

New Sequences Design from Weil Representation with Low Two-Dimensional Correlation in Both Time and Phase Shifts

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September 14, 2018

Abstract

A new elementary expression of the construction first proposed by Gurevich, Hadani, and Sochen is given, which avoids the explicit use of the Weil representation. The sequences in this signal set are given by both multiplicative character and additive character of finite field \mathbb{F}_p . Such a signal set consists of $p^2(p-2)$ time-shift distinct sequences, the magnitude of the two-dimensional autocorrelation function (i.e., the ambiguity function) in both time and phase of each sequence is upper bounded by $2\sqrt{p}$ at any shift not equal to $(0, 0)$. Furthermore, the magnitude of their Fourier transform spectrum is less than or equal to 2. For a subset consisting of $p(p-2)$ phase-shift distinct sequences in this signal set, the magnitude of the ambiguity function of any pair is upper bounded by $4\sqrt{p}$. A proof is given through finding a new expression of the sequences in the finite harmonic oscillator system. An open problem for directly establishing these assertions without involving the Weil representation is addressed.

Index Terms. Sequence, autocorrelation, cross correlation, ambiguity function, Fourier transform, and Weil representation.

1 Introduction

Sequence design for good correlation finds many important applications in various transmission systems in communication networks, and radar systems.

A. Low Correlation

In code division multiple access (CDMA) applications of spread spectral communication, multiple users share a common channel. Each user is assigned a different spreading sequence (or spread code) for transmission. At an intended receiver, despreading (recovering the original data) is accomplished by

^{0*} The work was conducted when Zilong Wang was a visiting Ph.D student at the Department of ECE in University of Waterloo from September 2008 to August 2009.

the correlation of the received spread signal with a synchronized replica of the spreading sequence used to spread the information where the spreading sequences used by other users are treated as interference, which is referred to as *multiple access interference*. This type of interference, which is different from interference that arises in radio-frequency (RF) communication channels, can be reduced by proper design of a spreading signal set. The performance of a signal (or sequence) set used in a CDMA system is measured by the parameters L , the length or period of a sequence in the set, r , the number of time-shift distinct sequences, and ρ , the maximum magnitude of the out-of-phase autocorrelation of any sequence and cross correlation of any pair of the sequences in the set. This is referred to as an (L, r, ρ) *signal set*. The trade-off of these three parameters is bounded by the Welch bound, established in 1974 by Welch [43]. The research for constructing good signal sets has flourished in the literature. The reader is referred to [6, 1, 39, 32, 5, 9] for polyphase sequences with large alphabet sizes, [26, 20] for \mathbb{Z}_4 sequences, [12, 30] for interleaved sequences, and [36, 24, 13] in general, for example.

B. Minimized Fourier Spectrum

The orthogonal frequency division multiplexing (OFDM) utilizes the concept of parsing the input data into N symbol streams, and each of which in turn is used to modulate parallel, synchronous subcarriers. With an OFDM system having N subchannels, the symbol rate on each subcarrier is reduced by a factor of N relative to the symbol rate on a single carrier system that employs the entire bandwidth and transmits data at the same rate as OFDM. An OFDM signal can be implemented by computing an inverse Fourier transform and Fourier transform at the transmitter side and receiver side, respectively. A major problem with the multicarrier modulation in general and OFDM system in particular is the high peak-to-average power ratio (PAPR) that is inherent in the transmitted signal. A bound on PAPR through the magnitude of the discrete Fourier transform (DFT) spectrum of employed signals is shown in [28, 31]. (See [40] for details of Fourier transform.) One way to achieve low PAPR is to employ Golay complementary sequences, as first shown by Davis and Jedwab in [8]. A tremendous amount of work has been done along this line since then.

C. Low Valued Ambiguity Functions

In radar or sonar applications, a sequence should be designed in such a way that the *ambiguity function* (the two-dimensional autocorrelation function in both time and frequency or equivalently phase, will be formally defined later), having the value of the length of the sequence at $(0, 0)$, and small values at any shift not $(0, 0)$. The *ambiguity function* is required for determining the *range* (proportional to the time-shift) and *Doppler* (the velocity to or from the observer, proportional to the frequency shift) of a target. Sequences with low ambiguity function can be achieved by Costas arrays, which yield the so-called *ideal* or *thumb-tack* ambiguity function (which only takes the values 0 or 1 at any shift not $(0, 0)$) [7, 14].

It is interesting to see whether there exists a signal set which simultaneously satisfies the requirements that arise from the above three transmission scenarios, i.e., having low correlation, low PAPR, low ambiguity function, but with large size and moderate implementation cost. It is anticipated that employing those sequences will improve the performance of communication systems with multi-carrier CDMA transmission [33], radar networks, and transmission systems in future cognitive radio networks [34].

Gurevich, Hadani, and Sochen [16, 18] proposed a signal set called finite oscillator system \mathfrak{S} which gives a positive answer to the above question except for the implementation cost. Their construction makes use of the group-theoretic Weil representation and the sequences are described in algorithmic terms by the end of [16, 17]. The main contribution of this paper is to propose a simple elementary expression for those sequences, which avoids the need to explicitly employ relatively costly group-theoretic computations.

It is interesting to observe that to date, almost all sequences with low correlation in the literature are related to the use of additive or multiplicative characters of the finite field or Galois rings together with functions. Recently, inspired by mutually unbiased bases discussed by Howe in [22], Howard, Calderbank, and Moran [21] investigated sequences constructed from the Heisenberg representation in 2006, then Gurevich, Hadani, and Sochen [16, 18] introduced sequences from the Weil representation in 2008, which are referred to as a finite oscillator system \mathfrak{S} .

In fact, sequences from the Heisenberg representation are related to extended a Frank-Zadoff-Chu (FZC) sequence [10, 6, 11], being complex valued sequences with period p . After normalization by the energy, the values of their ambiguity functions (precisely defined in the next section) is bounded by $\frac{1}{\sqrt{p}}$ except for some special case. While the sequences from the Weil representation, which will be introduced later, have the desired properties in the above mentioned three application scenarios, but have a complicated form. Gurevich, Hadani, and Sochen investigated how to implement their sequences in terms of an algorithm. The goal of this paper is to find a simple elementary expression for the finite harmonic oscillator system. We show that there are two types of the sequences in the finite harmonic oscillator system of the splitting case (we will formally define it later). Sequences of the first type can be given as product sequences using both multiplicative characters and additive characters of the finite field \mathbb{F}_p , and sequences of the second type are involved the summations of sequences of the first type. We construct a new signal set from the set consisting of the sequences of the first type with some extension.

The rest of the paper is organized as follows. In Section 2, we introduce some basic concepts and notations. In Section 3, we present our new constructions and the main results. In Section 4, we introduce Weil representation and the finite oscillator system constructed by Gurevich, Hadani and Sochen in [16, 18]. We show a simple elementary expression for this finite oscillator system, and present

a proof for the new constructions in Section 5. Comparisons of the new constructions with some known constructions are made in Section 6. Section 7 is for concluding remarks and addressing an open problem.

2 Basic Concepts and Definitions

In this section, we introduce some basic concepts and notations which are frequently used in this paper. For a given prime p , let θ and η denote the $(p-1)$ th and p th primitive roots of unity in the complex field respectively, i.e.,

$$\theta = \exp\left(\frac{2\pi i}{p-1}\right) \quad \text{and} \quad \eta = \exp\left(\frac{2\pi i}{p}\right).$$

We denote \mathbb{F}_p as the finite field with p elements, and \mathbb{F}_p^* as the multiplicative group of \mathbb{F}_p with a generator α . Then for every element $\beta \in \mathbb{F}_p^*$, there exists i with $0 \leq i \leq p-2$, such that $\beta = \alpha^i$. In other words, $i = \log_\alpha \beta$. We set $\theta^{\log_\alpha 0} = 0$ throughout this paper.

