# Combinatorial constructions for optimal two-dimensional optical orthogonal codes with $\lambda=2^{1}$ 

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#### Abstract

In this paper, we are concerned about optimal two-dimensional optical orthogonal codes with $\lambda=2$. Some combinatorial constructions are presented and many infinite families of optimal two-dimensional optical orthogonal codes with weight 4 and $\lambda=2$ are obtained. Especially, we shall see that in many cases an optimal twodimensional optical orthogonal code can not achieve the Johnson bound.


Keywords: optical orthogonal code; two-dimensional; optimal; OCDMA; 3-design; sfan design

## 1 Introduction

An optical orthogonal code is a family of sequences with good auto- and cross-correlation properties. Its study has been motivated by an application in an optical code-division multiple access (OCDMA) system. In a bursty traffic environment of a multiple access local area network (LAN), asynchronous multiplexing schemes are more efficient than synchronous schemes. OCDMA is such one asynchronous multiplexing scheme suitable for high speed LANs. For more information, the interested reader may refer to $[32,40$, 41,52,53].

In an OCDMA system different users share both time and frequency, and are distinguished by using a unique spreading sequence. Each user's data is multiplied by its spreading sequence, and then all the users are coupled into the shared channel. Optical orthogonal codes can be taken as the spreading sequences used in an OCDMA system.

Let $u, v, k$ and $\lambda$ be positive integers. A two-dimensional $(u \times v, k, \lambda)$ optical orthogonal code (briefly 2 -D $(u \times v, k, \lambda)$-OOC), $\mathcal{C}$, is a family of $u \times v(0,1)$-matrices (called codewords) of Hamming weight $k$ satisfying: for any matrix $\mathbf{A}=\left(a_{i j}\right)_{u \times v} \in \mathcal{C}$, $\mathbf{B}=\left(b_{i j}\right)_{u \times v} \in \mathcal{C}$ and any integer $r$ :

$$
\sum_{i=0}^{u-1} \sum_{j=0}^{v-1} a_{i j} b_{i, j+r} \leq \lambda
$$

where either $\mathbf{A} \neq \mathbf{B}$ or $r \neq 0$, and the arithmetic $j+r$ is reduced modulo $v$. Especially, when $u=1$, a two-dimensional $(1 \times v, k, \lambda)$ optical orthogonal code is said to be a one-dimensional $(v, k, \lambda)$-optical orthogonal code, denoted by 1-D $(v, k, \lambda)$-OOC.

1-D OOC was first suggested in 1989 [17]. Since then much work has been done on 1D OOCs. The interested reader may refer to $[1-3,7,8,10-16,18,21-23,36,37,44-47,59,63]$.

[^0]One limitation in applying 1-D OOCs is that the length of the sequences increases rapidly when the number of users or the weight of codes is increased, which means a large bandwidth expansion is required. Thus the bandwidth utilization is reduced. And a large code length causes the chip rate of the OCDMA system to exceed the maximum chip rate currently attainable in practice.

1-D OOCs spread the input data bits only in the time domain. By spreading in both time and wavelength domain, the chip rate can be reduced considerably. Technologies such as wavelength-division-multiplexing (WDM) and dense-WDM have made it possible to spread codes in both time and wavelength domain [61]. These codes are referred to wavelength-time hopping codes, multiple-wavelength codes, two-dimensional optical orthogonal codes, etc., which tend to require smaller code length and hence lower chip rate. Here we always refer to these codes as two-dimensional optical orthogonal codes.

The number of codewords of a 2-D OOC is called the size of the 2-D OOC. From a practical point of view, a code with a large size is required [53]. For fixed values of $u, v$, $k$ and $\lambda$, the largest possible size of a 2-D $(u \times v, k, \lambda)$-OOC is denoted by $\Phi(u \times v, k, \lambda)$. A 2-D $(u \times v, k, \lambda)$-OOC with $\Phi(u \times v, k, \lambda)$ codewords is said to be optimal. Generally speaking, it is difficult to determine the exact value of $\Phi(u \times v, k, \lambda)$. Based on the Johnson bound [31] for constant weight codes, the size of a 2-D $(u \times v, k, \lambda)$-OOC is upper bounded [61] by

$$
\Phi(u \times v, k, \lambda) \leq J(u \times v, k, \lambda),
$$

where

$$
J(u \times v, k, \lambda)=\left\lfloor\frac{u}{k}\left\lfloor\frac{u v-1}{k-1}\left\lfloor\frac{u v-2}{k-2}\left\lfloor\cdots\left\lfloor\frac{u v-\lambda}{k-\lambda}\right\rfloor \cdots\right\rfloor\right\rfloor\right\rfloor\right\rfloor .
$$

In optical code-division multiple-access applications, performance analysis shows that codes with $\lambda \leq 3$ are the most desirable. As pointed out by [38], from a multiple-access and synchronization point of view, the most desirable on-off signature sequences are OOCs with $\lambda=1$. However, these families of codes may suffer from low cardinality in some applications. It was hinted that in [5] OOCs with $\lambda=2$ could, under certain conditions, have better performance than that of OOCs with $\lambda=1$. In this paper, we are concerned about OPTIMAL 2-D $(u \times v, k, \lambda)$-OOCs with $\lambda=2$ and $k=4$.

We will neither try to explore the applications of 2-D OOCs, nor try to provide the performance analysis of a code-division multiple-access system which uses 2-D OOCs. Mathematically, combinatorial design theory, projective geometry and finite field theory are three main tools to investigate the constructions for 2-D OOCs. In this paper, we focus our attention on the combinatorial structures of 2-D OOCs. Many terminologies and results related to combinatorial design theory will be used. To ensure smooth reading of the paper, most of the proofs related to design theory have been moved to the Appendices. For more information on design theory, the interested reader may refer to [6].

### 1.1 Literature review

There is a considerable literature on 2-D OOC constructions. Yang and Kwong [61] used a 1-D OOC to achieve spreading in the wavelength and time domains to construct a 2-D OOC. The construction by Lee and Seo [34] spreads in the wavelength and the time domain by using two different 1-D OOCs. Sun et al. [56] constructed a 2-D OOC by employing a frequency hopping code and a 1-D OOC to spread in the wavelength
domain and the time axis, respectively. The construction by Alderson and Mellinger [4] are based on certain point sets in finite projective spaces of dimension $k$ over $\operatorname{GF}(q)$. Omrani et al. [49] constructed some 2-D OOCs using polynomials over finite fields and rational functions. Cao and Wei [9] first gave a combinatorial description of 2-D OOCs. Wang et al. [57] discussed the existence of optimal 2-D OOCs with weight 3 and index $\lambda=1$ using combinatorial design theory.

