Combinatorial constructions for optimal two-dimensional optical orthogonal codes with $\lambda = 2^{1}$

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Abstract: In this paper, we are concerned about optimal two-dimensional optical orthogonal codes with $\lambda = 2$. Some combinatorial constructions are presented and many infinite families of optimal two-dimensional optical orthogonal codes with weight 4 and $\lambda = 2$ are obtained. Especially, we shall see that in many cases an optimal two-dimensional optical orthogonal code can not achieve the Johnson bound.

Keywords: optical orthogonal code; two-dimensional; optimal; OCDMA; 3-design; s-fan design

1 Introduction

An optical orthogonal code is a family of sequences with good auto- and cross-correlation properties. Its study has been motivated by an application in an optical code-division multiple access (OCDMA) system. In a bursty traffic environment of a multiple access local area network (LAN), asynchronous multiplexing schemes are more efficient than synchronous schemes. OCDMA is such one asynchronous multiplexing scheme suitable for high speed LANs. For more information, the interested reader may refer to [32, 40, 41, 52, 53].

In an OCDMA system different users share both time and frequency, and are distinguished by using a unique spreading sequence. Each user's data is multiplied by its spreading sequence, and then all the users are coupled into the shared channel. Optical orthogonal codes can be taken as the spreading sequences used in an OCDMA system.

Let u, v, k and λ be positive integers. A two-dimensional $(u \times v, k, \lambda)$ optical orthogonal code (briefly 2-D $(u \times v, k, \lambda)$ -OOC), C, is a family of $u \times v$ (0, 1)-matrices (called codewords) of Hamming weight k satisfying: for any matrix $\mathbf{A} = (a_{ij})_{u \times v} \in C$, $\mathbf{B} = (b_{ij})_{u \times v} \in C$ and any integer r:

$$\sum_{i=0}^{u-1} \sum_{j=0}^{v-1} a_{ij} b_{i,j+r} \le \lambda,$$

where either $\mathbf{A} \neq \mathbf{B}$ or $r \neq 0$, and the arithmetic j + r is reduced modulo v. Especially, when u = 1, a two-dimensional $(1 \times v, k, \lambda)$ optical orthogonal code is said to be a *one-dimensional* (v, k, λ) -optical orthogonal code, denoted by 1-D (v, k, λ) -OOC.

1-D OOC was first suggested in 1989 [17]. Since then much work has been done on 1-D OOCs. The interested reader may refer to [1-3,7,8,10-16,18,21-23,36,37,44-47,59,63].

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One limitation in applying 1-D OOCs is that the length of the sequences increases rapidly when the number of users or the weight of codes is increased, which means a large bandwidth expansion is required. Thus the bandwidth utilization is reduced. And a large code length causes the chip rate of the OCDMA system to exceed the maximum chip rate currently attainable in practice.

1-D OOCs spread the input data bits only in the time domain. By spreading in both time and wavelength domain, the chip rate can be reduced considerably. Technologies such as wavelength-division-multiplexing (WDM) and dense-WDM have made it possible to spread codes in both time and wavelength domain [61]. These codes are referred to wavelength-time hopping codes, multiple-wavelength codes, two-dimensional optical orthogonal codes, etc., which tend to require smaller code length and hence lower chip rate. Here we always refer to these codes as two-dimensional optical orthogonal codes.

The number of codewords of a 2-D OOC is called the *size* of the 2-D OOC. From a practical point of view, a code with a large size is required [53]. For fixed values of u, v, k and λ , the largest possible size of a 2-D $(u \times v, k, \lambda)$ -OOC is denoted by $\Phi(u \times v, k, \lambda)$. A 2-D $(u \times v, k, \lambda)$ -OOC with $\Phi(u \times v, k, \lambda)$ codewords is said to be *optimal*. Generally speaking, it is difficult to determine the exact value of $\Phi(u \times v, k, \lambda)$. Based on the Johnson bound [31] for constant weight codes, the size of a 2-D $(u \times v, k, \lambda)$ -OOC is upper bounded [61] by

$$\Phi(u \times v, k, \lambda) \le J(u \times v, k, \lambda),$$

where

$$J(u \times v, k, \lambda) = \lfloor \frac{u}{k} \lfloor \frac{uv - 1}{k - 1} \lfloor \frac{uv - 2}{k - 2} \lfloor \cdots \lfloor \frac{uv - \lambda}{k - \lambda} \rfloor \cdots \rfloor \rfloor \rfloor \rfloor.$$

In optical code-division multiple-access applications, performance analysis shows that codes with $\lambda \leq 3$ are the most desirable. As pointed out by [38], from a multiple-access and synchronization point of view, the most desirable on-off signature sequences are OOCs with $\lambda = 1$. However, these families of codes may suffer from low cardinality in some applications. It was hinted that in [5] OOCs with $\lambda = 2$ could, under certain conditions, have better performance than that of OOCs with $\lambda = 1$. In this paper, we are concerned about OPTIMAL 2-D ($u \times v, k, \lambda$)-OOCs with $\lambda = 2$ and k = 4.

We will neither try to explore the applications of 2-D OOCs, nor try to provide the performance analysis of a code-division multiple-access system which uses 2-D OOCs. Mathematically, combinatorial design theory, projective geometry and finite field theory are three main tools to investigate the constructions for 2-D OOCs. In this paper, we focus our attention on the combinatorial structures of 2-D OOCs. Many terminologies and results related to combinatorial design theory will be used. To ensure smooth reading of the paper, most of the proofs related to design theory have been moved to the Appendices. For more information on design theory, the interested reader may refer to [6].

1.1 Literature review

There is a considerable literature on 2-D OOC constructions. Yang and Kwong [61] used a 1-D OOC to achieve spreading in the wavelength and time domains to construct a 2-D OOC. The construction by Lee and Seo [34] spreads in the wavelength and the time domain by using two different 1-D OOCs. Sun et al. [56] constructed a 2-D OOC by employing a frequency hopping code and a 1-D OOC to spread in the wavelength

domain and the time axis, respectively. The construction by Alderson and Mellinger [4] are based on certain point sets in finite projective spaces of dimension k over GF(q). Omrani et al. [49] constructed some 2-D OOCs using polynomials over finite fields and rational functions. Cao and Wei [9] first gave a combinatorial description of 2-D OOCs. Wang et al. [57] discussed the existence of optimal 2-D OOCs with weight 3 and index $\lambda = 1$ using combinatorial design theory.

For more information on 2-D OOC constructions, the interested reader may refer to [4,9,24,33–35,39,49,51,54–58,60–62] and the references therein. However, in this paper, we only focus our attention on OPTIMAL 2-D OOC constructions. In applications, optimal OOCs facilitate the largest possible number of asynchronous users to transmit information effectively and reliably. A quick review of the majority of constructions for optimal 2-D OOCs in the literature is presented in Table I.

Parameters	Conditions	Code size	Reference
$(u \times v, 3, 1)$	$u, v \ge 1$ and $v \equiv 1 \pmod{2}$	if $u \equiv 5 \pmod{6}$ and $v = 1$	[57]
		$J(u \times v, 3, 1) - 1;$	
		otherwise, $J(u \times v, 3, 1)$	
$(u \times u, k, 1)$	$u \equiv 1 \pmod{k(k-1)}$ and u a prime	J(u imes u, k, 1)	[61]
$(p^n \times p, p, 1)$	p a prime and $n \ge 1$	$J(p^n \times p, p, 1)$	[9]
$(u \times v, q, 1)$	$uv = q^n - 1, n \ge 1$, and q a prime power	J(u imes v, q, 1)	[4]
$(u \times v, q+1, 1)$	$uv = (q^{n+1} - 1)/(q - 1),$	J(u imes v, q+1, 1)	[4]
	q a prime power, $n \ge 1$,		
	either $n \equiv 0 \pmod{2}$,		
	or $n \equiv 1 \pmod{2}$ and $gcd(q+1, v) = 1$		
$(u \times v, 4, 2)$	$uv = 2^n - 1$ and $n \ge 3$	$J(u \times v, 4, 2)$	[4]
$(u \times v, 6, 2)$	$uv = (4^n - 1)/3, n \ge 3,$	J(u imes v, 6, 2)	[4]
	either $n \equiv 0, 1 \pmod{3}$,		
	or $n \equiv 2 \pmod{3}$ and $gcd(21, v) = 1$		
$(u \times v, q+1, 2)$	$uv = q^n + 1, q$ a prime power,	$J(u \times v, q+1, 2)$	[4]
	$n \ge 1$, either $n \equiv 0 \pmod{2}$,		
	or $n \equiv 1 \pmod{2}$ and $gcd(q+1, v) = 1$		

Table IOptimal 2-D OOCs in the literature

1.2 Outline of the paper

The rest of this paper is structured as follows. In Section 2 based on the relationship between 1-D OOCs and 2-D OOCs, many optimal 2-D $(u \times v, 4, 2)$ -OOCs are derived. Cao and Wei [9] showed that an optimal 2-D $(u \times v, k, t - 1)$ -OOC is equivalent to an optimal strictly v-cyclic t- $(u \times v, k, 1)$ -packing, provided that $t \leq k$ holds. We restate this combinatorial equivalence in Section 3. In this section perfect 2-D OOCs are defined as a special case of optimal 2-D OOCs. We point out in Remark 3.5 that the problem for the existence of perfect 2-D $(u \times v, 4, 2)$ -OOCs can be reduced to the problem for the existence of perfect 2-D $(w \times v, 4, 2)$ -OOCs, $w \in \{1, 2\}$. When w = 1, perfect 2-D $(1 \times v, 4, 2)$ -OOCs have been widely investigated as a kind of combinatorial object called strictly cyclic Steiner quadruple system. Thus we pay our attention to the case of w = 2in Section 4. We give a construction for perfect 2-D $(2 \times v, 4, 2)$ -OOCs. In Section 5 we improve the upper bound for optimal 2-D $(u \times v, 4, 2)$ -OOCs (not only focus on perfect), which is tighter than the well-known Johnson bound in many cases. In Sections 6 and 7 some auxiliary designs are introduced to establish recursive constructions for 2-D $(u \times v, k, 2)$ -OOCs with general k. Using these recursive constructions and some direct constructions, we obtain many infinite families of optimal 2-D $(u \times v, 4, 2)$ -OOCs in Sections 8 and 9. Finally, Section 10 gives a brief conclusion.

Our main results are summarized in Table II (in Section 9), Tables III and IV (in Section 10).

2 2-D OOCs from 1-D OOCs

2-D OOCs are very closely related to 1-D OOCs. A 2-D $(1 \times v, k, \lambda)$ -OOC is just a 1-D (v, k, λ) -OOC. A 1-D (v, k, λ) -OOC with $\Phi(1 \times v, k, \lambda)$ codewords is said to be *optimal*. In this section we shall derive some optimal 2-D OOCs from the known results on optimal 1-D OOCs. First we quote the following result from Alderson and Mellinger [4].

Theorem 2.1 ([4]) Suppose that there exists an optimal 2-D $(u \times v, k, \lambda)$ -OOC with $\Phi(u \times v, k, \lambda)$ codewords. Then for any integer factorization $v = v_1v_2$, there exists a 2-D $(uv_1 \times v_2, k, \lambda)$ -OOC with $v_1 \Phi(u \times v, k, \lambda)$ codewords.

As a corollary of Theorem 2.1, we have

Corollary 2.2 If there is an optimal 1-D (uv, k, λ) -OOC with $\Phi(1 \times uv, k, \lambda)$ codewords, then a 2-D $(u \times v, k, \lambda)$ -OOC with $u\Phi(1 \times uv, k, \lambda)$ codewords exists.

Therefore by Corollary 2.2, if $u\Phi(1 \times uv, k, \lambda)$ is just equal to $\Phi(u \times v, k, \lambda)$, then the resulting 2-D $(u \times v, k, \lambda)$ -OOC is optimal. In the following we shall give some analysis on $u\Phi(1 \times uv, k, \lambda) = \Phi(u \times v, k, \lambda)$. According to the Johnson bound, $\Phi(u \times v, k, \lambda) \leq J(u \times v, k, \lambda)$ and $\Phi(1 \times uv, k, \lambda) \leq J(1 \times uv, k, \lambda)$. Assume that

$$J_1(1 \times uv, k, \lambda) = \frac{1}{k} \lfloor \frac{uv - 1}{k - 1} \lfloor \frac{uv - 2}{k - 2} \lfloor \cdots \lfloor \frac{uv - \lambda}{k - \lambda} \rfloor \cdots \rfloor \rfloor \rfloor.$$

We have the following lemma.

Lemma 2.3 $J(u \times v, k, \lambda) = uJ(1 \times uv, k, \lambda)$ if and only if $J_1(1 \times uv, k, \lambda) - J(1 \times uv, k, \lambda) < 1/u$.

Proof Let $x = \lfloor \frac{uv-1}{k-1} \lfloor \frac{uv-2}{k-2} \lfloor \cdots \lfloor \frac{uv-\lambda}{k-\lambda} \rfloor \cdots \rfloor \rfloor$ and x = ak + b, where $0 \le b < k$. Then $J(u \times v, k, \lambda) = \lfloor \frac{u}{k}x \rfloor$ and $J(1 \times uv, k, \lambda) = \lfloor \frac{1}{k}x \rfloor$. It is easy to verify that

$$J(u \times v, k, \lambda) = uJ(1 \times uv, k, \lambda) \iff \lfloor \frac{u}{k}x \rfloor = u\lfloor \frac{1}{k}x \rfloor \iff ua + \lfloor \frac{ub}{k} \rfloor = ua$$
$$\iff \lfloor \frac{ub}{k} \rfloor = 0 \qquad \Longleftrightarrow \qquad ub < k$$
$$\iff \frac{b}{k} < \frac{1}{u}.$$
Note that $J_1(1 \times uv, k, \lambda) - J(1 \times uv, k, \lambda) = \frac{b}{k}.$

Theorem 2.4 If there exists an optimal 1-D (uv, k, λ) -OOC with $J(1 \times uv, k, \lambda)$ codewords and $J_1(1 \times uv, k, \lambda) - J(1 \times uv, k, \lambda) < 1/u$, then there exists an optimal 2-D $(u \times v, k, \lambda)$ -OOC with $J(u \times v, k, \lambda)$ codewords.

Proof By Corollary 2.2, if there is an optimal 1-D (uv, k, λ) -OOC with $J(1 \times uv, k, \lambda)$ codewords, then there is a 2-D $(u \times v, k, \lambda)$ -OOC with $uJ(1 \times uv, k, \lambda)$ codewords. Since $J_1(1 \times uv, k, \lambda) - J(1 \times uv, k, \lambda) < 1/u$, by Lemma 2.3, $uJ(1 \times uv, k, \lambda) = J(u \times v, k, \lambda)$. Thus the resulting 2-D OOC is optimal.

Corollary 2.5 If there exists an optimal 1-D (n, 4, 2)-OOC with $J(1 \times n, 4, 2)$ codewords, then

(1) for any integer $n \equiv 1, 3 \pmod{6}$ or $n \equiv 2, 10 \pmod{24}$, and for any integer factorization n = uv, there exists an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J(u \times v, 4, 2)$ codewords;

(2) for any integer $n \equiv 4, 20 \pmod{24}$ and $n = 2n_1$, there exists an optimal 2-D (2 × $n_1, 4, 2$)-OOC with $J(2 \times n_1, 4, 2)$ codewords.

Proof When $n \equiv 1,3 \pmod{6}$ or $n \equiv 2,10 \pmod{24}$, it is readily checked that $J_1(1 \times n, 4, 2) - J(1 \times n, 4, 2) = 0$. When $n \equiv 4,20 \pmod{24}$, it is readily checked that $J_1(1 \times n, 4, 2) - J(1 \times n, 4, 2) = 1/4$. The assertion then follows from Theorem 2.4. \Box

As a special topic, 1-D (n, 4, 2)-OOCs have been extensively studied, for example Alderson and Mellinger [3], Chu and Colbourn [14, 15], Feng, Chang and Ji [18, 19]. We only quote partial known results on optimal 1-D (n, 4, 2)-OOCs, which are essential for our work.

Lemma 2.6

(1) ([19]) There exists an optimal 1-D (uv, 4, 2)-OOC with $J(1 \times uv, 4, 2)$ codewords for any $u \in \{4^n - 1 : \text{integer } n \ge 1\} \cup \{1, 27, 33, 39, 51, 87, 123, 183\}$ and v an integer taken from the set $\{p \equiv 7 \pmod{12} : p \text{ is a prime}\} \cup \{2^n - 1 : \text{odd integer } n \ge 1\} \cup \{25, 37, 61, 73, 109, 157, 181, 229, 277\}$, or a product of such integers.

(2) ([18]) Let n be a positive integer. If $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$, where $p_i = 13$ or p_i is a prime, $p_i \equiv 5 \pmod{12}$ and $p_i < 1500000$, $r_i \ge 1$ for $1 \le i \le s$, then there is an optimal 1-D (4n, 4, 2)-OOC with $J(1 \times 4n, 4, 2)$ codewords.

(3) ([18]) There exists an optimal 1-D (n, 4, 2)-OOC with $J(1 \times n, 4, 2)$ codewords for all $7 \le n \le 100$ with the definite exceptions of $n \in \{9, 12, 13, 24, 48, 72, 96\}$ and possible exceptions of $n \in \{45, 47, 53, 55, 59, 60, 65, 66, 69, 71, 76, 77, 81, 83, 84, 85, 89, 91, 92, 95, 97, 99\}.$

(4) ([14,18]) There exists an optimal 1-D (n, 4, 2)-OOC with $J(1 \times n, 4, 2) - 1$ codewords for $n \in \{9, 12, 13, 24, 48, 72, 96\}$.

Theorem 2.7

(1) Let m = uv, where $u \in \{4^n - 1 : n \ge 1\} \cup \{1, 27, 33, 39, 51, 87, 123, 183\}$ and v is an integer taken from the set $\{p \equiv 7 \pmod{12} : p \text{ is a prime}\} \cup \{2^n - 1 : \text{odd integer } n \ge 1\} \cup \{25, 37, 61, 73, 109, 157, 181, 229, 277\}$, or a product of such integers. Then for any integer factorization $m = n_1n_2$, there exists an optimal 2-D $(n_1 \times n_2, 4, 2)$ -OOC with $J(n_1 \times n_2, 4, 2)$ codewords.

(2) Let $n \in \{10, 15, 21, 25, 26, 27, 33, 34, 39, 49, 50, 51, 57, 58, 63, 74, 75, 82, 87, 93, 98\}$. Then for any integer factorization $n = n_1 n_2$, there is an optimal 2-D $(n_1 \times n_2, 4, 2)$ -OOC with $J(n_1 \times n_2, 4, 2)$ codewords.

(3) Let n be a positive integer. If $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$, where $p_i = 13$ or p_i is a prime, $p_i \equiv 5 \pmod{12}$ and $p_i < 1500000$, $r_i \ge 1$ for $1 \le i \le s$, then there is an optimal 2-D $(2 \times 2n, 4, 2)$ -OOC with $J(2 \times 2n, 4, 2)$ codewords.

(4) Let $2n \in \{20, 28, 44, 52, 68, 100\}$. Then there is an optimal 2-D $(2 \times n, 4, 2)$ -OOC with $J(2 \times n, 4, 2)$ codewords.

Proof It is readily checked that the number n described in (1) is congruent to 1 or 3 modulo 6; the number n described in (2) is congruent to 1, 3 modulo 6, or 2, 10 modulo 24; the number 4n described in (3) and the number 2n described in (4) are congruent to 4 or 20 modulo 24. Combining the results of Lemma 2.6, the assertion then follows from Corollary 2.5.

3 Combinatorial descriptions

Two-dimensional optical orthogonal codes are closely related to some combinatorial configurations called strictly v-cyclic packings. Throughout this paper we always assume that $I_u = \{0, 1, \dots, u-1\}$ and denote by Z_v the additive group of integers modulo v.

3.1 Combinatorial equivalence

A t-(v, k, 1) packing is a pair (X, \mathcal{B}) , where X is a set of v points and \mathcal{B} is a set of subsets of X (called *blocks*), each of cardinality k, such that every t-subset of X occurs in at most one block. The set of all uncovered t-subsets by \mathcal{B} is said to be the *leave* of the packing.

An automorphism α of a packing (X, \mathcal{B}) is a permutation on X such that

$$\{\{\alpha(x): x \in B\} : B \in \mathcal{B}\} = \mathcal{B}.$$

In other words, a block of the packing is mapped to a block under an automorphism. A t- $(u \times v, k, 1)$ packing is said to be v-cyclic if it admits an automorphism π consisting of u cycles of length v. Without loss of generality identify X with $I_u \times Z_v$ and the automorphism π can be taken as $(i, x) \mapsto (i, x + 1) \pmod{(-, v)}$, $i \in I_u$ and $x \in Z_v$. Then all blocks of this packing can be partitioned into some orbits under π . Choose any fixed block from each orbit and then call it a *base block*.

All automorphisms of a packing form a group, called the *full automorphism group* of the packing. Any subgroup of the full automorphism group is called an *automorphism group* of the packing. Let G be an automorphism group of a packing. For any block B of the packing, the subgroup

$$\{\pi \in G : B^{\pi} = B\}$$

is called the *stabilizer* of B in G. If the stabilizer of each block of a v-cyclic $t-(u \times v, k, 1)$ packing is trivial in Z_v , i.e., for each block B, $\{\delta \in Z_v : B + \delta = B\} = \{0\}$, where $B + \delta = \{(i, x + \delta) : (i, x) \in B\}$, then the packing is called *strictly v-cyclic*. When u = 1, a (strictly) v-cyclic $t-(1 \times v, k, 1)$ packing is often simply referred to as a (*strictly*) cyclic t-(v, k, 1) packing. When v = 1, a (strictly) 1-cyclic $t-(u \times 1, k, 1)$ packing is just a t-(u, k, 1) packing.

A strictly v-cyclic t- $(u \times v, k, 1)$ packing is called *optimal* if it contains the largest possible block number. The main purpose of this paper is to construct optimal 2-D OOCs. Cao and Wei [9] established the equivalence between optimal 2-D OOCs and optimal strictly v-cyclic packings. Suppose (X, \mathcal{B}) is a strictly v-cyclic t- $(u \times v, k, 1)$ packing. Denote the family of base blocks of this packing by \mathcal{F} . For each base block Bof \mathcal{F} , construct an $u \times v$ (0,1)-matrix M_B whose rows are indexed by I_u and columns are indexed by Z_v , such that its (i, j) cell equals 1 if and only if $(i, j) \in B$. Since any two blocks intersect at most t - 1 points and all the blocks can be generated by developing cyclically the base blocks, $\{M_B : B \in \mathcal{F}\}$ forms a 2-D $(u \times v, k, t - 1)$ -OOC with $|\mathcal{F}|$ codewords. Conversely, given a 2-D $(u \times v, k, t - 1)$ -OOC, \mathcal{C} , for each $u \times v$ (0, 1)-matrix $M \in \mathcal{C}$ whose rows are indexed by I_u and columns are indexed by Z_v , construct a k-subset B_M of $I_u \times Z_v$ such that $(i, j) \in B_M$ if and only if M's (i, j) cell equals 1. Then $\{B_M : M \in \mathcal{C}\}$ is the family of base blocks of a strictly v-cyclic t- $(u \times v, k, 1)$ packing.

Theorem 3.1 ([9]) An optimal 2-D $(u \times v, k, t-1)$ -OOC is equivalent to an optimal strictly v-cyclic t- $(u \times v, k, 1)$ -packing, provided that $t \leq k$ holds.

Since a strictly 1-cyclic $3-(u \times 1, 4, 1)$ packing is just a 3-(u, 4, 1) packing, and the existence of an optimal 3-(u, 4, 1) packing has been investigated by Ji [29], we can have the following result.

Theorem 3.2 ([29]) There exists an optimal 2-D $(u \times 1, 4, 2)$ -OOC (i.e., an optimal 3-(u, 4, 1) packing) with ϕ codewords, where

$$\phi = \begin{cases} \lfloor \frac{u}{4} \lfloor \frac{u-1}{3} \lfloor \frac{u-2}{2} \rfloor \rfloor & u \not\equiv 0 \pmod{6}, \\\\ \lfloor \frac{u}{4} (\lfloor \frac{u-1}{3} \lfloor \frac{u-2}{2} \rfloor \rfloor - 1) \rfloor & u \equiv 0 \pmod{6}, \end{cases}$$

with the exception of 21 undecided values u = 6r + 5, $r \in \{m : m \text{ is odd}, 3 \le m \le 35, m \ne 17, 21\} \cup \{45, 47, 75, 77, 79, 159\}.$

Example 3.3 There is a trivial optimal 2-D $(1 \times 6, 4, 2)$ -OOC, whose number of codewords is $\lfloor \frac{1}{4} \lfloor \frac{5}{3} \lfloor \frac{4}{2} \rfloor \rfloor \rfloor = 0$. By Theorem 3.2, there is an optimal 2-D $(6 \times 1, 4, 2)$ -OOC with 3 codewords. An optimal 2-D $(2 \times 3, 4, 2)$ -OOC has only $J(2 \times 3, 4, 2) = 1$ codeword

$$\left(\begin{array}{rrr}1 & 1 & 0\\ 1 & 1 & 0\end{array}\right),$$

whose corresponding base block of the optimal strictly 3-cyclic $3-(2 \times 3, 4, 1)$ -packing is $\{(0,0), (1,0), (0,1), (1,1)\}$. A 2-D $(3 \times 2, 4, 2)$ -OOC can not contain $J(3 \times 2, 4, 2) = 2$ codewords. Otherwise, there were a 2-D $(6 \times 1, 4, 2)$ -OOC with 4 codewords by Theorem 2.1, which would be contradict to Theorem 3.2. Thus an optimal 2-D $(3 \times 2, 4, 2)$ -OOC has only one codeword

$$\left(\begin{array}{rrr}1&1\\1&0\\1&0\end{array}\right),$$

whose corresponding base block of the optimal strictly 2-cyclic $3-(3 \times 2, 4, 1)$ -packing is $\{(0,0), (1,0), (2,0), (0,1)\}$.