Every sequence with period p can be denoted by $\varphi = (\varphi(0), \varphi(1), \dots, \varphi(p-1))$, and also considered as a vector in the Hilbert space $\mathcal{H} = \mathbb{C}(\mathbb{F}_p)$ with the inner product given by the standard formula: $\langle \varphi, \psi \rangle = \sum_{i \in \mathbb{F}_p} \varphi(i) \overline{\psi(i)}$ where \bar{x} is the complex conjugate of x . We denote $U(\mathcal{H})$ (Appendix 7.3) as the group of unitary operators on \mathcal{H} . Let L_t, M_w and F be unitary operators of the time-shift, phase-shift and DFT respectively, which are defined as follows,

$$L_t[\varphi](i) = \varphi(i+t) \quad M_w[\varphi](i) = \eta^{wi} \varphi(i) \quad \text{and} \quad F[\varphi](j) = \frac{1}{\sqrt{p}} \sum_{i \in \mathbb{F}_p} \eta^{ji} \varphi(i), \quad \varphi \in \mathcal{H}. \quad (1)$$

We also use the notation $\widehat{\varphi}$ for $F[\varphi]$ for simplicity. If $\psi = L_t\varphi$ or $\psi = M_w\varphi$, then we say that φ and ψ are *time-shift equivalent* or *phase-shift equivalent*. Otherwise, they are *time-shift distinct* or *phase-shift distinct*.

We denote $C_\varphi(t)$ and $C_{\varphi,\psi}(t)$ their respective *autocorrelation* and *cross correlation* functions, which are defined by

$$C_\varphi(t) = \sum_{i \in \mathbb{F}_p} \varphi(i) \overline{\varphi(i+t)} \quad \text{and} \quad C_{\varphi,\psi}(t) = \sum_{i \in \mathbb{F}_p} \varphi(i) \overline{\psi(i+t)}. \quad (2)$$

Definition 1 We say that S is a (p, r, σ, ρ) signal set if each sequence in S has period p , there are r time-shift distinct sequences in S , and the maximum magnitude of out-of-phase autocorrelation values and cross correlation values are upper bounded by σ and ρ respectively, i.e.,

$$\begin{aligned} |C_\varphi(t)| &\leq \sigma, \quad t \neq 0, \varphi \in S, \\ |C_{\varphi,\psi}(t)| &\leq \rho, \quad t \in \mathbb{F}_p, \varphi \neq \psi \in S. \end{aligned} \quad (3)$$

In this paper, we also say that auto and cross correlation of S is upper bounded by σ and ρ respectively.

We say that a sequence φ is a *perfect sequence* if

$$C_\varphi(t) = \begin{cases} p & t \equiv 0 \pmod{p}, \\ 0 & t \not\equiv 0 \pmod{p}. \end{cases}$$

The *auto and cross ambiguity functions* of sequences are defined as two-dimensional autocorrelation and cross correlation functions in both time and phase, and are given by

$$A_\varphi(t, w) = \langle \varphi, M_w L_t \varphi \rangle \quad \text{and} \quad A_{\varphi, \psi}(t, w) = \langle \varphi, M_w L_t \psi \rangle. \quad (4)$$

The definitions of the auto and cross correlation functions are equal to their respective auto and cross ambiguity functions for the case $w = 0$.

Definition 2 We say that S is a (p, r, σ, ρ) ambiguity signal set if each sequence in S has period p , there are r time-shift distinct sequences in S , and the maximum magnitude of ambiguity out-of-phase autocorrelation values and cross correlation values are upper bounded by σ and ρ respectively, i.e.,

$$\begin{aligned} |A_\varphi(t, w)| &\leq \sigma, \quad (t, w) \neq (0, 0), \\ |A_{\varphi, \psi}(t, w)| &\leq \rho, \quad \varphi \neq \psi \in S. \end{aligned} \quad (5)$$

Property 1 Let S_1 be a (p, r, σ, ρ) ambiguity signal set, and $S_2 = \{M_w \varphi | w \in \mathbb{F}_p, \varphi \in S_1\}$. Then S_2 is a (p, pr, σ, ρ) signal set.

Remark 1 All the definitions and notations are stated for sequences with period p in this section. However, they are also valid for sequences with period n when p and \mathbb{F}_p are replaced by n and \mathbb{Z}_n respectively.

3 Main Results

There are two types of sequences in the set of the *finite oscillator system* \mathfrak{S} [16]. One is from the split case, denoted as \mathfrak{S}^s , and the other is from the non-split case, denoted as \mathfrak{S}^{ns} . In other words,

$$\mathfrak{S} = \mathfrak{S}^s \cup \mathfrak{S}^{ns}.$$

Gurevich, Hadani, and Sochen investigated how to implement the sequences in \mathfrak{S}^s by an algorithm [16]. Here we found a simple elementary construction for the sequences in \mathfrak{S}^s , which is presented as follows.

Theorem 1 Let α be a generator of \mathbb{F}_p^* .

$$\mathfrak{S}^s = \{\varphi_{x,y,z} \mid 1 \leq x \leq p-2, 0 \leq y \leq p-1, 0 \leq z \leq (p-1)/2\}$$

where $\varphi_{x,y,z} = \{\varphi_{x,y,z}(i)\}$ is a normalized sequence with period p whose elements are given by

$$\varphi_{x,y,0}(i) = \frac{1}{\sqrt{p-1}} \theta^{x \cdot \log_\alpha i} \eta^{yi^2},$$

and

$$\varphi_{x,y,z}(i) = \frac{\eta^{yi^2}}{\sqrt{p(p-1)}} \sum_{j=1}^{p-1} \theta^{x \cdot \log_\alpha j} \eta^{-(2z)^{-1}(j-i)^2} \text{ for } z \neq 0.$$

If $z \neq 0$, it is clearly every element in $\varphi_{x,y,z}$ has complicated form and does not lie on the unit circle, so we only consider the sequences where $z = 0$.

Construction of Ω_0 . Let α be a generator of \mathbb{F}_p^* . For a given prime p ($p \geq 5$), $n \in \mathbb{Z}$ and $0 \leq n < p(p-2)$, n has a p -adic decomposition given by: $n = (x-1)p + y$ where $1 \leq x \leq p-2$, $0 \leq y \leq p-1$. Let $\varphi_n = \{\varphi_n(i)\}$ be a sequence whose elements are defined as

$$\varphi_n(i) = \theta^{x \cdot \log_\alpha i} \cdot \eta^{yi^2}, \quad 0 \leq i \leq p-1,$$

and

$$\Omega_0 = \{\varphi_n : 0 \leq n < p(p-2)\}.$$

Then from the main results of [16] (also Theorem 3 in this paper), we have

Theorem 2 *The Signal set Ω_0 satisfies the following properties.*

- (a) Ω_0 is a $(p, p(p-2), 2\sqrt{p}, 4\sqrt{p})$ ambiguity signal set.
- (b) DFT of φ is bounded by $|\widehat{\varphi}(i)| < 2$, for $\varphi \in \Omega_0, i \in \mathbb{F}_p$.
- (c) The elements of each sequence φ in Ω lie on the unit circle of the complex plane except $\varphi(0) = 0$.

We can extend Ω_0 by the phase shift operator as follows.

Construction of Ω . Let α be a generator of \mathbb{F}_p^* . For a given prime p ($p \geq 5$), $n \in \mathbb{Z}$ and $0 \leq n < p^2(p-2)$, n has a p -adic decomposition given by: $n = (x-1)p^2 + yp + z$ where $1 \leq x \leq p-2$, $0 \leq y, z \leq p-1$. Let $\varphi_n = \{\varphi_n(i)\}$ be a sequence whose elements are defined as

$$\varphi_n(i) = \theta^{x \cdot \log_\alpha i} \cdot \eta^{yi^2+zi}, \quad 0 \leq i \leq p-1,$$

and

$$\Omega = \{\varphi_n : 0 \leq n < p^2(p-2)\}.$$

Then from Property 1, we have

Corollary 1 *The signal set Ω satisfies the following properties.*

- (a) Ω is a $(p, p^2(p-2), 2\sqrt{p}, 4\sqrt{p})$ signal set.
- (b) DFT of φ is bounded by $|\widehat{\varphi}(i)| < 2$, for $\varphi \in \Omega, i \in \mathbb{F}_p$.
- (c) The elements of each sequence φ in Ω lie on the unit circle of the complex plane except $\varphi(0) = 0$.
- (d) The magnitude of auto ambiguity function of every sequence in Ω is upper bounded by $2\sqrt{p}$ at any shift not equal to $(0, 0)$.

