Fast Algebraic Attacks and Decomposition of Symmetric Boolean Functions *

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Abstract Algebraic and fast algebraic attacks are power tools to analyze stream ciphers. A class of symmetric Boolean functions with maximum algebraic immunity were found vulnerable to fast algebraic attacks at EU-ROCRYPT'06. Recently, the notion of \mathcal{AAR} (algebraic attack resistant) functions was introduced as a unified measure of protection against both classical algebraic and fast algebraic attacks. In this correspondence, we first give a decomposition of symmetric Boolean functions, then we show that almost all symmetric Boolean functions, including these functions with good algebraic immunity, behave badly against fast algebraic attacks, and we also prove that no symmetric Boolean functions are \mathcal{AAR} functions. Besides, we improve the relations between algebraic degree and algebraic immunity of symmetric Boolean functions.

Key Words stream cipher, symmetric Boolean function, algebraic attacks, algebraic immunity, algebraic degree.

1 Introduction

Boolean functions are frequently used in the design of stream ciphers, block ciphers and hash functions. One of the most vital roles in cryptography of Boolean functions is to be used as filter and combination generators of stream ciphers based on linear feedback shift registers (LFSRs). Symmetric functions are an interesting subclass of Boolean functions for their advantage in both implementation complexity and storage space (see [5]).

In recent years, algebraic and fast algebraic attacks [7, 15, 6, 1] have been regarded as a great threat against LFSR-based stream ciphers. These attacks use cleverly overdefined systems of multi-variable nonlinear equations to recover the secret key. Algebraic

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attacks lower the degree of the equations by multiplying a nonzero function while fast algebraic attacks by linear combination. Thus algebraic immunity (AI) was introduced in [15] to measure the ability of Boolean functions to resist algebraic attacks while the notion of \mathcal{AAR} (algebraic attack resistant) functions in [11] as a unified measure of protection against both classical algebraic and fast algebraic attacks.

The maximum algebraic immunity (MAI) of *n*-variable Boolean functions is $\lceil \frac{n}{2} \rceil$ [7]. The majority function achieves MAI [8, 3]. For odd *n*, the majority function is the only symmetric MAI functions, up to addition of a constant [12, 18]. However, the majority function was found vulnerable to fast algebraic attacks in [2] at EUROCRYPT'06.

All the symmetric MAI functions on 2^m variables were obtained in [17, 14] and were proven having algebraic degree 2^{m-1} or 2^m . Moreover, all the symmetric functions on $2^m + 1$ variables with sub-MAI 2^{m-1} were derived in [13]. A general method to construct symmetric MAI functions was further provided in [16]. Nevertheless, we find all these functions but few very vulnerable to fast algebraic attacks despite their resistance against classical algebraic attacks.

A preprocessing of fast algebraic attacks on LFSR-based stream ciphers, which use a Boolean function f as the filter or combination generator, is to find a function g of small degree such that the multiple gf has degree not too large. For any pair of integers (e, d)such that $e + d \ge n$, there is a nonzero function g of degree at most e such that gf has degree at most d [6]. We concentrate on the minimum of e + d and introduce the notion of fast algebraic immunity (FAI), which generalizes the notion of \mathcal{AAR} . The full fast algebraic immunity (FFAI) of Boolean functions is n and FFAI functions are equivalent to \mathcal{AAR} functions.

In this correspondence, the fast algebraic immunity of symmetric Boolean functions is studied. It's found that any symmetric function is a composition of a Boolean function and elementary symmetric functions with degree equal to a power of 2 and that the set of all symmetric functions with degree at most $2^k - 1$ is a ring generated by $\sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^{k-1}}$ and isomorphic to \mathbf{B}_k . It's further shown that almost all symmetric Boolean functions behave badly against fast algebraic attacks. For one thing, any symmetric function with degree not equal to a power of 2 has FAI strictly less than its degree, for another, in the case n close to $2^{\lfloor \log_2 n \rfloor}$, symmetric functions with AI at least $2^{\lfloor \log_2 n \rfloor}/2$ have FAI close to n/2, which is almost the worst case against fast algebraic attacks. Unfortunately, all but few symmetric functions shown to be immune to classical algebraic attacks in the previous literatures, such as [17, 14, 13, 16], fall into the case that n is either equal to or a little more than $2^{\lfloor \log_2 n \rfloor}$. One (or more) function g with small degree, such that gfhas degree not large, is straightway derived from the SANFV of f, while the algorithm proposed in [2] at EUROCRYPT'06 to determine g and gf for a symmetric function fhas complexity $\mathcal{O}(n^3)$. Furthermore, it's proven that there exist no symmetric FFAI (i.e. \mathcal{AAR}) functions. Lastly, the relations between algebraic degree and algebraic immunity of symmetric Boolean functions are improved. $2^{\lfloor \log_2(2a-1) \rfloor}$ is the lower degree of symmetric functions with AI a. This bound is tight for symmetric MAI functions.