For more information on 2-D OOC constructions, the interested reader may refer to [ $4,9,24,33-35,39,49,51,54-58,60-62]$ and the references therein. However, in this paper, we only focus our attention on OPTIMAL 2-D OOC constructions. In applications, optimal OOCs facilitate the largest possible number of asynchronous users to transmit information effectively and reliably. A quick review of the majority of constructions for optimal 2-D OOCs in the literature is presented in Table I.

Table I
Optimal 2-D OOCs in the literature

| Parameters | Conditions | Code size | Reference |
| :---: | :---: | :---: | :---: |
| ( $u \times v, 3,1$ ) | $u, v \geq 1$ and $v \equiv 1(\bmod 2)$ | $\begin{gathered} \text { if } u \equiv 5(\bmod 6) \text { and } v=1 \\ J(u \times v, 3,1)-1 ; \\ \text { otherwise, } J(u \times v, 3,1) \end{gathered}$ | [57] |
| $(u \times u, k, 1)$ | $u \equiv 1(\bmod k(k-1))$ and $u$ a prime | $J(u \times u, k, 1)$ | [61] |
| $\left(p^{n} \times p, p, 1\right)$ | $p$ a prime and $n \geq 1$ | $J\left(p^{n} \times p, p, 1\right)$ | 9] |
| $(u \times v, q, 1)$ | $u v=q^{n}-1, n \geq 1$, and $q$ a prime power | $J(u \times v, q, 1)$ | [4] |
| $(u \times v, q+1,1)$ | $u v=\left(q^{n+1}-1\right) /(q-1)$, <br> $q$ a prime power, $n \geq 1$, either $n \equiv 0(\bmod 2)$, <br> or $n \equiv 1(\bmod 2)$ and $\operatorname{gcd}(q+1, v)=1$ | $J(u \times v, q+1,1)$ | [4] |
| ( $u \times v, 4,2)$ | $u v=2^{n}-1$ and $n \geq 3$ | $J(u \times v, 4,2)$ | [4] |
| (u×v,6,2) | $\begin{gathered} u v=\left(4^{n}-1\right) / 3, n \geq 3, \\ \text { either } n \equiv 0,1(\bmod 3), \\ \text { or } n \equiv 2(\bmod 3) \text { and } \operatorname{gcd}(21, v)=1 \end{gathered}$ | $J(u \times v, 6,2)$ | [4] |
| $(u \times v, q+1,2)$ | $u v=q^{n}+1, q$ a prime power, <br> $n \geq 1$, either $n \equiv 0(\bmod 2)$, <br> or $n \equiv 1(\bmod 2)$ and $\operatorname{gcd}(q+1, v)=1$ | $J(u \times v, q+1,2)$ | [4] |

### 1.2 Outline of the paper

The rest of this paper is structured as follows. In Section 2 based on the relationship between 1-D OOCs and 2-D OOCs, many optimal 2-D $(u \times v, 4,2)$-OOCs are derived. Cao and Wei [9] showed that an optimal 2-D $(u \times v, k, t-1)$-OOC is equivalent to an optimal strictly $v$-cyclic $t$ - $(u \times v, k, 1)$-packing, provided that $t \leq k$ holds. We restate this combinatorial equivalence in Section 3. In this section perfect 2-D OOCs are defined as a special case of optimal 2-D OOCs. We point out in Remark 3.5 that the problem for the existence of perfect 2-D $(u \times v, 4,2)$-OOCs can be reduced to the problem for the existence of perfect 2-D $(w \times v, 4,2)$-OOCs, $w \in\{1,2\}$. When $w=1$, perfect 2-D $(1 \times v, 4,2)$-OOCs have been widely investigated as a kind of combinatorial object called strictly cyclic Steiner quadruple system. Thus we pay our attention to the case of $w=2$ in Section 4. We give a construction for perfect 2-D $(2 \times v, 4,2)$-OOCs. In Section 5 we improve the upper bound for optimal 2-D ( $u \times v, 4,2$ )-OOCs (not only focus on perfect), which is tighter than the well-known Johnson bound in many cases. In Sections 6 and 7 some auxiliary designs are introduced to establish recursive constructions for 2-D ( $u \times v, k, 2$ )-OOCs with general $k$. Using these recursive constructions and some
direct constructions, we obtain many infinite families of optimal $2-\mathrm{D}(u \times v, 4,2)$-OOCs in Sections 8 and 9. Finally, Section 10 gives a brief conclusion.

Our main results are summarized in Table II (in Section 9), Tables III and IV (in Section 10).

## 2 2-D OOCs from 1-D OOCs

$2-\mathrm{D}$ OOCs are very closely related to $1-\mathrm{D}$ OOCs. A $2-\mathrm{D}(1 \times v, k, \lambda)$-OOC is just a $1-\mathrm{D}$ $(v, k, \lambda)$-OOC. A 1-D $(v, k, \lambda)$-OOC with $\Phi(1 \times v, k, \lambda)$ codewords is said to be optimal. In this section we shall derive some optimal 2-D OOCs from the known results on optimal 1-D OOCs. First we quote the following result from Alderson and Mellinger [4].

Theorem 2.1 ( [4]) Suppose that there exists an optimal 2-D $(u \times v, k, \lambda)$-OOC with $\Phi(u \times v, k, \lambda)$ codewords. Then for any integer factorization $v=v_{1} v_{2}$, there exists a 2-D $\left(u v_{1} \times v_{2}, k, \lambda\right)-O O C$ with $v_{1} \Phi(u \times v, k, \lambda)$ codewords.

As a corollary of Theorem 2.1, we have
Corollary 2.2 If there is an optimal 1-D (uv, $k, \lambda)-O O C$ with $\Phi(1 \times u v, k, \lambda)$ codewords, then a 2-D $(u \times v, k, \lambda)$-OOC with $u \Phi(1 \times u v, k, \lambda)$ codewords exists.