3.2 Perfect 2-D OOCs

Let K be a set of positive integers. A t-wise balanced design (briefly t-design) is a pair (X, \mathcal{B}) , where X is a set of v points and \mathcal{B} is a set of subsets of X (called *blocks*), each of cardinality from K, such that every t-subset of X is contained in a unique block. Such a design is denoted by S(t, K, v). If $K = \{k\}$, we write S(t, K, v) by S(t, k, v). An S(2, 3, v) is called a *Steiner triple system* and denoted by STS(v). An S(3, 4, v) is called a *Steiner quadruple system* and denoted by SQS(v).

Evidently an S(t, k, v) is a special t-(v, k, 1)-packing, whose leave is an empty set. Thus one can similarly define (strictly) v-cyclic $S(t, k, u \times v)$ as we have done for (strictly) v-cyclic t- $(u \times v, k, 1)$ -packing. A strictly v-cyclic $SQS(1 \times v)$ is often simply written as an sSQS(v) (cf. [18]).

If a 2-D $(u \times v, k, t-1)$ -OOC is equivalent to a strictly v-cyclic $S(t, k, u \times v)$, then the OOC is said to be *perfect*. It is easy to verify that a perfect OOC is an optimal OOC that attains the Johnson bound without using the brackets (cf. [48]).

Lemma 3.4 The necessary conditions for the existence of a strictly v-cyclic $SQS(u \times v)$ (or equivalently, a perfect 2-D ($u \times v, 4, 2$)-OOC) are $uv \equiv 2, 4 \pmod{6}$, $u(uv - 1)(uv - 2) \equiv 0 \pmod{24}$. Specifically, the necessary conditions can be classified as follows:

- (1) $u \equiv 1,5 \pmod{12}$ and $v \equiv 2,10 \pmod{24}$;
- (2) $u \equiv 7, 11 \pmod{12}$ and $v \equiv 14, 22 \pmod{24}$;
- (3) $u \equiv 2, 4 \pmod{6}$ and $v \equiv 1, 5 \pmod{6}$;
- (4) $u \equiv 4, 8 \pmod{12}$ and $v \equiv 2, 4 \pmod{6}$.

Proof It is well known that an SQS(uv) exists if and only if $uv \equiv 2, 4 \pmod{6}$ [25]. Count the number of base blocks of a strictly *v*-cyclic $SQS(u \times v)$. It follows that $u(uv-1)(uv-2) \equiv 0 \pmod{24}$.

A natural question from Lemma 3.4 is whether the necessary conditions for the existence of a perfect 2-D $(u \times v, 4, 2)$ -OOC are sufficient. In Section 5, by Corollary 5.5, we show that for $u \equiv 4,8 \pmod{12}$ and $v \equiv 2,4 \pmod{6}$, there is no perfect 2-D $(u \times v, 4, 2)$ -OOC. In Section 9, by Proposition 9.2, if there exists a perfect 2-D $(2 \times v, 4, 2)$ -OOC with $v \equiv 1,5 \pmod{6}$, then a perfect 2-D $(u \times v, 4, 2)$ -OOC exists for any $u \equiv 2, 4 \pmod{6}$. When u and v satisfy Conditions (1) and (2) in Lemma 3.4, $uv \equiv 2, 10 \pmod{24}$. Then by Corollary 2.5(1), if there exists an optimal 1-D (uv, 4, 2)-OOC with $J(1 \times uv, 4, 2)$ codewords, a perfect 2-D $(u \times v, 4, 2)$ -OOC exists. Note that when $uv \equiv 2, 10 \pmod{24}$, an optimal 1-D (uv, 4, 2)-OOC with $J(1 \times uv, 4, 2)$ codewords is just a perfect 2-D $(1 \times uv, 4, 2)$ -OOC. Thus

Remark 3.5 The existence problem of perfect 2-D ($u \times v, 4, 2$)-OOCs can be reduced to the existence problems of perfect 2-D ($1 \times v, 4, 2$)-OOCs and perfect 2-D ($2 \times v, 4, 2$)-OOCs.

4 A construction for perfect 2-D $(2 \times v, 4, 2)$ -OOCs

According to Remark 3.5, it is important to consider the existences of perfect 2-D $(1 \times v, 4, 2)$ -OOCs and perfect 2-D $(2 \times v, 4, 2)$ -OOCs. A perfect 2-D $(1 \times v, 4, 2)$ -OOC is equivalent to an sSQS(v). Much work has been done on sSQSs in the literature. The interested reader may refer to [18] and the references therein. In this section, we shall present a construction for perfect 2-D $(2 \times v, 4, 2)$ -OOCs.

The idea of this construction is originally from Hartman [27]. In 1980 Hartman [27] gave a construction for an SQS(2p), which can be obtained from an SQS(p+1) with a cyclic derived Steiner triple system, where $p \equiv 1 \pmod{6}$ is a prime. Here, we shall generalize Hartman's method to obtain a construction for strictly *p*-cyclic $SQS(2 \times p)$ s.

The existence of a strictly *p*-cyclic $SQS(2 \times p)$ implies the existence of a perfect 2-D $(2 \times p, 4, 2)$ -OOC.

Our construction are based on the concept of rotational SQSs. A rotational SQS(n) is an SQS(n) with an automorphism consisting of one fixed point and a cycle of length n-1. Such a design is denoted by RoSQS(n).

Assume that (X, \mathcal{B}) is an RoSQS(n). We can identify X with $Z_{n-1} \cup \{\infty\}$, and let the permutation α fixing ∞ and mapping i to $i + 1 \pmod{n-1}$, $i \in Z_{n-1}$, be an automorphism of the RoSQS. Let G be a cyclic group generated by α under the compositions of permutations. Then all blocks of the RoSQS can be partitioned into some orbits under G. Choose any fixed block from each orbit and then call it a *base block*.

Example 4.1 An RoSQS(8) (X, \mathcal{B}) is constructed on $X = Z_7 \cup \{\infty\}$. All blocks of \mathcal{B} are listed below:

$$\{i, i+1, i+2, i+5\}, \{i, i+1, i+3, \infty\}, 0 \le i \le 6.$$

Obviously, all blocks of \mathcal{B} can be obtained by developing the two base blocks $\{0, 1, 2, 5\}$, $\{0, 1, 3, \infty\}$ by +1 modulo 7, where $\infty + 1 = \infty$.

Construction 4.2 Let $p \equiv 1 \pmod{6}$ be a prime. If there exists an RoSQS(p+1), then there exists a strictly p-cyclic $SQS(2 \times p)$.

Proof Here we only exhibit the algorithm in Figure 1. The detailed proof of this construction has been moved to Appendix I. \Box

Step 1: Start from an RoSQS(p+1), which is constructed on $Z_p \cup \{\infty\}$. Denote the set of base blocks of this design by $\mathcal{B}_1 \cup \mathcal{B}_2$, where \mathcal{B}_1 and \mathcal{B}_2 generate all the blocks containing and not containing ∞ , respectively.

Step 2: We write the element (i, x) of $I_2 \times Z_p$ as x_i for short. Let

 $\begin{aligned} \mathcal{A}_1 &= \{ \{x_0, y_0, z_0, u_0\} : \{x, y, z, u\} \in \mathcal{B}_2 \}, \\ \mathcal{A}_2 &= \{ \{0_1, x_0, y_0, z_0\} : \{\infty, x, y, z\} \in \mathcal{B}_1 \}, \\ \mathcal{A}_3 &= \{ \{x_0, y_0, (2r-1)(y-x)_1, 2r(y-x)_1\} : \\ \{\infty, x, y\} \subseteq B \in \mathcal{B}_1, 1 \le r \le (p-1)/2 \}. \end{aligned}$

Step 3: Define a mapping τ from $I_2 \times Z_p$ to $I_2 \times Z_p : x_i \longmapsto (-x)_{1-i}$. For j = 1, 2,

 $\mathcal{A}'_{j} = \{\{\tau(a) : a \in A\} : A \in \mathcal{A}_{j}\}.$

Step 4: Take

 $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_1' \cup \mathcal{A}_2 \cup \mathcal{A}_2' \cup \mathcal{A}_3.$

Then \mathcal{A} is the set of base blocks of the required strictly *p*-cyclic $SQS(2 \times p)$, which is constructed on $I_2 \times Z_p$.

Figure 1: Algorithm to construct a strictly *p*-cyclic $SQS(2 \times p)$

The following example illustrates the algorithm presented in Figure 1.

Example 4.3 In this example we shall show how to construct a strictly 7-cyclic $SQS(2 \times 7)$ from an RoSQS(8). It is equivalent to a perfect 2-D $(2 \times 7, 4, 2)$ -OOC by Theorem 3.1.

• Step 1: Start from an RoSQS(8), which is given by Example 4.1. Take

$$\mathcal{B}_1 = \{\{0, 1, 3, \infty\}\}, \qquad \mathcal{B}_2 = \{\{0, 1, 2, 5\}\}$$

• Step 2: Construct the required strictly 7-cyclic $SQS(2 \times 7)$ on $I_2 \times Z_7$. Let

 $\begin{aligned} \mathcal{A}_1 &= \{\{0_0, 1_0, 2_0, 5_0\}\}, \qquad \mathcal{A}_2 &= \{\{0_1, 0_0, 1_0, 3_0\}\}, \\ \mathcal{A}_3 &= \{\{0_0, 1_0, 1_1, 2_1\} \cup \{0_0, 1_0, 3_1, 4_1\} \cup \{0_0, 1_0, 5_1, 6_1\} \\ &\cup \{0_0, 3_0, 3_1, 6_1\} \cup \{0_0, 3_0, 2_1, 5_1\} \cup \{0_0, 3_0, 1_1, 4_1\} \\ &\cup \{1_0, 3_0, 2_1, 4_1\} \cup \{1_0, 3_0, 6_1, 1_1\} \cup \{1_0, 3_0, 3_1, 5_1\}. \end{aligned}$

• Step 3: Under the action of the mapping $\tau: x_i \mapsto (-x)_{1-i}$, we have

$$\mathcal{A}'_1 = \{\{0_1, 6_1, 5_1, 2_1\}\}, \quad \mathcal{A}'_2 = \{\{0_0, 0_1, 6_1, 4_1\}\}.$$

• Step 4: Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}'_1 \cup \mathcal{A}_2 \cup \mathcal{A}'_2 \cup \mathcal{A}_3$. Then $|\mathcal{A}| = 13$ and \mathcal{A} is the set of base blocks of the required strictly 7-cyclic $SQS(2 \times 7)$.

By Theorem 3.1, a strictly *p*-cyclic $SQS(2 \times p)$ is equivalent to a perfect 2-D (2 × p, 4, 2)-OOC. Thus by Construction 4.2, for obtaining some perfect 2-D (2×p, 4, 2)-OOCs, we need some results on RoSQSs.

Theorem 4.4 ([19]) There exists an RoSQS(uv + 1) for any $u \in \{4^n - 1 : \text{integer } n \ge 1\} \cup \{1, 27, 33, 39, 51, 87, 123, 183\}$ and v is an integer taken from the set $\{p \equiv 7 \pmod{12} : p \text{ is a prime}\} \cup \{2^n - 1 : \text{odd integer } n \ge 1\} \cup \{25, 37, 61, 73, 109, 157, 181, 229, 277\}$, or a product of such integers.

Combining the results of Theorem 3.1, Construction 4.2 and Theorem 4.4, we have

Theorem 4.5 There exist a strictly p-cyclic $SQS(2 \times p)$ and a perfect 2-D $(2 \times p, 4, 2)$ -OOC for any prime $p \equiv 7 \pmod{12}$ or $p \in \{37, 61, 73, 109, 157, 181, 229, 277\}$.

5 Tighter upper bound for 2-D $(u \times v, 4, 2)$ -OOCs

In most cases an optimal 2-D $(u \times v, 4, 2)$ -OOC is not a perfect 2-D $(u \times v, 4, 2)$ -OOC. Thus the determination of the largest possible size $\Phi(u \times v, 4, 2)$ of a optimal 2-D $(u \times v, 4, 2)$ -OOC is of interest. Recall that in Section 1, we mention that $\Phi(u \times v, 4, 2) \leq J(u \times v, 4, 2)$, where $J(u \times v, 4, 2) = \lfloor \frac{u}{4} \lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor \rfloor$ is the famous Johnson bound. Here we shall give a tighter upper bound for 2-D $(u \times v, 4, 2)$ -OOCs than the Johnson bound.

Lemma 5.1 Let $uv \equiv 0 \pmod{6}$. Then $\Phi(u \times v, 4, 2) \leq \lfloor \frac{u}{4} \lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1 \rfloor$.

Proof By Theorem 2.1, an optimal 2-D $(u \times v, 4, 2)$ -OOC with $\Phi(u \times v, 4, 2)$ codewords implies a 2-D $(uv \times 1, 4, 2)$ -OOC with $v\Phi(u \times v, 4, 2)$ codewords. Since a 2-D $(uv \times 1, 4, 2)$ -OOC is equivalent to a strictly 1-cyclic 3- $(uv \times 1, 4, 1)$ -packing, by Theorem 3.2, when $uv \equiv 0 \pmod{6}$, it has at most $\lfloor \frac{uv}{4}(\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1)\rfloor$ blocks. Thus we have $v\Phi(u \times v, 4, 2) \leq \lfloor \frac{uv}{4}(\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1)\rfloor$. It is readily checked that $\Phi(u \times v, 4, 2) \leq \lfloor \frac{1}{v} \lfloor \frac{uv-2}{4} \lfloor \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1) \rfloor \rfloor = \lfloor \frac{u}{24}(u^2v^2 - 3uv - 6) \rfloor = \lfloor \frac{u}{4}(\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1) \rfloor$. \Box

The following two lemmas shows that in some cases of $uv \equiv 0 \pmod{6}$, the bound for $\Phi(u \times v, 4, 2)$ in Lemma 5.1 is not tight enough. Their proofs are lengthy. To ensure smooth reading of the paper, their proofs have been moved to Appendix II.

Lemma 5.2 Let $u \equiv 0 \pmod{12}$ and $v \equiv 2,4 \pmod{6}$. Then $\Phi(u \times v, 4, 2) \leq \lfloor \frac{u}{4} \lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1 \rfloor \rfloor - 1$.

Lemma 5.3 Let $uv \equiv 0 \pmod{12}$ and $v \equiv 0 \pmod{6}$. Then $\Phi(u \times v, 4, 2) \leq \lfloor \frac{u}{4}(\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 2) \rfloor$.

Lemma 5.4 Let $uv \equiv 4,8 \pmod{12}$ and $v \equiv 0 \pmod{2}$. Then $\Phi(u \times v, 4, 2) \leq \lfloor \frac{u}{4} \lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1 \rfloor \rfloor$.

Proof For each $a \in I_u$ and each $0 \le i < v/2$, consider the number n of the base blocks containing the two points (a, i), (a, v/2 + i) in a strictly v-cyclic 3- $(u \times v, 4, 1)$ -packing. Since each 3-subset of $I_u \times Z_v$ occurs in at most one block and each base block containing the two points (a, i), (a, v/2 + i) generates exactly two different blocks containing the same two points, the number n is at most $\lfloor (uv - 2)/4 \rfloor = (uv - 4)/4$. Thus there are at least two 3-subsets of the form $\{(a, i), (a, v/2 + i), (*, *)\}$ in the leave. Note that the above conclusion holds for each $a \in I_u$ and each $0 \le i < v/2$. It follows that there are at least uv 3-subsets in the leave. It implies that $\Phi(u \times v, 4, 2) \le \lfloor (\binom{uv}{3} - uv)/(4v) \rfloor = \lfloor \frac{1}{24}u(u^2v^2 - 3uv - 4) \rfloor$. It is readily checked that $\lfloor \frac{u}{4} \lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1) \rfloor = \lfloor \frac{1}{24}u(u^2v^2 - 3uv - 4) \rfloor$. This completes the proof.

Corollary 5.5 For any $u \equiv 4, 8 \pmod{12}$ and $v \equiv 2, 4 \pmod{6}$, there is no perfect 2-D $(u \times v, 4, 2)$ -OOC.

Proof If there were a perfect 2-D $(u \times v, 4, 2)$ -OOC for $u \equiv 4, 8 \pmod{12}$ and $v \equiv 2, 4 \pmod{6}$, then it should have u(uv-1)(uv-2)/24 codewords. By Lemma 5.4, the largest possible size of the perfect 2-D $(u \times v, 4, 2)$ -OOC should be u(uv+1)(uv-4)/24. A contradiction occurs.

Lemma 5.6 Let $u \equiv 7, 11 \pmod{12}$. Then $\Phi(u \times 2, 4, 2) \leq \lfloor \frac{u}{4} \lfloor \frac{2u-1}{3} \lfloor \frac{2u-2}{2} \rfloor \rfloor \rfloor - 1$.

Proof It is known that $\Phi(u \times 2, 4, 2) \leq J(u \times 2, 4, 2) = \lfloor \frac{u}{4} \lfloor \frac{2u-1}{3} \lfloor \frac{2u-2}{2} \rfloor \rfloor \rfloor$. Suppose that $\Phi(u \times 2, 4, 2) = J(u \times 2, 4, 2)$. Then there were a strictly 2-cyclic 3- $(u \times 2, 4, 1)$ -packing with $J(u \times 2, 4, 2)$ base blocks. Count the number of 3-subsets in the leave \mathcal{L} of the strictly 2-cyclic 3- $(u \times 2, 4, 1)$ -packing. It is $\binom{2u}{3} - J(u \times 2, 4, 2) \cdot 2 \cdot 4 = 4$. Each 3-subset in the leave is of the form $\{(a, i), (b, j), (c, k)\}$ or $\{(a, i), (b, j), (a, i+1)\}$, where a, b, c are distinct elements in I_u , and $i, j, k \in \mathbb{Z}_2$.

Assume that $\{(a, i), (b, j), (x, k)\}$ is a 3-subset in the leave, where $a, b, x \in I_u, a \neq b$ and $i, j, k \in Z_2$. Consider the number n of the blocks containing the two points (a, i), (b, j). Since each 3-subset of $I_u \times Z_2$ occurs in at most one block, the number n is at most $\lfloor (2u-3)/2 \rfloor = (2u-4)/2$. Thus there must be another 3-subset $\{(a, i), (b, j), (y, l)\}$ in the leave, where $(y, l) \neq (x, k)$. Due to $|\mathcal{L}| = 4$, we have $\mathcal{L} = \{\{(a, i), (b, j), (x, k)\}, \{(a, i + 1), (b, j + 1), (x, k + 1)\}, \{(a, i), (b, j), (y, l)\}, \{(a, i + 1), (b, j + 1), (y, l + 1)\}\}$.

If $x \neq a$ and $x \neq b$, since each 3-subset of $I_u \times Z_2$ occurs in at most one block, the number of blocks containing the two points (a, i), (x, k) is exactly (2u - 3)/2, which is

not an integer. A contradiction. If x = a, then (x, k) = (a, i + 1) and there are exactly (2u-4)/4 base blocks containing the points (a, i), (x, k). If x = b, then (x, k) = (b, j+1) and there are also exactly (2u-4)/4 base blocks containing the points (b, j), (x, k). The number (2u-4)/4 is not an integer. A contradiction. Hence $|\mathcal{L}| \neq 4$ and $\Phi(u \times 2, 4, 2) \leq J(u \times 2, 4, 2) - 1$.

Combine the results of Lemmas 5.1-5.6. Let $A = \{(u, v) : u \equiv 0 \pmod{12}, v \equiv 2, 4 \pmod{6}\}$ and $B = \{(u, v) : uv \equiv 0 \pmod{12}, v \equiv 0 \pmod{6}\}$. In the rest of this paper, we always assume that

$$J^*(u \times v) = \begin{cases} \lfloor \frac{u}{4} \lfloor \frac{2u-1}{3} \lfloor \frac{2u-2}{2} \rfloor \rfloor \rfloor - 1, & \text{if } u \equiv 7, 11 \pmod{12} \text{ and } v = 2; \\ \lfloor \frac{u}{4} (\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1) \rfloor, & \text{if } uv \equiv 0 \pmod{6} \text{ and } (u, v) \notin A \cup B, \\ or \ uv \equiv 4, 8 \pmod{12} \text{ and } v \equiv 0 \pmod{2}; \\ \lfloor \frac{u}{4} (\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1) \rfloor - 1, & \text{if } (u, v) \in A; \\ \lfloor \frac{u}{4} (\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 2) \rfloor, & \text{if } (u, v) \in B; \\ \lfloor \frac{u}{4} \lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor \rfloor, & \text{otherwise.} \end{cases}$$

We have the following theorem.

Theorem 5.7 $\Phi(u \times v, 4, 2) \leq J^*(u \times v).$

Now the question is whether there are optimal 2-D $(u \times v, 4, 2)$ -OOCs to achieve the upper bounds established in Theorem 5.7. In Section 9 we shall give many infinite families for optimal 2-D $(u \times v, 4, 2)$ -OOCs, which achieve the upper bound in Theorem 5.7.

6 Auxiliary designs and filling constructions

In this section and the next section, some recursive constructions for optimal 2-D OOCs will be given, called filling constructions and weighting constructions, respectively. These constructions are the generalization of standard constructions for 3-designs in combinatorial design theory. So far the research on combinatorial constructions for 2-D OOCs mainly focuses on $\lambda = 1$ [9,57], which corresponds to the theory of 2-designs. However, when $\lambda = 2$, the research is related to the theory of 3-designs. Compared to 2-designs, the known results on 3-designs are limited, and the auxiliary structures to construct 3-designs are more complex. Thus the following auxiliary designs will be a little strange for the reader who first meets them. If one is familiar with 2-designs, it is useful to notice that the concepts of s-fan designs and H designs are two possible generalizations of group divisible designs. Group divisible design is one of the most basic research objects in combinatorial design theory [6].

6.1 *s*-fan designs

Hartman [28] first introduced the concept of s-fan designs in 1994. Let s be a non-negative integer. An s-fan design is an (s+3)-tuple $(X, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s, \mathcal{T})$ satisfying that (X, \mathcal{G}) is a 1-design, $(X, \mathcal{G} \cup \mathcal{B}_i)$ is a 2-design for each $1 \leq i \leq s$ and $(X, \mathcal{G} \cup (\bigcup_{i=1}^s \mathcal{B}_i) \cup \mathcal{T})$ is a 3-design. The elements of \mathcal{G} and $(\bigcup_{i=1}^s \mathcal{B}_i) \cup \mathcal{T}$ are called groups and blocks, respectively.

For understanding the concept of s-fan designs, we first consider the case of s = 0. A 0-fan design is a 3-tuple $(X, \mathcal{G}, \mathcal{T})$ satisfying that (X, \mathcal{G}) is a 1-design and $(X, \mathcal{G} \cup \mathcal{T})$ is a 3-design.

Example 6.1 Take $X = I_8$ and $\mathcal{G} = \{\{0, 2, 4, 6\}, \{1, 3, 5, 7\}\}$. Then (X, \mathcal{G}) is a 1-design. Let \mathcal{T} consists of the following 12 blocks

It is readily checked that each 3-subset of I_8 is either contained in exactly one block of \mathcal{T} or in exactly one group of \mathcal{G} , but not in both. Hence, $(X, \mathcal{G} \cup \mathcal{T})$ is a 3-design. This is an example of 0-fan designs.

Next we give an example of 1-fan designs. A 1-fan design is a 4-tuple $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ satisfying that (X, \mathcal{G}) is a 1-design, $(X, \mathcal{G} \cup \mathcal{B})$ is a 2-design and $(X, \mathcal{G} \cup \mathcal{B} \cup \mathcal{T})$ is a 3-design.

Example 6.2 Take $X = I_9$ and $\mathcal{G} = \{\{0, 1, 8\}, \{2, 3, 6\}, \{4, 5, 7\}\}$. Then (X, \mathcal{G}) is a 1-design. Let \mathcal{B} consists of the following 9 blocks

It is readily checked that each 2-subset of I_9 is either contained in exactly one block of \mathcal{B} or in exactly one group of \mathcal{G} , but not in both. Hence, $(X, \mathcal{G} \cup \mathcal{B})$ is a 2-design. Let \mathcal{T} consists of the following 18 blocks

$\{0, 1, 2, 3\},\$	$\{0, 1, 4, 5\},\$	$\{0, 1, 6, 7\},\$	$\{0, 2, 4, 6\},\$	$\{0, 2, 5, 8\},\$	$\{0, 3, 5, 7\},\$
$\{0, 3, 6, 8\},\$	$\{0, 4, 7, 8\},\$	$\{1, 2, 4, 7\},\$	$\{1, 2, 6, 8\},\$	$\{1, 3, 4, 8\},\$	$\{1, 3, 5, 6\}$
$\{1, 5, 7, 8\},\$	$\{2, 3, 4, 5\},\$	$\{2, 3, 7, 8\},\$	$\{2, 5, 6, 7\},\$	$\{3, 4, 6, 7\},\$	$\{4, 5, 6, 8\}.$

It is readily checked that $(X, \mathcal{G} \cup \mathcal{B} \cup \mathcal{T})$ is a 3-design. This is an example of 1-fan designs.