The remainder of this correspondence is organized as follows. In Section 2, some basic concepts are provided and the notion of fast algebraic immunity is introduced, while Section 3 presents the decomposition of symmetric functions. Section 4 studies the fast algebraic immunity of symmetric functions and Section 5 discusses the relations between algebraic degree and algebraic immunity of symmetric functions. Section 6 concludes the correspondence.

2 Preliminary

An *n*-variable Boolean function is a mapping from \mathbb{F}_2^n into \mathbb{F}_2 , where \mathbb{F}_2 denote the binary field. A Boolean function is said to be symmetric if its output is invariant under any permutation of its input bits. Denote by \mathbf{B}_n (resp. \mathbf{SB}_n) the set of all Boolean functions (resp. symmetric Boolean functions) on *n* variables. Any function $f \in \mathbf{B}_n$ can be uniquely represented as a truth table

$$f = [f(0, 0, \dots, 0), f(1, 0, \dots, 0), \dots, f(1, 1, \dots, 1)] \in \mathbb{F}_2^{2^n},$$

or as a multivariate polynomial over \mathbb{F}_2 , called the algebraic normal form (ANF),

$$f(x) = \sum_{c = (c_1, c_2, \cdots, c_n) \in \mathbb{F}_2^n} a_c x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}, \ a_c \in \mathbb{F}_2.$$

The algebraic degree of f, denoted by deg(f), is given by $\max_{a_c \neq 0} wt(c)$, where wt(c) denote the Hamming weight of c. Any function $f \in \mathbf{SB}_n$ can be uniquely represented as a vector

$$v_f = (v_f(0), v_f(1), \cdots, v_f(n)) \in \mathbb{F}_2^{n+1},$$

where $v_f(i)$ represents the function value for vectors of weight *i*. Let σ_i be the *i*-th elementary symmetric function of x_1, x_2, \dots, x_n . The symmetric function *f* can also be uniquely represented as

$$f(x) = \sum_{i=0}^{n} \lambda_f(i)\sigma_i, \ \lambda_f(i) \in \mathbb{F}_2.$$

The vector $\lambda_f = (\lambda_f(0), \lambda_f(1), \dots, \lambda_f(n))$ is called the simplified algebraic normal form vector (SANFV) of f. More properties of symmetric Boolean functions can be found in [5].

The algebraic immunity of Boolean functions is defined as follows.

Definition 1. [15] Let f be an n-variable Boolean function. The algebraic immunity (AI) of f, denoted by $\mathcal{AI}(f)$, is defined as

$$\mathcal{AI}(f) = \min_{g \neq 0} \{ \deg(g) | gf = 0 \text{ or } g(f+1) = 0 \}.$$

To resist fast algebraic attacks, the Boolean function f shouldn't admit a function g of small degree such that the multiple gf has degree not too large. There are several notions of the immunity of Boolean functions against fast algebraic attacks in previous literatures, such as [10], but they separately treat the two parameters deg(g) and deg(gf). Recently, the notion of \mathcal{AAR} (algebraic attack resistant) functions was introduced in [11] as a unified measure of protection against classical algebraic attacks as well as fast algebraic attacks. **Definition 2.** [11] Let f be an n-variable Boolean function. The function f is called \mathcal{AAR} if f has MAI and $\deg(g) + \deg(gf) \ge n$ for any function g with $1 \le \deg(g) < n/2$.

However, \mathcal{AAR} is too restrictive to achieve. A 9-variable \mathcal{AAR} function was observed in [4]. A class of almost \mathcal{AAR} functions were constructed in [11] by iteration. While it's still unknown whether there are \mathcal{AAR} functions for any n.

For any Boolean function f, from the definition of AI there always exists a function g of degree equal to $\mathcal{AI}(f)$ such that gf = 0 or gf = g. Therefore the minimum of deg(g) + deg(gf) is smaller than or equal to $2\mathcal{AI}(f)$. The notion of fast algebraic immunity is introduced as follows.

Definition 3. Let f be an n-variable Boolean function. The fast algebraic immunity *(FAI)* of the function f, denoted by $\mathcal{FAI}(f)$, is defined as

$$\mathcal{FAI}(f) = \min_{g:1 \le \deg(g) < \mathcal{AI}(f)} \{ 2\mathcal{AI}(f), \deg(g) + \deg(gf) \}.$$

From the above definition, we know that $\mathcal{AI}(f) + 1 \leq \mathcal{FAI}(f) \leq \deg(f) + 2$ since $\mathcal{AI}(f) + 1 \leq \deg(g) + \deg(gf)$ for any nonconstant function g with degree less than $\mathcal{AI}(f)$ and $\deg(l) + \deg(lf) \leq \deg(f) + 2$ for any affine function l.

For any pair of integers (e, d) such that $e + d \ge n$, there is a nonzero function g of degree at most e such that gf has degree at most d [6]. Hence, the full fast algebraic immunity (FFAI) of *n*-variable Boolean functions is n. It is clear that any Boolean function has AI greater than or equal to a half of its FAI. Therefore FFAI functions are also MAI functions and are equivalent to \mathcal{AAR} functions. By almost FFAI or almost \mathcal{AAR} functions we mean Boolean functions with FAI n - 1.