Example 1 For $p = 5$, $a = 2$ is a generator of \mathbb{F}_5 , the elements of the sequences φ_x, φ_y , and φ_z are defined as $\varphi_x(i) = \theta^{x \cdot \log_\alpha i}$, $\varphi_y(i) = \eta^{yi^2}$, and $\varphi_z(i) = \eta^{zi}$ respectively, which are given as follows.

x	$\varphi_x(i) = \theta^{x \cdot \log_\alpha i}$
1	$\{0, 1, \theta, \theta^3, \theta^2\}$
2	$\{0, 1, \theta^2, \theta^2, 1\}$
3	$\{0, 1, \theta^3, \theta, \theta^2\}$

y	$\varphi_y(i) = \eta^{yi^2}$
0	$\{1, 1, 1, 1, 1\}$
1	$\{1, \eta, \eta^4, \eta^4, \eta\}$
2	$\{1, \eta^2, \eta^3, \eta^3, \eta^2\}$
3	$\{1, \eta^3, \eta^2, \eta^2, \eta^3\}$
4	$\{1, \eta^4, \eta, \eta, \eta^4\}$

z	$\varphi_z(i) = \eta^{zi}$
0	$\{1, 1, 1, 1, 1\}$
1	$\{1, \eta, \eta^2, \eta^3, \eta^4\}$
2	$\{1, \eta^2, \eta^4, \eta, \eta^3\}$
3	$\{1, \eta^3, \eta, \eta^4, \eta^2\}$
4	$\{1, \eta^4, \eta^3, \eta^2, \eta\}$

Then the elements of each sequence in the signal set Ω are constructed by term-by-term products of the elements of φ_x, φ_y , and φ_z . The first three sequences and last two sequences are given as follows.

$$\begin{aligned}
\varphi_0 &= \varphi_{1,0,0} = (0, 1, \theta, \theta^3, \theta^2), \\
\varphi_1 &= \varphi_{1,0,1} = (0, \eta, \theta\eta^2, \theta^3\eta^3, \theta^2\eta^4), \\
\varphi_2 &= \varphi_{1,0,2} = (0, \eta^2, \theta\eta^4, \theta^3\eta, \theta^2\eta^3), \\
&\vdots \\
\varphi_{73} &= \varphi_{3,4,3} = (0, \eta^2, \theta^3\eta^2, \theta, \theta^2\eta), \\
\varphi_{74} &= \varphi_{3,4,4} = (0, \eta^3, \theta^3\eta^4, \theta\eta^3, \theta^2).
\end{aligned}$$

In the rest of the sections, we first prove that Theorem 1 is the split case of the finite oscillator system, and then complete proofs for Theorem 2 and Corollary 1. In order to do so, in the next section, we first introduce some basic concepts and definitions on Weil representations, and then present the oscillator system signal set.

4 The Weil Representation and Finite Oscillator System

For more details about the representation theory and the Weil representation, we refer the reader to [16, 21, 22] as well as the appendix in this paper.

4.1 Weil Representation

The Weil representation is a unitary representation from $SL_2(\mathbb{F}_p)$ to $U(\mathcal{H})$ (see the details in Appendix).

$SL_2(\mathbb{F}_p)$ can be generated by $g_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $g_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$, and Weyl element $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

where $a \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_p$. The Weil representations for g_a, g_b and w are given in [17] as follows

$$\rho(g_a)[\varphi](i) = \sigma(a)\varphi(a^{-1}i) \quad (6)$$

$$\rho(g_b)[\varphi](i) = \eta^{-2^{-1}bi^2}\varphi(i) \quad (7)$$

$$\rho(w)[\varphi](j) = \frac{1}{\sqrt{p}} \sum_{i \in \mathbb{F}_p} \eta^{ji} \varphi(i) \quad (8)$$

where $\sigma : \mathbb{F}_p^* \rightarrow \{\pm 1\}$ is the Legendre character, i.e., $\sigma(a) = a^{\frac{p-1}{2}}$ in \mathbb{F}_p .

Obviously, $\rho(w)$ is equal to F defined in (1). Here we denote $\rho(g_a) = S_a, \rho(g_b) = N_b, \rho(w) = F$ for simplicity. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_p)$, if $b \neq 0$, we have

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ (ad-1)b^{-1} & d \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ bd & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ ab^{-1} & 1 \end{pmatrix}.$$

Thus the Weil representation of g is given by

$$\rho(g) = S_b \circ N_{bd} \circ F \circ N_{ab^{-1}}. \quad (9)$$

If $b = 0$, then

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ ac & 1 \end{pmatrix}.$$

Hence the Weil representation of g is as follows

$$\rho(g) = S_a \circ N_{ac}. \quad (10)$$

4.2 The Finite Oscillator System

In this subsection, we introduce the main results of [16].

A. Maximal Algebraic Tori

A *maximal algebraic torus* [4] in $SL_2(\mathbb{F}_p)$ is a maximal commutative subgroup which becomes diagonalizable over the original field or quadratic extension of the field. One example of a maximal algebraic torus in $SL_2(\mathbb{F}_p)$ is the standard diagonal torus

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_p^* \right\}.$$

Up to conjugation, there are two classes of the maximal algebraic tori in $SL_2(\mathbb{F}_p)$. The first class, called *split tori*, consists of those tori which are diagonalizable over \mathbb{F}_p . Every split torus T is conjugated to the standard diagonal torus A , i.e., there exists an element $g \in SL_2(\mathbb{F}_p)$ such that $g \cdot T \cdot g^{-1} = A$. The second class, called *non-split tori*, consists of those tori which are not diagonalizable over \mathbb{F}_p , but become diagonalizable over the quadratic extension \mathbb{F}_{p^2} . In fact, a split torus is a cyclic subgroup of $SL_2(\mathbb{F}_p)$ with order $p - 1$, while a non-split torus is a cyclic subgroup of $SL_2(\mathbb{F}_p)$ with order $p + 1$.

All split (non-split) tori are conjugated to one another, so the number of split (non-split) tori is the number of elements in the coset space $SL_2(\mathbb{F}_p)/N$ ($SL_2(\mathbb{F}_p)/M$) (see [41] for basics of group theory), where N (M) is the *normalizer group* of a non-split torus A . Thus

$$\#(SL_2(\mathbb{F}_p)/N) = \frac{1}{2}p(p+1) \quad \text{and} \quad \#(SL_2(\mathbb{F}_p)/M) = \frac{1}{2}p(p-1). \quad (11)$$

Remark 2 A direct calculation shows that the number of non-split tori is equal to $\frac{1}{2}p(p-1)$ instead of $p(p-1)$, which is a mistake made in [16].

B. Decomposition of Weil Representation Associated with Maximal Tori

Because every maximal torus $T \in SL_2(\mathbb{F}_p)$ is a cyclic group, restricting the Weil representation to T : $\rho|_T : T \rightarrow U(\mathcal{H})$, we obtain a decomposition of $\rho|_T$ corresponding to an orthogonal decomposition of \mathcal{H} .

$$\rho|_T = \bigoplus_{\chi \in \Lambda_T} \chi \quad \text{and} \quad \mathcal{H} = \bigoplus_{\chi \in \Lambda_T} \mathcal{H}_\chi \quad (12)$$

where Λ_T is a collection of all the one dimensional subrepresentations (characters) $\chi : T \rightarrow \mathbb{C}$ in the decomposition of the Weil representation restricted on the torus T .

The decomposition (12) depends on the type of T . In the case where T is a split torus, χ is a character given by $\chi : \mathbb{Z}_{p-1} \rightarrow \mathbb{C}$. We have $\dim \mathcal{H}_\chi = 1$ unless $\chi = \sigma$ where σ is the Legendre character

of T , and $\dim \mathcal{H}_\sigma = 2$. In the case where T is a non-split torus, χ is the character given by $\chi : \mathbb{Z}_{p+1} \rightarrow \mathbb{C}$. There is only one character σ with order 2 that does not appear in the decomposition. For the other characters $\chi \neq \sigma$, $\dim \mathcal{H}_\chi = 1$.