If there are a_i groups of size g_i in an s-fan design, $1 \leq i \leq m$, then the type of the s-fan design is defined to be $g_1^{a_1}g_2^{a_2}\cdots g_m^{a_m}$. Let $K_1, K_2, \ldots, K_s, K_T$ be sets of positive integers. If block sizes of \mathcal{B}_i and \mathcal{T} are from K_i $(1 \leq i \leq s)$ and K_T , respectively, then the s-fan design is denoted by s-FG(3, $(K_1, K_2, \ldots, K_s, K_T), \sum_{i=1}^m g_i a_i)$ of type $g_1^{a_1}g_2^{a_2}\cdots g_m^{a_m}$. Example 6.1 shows a 0-FG (3, $(\emptyset, 4), 8$) of type 4^2 . Example 6.2 presents a 1-FG(3, (3, 4), 9) of type 3^3 .

Lemma 6.3 ([18]) The necessary conditions for the existence of a 0-FG(3, $(\emptyset, K_T), gn)$ of type g^n $(n \ge 2)$ are

(1) $g^2n(n-1)(gn+g-3) \equiv 0 \pmod{\alpha}$, where $\alpha = gcd\{k(k-1)(k-2) : k \in K_T\}$;

- (2) $g(n-1)(gn+g-3) \equiv 0 \pmod{\beta}$, where $\beta = gcd\{(k-1)(k-2) : k \in K_T\}$;
- (3) if g = 1, then $n \equiv 2 \pmod{\gamma}$; if g > 1, then $gn \equiv g \equiv 2 \pmod{\gamma}$, where $\gamma = gcd\{k-2: k \in K_T\}$.

Theorem 6.4 ([64]) There exists a 0-FG(3, (\emptyset , 4), gn) of type g^n if and only if either g = 1 and $n \equiv 2, 4 \pmod{6}$, or g is even and $g(n-1)(n-2) \equiv 0 \pmod{3}$.

6.1.1 The basic idea

Since an optimal 2-D $(u \times v, k, 2)$ -OOC is equivalent to an optimal strictly v-cyclic 3- $(u \times v, k, 1)$ -packing, we first consider how to construct a 3-packing without the restriction of automorphism groups.

- Step 1: Start from a 0-FG(3, $(\emptyset, k), gn$) of type $g^n(X, \mathcal{G}, \emptyset, \mathcal{T})$. By the definition of s-fan designs, $(X, \mathcal{G} \cup \mathcal{T})$ is a 3-design. (X, \mathcal{T}) satisfies that each 3-subset of X not contained in some group of \mathcal{G} occurs in exactly one block of \mathcal{T} , and each 3-subset of X contained in some group of \mathcal{G} never occur in any block of \mathcal{T} .
- Step 2: If a 3-(g, k, 1)-packing exists, then one can construct a 3-(g, k, 1)-packing on the set G for each $G \in \mathcal{G}$. Denote its block set by \mathcal{A}_G .
- Step 3: Let $\mathcal{A} = \bigcup_{G \in \mathcal{G}} \mathcal{A}_G$. It follows that each 3-subset of X contained in some group of \mathcal{G} occurs in at most one block of \mathcal{A} .
- Step 4: Let $\mathcal{C} = \mathcal{A} \cup \mathcal{T}$. We have that (X, \mathcal{C}) is a 3-(gn, k, 1)-packing.

The main idea of the above construction is to fill in the groups of a 0-fan design with a 3-packing. So this construction is termed as "Filling Construction". Furthermore, if one hope to obtain an optimal 3-(gn, k, 1)-packing, it is necessary to input an optimal 3-(g, k, 1)-packing. Note that the reverse is not always correct. Now our purpose is to construct strictly v-cyclic $3-(u \times v, k, 1)$ -packings. We need to modify the above "Filling Construction" such that the initial 0-fan design admits some special automorphisms.

An *automorphism* of an s-fan design $(X, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s, \mathcal{T})$ is a permutation on X leaving $\mathcal{G}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s, \mathcal{T}$ invariant, respectively. All automorphisms of an s-fan design form a group, called the *full automorphism group* of the s-fan design. Any subgroup of the full automorphism group is called an *automorphism group* of the s-fan design.

Let G be an automorphism group of an s-fan design. All blocks of the s-fan design can be partitioned into some orbits under G. Choose any fixed block from each orbit and then call it a *base block* of this s-fan design. For any block B of the s-fan design, the subgroup $\{\pi \in G : B^{\pi} = B\}$ is called the *stabilizer* of B in G.

Example 6.5 Observe the 0-FG $(3, (\emptyset, 4), 8)$ of type 4^2 from Example 6.1. Consider the permutation $\alpha = (0 \ 1 \ 2 \ 3)(4 \ 5 \ 6 \ 7)$ on I_8 . It is easy to checked that α is an automorphism of this 0-FG. All blocks are partitioned into 5 orbits under the action of α . The 5 base blocks are $\{0, 1, 2, 3\}^*$, $\{4, 5, 6, 7\}^*$, $\{1, 2, 5, 6\}$, $\{0, 1, 6, 7\}$ and $\{0, 2, 5, 7\}$, where the stabilizer of each base block marked with a^* is trivial, i.e., it contains only the identity permutation.

In the following we introduce two kinds of s-fan designs with special automorphism groups.

6.1.2 *h*-cyclic *s*-fan designs

Construct an s-fan design of type $(hg_1)^{a_1}(hg_2)^{a_2}\cdots(hg_m)^{a_m}$ on $(\bigcup_{i=1}^m (I_{a_i}\times I_{g_i}))\times Z_h$ with the group set $\{\{x\}\times I_{g_i}\times Z_h: x\in I_{a_i}, 1\leq i\leq m\}$. If this s-fan design admits an automorphism π mapping $(x, y, j) \longmapsto (x, y, j+1) \pmod{(-, -, h)}, x \in I_{a_i}, y \in I_{g_i}$ and $j \in Z_h$, then the s-fan design is said to be h-cyclic. For each block B of an h-cyclic s-fan design of type $(hg_1)^{a_1}(hg_2)^{a_2}\cdots(hg_m)^{a_m}$, if the stabilizer of B in Z_h is trivial, i.e., $\{\delta \in Z_h : B + \delta = B\} = \{0\}$, where $B + \delta =$ $\{(x, y, j + \delta) : (x, y, j) \in B\}$, then the s-fan design is called *strictly h-cyclic*. A (strictly) h-cyclic s-fan design of type h^n is often referred to as a (strictly) semi-cyclic s-fan design of type h^n (cf. [18]).

The following construction is straightforward.

Construction 6.6 (Filling Construction-I) Suppose that the following exist:

- (1) a strictly h-cyclic 0-FG(3, $(\emptyset, k), \Sigma_{i=1}^m g_i a_i h)$ of type $(hg_1)^{a_1} (hg_2)^{a_2} \cdots (hg_m)^{a_m}$ with b_0 base blocks;
- (2) a strictly h-cyclic 3- $(g_i \times h, k, 1)$ packing with b_i base blocks for each $1 \le i \le m$.

Then there exists a strictly h-cyclic 3- $((\sum_{i=1}^{m} g_i a_i) \times h, k, 1)$ packing with $b_0 + \sum_{i=1}^{m} a_i b_i$ base blocks, which is a 2-D $((\sum_{i=1}^{m} g_i a_i) \times h, k, 1)$ -OOC.

Furthermore, if the given strictly h-cyclic $3 \cdot (g_i \times h, k, 1)$ packing is a strictly h-cyclic $S(3, k, g_i \times h)$ for each $1 \le i \le m$, then we obtain a strictly h-cyclic $S(3, k, (\sum_{i=1}^m g_i a_i) \times h)$, which is a perfect 2-D $((\sum_{i=1}^m g_i a_i) \times h, k, 2)$ -OOC.

Example 6.7 In this example, we construct an optimal 2-D $(4 \times 2, 4, 2)$ -OOC.

Step 1: First we construct a strictly 2-cyclic 0-FG(3, (∅, 4), 8) of type 4² on I₂ × I₂ × Z₂ with the group set {{x} × I₂ × Z₂ : x ∈ I₂}. All the 6 base blocks are listed below.

 $\begin{array}{ll} \{(0,0,0),(0,0,1),(1,0,0),(1,1,0)\}, & \{(0,0,0),(1,0,0),(1,0,1),(0,1,0)\}, \\ \{(0,0,0),(1,0,0),(1,1,1),(0,1,1)\}, & \{(0,0,0),(1,0,1),(1,1,0),(0,1,1)\}, \\ \{(0,0,0),(1,1,0),(1,1,1),(0,1,0)\}, & \{(1,0,0),(1,1,0),(0,1,0),(0,1,1)\}. \end{array}$

- Step 2: Take an optimal strictly 2-cyclic 3-(2 × 2,4,1) packing, which is trivial without base blocks.
- Step 3: Apply Construction 6.6 to obtain a strictly 2-cyclic 3-(4 × 2, 4, 1) packing with 6 base blocks, which achieves the upper bound in Theorem 5.7 and is an optimal 2-D (4 × 2, 4, 2)-OOC with 6 codewords. Note that I₂ × I₂ × Z₂ ≃ I₄ × Z₂. Hence Φ(4 × 2, 4, 2) = J*(4 × 2) = 6.

Example 6.8 In this example, we construct an optimal 2-D $(4 \times 3, 4, 2)$ -OOC.

Step 1: First we construct a strictly 3-cyclic 0-FG(3, (∅, 4), 12) of type 6² on I₂ × I₂ × Z₃ with the group set {{x} × I₂ × Z₃ : x ∈ I₂}. All the 15 base blocks are listed below:

$$\begin{split} \{(0,0,0),(0,0,1),(1,0,0),(1,0,1)\}, & \{(0,0,0),(0,0,1),(1,0,2),(1,1,0)\}, \\ \{(0,0,0),(0,0,1),(1,1,1),(1,1,2)\}, & \{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\}, \\ \{(0,0,0),(0,1,0),(1,0,1),(1,1,1)\}, & \{(0,0,0),(0,1,0),(1,0,2),(1,1,2)\}, \\ \{(0,0,0),(0,1,1),(1,0,0),(1,1,1)\}, & \{(0,0,0),(0,1,1),(1,0,2),(1,1,2)\}, \\ \{(0,0,0),(0,1,1),(1,1,0),(1,1,2)\}, & \{(0,0,0),(0,1,2),(1,0,0),(1,1,2)\}, \\ \{(0,0,0),(0,1,2),(1,0,1),(1,1,0)\}, & \{(0,0,0),(0,1,2),(1,0,2),(1,1,1)\}, \\ \{(0,1,0),(0,1,1),(1,1,0),(1,1,1)\}, & \{(0,1,0),(0,1,1),(1,0,1),(1,1,2)\}, \\ \{(0,1,0),(0,1,1),(1,1,0),(1,1,1)\}. \end{split}$$

 Step 2: Construct an optimal strictly 3-cyclic 3-(2×3,4,1) packing on {x}×I₂×Z₃ for each x ∈ I₂, which has 1 base block and exists by Example 3.3. Then this step contributes 2 base blocks as follows

 $\{(0,0,0),(0,1,0),(0,0,1),(0,1,1)\}, \quad \{(1,0,0),(1,1,0),(1,0,1),(1,1,1)\}.$

• Step 3: Apply Construction 6.6 to obtain a strictly 3-cyclic $(4 \times 3, 4, 1)$ packing with 17 base blocks, which achieves the upper bound in Theorem 5.7 and is an optimal 2-D $(4 \times 3, 4, 2)$ -OOC with 17 codewords. Note that $I_2 \times I_2 \times Z_3 \cong I_4 \times Z_3$. Hence $\Phi(4 \times 3, 4, 2) = J^*(4 \times 3) = 17$.

Lemma 6.9 For any $v \equiv 1 \pmod{2}$, there exists a strictly v-cyclic 0-FG $(3, (\emptyset, 4), 4v)$ of type $(2v)^2$.

Proof By Lemma 2.11 in [20], there is a semi-cyclic 0-FG(3, $(\emptyset, 4), 4v$) of type $(2v)^2$ on the point set $X = I_2 \times Z_{2v}$ and the group set $\mathcal{G} = \{\{x\} \times Z_{2v} : x \in I_2\}$. Denote the family of its blocks by \mathcal{T} . For each $(x, i) \in X$, define a mapping

$$\tau: \quad (x,i)\longmapsto (x,i-2\lfloor i/2\rfloor,\lfloor i/2\rfloor).$$

Let $X' = I_2 \times I_2 \times Z_v$ and $\mathcal{G}' = \{\{x\} \times I_2 \times Z_v : x \in I_2\}$. Let $\mathcal{T}' = \bigcup_{T \in \mathcal{T}} \tau(T)$, where $\tau(T) = \{\tau(r) : r \in T\}$. Since $v \equiv 1 \pmod{2}$, it is readily checked that $(X', \mathcal{G}', \emptyset, \mathcal{T}')$ is a strictly v-cyclic 0-FG $(3, (\emptyset, 4), 4v)$ of type $(2v)^2$.

Remark 6.10 In Construction 6.6, even if the given strictly h-cyclic 3- $(g_i \times h, k, 1)$ packing is optimal for each $1 \le i \le m$, the resulting strictly h-cyclic 3- $((\sum_{i=1}^m g_i a_i) \times h, k, 1)$ packing may not be optimal.

6.1.3 (u, h)-regular s-fan designs

Let *h* divide *v* and *H* be a subgroup of order *h* in Z_v , i.e., $H = \{0, v/h, \ldots, (h-1)v/h\}$. Let $H_i = H + i$ be a coset of *H* in Z_v , $0 \le i < v/h$. Construct an *s*-fan design of type $(uh)^{v/h}$ on $I_u \times Z_v$ with the group set $\{I_u \times H_i : 0 \le i < v/h\}$. If this *s*-fan design admits an automorphism π mapping $(x, j) \mapsto (x, j+1) \pmod{(-, v)}$, $x \in I_u$ and $j \in Z_v$, then the *s*-fan design is said to be (u, h)-regular.

For each block B of a (u, h)-regular s-fan design of type $(uh)^{v/h}$, if the stabilizer of B in Z_v is trivial, i.e., $\{\delta \in Z_v : B + \delta = B\} = \{0\}$, where $B + \delta = \{(x, j + \delta) : (x, j) \in B\}$, then the s-fan design is called strictly (u, h)-regular.

Example 6.11 By Example 6.5, the 0-FG $(3, (\emptyset, 4), 8)$ of type 4^2 from Example 6.1 admits an automorphism (0 1 2 3)(4 5 6 7). Actually the reader may check that this 0-FG is isomorphic to a (2,2)-regular 0-FG under the mapping $\tau : v \to (\lfloor v/4 \rfloor, v \pmod{4})$ from I_8 to $I_2 \times Z_4$. But it is not strictly (2,2)-regular.

When u = 1, a (strictly) (1, h)-regular s-fan design of type $h^{v/h}$ is often referred to as a (strictly) cyclic s-fan design of type $h^{v/h}$ (cf. [18]). We quote the following results for later use.

Lemma 6.12 ([18])

- (1) There exists a strictly cyclic 0-FG $(3, (\emptyset, 4), 2h)$ of type h^2 for any $h \equiv 0 \pmod{8}$.
- (2) There exists a strictly cyclic 0-FG $(3, (\emptyset, 4), 3h)$ of type h^3 for any $h \equiv 0 \pmod{12}$.
- (3) There exists a strictly cyclic 0-FG(3, $(\emptyset, 4), 5h$) of type h^5 for any $h \equiv 0 \pmod{2}$.

Lemma 6.13 ([19]) An RoSQS(v+1) for $v \equiv 1 \pmod{6}$ is equivalent to a strictly cyclic 1-FG(3, (3, 4), v) of type 1^v . An RoSQS(v+1) for $v \equiv 3 \pmod{6}$ is equivalent to a strictly cyclic 1-FG(3, (3, 4), v) of type $3^{v/3}$.

Construction 6.14 (Filling Construction-II) Let $uh \ge k \ge 3$. Suppose that the following exist.

- (1) a strictly (u, h)-regular 0-FG $(3, (\emptyset, k), uv)$ of type $(uh)^{v/h}$ with b_0 base blocks;
- (2) a strictly h-cyclic 3- $(u \times h, k, 1)$ packing with b_1 base blocks.

Then there exists a strictly v-cyclic $3(u \times v, k, 1)$ packing with $b_0 + b_1$ base blocks.

Furthermore, if the given strictly h-cyclic 3- $(u \times h, k, 1)$ packing is optimal with $J(u \times h, k, 2)$ base blocks, then the derived strictly v-cyclic 3- $(u \times v, k, 1)$ packing is also optimal with $J(u \times v, k, 2)$ base blocks, which is an optimal 2- $D(u \times v, k, 2)$ -OOC with $J(u \times v, k, 2)$ codewords.

Proof First we prove the first part of this construction.

- Step 1: Start from a strictly (u, h)-regular 0-FG $(3, (\emptyset, k), uv)$ of type $(uh)^{v/h}$. Denote the family of base blocks of this design by \mathcal{F} .
- Step 2: Let \mathcal{E} be the family of base blocks of a strictly *h*-cyclic 3- $(u \times h, k, 1)$ packing. For each $B = \{(x_1, j_1), (x_2, j_2), \dots, (x_k, j_k)\} \in \mathcal{E}$ we take

$$\frac{v}{h}B = \{(x_1, \frac{v}{h}j_1), (x_2, \frac{v}{h}j_2), \dots, (x_k, \frac{v}{h}j_k)\}.$$

• Step 3: Then $\mathcal{F} \cup \{\frac{v}{h}B : B \in \mathcal{E}\}$ forms the family of base blocks of the desired strictly v-cyclic $(u \times v, k, 1)$ packing.

For checking optimality of the required design in the second part, it suffices to show that

$$\frac{u((uv-1)(uv-2) - (uh-1)(uh-2))}{k(k-1)(k-2)} + \lfloor \frac{u}{k} \lfloor \frac{uh-1}{k-1} \lfloor \frac{uh-2}{k-2} \rfloor \rfloor \rfloor$$

$$= \lfloor \frac{u}{k} \lfloor \frac{uv-1}{k-1} \lfloor \frac{uv-2}{k-2} \rfloor \rfloor \rfloor.$$
(1)

By Lemma 6.3 (3), since uh > 1, the existence of a strictly (u, h)-regular 0-FG(3, (\emptyset, k) , uv) of type $(uh)^{v/h}$ implies that $uv - 2 \equiv uh - 2 \equiv 0 \pmod{k-2}$. By Lemma 6.3 (2), one can verify that $(uv - 1)(uv - 2) \equiv (uh - 1)(uh - 2) \pmod{(k-1)(k-2)}$. Let $(uv - 1)(uv - 2) \equiv a_1(k-1)(k-2) + r$ and $(uh - 1)(uh - 2) \equiv a_2(k-1)(k-2) + r$, where $0 \leq r < (k-1)(k-2)$. Thus for obtaining the equation (1), it suffices to prove that

$$\frac{u(a_1 - a_2)}{k} + \lfloor \frac{ua_2}{k} \rfloor = \lfloor \frac{ua_1}{k} \rfloor.$$
⁽²⁾

Note that $u(a_1 - a_2) \equiv 0 \pmod{k}$. Let $ua_1 = b_1k + r_1$ and $ua_2 = b_2k + r_1$, where $0 \leq r_1 < k$. It is readily checked that the equation (2) holds.

Example 6.15 In this example, we construct an optimal 2-D $(2 \times 4, 4, 2)$ -OOC.

• Step 1: First we construct a strictly (2,2)-regular 0-FG $(3, (\emptyset, 4), 8)$ of type 4^2 on $I_2 \times Z_4$ with the group set $\{I_2 \times H_i : 0 \le i \le 1\}$, where $H_0 = \{0,2\}$ is a subgroup of order 2 in Z_4 and $H_1 = \{1,3\}$. All the 3 base blocks are listed below:

 $\{(0,0), (0,1), (0,2), (1,1)\}, \quad \{(0,0), (0,1), (1,2), (1,3)\}, \quad \{(0,0), (1,0), (1,1), (1,3)\}.$

- Step 2: Take an optimal strictly 2-cyclic 3-(2 × 2, 4, 1) packing, which is trivial without base blocks.
- Step 3: Apply Construction 6.14 to obtain a strictly 4-cyclic 3-(2 × 4, 4, 1) packing with 3 base blocks, which achieves the upper bound in Theorem 5.7 and is an optimal 2-D (2 × 4, 4, 2)-OOC with 3 codewords. Hence Φ(2 × 4, 4, 2) = J*(2 × 4) = 3.

Example 6.16 In this example, we construct an optimal 2-D $(2 \times 8, 4, 2)$ -OOC.

• Step 1: First we construct a strictly (2, 4)-regular 0-FG $(3, (\emptyset, 4), 16)$ of type 8^2 on $I_2 \times Z_8$ with the group set $\{I_2 \times H_i : 0 \le i \le 1\}$, where $H_0 = \{0, 2, 4, 6\}$ is a subgroup of order 4 in Z_8 and $H_1 = \{1, 3, 5, 7\}$. All the 14 base blocks are listed below:

• Step 2: Construct an optimal strictly 4-cyclic 3-(2 × 4,4,1) packing with 3 base blocks, which exists by Example 6.15. Then this step contributes 3 base blocks as follows

 $\{(0,0), (0,2), (0,4), (1,2)\}, \{(0,0), (0,2), (1,4), (1,6)\}, \{(0,0), (1,0), (1,2), (1,6)\}.$

Step 3: Apply Construction 6.14 to obtain a strictly 8-cyclic (2×8,4,1) packing with 17 base blocks, which achieves the upper bound in Theorem 5.7 and is an optimal 2-D (2×8,4,2)-OOC with 17 codewords. Hence Φ(2×8,4,2) = J*(2×8) = 17.

The following result is simple but very useful.

Lemma 6.17 If there exists a strictly (u, h)-regular 0- $FG(3, (\emptyset, k), uv)$ of type $(uh)^{v/h}$, then for any integer divisor h_1 of h, there exists a strictly h_1 -cyclic 0- $FG(3, (\emptyset, k), uv)$ of type $(uh)^{v/h}$.

Corollary 6.18 For any $v \not\equiv 2 \pmod{4}$, there exists a strictly v-cyclic 0-FG(3, $(\emptyset, 4), 4v$) of type $(2v)^2$.

Proof When $v \equiv 0 \pmod{4}$, by Lemma 6.12 there is a strictly cyclic 0-FG(3, $(\emptyset, 4), 4v$) of type $(2v)^2$. Apply Lemma 6.17 to obtain a strictly *v*-cyclic 0-FG(3, $(\emptyset, 4), 4v$) of type $(2v)^2$. When $v \equiv 1 \pmod{2}$, the conclusion follows from Lemma 6.9.

Lemma 6.19 If there is a perfect 2-D $(2 \times v, 4, 2)$ -OOC with $v \equiv 1, 5 \pmod{6}$, then there is a strictly (2, 1)-regular 1-FG(3, (2, 4), 2v) of type 2^v .

Proof By Lemma 3.4, the necessary condition for the existence of a perfect 2-D $(2 \times v, 4, 2)$ -OOC is $v \equiv 1, 5 \pmod{6}$. Suppose that (X, \mathcal{T}) is a strictly *v*-cyclic SQS $(2 \times v)$ with $X = I_2 \times Z_v$, which is equivalent to a perfect 2-D $(2 \times v, 4, 2)$ -OOC. Let $\mathcal{G} = \{I_2 \times \{x\} : x \in Z_v\}$. Then $(X, \mathcal{G}, \emptyset, \mathcal{T})$ is a strictly (2, 1)-regular 0-FG $(3, (\emptyset, 4), 2v)$ of type 2^v . Collect all 2-subsets of X from distinct groups of \mathcal{G} into a set \mathcal{B} . Since v is odd, $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ is a strictly (2, 1)-regular 1-FG(3, (2, 4), 2v) of type 2^v .

6.2 *H* designs

Mills first used the terminology of H designs in [42]. Let n, g, t be positive integers and K be a set of positive integers. An H design is a triple $(X, \mathcal{G}, \mathcal{B})$, where \mathcal{G} is a partition of a set of points X into n subsets (called groups), each of cardinality g, and \mathcal{B} is a collection of subsets of X (called *blocks*), each of cardinality from K, such that each block intersects any given group in at most one point, and each t-subset of X from tdistinct groups is contained in a unique block. Such a design is denoted by H(n, g, K, t).

Example 6.20 Take $X = I_8$ and $\mathcal{G} = \{\{0, 4\}, \{1, 5\}, \{2, 6\}, \{3, 7\}\}$. Let \mathcal{B} consists of the following 8 blocks

 $\{ 0,1,2,3 \}, \quad \{ 4,5,6,7 \}, \quad \{ 0,1,6,7 \}, \quad \{ 2,3,4,5 \}, \quad \{ 0,2,5,7 \}, \quad \{ 1,3,4,6 \}, \\ \{ 0,3,5,6 \}, \quad \{ 1,2,4,7 \}.$

It is easy to see that each 3-subset of I_8 from three distinct groups of \mathcal{G} is contained in a unique block of \mathcal{B} . Then $(X, \mathcal{G}, \mathcal{B})$ is an H(4, 2, 4, 3).

Example 6.21 Let $(X, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s, \mathcal{T})$ be an s- $FG(3, (K_1, K_2, \ldots, K_s, K_T), gn)$ of type g^n . Then for each $1 \leq i \leq s$, $(X, \mathcal{G}, \mathcal{B}_i)$ is an $H(n, g, K_i, 2)$ (called the *i*-th subdesign of the *s*-fan design).

Lemma 6.22 ([30,43]) For any $n \ge 4$, $n \ne 5$, an H(n, g, 4, 3) exists if and only if gn is even and g(n-1)(n-2) is divisible by 3. For n = 5, an H(5, g, 4, 3) exists if g is even, $g \ne 2$ and $g \not\equiv 10, 26 \pmod{48}$.