3 Decomposition of symmetric Boolean functions

Thereinafter, $\binom{k}{i}$ may be regarded as $\binom{k}{i}$ mod $2 \in \mathbb{F}_2$ if there is no ambiguousness.

Lemma 1. Let σ_i and σ_j be *i*-th and *j*-th elementary symmetric Boolean functions on n variables, $0 \le i, j \le n$. Then we have

$$\sigma_i \sigma_j = \sum_{k=j}^{j+i} \binom{k}{i} \binom{i}{k-j} \sigma_k = \sum_{k=0}^n \binom{k}{i} \binom{i}{k-j} \sigma_k.$$

In particular, $\sigma_j^2 = \sigma_j$.

Proof. Expanding the product $\sigma_i \sigma_j$ gives $\binom{n}{i} \binom{n-i}{k-j} \binom{i}{k-j}$ monomials with degree k for $j \leq k \leq j+i$ and $0 \leq k \leq n$. Since the product $\sigma_i \sigma_j$ is also a symmetric function and σ_k consists of $\binom{n}{k}$ monomials with degree k, the coefficient of σ_k in $\sigma_i \sigma_j$ equals to $\binom{n}{i}\binom{n-i}{k-i}\binom{i}{k-j}/\binom{n}{k} = \binom{k}{i}\binom{i}{k-j}$. Since $f^2 = f$ for any Boolean function f, we also have $\sigma_j^2 = \sigma_j$.

Corollary 2. 1. $\sigma_{2^{s_1}+\cdots+2^{s_k}} = \sigma_{2^{s_1}}\sigma_{2^{s_2}}\cdots\sigma_{2^{s_k}}$ for pairwise different s_1, s_2, \cdots, s_k .

2. If $t \geq 1$ and $j < 2^s$ then $\sigma_{t \cdot 2^s + j} = \sigma_{t \cdot 2^s} \sigma_j$.

Proof. 1) Consider the polynomial $(1 + x)^{2^s} \in \mathbb{F}_2[x]$. Since $1 + x^{2^s} = (1 + x)^{2^s} = \sum_{k=0}^{2^s} {2^s \choose k} x^k$, we have ${2^s \choose k} = 1$ if and only if $k = 0, 2^s$. By Lemma 1 we have

$$\sigma_{2^s}\sigma_j = \sum_{k=j}^{2^s+j} \binom{k}{2^s} \binom{2^s}{k-j} \sigma_k = \binom{j}{2^s} \sigma_j + \binom{2^s+j}{2^s} \sigma_{2^s+j}.$$
 (1)

If $j < 2^s$, then $\binom{j}{2^s} = 0$ and $\binom{2^s+j}{2^s} = 1$ (considering the polynomial $(1+x)^{2^s+j} \in \mathbb{F}_2[x]$), and by Eq.(1) we have $\sigma_{2^s}\sigma_j = \sigma_{2^s+j}$. Assuming $s_1 < s_2 < \cdots < s_k$ without loss of generality, since $2^{s_{i-1}} + \cdots + 2^{s_1} < 2^{s_i}$ for $2 \le i \le k$, we have $\sigma_{2^{s_i}}\sigma_{2^{s_{i-1}}+\cdots+2^{s_1}} = \sigma_{2^{s_i}+\cdots+2^{s_1}}$ and therefore $\sigma_{2^{s_k}+\cdots+2^{s_1}} = \sigma_{2^{s_k}}\sigma_{2^{s_{k-1}}+\cdots+2^{s_1}} = \cdots = \sigma_{2^{s_k}}\sigma_{2^{s_{k-1}}}\cdots\sigma_{2^{s_1}}$.

2) Let $t \cdot 2^{s} = 2^{s} + 2^{s_{1}} + \dots + 2^{s_{k}}$, $s < s_{1} < \dots < s_{k}$. From 1) we know $\sigma_{t \cdot 2^{s}} = \sigma_{2^{s_{k}}} \cdots \sigma_{2^{s_{1}}} \sigma_{2^{s}}$, and hence $\sigma_{t \cdot 2^{s}} \sigma_{j} = \sigma_{2^{s_{k}}} \cdots \sigma_{2^{s_{1}}} \sigma_{2^{s}} \sigma_{j} = \sigma_{2^{s_{k}}} \cdots \sigma_{2^{s_{1}}} \sigma_{2^{s+j}} = \dots = \sigma_{2^{s_{k}+\dots+2^{s}+j}} = \sigma_{t \cdot 2^{s+j}}$.

From the above corollary we obtain the following results proven in [3].

Corollary 3. Let
$$m = \lfloor \log_2 n \rfloor$$
 and $j = \sum_{k=0}^m j_k 2^k$, $j_k \in \{0, 1\}$. Then we have

- 1. $\sigma_j = \sigma_1^{j_0} \sigma_2^{j_1} \sigma_4^{j_2} \cdots \sigma_{2^m}^{j_m}$.
- 2. If $n 2^m < j < 2^m$ then $\sigma_{2^m} \sigma_j = 0$.
- 3. $\sigma_i \sigma_j = \sigma_{i \lor j}$ where \lor means OR operation.