Therefore by Corollary 2.2, if $u \Phi(1 \times u v, k, \lambda)$ is just equal to $\Phi(u \times v, k, \lambda)$, then the resulting 2-D $(u \times v, k, \lambda)$-OOC is optimal. In the following we shall give some analysis on $u \Phi(1 \times u v, k, \lambda)=\Phi(u \times v, k, \lambda)$. According to the Johnson bound, $\Phi(u \times v, k, \lambda) \leq$ $J(u \times v, k, \lambda)$ and $\Phi(1 \times u v, k, \lambda) \leq J(1 \times u v, k, \lambda)$. Assume that

$$
J_{1}(1 \times u v, k, \lambda)=\frac{1}{k}\left\lfloor\frac{u v-1}{k-1}\left\lfloor\frac{u v-2}{k-2}\left\lfloor\cdots\left\lfloor\frac{u v-\lambda}{k-\lambda}\right\rfloor \cdots\right\rfloor\right\rfloor\right\rfloor .
$$

We have the following lemma.
Lemma 2.3 $J(u \times v, k, \lambda)=u J(1 \times u v, k, \lambda)$ if and only if $J_{1}(1 \times u v, k, \lambda)-J(1 \times$ $u v, k, \lambda)<1 / u$.

Proof Let $x=\left\lfloor\frac{u v-1}{k-1}\left\lfloor\frac{u v-2}{k-2}\left\lfloor\cdots\left\lfloor\frac{u v-\lambda}{k-\lambda}\right\rfloor \cdots\right\rfloor\right\rfloor\right\rfloor$ and $x=a k+b$, where $0 \leq b<k$. Then $J(u \times v, k, \lambda)=\left\lfloor\frac{u}{k} x\right\rfloor$ and $J(1 \times u v, k, \lambda)=\left\lfloor\frac{1}{k} x\right\rfloor$. It is easy to verify that

$$
\begin{aligned}
J(u \times v, k, \lambda)=u J(1 \times u v, k, \lambda) & \Longleftrightarrow\left\lfloor\frac{u}{k} x\right\rfloor=u\left\lfloor\frac{1}{k} x\right\rfloor
\end{aligned} \Longleftrightarrow \Longleftrightarrow u a+\left\lfloor\frac{u b}{k}\right\rfloor=u a
$$

Note that $J_{1}(1 \times u v, k, \lambda)-J(1 \times u v, k, \lambda)=\frac{b}{k}$.

Theorem 2.4 If there exists an optimal $1-D(u v, k, \lambda)-O O C$ with $J(1 \times u v, k, \lambda)$ codewords and $J_{1}(1 \times u v, k, \lambda)-J(1 \times u v, k, \lambda)<1 / u$, then there exists an optimal $2-D$ $(u \times v, k, \lambda)-O O C$ with $J(u \times v, k, \lambda)$ codewords.

Proof By Corollary 2.2, if there is an optimal 1-D $(u v, k, \lambda)$-OOC with $J(1 \times u v, k, \lambda)$ codewords, then there is a 2-D $(u \times v, k, \lambda)$-OOC with $u J(1 \times u v, k, \lambda)$ codewords. Since $J_{1}(1 \times u v, k, \lambda)-J(1 \times u v, k, \lambda)<1 / u$, by Lemma $2.3, u J(1 \times u v, k, \lambda)=J(u \times v, k, \lambda)$. Thus the resulting $2-\mathrm{D}$ OOC is optimal.

Corollary 2.5 If there exists an optimal 1-D ( $n, 4,2$ )-OOC with $J(1 \times n, 4,2)$ codewords, then
(1) for any integer $n \equiv 1,3(\bmod 6)$ or $n \equiv 2,10(\bmod 24)$, and for any integer factorization $n=u v$, there exists an optimal $2-D(u \times v, 4,2)$-OOC with $J(u \times v, 4,2)$ codewords;
(2) for any integer $n \equiv 4,20(\bmod 24)$ and $n=2 n_{1}$, there exists an optimal $2-D(2 \times$ $\left.n_{1}, 4,2\right)$-OOC with $J\left(2 \times n_{1}, 4,2\right)$ codewords.

Proof When $n \equiv 1,3(\bmod 6)$ or $n \equiv 2,10(\bmod 24)$, it is readily checked that $J_{1}(1 \times n, 4,2)-J(1 \times n, 4,2)=0$. When $n \equiv 4,20(\bmod 24)$, it is readily checked that $J_{1}(1 \times n, 4,2)-J(1 \times n, 4,2)=1 / 4$. The assertion then follows from Theorem 2.4.

As a special topic, 1-D $(n, 4,2)$-OOCs have been extensively studied, for example Alderson and Mellinger [3], Chu and Colbourn [14, 15], Feng, Chang and Ji [18, 19]. We only quote partial known results on optimal 1-D ( $n, 4,2$ )-OOCs, which are essential for our work.

## Lemma 2.6

(1) ( [19]) There exists an optimal 1-D (uv,4,2)-OOC with $J(1 \times u v, 4,2)$ codewords for any $u \in\left\{4^{n}-1\right.$ : integer $\left.n \geq 1\right\} \cup\{1,27,33,39,51,87,123,183\}$ and $v$ an integer taken from the set $\{p \equiv 7(\bmod 12): p$ is a prime $\} \cup\left\{2^{n}-1\right.$ : odd integer $\left.n \geq 1\right\} \cup\{25$, 37, 61, 73, 109, 157, 181, 229, 277\}, or a product of such integers.
(2) ( [18]) Let $n$ be a positive integer. If $n=p_{1}^{r_{1}} p_{2}^{r_{2}} \cdots p_{s}^{r_{s}}$, where $p_{i}=13$ or $p_{i}$ is a prime, $p_{i} \equiv 5(\bmod 12)$ and $p_{i}<1500000, r_{i} \geq 1$ for $1 \leq i \leq s$, then there is an optimal 1-D (4n, 4, 2)-OOC with $J(1 \times 4 n, 4,2)$ codewords.
(3) ( [18]) There exists an optimal 1-D (n,4,2)-OOC with $J(1 \times n, 4,2)$ codewords for all $7 \leq n \leq 100$ with the definite exceptions of $n \in\{9,12,13,24,48,72,96\}$ and possible exceptions of $n \in\{45,47,53,55,59,60,65,66,69,71,76,77,81,83,84,85,89,91$, 92, 95, 97, 99\}.
(4) ( $[14,18])$ There exists an optimal 1-D $(n, 4,2)$-OOC with $J(1 \times n, 4,2)-1$ codewords for $n \in\{9,12,13,24,48,72,96\}$.

