\|_1 &= \frac{|D|}{|I_{10}|} \left\| W_{10} - \frac{1}{|D|} \sum_{k \in D} W_{z_k} \right\|_1 \\ &= \frac{|D|}{|I_{10}|} \delta \ge \frac{1}{8} k_1 \delta/n. \end{aligned}$$

We assume $\tilde{z}^n \in \mathcal{T}^n_{Q,\epsilon}$. Then it follows as in the proof of Lemma 8 that $W^n_{z^n}(\mathcal{T}^n_{W,\epsilon}(\tilde{z}^n))$ is negligible. The same holds if $|\{i \in I_{11} : z_i = 10\}| \geq \frac{1}{8}k_1$. In all other cases it follows that $\tilde{z}^n \notin \mathcal{T}^n_{Q,\epsilon}$.

Theorem 3. Bit Commitment can be reduced to any nonsignaling box with binary outputs that cannot be securely implemented from shared randomness.

Proof: If $|\mathcal{U}| \leq 2$, then the statement follows from the proof of Theorem 2. Otherwise, we first eliminate the cases where there exists (x_0, u_0) such that $W^A(x_0|u_0) = 0$ or $\hat{W}_{x_0u_0} = \hat{W}_{x_1u_1}$ for some $(x_1, u_1) \neq (x_0, u_0)$ by using Lemma 19 to reduce the box. Then we consider $D := \operatorname{extr}(\operatorname{conv}(\hat{W}))$: In the case |D| = 2 the statement is proven in the same way as in the proof of Theorem 2. The case $|D| \ge 3$ is a little bit more involved: If there is $\hat{W}_{1u}, \hat{W}_{0u} \in D$, then Condition 2 is fulfilled and we can implement bit commitment using Protocol II. Otherwise, we can either implement bit commitment using Protocol I or for every input u corresponding to an element of D there is an input v for Bob such that the box is perfectly correlated or anti-correlated. Let $\hat{W}_{x_0u_0} \in D$. Without loss of generality we assume that W is perfectly correlated for input (u_0, v_0) . Then there exist λ_z with $\sum_{z:\hat{W}_z \in D} \lambda_z = 1$ such that

$$\hat{W}_{(1-x_0)u_0} = \sum_{z:\hat{W}_z \in D} \lambda_z W_z.$$

There exists (x_1, u_1) with $u_1 \neq u_0$ such that $\lambda_{x_1u_1} > 0$. We assume $x_0 = x_1 = 0$. We have $W(10|u_0v_0) = 0$. This implies $W(00|u_1v_0) = 0$. From the non-signaling conditions follows that $W(10|u_1v_0) = W(00|u_0v_0) > 0$. There exists $v_1 \in \mathcal{V}$ such that (u_1, v_1) is perfectly correlated or anti-correlated. We assume without loss of generality that (u_1, v_1) is perfectly correlated. We correlated. This implies $W(00|u_1v_1) > 0$ and $W(10|u_1v_1) = 0$. From $\lambda_{x_1u_1} > 0$ follows that $W(10|u_0v_1) > 0$. So we have $\hat{W}_{0u_0}, \hat{W}_{0u_1} \in D$, $W(10|u_0v_1) = W(10|u_1v_1) = 0$, $W(10|u_1v_0) > 0$ and $W(10|u_0v_1) > 0$. Thus, Condition 3 is fulfilled.

VI. CONCLUDING REMARKS

We have shown that any bipartite non-signaling system with binary outputs can either be securely realized from shared randomness or allows for bit commitment.

An obvious open question is whether a similar result holds for arbitrary output alphabets. Furthermore, it would be interesting to know whether oblivious transfer can be implemented from the same set of non-signaling systems.

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REFERENCES

- M. Blum, "Coin flipping by telephone a protocol for solving impossible problems," *SIGACT News*, vol. 15, no. 1, pp. 23–27, 1983.
- [2] D. Mayers, "Unconditionally secure quantum bit commitment is impossible," *Physical Review Letters*, vol. 78, pp. 3414–3417, 1997.
- [3] H. K. Lo and H. F. Chau, "Is quantum bit commitment really possible?" *Physical Review Letters*, vol. 78, pp. 3410–3413, 1997.
- [4] C. Crépeau, "Efficient cryptographic protocols based on noisy channels," in Advances in Cryptology EUROCRYPT 97, ser. Lecture Notes in Computer Science, W. Fumy, Ed., vol. 1233. Springer, 1997, pp. 306– 317.
- [5] A. Winter, A. C. A. Nascimento, and H. Imai, "Commitment capacity of discrete memoryless channels," in *IMA Int. Conf.*, K. G. Paterson, Ed., 2003, pp. 35–51.
- [6] H. Imai, J. Müller-Quade, A. Nascimento, and A. Winter, "Rates for bit commitment and coin tossing from noisy correlation," in *Proceedings* of the IEEE International Symposium on Information Theory (ISIT '04), 2004.
- [7] S. Wolf and J. Wullschleger, "Zero-error information and applications in cryptography," in *Proceedings of 2004 IEEE Information Theory Workshop (ITW '04)*, 2004.

- [8] H. Imai, K. Morozov, A. C. A. Nascimento, and A. Winter, "Efficient protocols achieving the commitment capacity of noisy correlations," *Information Theory, 2006 IEEE International Symposium on*, pp. 1432– 1436, 2006.
- [9] S. Popescu and D. Rohrlich, "Quantum nonlocality as an axiom," Foundations of Physics, vol. 24, pp. 379–385.
- [10] H. Buhrman, M. Christandl, F. Unger, S. Wehner, and A. Winter, "Implications of superstrong nonlocality for cryptography," *Proceedings* of the Royal Society A, vol. 462, pp. 1919–1932, 2006.
- [11] B. S. Tsirelson, "Some results and problems on quantum Bell-type inequalities," *Hadronic J. Suppl.*, vol. 8, no. 4, pp. 329–345, 1993.
- [12] H. Chernoff, "A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations," *Annals of Mathematical Statistics*, vol. 23, pp. 493–507, 1952.
- [13] W. Hoeffding, "Probability inequalities for sums of bounded random variables," *Journal of the American Statistical Association*, vol. 58, no. 301, pp. 13–30, 1963.
- [14] R. Renner and S. Wolf, "Simple and tight bounds for information reconciliation and privacy amplification," in ASIACRYPT, ser. Lecture Notes in Computer Science, B. K. Roy, Ed., vol. 3788. Springer, 2005, pp. 199–216.
- [15] T. Holenstein and R. Renner, "On the randomness of independent experiments," arXiv:cs/0608007v1, 2006.
- [16] J. L. Carter and M. N. Wegman, "Universal classes of hash functions," *Journal of Computer and System Sciences*, vol. 18, pp. 143–154, 1979.
- [17] C. H. Bennett, G. Brassard, and J.-M. Robert, "Privacy amplification by public discussion," *SIAM Journal on Computing*, vol. 17, no. 2, pp. 210–229, 1988.
- [18] R. Impagliazzo, L. A. Levin, and M. Luby, "Pseudo-random generation from one-way functions," in *Proceedings of the 21st Annual ACM Symposium on Theory of Computing (STOC '89)*. ACM Press, 1989, pp. 12–24.
- [19] I. Csiszár and J. Körner, Information Theory: Coding Theorems for Discrete Memoryless Systems. New York: Academic Press, 1981.
- [20] S. Wolf and J. Wullschleger, "Bit commitment from weak non-locality," in Proceedings of 2005 IEEE Information Theory Workshop on Theory and Practice in Information-Theoretic Security, 2005.