Proof. 1) It is affirmed by Corollary 2(1).

2) It is derived from the fact that $\sigma_{2^m}\sigma_j = \sigma_{2^m+j}$ for $j < 2^m$ by Corollary 2 (2) and there is no σ_{2^m+j} for $2^m + j > n$.

3) It can be deduced from 1) since $\sigma_{2^s}^2 = \sigma_{2^s}$.

Thanks to the work above, the decomposition of symmetric Boolean functions is given as follows.

Theorem 4 (Decomposition of symmetric Boolean functions). Let $f \in SB_n$ and $m = \lfloor \log_2 n \rfloor$.

1. The symmetric function f is a composition of an (m+1)-variable Boolean function F_{m+1} and elementary symmetric functions $\sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^m}$:

$$f(x) = F_{m+1}(\sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^m}).$$

In particular, if f has degree at most $2^k - 1$, then $f(x) = F_k(\sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^{k-1}})$, $F_k \in \mathbf{B}_k$.

2. Furthermore we have

$$f(x) = \sum_{i=k}^{m} \sigma_{2^{i}} f_{i}(x) + f_{k}^{-}(x),$$

where f_i $(k \le i \le m)$ and f_k^- are symmetric functions of degree at most $2^i - 1$ and $2^k - 1$.

Proof. 1) Let $f(x) = \sum_{j=0}^{n} \lambda_f(j) \sigma_j$, $\lambda_f(j) \in \mathbb{F}_2$. By Corollary 3, we have $\sigma_j = \sigma_1^{j_0} \sigma_2^{j_1} \sigma_4^{j_2} \cdots \sigma_{2^m}^{j_m}$ and hence

$$f(x) = \sum_{j=0}^{n} \lambda_f(j) \sigma_j = \sum_{j=0}^{n} \lambda_f(j) \sigma_1^{j_0} \sigma_2^{j_1} \sigma_4^{j_2} \cdots \sigma_{2^m}^{j_m}.$$

Let $F_{m+1} \in \mathbf{B}_{m+1}$ and

$$F_{m+1}(y_1, y_2, \cdots, y_{m+1}) = \sum_{0 \le j \le n} \lambda_f(j) y_1^{j_0} y_2^{j_1} \cdots y_{m+1}^{j_m}$$

Then

$$f(x) = F_{m+1}(\sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^m}).$$

The same proof shows $f(x) = F_k(\sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^{k-1}})$ with $F_k \in \mathbf{B}_k$ when $\deg(f) \le 2^k - 1$.

2) Since F_{m+1} is a Boolean function, we can write $F_{m+1}(y_1, \dots, y_{m+1}) = \sum_{i=k}^m y_{i+1}F_i(y_1, \dots, y_i) + F_k^-(y_1, \dots, y_k), k \ge 1$. Therefore

$$f(x) = F_{m+1}(\sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^m})$$

= $\sum_{i=k}^m \sigma_{2^i} F_i(\sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^{i-1}})$
+ $F_k^-(\sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^{k-1}}).$

Let $f_i(x) = F_i(\sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^{i-1}})$ $(k \leq i \leq m)$ and $f_k^-(x) = F_k^-(\sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^{k-1}})$. The symmetric function f_i has degree at most $2^i - 1$ since the degree of $\sigma_1^{j_0} \sigma_2^{j_1} \sigma_4^{j_2} \cdots \sigma_{2^{i-1}}^{j_{i-1}}$ can not exceed $1 + 2 + 4 + \cdots + 2^{i-1} = 2^i - 1$. Similarly, f_k^- has degree at most $2^k - 1$.

Note that $\lceil \log_2(n+1) \rceil = \lfloor \log_2 n \rfloor + 1$. Theorem 4 shows that an *n*-variable symmetric Boolean function corresponds to a $\lceil \log_2(n+1) \rceil$ -variable Boolean function. Furthermore, a symmetric Boolean function of degree at most $2^k - 1$ corresponds to a *k*-variable Boolean function.

Theorem 5. Let $1 \leq 2^k \leq n+1$. Then the set of all functions in \mathbf{SB}_n with degree at most $2^k - 1$, denoted by $\mathbf{SB}_n^{2^k-1}$, is the ring $\langle \sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^{k-1}} \rangle$ and isomorphic to \mathbf{B}_k .