C. Sequences Associated with Finite Oscillator System

For a given torus T and each character $\chi \in \Lambda_T$, choosing a vector $\varphi_\chi \in \mathcal{H}_\chi$ of unit norm, we obtain a collection of orthonormal vectors

$$\mathcal{B}_T = \{\varphi_\chi : \chi \in \Lambda_T, \chi \neq \sigma\}. \quad (13)$$

Considering the union of these collections, then the finite oscillator system

$$\mathfrak{S} = \{\varphi \in \mathcal{B}_T : T \subset SL_2(\mathbb{F}_p)\}. \quad (14)$$

\mathfrak{S} is naturally separated into two sub-systems \mathfrak{S}^s and \mathfrak{S}^{ns} which correspond to the split tori and the non-split tori respectively. The sub-system \mathfrak{S}^s (\mathfrak{S}^{ns}) consists of the union of \mathcal{B}_T , where T runs through all the split tori (non-split tori) in $SL_2(\mathbb{F}_p)$. Totally there are $\frac{1}{2}p(p+1)$ ($\frac{1}{2}p(p-1)$) tori consisting of $p-2$ (p) orthonormal sequences. Hence

$$\#\mathfrak{S}^s = \frac{1}{2}p(p+1)(p-2) \quad \text{and} \quad \#\mathfrak{S}^{ns} = \frac{1}{2}p^2(p-1). \quad (15)$$

Theorem 3 *Sequences in the set \mathfrak{S} satisfy the following properties. For $\varphi, \psi \in \mathfrak{S}$ and $(t, w) \in V = \mathbb{F}_p \times \mathbb{F}_p$,*

(a) \mathfrak{S} is a $(p, p(p^2 - p - 1), \frac{2\sqrt{p}}{p-1}, \frac{4\sqrt{p}}{p-1})$ ambiguity signal set.

(b) Supremum of φ is given by $\max\{|\varphi(i)| : i \in \mathbb{F}_p\} \leq \frac{2}{\sqrt{p}}$.

(c) For every sequences $\varphi \in \mathfrak{S}$, its DFT $\hat{\varphi}$ is (up to multiplication by a unitary scalar) also in \mathfrak{S} .

5 Proof of Main Results

An efficient method to specify the decomposition (12) is by choosing a generator $t \in T$, the character which is generated by the eigenvalue of linear operator $\rho(t)$, and the character space \mathcal{H}_χ that naturally corresponds to the eigenspace. Below are three steps to construct the sequences in the split case of finite oscillator system \mathfrak{S}^s .

Step 1 Compute the generator g_α for the standard torus A and \mathcal{B}_A . In other words, the collection of the eigenvectors of $\rho(g_\alpha)$ which do not correspond to eigenvalue -1 .

Step 2 Compute all representative elements g in the coset $\{gN(A) : g \in SL_2(\mathbb{F}_p)\}$ where $N(A)$ is the normalizer group of A .

Step 3 Compute all sequences $\rho(g)\varphi$ where g is the representative element presented in Step 2 and $\varphi \in \mathcal{B}_A$ is calculated in Step 1.

Considering $\{\delta_i : i \in \mathbb{F}_p\}$ as the Kronecker delta function of Hilbert space $\mathcal{H} = \mathbb{C}(\mathbb{F}_p)$ (i.e., δ_i is defined as $\delta_i(j) = \delta_{ij}$ for $\forall j \in \mathbb{F}_p$), every sequence $\varphi = \{\varphi(i)\}$ with period p can be written as $\varphi = \sum_{i=0}^{p-1} \varphi(i)\delta_i$. Recall that $SL_2(\mathbb{F}_p)$ can be generated by $g_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $g_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ where $a \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_p$, then their respective Weil representations (6), (7), and (8) of g_a , g_b , and w can be rewritten as follows

$$\rho(g_a)\delta_i = S_a\delta_i = \sigma(a)\delta_{ai} \quad (16)$$

$$\rho(g_b)\delta_i = N_b\delta_i = \eta^{-2^{-1}bi^2}\delta_i \quad (17)$$

$$\rho(w)\delta_j = F\delta_j = \frac{1}{\sqrt{p}} \sum_{i \in \mathbb{F}_p} \eta^{ji}\delta_i. \quad (18)$$

Lemma 1 Let α be a generator of \mathbb{F}_p^* , and $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_p^* \right\}$ be the standard diagonal torus. Then

$$\mathcal{B}_A = \left\{ \varphi_x = \frac{1}{\sqrt{p-1}} \sum_{i=1}^{p-1} \theta^{x \cdot \log_\alpha i} \delta_i : 1 \leq x \leq p-2 \right\}.$$

Proof. The set \mathcal{B}_A is a collection of φ_χ with unit norm where $\varphi_\chi \in \mathcal{H}_\chi$ for every character $\chi \neq \sigma$. In other words, the set \mathcal{B}_A is a collection of unit eigenvectors (not belonging to eigenvalue -1) of $\rho(g_\alpha)$ where g_α is a generator of Torus A .

Let α be a generator of \mathbb{F}_p^* . Then $g_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ is a generator of torus A . From (16), we have

$$\rho(g_\alpha)\delta_i = \sigma(\alpha)\delta_{\alpha i} = -\delta_{\alpha i}.$$

The eigenfunction of $\rho(g_\alpha)$ is $(x+1)(x^{p-1}-1)$, so the eigenvalues of $\rho(g_\alpha)$ are $-1, \theta^0, \theta^1, \theta^2, \dots, \theta^{p-2}$. Obviously, $-1 = \theta^{\frac{p-1}{2}}$ occurs twice in the eigenvalues set. We assert that $\sum_{i=1}^{p-1} \theta^{(\frac{p-1}{2}-j) \log_\alpha i} \delta_i$ is an

eigenvector associated to the eigenvalue θ^j ($0 \leq j \leq p-2$), and it can be verified as follows

$$\begin{aligned}
\rho(g_\alpha)\left(\sum_{i=1}^{p-1} \theta^{(\frac{p-1}{2}-j) \log_\alpha i} \delta_i\right) &= -\sum_{i=1}^{p-1} \theta^{(\frac{p-1}{2}-j) \log_\alpha i} \delta_{ai} \\
&= -\sum_{i=1}^{p-1} \theta^{(\frac{p-1}{2}-j) \log_\alpha (a^{-1}i)} \delta_i \\
&= \theta^{\frac{p-1}{2}} \sum_{i=1}^{p-1} \theta^{(\frac{p-1}{2}-j)(\log_\alpha i-1)} \delta_i \\
&= \theta^{\frac{p-1}{2}} \theta^{j-\frac{p-1}{2}} \sum_{i=1}^{p-1} \theta^{(\frac{p-1}{2}-j) \log_\alpha i} \delta_i \\
&= \theta^j \sum_{i=1}^{p-1} \theta^{(\frac{p-1}{2}-j) \log_\alpha i} \delta_i.
\end{aligned}$$

Let $x = \frac{p-1}{2} - j$. Then $\{\sum_{i=1}^{p-1} \theta^{x \cdot \log_\alpha i} \delta_i \mid 1 \leq x \leq p-2\}$ is a set of the eigenvectors corresponding to all the eigenvalues not equal to -1 . By normalizing the eigenvectors, we complete the proof. \square

Lemma 2 Let $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_p^* \right\}$ be the standard diagonal torus, and $N(A)$ be the normalizer group of A . Then

$$R = \left\{ \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix} : 0 \leq b \leq \frac{p-1}{2}, c \in \mathbb{F}_p \right\}$$

is a collection of coset representatives of $\{gN(A) : g \in SL_2(\mathbb{F}_p)\}$.