An *automorphism* of an H design $(X, \mathcal{G}, \mathcal{B})$ is a permutation on X leaving \mathcal{G}, \mathcal{B} invariant, respectively. All automorphisms of an H design form a group, called the *full automorphism group* of the H design. Any subgroup of the full automorphism group is called an *automorphism group* of the H design.

Let G be an automorphism group of an H design. All blocks of the H design can be partitioned into some orbits under G. Choose any fixed block from each orbit and then call it a *base block* of this H design. For any block B of the H design, the subgroup $\{\pi \in G : B^{\pi} = B\}$ is called the *stabilizer* of B in G.

Example 6.23 Observe the H(4, 2, 4, 3) from Example 6.20. Consider the permutation $(0 \ 4)(1 \ 5)(2 \ 6)(3 \ 7)$ on I_8 . It is readily checked that α is an automorphism of this H design. All blocks are partitioned into 4 orbits under the action of α . The 4 base blocks are $\{0, 1, 2, 3\}, \{0, 1, 6, 7\}, \{0, 2, 5, 7\}, \{0, 3, 5, 6\}.$

Construct an H(n, lh, K, t) on $I_n \times I_l \times Z_h$ with the group set $\{\{x\} \times I_l \times Z_h : x \in I_n\}$. If this H design admits an automorphism π mapping $(x, y, j) \mapsto (x, y, j + 1) \pmod{(-, -, h)}$, $x \in I_n$, $y \in I_l$ and $j \in Z_h$, then the H design is said to be h-cyclic. If the stabilizer of each block of an h-cyclic H(n, lh, K, t) in Z_h is trivial, i.e., for any block B, $\{\delta \in Z_h : B + \delta = B\} = \{0\}$, where $B + \delta = \{(x, y, j + \delta) : (x, y, j) \in B\}$, then the H design is called *strictly h*-cyclic. Note that one can verify that an h-cyclic H design is always strictly h-cyclic.

Example 6.24 By Example 6.23, the H(4, 2, 4, 3) from Example 6.20 admits an automorphism (0 4)(1 5)(2 6)(3 7). Actually the reader may check that this H design is isomorphic to a 2-cyclic H(4, 2, 4, 3) under the mapping $\tau : v \to (v \pmod{4}, 0, \lfloor v/4 \rfloor)$ from I_8 to $I_4 \times I_1 \times Z_2$.

When l = 1, an h-cyclic H(n, h, K, t) is often referred to as a semi-cyclic H(n, h, K, t).

Lemma 6.25 ([18]) For any $h \ge 1$, there exists a semi-cyclic H(4, h, 4, 3).

7 Weighting constructions

For applying Constructions 6.6 and 6.14, we need some strictly *h*-cyclic 0-FGs and strictly (u, h)-regular *s*-FGs. Construction 7.1 shows that if one has a strictly h_1 -cyclic 1-FG of type $(g_1h_1)^n$, and gives each point of the 1-FG a weight g_2h_2 , then a strictly h_1h_2 -cyclic 0-FG of type $(g_1g_2h_1h_2)^n$ can be obtained; Construction 7.3 shows that if one has a strictly (g_1, h_1) -regular 1-FG of type $(g_1h_1)^n$, and gives each point of the 1-FG a weight g_2h_2 , then a strictly (g_1g_2, h_1h_2) -regular 0-FG of type $(g_1g_2h_1h_2)^n$ can be obtained. So Constructions 7.1 and 7.3 give an approach to find some infinite families of strictly *h*-cyclic 0-FGs and strictly (u, h)-regular *s*-FGs. Then apply Constructions 6.6 and 6.14 to fill in the groups of these infinite families. We can obtain many optimal 2-D OOCs, which will be presented in Sections 8 and 9.

Condition (3) in Constructions 7.1 and 7.3 implies that h-cyclic H designs are important. Thus a recursive construction for h-cyclic H designs is presented in Construction 7.5. The proofs of all constructions in this section are of design theory. Here we only focus on how these constructions work. The detailed proofs of Constructions 7.1 and 7.5 have been moved to Appendix III. The detailed proof of Construction 7.3 is omitted, which is similar to that of Construction 7.1.

Construction 7.1 (Weighting Construction-I) Let K and L_i for each $1 \le i \le s$ be all sets of positive integers greater than 1. Let K_T and L_T be both sets of positive integers greater than 2. Suppose that the following exist:

- (1) a strictly h_1 -cyclic 1-FG(3, (K, K_T) , ng_1h_1) of type $(g_1h_1)^n$ (called the master design);
- (2) a strictly h_2 -cyclic s-FG(3, $(L_1, L_2, \ldots, L_s, L_T)$, kg_2h_2) of type $(g_2h_2)^k$ for each $k \in K$;
- (3) an h_2 -cyclic $H(k, g_2h_2, L_T, 3)$ for each $k \in K_T$.

Then there exists a strictly h_1h_2 -cyclic s-FG $(3, (L_1, L_2, \ldots, L_s, L_T), ng_1g_2h_1h_2)$ of type $(g_1g_2h_1h_2)^n$.

Step 1: Start from

a strictly h_1 -cyclic 1-FG(3, (K, K_T) , ng_1h_1) of type $(g_1h_1)^n$ $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$,

where $X = I_n \times I_{g_1} \times Z_{h_1}$ and $\mathcal{G} = \{\{x\} \times I_{g_1} \times Z_{h_1} : x \in I_n\}.$

• Denote the family of base blocks of this design by $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where \mathcal{F}_1 and \mathcal{F}_2 generate all the blocks of \mathcal{B} and \mathcal{T} respectively.

Step 2 (input): For any base block $B \in \mathcal{F}_1$, construct

a strictly h_2 -cyclic s-FG(3, $(L_1, L_2, \ldots, L_s, L_T)$, $|B|g_2h_2)$ of type $(g_2h_2)^{|B|}$

on $B \times I_{g_2} \times Z_{h_2}$ with the group set $\{\{x\} \times I_{g_2} \times Z_{h_2} : x \in B\}$.

• Denote the family of base blocks of the *j*-th subdesign $H(|B|, g_2h_2, L_j, 2)$ by \mathcal{A}_B^j for $1 \leq j \leq s$. Denote the family of all the other base blocks by \mathcal{D}_B .

Step 3 (input): For any base block $B \in \mathcal{F}_2$, construct

au

an
$$h_2$$
-cyclic $H(|B|, g_2h_2, L_T, 3)$

on $B \times I_{g_2} \times Z_{h_2}$ with the group set $\{\{x\} \times I_{g_2} \times Z_{h_2} : x \in B\}$.

• Denote the family of base blocks of this design by \mathcal{D}'_B .

Step 4 (mapping): Let

$$\mathcal{A}_j = \bigcup_{B \in \mathcal{F}_1} \mathcal{A}_B^j \text{ for } 1 \le j \le s, \qquad \mathcal{D} = (\bigcup_{B \in \mathcal{F}_1} \mathcal{D}_B) \bigcup (\bigcup_{B \in \mathcal{F}_2} \mathcal{D}_B').$$

For each $C \in (\bigcup_{1 \le j \le s} \mathcal{A}_j) \bigcup \mathcal{D}$ and each $(x, y, z, u, v) \in C$, define a mapping

:
$$(x, y, z, u, v) \longmapsto (x, y + ug_1, z + vh_1).$$

Define $\tau(C) = \{\tau(c) : c \in C\}$. Let

$$\mathcal{A}_j^* = \bigcup_{C \in \mathcal{A}_j} \tau(C), \ 1 \le j \le s, \qquad \mathcal{D}^* = \bigcup_{C \in \mathcal{D}} \tau(C).$$

Step 5 (final): Take

$$\mathcal{A}'_{j} = \{A + \delta : A \in \mathcal{A}^{*}_{j}, \delta \in Z_{h_{1}h_{2}}\}, \qquad \mathcal{D}' = \{A + \delta : A \in \mathcal{D}^{*}, \delta \in Z_{h_{1}h_{2}}\},$$

where $A + \delta = \{(x, y, z + \delta) \pmod{(-, -, h_1 h_2)} : (x, y, z) \in A\}$. Take

$$X' = I_n \times I_{g_1g_2} \times Z_{h_1h_2}, \qquad \mathcal{G}' = \{\{x\} \times I_{g_1g_2} \times Z_{h_1h_2} : x \in I_n\}$$

Then $(X', \mathcal{G}', \mathcal{A}_1', \ldots, \mathcal{A}_s', \mathcal{D}')$ is the required strictly h_1h_2 -cyclic s-FG $(3, (L_1, L_2, \ldots, L_s, L_T), ng_1g_2h_1h_2)$ of type $(g_1g_2h_1h_2)^n$.

Figure 2: Algorithm in Construction 7.1

The following example illustrates the algorithm presented in Figure 2.

Example 7.2 In this example, we construct an optimal 2-D $(8 \times 2, 4, 2)$ -OOC.

• Step 1: First construct a strictly 1-cyclic 1-FG(3, (2,4), 4) of type 1⁴ (X, $\mathcal{G}, \mathcal{B}, \mathcal{T}$) on $X = I_4 \times I_1 \times Z_1$ with the group set $\mathcal{G} = \{\{x\} \times I_1 \times Z_1 : x \in I_4\}$, which is trivial. Take

 $\mathcal{F}_1 = \{\{(i,0,0), (j,0,0)\} : \{i,j\} \in \{\{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}\}\},\$

which generates 6 blocks of \mathcal{B} under Z_1 such that $(X, \mathcal{G} \cup \mathcal{B})$ is a 2-design. Take $\mathcal{F}_2 = \{\{(0,0,0), (1,0,0), (2,0,0), (3,0,0)\}\},\$

which generates the unique block of \mathcal{T} such that $(X, \mathcal{G} \cup \mathcal{B} \cup \mathcal{T})$ is a 3-design.

• Step 2: For each $B = \{(i,0,0), (j,0,0)\} \in \mathcal{F}_1$, construct a strictly 2-cyclic 0- $FG(3, (\emptyset, 4), 8)$ of type 4^2 on $B \times I_2 \times Z_2$ with the group set $\{\{x\} \times I_2 \times Z_2 : x \in B\}$, which exists by Example 6.7. All the 6 base blocks of \mathcal{D}_B are listed below.

$$\begin{split} & \{(i,0,0,0,0),(i,0,0,0,1),(j,0,0,0,0),(j,0,0,1,0)\}, \\ & \{(i,0,0,0,0),(j,0,0,0,0),(j,0,0,0,1),(i,0,0,1,0)\}, \\ & \{(i,0,0,0,0),(j,0,0,0),(j,0,0,1,1),(i,0,0,1,1)\}, \\ & \{(i,0,0,0,0),(j,0,0,1),(j,0,0,1,0),(i,0,0,1,1)\}, \\ & \{(i,0,0,0,0),(j,0,0,1,0),(j,0,0,1,1),(i,0,0,1,0)\}, \\ & \{(j,0,0,0,0),(j,0,0,1,0),(i,0,0,1,0),(i,0,0,1,1)\}. \end{split}$$

- Step 3: For the unique $B \in \mathcal{F}_2$, construct a 2-cyclic H(4,4,4,3) on $B \times I_2 \times Z_2$ with the group set $\{\{x\} \times I_2 \times Z_2 : x \in B\}$, which exists by Corollary 7.6. Denote the family of base blocks of this design by \mathcal{D}'_B , and $|\mathcal{D}'_B| = 32$.
- Step 4: Let $\mathcal{D} = (\bigcup_{B \in \mathcal{F}_1} \mathcal{D}_B) \bigcup (\bigcup_{B \in \mathcal{F}_2} \mathcal{D}'_B)$. For each $C \in \mathcal{D}$ and each $(x, y, z, u, v) \in C$, define a mapping $\tau : (x, y, z, u, v) \longmapsto (x, y + u, z + v)$. Define $\tau(C) = \{\tau(c) : c \in C\}$. Let $\mathcal{D}^* = \bigcup_{C \in \mathcal{D}} \tau(C)$. Then $|\mathcal{D}^*| = 68$, which is just the number of base blocks in a strictly 2-cyclic 0-FG(3, (\emptyset , 4), 16) of type 4⁴.
- Step 5: Let $\mathcal{D}' = \{A+\delta : A \in \mathcal{D}^*, \delta \in Z_2\}$, where $A+\delta = \{(x, y, z+\delta) \pmod{(-, -, 2)} : (x, y, z) \in A\}$. Take $X' = I_4 \times I_2 \times Z_2$ and $\mathcal{G}' = \{\{x\} \times I_2 \times Z_2 : x \in I_4\}$. Then $(X', \mathcal{G}', \emptyset, \mathcal{D}')$ is the required strictly 2-cyclic 0-FG(3, $(\emptyset, 4), 16)$ of type 4^4 .
- Step 6: Apply Construction 6.6. Fill in the groups of the resulting strictly 2-cyclic 0-FG(3, (Ø, 4), 16) of type 4⁴ with a trivial optimal strictly 2-cyclic 3-(2 × 2, 4, 1) packing without base blocks. We have an optimal strictly 2-cyclic 3-(8 × 2, 4, 1) packing with 68 base blocks, which achieves the upper bound in Theorem 5.7 and is an optimal 2-D (8 × 2, 4, 2)-OOC. Hence Φ(8 × 2, 4, 2) = J*(8 × 2) = 68.

Construction 7.3 (Weighting Construction-II) Let K and L_i for each $1 \le i \le s$ be all sets of positive integers greater than 1. Let K_T and L_T be both sets of positive integers greater than 2. Suppose that the following exist:

- (1) a strictly (g_1, h_1) -regular 1-FG(3, (K, K_T) , $g_1h_1n)$ of type $(g_1h_1)^n$;
- (2) a strictly h_2 -cyclic s-FG(3, $(L_1, L_2, ..., L_s, L_T)$, kg_2h_2) of type $(g_2h_2)^k$ for each $k \in K$;
- (3) an h_2 -cyclic $H(k, g_2h_2, L_T, 3)$ for each $k \in K_T$.

Then there exists a strictly (g_1g_2, h_1h_2) -regular s- $FG(3, (L_1, L_2, ..., L_s, L_T), g_1g_2h_1h_2n)$ of type $(g_1g_2h_1h_2)^n$. Step 1: Start from

a strictly (g_1, h_1) -regular 1-FG $(3, (K, K_T), g_1h_1n)$ of type $(g_1h_1)^n (X, \mathcal{G}, \mathcal{B}, \mathcal{T})$,

on $X = I_{g_1} \times Z_{h_1n}$ with the group set $\mathcal{G} = \{I_{g_1} \times H_i : 0 \le i < n\}$, where $H = \{0, n, \dots, (h_1 - 1)n\}$ is a subgroup of order h_1 in Z_{h_1n} , and $H_i = H + i$ be a coset of H in Z_{h_1n} , $0 \le i < n$.

• Denote the family of base blocks of this design by $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where \mathcal{F}_1 and \mathcal{F}_2 generate all the blocks of \mathcal{B} and \mathcal{T} respectively.

Step 2 (input): For any base block $B \in \mathcal{F}_1$, construct

a strictly h_2 -cyclic s-FG(3, $(L_1, L_2, \ldots, L_s, L_T)$, $|B|g_2h_2$) of type $(g_2h_2)^{|B|}$

on $B \times I_{g_2} \times Z_{h_2}$ with the group set $\{\{x\} \times I_{g_2} \times Z_{h_2} : x \in B\}$.

• Denote the family of base blocks of the *j*-th subdesign $H(|B|, g_2h_2, L_j, 2)$ by \mathcal{A}_B^j for $1 \leq j \leq s$, and denote the family of all the other base blocks by \mathcal{D}_B .

Step 3 (input): For any base block $B \in \mathcal{F}_2$, construct

an
$$h_2$$
-cyclic $H(|B|, g_2h_2, L_T, 3)$

on $B \times I_{g_2} \times Z_{h_2}$ with the group set $\{\{x\} \times I_{g_2} \times Z_{h_2} : x \in B\}$.

• Denote the family of base blocks of this design by \mathcal{D}'_B .

Step 4 (mapping): Let

$$\mathcal{A}_{j} = \bigcup_{B \in \mathcal{F}_{1}} \mathcal{A}_{B}^{j} \text{ for } 1 \leq j \leq s, \qquad \mathcal{D} = (\bigcup_{B \in \mathcal{F}_{1}} \mathcal{D}_{B}) \bigcup (\bigcup_{B \in \mathcal{F}_{2}} \mathcal{D}_{B}')$$

For each $C \in (\bigcup_{1 \le j \le s} \mathcal{A}_j) \bigcup \mathcal{D}$ and each $(x, y, z, u) \in C$, define a mapping

:
$$(x, y, z, u) \longmapsto (x + zg_1, y + uh_1n).$$

Define $\tau(C) = \{\tau(c) : c \in C\}$. Let

$$\mathcal{A}_j^* = \bigcup_{C \in \mathcal{A}_j} \tau(C), \ 1 \le j \le s, \qquad \mathcal{D}^* = \bigcup_{C \in \mathcal{D}} \tau(C).$$

Step 5 (final): Take

$$\mathcal{A}'_{j} = \{A + \delta : A \in \mathcal{A}^{*}_{j}, \delta \in Z_{h_{1}h_{2}n}\}, \qquad \mathcal{D}' = \{A + \delta : A \in \mathcal{D}^{*}, \delta \in Z_{h_{1}h_{2}n}\},$$

where $A + \delta = \{(x, y + \delta) \pmod{(-, h_1 h_2 n)} : (x, y) \in A\}$. Let $H' = \{0, n, \dots, (h_1 h_2 - 1)n\}$ be a subgroup of order $h_1 h_2$ in $Z_{h_1 h_2 n}$, and $H'_i = H' + i$ be a coset of H' in $Z_{h_1 h_2 n}$, $0 \le i < n$. Take

 $X' = I_{g_1g_2} \times Z_{h_1h_2n}, \qquad \mathcal{G}' = \{I_{g_1g_2} \times H'_i : 0 \le i < n\}.$

Then $(X', \mathcal{G}', \mathcal{A}_1', \ldots, \mathcal{A}_s', \mathcal{D}')$ is the required strictly (g_1g_2, h_1h_2) -regular s-FG $(3, (L_1, L_2, \ldots, L_s, L_T), g_1g_2h_1h_2n)$ of type $(g_1g_2h_1h_2)^n$.

Figure 3: Algorithm in Construction 7.3

The following example illustrates the algorithm presented in Figure 3.

Example 7.4 In this example, we construct an optimal 2-D $(8 \times 4, 4, 2)$ -OOC.

• Step 1: First we construct a strictly (2,2)-regular 1-FG(3,(2,4),8) of type 4² as follows.

(1) Take a strictly (2,2)-regular 0-FG(3, $(\emptyset, 4), 8$) of type 4^2 $(X, \mathcal{G}, \emptyset, \mathcal{T})$ on $X = I_2 \times Z_4$ with the group set $\mathcal{G} = \{I_2 \times H_i : 0 \leq i < 2\}$, where $H_0 = \{0, 2\}$ is a subgroup of order 2 in Z_4 and $H_1 = \{1, 3\}$. It exists by Example 6.15. Denote the

family of base blocks of this design by \mathcal{F}_2 . It follows that \mathcal{F}_2 generates all the blocks of \mathcal{T} and $|\mathcal{F}_2| = 3$.

(2) Collect all 2-subsets from distinct groups of \mathcal{G} into a set \mathcal{B} . Then $(X, \mathcal{G} \cup \mathcal{B})$ is a 2-design. Hence, $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ is a strictly (2,2)-regular 1-FG(3, (2,4),8) of type 4^2 . Take

 $\mathcal{F}_1 = \{\{(0,0), (0,1)\}, \{(0,0), (1,1)\}, \{(1,0), (1,1)\}, \{(0,0), (1,3)\}\}.$

 \mathcal{F}_1 generates all the blocks of \mathcal{B} .

- Step 2: For each $B \in \mathcal{F}_1$, construct a strictly 1-cyclic 0-FG(3, $(\emptyset, 4), 8$) of type 4^2 on $B \times I_4 \times Z_1$ with the group set $\{\{x\} \times I_4 \times Z_1 : x \in B\}$, which can be taken from Example 6.1. Denote the family of base blocks of this design by \mathcal{D}_B , and $|\mathcal{D}_B| = 12$.
- Step 3: For each $B \in \mathcal{F}_2$, construct a 1-cyclic H(4, 4, 4, 3) on $B \times I_4 \times Z_1$ with the group set $\{\{x\} \times I_4 \times Z_1 : x \in B\}$, which exists by Corollary 7.6. Denote the family of base blocks of this design by \mathcal{D}'_B , and $|\mathcal{D}'_B| = 64$.
- Step 4: Let $\mathcal{D} = (\bigcup_{B \in \mathcal{F}_1} \mathcal{D}_B) \bigcup (\bigcup_{B \in \mathcal{F}_2} \mathcal{D}'_B)$. For each $C \in \mathcal{D}$ and each $(x, y, z, u) \in C$, define a mapping $\tau : (x, y, z, u) \longmapsto (x + 2z, y + 4u)$. Define $\tau(C) = \{\tau(c) : c \in C\}$. Let $\mathcal{D}^* = \bigcup_{C \in \mathcal{D}} \tau(C)$. Then $|\mathcal{D}^*| = 240$, which is just the number of base blocks in a strictly (8, 2)-regular 0-FG(3, $(\emptyset, 4), 32)$ of type 16².
- Step 5: Let $\mathcal{D}' = \{A + \delta : A \in \mathcal{D}^*, \delta \in Z_4\}$, where $A + \delta = \{(x, y + \delta) \pmod{(-, 4)} : (x, y) \in A\}$. Let $H' = \{0, 2\}$ be a subgroup of order 2 in Z_4 , and $H'_1 = \{1, 3\}$. Take $X' = I_8 \times Z_4$ and $\mathcal{G}' = \{I_8 \times H'_i : 0 \le i < 2\}$. Then $(X', \mathcal{G}', \emptyset, \mathcal{D}')$ is the required strictly (8, 2)-regular 0-FG $(3, (\emptyset, 4), 32)$ of type 16².
- Step 6: Apply Construction 6.14. Fill in the groups of the resulting strictly (8,2)-regular 0-FG(3, (∅, 4), 32) of type 16² with an optimal strictly 2-cyclic 3-(8×2, 4, 1) packing with 68 base blocks, which exists by Example 7.2. We have an optimal strictly 4-cyclic 3-(8×4, 4, 1) packing with 308 base blocks, which achieves the upper bound in Theorem 5.7 and is an optimal 2-D (8×4, 4, 2)-OOC with 308 codewords. Hence Φ(8×4, 4, 2) = J^{*}(8×4) = 308.

Step 1: Start from an h_1 -cyclic $H(n, g_1h_1, K, t)$ $(X, \mathcal{G}, \mathcal{B})$, where $X = I_n \times I_{g_1} \times Z_{h_1}$ and $\mathcal{G} = \{\{x\} \times I_{g_1} \times Z_{h_1} : x \in I_n\}.$

• Denote the family of base blocks of this design by \mathcal{F} .

Step 2 (input): For any base block $B \in \mathcal{F}$, construct an h_2 -cyclic $H(|B|, g_2h_2, L, t)$ on $B \times I_{g_2} \times Z_{h_2}$ with the group set $\{\{x\} \times I_{g_2} \times Z_{h_2} : x \in B\}$.

• Denote the family of base blocks of this design by \mathcal{D}_B .

Step 3 (mapping): Let $\mathcal{D} = \bigcup_{B \in \mathcal{F}} \mathcal{D}_B$. For each $C \in \mathcal{D}$ and each $(x, y, z, u, v) \in C$, define a mapping

 $\tau: \quad (x, y, z, u, v) \longmapsto (x, y + ug_1, z + vh_1).$

Define $\tau(C) = \{\tau(c) : c \in C\}$. Let $\mathcal{D}^* = \bigcup_{C \in \mathcal{D}} \tau(C)$. Step 4 (final): Take

 $\mathcal{D}' = \{ D + \delta : D \in \mathcal{D}^*, \delta \in Z_{h_1 h_2} \},\$

where $D + \delta = \{(x, y, z + \delta) \pmod{(-, -, h_1 h_2)} : (x, y, z) \in D\}$. Take

 $X' = I_n \times I_{g_1g_2} \times Z_{h_1h_2}, \qquad \mathcal{G}' = \{\{x\} \times I_{g_1g_2} \times Z_{h_1h_2} : x \in I_n\}.$

Then $(X', \mathcal{G}', \mathcal{D}')$ is the required h_1h_2 -cyclic $H(n, g_1g_2h_1h_2, L, t)$.

Figure 4: Algorithm in Construction 7.5

Construction 7.5 (Weighting Construction-III) Suppose that the following exist:

- (1) an h_1 -cyclic $H(n, g_1h_1, K, t)$;
- (2) an h_2 -cyclic $H(k, g_2h_2, L, t)$ for each $k \in K$.

Then there exists an h_1h_2 -cyclic $H(n, g_1g_2h_1h_2, L, t)$.

Corollary 7.6 For any $h \ge 1$ and $n \ge 4$, $n \ne 5$, if gn is even and g(n-1)(n-2) is divisible by 3, then there is an h-cyclic H(n, gh, 4, 3). For any $h \ge 1$ and n = 5, an h-cyclic H(5, gh, 4, 3) exists if g is even, $g \ne 2$ and $g \not\equiv 10, 26 \pmod{48}$.

Proof By Lemma 6.25, for any $h \ge 1$, there exists a semi-cyclic H(4, h, 4, 3) (i.e., an *h*-cyclic H(4, h, 4, 3)). Apply Construction 7.5 with $h_1 = g_2 = 1$, $g_1 = g$ and $h_2 = h$. Combine the results of Lemma 6.22 to complete the proof.