Proof. From Theorem 4 we know $\mathbf{SB}_n^{2^{k-1}}$ is contained in $R = \langle \sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^{k-1}} \rangle$. We just check that R is an isomorphism of \mathbf{B}_k since $|\mathbf{B}_k| = |\mathbf{SB}_n^{2^{k-1}}|$. Let

$$\tau: \mathbf{B}_k \to R, \ f(y_1, y_2, \cdots, y_k) \mapsto f(\sigma_1, \sigma_2, \sigma_4, \cdots, \sigma_{2^{k-1}}).$$

From Theorem 4 we know τ is a injection. By Lemma 1 we know $\sigma_{2s}^2 = \sigma_{2s}$, and hence τ is a surjection. Let $f, g \in \mathbf{B}_k$. It's clear that $\tau(f+g) = \tau(f) + \tau(g)$. Since $\tau(y_s) = \sigma_{2s-1}$ for $1 \leq s \leq k$, we have $\tau(\prod_{s=1}^k y_s^{c_s}) = \prod_{s=1}^k \sigma_{2s-1}^{c_s} = \prod_{s=1}^k \tau^{c_s}(y_s)$ and therefore $\tau(fg) = \tau(f)\tau(g)$.

Corollary 6. SB_{2^m-1} is an isomorphism of B_m .

The upper degree of the product of two symmetric Boolean functions is given as below.

Corollary 7. Let $g, f \in \mathbf{SB}_n$, $\deg(g) \leq 2^k - 1$ and $\deg(f) \leq t \cdot 2^k - 1$ $(t \geq 1)$. Then $\deg(gf) \leq t \cdot 2^k - 1$.

Proof. It holds for t = 1 since $\mathbf{SB}_n^{2^k - 1}$ is a ring by Theorem 5. Consider the product $\sigma_i \sigma_j$ with $i \leq 2^k - 1$ and $2^k \leq j \leq t \cdot 2^k - 1$ when t > 1. Let $j = t' \cdot 2^k + j'$, $1 \leq t' \leq t - 1$, $j' \leq 2^k - 1$. From Corollary 2 we have $\sigma_i \sigma_j = \sigma_i \sigma_{t' \cdot 2^k + j'} = \sigma_i \sigma_{j'} \sigma_{t' \cdot 2^k}$, which has degree at most $t' \cdot 2^k + 2^k - 1 \leq t \cdot 2^k - 1$ since $\sigma_i \sigma_{j'} \in \mathbf{SB}_n^{2^k - 1}$.

From the above corollary we know $\mathbf{SB}_n^{2^k-1}\mathbf{SB}_n^{t\cdot 2^k-1} = \mathbf{SB}_n^{t\cdot 2^k-1}$ for $t \ge 1$.

4 Fast algebraic attacks on symmetric Boolean functions

In this section, we will first show that fast algebraic attacks on symmetric Boolean functions work efficiently, and then prove the nonexistence of symmetric FFAI functions.

Theorem 8. Let $f \in \mathbf{SB}_n$, $f(x) = \sigma_{2t+1} + \sum_{i=0}^{2t} \lambda_f(i)\sigma_i$ and $g(x) = \sigma_1 + \lambda_f(2t) + 1$. Then the multiple gf has degree at most 2t - 1. Moreover, if $\lambda_f(2t) = 0$, then $(\sigma_1 + 1)f$, if nonzero, has degree 2s + 1 where s is maximum such that $\lambda_f(2s) = 1$; if $\lambda_f(2t) = 1$, then $\sigma_1 f$, if nonzero, has degree 2s + 1 where s is maximum such that $\lambda_f(2s) + \lambda_f(2s + 1) = 1$.

Proof. It's trivial for t = 0. Then assume $t \ge 1$.

By Corollary 2, we have $\sigma_1 \sigma_{2i} = \sigma_{2i+1}$ and therefore $\sigma_1 \sigma_{2i+1} = \sigma_{2i+1}$. Hence

$$h(x) = (\sigma_1 + \lambda_f(2t) + 1)f(x)$$

= $(\sigma_1 + \lambda_f(2t) + 1)(\sigma_{2t+1} + \lambda_f(2t)\sigma_{2t} + \sum_{i=0}^{2t-1} \lambda_f(i)\sigma_i)$
= $\sum_{i=0}^{t-1} [\lambda_f(2t)\lambda_f(2i+1) + \lambda_f(2i)]\sigma_{2i+1}$
+ $\sum_{i=0}^{t-1} [(\lambda_f(2t) + 1)\lambda_f(2i)]\sigma_{2i},$

showing $\deg(h) \leq 2t - 1$. If $\lambda_f(2t) = 0$, then

$$h(x) = \sum_{i=0}^{t-1} \lambda_f(2i)\sigma_{2i+1} + \sum_{i=0}^{t-1} \lambda_f(2i)\sigma_{2i}$$

and therefore $\deg(h) = 2s + 1$ when s is maximum such that $\lambda_f(2s) = 1$.

If $\lambda_f(2t) = 1$, then

$$h(x) = \sum_{i=0}^{t-1} [\lambda_f(2i+1) + \lambda_f(2i)]\sigma_{2i+1}$$

and therefore deg(h) = 2s + 1 when s is maximum such that $\lambda_f(2s) + \lambda_f(2s + 1) = 1$. \Box

Remark. If $\lambda_f(2s) = 0$ (resp. $\lambda_f(2s) = \lambda_f(2s+1)$) for any s with $0 \le s \le t$, then we have $(\sigma_1 + 1)f = 0$ (resp. $\sigma_1 f = 0$).