Proof. Denote $B = \left\{ \begin{pmatrix} 0 & -b \\ b^{-1} & 0 \end{pmatrix} : b \in \mathbb{F}_p^* \right\}$. Then it's not hard to verify

$$N(A) = \{g : gAg^{-1} = A, g \in SL_2(\mathbb{F}_p)\} = AB.$$

Thus every representative element g can be written as the form

$$g = \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix} \quad b, c \in \mathbb{F}_p.$$

Note that $g = \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix}$ and $g' = \begin{pmatrix} 1 & b' \\ c' & 1+b'c' \end{pmatrix}$ are in the same coset, i.e., $g^{-1}g' \in N(A)$, if and only if

$$\begin{aligned} \begin{pmatrix} 1 & b' \\ c' & 1+b'c' \end{pmatrix} &= \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix} \begin{pmatrix} 0 & -b \\ b^{-1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -b \\ b^{-1}+c & -bc \end{pmatrix} \\ &= \begin{pmatrix} 1 & -b \\ b^{-1}+c & 1+(-b)(b^{-1}+c) \end{pmatrix} \end{aligned}$$

if and only if $b' = -b$ and $c' = b^{-1} + c$. Therefore R contains all representative elements in the coset $\{gN(A) : g \in SL_2(\mathbb{F}_p)\}$. □

By Lemmas 1 and 2, we can now prove Theorem 1, which is a direct consequence of the following result.

Proposition 1 *There are two types of vectors in \mathfrak{S}^s .*

The first type is

$$\varphi_{x,y,0} = \frac{1}{\sqrt{p-1}} \sum_{i=1}^{p-1} \theta^{x \cdot \log_{\alpha} i} \eta^{yi^2} \delta_i$$

where $1 \leq x \leq p-2, 0 \leq y \leq p-1$.

The second type is

$$\varphi_{x,y,z} = \frac{1}{\sqrt{p(p-1)}} \sum_{i=0}^{p-1} \sum_{j=1}^{p-1} \theta^{x \cdot \log_{\alpha} j} \eta^{yi^2 - (2z)^{-1}(j-i)^2} \delta_i$$

where $1 \leq x \leq p-2, 0 \leq y \leq p-1, 1 \leq z \leq \frac{p-1}{2}$.

Proof. Every split torus $T \subset SL_2(\mathbb{F}_p)$ can be written as the form gAg^{-1} where A is the diagonal torus and $g = \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix} \in R$ in Lemma 2. Then

$$\mathcal{B}_T = \mathcal{B}_{gAg^{-1}} = \{\rho(g)\varphi : \varphi \in \mathcal{B}_A\},$$

and

$$\mathfrak{S}^s = \bigcup_{g \in R} \mathcal{B}_{gTg^{-1}} = \{\rho(g)\varphi : g \in R, \varphi \in \mathcal{B}_A\}.$$

If $b = 0$, $g = \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix}$ has the form $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ ($0 \leq c \leq p-1$), then from (17), we have

$$\begin{aligned} \rho(g)\varphi_x &= N_c \left(\frac{1}{\sqrt{p-1}} \sum_{i=1}^{p-1} \theta^{x \cdot \log_\alpha i} \delta_i \right) \\ &= \frac{1}{\sqrt{p-1}} \sum_{i=1}^{p-1} \theta^{x \cdot \log_\alpha i} N_c \delta_i \\ &= \frac{1}{\sqrt{p-1}} \sum_{i=1}^{p-1} \theta^{x \cdot \log_\alpha i} \eta^{-2^{-1}ci^2} \delta_i. \end{aligned}$$

If $b \neq 0$, g has the following decomposition

$$g = \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b(1+bc) & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b^{-1} & 1 \end{pmatrix}.$$

Then applying (16),(17), and (18), for $1 \leq x \leq p-1$, we have

$$\begin{aligned} \rho(g)\varphi_x &= S_b \circ N_{b(1+bc)} \circ F \circ N_{b^{-1}} \left(\frac{1}{\sqrt{p-1}} \sum_{j=1}^{p-1} \theta^{x \cdot \log_\alpha j} \delta_j \right) \\ &= S_b \circ N_{b(1+bc)} \circ F \left(\frac{1}{\sqrt{p-1}} \sum_{j=1}^{p-1} \theta^{x \cdot \log_\alpha j} \eta^{-2^{-1}b^{-1}j^2} \delta_j \right) \\ &= S_b \circ N_{b(1+bc)} \left(\frac{1}{\sqrt{p(p-1)}} \sum_{i=0}^{p-1} \sum_{j=1}^{p-1} \theta^{x \cdot \log_\alpha j} \eta^{-2^{-1}b^{-1}j^2} \eta^{ij} \delta_i \right) \\ &= S_b \left(\frac{1}{\sqrt{p(p-1)}} \sum_{i=0}^{p-1} \sum_{j=1}^{p-1} \theta^{x \cdot \log_\alpha j} \eta^{-2^{-1}b^{-1}j^2} \eta^{ij} \eta^{-2^{-1}b(1+bc)i^2} \delta_i \right) \\ &= \sigma(b) \left(\frac{1}{\sqrt{p(p-1)}} \sum_{i=0}^{p-1} \sum_{j=1}^{p-1} \theta^{x \cdot \log_\alpha j} \eta^{-2^{-1}b^{-1}j^2} \eta^{ij} \eta^{-2^{-1}b(1+bc)i^2} \delta_{bi} \right) \\ &= \sigma(b) \left(\frac{1}{\sqrt{p(p-1)}} \sum_{i=0}^{p-1} \sum_{j=1}^{p-1} \theta^{x \cdot \log_\alpha j} \eta^{-2^{-1}b^{-1}j^2} \eta^{b^{-1}ij} \eta^{-2^{-1}b^{-1}(1+bc)i^2} \delta_i \right) \\ &= \frac{\sigma(b)}{\sqrt{p(p-1)}} \sum_{i=0}^{p-1} \sum_{j=1}^{p-1} \theta^{x \cdot \log_\alpha j} \eta^{-(2b)^{-1}(j-i)^2 - 2^{-1}ci^2} \delta_i. \end{aligned}$$

Let $y = -2^{-1}c$, $z = b$. Then y ranges over \mathbb{F}_p as c ranges over \mathbb{F}_p . Note that $\sigma(z) = \pm 1$ is a constant, so $\frac{1}{\sqrt{p-1}} \sum_{i=1}^{p-1} \theta^{x \cdot \log_\alpha i} \eta^{yi^2} \delta_i$ and $\frac{1}{\sqrt{p(p-1)}} \sum_{i=0}^{p-1} \eta^{yi^2} \sum_{j=1}^{p-1} \theta^{x \cdot \log_\alpha j} \eta^{-(2z)^{-1}(j-i)^2} \delta_i$ with $1 \leq x \leq p-2$,

$0 \leq y \leq p-1, 1 \leq z \leq \frac{p-1}{2}$ are all the vectors in \mathfrak{S}^s , which completes the proof. \square

Thus, we have found a simple elementary representation for the split case of the finite oscillator system.

The following lemma gives the relationship of the correlation function, ambiguity function, and unitary operator L_t, M_w, F defined in (1), which is easy to verify.

Lemma 3 $\forall \varphi, \psi$ sequences with period $p, \forall t, w, z \in \mathbb{F}_p$, and where L_t, M_w, F are defined in (1), we have:

- (a) $C_\varphi(t) = \langle \varphi, L_t \varphi \rangle$ and $C_{\varphi, \psi}(t) = \langle \varphi, L_t \psi \rangle$.
- (b) $|\langle \varphi, \pi(t, w, z) \psi \rangle| = |\langle \varphi, M_w \cdot L_t \psi \rangle| = |\langle \varphi, L_t \cdot M_w \psi \rangle|$.
- (c) $L_t \cdot F = F \cdot M_t$ and $FL_{-t} = M_t \cdot F$.
- (d) $\langle \widehat{\varphi}, L_t \widehat{\psi} \rangle = \langle \varphi, M_{-t} \psi \rangle$ and $\langle \widehat{\varphi}, M_w \widehat{\psi} \rangle = \langle \varphi, L_t \psi \rangle$ (Parseval Formulae).