8 Small orders of optimal 2-D $(u \times v, 4, 2)$ -OOCs

In this section, we obtain some small orders of optimal 2-D OOCs. Some of them are obtained by computer search, and some of them are obtained by applying filling constructions in Section 6.

Lemma 8.1 There exists an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords for each $(u, v) \in \{(3, 3), (2, 6), (3, 4), (6, 2), (7, 2), (2, 11)\}$.

Proof We here give a construction of a 3-(uv, 4, 1)-packing on I_{uv} . Let $\alpha = (0 \ 1 \ \cdots \ v - 1)(v \ v + 1 \ \cdots \ 2v - 1) \cdots ((u - 1)v \ \cdots \ uv - 1)$ be a permutation on I_{uv} , which consists of u cycles of length v. Let G be the group generated by α . Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of G. Obviously this design is isomorphic to a strictly v-cyclic 3- $(u \times v, 4, 1)$ -packing, which achieves the upper bound in Theorem 5.7, and is an optimal 2-D $(u \times v, 4, 2)$ -OOC.

(u, v) = (3, 3):	$\{0, 1, 3, 4\}$	$\{0, 1, 5, 6\}$	$\{0, 1, 7, 8\}$	$\{0, 3, 6, 8\}$	$\{0, 4, 5, 7\}$	$\{3, 4, 7, 8\}$
(u, v) = (2, 6):	$\{0, 1, 2, 6\}$	$\{0, 1, 3, 8\}$	$\{0, 1, 4, 7\}$	$\{0, 1, 9, 10\}$	$\{0, 2, 8, 10\}$	$\{0, 6, 7, 9\}$
	$\{0, 6, 10, 11\}$	$\{0, 7, 8, 11\}$				
(u, v) = (3, 4):	$\{0, 2, 4, 5\}$	$\{0, 1, 2, 8\}$	$\{0, 1, 4, 9\}$	$\{0, 1, 5, 7\}$	$\{0, 1, 6, 10\}$	$\{0, 4, 7, 10\}$
	$\{0, 4, 8, 11\}$	$\{0, 5, 6, 11\}$	$\{0, 5, 8, 10\}$	$\{0, 6, 8, 9\}$	$\{0, 7, 9, 11\}$	$\{4, 5, 6, 9\}$
(u, v) = (6, 2):	$\{0, 1, 2, 4\}$	$\{0, 1, 6, 8\}$	$\{0, 2, 5, 9\}$	$\{0, 2, 3, 6\}$	$\{0, 2, 7, 8\}$	$\{0, 2, 10, 11\}$
	$\{0, 3, 4, 7\}$	$\{0, 3, 8, 9\}$	$\{0, 4, 5, 6\}$	$\{0, 4, 8, 10\}$	$\{0, 4, 9, 11\}$	$\{0, 5, 7, 10\}$
	$\{0, 5, 8, 11\}$	$\{0, 6, 7, 11\}$	$\{0, 6, 9, 10\}$	$\{2, 3, 4, 9\}$	$\{2, 4, 5, 10\}$	$\{2, 4, 6, 7\}$
	$\{2, 4, 8, 11\}$	$\{2, 5, 7, 11\}$	$\{2, 6, 8, 10\}$	$\{2, 6, 9, 11\}$	$\{2, 7, 9, 10\}$	$\{4, 6, 8, 9\}$
	$\{4, 7, 10, 11\}$					
(u, v) = (7, 2):	$\{0, 1, 2, 4\}$	$\{0, 1, 6, 8\}$	$\{0, 1, 10, 12\}$	$\{0, 2, 3, 7\}$	$\{0, 2, 5, 10\}$	$\{0, 2, 6, 13\}$
	$\{0, 2, 8, 11\}$	$\{0, 2, 9, 12\}$	$\{0, 3, 4, 6\}$	$\{0, 3, 8, 10\}$	$\{0, 3, 9, 11\}$	$\{0, 3, 12, 13\}$
	$\{0, 4, 5, 13\}$	$\{0, 4, 7, 10\}$	$\{0, 4, 8, 9\}$	$\{0, 4, 11, 12\}$	$\{0, 5, 6, 9\}$	$\{0, 5, 7, 11\}$
	$\{0, 5, 8, 12\}$	$\{0, 6, 7, 12\}$	$\{0, 6, 10, 11\}$	$\{0, 7, 8, 13\}$	$\{0, 9, 10, 13\}$	$\{2, 3, 4, 9\}$
	$\{2, 3, 10, 12\}$	$\{2, 4, 5, 6\}$	$\{2, 4, 7, 12\}$	$\{2, 4, 8, 13\}$	$\{2, 4, 10, 11\}$	$\{2, 5, 9, 13\}$
	$\{2, 5, 11, 12\}$	$\{2, 6, 7, 11\}$	$\{2, 6, 8, 12\}$	$\{2, 6, 9, 10\}$	$\{2, 7, 8, 9\}$	$\{2, 7, 10, 13\}$
	$\{4, 5, 8, 10\}$	$\{4, 6, 7, 9\}$	$\{4, 6, 8, 11\}$	$\{4, 6, 12, 13\}$	$\{4, 7, 11, 13\}$	$\{4, 9, 10, 12\}$
	$\{6, 8, 10, 13\}$	$\{6, 9, 11, 13\}$				
(u, v) = (2, 11):	$\{0, 1, 2, 4\}$	$\{0, 1, 5, 7\}$	$\{0, 1, 6, 9\}$	$\{0, 1, 8, 11\}$	$\{0, 1, 12, 13\}$	$\{0, 1, 14, 15\}$
	$\{0, 1, 16, 17\}$	$\{0, 1, 18, 19\}$	$\{0, 1, 20, 21\}$	$\{0, 2, 5, 11\}$	$\{0, 2, 12, 14\}$	$\{0, 2, 13, 15\}$
	$\{0, 2, 16, 19\}$	$\{0, 2, 17, 21\}$	$\{0, 2, 18, 20\}$	$\{0, 3, 7, 12\}$	$\{0, 3, 11, 15\}$	$\{0, 3, 13, 16\}$
	$\{0, 3, 17, 19\}$	$\{0, 3, 18, 21\}$	$\{0, 4, 11, 18\}$	$\{0, 4, 12, 15\}$	$\{0, 4, 13, 17\}$	$\{0, 4, 19, 21\}$
	$\{0, 5, 12, 17\}$	$\{0, 5, 13, 18\}$	$\{0, 5, 14, 19\}$	$\{0, 5, 15, 20\}$	$\{0, 5, 16, 21\}$	$\{0, 11, 14, 21\}$
	$\{0, 11, 17, 20\}$	$\{0, 12, 16, 20\}$	$\{11, 12, 13, 17\}$	$\{11, 12, 14, 19\}$	$\{11, 12, 18, 20\}$	

Lemma 8.2 There exists a strictly (2,6)-regular 0-FG $(3,(\emptyset,4),24)$ of type 12^2 .

Proof We here give a construction of a 0-FG(3, $(\emptyset, 4), 24$) of type 12^2 on I_{24} with the group set $\{\{2i + j : 0 \le i \le 11\} : 0 \le j \le 1\}$. Let $\alpha = (0 \ 1 \ \cdots \ 11)(12 \ 13 \ \cdots \ 23)$ be a permutation on I_{24} , which consists of 2 cycles of length 12. Let G be the group generated by α . Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of G. Obviously this design is isomorphic to a strictly (2, 6)-regular 0-FG $(3, (\emptyset, 4), 24)$ of type 12^2 .

 $\{0, 1, 2, 5\}$ $\{0, 1, 3, 8\}$ $\{0, 1, 6, 9\}$ $\{0, 1, 7, 12\}$ $\{0, 1, 10, 13\}$ $\{0, 1, 14, 15\}$ $\{0, 1, 16, 17\}$ $\{0, 1, 18, 19\}$ $\{0, 1, 20, 21\}$ $\{0, 1, 22, 23\}$ $\{0, 2, 9, 13\}$ $\{0, 2, 17, 19\}$ $\{0, 2, 21, 23\}$ $\{0, 3, 12, 19\}$ $\{0, 3, 13, 18\}$ $\{0, 3, 14, 21\}$ $\{0, 3, 17, 22\}$ $\{0, 3, 20, 23\}$ $\{0, 4, 13, 19\}$ $\{0, 4, 15, 23\}$ $\{0, 4, 17, 21\}$ $\{0, 5, 13, 20\}$ $\{0, 5, 14, 23\}$ $\{0, 5, 19, 22\}$ $\{0, 6, 13, 21\}$ $\{0, 12, 13, 15\}$ $\{0, 5, 12, 21\}$ $\{0, 5, 15, 18\}$ $\{0, 12, 17, 23\}$ $\{0, 13, 16, 23\}$ $\{12, 13, 14, 21\}$ $\{12, 13, 16, 19\}$ $\{12, 13, 17, 22\}$

Lemma 8.3 There exists an optimal 2-D $(2 \times 12, 4, 2)$ -OOC with $J^*(2 \times 12) = 41$ codewords.

Proof Start from a strictly (2, 6)-regular 0-FG $(3, (\emptyset, 4), 24)$ of type 12^2 , which exists by Lemma 8.2. Apply Construction 6.14 with an optimal strictly 6-cyclic $3-(2 \times 6, 4, 1)$ packing from Lemma 8.1 to obtain a strictly 12-cyclic $3-(2 \times 12, 4, 1)$ packing with 41 base blocks, which achieves the upper bound in Theorem 5.7, and is an optimal 2-D $(2 \times 12, 4, 2)$ -OOC with 41 codewords.

Lemma 8.4 There exists a strictly 2-cyclic 0-FG $(3, (\emptyset, 4), 24)$ of type 12^2 .

Proof We here give a construction of a 0-FG(3, $(\emptyset, 4), 24$) of type 12^2 on I_{24} with the group set $\{\{0, 1, \ldots, 11\} + i : i \in \{0, 12\}\}$. Let $\alpha = (0 \ 1)(2 \ 3) \cdots (22 \ 23)$ be a permutation on I_{24} and G be the group generated by α . Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of G. Obviously this design is isomorphic to a strictly 2-cyclic 0-FG(3, $(\emptyset, 4), 24)$ of type 12^2 .

$\{0, 1, 12, 14\}$	$\{0, 1, 16, 18\}$	$\{0, 1, 20, 22\}$	$\{0, 2, 12, 13\}$	$\{0, 2, 14, 15\}$	$\{0, 2, 16, 17\}$	$\{0, 2, 18, 19\}$
$\{0, 2, 20, 21\}$	$\{0, 2, 22, 23\}$	$\{0, 3, 12, 15\}$	$\{0, 3, 13, 14\}$	$\{0, 3, 16, 19\}$	$\{0, 3, 17, 18\}$	$\{0, 3, 20, 23\}$
$\{0, 3, 21, 22\}$	$\{0, 4, 12, 16\}$	$\{0, 4, 13, 17\}$	$\{0, 4, 14, 20\}$	$\{0, 4, 15, 21\}$	$\{0, 4, 18, 22\}$	$\{0, 4, 19, 23\}$
$\{0, 5, 12, 17\}$	$\{0, 5, 13, 20\}$	$\{0, 5, 14, 22\}$	$\{0, 5, 15, 18\}$	$\{0, 5, 16, 23\}$	$\{0, 5, 19, 21\}$	$\{0, 6, 12, 19\}$
$\{0, 6, 13, 16\}$	$\{0, 6, 14, 21\}$	$\{0, 6, 15, 20\}$	$\{0, 6, 17, 22\}$	$\{0, 6, 18, 23\}$	$\{0, 7, 12, 20\}$	$\{0, 7, 13, 21\}$
$\{0, 7, 14, 18\}$	$\{0, 7, 15, 19\}$	$\{0, 7, 16, 22\}$	$\{0, 7, 17, 23\}$	$\{0, 8, 12, 18\}$	$\{0, 8, 13, 19\}$	$\{0, 8, 14, 23\}$
$\{0, 8, 15, 22\}$	$\{0, 8, 16, 20\}$	$\{0, 8, 17, 21\}$	$\{0, 9, 12, 21\}$	$\{0, 9, 13, 18\}$	$\{0, 9, 14, 16\}$	$\{0, 9, 15, 23\}$
$\{0, 9, 17, 20\}$	$\{0, 9, 19, 22\}$	$\{0, 10, 12, 22\}$	$\{0, 10, 13, 23\}$	$\{0, 10, 14, 17\}$	$\{0, 10, 15, 16\}$	$\{0, 10, 18, 21\}$
$\{0, 10, 19, 20\}$	$\{0, 11, 12, 23\}$	$\{0, 11, 13, 22\}$	$\{0, 11, 14, 19\}$	$\{0, 11, 15, 17\}$	$\{0, 11, 16, 21\}$	$\{0, 11, 18, 20\}$
$\{2, 3, 12, 14\}$	$\{2, 3, 16, 18\}$	$\{2, 3, 20, 22\}$	$\{2, 4, 12, 17\}$	$\{2, 4, 13, 18\}$	$\{2, 4, 14, 21\}$	$\{2, 4, 15, 20\}$
$\{2, 4, 16, 23\}$	$\{2, 4, 19, 22\}$	$\{2, 5, 12, 18\}$	$\{2, 5, 13, 19\}$	$\{2, 5, 14, 23\}$	$\{2, 5, 15, 22\}$	$\{2, 5, 16, 20\}$
$\{2, 5, 17, 21\}$	$\{2, 6, 12, 16\}$	$\{2, 6, 13, 17\}$	$\{2, 6, 14, 20\}$	$\{2, 6, 15, 21\}$	$\{2, 6, 18, 22\}$	$\{2, 6, 19, 23\}$
$\{2, 7, 12, 19\}$	$\{2, 7, 13, 20\}$	$\{2, 7, 14, 22\}$	$\{2, 7, 15, 17\}$	$\{2, 7, 16, 21\}$	$\{2, 7, 18, 23\}$	$\{2, 8, 12, 22\}$
$\{2, 8, 13, 23\}$	$\{2, 8, 14, 16\}$	$\{2, 8, 15, 18\}$	$\{2, 8, 17, 20\}$	$\{2, 8, 19, 21\}$	$\{2, 9, 12, 23\}$	$\{2, 9, 13, 22\}$
$\{2, 9, 14, 17\}$	$\{2, 9, 15, 16\}$	$\{2, 9, 18, 21\}$	$\{2, 9, 19, 20\}$	$\{2, 10, 12, 21\}$	$\{2, 10, 13, 16\}$	$\{2, 10, 14, 19\}$
$\{2, 10, 15, 23\}$	$\{2, 10, 17, 22\}$	$\{2, 10, 18, 20\}$	$\{2, 11, 12, 20\}$	$\{2, 11, 13, 21\}$	$\{2, 11, 14, 18\}$	$\{2, 11, 15, 19\}$
$\{2, 11, 16, 22\}$	$\{2, 11, 17, 23\}$	$\{4, 5, 12, 14\}$	$\{4, 5, 16, 18\}$	$\{4, 5, 20, 22\}$	$\{4, 6, 12, 22\}$	$\{4, 6, 13, 23\}$
$\{4, 6, 14, 17\}$	$\{4, 6, 15, 16\}$	$\{4, 6, 18, 21\}$	$\{4, 6, 19, 20\}$	$\{4, 7, 12, 23\}$	$\{4, 7, 13, 22\}$	$\{4, 7, 14, 16\}$
$\{4, 7, 15, 18\}$	$\{4, 7, 17, 20\}$	$\{4, 7, 19, 21\}$	$\{4, 8, 12, 13\}$	$\{4, 8, 14, 15\}$	$\{4, 8, 16, 17\}$	$\{4, 8, 18, 19\}$
$\{4, 8, 20, 21\}$	$\{4, 8, 22, 23\}$	$\{4, 9, 12, 20\}$	$\{4, 9, 13, 21\}$	$\{4, 9, 14, 18\}$	$\{4, 9, 15, 19\}$	$\{4, 9, 16, 22\}$
$\{4, 9, 17, 23\}$	$\{4, 10, 12, 19\}$	$\{4, 10, 13, 20\}$	$\{4, 10, 14, 22\}$	$\{4, 10, 15, 17\}$	$\{4, 10, 16, 21\}$	$\{4, 10, 18, 23\}$
$\{4, 11, 12, 15\}$	$\{4, 11, 13, 14\}$	$\{4, 11, 16, 19\}$	$\{4, 11, 17, 18\}$	$\{4, 11, 20, 23\}$	$\{4, 11, 21, 22\}$	$\{6, 7, 12, 14\}$
$\{6, 7, 16, 18\}$	$\{6, 7, 20, 22\}$	$\{6, 8, 12, 15\}$	$\{6, 8, 13, 14\}$	$\{6, 8, 16, 19\}$	$\{6, 8, 17, 18\}$	$\{6, 8, 20, 23\}$
$\{6, 8, 21, 22\}$	$\{6, 9, 12, 17\}$	$\{6, 9, 13, 20\}$	$\{6, 9, 14, 22\}$	$\{6, 9, 15, 18\}$	$\{6, 9, 16, 23\}$	$\{6, 9, 19, 21\}$
$\{6, 10, 12, 13\}$	$\{6, 10, 14, 15\}$	$\{6, 10, 16, 17\}$	$\{6, 10, 18, 19\}$	$\{6, 10, 20, 21\}$	$\{6, 10, 22, 23\}$	$\{6, 11, 12, 18\}$
$\{6, 11, 13, 19\}$	$\{6, 11, 14, 23\}$	$\{6, 11, 15, 22\}$	$\{6, 11, 16, 20\}$	$\{6, 11, 17, 21\}$	$\{8, 9, 12, 14\}$	$\{8, 9, 16, 18\}$
$\{8, 9, 20, 22\}$	$\{8, 10, 12, 17\}$	$\{8, 10, 13, 18\}$	$\{8, 10, 14, 20\}$	$\{8, 10, 15, 21\}$	$\{8, 10, 16, 23\}$	$\{8, 10, 19, 22\}$
$\{8, 11, 12, 16\}$	$\{8, 11, 13, 17\}$	$\{8, 11, 14, 21\}$	$\{8, 11, 15, 20\}$	$\{8, 11, 18, 22\}$	$\{8, 11, 19, 23\}$	$\{10, 11, 12, 14\}$
$\{10, 11, 16, 18\}$	$\{10, 11, 20, 22\}$					

Lemma 8.5 There exists an optimal 2-D $(12 \times 2, 4, 2)$ -OOC with $J^*(12 \times 2) = 248$ codewords.

Proof Start from a strictly 2-cyclic 0-FG(3, $(\emptyset, 4), 24$) of type 12², which exists by Lemma 8.4. Applying Construction 6.6 with an optimal strictly 2-cyclic 3-(6 × 2, 4, 1) packing from Lemma 8.1, we have a strictly 2-cyclic 3-(12 × 2, 4, 1) packing with 248 base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D (12 × 2, 4, 2)-OOC with 248 codewords exists.

Lemma 8.6 There exists a strictly (2,3)-regular 0-FG $(3, (\emptyset, 4), 30)$ of type 6^5 .

Proof We here give a construction of a 0-FG(3, $(\emptyset, 4), 30$) of type 6^5 on I_{30} with the group set $\{\{5i + j : 0 \le i \le 5\} : 0 \le j \le 4\}$. Let $\alpha = (0 \ 1 \ \cdots \ 14)(15 \ 16 \ \cdots \ 29)$ be a permutation on I_{30} , which consists of 2 cycles of length 15. Let G be the group generated by α . Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of G. Obviously this design is isomorphic to a strictly (2, 3)-regular 0-FG(3, $(\emptyset, 4), 30)$ of type 6^5 .

$\{0, 1, 2, 4\}$	$\{0, 1, 5, 6\}$	$\{0, 1, 7, 9\}$	$\{0, 1, 8, 12\}$	$\{0, 1, 13, 15\}$	$\{0, 1, 16, 17\}$	$\{0, 1, 18, 19\}$
$\{0, 1, 20, 21\}$	$\{0, 1, 22, 23\}$	$\{0, 1, 24, 25\}$	$\{0, 1, 26, 27\}$	$\{0, 1, 28, 29\}$	$\{0, 2, 5, 12\}$	$\{0, 2, 6, 11\}$
$\{0, 2, 7, 15\}$	$\{0, 2, 10, 16\}$	$\{0, 2, 18, 20\}$	$\{0, 2, 19, 21\}$	$\{0, 2, 22, 24\}$	$\{0, 2, 23, 25\}$	$\{0, 2, 26, 28\}$
$\{0, 2, 27, 29\}$	$\{0, 3, 6, 15\}$	$\{0, 3, 7, 16\}$	$\{0, 3, 9, 19\}$	$\{0, 3, 18, 21\}$	$\{0, 3, 20, 23\}$	$\{0, 3, 22, 26\}$
$\{0, 3, 24, 29\}$	$\{0, 3, 25, 28\}$	$\{0, 4, 15, 19\}$	$\{0, 4, 16, 20\}$	$\{0, 4, 17, 24\}$	$\{0, 4, 18, 22\}$	$\{0, 4, 21, 27\}$
$\{0, 4, 23, 26\}$	$\{0, 4, 25, 29\}$	$\{0, 5, 16, 22\}$	$\{0, 5, 17, 27\}$	$\{0, 5, 18, 23\}$	$\{0, 5, 19, 24\}$	$\{0, 5, 26, 29\}$
$\{0, 6, 17, 22\}$	$\{0, 6, 18, 24\}$	$\{0, 6, 19, 28\}$	$\{0, 6, 20, 26\}$	$\{0, 6, 21, 29\}$	$\{0, 6, 23, 27\}$	$\{0, 7, 17, 25\}$
$\{0, 7, 18, 26\}$	$\{0, 7, 19, 27\}$	$\{0, 7, 20, 24\}$	$\{0, 7, 22, 29\}$	$\{0, 7, 23, 28\}$	$\{0, 15, 17, 29\}$	$\{0, 15, 21, 28\}$
$\{0, 15, 24, 27\}$	$\{0, 16, 19, 25\}$	$\{0, 16, 23, 29\}$	$\{0, 16, 24, 28\}$	$\{15, 16, 17, 28\}$	$\{15, 16, 19, 23\}$	$\{15, 16, 20, 22\}$
$\{15, 16, 21, 25\}$	$\{15, 16, 24, 26\}$	$\{15, 17, 20, 27\}$				

Lemma 8.7 There exists an optimal 2-D $(2 \times 15, 4, 2)$ -OOC with $J^*(2 \times 15) = 67$ codewords.

Proof Start from a strictly (2,3)-regular 0-FG $(3, (\emptyset, 4), 30)$ of type 6^5 , which exists by Lemma 8.6. Apply Construction 6.14 with an optimal strictly 3-cyclic $3-(2 \times 3, 4, 1)$ packing from Example 3.3 to obtain a strictly 15-cyclic $3-(2 \times 15, 4, 1)$ packing with 67 base blocks, which achieves the upper bound in Theorem 5.7, and is an optimal 2-D $(2 \times 15, 4, 2)$ -OOC with 67 codewords.

Lemma 8.8 There exists a strictly (3, 2)-regular 0-FG $(3, (\emptyset, 4), 30)$ of type 6^5 .

Proof We here give a construction of a 0-FG(3, $(\emptyset, 4), 30$) of type 6⁵ on I_{30} with the group set $\{\{5i + j : 0 \le i \le 5\} : 0 \le j \le 4\}$. Let $\alpha = (0 \ 1 \ \cdots \ 9)(10 \ 11 \ \cdots \ 19)(20 \ 21 \ \cdots \ 29)$ and $\beta = (0 \ 10 \ 20)(1 \ 11 \ 21) \cdots (9 \ 19 \ 29)$ be two permutations on I_{30} and G be the group generated by α and β . Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of G. Obviously this design is isomorphic to a (3, 2)-regular 0-FG $(3, (\emptyset, 4), 30)$ of type 6⁵.

Lemma 8.9 There exists an optimal 2-D $(3 \times 10, 4, 2)$ -OOC with $J^*(3 \times 10) = 100$ codewords.

Proof Start from a strictly (3, 2)-regular 0-FG $(3, (\emptyset, 4), 30)$ of type 6⁵, which exists by Lemma 8.8. Apply Construction 6.14 with an optimal strictly 2-cyclic 3- $(3 \times 2, 4, 1)$ packing from Example 3.3 to obtain a strictly 10-cyclic 3- $(3 \times 10, 4, 1)$ packing with 100 base blocks, which achieves the upper bound in Theorem 5.7, and is an optimal 2-D $(3 \times 10, 4, 2)$ -OOC with 100 codewords.

Lemma 8.10 Let $n \equiv 18 \pmod{24}$. If there is an optimal 1-D (n, 4, 2)-OOC, which achieves the Johnson bound $J(1 \times n, 4, 2) = \lfloor \frac{1}{4} \lfloor \frac{n-1}{3} \lfloor \frac{n-2}{2} \rfloor \rfloor$, then for any integer factorization n = uv, there is an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v) = \lfloor \frac{u}{4} \lfloor \lfloor \frac{n-1}{3} \lfloor \frac{n-2}{2} \rfloor \rfloor - 1 \rfloor$ codewords.

Proof By Corollary 2.2, if there exists an optimal 1-D (uv, 4, 2)-OOC with $J(1 \times uv, 4, 2)$ codewords, then there exists a 2-D $(u \times v, 4, 2)$ -OOC with $uJ(1 \times uv, 4, 2)$ codewords. It is readily checked that $uJ(1 \times uv, 4, 2) = u(u^2v^2 - 3uv - 6)/24 = \lfloor \frac{u}{4}(\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1) \rfloor = J^*(u \times v)$. This number achieves the upper bound in Theorem 5.7. This completes the proof.