Theorem 8 gives an affine function g such that gf has degree at most $\deg(f) - 2$ for odd $\deg(f)$. In other words, any symmetric function with odd degree has FAI strictly smaller than its degree. Although the product gf has odd degree if $gf \neq 0$, we cannot apply the theorem recursively to gf to lower the degree of f since $g \cdot gf = g \cdot f$ and $(g+1) \cdot gf = 0 \cdot f$.

From Theorem 8, we know that 2^{-t} is the probability that gf = 0 and 2^{-i} the probability that gf has degree $\deg(f) - 2i$. Consequently, the expectation of the degree of gf is $\deg(f) - 4$ when $\deg(f)$ is large.

Corollary 9. The expectation of the degree of the product gf of Theorem 8 is $\deg(f) - 4$ when $\deg(f)$ tends to infinity.

Now we consider symmetric Boolean functions with degree not equal to a power of 2.

Theorem 10. Let $f \in \mathbf{SB}_n$ and $\deg(f) \ge 2^k > 1$. If 2^k does not divide $\deg(f)$, then there exists a nonconstant function g of degree at most e with $e = \deg(f) \mod 2^k$ such that the product gf has degree at most $\deg(f) - e - 1$.

Proof. Let deg $(f) = t \cdot 2^k + e, t \ge 1, 0 < e < 2^k$. Let $f(x) = \sigma_{t \cdot 2^k + e} + \sum_{i=0}^{t \cdot 2^k + e^{-1}} \lambda_f(i) \sigma_i$. By Corollary 2, we have $\sigma_{t \cdot 2^k + i} = \sigma_{t \cdot 2^k} \sigma_i$ for $0 \le i < 2^k$, and therefore

$$f(x) = \sigma_{t \cdot 2^k} (\sigma_e + \sum_{i=0}^{e-1} \lambda_f (t \cdot 2^k + i) \sigma_i) + \sum_{i=0}^{t \cdot 2^k - 1} \lambda_f (i) \sigma_i.$$

Let $g(x) = \sigma_e + \sum_{i=0}^{e-1} \lambda_f(t \cdot 2^k + i)\sigma_i + 1$ and $f^-(x) = \sum_{i=0}^{t \cdot 2^k - 1} \lambda_f(i)\sigma_i$. Then

$$f(x) = \sigma_{t \cdot 2^k}(g(x) + 1) + f^{-}(x),$$

and hence $gf = gf^-$. On one hand, the symmetric function g has degree e; on the other hand, by Corollary 7, the function gf^- has degree at most $t \cdot 2^k - 1 = \deg(f) - e - 1$.

The theorem not only proves the existence of the function g but also explicitly identifies several such functions. More exactly, the number of g's is 1 less than the weight of $\deg(f)$.

Taking $k = \lfloor \log_2 \deg(f) \rfloor$, if $\deg(f) \neq 2^k$ then there is a nonconstant function g such that $\deg(g) + \deg(gf) \leq \deg(f) - 1$ and therefore the following result is obtained.

Corollary 11. Let $f \in \mathbf{SB}_n$ and $\deg(f) > 1$ is not a power of 2. Then $\mathcal{FAI}(f) \leq \deg(f) - 1$.

Theorem 10 and Corollary 11 show that symmetric functions with degree not equal to a power of 2 do not behave well against fast algebraic attacks. Then we consider the symmetric functions with degree $2^{\lfloor \log_2 n \rfloor}$. For the case $n - 2^{\lfloor \log_2 n \rfloor}$ large, $2^{\lfloor \log_2 n \rfloor}$ is very small compared with n and therefore the symmetric functions with degree $2^{\lfloor \log_2 n \rfloor}$ naturally behave badly against fast algebraic attacks. For $n - 2^{\lfloor \log_2 n \rfloor}$ not too large, we will show fast algebraic attacks on the symmetric functions with any degree are also very efficient. These imply that almost all symmetric Boolean functions are vulnerable to fast algebraic attacks.

Now we consider the symmetric functions on n variables, including the functions of degree equal to a power of 2, for the case $n - 2^{\lfloor \log_2 n \rfloor}$ smaller than $2^{\lfloor \log_2 n \rfloor}/2 - 1$.

Theorem 12. Let $f \in SB_n$ and $2^m \le n < 2^m + 2^{m-1} - 1$. Then $\mathcal{AI}(f) \le 2^{m-1} - 1$ or $deg(\sigma_e f) = 2^{m-1} + e$ with $e = n - 2^m + 1$.