Now we extend signal set from \mathfrak{S} to $\overline{\mathfrak{S}}$ by the phase shift operator, i.e.,

$$\overline{\mathfrak{S}} = \{M_w \varphi : \forall \varphi \in \mathfrak{S}, w \in \mathbb{F}_p\}.$$

Then $\overline{\mathfrak{S}}$ satisfy the following property.

Property 2 *With the above notation,*

- (a) $\overline{\mathfrak{S}}$ is a $(p, p^2(p^2 - p - 1), \frac{2\sqrt{p}}{p-1}, \frac{4\sqrt{p}}{p-1})$ signal set.
- (b) Supremum of ψ is given by $\max\{|\psi(i)| : i \in \mathbb{F}_p\} \leq \frac{2}{\sqrt{p}}, \psi \in \overline{\mathfrak{S}}$.
- (c) DFT of ψ is bounded by $|\widehat{\psi}(i)| \leq \frac{2}{\sqrt{p}}, \forall i \in \mathbb{F}_p, \psi \in \overline{\mathfrak{S}}$.

Proof.

- (a) By Property 1, \mathfrak{S} is a $(p, p(p^2 - p - 1), \frac{2\sqrt{p}}{p-1}, \frac{4\sqrt{p}}{p-1})$ ambiguity signal set, so $\overline{\mathfrak{S}}$ is a $(p, p^2(p^2 - p - 1), \frac{2\sqrt{p}}{p-1}, \frac{4\sqrt{p}}{p-1})$ signal set.
- (b) $\forall M_w \varphi \in \overline{\mathfrak{S}}$, it is clear that the magnitude of $M_w \varphi(i)$ is as same as that of $\varphi(i)$.
- (c) Applying Lemma 3-(c), the DFT of $M_w \varphi$ can be written as $F \cdot M_w \varphi = L_w \cdot F \varphi$. We can see that $F \varphi$ is also in \mathfrak{S} from Lemma 3-(c), and $|F \varphi(i)| \leq \frac{2}{\sqrt{p}}$ from Theorem 3-(b). Thus $|F \cdot M_w \varphi(i)| = |L_w \cdot F \varphi(i)| \leq \frac{2}{\sqrt{p}}$, which completes the proof.

□

Proof of Theorem 2. Considering Ω_0 and \mathfrak{S}^s , it is obvious that Ω_0 is a subset of \mathfrak{S}^s up to multiplication by $\sqrt{p-1}$. Thus Ω_0 is a $(p, p^2(p-2), 2\sqrt{p}, 4\sqrt{p})$ ambiguity signal set, and the DFT of $\varphi \in \Omega_0$ is bounded by $|\widehat{\varphi}(i)| \leq 2\sqrt{\frac{p-1}{p}} < 2 \forall \varphi \in \Omega$ and $\forall i \in \mathbb{F}_p$. □

From Theorem 2, Property 1 and Lemma 3, Corollary 1 holds.

6 Comparisons of the New Constructions with Some Known Constructions

The split case of the finite oscillator system \mathfrak{S}^s and the extended construction $\overline{\mathfrak{S}}$ can be efficiently implemented for moderate p . However, for large p , since one needs to compute the exponential sum of p elements, they are not so efficient. Therefore, in this section, we only make some comparisons for the set Ω or Ω_0 with some known constructions.

A. Compared with Complex Valued Sequences with Good Ambiguity Function or DFT

Let n be a positive integer and ω_n be an n th primitive root of unit in the complex field, i.e., $\omega_n = e^{-\frac{2\pi j}{n}}$ where $j = \sqrt{-1}$. For fixed $0 < y < n, 0 \leq z < n$ where y is relatively prime to n , a Frank-Zadoff-Chu (FZC) sequence [10, 6, 11] $\{\varphi_{y,z}(i)\}$ is given by

$$\varphi_{y,z}(i) = \begin{cases} \omega_n^{(1/2)yi^2+zi} & n \text{ is even,} \\ \omega_n^{(1/2)yi(i+1)+zi} & n \text{ is odd.} \end{cases} \quad (19)$$

Any FZC sequence is a perfect sequence, i.e., its out-of-phase autocorrelation is zero. Note that $\omega_n^{1/2}$ is a $(2n)$ th primitive root of unit in the complex field. For n odd, $\varphi_{y,z}(i)$ can be given by an equivalent expression: $\omega_n^{y'i^2+z'i}$ where $0 < y' < n, 0 \leq z' < n$. This form will be used below.

1. For a fixed z , a FZC signal set is a set consisting of the $\varphi(n)$ sequences defined by (19) where $\varphi(n)$ is the Euler function. When $n = p$ a prime, the FZC set is a $(p, p-1, 0, \sqrt{p})$ signal set. The magnitude of the DFT of these sequences is bounded by 1.
2. The elements in Alltop cubic sequences [1] with period p are given by $\varphi_y(i) = \omega_p^{i^3+yi}$ where $0 \leq y \leq p-1$. The auto and cross ambiguity function can reach p with $\frac{1}{p}$ probability, and the magnitude of the DFT of these sequences is bounded by 2.
3. Sequences from Heisenberg representation: The elements in a sequence from the Heisenberg representation [21] have the form $\varphi_{y,z}(i) = \omega_p^{yi^2+zi}$ where $0 \leq y, z \leq p-1$. (Note that the sequences

from Heisenberg representation are the same as the FZC sequences with period $n = p$, a prime.) Here the magnitude of the auto ambiguity function of such sequences can reach p with $\frac{1}{p}$ probability, while the upper bound of the cross ambiguity function between two phase-shift distinct sequences is given by \sqrt{p} , and the magnitude of the DFT of these sequences is bounded by 1.

4. Modulatable orthogonal sequences [39]: An $h \times h$ discrete Fourier transform (DFT) matrix is defined by the j th row and the k th column elements of

$$d_{z,j,k} = \omega_h^{zjk} \quad (20)$$

where z is a fixed number with $0 < z < h$ and $\gcd(z, h) = 1$, and $0 \leq j, k < h$. Let a sequence $\{a_z(i)\}$ be given by concatenation of the rows of DFT matrix starting from the first row, second row, and so on, i.e., $a_z(i = jh + k) = d_{z,j,k}, 0 \leq j, k < h$. (Note that $\{a_z(i)\}$ can be considered as an interleaved sequence associated with the DFT matrix [13].) Let $\{b(i)\}_{i \geq 0}$ be a complex valued sequence with period h and $|b_i| = 1$, i.e., the magnitude of b_i is equal to 1. A modulatable orthogonal (MO) sequence $\{c_z(i)\}$ of period $n = h^2$ is given by

$$c_z(i) = a_z(i)b(i), i = 0, 1, \dots$$

For each h , an MO sequence is a perfect sequence. An MO signal set consists of the sequences for all z . When $h = p$, a prime, this set is a signal set with parameters $(p^2, p - 1, 0, p)$.

5. Generalized chirp-like (GCL) sequences [32]: Popović, generalized the construction of the modulatable orthogonal sequences in 1992 as follows. Let $\{\varphi_{y,z}(i)\}$ be a FZC sequence with period $n = th^2$ where both t and h are arbitrary positive integers, and $\{b(i)\}$ be the same as defined for MO sequences. A generalized chirp-like sequence $\{c_{y,z}(i)\}$ is given by

$$c_{y,z}(i) = \varphi_{y,z}(i)b(i), i = 0, 1, \dots$$

where the index i of $\varphi_{y,z}(i)$ is reduced modular n and the index of $b(i)$ is reduced modular h . Each generalized chirp-like sequence is a perfect sequence. For a fixed z , a GCL signal set consists of all $\{c_{y,z}(i)\}$ for $\text{GCD}(y, n) = 1$. When $n = p^2$ where p is a prime, a GCL signal set is a $(p^2, p - 1, 0, p)$ signal set.

Note that their respective auto/cross ambiguity functions and the DFT of MO and GCL sequences are not reported in the literature. A more recent work [5] using the Zak transform showed that the above perfect sequences, i.e., FZC, MO and GCL sequences, can be considered as subsets of the sequences constructed from the Zak transform for some special parameters.