## Theorem 2.7

(1) Let $m=u v$, where $u \in\left\{4^{n}-1: n \geq 1\right\} \cup\{1,27,33,39,51,87,123,183\}$ and $v$ is an integer taken from the set $\{p \equiv 7(\bmod 12): p$ is a prime $\} \cup\left\{2^{n}-1\right.$ : odd integer $n \geq$ $1\} \cup\{25,37,61,73,109,157,181,229,277\}$, or a product of such integers. Then for any integer factorization $m=n_{1} n_{2}$, there exists an optimal $2-D\left(n_{1} \times n_{2}, 4,2\right)$-OOC with $J\left(n_{1} \times n_{2}, 4,2\right)$ codewords.
(2) Let $n \in\{10,15,21,25,26,27,33,34,39,49,50,51,57,58,63,74,75,82,87,93,98\}$. Then for any integer factorization $n=n_{1} n_{2}$, there is an optimal $2-D\left(n_{1} \times n_{2}, 4,2\right)$-OOC with $J\left(n_{1} \times n_{2}, 4,2\right)$ codewords.
(3) Let $n$ be a positive integer. If $n=p_{1}^{r_{1}} p_{2}^{r_{2}} \cdots p_{s}^{r_{s}}$, where $p_{i}=13$ or $p_{i}$ is a prime, $p_{i} \equiv 5(\bmod 12)$ and $p_{i}<1500000, r_{i} \geq 1$ for $1 \leq i \leq s$, then there is an optimal $2-D$ $(2 \times 2 n, 4,2)$-OOC with $J(2 \times 2 n, 4,2)$ codewords.
(4) Let $2 n \in\{20,28,44,52,68,100\}$. Then there is an optimal $2-D(2 \times n, 4,2)$-OOC with $J(2 \times n, 4,2)$ codewords.

Proof It is readily checked that the number $n$ described in (1) is congruent to 1 or 3 modulo 6 ; the number $n$ described in (2) is congruent to 1,3 modulo 6 , or 2,10 modulo 24 ; the number $4 n$ described in (3) and the number $2 n$ described in (4) are congruent to 4 or 20 modulo 24. Combining the results of Lemma 2.6, the assertion then follows from Corollary 2.5.

## 3 Combinatorial descriptions

Two-dimensional optical orthogonal codes are closely related to some combinatorial configurations called strictly $v$-cyclic packings. Throughout this paper we always assume that $I_{u}=\{0,1, \ldots, u-1\}$ and denote by $Z_{v}$ the additive group of integers modulo $v$.

### 3.1 Combinatorial equivalence

A $t-(v, k, 1)$ packing is a pair $(X, \mathcal{B})$, where $X$ is a set of $v$ points and $\mathcal{B}$ is a set of subsets of $X$ (called blocks), each of cardinality $k$, such that every $t$-subset of $X$ occurs in at most one block. The set of all uncovered $t$-subsets by $\mathcal{B}$ is said to be the leave of the packing.

An automorphism $\alpha$ of a packing $(X, \mathcal{B})$ is a permutation on $X$ such that

$$
\{\{\alpha(x): x \in B\}: B \in \mathcal{B}\}=\mathcal{B} .
$$

In other words, a block of the packing is mapped to a block under an automorphism. A $t$-( $u \times v, k, 1$ ) packing is said to be $v$-cyclic if it admits an automorphism $\pi$ consisting of $u$ cycles of length $v$. Without loss of generality identify $X$ with $I_{u} \times Z_{v}$ and the automorphism $\pi$ can be taken as $(i, x) \longmapsto(i, x+1)(\bmod (-, v)), i \in I_{u}$ and $x \in Z_{v}$. Then all blocks of this packing can be partitioned into some orbits under $\pi$. Choose any fixed block from each orbit and then call it a base block.

All automorphisms of a packing form a group, called the full automorphism group of the packing. Any subgroup of the full automorphism group is called an automorphism group of the packing. Let $G$ be an automorphism group of a packing. For any block $B$ of the packing, the subgroup

$$
\left\{\pi \in G: B^{\pi}=B\right\}
$$

is called the stabilizer of $B$ in $G$. If the stabilizer of each block of a $v$-cyclic $t$ - $(u \times v, k, 1)$ packing is trivial in $Z_{v}$, i.e., for each block $B,\left\{\delta \in Z_{v}: B+\delta=B\right\}=\{0\}$, where $B+\delta=\{(i, x+\delta):(i, x) \in B\}$, then the packing is called strictly $v$-cyclic. When $u=1$, a (strictly) $v$-cyclic $t$-( $1 \times v, k, 1$ ) packing is often simply referred to as a (strictly) cyclic $t-(v, k, 1)$ packing. When $v=1$, a (strictly) 1-cyclic $t$ - $(u \times 1, k, 1)$ packing is just a $t-(u, k, 1)$ packing.

A strictly $v$-cyclic $t$ - $(u \times v, k, 1)$ packing is called optimal if it contains the largest possible block number. The main purpose of this paper is to construct optimal 2-D OOCs. Cao and Wei [9] established the equivalence between optimal 2-D OOCs and optimal strictly $v$-cyclic packings. Suppose $(X, \mathcal{B})$ is a strictly $v$-cyclic $t$ - $(u \times v, k, 1)$ packing. Denote the family of base blocks of this packing by $\mathcal{F}$. For each base block $B$ of $\mathcal{F}$, construct an $u \times v(0,1)$-matrix $M_{B}$ whose rows are indexed by $I_{u}$ and columns are indexed by $Z_{v}$, such that its $(i, j)$ cell equals 1 if and only if $(i, j) \in B$. Since any two blocks intersect at most $t-1$ points and all the blocks can be generated by developing
cyclically the base blocks, $\left\{M_{B}: B \in \mathcal{F}\right\}$ forms a 2-D $(u \times v, k, t-1)$-OOC with $|\mathcal{F}|$ codewords. Conversely, given a 2-D $(u \times v, k, t-1)$-OOC, $\mathcal{C}$, for each $u \times v(0,1)$-matrix $M \in \mathcal{C}$ whose rows are indexed by $I_{u}$ and columns are indexed by $Z_{v}$, construct a $k$ subset $B_{M}$ of $I_{u} \times Z_{v}$ such that $(i, j) \in B_{M}$ if and only if $M$ 's $(i, j)$ cell equals 1 . Then $\left\{B_{M}: M \in \mathcal{C}\right\}$ is the family of base blocks of a strictly $v$-cyclic $t$ - $(u \times v, k, 1)$ packing.

Theorem 3.1 ( [9]) An optimal 2-D $(u \times v, k, t-1)$-OOC is equivalent to an optimal strictly $v$-cyclic $t$ - $(u \times v, k, 1)$-packing, provided that $t \leq k$ holds.

Since a strictly 1 -cyclic 3 - $(u \times 1,4,1)$ packing is just a $3-(u, 4,1)$ packing, and the existence of an optimal 3-( $u, 4,1$ ) packing has been investigated by Ji [29], we can have the following result.