Proof. By Theorem 4, we have

$$f(x) = \sigma_{2^m} f_m(x) + \sigma_{2^{m-1}} f_{m-1}(x) + f_{m-1}^-(x),$$

where f_m is a symmetric function of degree at most $2^m - 1$, and f_{m-1}, f_{m-1}^- are of degree at most $2^{m-1} - 1$. Let $g = \sigma_e(f_{m-1} + 1)$. Since $n < 2^m + 2^{m-1} - 1$, we have $e = n - 2^m + 1 < 2^{m-1}$ and therefore deg $(g) \le 2^{m-1} - 1$ by Corollary 7. By Corollary 3, we have $\sigma_e \sigma_{2^m} = 0$ since $n - 2^m < e < 2^m$. If $g \ne 0$, then $gf = gf_{m-1}^-$ which is again of degree at most $2^{m-1} - 1$ by Corollary 7. This means f or f + 1 admits an annihilator of degree at most $2^{m-1} - 1$, that is, $\mathcal{AI}(f) \le 2^{m-1} - 1$. Otherwise g = 0, then $\sigma_e f_{m-1} = \sigma_e$ and hence $\sigma_e f = \sigma_{2^{m-1} + \sigma_e} + \sigma_e f_{m-1}^-$, which is of degree $2^{m-1} + e$.

Remark. The same proof shows that the theorem applies to e with $n - 2^m < e < 2^{m-1}$, but $e = n - 2^m + 1$ is minimum.

Theorem 12 shows that symmetric functions on n variables with $n - 2^{\lfloor \log_2 n \rfloor}$ not large are vulnerable to fast algebraic attacks. Especially if n is close to $2^{\lfloor \log_2 n \rfloor}$, then $e = n - 2^{\lfloor \log_2 n \rfloor} + 1$ is close to 1 and $d = 2^{\lfloor \log_2 n \rfloor}/2 + e$ is close to n/2, so e + d is close to n/2, and therefore the symmetric functions with AI at least $2^{\lfloor \log_2 n \rfloor}/2$ are very vulnerable to fast algebraic attacks. For example, any symmetric MAI function f on 2^m variables admits the linear function σ_1 such that $\sigma_1 f$ has degree $2^{m-1} + 1$ while any symmetric function f on $2^m + 1$ variables with MAI $2^{m-1} + 1$ or sub-MAI 2^{m-1} admits the quadratic function σ_2 such that $\sigma_2 f$ has degree $2^{m-1} + 2$. They are almost the worst cases against fast algebraic attacks since any function with AI a has FAI at least a + 1. Unfortunately, the symmetric MAI functions obtained in [17, 14] are in the case $n = 2^m$ and the symmetric sub-MAI functions derived in [13] have $n = 2^m + 1$. Moreover, the symmetric MAI functions constructed in [16, Theorem 2.4] have $n \in [2^m, 5/4 \cdot 2^m]$, and therefore these functions admit σ_e with $e \leq n/5$ such that $\sigma_e f$ has degree at most 3n/5. Theorem 12 also gives $g = \sigma_e(f_{m-1}+1) \neq 0$ and $h = \sigma_e(f_{m-1}+1)f_{m-1}^-$ both with degree at most $2^{m-1} - 1$ or $g = \sigma_e$ with degree e and $h = \sigma_{2^{m-1}+e} + \sigma_e f_{m-1}^-$ with degree $2^{m-1} + e$ such that gf = h. The symmetric functions with AI at most $2^{m-1} - 1 = 2^{\lfloor \log_2 n \rfloor}/2 - 1$ naturally have FAI at most $2^m - 2 = 2^{\lfloor \log_2 n \rfloor} - 2$. And these symmetric functions with AI at least $2^{m-1} = 2^{\lfloor \log_2 n \rfloor}/2$ have FAI smaller than or equal to $2^{m-1} + 2e = 2^{m-1} + 2(n-2^m+1) = 2n - 3 \cdot 2^{m-1} + 2 = 2n - 3 \cdot 2^{\lfloor \log_2 n \rfloor}/2 + 2$.

Corollary 13. Let $f \in SB_n$ and $2^m \le n < 2^m + 2^{m-1} - 1$. Then $\mathcal{FAI}(f) \le \max\{2^m - 2, 2n - 3 \cdot 2^{m-1} + 2\}$.

The following theorem proves the nonexistence of symmetric FFAI functions.

Theorem 14. Let $n \geq 5$ and $f \in SB_n$. Then $\mathcal{FAI}(f) < n$.

Proof. Corollary 11 has proven the case that f has degree not equal to 2^m . When f has degree 2^m , we just check the cases $n = 2^m$, $2^m + 1$ or $2^m + 2$. These cases have been proven in Corollary 13 if $2n - 3 \cdot 2^{m-1} + 2 < n < 2^m + 2^{m-1} - 1$, i.e. $m \ge 3$ for $n = 2^m$, $2^m + 1$ and $m \ge 4$ for $n = 2^m + 2$. The rest cases n = 5, 6, 10 are confirmed by computing all possible values of FAI for the symmetric functions on 5, 6 or 10 variables.

5 Relations between algebraic degree and algebraic immunity of symmetric Boolean functions

In this section, we will study the relations between algebraic degree and algebraic immunity of symmetric functions. It's well known that for any Boolean function f the algebraic immunity is less than or equal to its algebraic degree since f(f + 1) = 0, whereas the relations between algebraic degree and algebraic immunity can be improved for symmetric functions.