6. Power residue sequences [38, 27, 35]: Let k be a proper factor of $p - 1$. A power residue sequence $\{\varphi_x(i)\}$ of period p is defined as

$$\varphi_x(i) = \omega_k^{x \cdot \log_a i}, i = 0, 1, \dots, \quad (21)$$

where $0 < x < k$. A power residue sequence is a polyphase sequence with period p and k different phases, which is represented by multiplicative characters. A k -ary *power residue sequence* of period p has the out-of-phase autocorrelation magnitude of at most 3, which is also studied in [15]. Moreover, it is shown in [23] that the magnitude of the cross-correlation of distinct k -ary power residue sequences of period p is bounded by $\sqrt{p} + 2$. Thus, the set consisting of the power residue sequences defined by (21) for all $x : 0 < x < k$ is a signal set with parameters $(p, k - 1, 3, \sqrt{p} + 2)$ where k can be up to $k = p - 1$. When $k = p - 1$, it can be seen that this is a subset of Ω , the new expression of the sequences from the Weil representation. Thus the ambiguity and the DFT are bounded with the same values as for Ω . Furthermore, this signal set can be enlarged using the shift-and-add operators. For details, see a recent paper [45].

7. For the new construction Ω , there are $p^2(p - 2)$ time-shift distinct sequences, and the elements in every sequence have the expression $\varphi_{x,y,z}(i) = \omega_{p-1}^{x \cdot \log_a i} \cdot \omega_p^{yi^2 + zi}$ (note $\theta = \omega_{p-1}$ and $\eta = \omega_p$ in the previous notation for the new construction). The magnitude of auto and cross correlation of sequences in the set are upper bounded by $2\sqrt{p}$ and $4\sqrt{p}$, respectively, and the magnitude of the DFT of these sequences is upper bounded by 2. The subset Ω_0 where $z = 0$ is an ambiguity signal set with parameters $(p, p(p - 2), 2\sqrt{p}, 4\sqrt{p})$. However, there is a possible drawback of those sequences in practice. The alphabet for a sequence of length p grows roughly as $O(p^2)$.

We summarize the above discussions in Table 1. We use the notation $\eta = \omega_p$ as we used in the previous sections except for the case of GCL where we use ω_{p^2} .

B. Signal Sets with Sizes in the Order of p^3 and Low Correlation

Signal sets with family size in the order of p^3 , and with low correlation are known in the literature and are shown in Table 2. The bounds of auto and cross correlation function for construction Ω are better than or as good as the sequences in [3], \mathbb{Z}_4 sequences $S(2)$ [25], and the sequences in [44], while the maximum magnitudes of DFT are only known for Ω , and \mathbb{Z}_4 sequences $S(2)$.

C. Implementation Cost

Note that the i th element of a sequence in Ω is a product of the i th element of a $(p - 1)$ -ary power residue sequence of period p and the i th element of an FZC sequence of period p . Thus, the implementation cost of construction Ω is equal to the sum of the cost of those two types of sequences. Since both power residue sequences and FZC sequences can be implemented efficiently at both hardware

Table 1: The Comparison with Well-known Complex Valued Sequences

Family	i th element	Period L	Size	Ambiguity and DFT
FZC ⁽¹⁾ [10] [6] [11]	$\varphi_y(i) = \eta^{yi^2}$ ($0 \leq y \leq p-1$)	p	p	$ AA : p.$ $ CA \leq \sqrt{p}.$ $ DFT \leq 1.$
Alltop cubic [1]	$\varphi_y(i) = \eta^{i^3+yi^2}$ ($0 \leq y \leq p-1$)	p	p	$ AA : p.$ $ CA : p.$ $ DFT \leq 2.$
Sequences from Heisenberg representation [21]	$\varphi_y(i) = \eta^{yi^2+zi}$ ($0 \leq y \leq p-1$)	p	p	$ AA : p.$ $ CA \leq \sqrt{p}.$ $ DFT \leq 1.$
MO ⁽¹⁾ [39]	$c_z(ip+k) = \eta^{zik}b(k)$ ($1 \leq z \leq p-1$)	p^2	$p-1$	AA, CA, DFT are unknown.
GCL ⁽¹⁾ [32]	$c_y(i) = \omega_{p^2}^{yi^2+zi}b(i)$ ($1 \leq y \leq p-1$)	p^2	$p-1$	AA, CA, DFT are unknown.
Power residue sequences [38][23]	$\varphi_x(i) = \theta^{x \cdot \log_a i}$ ($1 \leq x \leq p-2$)	p	$p-2$	The same as Ω_0 .
Sequences from Weil representation Ω_0 (this paper)	$\varphi_{x,y,z}(i) = \theta^{x \cdot \log_a i} \eta^{yi^2}$ ($1 \leq x \leq p-2,$ $0 \leq y \leq p-1$)	p	$p(p-2)$	$ AA \leq 2\sqrt{p}.$ $ CA \leq 4\sqrt{p}.$ $ DFT \leq 2$

- AA =Auto ambiguity, CA = Cross ambiguity.

- ⁽¹⁾ Those are perfect sequences.

and software level, so do the sequences in Ω . Furthermore, the new expression of Weil representation sequences provides a trade-off among the alphabet size and good ambiguity.

7 Concluding Remarks and An Open Problem

We have discovered a simple elementary representation of the sequences in the finite oscillator system from the Weil representation, introduced by Gurevich, Hadani, and Sochen. From this, we have shown a construction Ω of families of complex valued sequences of period p having low valued correlation functions. This construction produces a signal set with $p^2(p-2)$ shift distinct sequences. The magnitude of the auto and cross correlation functions are upper bounded by $2\sqrt{p}$ and $4\sqrt{p}$, respectively. The DFT of every sequence in the signal set is upper bounded by 2. The signal set Ω_0 , a subset of Ω , possesses all the properties of Ω as well as the magnitude of their auto and cross ambiguity functions are bounded

Table 2: The Comparison with Sequences with Low Correlation

Family	Period L	Size	Correlation	DFT	Ambiguity
Blake and Mark [3] ⁽²⁾	$p - 1$	$(L + 1)^3$	$4\sqrt{L+1} + 1$	N	N
\mathbb{Z}_4 sequences $S(2)$ [25]	$2^k - 1$	$L^3 + 4L^2 + 5L + 2$	$4\sqrt{L+1} + 1$	5 [31]	N
Yu and Gong [44]	$2^k - 1$	$(L + 1)^3$	$2^{2.5}\sqrt{L+1}$	N	N
Ω	p	$L^2(L - 2)$	$2\sqrt{L}, 4\sqrt{L}$	2	Not good
Ω_0	p	$L(L - 2)$	$2\sqrt{L}, 4\sqrt{L}$	2	$ AA \leq 2\sqrt{L}, CA \leq 4\sqrt{L}$

- ⁽²⁾ This family can be easily extended to sequences over the finite field \mathbb{F}_p with period $p^n - 1$ and the same correlation property from the work in [29].
- AA =Auto ambiguity, CA = Cross ambiguity.
- N: no reported results in the literature.

by $2\sqrt{p}$ and $4\sqrt{p}$. However, there is a drawback of this construction in practice, since the alphabet for a sequence of length p grows roughly as $O(p^2)$.

If we look at the construction Ω again, we find that each sequence $\varphi_n = \{\varphi_n(i)\}_{i \geq 0}$ is the term-by-term product of sequences $\{\theta^{x \log_\alpha i}\}_{i \geq 0}$ and $\{\eta^{y i^2 + z i}\}_{i \geq 0}$ which are related to power residue sequences and FZC sequences, respectively. Going back to the literature, all the known constructions only involve one type of character from finite field \mathbb{F}_p , While here we use both multiplicative and additive characters of finite field \mathbb{F}_p . The proof of those results requires very deep mathematics, i.e., the representation theory and l -adic algebraic geometry. This suggests that it is worth looking for a direct proof for the construction, which will have a two-fold effect. One is for better promotion of those sequences in practice without introducing the Weil representation theory. The other is that it may lead to more discoveries of new signal sets with good auto and cross ambiguity functions as well as with low magnitude of the DFT spectrum.