Theorem 3.2 ( [29]) There exists an optimal 2-D ( $u \times 1,4,2$ )-OOC (i.e., an optimal $3-(u, 4,1)$ packing) with $\phi$ codewords, where

$$
\phi= \begin{cases}\left\lfloor\frac{u}{4}\left\lfloor\frac{u-1}{3}\left\lfloor\frac{u-2}{2}\right\rfloor\right\rfloor\right\rfloor & u \not \equiv 0(\bmod 6), \\ \left\lfloor\frac{u}{4}\left\lfloor\left\lfloor\frac{u-1}{3}\left\lfloor\frac{u-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor & u \equiv 0(\bmod 6),\end{cases}
$$

with the exception of 21 undecided values $u=6 r+5, r \in\{m: m$ is odd, $3 \leq m \leq 35$, $m \neq 17,21\} \cup\{45,47,75,77,79,159\}$.

Example 3.3 There is a trivial optimal 2-D ( $1 \times 6,4,2$ )-OOC, whose number of codewords is $\left\lfloor\frac{1}{4}\left\lfloor\frac{5}{3}\left\lfloor\frac{4}{2}\right\rfloor\right\rfloor\right\rfloor=0$. By Theorem 3.2, there is an optimal $2-D(6 \times 1,4,2)$-OOC with 3 codewords. An optimal $2-D(2 \times 3,4,2)$-OOC has only $J(2 \times 3,4,2)=1$ codeword

$$
\left(\begin{array}{lll}
1 & 1 & 0 \\
1 & 1 & 0
\end{array}\right),
$$

whose corresponding base block of the optimal strictly 3 -cyclic 3 - $(2 \times 3,4,1)$-packing is $\{(0,0),(1,0),(0,1),(1,1)\}$. A 2-D $(3 \times 2,4,2)$-OOC can not contain $J(3 \times 2,4,2)=2$ codewords. Otherwise, there were a $2-D(6 \times 1,4,2)$-OOC with 4 codewords by Theorem 2.1, which would be contradict to Theorem 3.2. Thus an optimal $2-D(3 \times 2,4,2)$-OOC has only one codeword

$$
\left(\begin{array}{ll}
1 & 1 \\
1 & 0 \\
1 & 0
\end{array}\right)
$$

whose corresponding base block of the optimal strictly 2 -cyclic 3 - $(3 \times 2,4,1)$-packing is $\{(0,0),(1,0),(2,0),(0,1)\}$.

### 3.2 Perfect 2-D OOCs

Let $K$ be a set of positive integers. A $t$-wise balanced design (briefly $t$-design) is a pair $(X, \mathcal{B})$, where $X$ is a set of $v$ points and $\mathcal{B}$ is a set of subsets of $X$ (called blocks), each of cardinality from $K$, such that every $t$-subset of $X$ is contained in a unique block. Such a design is denoted by $S(t, K, v)$. If $K=\{k\}$, we write $S(t, K, v)$ by $S(t, k, v)$. An $S(2,3, v)$ is called a Steiner triple system and denoted by $S T S(v)$. An $S(3,4, v)$ is called a Steiner quadruple system and denoted by $S Q S(v)$.

Evidently an $S(t, k, v)$ is a special $t-(v, k, 1)$-packing, whose leave is an empty set. Thus one can similarly define (strictly) $v$-cyclic $S(t, k, u \times v)$ as we have done for (strictly) $v$-cyclic $t$ - $(u \times v, k, 1)$-packing. A strictly $v$-cyclic $S Q S(1 \times v)$ is often simply written as an $s S Q S(v)$ (cf. [18]).

If a 2 -D $(u \times v, k, t-1)$-OOC is equivalent to a strictly $v$-cyclic $S(t, k, u \times v)$, then the OOC is said to be perfect. It is easy to verify that a perfect OOC is an optimal OOC that attains the Johnson bound without using the brackets (cf. [48]).

Lemma 3.4 The necessary conditions for the existence of a strictly v-cyclic $S Q S(u \times v)$ (or equivalently, a perfect $2-D(u \times v, 4,2)$-OOC) are $u v \equiv 2,4(\bmod 6), u(u v-1)(u v-$ $2) \equiv 0(\bmod 24)$. Specifically, the necessary conditions can be classified as follows:
(1) $u \equiv 1,5(\bmod 12)$ and $v \equiv 2,10(\bmod 24)$;
(2) $u \equiv 7,11(\bmod 12)$ and $v \equiv 14,22(\bmod 24)$;
(3) $u \equiv 2,4(\bmod 6)$ and $v \equiv 1,5(\bmod 6)$;
(4) $u \equiv 4,8(\bmod 12)$ and $v \equiv 2,4(\bmod 6)$.

Proof It is well known that an $S Q S(u v)$ exists if and only if $u v \equiv 2,4(\bmod 6)[25]$. Count the number of base blocks of a strictly $v$-cyclic $S Q S(u \times v)$. It follows that $u(u v-1)(u v-2) \equiv 0(\bmod 24)$.

A natural question from Lemma 3.4 is whether the necessary conditions for the existence of a perfect 2-D $(u \times v, 4,2)$-OOC are sufficient. In Section 5, by Corollary 5.5, we show that for $u \equiv 4,8(\bmod 12)$ and $v \equiv 2,4(\bmod 6)$, there is no perfect 2-D $(u \times v, 4,2)$-OOC. In Section 9, by Proposition 9.2, if there exists a perfect 2 -D $(2 \times v, 4,2)$ OOC with $v \equiv 1,5(\bmod 6)$, then a perfect 2-D $(u \times v, 4,2)$-OOC exists for any $u \equiv 2,4$ $(\bmod 6)$. When $u$ and $v$ satisfy Conditions $(1)$ and $(2)$ in Lemma $3.4, u v \equiv 2,10(\bmod 24)$. Then by Corollary 2.5(1), if there exists an optimal 1-D (uv, 4, 2)-OOC with $J(1 \times u v, 4,2)$ codewords, a perfect 2-D $(u \times v, 4,2)$-OOC exists. Note that when $u v \equiv 2,10(\bmod$ 24), an optimal 1-D ( $u v, 4,2$ )-OOC with $J(1 \times u v, 4,2)$ codewords is just a perfect 2-D $(1 \times u v, 4,2)$-OOC. Thus

Remark 3.5 The existence problem of perfect 2-D $(u \times v, 4,2)$-OOCs can be reduced to the existence problems of perfect $2-D(1 \times v, 4,2)$-OOCs and perfect $2-D(2 \times v, 4,2)$-OOCs.