Note that when $n \equiv 18 \pmod{24}$, $J(1 \times n, 4, 2) = \lfloor \frac{1}{4} \lfloor \frac{n-1}{3} \lfloor \frac{n-2}{2} \rfloor \rfloor = \lfloor \frac{1}{4} \lfloor \lfloor \frac{n-1}{3} \lfloor \frac{n-2}{2} \rfloor \rfloor - 1 \rfloor$. Hence no confusion occurs in Lemma 8.10. By Lemma 2.6(3), there is an optimal 1-D (n, 4, 2)-OOC with $J(1 \times n, 4, 2)$ codewords for each $n \in \{18, 42, 90\}$. Then we have

Corollary 8.11 Let $n \in \{18, 42, 90\}$. For any integer factorization $n = n_1n_2$, there is an optimal 2-D $(n_1 \times n_2, 4, 2)$ -OOC with $J^*(n_1 \times n_2)$ codewords.

Lemma 8.12 There exists an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords for each $(u, v) \in \{(5, 4), (7, 4), (6, 5)\}$.

Proof Apply Theorem 2.1 with some known optimal 2-D $(u_1 \times v_1, 4, 2)$ -OOCs. One can have all the required optimal 2-D $(u \times v, 4, 2)$ -OOCs. For illustrating the details, we give the following table.

(u_1, v_1)	Source	number of codewords	\Rightarrow	(u, v)	number of codewords	$J^*(u \times v)$
(1, 20)	Lemma 2.6	14		(5, 4)	70	70
(1, 28)	Lemma 2.6	29		(7, 4)	203	203
(2, 15)	Lemma 8.7	67		(6, 5)	201	201

9 Infinite families of optimal 2-D $(u \times v, 4, 2)$ -OOCs

In this section, on one hand we shall give some infinite families of optimal 2-D $(u \times v, 4, 2)$ -OOCs, which will be presented as Theorems. On the other hand, although we can not complete the existence of optimal 2-D $(u \times v, 4, 2)$ -OOCs, we hope to present some possible approaches to complete it, which will be presented as Propositions.

Lemma 9.1 There exists an optimal 2-D $(u \times 2, 4, 2)$ -OOC with $J^*(u \times 2)$ codewords for any $u \equiv 2, 4 \pmod{6}$.

Proof Let n = u/2. Then $n \equiv 1, 2 \pmod{3}$. When n = 1, an optimal 2-D $(2 \times 2, 4, 2)$ -OOC is trivial without base blocks. In the following consider $n \ge 2$. First we shall show that there is a strictly 2-cyclic 0-FG $(3, (\emptyset, 4), 4n)$ of type 4^n for any $n \equiv 1, 2 \pmod{3}$ and $n \ge 2$. When n = 2, a strictly 2-cyclic 0-FG $(3, (\emptyset, 4), 8)$ of type 4^2 exists by Example 6.7. When $n \equiv 1, 2 \pmod{3}$, $n \ge 4$ and $n \ne 5$, start from a 1-FG(3, (2, n), n) of type 1^n , which contains one block of size n and all 2-subsets of n points. Apply Construction 7.1 with $h_1 = 1$ and $h_2 = 2$ to obtain a strictly 2-cyclic 0-FG $(3, (\emptyset, 4), 4n)$ of type 4^n , where the needed 2-cyclic H(n, 4, 4, 3) is from Corollary 7.6. When n = 5, there is a strictly cyclic 0-FG $(3, (\emptyset, 4), 20)$ of type 4^5 .

Next applying Construction 6.6 with an optimal strictly 2-cyclic 3- $(2 \times 2, 4, 1)$ packing, which is trivial without base blocks, we have a strictly 2-cyclic 3- $(2n \times 2, 4, 1)$ packing, which contains $\lfloor \frac{2n}{4} (\lfloor \frac{4n-1}{3} \lfloor \frac{4n-2}{2} \rfloor \rfloor - 1) \rfloor = n(n-1)(4n+1)/3$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(2n \times 2, 4, 2)$ -OOC with $J^*(2n \times 2)$ codewords exists. It is an optimal 2-D $(u \times 2, 4, 2)$ -OOC with $J^*(u \times 2)$ codewords.

Proposition 9.2 Let $v \equiv 1 \pmod{2}$ or $v \equiv 0 \pmod{4}$. Suppose that there is an optimal 2-D $(2 \times v, 4, 2)$ -OOC with $J^*(2 \times v)$ codewords. Then there is an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords for any $u \equiv 2, 4 \pmod{6}$. Especially when $v \equiv 1, 5 \pmod{6}$, the resulting optimal 2-D $(u \times v, 4, 2)$ -OOC is perfect.

Proof Let n = u/2. Then $n \equiv 1, 2 \pmod{3}$. When n = 1, the conclusion follows from the assumption. In the following consider $n \geq 2$. First we shall show that there is a strictly v-cyclic 0-FG(3, $(\emptyset, 4), 2vn$) of type $(2v)^n$ for any $n \equiv 1, 2 \pmod{3}$ and $n \geq 2$. When n = 2, a strictly v-cyclic 0-FG(3, $(\emptyset, 4), 4v$) of type $(2v)^2$ is from Corollary 6.18. When $n \equiv 1, 2 \pmod{3}$, $n \geq 4$ and $n \neq 5$, start from a 1-FG(3, (2, n), n) of type 1^n , which contains a block of size n and all 2-subsets of n points. Apply Construction 7.1 with $h_1 = 1$ and $h_2 = v$ to obtain a strictly v-cyclic 0-FG(3, $(\emptyset, 4), 2vn$) of type $(2v)^n$, where the needed v-cyclic H(n, 2v, 4, 3) is from Corollary 7.6. When n = 5, there is a strictly cyclic 0-FG(3, $(\emptyset, 4), 10v)$ of type $(2v)^5$ from Lemma 6.12. By Lemma 6.17, it implies a strictly v-cyclic 0-FG(3, $(\emptyset, 4), 10v)$ of type $(2v)^5$.

Next apply Construction 6.6 with an optimal strictly v-cyclic $3-(2 \times v, 4, 1)$ packing with $J^*(2 \times v)$ base blocks, which exists by assumption. Note that by Theorem 5.7,

$$J^{*}(2 \times v) = \begin{cases} \lfloor \frac{2}{4} \lfloor \frac{2v-1}{3} \lfloor \frac{2v-2}{2} \rfloor \rfloor \rfloor, & \text{if } v \equiv 1, 5 \pmod{6}, \\ \lfloor \frac{2}{4} (\lfloor \frac{2v-1}{3} \lfloor \frac{2v-2}{2} \rfloor \rfloor - 1) \rfloor, & \text{if } v \equiv 3 \pmod{6} \text{ or } v \equiv 4, 8 \pmod{12}, \\ \lfloor \frac{2}{4} (\lfloor \frac{2v-1}{3} \lfloor \frac{2v-2}{2} \rfloor \rfloor - 2) \rfloor, & \text{if } v \equiv 0 \pmod{12}. \end{cases}$$

Then we have a strictly v-cyclic $3-(2n \times v, 4, 1)$ packing, which contains

$$n(nv-1)(2nv-1)/6 = \lfloor \frac{2n}{4} \lfloor \frac{2nv-1}{3} \lfloor \frac{2nv-2}{2} \rfloor \rfloor \rfloor, \quad \text{if } v \equiv 1,5 \pmod{6},$$

$$n(2n^2v^2 - 3nv - 3)/6 = \lfloor \frac{2n}{4} (\lfloor \frac{2nv-1}{3} \lfloor \frac{2nv-2}{2} \rfloor \rfloor - 1) \rfloor, \quad \text{if } v \equiv 3 \pmod{6},$$

$$n(nv-2)(2nv+1)/6 = \lfloor \frac{2n}{4} (\lfloor \frac{2nv-1}{3} \lfloor \frac{2nv-2}{2} \rfloor \rfloor - 1) \rfloor, \quad \text{if } v \equiv 4,8 \pmod{12},$$

$$n(2n^2v^2 - 3nv - 6)/6 = \lfloor \frac{2n}{4} (\lfloor \frac{2nv-1}{3} \lfloor \frac{2nv-2}{2} \rfloor \rfloor - 2) \rfloor, \quad \text{if } v \equiv 0 \pmod{12},$$

base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(2n \times v, 4, 2)$ -OOC with $J^*(2n \times v)$ codewords exists. It is an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords. Especially by Lemma 3.4, when $v \equiv 1,5 \pmod{6}$, the resulting optimal 2-D $(u \times v, 4, 2)$ -OOC is perfect.

Lemma 9.3 There is an optimal 2-D $(2 \times 2^n, 4, 2)$ -OOC with $J^*(2 \times 2^n)$ codewords for any positive integer n.

Proof When n = 1, an optimal 2-D (2×2, 4, 2)-OOC is trivial without codewords. When n=2, the conclusion follows from Example 6.15. When $n\geq 3$, by Lemma 6.12 there exists a strictly cyclic 0-FG(3, $(\emptyset, 4), 2^{n+1}$) of type $(2^n)^2$, denoted by $(X, \mathcal{G}, \emptyset, \mathcal{T})$, which is also a strictly $(1, 2^n)$ -regular 0-FG $(3, (\emptyset, 4), 2^{n+1})$ of type $(2^n)^2$. Collect all 2-subsets from distinct groups of \mathcal{G} into a set \mathcal{B} . Then $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ is a strictly $(1, 2^n)$ -regular $1-FG(3,(2,4),2^{n+1})$ of type $(2^n)^2$. Start from this 1-FG and apply Construction 7.3 with $h_1 = 2^n$ and $h_2 = 1$ to obtain a strictly $(2, 2^n)$ -regular 0-FG $(3, (\emptyset, 4), 2^{n+2})$ of type $(2^{n+1})^2$, where the needed strictly 1-cyclic 0-FG(3, $(\emptyset, 4), 4)$ of type 2^2 is from Theorem 6.4, and the needed 1-cyclic H(4, 2, 4, 3) is from Corollary 7.6. Now use induction on n. When n = 3, there is an optimal 2-D $(2 \times 2^3, 4, 2)$ -OOC with $J^*(2 \times 2^3)$ codewords from Example 6.16. Assume that an optimal 2-D $(2 \times 2^n, 4, 2)$ -OOC with $J^*(2 \times 2^n)$ codewords exists for some $n \geq 3$. Then start from a strictly $(2, 2^n)$ -regular 0-FG $(3, (\emptyset, 4), 2^{n+2})$ of type $(2^{n+1})^2$, and apply Construction 6.14 with an optimal 2-D $(2 \times 2^n, 4, 2)$ -OOC with $J^*(2 \times 2^n)$ codewords to obtain an 2-D $(2 \times 2^{n+1}, 4, 2)$ -OOC, which contains $(2^n - 1)(2^{n+2} + 1)$ $1)/3 = \lfloor \frac{2}{4} \lfloor \lfloor \frac{2^{n+2}-1}{3} \lfloor \frac{2^{n+2}-2}{2} \rfloor \rfloor - 1 \rfloor$ codewords. This number achieves the upper bound in Theorem 5.7 (note that for any integer $n \ge 2$, $2^n \equiv 4,8 \pmod{12}$). Thus an optimal 2-D $(2 \times 2^{n+1}, 4, 2)$ -OOC with $J^*(2 \times 2^{n+1})$ codewords exists.

Combining the results of Proposition 9.2 and Lemmas 9.1, 9.3, we have

Theorem 9.4 There is an optimal 2-D $(u \times 2^n, 4, 2)$ -OOC with $J^*(u \times 2^n)$ codewords for any $u \equiv 2, 4 \pmod{6}$ and any positive integer n.

Proposition 9.2 can only deal with the case of $u \equiv 2, 4 \pmod{6}$ and $v \not\equiv 2 \pmod{4}$. When $v \equiv 2 \pmod{4}$, we have the following proposition.

Proposition 9.5 Let $u \equiv 2, 4 \pmod{6}$ and $v \equiv 2 \pmod{4}$. Suppose that there is an optimal 2-D $(u/2 \times 2v, 4, 2)$ -OOC with $J^*(u/2 \times 2v)$ codewords. Then there is an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords.

Proof By Theorem 2.1, if there exists an optimal 2-D $(u/2 \times 2v, 4, 2)$ -OOC with $J^*(u/2 \times 2v)$ codewords, then there exits a 2-D $(u \times v, 4, 2)$ -OOC with $2J^*(u/2 \times 2v)$ codewords. Note that by Theorem 5.7,

$$J^*(u/2 \times 2v) = \begin{cases} \lfloor \frac{u}{8} \lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1 \rfloor \rfloor, & \text{if } v \equiv 2, 10 \pmod{12}, \\ \lfloor \frac{u}{8} \lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 2 \rfloor \rfloor, & \text{if } v \equiv 6 \pmod{12}. \end{cases}$$

It is readily checked that $2J^*(u/2 \times 2v) =$

$$\begin{cases} u(uv+1)(uv-4)/24 = \lfloor \frac{u}{4}(\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1) \rfloor, & \text{if } v \equiv 2, 10 \pmod{12}, \\ u(u^2v^2 - 3uv - 12)/24 = \lfloor \frac{u}{4}(\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 2) \rfloor, & \text{if } v \equiv 6 \pmod{12}. \end{cases}$$

This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords exists.

The following proposition shows another approach to obtain some optimal 2-D ($u \times v, 4, 2$)-OOCs with $v \equiv 2 \pmod{4}$.

Proposition 9.6 If there is a perfect 2-D $(2 \times v, 4, 2)$ -OOC with $v \equiv 1, 5 \pmod{6}$, then there is an optimal 2-D $(u \times 2v, 4, 2)$ -OOC for any $u \equiv 8, 16 \pmod{24}$ with $J^*(u \times 2v)$ codewords.

Proof Let n = u/8. Then $n \equiv 1, 2 \pmod{3}$. There is a strictly cyclic 0-FG(3, $(\emptyset, 4)$, 16n) of type $(8n)^2$, which exists by Lemma 6.12. By Lemma 6.17, it implies a strictly 2-cyclic 0-FG(3, $(\emptyset, 4), 16n$) of type $(8n)^2$. By Lemma 6.19, if there is a perfect 2-D $(2 \times v, 4, 2)$ -OOC with $v \equiv 1, 5 \pmod{6}$, then there is a strictly (2, 1)-regular 1-FG(3, (2, 4), 2v) of type 2^v . Start from this 1-FG and apply Construction 7.3 with $h_1 = 1$ and $h_2 = 2$ to obtain a strictly (8n, 2)-regular 0-FG(3, $(\emptyset, 4), 16nv$) of type $(16n)^v$, where the needed 2-cyclic H(4, 8n, 4, 3) is from Corollary 7.6. Applying Construction 6.14 with an optimal strictly 2-cyclic $3-(8n \times 2, 4, 1)$ packing with $J^*(8n \times 2) = \lfloor \frac{8n}{4}(\lfloor \frac{16n-1}{3} \lfloor \frac{16n-2}{2} \rfloor \rfloor - 1)\rfloor$ base blocks, which exists by Lemma 9.1, we have a strictly 2v-cyclic $3-(8n \times 2v, 4, 1)$ packing, which contains $4n(4nv-1)(16nv+1)/3 = \lfloor \frac{8n}{4}(\lfloor \frac{16nv-1}{3} \lfloor \frac{16nv-2}{2} \rfloor \rfloor - 1)\rfloor$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(8n \times 2v, 4, 2)$ -OOC with $J^*(8n \times 2v)$ codewords exists. It is an optimal 2-D $(u \times 2v, 4, 2)$ -OOC with $J^*(u \times 2v)$ codewords.

Theorem 9.7 Let $p \equiv 7 \pmod{12}$ be a prime or $p \in \{37, 61, 73, 109, 157, 181, 229, 277\}$. There exists an optimal $(u \times 2p, 4, 2)$ -OOC with $J^*(u \times 2p)$ codewords for any $u \equiv 8, 16 \pmod{24}$.

Proof Start from a perfect 2-D $(2 \times p, 4, 2)$ -OOC, which exists by Theorem 4.5. Apply Proposition 9.6 to complete the proof.

Lemma 9.8 Let $v \equiv 1 \pmod{2}$ or $v \equiv 0 \pmod{12}$. Suppose that there is an optimal 2-D $(12 \times v, 4, 2)$ -OOC with $J^*(12 \times v)$ codewords. Then there is an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords for any $u \equiv 0 \pmod{12}$.

Proof Let n = u/12. When n = 1, the conclusion follows from the assumption. When $n \ge 2$, by Theorem 6.4, there exists a 0-FG(3, $(\emptyset, 4), 6n$) of type 6^n $(X, \mathcal{G}, \emptyset, \mathcal{T})$. Collect all 2-subsets from distinct groups of \mathcal{G} into a set \mathcal{B} . Then $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ is a 1-FG(3, (2, 4), 6n) of type 6^n . Apply Construction 7.1 with $h_1 = 1$ and $h_2 = v$ to obtain a strictly v-cyclic 0-FG(3, $(\emptyset, 4), 12nv$) of type $(12v)^n$, where the needed strictly v-cyclic 0-FG(3, $(\emptyset, 4), 4v$) of type $(2v)^2$ is from Corollary 6.18, and the needed v-cyclic H(4, 2v, 4, 3) is from Corollary 7.6. Apply Construction 6.6 with an optimal strictly v-cyclic 3- $(12 \times v, 4, 1)$ packing with $J^*(12 \times v)$ base blocks, which exists by assumption. Note that by Theorem 5.7,

$$J^*(12 \times v) = \begin{cases} \lfloor \frac{12}{4} (\lfloor \frac{12v-1}{3} \lfloor \frac{12v-2}{2} \rfloor \rfloor - 1) \rfloor, & \text{if } v \equiv 1 \pmod{2}, \\ \\ \lfloor \frac{12}{4} (\lfloor \frac{12v-1}{3} \lfloor \frac{12v-2}{2} \rfloor \rfloor - 2) \rfloor, & \text{if } v \equiv 0 \pmod{12}. \end{cases}$$

Then we have a strictly v-cyclic $3-(12n \times v, 4, 1)$ packing, which contains

$$\begin{cases} 3n(24n^2v^2 - 6nv - 1) = \lfloor \frac{12n}{4}(\lfloor \frac{12nv-1}{3} \lfloor \frac{12nv-2}{2} \rfloor \rfloor - 1) \rfloor, & \text{if } v \equiv 1 \pmod{2}, \\ 6n(12n^2v^2 - 3nv - 1) = \lfloor \frac{12n}{4}(\lfloor \frac{12nv-1}{3} \lfloor \frac{12nv-2}{2} \rfloor \rfloor - 2) \rfloor, & \text{if } v \equiv 0 \pmod{12} \end{cases}$$

base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(12n \times v, 4, 2)$ -OOC with $J^*(12n \times v)$ codewords exists. It is an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords.

The use of Lemma 9.8 depends on the existence of optimal 2-D $(12 \times v, 4, 2)$ -OOCs with $J^*(12 \times v)$ codewords. The following lemma shows an approach to obtain some optimal 2-D $(12 \times v, 4, 2)$ -OOCs from perfect 2-D $(2 \times v, 4, 2)$ -OOCs.

Lemma 9.9 Suppose that there is a perfect 2-D $(2 \times v, 4, 2)$ -OOC with $v \equiv 1, 5 \pmod{6}$. Then there is an optimal 2-D $(12 \times v, 4, 2)$ -OOC with $J^*(12 \times v)$ codewords.

Proof By Lemma 6.19, if there is a perfect 2-D $(2 \times v, 4, 2)$ -OOC with $v \equiv 1, 5 \pmod{6}$, then there is a strictly (2, 1)-regular 1-FG(3, (2, 4), 2v) of type 2^v . Start from this 1-FG and apply Construction 7.3 with $h_1 = 1$ and $h_2 = 1$ to obtain a strictly (12, 1)-regular 0-FG $(3, (\emptyset, 4), 12v)$ of type 12^v , where the needed strictly 1-cyclic 0-FG $(3, (\emptyset, 4), 12)$ of type 6^2 is from Theorem 6.4, and the needed 1-cyclic H(4, 6, 4, 3) is from Corollary 7.6. Applying Construction 6.14 with an optimal strictly 1-cyclic $3-(12 \times 1, 4, 1)$ packing with 51 base blocks from Theorem 3.2, we have a strictly v-cyclic $3-(12 \times v, 4, 1)$ packing, which contains $72v^2 - 18v - 3 = \lfloor \frac{12}{4}(\lfloor \frac{12v-1}{3} \lfloor \frac{12v-2}{2} \rfloor \rfloor - 1)\rfloor$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(12 \times v, 4, 2)$ -OOC with $J^*(12 \times v)$ codewords exists.

Lemma 9.10 Let $v \equiv 3 \pmod{6}$ or $v \equiv 0 \pmod{12}$. Suppose that there is an optimal 2-D $(2 \times v, 4, 2)$ -OOC with $J^*(2 \times v)$ codewords. Then there is an optimal 2-D $(12 \times v, 4, 2)$ -OOC with $J^*(12 \times v)$ codewords.

Proof By Proposition 9.2, if there is an optimal 2-D $(2 \times v, 4, 2)$ -OOC with $J^*(2 \times v)$ codewords for $v \equiv 3 \pmod{6}$ or $v \equiv 0 \pmod{12}$, then there is an optimal 2-D $(4 \times v, 4, 2)$ -OOC with $J^*(4 \times v)$ codewords. Note that by Theorem 5.7,

$$J^*(4 \times v) = \begin{cases} \lfloor \frac{4}{4} (\lfloor \frac{4v-1}{3} \lfloor \frac{4v-2}{2} \rfloor \rfloor - 1) \rfloor, & \text{if } v \equiv 3 \pmod{6}, \\ \\ \lfloor \frac{4}{4} (\lfloor \frac{4v-1}{3} \lfloor \frac{4v-2}{2} \rfloor \rfloor - 2) \rfloor, & \text{if } v \equiv 0 \pmod{12}. \end{cases}$$

By Lemma 6.12, there is a strictly cyclic 0-FG(3, $(\emptyset, 4), 12v$) of type $(4v)^3$. By Lemma 6.17, it implies a strictly *v*-cyclic 0-FG(3, $(\emptyset, 4), 12v$) of type $(4v)^3$. Start from this strictly *v*-cyclic 0-FG and apply Construction 6.6 with an optimal strictly *v*-cyclic 3- $(4 \times v, 4, 1)$ packing with $J^*(4 \times v)$ base blocks, which is equivalent to an optimal 2-D ($4 \times v, 4, 2$)-OOC with $J^*(4 \times v)$ codewords, to obtain a strictly *v*-cyclic 3- $(12 \times v, 4, 1)$ packing, which contains

$$\begin{cases} 72v^2 - 18v - 3 = \lfloor \frac{12}{4} (\lfloor \frac{12v - 1}{3} \lfloor \frac{12v - 2}{2} \rfloor \rfloor - 1) \rfloor, & \text{if } v \equiv 3 \pmod{6}, \\ 72v^2 - 18v - 6 = \lfloor \frac{12}{4} (\lfloor \frac{12v - 1}{3} \lfloor \frac{12v - 2}{2} \rfloor \rfloor - 2) \rfloor, & \text{if } v \equiv 0 \pmod{12}, \end{cases}$$

base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(12 \times v, 4, 2)$ -OOC with $J^*(12 \times v)$ codewords exists.

Combining the results of Lemmas 9.8, 9.9 and 9.10, we have the following proposition.

Proposition 9.11 Let $v \equiv 1 \pmod{2}$ or $v \equiv 0 \pmod{12}$. Suppose that there is an optimal 2-D $(2 \times v, 4, 2)$ -OOC with $J^*(2 \times v)$ codewords. Then there is an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords for any $u \equiv 0 \pmod{12}$.

Lemma 9.12 There exists a strictly 3-cyclic 0-FG $(3, (\emptyset, 4), 18)$ of type 6^3 .

Proof We here give a construction of a 0-FG(3, $(\emptyset, 4)$, 18) of type 6^3 on I_{18} with the group set { $\{0, 1, 2, 3, 4, 5\} + i : i \in \{0, 6, 12\}$ }. Let $\alpha = (0 \ 1 \ 2)(3 \ 4 \ 5)(6 \ 7 \ 8)(9 \ 10 \ 11)(12 \ 13 \ 14)(15 \ 16 \ 17)$ be a permutation on I_{18} and G be the group generated by α . Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of G. Obviously this design is isomorphic to a strictly 3-cyclic 0-FG(3, $(\emptyset, 4), 18)$ of type 6^3 .