Open Problem. For $\Omega = \{\varphi_n \mid 0 \leq n \leq p^2(p - 2)\}$, directly show that Ω is a $(p, p^2(p - 2), 2\sqrt{p}, 4\sqrt{p})$ signal set and that the DFT of every sequence is upper bounded by 2 without introducing Weil representation and finite oscillator system.

Acknowledgment

The authors would like to thank Grevich, Hadani and Sochen for their help during the course of conducting this work. The authors are deeply grateful to the Associate Editor and reviewers for their many valuable and helpful suggestions which greatly improved the presentation of the work.

Appendix

The Heisenberg Representation

Let $V = \mathbb{F}_p^2$ be a two-dimensional vector space over the finite field \mathbb{F}_p . Then (V, ω) is symplectic if the symplectic form ω is given by

$$\omega((t_1, w_1), (t_2, w_2)) = t_1 w_2 - t_2 w_1,$$

for $(t_i, w_i) \in V$, $i = 1, 2$.

Considering V as an Abelian group, it admits a non-trivial central extension called the *Heisenberg group* H ($p \neq 2$). The group H can be presented as $H = V \times F_p$ with the multiplication given by

$$(t_1, w_1, z_1) \cdot (t_2, w_2, z_2) = (t_1 + t_2, w_1 + w_2, z_1 + z_2 + 2^{-1}\omega((t_1, w_1), (t_2, w_2))).$$

It is easy to verify that the center of H is $Z = Z(H) = \{(0, 0, z) : z \in \mathbb{F}_p\}$.

Theorem 4 (*Stone-Von Neuman*) *Up to isomorphism, there exists a unique irreducible unitary representation $\pi : H \rightarrow U(\mathcal{H})$ with central character ϕ , that is, $\pi|_Z = \phi \cdot Id_{\mathcal{H}}$.*

The representation π in the above theorem is called the *Heisenberg representation*. In this paper, we take a character of Z as $\phi((0, 0, z)) = \eta^z$. Then the unique irreducible unitary representation π corresponding to ϕ has the following formula

$$\pi(t, w, z)[\varphi](i) = \eta^{2^{-1}tw + z + wi} \varphi(i + t) \tag{22}$$

for $\varphi \in \mathbb{C}(\mathbb{F}_p)$, $(t, w, z) \in H$. Consequently, we have

$$\pi(t, 0, 0)[\varphi](i) = \varphi(i + t)$$

$$\pi(0, w, 0)[\varphi](i) = \eta^{wi} \varphi(i)$$

$$\pi(0, 0, z)[\varphi](i) = \eta^z \varphi(i).$$

Thus $\pi(t, 0, 0), \pi(0, w, 0)$ are equal to the unitary operators time-shift L_t and phase-shift M_w , respectively, defined in (1).

The Weil Representation

The symplectic group $Sp = Sp(V, \omega)$, which is isomorphic to $SL_2(\mathbb{F}_p)$, acts by automorphism of H through its action on the V -coordinate, i.e., $\forall (t, w, z) \in H$ and a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_p)$, the action g on (t, w, z) as

$$g \cdot (t, w, z) = (at + bw, ct + dw, z). \tag{23}$$

Due to Weil [42], a projective unitary representation $\tilde{\rho} : Sp \rightarrow PGL(\mathcal{H})$ is constructed as follows. Considering the Heisenberg representation $\pi : H \rightarrow U(\mathcal{H})$ and $g \in Sp$, a new representation is defined as: $\pi^g : H \rightarrow U(\mathcal{H})$ by $\pi^g(h) = \pi(g(h))$. Because both π and π^g have the same central character ϕ , they are isomorphic by Theorem 4. By Schur's Lemma [37], $Hom_H(\pi, \pi^g) \cong \mathbb{C}^*$, so there exist a projective representation $\tilde{\rho} : Sp \rightarrow PGL(\mathcal{H})$. This projective representation $\tilde{\rho}$ is characterized by the formula:

$$\tilde{\rho}(g)\pi(h)\tilde{\rho}(g^{-1}) = \pi(g(h)) \quad (24)$$

for every $g \in Sp$ and $h \in H$. Moreover, $\tilde{\rho}(g)$ uniquely lifts to a unitary representation

$$\rho : Sp \rightarrow U(\mathcal{H})$$

that satisfies equation (24). The existence of ρ follows from the fact [2] that any projective representation of $SL_2(\mathbb{F}_p)$ can be lifted to an honest representation, while the uniqueness of ρ follows from the fact [19] that the group $SL_2(\mathbb{F}_p)$ has no non-trivial characters for $p \neq 3$.

Thus the Weil representation, specified in Section 4.1, follows.

Notion of an Unitary Representation

Let \mathcal{H} be a Hilbert space. A unitary operator on \mathcal{H} is an operator $A : \mathcal{H} \rightarrow \mathcal{H}$ which preserves the inner product, that is, $\langle A\varphi, A\psi \rangle = \langle \varphi, \psi \rangle$ for every $\varphi, \psi \in \mathcal{H}$. The set of unitary operators forms a group under composition of operators, which is denoted by $U(\mathcal{H})$.

Definition 3 A unitary representation of a group G on the Hilbert space \mathcal{H} is a homomorphism $\pi : G \rightarrow U(\mathcal{H})$, i.e., π is map which satisfies the condition

$$\pi(g \cdot h) = \pi(g) \cdot \pi(h), \quad \forall g, h \in G.$$

Definition 4 A unitary representation $\pi : G \rightarrow U(\mathcal{H})$ is called irreducible if there is no proper subspace $\mathcal{H}' \subset \mathcal{H}$ invariant under G , i.e., such that

$$\pi(g)\varphi \in \mathcal{H}', \quad \forall \varphi \in \mathcal{H}'.$$

All unitary representations $\pi : G \rightarrow U(\mathcal{H})$ can be decomposed into a direct sum of irreducible unitary representations. In other words, there exists is a decomposition of the Hilbert space \mathcal{H} into a direct sum

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i,$$

such that each subspace \mathcal{H}_i is closed under the action of G , that is $\pi(g)\varphi \in \mathcal{H}_i, \forall \varphi \in \mathcal{H}_i$, and such that the restricted unitary representations $\pi_i : G \rightarrow U(\mathcal{H}_i)$ are irreducible.

A unitary operator $A : \mathcal{H} \rightarrow \mathcal{H}$ can be diagonalized, which means that there exists an orthogonal decomposition of \mathcal{H} into a direct sum of eigenspaces

$$\mathcal{H} = \bigoplus_{\lambda_i} \mathcal{H}_{\lambda_i},$$

where $\dim(\mathcal{H}_{\lambda_i}) = 1$, and $\forall \varphi \in \mathcal{H}_{\lambda_i}, A\varphi = \lambda_i\varphi$.

Now we consider a unitary representation $\pi : G \rightarrow U(\mathcal{H})$ of a commutative group G , which yields a commutative group $\{\pi(g) : g \in G\}$ of unitary operators. Then the unitary operators $\{\pi(g) : g \in G\}$ can be diagonalized simultaneously, i.e., there exists an orthogonal decomposition of \mathcal{H} into common eigenspaces

$$\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_{\chi_i}.$$

Here the eigenspaces are indexed by the characters $\chi_i : G \rightarrow S^1$ where we have $\pi(g)\varphi = \chi_i(g)\varphi$ for every $g \in G, \varphi \in \mathcal{H}_{\chi_i}$.

Then we achieve the decomposition of the Weil representation associated with maximal tori in Section 4.2.

PAPR and Discrete Fourier Transform

The following theorem gives a relationship among the continuous Fourier transform, discrete Fourier transform and PAPR.

Theorem 5 ([28]) *Let φ be a sequence with period n , and define continuous Fourier transform of φ as $S_\varphi(z) = \sum_{i=0}^{n-1} \varphi(i)z^i$, then*

$$\max_{|z|=1} |S_\varphi(z)| \leq \left(\frac{2}{\pi} \ln N + 2\right) \max_{0 \leq i \leq n-1} |F\varphi(i)|.$$

Thus

$$PAPR(\varphi) \leq \left(\frac{2}{\pi} \ln N + 2\right) \max_{0 \leq i \leq n-1} |F\varphi(i)|.$$

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