## 4 A construction for perfect 2-D $(2 \times v, 4,2)$-OOCs

According to Remark 3.5, it is important to consider the existences of perfect 2-D ( $1 \times$ $v, 4,2)$-OOCs and perfect 2-D $(2 \times v, 4,2)$-OOCs. A perfect 2 - $\mathrm{D}(1 \times v, 4,2)$-OOC is equivalent to an $s S Q S(v)$. Much work has been done on $s S Q S \mathrm{~s}$ in the literature. The interested reader may refer to [18] and the references therein. In this section, we shall present a construction for perfect 2-D $(2 \times v, 4,2)$-OOCs.

The idea of this construction is originally from Hartman [27]. In 1980 Hartman [27] gave a construction for an $S Q S(2 p)$, which can be obtained from an $S Q S(p+1)$ with a cyclic derived Steiner triple system, where $p \equiv 1(\bmod 6)$ is a prime. Here, we shall generalize Hartman's method to obtain a construction for strictly $p$-cyclic $S Q S(2 \times p)$ s.

The existence of a strictly $p$-cyclic $S Q S(2 \times p)$ implies the existence of a perfect 2-D ( $2 \times p, 4,2$ )-OOC.

Our construction are based on the concept of rotational SQSs. A rotational $S Q S(n)$ is an $S Q S(n)$ with an automorphism consisting of one fixed point and a cycle of length $n-1$. Such a design is denoted by $\operatorname{RoSQS}(n)$.

Assume that $(X, \mathcal{B})$ is an $\operatorname{RoSQS}(n)$. We can identify $X$ with $Z_{n-1} \cup\{\infty\}$, and let the permutation $\alpha$ fixing $\infty$ and mapping $i$ to $i+1(\bmod n-1), i \in Z_{n-1}$, be an automorphism of the RoSQS. Let $G$ be a cyclic group generated by $\alpha$ under the compositions of permutations. Then all blocks of the $R o S Q S$ can be partitioned into some orbits under $G$. Choose any fixed block from each orbit and then call it a base block.

Example 4.1 An $\operatorname{RoSQS}(8)(X, \mathcal{B})$ is constructed on $X=Z_{7} \cup\{\infty\}$. All blocks of $\mathcal{B}$ are listed below:

$$
\{i, i+1, i+2, i+5\},\{i, i+1, i+3, \infty\}, 0 \leq i \leq 6
$$

Obviously, all blocks of $\mathcal{B}$ can be obtained by developing the two base blocks $\{0,1,2,5\}$, $\{0,1,3, \infty\}$ by +1 modulo 7 , where $\infty+1=\infty$.

Construction 4.2 Let $p \equiv 1(\bmod 6)$ be a prime. If there exists an $\operatorname{RoSQS}(p+1)$, then there exists a strictly p-cyclic $S Q S(2 \times p)$.

Proof Here we only exhibit the algorithm in Figure 1. The detailed proof of this construction has been moved to Appendix I.

Step 1: Start from an $\operatorname{RoSQS}(p+1)$, which is constructed on $Z_{p} \cup\{\infty\}$. Denote the set of base blocks of this design by $\mathcal{B}_{1} \cup \mathcal{B}_{2}$, where $\mathcal{B}_{1}$ and $\mathcal{B}_{2}$ generate all the blocks containing and not containing $\infty$, respectively.
Step 2: We write the element $(i, x)$ of $I_{2} \times Z_{p}$ as $x_{i}$ for short. Let

$$
\begin{aligned}
\mathcal{A}_{1}= & \left\{\left\{x_{0}, y_{0}, z_{0}, u_{0}\right\}:\{x, y, z, u\} \in \mathcal{B}_{2}\right\}, \\
\mathcal{A}_{2}= & \left\{\left\{0_{1}, x_{0}, y_{0}, z_{0}\right\}:\{\infty, x, y, z\} \in \mathcal{B}_{1}\right\} \\
\mathcal{A}_{3}= & \left\{\left\{x_{0}, y_{0},(2 r-1)(y-x)_{1}, 2 r(y-x)_{1}\right\}:\right. \\
& \left.\quad\{\infty, x, y\} \subseteq B \in \mathcal{B}_{1}, 1 \leq r \leq(p-1) / 2\right\} .
\end{aligned}
$$

Step 3: Define a mapping $\tau$ from $I_{2} \times Z_{p}$ to $I_{2} \times Z_{p}: x_{i} \longmapsto(-x)_{1-i}$. For $j=1,2$,

$$
\mathcal{A}_{j}^{\prime}=\left\{\{\tau(a): a \in A\}: A \in \mathcal{A}_{j}\right\} .
$$

Step 4: Take

$$
\mathcal{A}=\mathcal{A}_{1} \cup \mathcal{A}_{1}^{\prime} \cup \mathcal{A}_{2} \cup \mathcal{A}_{2}^{\prime} \cup \mathcal{A}_{3} .
$$

Then $\mathcal{A}$ is the set of base blocks of the required strictly $p$-cyclic $S Q S(2 \times p)$, which is constructed on $I_{2} \times Z_{p}$.

Figure 1: Algorithm to construct a strictly $p$-cyclic $S Q S(2 \times p)$
The following example illustrates the algorithm presented in Figure 1.
Example 4.3 In this example we shall show how to construct a strictly 7 -cyclic $S Q S(2 \times$ 7) from an RoSQS(8). It is equivalent to a perfect $2-D(2 \times 7,4,2)$-OOC by Theorem 3.1.