$\{0, 1, 6, 7\}$	$\{0, 1, 8, 11\}$	$\{0, 1, 9, 10\}$	$\{0, 1, 12, 13\}$	$\{0, 1, 14, 15\}$	$\{0, 1, 16, 17\}$	$\{0, 3, 6, 9\}$
$\{0, 3, 7, 8\}$	$\{0, 3, 10, 11\}$	$\{0, 3, 12, 15\}$	$\{0, 3, 13, 16\}$	$\{0, 3, 14, 17\}$	$\{0, 4, 6, 12\}$	$\{0, 4, 7, 14\}$
$\{0, 4, 8, 16\}$	$\{0, 4, 9, 13\}$	$\{0, 4, 10, 15\}$	$\{0, 4, 11, 17\}$	$\{0, 5, 6, 16\}$	$\{0, 5, 7, 12\}$	$\{0, 5, 8, 17\}$
$\{0, 5, 9, 14\}$	$\{0, 5, 10, 13\}$	$\{0, 5, 11, 15\}$	$\{0, 6, 10, 17\}$	$\{0, 6, 11, 14\}$	$\{0, 6, 13, 15\}$	$\{0, 7, 9, 16\}$
$\{0, 7, 11, 13\}$	$\{0, 7, 15, 17\}$	$\{0, 8, 9, 15\}$	$\{0, 8, 10, 12\}$	$\{0, 8, 13, 14\}$	$\{0, 9, 12, 17\}$	$\{0, 10, 14, 16\}$
$\{0, 11, 12, 16\}$	$\{3, 4, 6, 7\}$	$\{3, 4, 8, 11\}$	$\{3, 4, 9, 10\}$	$\{3, 4, 12, 17\}$	$\{3, 4, 13, 14\}$	$\{3, 4, 15, 16\}$
$\{3, 6, 10, 14\}$	$\{3, 6, 11, 17\}$	$\{3, 6, 12, 16\}$	$\{3, 7, 9, 13\}$	$\{3, 7, 11, 16\}$	$\{3, 7, 12, 14\}$	$\{3, 8, 9, 12\}$
$\{3, 8, 10, 15\}$	$\{3, 8, 16, 17\}$	$\{3, 9, 14, 15\}$	$\{3, 10, 13, 17\}$	$\{3, 11, 13, 15\}$	$\{6, 7, 12, 17\}$	$\{6, 7, 13, 16\}$
$\{6, 7, 14, 15\}$	$\{6, 9, 12, 13\}$	$\{6, 9, 14, 17\}$	$\{6, 9, 15, 16\}$	$\{9, 10, 12, 14\}$	$\{9, 10, 13, 16\}$	$\{9, 10, 15, 17\}$

Lemma 9.13 There exists an optimal 2-D $(u \times 3, 4, 2)$ -OOC with $J^*(u \times 3)$ codewords for any $u \equiv 6 \pmod{12}$.

Proof Let n = (u+2)/2. Then $n \equiv 4 \pmod{6}$. There is an SQS(n) [25]. Delete one point to obtain a 1-FG(3, (3, 4), n-1) of type 1^{n-1} . Start from this 1-FG and apply Construction 7.1 with $h_1 = 1$ and $h_2 = 3$ to obtain a strictly 3-cyclic 0-FG $(3, (\emptyset, 4), 6(n-1))$ of type 6^{n-1} , where the needed strictly 3-cyclic 0-FG $(3, (\emptyset, 4), 18)$ of type 6^3 is from Lemma 9.12, and the needed 3-cyclic H(4, 6, 4, 3) is from Corollary 7.6. Applying Construction 6.6 with an optimal strictly 3-cyclic $3-(2 \times 3, 4, 1)$ packing with $J^*(2 \times 3) = 1$ base block from Example 3.3, we have a strictly 3-cyclic $3-(2(n-1) \times 3, 4, 1)$ packing, which contains $(n-1)(6n^2 - 15n + 8)/2$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(2(n-1) \times 3, 4, 2)$ -OOC with $J^*(2(n-1) \times 3)$ codewords exists. It is an optimal 2-D $(u \times 3, 4, 2)$ -OOC with $J^*(u \times 3)$ codewords.

Theorem 9.14 There exists an optimal 2-D $(u \times 3, 4, 2)$ -OOC with $J^*(u \times 3)$ codewords for any $u \equiv 0 \pmod{2}$.

Proof When $u \equiv 2, 4 \pmod{6}$, apply Proposition 9.2 with an optimal 2-D $(2 \times 3, 4, 2)$ -OOC with $J^*(2 \times 3) = 1$ base block from Example 3.3 to obtain an optimal 2-D $(u \times 3, 4, 2)$ -OOC with $J^*(u \times 3)$ codewords. When $u \equiv 0 \pmod{12}$, apply Proposition 9.11 with an optimal 2-D $(2 \times 3, 4, 2)$ -OOC with $J^*(2 \times 3) = 1$ base block from Example 3.3 to obtain an optimal 2-D $(u \times 3, 4, 2)$ -OOC with $J^*(u \times 3)$ codewords. When $u \equiv 6 \pmod{12}$, the conclusion follows from Lemma 9.13.

Proposition 9.15 If there is an RoSQS(v+1), then there is an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords for any $u \equiv 0 \pmod{6}$.

Proof By Lemma 6.13, when $v \equiv 1 \pmod{6}$, an $\operatorname{RoSQS}(v+1)$ is equivalent to a strictly cyclic 1-FG(3, (3, 4), v) of type 1^v , which is also a strictly (1, 1)-regular 1-FG(3, (3, 4), v) of type 1^v . Start from this 1-FG and apply Construction 7.3 with $h_1 = 1$ and $h_2 = 1$ to obtain a strictly (u, 1)-regular 0- $FG(3, (\emptyset, 4), uv)$ of type u^v for any $u \equiv 0 \pmod{6}$, where the needed strictly 1-cyclic 0- $FG(3, (\emptyset, 4), 3u)$ of type u^3 exists from Theorem 6.4, and the needed 1-cyclic H(4, u, 4, 3) is from Corollary 7.6. Applying Construction 6.14 with an optimal strictly 1-cyclic $3 - (u \times 1, 4, 1)$ packing with $J^*(u \times 1)$ base blocks from Theorem 3.2, we have a strictly v-cyclic $3 - (u \times v, 4, 1)$ packing, which contains $u(u^2v^2 - 3uv - 6)/24 = \lfloor \frac{u}{4}(\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1) \rfloor$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords exists.

By Lemma 6.13, when $v \equiv 3 \pmod{6}$, an $\operatorname{RoSQS}(v+1)$ is equivalent to a strictly cyclic 1-FG(3, (3, 4), v) of type $3^{v/3}$, which is also a strictly (1, 3)-regular 1-FG(3, (3, 4), v) of type $3^{v/3}$. Start from this 1-FG and apply Construction 7.3 with $h_1 = 3$ and $h_2 = 1$ to obtain a strictly (u, 3)-regular 0-FG $(3, (\emptyset, 4), uv)$ of type $(3u)^{v/3}$ for any $u \equiv 0 \pmod{6}$. Applying Construction 6.14 with an optimal strictly 3-cyclic $3-(u \times 3, 4, 1)$ packing with $J^*(u \times 3)$ base blocks from Theorem 9.14, we have a strictly v-cyclic $3-(u \times v, 4, 1)$ packing, which contains $u(u^2v^2 - 3uv - 6)/24 = \lfloor \frac{u}{4}(\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1)\rfloor$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords exists.

Combining Theorem 4.4 and Proposition 9.15, many infinite families of optimal 2-D $(u \times v, 4, 2)$ -OOCs with $J^*(u \times v)$ codewords will be obtained. As an example, we have

Theorem 9.16 Let $p \equiv 7 \pmod{12}$ be a prime or $p \in \{37, 61, 73, 109, 157, 181, 229, 277\}$. There exist a perfect 2-D $(u \times p, 4, 2)$ -OOC for any $u \equiv 2, 4 \pmod{6}$, and an optimal 2-D $(u \times p, 4, 2)$ -OOC with $J^*(u \times p)$ codewords for any $u \equiv 0 \pmod{6}$.

Proof When $u \equiv 2, 4 \pmod{6}$, start from a perfect 2-D $(2 \times p, 4, 2)$ -OOC, which exists by Theorem 4.5, and apply Proposition 9.2 to obtain a perfect 2-D $(u \times p, 4, 2)$ -OOC. When $u \equiv 0 \pmod{6}$, start from an $\operatorname{RoSQS}(p+1)$, which exists by Theorem 4.4, and apply Proposition 9.15 to obtain an optimal 2-D $(u \times p, 4, 2)$ -OOC with $J^*(u \times p)$ codewords.

Lemma 9.17 If there is a perfect 2-D $(2 \times v, 4, 2)$ -OOC with $v \equiv 1, 5 \pmod{6}$, then there is an optimal 2-D $(12 \times 2v, 4, 2)$ -OOC with $J^*(12 \times 2v)$ codewords.

Proof By Lemma 6.19, if there is a perfect 2-D $(2 \times v, 4, 2)$ -OOC with $v \equiv 1, 5 \pmod{6}$, then there is a strictly (2, 1)-regular 1-FG(3, (2, 4), 2v) of type 2^v . Start from this 1-FG and apply Construction 7.3 with $h_1 = 1$ and $h_2 = 2$ to obtain a strictly (12, 2)-regular 0-FG $(3, (\emptyset, 4), 24v)$ of type 24^v , where the needed strictly 2-cyclic 0-FG $(3, (\emptyset, 4), 24)$ of type 12^2 is from Lemma 8.4, and the needed 2-cyclic H(4, 12, 4, 3) is from Corollary 7.6. Applying Construction 6.14 with an optimal strictly 2-cyclic $3-(12 \times 2, 4, 1)$ packing with $J^*(12 \times 2) = 248$ base blocks from Lemma 8.5, we have a strictly 2v-cyclic $3-(12 \times 2v, 4, 1)$ packing, which contains $4(72v^2 - 9v - 1) = \lfloor \frac{12}{4}(\lfloor \frac{24v-1}{3} \lfloor \frac{24v-2}{2} \rfloor \rfloor - 1) \rfloor - 1$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(12 \times 2v, 4, 2)$ -OOC with $J^*(12 \times 2v)$ codewords exists.

Theorem 9.18 Let $p \equiv 7 \pmod{12}$ be a prime or $p \in \{37, 61, 73, 109, 157, 181, 229, 277\}$. There exists an optimal $(12 \times 2p, 4, 2)$ -OOC with $J^*(12 \times 2p)$ codewords.

Proof Start from a perfect 2-D $(2 \times p, 4, 2)$ -OOC, which exists by Theorem 4.5. Apply Lemma 9.17 to complete the proof.

Table II Small orders of optimal 2-D $(u \times v, 4, 2)$ -OOCs with $\Phi(u \times v, 4, 2) = J^*(u \times v)$ codewords for $6 \le uv \le 34$

uv	$u \times v$	$\Phi(u \times v, 4, 2)$	Source	$u \times v$	$\Phi(u \times v, 4, 2)$	Source
6	2×3	1	Example 3.3	3×2	1	Example 3.3
8	2×4	3	Example 6.15	4×2	6	Example 6.7
ğ	3×3	6	Lemma 8.1	1	Ŭ	Enample off
10	2×5	6	Theorem $2.7(2)$	5×2	15	Theorem $2.7(2)$
12	2×6	8	Lemma 8.1	3×4	12	Lemma 8.1
	4×3	17	Example 6.8	6×2	25	Lemma 8.1
14	2×7	13	Theorem 4.5	7×2	44	Lemma 8.1
15	3×5	21	Theorem $2.7(2)$	5×3	35	Theorem $2.7(2)$
16	2×8	17	Example 6.16	4×4	34	Theorem 9.4
	8×2	68	Lemma 9.1			
18	2×9	22	Corollary 8.11	3×6	33	Corollary 8.11
	6×3	66	Corollary 8.11	9×2	99	Corollary 8.11
20	2×10	28	Theorem $2.7(4)$	4×5	57	Proposition 9.2
	5×4	70	Lemma 8.12	10×2	140	Lemma 9.1
21	3×7	45	Theorem $2.7(2)$	7×3	105	Theorem $2.7(2)$
22	2×11	35	Lemma 8.1	11×2		??
24	2×12	41	Lemma 8.3	3×8		??
	4×6	82	Proposition 9.5	6×4		??
	8×3	166	Theorem 9.14	12×2	248	Lemma 8.5
25	5×5	110	Theorem $2.7(2)$			
26	2×13	50	Theorem $2.7(2)$	13×2	325	Theorem $2.7(2)$
27	3×9	78	Theorem $2.7(2)$	9×3	234	Theorem $2.7(2)$
28	2×14	58	Theorem $2.7(4)$	4×7	117	Proposition 9.2
	7×4	203	Lemma 8.12	14×2	406	Lemma 9.1
30	2×15	67	Lemma 8.7	3×10	100	Lemma 8.9
	5×6		??	6×5	201	Lemma 8.12
	10×3	335	Theorem 9.14	15×2		??
32	2×16	77	Theorem 9.4	4×8	154	Theorem 9.4
	8×4	308	Theorem 9.4	16×2	616	Lemma 9.1
33	3×11	120	Theorem $2.7(2)$	11×3	440	Theorem $2.7(2)$
34	2×17	88	Theorem $2.7(2)$	17×2	748	Theorem $2.7(2)$

Finally we summarize the existence of small orders of optimal 2-D $(u \times v, 4, 2)$ -OOCs with $J^*(u \times v)$ codewords as follows.

Theorem 9.19 There exists an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords for each (u, v) satisfying $6 \le uv \le 34$ and $(u, v) \notin \{(1, 9), (1, 12), (1, 13), (11, 2), (23, 1), (3, 8), (6, 4), (5, 6), (15, 2)\}$. When $(u, v) \in \{(1, 9), (1, 12), (1, 13)\}$, there exists an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J(1 \times uv, 4, 2) - 1$ codewords.

Proof By Theorem 3.2, there exists an optimal 2-D $(u \times 1, 4, 2)$ -OOC with $J^*(u \times 1)$ codewords for each $6 \le u \le 34$ and $u \ne 23$. By Lemma 2.6(3), there exists an optimal 2-D $(1 \times v, 4, 2)$ -OOC with $J^*(1 \times v)$ codewords for each $7 \le v \le 34$ and $v \notin \{9, 12, 13\}$. Note that $J^*(1 \times v) = J(1 \times v, 4, 2)$ when $v \ne 0 \pmod{24}$, and $J^*(1 \times v) = J(1 \times v, 4, 2) - 1$ when $v \equiv 0 \pmod{24}$. When $v \in \{9, 12, 13\}$, by Lemma 2.6(4), there exists an optimal 2-D $(1 \times v, 4, 2)$ -OOC with $J(1 \times v, 4, 2) - 1$ codewords. An optimal 2-D $(1 \times 6, 4, 2)$ -OOC is trivial without codewords. For all other cases of (u, v) such that there is an optimal

2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords, we show the sources in Table II, where the question marks "??" indicates the orders for each of which the existence of an optimal 2-D $(u \times v, 4, 2)$ -OOC with $J^*(u \times v)$ codewords is still open.

10 Conclusion

In this paper, we gave some combinatorial constructions for optimal 2-D $(u \times v, k, 2)$ -OOCs. As applications, many infinite families of optimal 2-D $(u \times v, 4, 2)$ -OOCs are obtained. We summarize all infinite families obtained in this paper in Table III. Although we can not complete the existence of optimal 2-D $(u \times v, 4, 2)$ -OOCs, we hope to present some possible approaches to reduce the existence problem. We summarize these approaches in Table IV.

Table III New infinite families of optimal 2-D $(u \times v, 4, 2)$ -OOCs with $J^*(u \times v)$ codewords in this paper

Parameters	Conditions	Source
$(n_1 \times n_2, 4, 2)$	$\begin{array}{l} n_1 n_2 = uv, \\ u \in \{4^n - 1: n \ge 1\} \cup \{1, 27, 33, 39, 51, 87, 123, 183\} \text{ and} \\ v \in S = \{p \equiv 7 \pmod{12}: p \text{ is a prime}\} \cup \\ \{2^n - 1: \text{odd integer } n \ge 1\} \cup \{25, 37, 61, 73, 109, 157, 181, 229, 277\}, \\ \text{ or } v \text{ is a product of integers from } S \end{array}$	Theorem 2.7(1)
$(2 \times 2n, 4, 2)$	$n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ $p_i = 13 \text{ or } p_i \equiv 5 \pmod{12} \text{ is a prime and } p_i < 1500000,$ $r_i \ge 1 \text{ for } 1 \le i \le s$	Theorem $2.7(3)$
$(u \times 2^n, 4, 2)$	$u \equiv 2, 4 \pmod{6}$ and $n \ge 1$	Theorem 9.4
$(u \times 2p, 4, 2)$	$u \equiv 8, 16 \pmod{24}$ or $u = 12$	Theorem 9.7
	$p \equiv 7 \pmod{12}$ a prime or $p \in \{37, 61, 73, 109, 157, 181, 229, 277\}$	Theorem 9.18
$(u \times p, 4, 2)$	$u \equiv 0 \pmod{2}$	Theorem 9.14
	$p \equiv 7 \pmod{12}$ a prime or $p \in \{3, 37, 61, 73, 109, 157, 181, 229, 277\}$	Theorem 9.16

Input	\Rightarrow	Output	Source
optimal 1-D $(n, 4, 2)$ -OOC		optimal 2-D $(u \times v, 4, 2)$ -OOC	
with $J(1 \times n, 4, 2)$ codewords,		with $J(u \times v, 4, 2)$ codewords,	Corollary $2.5(1)$
$n \equiv 1,3 \pmod{6}$ or $n \equiv 2,10 \pmod{24}$		for any integer factorization $n = uv$	
optimal 2-D($2 \times v, 4, 2$)-OOC		optimal 2-D $(u \times v, 4, 2)$ -OOC	
with $J^*(2 \times v)$ codewords,		with $J^*(u \times v)$ codewords,	Proposition 9.2
$v \equiv 1 \pmod{2}$ or $v \equiv 0 \pmod{4}$		$u \equiv 2, 4 \pmod{6}$	
optimal 2-D $(u/2 \times 2v, 4, 2)$ -OOC		optimal 2-D $(u \times v, 4, 2)$ -OOC	
with $J^*(u/2 \times 2v)$ codewords,		with $J^*(u \times v)$ codewords	Proposition 9.5
$u \equiv 2, 4 \pmod{6}$ and $v \equiv 2 \pmod{4}$			
perfect 2-D $(2 \times v, 4, 2)$ -OOC		optimal 2-D $(u \times 2v, 4, 2)$ -OOC	
$v \equiv 1,5 \pmod{6}$		with $J^*(u \times 2v)$ codewords,	Proposition 9.6
		$u \equiv 8, 16 \pmod{24}$	
optimal 2-D $(2 \times v, 4, 2)$ -OOC		optimal 2-D $(u \times v, 4, 2)$ -OOC	
with $J^*(2 \times v)$ codewords,		with $J^*(u \times v)$ codewords,	Proposition 9.11
$v \equiv 1 \pmod{2}$ or $v \equiv 0 \pmod{12}$		$u \equiv 0 \pmod{12}$	
RoSQS(v+1)		optimal 2-D $(u \times v, 4, 2)$ -OOC	
$v \equiv 1,3 \pmod{6}$		with $J^*(u \times v)$ codewords,	Proposition 9.15
		$u \equiv 0 \pmod{6}$	

By Theorem 5.7, we see that in many cases the Johnson bound can not be achieved. A natural question is whether the bounds established in Theorem 5.7 is good enough to make each optimal 2-D $(u \times v, 4, 2)$ -OOC achieve it. Although many infinite families are given to achieve the upper bound in Theorem 5.7, we still tend to think it not true. For example we conjecture that when $u \equiv 0 \pmod{6}$ and $v \equiv 2, 4 \pmod{6}$, the upper bound is $\lfloor \frac{u}{4}(\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1) \rfloor - \lfloor \frac{u}{12} \rfloor$. If the conjecture is correct, the condition in Lemma 9.8 can be relaxed to $v \neq 2 \pmod{4}$, which implies that the condition in Proposition 9.11 can also be relaxed to $v \neq 2 \pmod{4}$.

Another question is to find more constructions for optimal 2-D $(2 \times v, 4, 2)$ -OOCs, which are very useful by Propositions 9.2, 9.6 and 9.11. In 1991 Phelps [50] constructed a class of 2-chromatic SQS(22) using cyclic large sets of 2-(11,3,1) packings. It seems that Phelps's method can be generalized to construct some strictly v-cyclic SQS(2 × v)s for $v \equiv 1, 5 \pmod{6}$, which are also perfect 2-D $(2 \times v, 4, 2)$ -OOCs. The interested reader may refer to the paper [50].

Appendix I

Proof of Construction 4.2: For checking the correctness of the algorithm shown in Figure 1, first count the number of base blocks in \mathcal{A} . It is clear that $|\mathcal{A}_1 \cup \mathcal{A}'_1| = (p-1)(p-3)/12$, $|\mathcal{A}_2 \cup \mathcal{A}'_2| = (p-1)/3$, $|\mathcal{A}_3| \le 3 \times (p-1)/6 \times (p-1)/2 = (p-1)^2/4$. Thus $|\mathcal{A}| \le (p-1)(2p-1)/6$.

Since the number (p-1)(2p-1)/6 is the right number of base blocks in a strictly *p*-cyclic $SQS(2 \times p)$, in the following it suffices to show that each triple of $I_2 \times Z_p$ appears in at least one block of the resulting design. (1) Each triple of $\{0\} \times Z_p$ appears in one block of $\mathcal{A}_1 \cup \mathcal{A}_2$ and their cyclic shifts. (2) Each triple of $\{1\} \times Z_p$ appears in one block of $\mathcal{A}'_1 \cup \mathcal{A}'_2$ and their cyclic shifts. (3) Each triple of the form $\{x_0, y_0, z_1\}$ appears in one block of $\mathcal{A}'_2 \cup \mathcal{A}_3$ and their cyclic shifts. (4) Each triple of the form $\{x_1, y_1, z_0\}$ appears in one block of $\mathcal{A}'_2 \cup \mathcal{A}_3$ and their cyclic shifts. \Box

Appendix II

Lemma 5.2 Let $u \equiv 0 \pmod{12}$ and $v \equiv 2,4 \pmod{6}$. Then $\Phi(u \times v, 4, 2) \leq \lfloor \frac{u}{4} \lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1 \rfloor \rfloor - 1$.

Proof First we shall show that $\Phi(u \times 2, 4, 2) \leq \lfloor \frac{u}{4}(\lfloor \frac{2u-1}{3} \lfloor \frac{2u-2}{2} \rfloor \rfloor - 1) \rfloor - 1$. By Lemma 5.1, $\Phi(u \times 2, 4, 2) \leq \lfloor \frac{u}{4}(\lfloor \frac{2u-1}{3} \lfloor \frac{2u-2}{2} \rfloor \rfloor - 1) \rfloor$. Suppose that $\Phi(u \times 2, 4, 2) = \lfloor \frac{u}{4}(\lfloor \frac{2u-1}{3} \lfloor \frac{2u-2}{2} \rfloor \rfloor - 1) \rfloor$ 1)]. Then there were a strictly 2-cyclic 3- $(u \times 2, 4, 1)$ -packing with $\lfloor \frac{u}{4}(\lfloor \frac{2u-1}{3} \lfloor \frac{2u-2}{2} \rfloor \rfloor - 1) \rfloor$ base blocks. Let \mathcal{L} be the leave of the strictly 2-cyclic 3- $(u \times 2, 4, 1)$ -packing. Count the number of 3-subsets in the leave \mathcal{L} . It is $\binom{2u}{3} - \lfloor \frac{u}{4}(\lfloor \frac{2u-1}{3} \lfloor \frac{2u-2}{2} \rfloor \rfloor - 1) \rfloor \cdot 2 \cdot 4 = 8u/3$. For each $a \in I_u$ and each $i \in Z_2$, consider the number n of 3-subsets containing the

For each $a \in I_u$ and each $i \in Z_2$, consider the number n of 3-subsets containing the point (a, i) in the leave \mathcal{L} . Delete one point from a strictly 2-cyclic 3- $(u \times 2, 4, 1)$ -packing to obtain a 2-(2u-1, 3, 1)-packing, which contains at most $\lfloor (2u-1)(2u-2)/6 \rfloor - 1$ blocks when $2u \equiv 0 \pmod{6}$ [26]. Since each 3-subset of $I_u \times Z_2$ occurs in at most one block, we have $n \ge \binom{2u-1}{2} - 3(\lfloor (2u-1)(2u-2)/6 \rfloor - 1) = 4$, which implies that $|\mathcal{L}| \ge 4 \cdot 2u/3$. Due to $|\mathcal{L}| = 8u/3$, n must be equal to 4. Note that the above conclusion holds for each $a \in I_u$ and each $i \in Z_2$.

For each $a \in I_u$, consider the number m of the base blocks containing the two points (a, 0), (a, 1). Since each 3-subset of $I_u \times Z_2$ occurs in at most one block and each

base block containing the two points (a, 0), (a, 1) generates exactly two different blocks containing the same two points, the number m is at most $\lfloor (2u-2)/4 \rfloor = (2u-4)/4$. Thus there are at least two 3-subsets containing the two points (a, 0), (a, 1) in the leave, denoted by $\{(a, 0), (a, 1), (b_a, 0)\}$ and $\{(a, 0), (a, 1), (b_a, 1)\}$, where $b_a \in I_u$ and $b_a \neq a$. Note that the above conclusion holds for each $a \in I_u$. We have that $\mathcal{L} \supset$ $\{\{(a, 0), (a, 1), (b_a, 0)\}, \{(a, 0), (a, 1), (b_a, 1)\}: a \in I_u\}$.

Given any $a \in I_u$, consider the number r of the blocks containing the two points $(a, 0), (b_a, 0)$. Since each 3-subset of $I_u \times Z_2$ occurs in at most one block, the number r is at most $\lfloor (2u-3)/2 \rfloor = (2u-4)/2$. Thus there is at least another one 3-subset in the leave containing the two points $(a, 0), (b_a, 0)$. Assume that $\{(a, 0), (b_a, 0), (x, k)\} \in \mathcal{L}$, where $(x, k) \neq (a, 1)$. Similarly, consider the blocks containing the two points $(a, 0), (b_a, 1), (y, l)\} \in \mathcal{L}$, where $(y, l) \neq (a, 1)$.