Proposition 15. Let $f \in \mathbf{SB}_n$. If f has degree not equal to a power of 2, then $\mathcal{AI}(f) < 2^{\lfloor \log_2 \deg(f) \rfloor}$. Consequently, we have $\mathcal{AI}(f) \leq 2^{\lfloor \log_2 \deg(f) \rfloor}$ for any $f \in \mathbf{SB}_n$.

Proof. Let $k = \lfloor \log_2 \deg(f) \rfloor$. By Theorem 4 we have $f = \sigma_{2^k} f_k + f_k^-$, where f_k is a symmetric function of degree $\deg(f) - 2^k$ and f_k^- of degree at most $2^k - 1$. Let $g = f_k + 1$ and $h = gf_k^-$. Then gf = h, $\deg(g) = \deg(f) - 2^k \leq 2^k - 1$ and $\deg(h) \leq 2^k - 1$. If $\deg(f) \neq 2^k$, then $g \neq 0$ and therefore $\mathcal{AI}(f) \leq 2^k - 1$. If $\deg(f) = 2^k$, then $\mathcal{AI}(f) \leq \deg(f) = 2^k$. \Box

Corollary 16. Let $f \in \mathbf{SB}_n$. Then $\deg(f) \geq 2^{\lceil \log_2 \mathcal{AI}(f) \rceil} = 2^{\lfloor \log_2(2\mathcal{AI}(f)-1) \rfloor}$.

Proof. We only check the case $\mathcal{AI}(f) > 1$. Let $a = \mathcal{AI}(f)$ and $d = \deg(f)$. Proposition 15 shows that $\log_2 a \leq \lfloor \log_2 d \rfloor$, i.e. $\lceil \log_2 a \rceil \leq \lfloor \log_2 d \rfloor$. Hence $d \geq 2^{\lceil \log_2 a \rceil} = 2^{\lfloor \log_2 (2a-1) \rfloor}$. \Box

Siegenthaler's inequality[19] states that any *m*-th order correlation-immune function has degree at most n-m and any *m*-resilient function $(0 \le m < n-1)$ has degree at most n-m-1. Therefore the order of correlation-immune (resp. resiliency) of any symmetric Boolean function with AI equal to a (a > 1) is smaller than or equal to $n - 2^{\lfloor \log_2(2a-1) \rfloor}$ (resp. $n - 2^{\lfloor \log_2(2a-1) \rfloor} - 1$).

Now we consider the lower bound of algebraic degree for symmetric MAI functions.

Corollary 17. Let $f \in \mathbf{SB}_n$ and $\mathcal{AI}(f) = \lceil \frac{n}{2} \rceil$. Then $\deg(f) \ge 2^{\lfloor \log_2(n-1) \rfloor}$. *Proof.* Since $\lceil \frac{n}{2} \rceil \ge \frac{n}{2}$, by Corollary 16, we have $\deg(f) \ge 2^{\lfloor \log_2(n-1) \rfloor}$.

For every *n*, there exist symmetric MAI functions on *n* variables of degree $2^{\lfloor \log_2 n \rfloor}$. For example, the majority function *f* achieves MAI and also has degree $2^{\lfloor \log_2 n \rfloor}$ [8]. When $n = 2^m$, the function $\sigma_{2^{m-1}}$ achieves MAI [3]. In addition, all the symmetric MAI functions on 2^m variables were obtained in [17, 14] and were proven having algebraic degree 2^{m-1} or 2^m . Notice that $2^{\lfloor \log_2(n-1) \rfloor} = 2^{\lfloor \log_2 n \rfloor}$ if $n \neq 2^m$, and $2^{\lfloor \log_2(n-1) \rfloor} = 2^{m-1}$ if $n = 2^m$. Therefore the bound of Corollary 17 is tight.

Table 1: The upper algebraic immunity of symmetric functions with designated degree

									128 - 255
Upper AI	$2^{\lfloor \log_2 d \rfloor}$	1	2	4	8	16	32	64	128

Table 2: The lower degree of symmetric functions with designated algebraic immunity

AI	a	1	2	3–4	5-8	9–16	17 - 32	33-64	65 - 128
Lower deg	$2^{\lceil \log_2 a \rceil}$	1	2	4	8	16	32	64	128

Notice that the relation between algebraic degree and algebraic immunity of symmetric Boolean functions doesn't relate to the number of variables. Therefore the bounds listed in Table 1 and Table 2 are true for any reasonable n. We leave a open problem whether the bound is tight when AI isn't MAI.

6 Conclusion

Symmetric Boolean functions, which can be considered as compositions of Boolean functions and elementary symmetric functions with power-of-2 degree, behave badly against fast algebraic attacks, so these functions are unfit to be used in stream ciphers. In other words, if symmetric functions are used in the design of ciphers, fast algebraic immunity should never be ignored, and the number n of variables had better be neither equal to nor a little more than 2^m . n approximating $3 \cdot 2^{m-1}$ seems to be a good choice but it still need further study.

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