- Step 1: Start from an RoSQS(8), which is given by Example 4.1. Take

$$
\mathcal{B}_{1}=\{\{0,1,3, \infty\}\}, \quad \mathcal{B}_{2}=\{\{0,1,2,5\}\} .
$$

- Step 2: Construct the required strictly 7 -cyclic $S Q S(2 \times 7)$ on $I_{2} \times Z_{7}$. Let

$$
\begin{aligned}
\mathcal{A}_{1} & \left\{\left\{0_{0}, 1_{0}, 2_{0}, 5_{0}\right\}\right\}, \quad \mathcal{A}_{2}=\left\{\left\{0_{1}, 0_{0}, 1_{0}, 3_{0}\right\}\right\}, \\
\mathcal{A}_{3}= & \left\{\left\{0_{0}, 1_{0}, 1_{1}, 2_{1}\right\} \cup\left\{0_{0}, 1_{0}, 3_{1}, 4_{1}\right\} \cup\left\{0_{0}, 1_{0}, 5_{1}, 6_{1}\right\}\right. \\
& \cup\left\{0_{0}, 3_{0}, 3_{1}, 6_{1}\right\} \cup\left\{0_{0}, 3_{0}, 2_{1}, 5_{1}\right\} \cup\left\{0_{0}, 3_{0}, 1_{1}, 4_{1}\right\} \\
& \cup\left\{1_{0}, 3_{0}, 2_{1}, 4_{1}\right\} \cup\left\{1_{0}, 3_{0}, 6_{1}, 1_{1}\right\} \cup\left\{1_{0}, 3_{0}, 3_{1}, 5_{1}\right\} .
\end{aligned}
$$

- Step 3: Under the action of the mapping $\tau: x_{i} \longmapsto(-x)_{1-i}$, we have

$$
\mathcal{A}_{1}^{\prime}=\left\{\left\{0_{1}, 6_{1}, 5_{1}, 2_{1}\right\}\right\}, \quad \mathcal{A}_{2}^{\prime}=\left\{\left\{0_{0}, 0_{1}, 6_{1}, 4_{1}\right\}\right\} .
$$

- Step 4: Let $\mathcal{A}=\mathcal{A}_{1} \cup \mathcal{A}_{1}^{\prime} \cup \mathcal{A}_{2} \cup \mathcal{A}_{2}^{\prime} \cup \mathcal{A}_{3}$. Then $|\mathcal{A}|=13$ and $\mathcal{A}$ is the set of base blocks of the required strictly 7 -cyclic $S Q S(2 \times 7)$.

By Theorem 3.1, a strictly $p$-cyclic $S Q S(2 \times p)$ is equivalent to a perfect 2-D $(2 \times$ $p, 4,2)$-OOC. Thus by Construction 4.2, for obtaining some perfect 2-D $(2 \times p, 4,2)$-OOCs, we need some results on RoSQSs.

Theorem 4.4 ( [19]) There exists an RoSQS(uv +1) for any $u \in\left\{4^{n}-1\right.$ : integer $n \geq$ $1\} \cup\{1,27,33,39,51,87,123,183\}$ and $v$ is an integer taken from the set $\{p \equiv$ $7(\bmod 12): p$ is a prime $\} \cup\left\{2^{n}-1\right.$ : odd integer $\left.n \geq 1\right\} \cup\{25,37,61,73,109,157$, 181, 229, 277\}, or a product of such integers.

Combining the results of Theorem 3.1, Construction 4.2 and Theorem 4.4, we have
Theorem 4.5 There exist a strictly p-cyclic $S Q S(2 \times p)$ and a perfect 2-D $(2 \times p, 4,2)$ OOC for any prime $p \equiv 7(\bmod 12)$ or $p \in\{37,61,73,109,157,181,229,277\}$.

## 5 Tighter upper bound for 2-D ( $u \times v, 4,2$ )-OOCs

In most cases an optimal 2-D $(u \times v, 4,2)$-OOC is not a perfect 2-D $(u \times v, 4,2)$-OOC. Thus the determination of the largest possible size $\Phi(u \times v, 4,2)$ of a optimal 2-D $(u \times v, 4,2)$ OOC is of interest. Recall that in Section 1, we mention that $\Phi(u \times v, 4,2) \leq J(u \times v, 4,2)$, where $J(u \times v, 4,2)=\left\lfloor\frac{u}{4}\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor\right\rfloor$ is the famous Johnson bound. Here we shall give a tighter upper bound for 2-D $(u \times v, 4,2)$-OOCs than the Johnson bound.

Lemma 5.1 Let $u v \equiv 0(\bmod 6)$. Then $\Phi(u \times v, 4,2) \leq\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$.
Proof By Theorem 2.1, an optimal 2-D $(u \times v, 4,2)$-OOC with $\Phi(u \times v, 4,2)$ codewords implies a 2-D $(u v \times 1,4,2)$-OOC with $v \Phi(u \times v, 4,2)$ codewords. Since a 2-D $(u v \times$ $1,4,2)$-OOC is equivalent to a strictly 1 -cyclic 3 - $(u v \times 1,4,1)$-packing, by Theorem 3.2, when $u v \equiv 0(\bmod 6)$, it has at most $\left\lfloor\frac{u v}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$ blocks. Thus we have $v \Phi(u \times v, 4,2) \leq\left\lfloor\frac{u v}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$. It is readily checked that $\Phi(u \times v, 4,2) \leq$ $\left\lfloor\frac{1}{v}\left\lfloor\frac{u v}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor\right\rfloor=\left\lfloor\frac{u}{24}\left(u^{2} v^{2}-3 u v-6\right)\right\rfloor=\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$.

The following two lemmas shows that in some cases of $u v \equiv 0(\bmod 6)$, the bound for $\Phi(u \times v, 4,2)$ in Lemma 5.1 is not tight enough. Their proofs are lengthy. To ensure smooth reading of the paper, their proofs have been moved to Appendix II.

Lemma 5.2 Let $u \equiv 0(\bmod 12)$ and $v \equiv 2,4(\bmod 6)$. Then $\Phi(u \times v, 4,2) \leq$ $\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor-1$.

Lemma 5.3 Let $u v \equiv 0(\bmod 12)$ and $v \equiv 0(\bmod 6)$. Then $\Phi(u \times v, 4,2) \leq\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\right.\right.\right.$ $\left.\left.\left.\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-2\right)\right\rfloor$.

Lemma 5.4 Let $u v \equiv 4,8(\bmod 12)$ and $v \equiv 0(\bmod 2)$. Then $\Phi(u \times v, 4,2) \leq$ $\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$.

Proof For each $a \in I_{u}$ and each $0 \leq i<v / 2$, consider the number $n$ of the base blocks containing the two points ( $a, i),(a, v / 2+i)$ in a strictly $v$-cyclic 3 - $(u \times v, 4,1)$-packing. Since each 3 -subset of $I_{u} \times Z_{v}$ occurs in at most one block and each base block containing the two points $(a, i),(a, v / 2+i)$ generates exactly two different blocks containing the same two points, the number $n$ is at most $\lfloor(u v-2) / 4\rfloor=(u v-4) / 4$. Thus there are at least two 3 -subsets of the form $\{(a, i),(a, v / 2+i),(*, *)\}$ in the leave. Note that the above conclusion holds for each $a \in I_{u}$ and each $0 \leq i<v / 2$. It follows that there are at least $u v$ 3 -subsets in the leave. It implies that $\Phi(u \times v, 4,2) \leq\left\lfloor\left(\binom{u v}{3}-u v\right) /(4 v)\right\rfloor=\left\lfloor\frac{1}{24} u\left(u^{2} v^{2}-\right.\right.$ $3 u v-4)\rfloor$. It is readily checked that $\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor=\left\lfloor\frac{1}{24} u\left(u^{2} v^{2}-3 u v-4\right)\right\rfloor$. This completes the proof.