If $(x,k) \neq (b_a, 1)$, then $\{(a,0), (b_a,0), (x,k)\} \neq \{(a,0), (b_a,1), (y,l)\}$. Since the number of 3-subsets containing the point (a,0) in the leave is exactly four, they must be $\{(a,0), (a,1), (b_a,0)\}$, $\{(a,0), (a,1), (b_a,1)\}$, $\{(a,0), (b_a,0), (x,k)\}$, $\{(a,0), (b_a,1), (y,l)\}$. Note that $(x,k) \neq (b_a,0)$. Consider the number s of the blocks containing the two points (a,0), (x,k). Since each 3-subset of $I_u \times Z_2$ occurs in at most one block, the number s is at most $\lfloor (2u-3)/2 \rfloor = (2u-4)/2$. Thus there is at least another one 3-subset in the leave containing the two points (a,0), (x,k). It implies that (y,l) = (x,k). Due to $\{(a,0), (b_a,1), (y,l)\} \in \mathcal{L}$, i.e., $\{(a,0), (b_a,1), (x,k)\} \in \mathcal{L}$, under the action of Z_2 we have $\{(a,1), (b_a,0), (x,k+1)\} \in \mathcal{L}$. It implies that there are at least five 3-subsets containing the point $(b_a,0), (a,1), (b_a,0)\}$, $\{(a,0), (b_a,1), (b_a,0)\}$, $\{(a,0), (b_a,1), (b_a,0)\}$, $\{(a,0), (b_a,1), (b_a,0)\}$, $\{(a,1), (b_a,0), (x,k+1)\}$, $\{(b_a,0), (b_a,1), (b_{b_a},0)\}$, $\{(b_a,0), (b_a,1), (b_{b_a},1)\}$. A contradiction.

If $(x, k) = (b_a, 1)$ and $(y, l) \neq (b_a, 0)$, then $\{(a, 0), (b_a, 0), (x, k)\} \neq \{(a, 0), (b_a, 1), (y, l)\}$. Note that $(y, l) \neq (b_a, 1)$, and hence $(y, l) \neq (x, k)$. Since the number of 3-subsets containing the point (a, 0) in the leave is exactly four, they must be $\{(a, 0), (a, 1), (b_a, 0)\}$, $\{(a, 0), (a, 1), (b_a, 1)\}$, $\{(a, 0), (b_a, 0), (x, k)\}$, $\{(a, 0), (b_a, 1), (y, l)\}$. It implies that there is only one 3-subset containing the two points (a, 0) and (y, l) in the leave. Similar arguments to those in the paragraph 4 of this proof, it is impossible.

If $(x,k) = (b_a, 1)$ and $(y,l) = (b_a, 0)$, then $\{(a,0), (b_a,0), (x,k)\} = \{(a,0), (b_a,1), (y,l)\}$. Since the number of 3-subsets containing the point (a,0) in the leave is exactly four, three of them must be $\{(a,0), (a,1), (b_a,0)\}$, $\{(a,0), (a,1), (b_a,1)\}$ and $\{(a,0), (b_a,0), (b_a,1)\}$. Assume that the 4th 3-subset containing the point (a,0) is $\{(a,0), (z,i), (w,j)\}$. Similar arguments to those in the paragraph 4 of this proof, we have that the number of 3-subsets containing the points (a,0), (z,i) in the leave must be even. A contradiction. Hence $\Phi(u \times 2, 4, 2) \leq \lfloor \frac{u}{4} (\lfloor \frac{2u-1}{3} \lfloor \frac{2u-2}{2} \rfloor \rfloor - 1) \rfloor - 1$.

Next consider the number of $\Phi(u \times v, 4, 2)$. If there is an optimal 2-D $(u \times v, 4, 2)$ -OOC with $\Phi(u \times v, 4, 2)$ codewords, then by Theorem 2.1, for integer factorization $v = 2v_1$, there exits a 2-D $(uv_1 \times 2, 4, 2)$ -OOC with $v_1 \Phi(u \times v, 4, 2)$ codewords. Since $uv_1 \equiv 0 \pmod{12}$, we have $v_1 \Phi(u \times v, 4, 2) \leq \lfloor \frac{uv_1}{4} (\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1) \rfloor - 1$. It is readily checked that $\Phi(u \times v, 4, 2) \leq \lfloor \frac{1}{v_1} (\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1) \rfloor - 1$. It is readily checked $\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 1) \rfloor - 1$. \Box

Lemma 5.3 Let $uv \equiv 0 \pmod{12}$ and $v \equiv 0 \pmod{6}$. Then $\Phi(u \times v, 4, 2) \leq \lfloor \frac{u}{4} \lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 2 \rfloor$.

Proof Let \mathcal{L} be the leave of a strictly *v*-cyclic 3- $(u \times v, 4, 1)$ -packing. Let $\mathcal{L}_1 = \{\{(a, i), (a, v/3 + i), (a, 2v/3 + i)\} : a \in I_u, 0 \leq i < v/3\}$. Since each orbit of 3-subsets in \mathcal{L}_1 is of length v/3 under the action of Z_v , each 3-subset in \mathcal{L}_1 must be contained in \mathcal{L} , i.e., $\mathcal{L}_1 \subset \mathcal{L}$.

For each $a \in I_u$ and each $i \in Z_v$, consider the number n of the blocks containing the two points (a, i), (a, v/3+i). Since each 3-subset of $I_u \times Z_v$ occurs in at most one block and $\{(a, i), (a, v/3+i), (a, 2v/3+i)\} \in \mathcal{L}$, the number n is at most $\lfloor (uv-3)/2 \rfloor = (uv-4)/2$. Thus there is at least another one 3-subset in the leave containing the two points $(a, i), (a, v/3+i), \text{ denoted by } \{(a, i), (a, v/3+i), (b_{a,i}, j_{a,i})\}$, where $(b_{a,i}, j_{a,i}) \neq (a, 2v/3+i)$. Note that the above conclusion holds for each $a \in I_u$ and each $i \in Z_v$. Thus we have that $\mathcal{L}_2 = \{\{(a, i), (a, v/3+i), (b_{a,i}, j_{a,i})\} : a \in I_u, i \in Z_v\} \subset \mathcal{L} \setminus \mathcal{L}_1$.

For each $a \in I_u$ and each $0 \leq i < v/2$, consider the number m of the base blocks containing the two points (a, i), (a, v/2 + i). Since each 3-subset of $I_u \times Z_v$ occurs in at most one block and each base block containing the two points (a, i), (a, v/2 + i) generates exactly two different blocks containing the same two points, the number m is at most $\lfloor (uv - 2)/4 \rfloor = (uv - 4)/4$. Thus there are at least two 3-subsets containing the two points (a, i), (a, v/2 + i) in the leave, denoted by $\{(a, i), (a, v/2 + i), (c_{a,i}, k_{a,i})\}$ and $\{(a, i), (a, v/2 + i), (c_{a,i}, v/2 + k_{a,i})\}$. Note that the above conclusion holds for each $a \in I_u$ and each $0 \leq i < v/2$. Thus we have that $\mathcal{L}_3 = \{\{(a, i), (a, v/2 + i), (c_{a,i}, v/2 + i), (c_{a,i}, v/2 + k_{a,i})\}$: $a \in I_u, 0 \leq i < v/2\} \subset \mathcal{L} \setminus \mathcal{L}_1$. For convenience assume that $\mathcal{L}_3 = \{\{(a, i), (a, v/2 + i), (c_{a,i}, v/2 + k_{a,i})\}$: $a \in I_u, i \in Z_v\}$, where $l_{a,i} = k_{a,i}$ when $0 \leq i < v/2$, and $l_{a,i} = v/2 + k_{a,i}$ when $v/2 \leq i < v$.

If $\mathcal{L}_2 \cap \mathcal{L}_3 = \emptyset$, then $|\mathcal{L}| \geq 7uv/3$. If $\mathcal{L}_2 \cap \mathcal{L}_3 \neq \emptyset$, assume that $\{(x, i_1), (x, v/3 + i_1), (b_{x,i_1}, j_{x,i_1})\} = \{(x, i_2), (x, v/2 + i_2), (c_{x,i_2}, l_{x,i_2})\}$ for some $x \in I_u$ and some $i_1, i_2 \in Z_v$. If $(x, i_1) \neq (c_{x,i_2}, l_{x,i_2})$, we have $(b_{x,i_1}, j_{x,i_1}) = (x, v/2 + i_1)$. If $(x, i_1) = (c_{x,i_2}, l_{x,i_2})$, we have $(b_{x,i_1}, j_{x,i_1}) = (x, 5v/6 + i_1)$. Thus each 3-subset in $\mathcal{L}_2 \cap \mathcal{L}_3$ is of the form $\{(x, i_1), (x, v/3 + i_1), (x, v/2 + i_1)\}$ or $\{(x, i_1), (x, v/3 + i_1), (x, 5v/6 + i_1)\}$. Let $\mathcal{L}_2 \cap \mathcal{L}_3 =$ $\{\{(x, i), (x, v/3 + i), (x, v/2 + i)\} : x \in A, i \in Z_v\} \cup \{\{(x, i), (x, v/3 + i), (x, 5v/6 + i)\} :$ $x \in B, i \in Z_v\} = \{\{(x, i), (x, v/3 + i), (x, v/2 + i)\} : x \in A, i \in Z_v\} \cup \{\{(x, v/2 + i), (x, 5v/6 + i), (x, v/3 + i)\} : x \in B, i \in Z_v\}$, where $A, B \subset I_u$ and $A \cap B = \emptyset$. Then $\mathcal{L}_2 \setminus (\mathcal{L}_2 \cap \mathcal{L}_3) = \{\{(a, i), (a, v/3 + i), (b_{a,i}, j_{a,i})\} : a \in I_u \setminus (A \cup B), i \in Z_v\}$ and $\mathcal{L}_3 \setminus (\mathcal{L}_2 \cap \mathcal{L}_3) = \{\{(a, i), (a, v/2 + i), (c_{a,i}, l_{a,i})\} : a \in I_u \setminus (A \cup B), i \in Z_v\}$.

Let $\{(x, v/3 + i), (x, v/2 + i)\} \subset T \in \mathcal{L}_2 \cap \mathcal{L}_3$. Consider the number of the blocks containing the two points (x, v/3 + i), (x, v/2 + i). It is at most $\lfloor (uv - 3)/2 \rfloor = (uv - 4)/2$. Thus there is at least another one 3-subset containing the two points (x, v/3 + i), (x, v/2 + i) in $\mathcal{L} \setminus (\mathcal{L}_2 \cap \mathcal{L}_3)$, denoted by $\{(x, v/3 + i), (x, v/2 + i), (d_{x,i}, r_{x,i})\}$, where $(d_{x,i}, r_{x,i}) \neq (x, i)$ if $x \in A$, and $(d_{x,i}, r_{x,i}) \neq (x, 5v/6 + i)$ if $x \in B$. Let $\mathcal{L}_4 = \{\{(x, v/3 + i), (x, v/2 + i), (d_{x,i}, r_{x,i})\} : \{(x, v/3 + i), (x, v/2 + i)\} \subset T \in \mathcal{L}_2 \cap \mathcal{L}_3\}$. Then $\mathcal{L}_4 \subset \mathcal{L}$ and $\mathcal{L}_4 \cap (\mathcal{L}_2 \cup \mathcal{L}_3) = \emptyset$. Since $|\mathcal{L}_4| = |\mathcal{L}_2 \cap \mathcal{L}_3|$, we have $|\mathcal{L}| \geq |\mathcal{L}_1| + |\mathcal{L}_2| + |\mathcal{L}_3 \setminus (\mathcal{L}_2 \cap \mathcal{L}_3)| + |\mathcal{L}_4| = 7uv/3$.

Thus there are at least 7uv/3 3-subsets in the leave. It implies that $\Phi(u \times v, 4, 2) \leq \lfloor \binom{uv}{3} - \frac{7}{3}uv \rfloor / (4v) \rfloor = \lfloor \frac{1}{24}u(u^2v^2 - 3uv - 12) \rfloor$. It is readily checked that $\lfloor \frac{u}{4}(\lfloor \frac{uv-1}{3} \lfloor \frac{uv-2}{2} \rfloor \rfloor - 2) \rfloor = \lfloor \frac{1}{24}u(u^2v^2 - 3uv - 12) \rfloor$. \Box

Appendix III

Proof of Construction 7.1: For checking the correctness of the algorithm shown in Figure 2, it suffices to show that: (1) the resulting design is strictly h_1h_2 -cyclic; (2) any

3-subset S of X' satisfying that $|S \cap G'| < 3$ for each $G' \in \mathcal{G}'$ is contained in a unique block of the resulting design; (3) any 2-subset R of X' satisfying that $|R \cap G'| < 2$ for each $G' \in \mathcal{G}'$ is contained in a unique block of \mathcal{A}'_i for each $1 \leq i \leq s$.

For convenience assume that $\mathcal{A}_B = \bigcup_{i=1}^s \mathcal{A}_B^i$ for each $B \in \mathcal{F}_1$.

(1) Suppose that $A = \{(x_l, y_l + u_l g_1, z_l + v_l h_1) : 1 \leq l \leq r\}$ is a base block of the resulting design, where $x_l \in I_n, y_l \in I_{g_1}, u_l \in I_{g_2}, z_l \in Z_{h_1}, v_l \in Z_{h_2}$. We need to show that the stabilizer of A is trivial, i.e., $A + \delta = A$ if and only if $\delta \equiv 0 \pmod{h_1 h_2}$. The sufficiency follows immediately, so we consider the necessity. Assume that $\delta = \delta_1 + \delta_2 h_1$, $\delta_1 \in Z_{h_1}, \delta_2 \in Z_{h_2}$. If $A + \delta = A$, we have

$$\{(x_l, y_l + u_l g_1, z_l + v_l h_1) : 1 \le l \le r\} = \{(x_l, y_l + u_l g_1, z_l + \delta_1 + (v_l + \delta_2)h_1) : 1 \le l \le r\},\$$

where the arithmetic is modulo $(-, -, h_1h_2)$. It follows that

$$\{(x_l, y_l, z_l) : 1 \le l \le r\} = \{(x_l, y_l, z_l + \delta_1) : 1 \le l \le r\},\$$

where the arithmetic is modulo $(-, -, h_1)$. Let $U = \{(x_l, y_l, z_l) : 1 \le l \le r\}$.

If $A \in \mathcal{A}'_j$, $1 \leq j \leq s$, then $|U| = r \geq 2$. Since the subdesign $(X, \mathcal{G}, \mathcal{B})$ of the master design 1-FG(3, $(K, K_T), ng_1h_1$) of type $(g_1h_1)^n$ $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ is strictly h_1 -cyclic and it requires that any 2-subset of X which intersects each group of \mathcal{G} in at most one point occurs in exactly one block, we have $\delta_1 = 0$.

If $A \in \mathcal{D}'$, without loss of generality assume that $A \in \mathcal{D}^*$. If $A = \tau(C)$ for some $C \in \bigcup_{B \in \mathcal{F}_2} \mathcal{D}'_B$, then $|U| = r \geq 3$. Since the master design 1-FG(3, $(K, K_T), ng_1h_1$) of type $(g_1h_1)^n$ is strictly h_1 -cyclic and it requires that any 3-subset of X which intersects each group of \mathcal{G} in at most two points occurs in exactly one block, we have $\delta_1 = 0$. If $A = \tau(C)$ for some $C \in \bigcup_{B \in \mathcal{F}_1} \mathcal{D}_B$, then $|U| \geq 2$. Note that in this case U may be a multiset, i.e., |U| may be not equal to r. By similar arguments to those in the case of $A \in \mathcal{A}'_i$, we have $\delta_1 = 0$.

Hence,

$$\{(x_l, y_l + u_l g_1, z_l + v_l h_1) : 1 \le l \le r\} = \{(x_l, y_l + u_l g_1, z_l + (v_l + \delta_2)h_1) : 1 \le l \le r\},\$$

where the arithmetic is modulo $(-, -, h_1h_2)$. It follows that

$$\{(x_l, y_l, z_l, u_l, v_l) : 1 \le l \le r\} = \{(x_l, y_l, z_l, u_l, v_l + \delta_2) : 1 \le l \le r\},\$$

where the arithmetic is modulo $(-, -, -, -, h_2)$. Since the input designs are all strictly h_2 -cyclic, we have $\delta_2 = 0$. Thus the resulting design is strictly h_1h_2 -cyclic.

(2) Take any triple $S = \{(x_l, y_l + u_l g_1, z_l + v_l h_1) : 1 \leq l \leq 3\} \subset X'$, where $x_l \in I_n$, $y_l \in I_{g_1}, u_l \in I_{g_2}, z_l \in Z_{h_1}, v_l \in Z_{h_2}$ and x_1, x_2, x_3 are not equal at the same time.

Case 1. Suppose that x_1, x_2, x_3 are pairwise distinct. Then there exist a unique base block F in \mathcal{F} and a unique element $\delta_1 \in Z_{h_1}$, such that $\{(x_l, y_l, z_l^*) : 1 \leq l \leq 3\} \subseteq F$ and $(x_l, y_l, z_l^*) + \delta_1 = (x_l, y_l, z_l^* + \delta_1) = (x_l, y_l, z_l), 1 \leq l \leq 3$, where the arithmetic is modulo $(-, -, h_1)$. It follows that $(x_l, y_l, z_l) - \delta_1 = (x_l, y_l, z_l^*)$.

If $F \in \mathcal{F}_1$, then there exist a unique base block $B \in \mathcal{A}_F \bigcup \mathcal{D}_F$ and a unique element $\delta_2 \in Z_{h_2}$, such that $\{(x_l, y_l, z_l^*, u_l, v_l^*) : 1 \leq l \leq 3\} \subseteq B$ and $(x_l, y_l, z_l^*, u_l, v_l^*) + \delta_2 = (x_l, y_l, z_l^*, u_l, v_l^* + \delta_2) = (x_l, y_l, z_l^*, u_l, v_l), 1 \leq l \leq 3$, where the arithmetic is modulo $(-, -, -, -, h_2)$. By the mapping τ , we have that $(x_l, y_l + u_l g_1, z_l^* + (v_l^* + \delta_2)h_1) = (x_l, y_l + u_l g_1, z_l^* + v_l h_1) = (x_l, y_l + u_l g_1, z_l - \delta_1 + v_l h_1)$, where the arithmetic is modulo $(-, -, h_1 h_2)$.

Let $\delta = \delta_1 + \delta_2 h_1$. It follows that $(x_l, y_l + u_l g_1, z_l^* + v_l^* h_1 + \delta) = (x_l, y_l + u_l g_1, z_l + v_l h_1)$. By (1) the resulting design is strictly $h_1 h_2$ -cyclic, so $\{(x_l, y_l + u_l g_1, z_l + v_l h_1) : 1 \le l \le 3\}$ is contained in the unique block $\tau(B) + \delta$, which is generated by $\tau(B)$. Similar arguments hold for $F \in \mathcal{F}_2$, where $B \in \mathcal{D}'_F$.

Case 2. Suppose that $x_1 = x_2, x_1 \neq x_3$, and $(y_1, z_1) \neq (y_2, z_2)$. Then there exist a unique base block F in \mathcal{F}_2 and a unique element $\delta_1 \in Z_{h_1}$, such that $\{(x_l, y_l, z_l^*) : 1 \leq l \leq 3\} \subseteq F$ and $(x_l, y_l, z_l^* + \delta_1) = (x_l, y_l, z_l), 1 \leq l \leq 3$, where the arithmetic is modulo $(-, -, h_1)$. There exist a unique base block $B \in \mathcal{D}'_F$ and a unique element $\delta_2 \in Z_{h_2}$, such that $\{(x_l, y_l, z_l^*, u_l, v_l^*) : 1 \leq l \leq 3\} \subseteq B$ and $(x_l, y_l, z_l^*, u_l, v_l^* + \delta_2) = (x_l, y_l, z_l^*, u_l, v_l),$ $1 \leq l \leq 3$, where the arithmetic is modulo $(-, -, -, -, h_2)$. By similar arguments to those in Case 1, $\{(x_l, y_l + u_l g_1, z_l + v_l h_1) : 1 \leq l \leq 3\}$ is contained in the unique block $\tau(B) + \delta$, where $\delta = \delta_1 + \delta_2 h_1$.

Case 3. Suppose that $(x_1, y_1, z_1) = (x_2, y_2, z_2)$, $(u_1, v_1) \neq (u_2, v_2)$ and $x_1 \neq x_3$. Then there exist a unique base block F in \mathcal{F}_1 and a unique element $\delta_1 \in Z_{h_1}$, such that $\{(x_l, y_l, z_l^*) : 1 \leq l \leq 3\} \subseteq F$ and $(x_l, y_l, z_l^* + \delta_1) = (x_l, y_l, z_l)$, $1 \leq l \leq 3$, where the arithmetic is modulo $(-, -, h_1)$. There exist a unique base block $B \in \mathcal{D}_F$ and a unique element $\delta_2 \in Z_{h_2}$, such that $\{(x_l, y_l, z_l^*, u_l, v_l^*) : 1 \leq l \leq 3\} \subseteq B$ and $(x_l, y_l, z_l^*, u_l, v_l^* + \delta_2) = (x_l, y_l, z_l^*, u_l, v_l)$, $1 \leq l \leq 3$, where the arithmetic is modulo $(-, -, -, -, h_2)$. By similar arguments to those in Case 1, $\{(x_l, y_l + u_l g_1, z_l + v_l h_1) : 1 \leq l \leq 3\}$ is contained in the unique block $\tau(B) + \delta$, where $\delta = \delta_1 + \delta_2 h_1$.

(3) Take any 2-subset $R = \{(x_l, y_l + u_l g_1, z_l + v_l h_1) : 1 \leq l \leq 2\} \subset X'$, where $x_l \in I_n$, $y_l \in I_{g_1}, u_l \in I_{g_2}, z_l \in Z_{h_1}, v_l \in Z_{h_2}$ and $x_1 \neq x_2$. Then there exist a unique base block F in \mathcal{F}_1 and a unique element $\delta_1 \in Z_{h_1}$, such that $\{(x_l, y_l, z_l^*) : 1 \leq l \leq 2\} \subseteq F$ and $(x_l, y_l, z_l^* + \delta_1) = (x_l, y_l, z_l), 1 \leq l \leq 2$, where the arithmetic is modulo $(-, -, h_1)$.

Given any $1 \leq j \leq s$. There exist a unique base block B in \mathcal{A}_j and a unique element $\delta_2 \in Z_{h_2}$, such that $\{(x_l, y_l, z_l^*, u_l, v_l^*) : 1 \leq l \leq 2\} \subseteq B$ and $(x_l, y_l, z_l^*, u_l, v_l^* + \delta_2) = (x_l, y_l, z_l^*, u_l, v_l), 1 \leq l \leq 2$, where the arithmetic is modulo $(-, -, -, -, h_2)$. By the mapping τ , we have that $(x_l, y_l + u_l g_1, z_l^* + (v_l^* + \delta_2)h_1) = (x_l, y_l + u_l g_1, z_l^* + v_l h_1) = (x_l, y_l + u_l g_1, z_l^* + v_l h_1) = (x_l, y_l + u_l g_1, z_l - \delta_1 + v_l h_1)$, where the arithmetic is modulo $(-, -, h_1 h_2)$. Let $\delta = \delta_1 + \delta_2 h_1$. It follows that $(x_l, y_l + u_l g_1, z_l^* + v_l^* h_1 + \delta) = (x_l, y_l + u_l g_1, z_l + v_l h_1)$. By (1) the resulting design is strictly $h_1 h_2$ -cyclic, so $\{(x_l, y_l + u_l g_1, z_l + v_l h_1) : 1 \leq l \leq 2\}$ is contained in the unique block $\tau(B) + \delta$, which is generated by $\tau(B)$.

Proof of Construction 7.5: For checking the correctness of the algorithm shown in Figure 4, take any t-subset $T = \{(x_l, y_l + u_l g_1, z_l + v_l h_1) : 1 \leq l \leq t\} \subset X'$, where $x_l \in I_n$, $y_l \in I_{g_1}, u_l \in I_{g_2}, z_l \in Z_{h_1}, v_l \in Z_{h_2}$ and $|\{x_l : 1 \leq l \leq t\}| = t$. Then there exist a unique base block F in \mathcal{F} and a unique element $\delta_1 \in Z_{h_1}$, such that $\{(x_l, y_l, z_l^*) : 1 \leq l \leq t\} \subseteq F$ and $(x_l, y_l, z_l^* + \delta_1) = (x_l, y_l, z_l), 1 \leq l \leq t$, where the arithmetic is modulo $(-, -, h_1)$. There exist a unique base block B in \mathcal{D}_F and a unique element $\delta_2 \in Z_{h_2}$, such that $\{(x_l, y_l, z_l^*, u_l, v_l^*) : 1 \leq l \leq t\} \subseteq B$ and $(x_l, y_l, z_l^*, u_l, v_l^* + \delta_2) = (x_l, y_l, z_l^*, u_l, v_l), 1 \leq l \leq t$, where the arithmetic is modulo $(-, -, -, -, h_2)$. By the mapping τ , we have that $(x_l, y_l + u_l g_1, z_l^* + (v_l^* + \delta_2)h_1) = (x_l, y_l + u_l g_1, z_l^* + v_lh_1) = (x_l, y_l + u_l g_1, z_l - \delta_1 + v_lh_1)$, where the arithmetic is modulo $(-, -, h_1h_2)$. Let $\delta = \delta_1 + \delta_2h_1$. It follows that $(x_l, y_l + u_l g_1, z_l^* + v_l^*h_1 + \delta) = (x_l, y_l + u_l g_1, z_l + v_lh_1)$. Thus $\{(x_l, y_l + u_l g_1, z_l + v_lh_1) : 1 \leq l \leq t\}$ is contained in the unique block $\tau(B) + \delta$, which is generated by $\tau(B)$.

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