Corollary 5.5 For any $u \equiv 4,8(\bmod 12)$ and $v \equiv 2,4(\bmod 6)$, there is no perfect $2-D$ $(u \times v, 4,2)-O O C$.

Proof If there were a perfect 2-D $(u \times v, 4,2)$-OOC for $u \equiv 4,8(\bmod 12)$ and $v \equiv 2,4$ $(\bmod 6)$, then it should have $u(u v-1)(u v-2) / 24$ codewords. By Lemma 5.4, the largest possible size of the perfect 2-D $(u \times v, 4,2)$-OOC should be $u(u v+1)(u v-4) / 24$. A contradiction occurs.

Lemma 5.6 Let $u \equiv 7,11(\bmod 12)$. Then $\Phi(u \times 2,4,2) \leq\left\lfloor\frac{u}{4}\left\lfloor\frac{2 u-1}{3}\left\lfloor\frac{2 u-2}{2}\right\rfloor\right\rfloor\right\rfloor-1$.
Proof It is known that $\Phi(u \times 2,4,2) \leq J(u \times 2,4,2)=\left\lfloor\frac{u}{4}\left\lfloor\frac{2 u-1}{3}\left\lfloor\frac{2 u-2}{2}\right\rfloor\right\rfloor\right\rfloor$. Suppose that $\Phi(u \times 2,4,2)=J(u \times 2,4,2)$. Then there were a strictly 2 -cyclic 3 - $(u \times 2,4,1)$-packing with $J(u \times 2,4,2)$ base blocks. Count the number of 3 -subsets in the leave $\mathcal{L}$ of the strictly 2 -cyclic 3 - $(u \times 2,4,1)$-packing. It is $\binom{2 u}{3}-J(u \times 2,4,2) \cdot 2 \cdot 4=4$. Each 3 -subset in the leave is of the form $\{(a, i),(b, j),(c, k)\}$ or $\{(a, i),(b, j),(a, i+1)\}$, where $a, b, c$ are distinct elements in $I_{u}$, and $i, j, k \in Z_{2}$.

Assume that $\{(a, i),(b, j),(x, k)\}$ is a 3 -subset in the leave, where $a, b, x \in I_{u}, a \neq b$ and $i, j, k \in Z_{2}$. Consider the number $n$ of the blocks containing the two points $(a, i)$, $(b, j)$. Since each 3 -subset of $I_{u} \times Z_{2}$ occurs in at most one block, the number $n$ is at most $\lfloor(2 u-3) / 2\rfloor=(2 u-4) / 2$. Thus there must be another 3 -subset $\{(a, i),(b, j),(y, l)\}$ in the leave, where $(y, l) \neq(x, k)$. Due to $|\mathcal{L}|=4$, we have $\mathcal{L}=\{\{(a, i),(b, j),(x, k)\},\{(a, i+$ $1),(b, j+1),(x, k+1)\},\{(a, i),(b, j),(y, l)\},\{(a, i+1),(b, j+1),(y, l+1)\}\}$.

If $x \neq a$ and $x \neq b$, since each 3 -subset of $I_{u} \times Z_{2}$ occurs in at most one block, the number of blocks containing the two points $(a, i),(x, k)$ is exactly $(2 u-3) / 2$, which is
not an integer. A contradiction. If $x=a$, then $(x, k)=(a, i+1)$ and there are exactly $(2 u-4) / 4$ base blocks containing the points $(a, i),(x, k)$. If $x=b$, then $(x, k)=(b, j+1)$ and there are also exactly $(2 u-4) / 4$ base blocks containing the points $(b, j),(x, k)$. The number $(2 u-4) / 4$ is not an integer. A contradiction. Hence $|\mathcal{L}| \neq 4$ and $\Phi(u \times 2,4,2) \leq$ $J(u \times 2,4,2)-1$.

Combine the results of Lemmas 5.1-5.6. Let $A=\{(u, v): u \equiv 0(\bmod 12), v \equiv 2,4$ $(\bmod 6)\}$ and $B=\{(u, v): u v \equiv 0(\bmod 12), v \equiv 0(\bmod 6)\}$. In the rest of this paper, we always assume that
$J^{*}(u \times v)= \begin{cases}\left\lfloor\frac{u}{4}\left\lfloor\frac{2 u-1}{3}\left\lfloor\frac{2 u-2}{2}\right\rfloor\right\rfloor\right\rfloor-1, & \text { if } u \equiv 7,11(\bmod 12) \text { and } v=2 ; \\ \left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor, & \text { if } u v \equiv 0(\bmod 6) \text { and }(u, v) \notin A \cup B, \\ & \text { or } u v \equiv 4,8(\bmod 12) \text { and } v \equiv 0(\bmod 2) ; \\ \left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor-1, & \text { if }(u, v) \in A ; \\ \left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-2\right)\right\rfloor, & \text { if }(u, v) \in B ; \\ \left\lfloor\frac{u}{4}\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor\right\rfloor, & \text { otherwise. }\end{cases}$
We have the following theorem.
Theorem 5.7 $\Phi(u \times v, 4,2) \leq J^{*}(u \times v)$.
Now the question is whether there are optimal 2-D $(u \times v, 4,2)$-OOCs to achieve the upper bounds established in Theorem 5.7. In Section 9 we shall give many infinite families for optimal 2-D ( $u \times v, 4,2$ )-OOCs, which achieve the upper bound in Theorem 5.7.

## 6 Auxiliary designs and filling constructions

In this section and the next section, some recursive constructions for optimal 2-D OOCs will be given, called filling constructions and weighting constructions, respectively. These constructions are the generalization of standard constructions for 3-designs in combinatorial design theory. So far the research on combinatorial constructions for 2-D OOCs mainly focuses on $\lambda=1[9,57]$, which corresponds to the theory of 2-designs. However, when $\lambda=2$, the research is related to the theory of 3 -designs. Compared to 2-designs, the known results on 3 -designs are limited, and the auxiliary structures to construct 3 -designs are more complex. Thus the following auxiliary designs will be a little strange for the reader who first meets them. If one is familiar with 2-designs, it is useful to notice that the concepts of $s$-fan designs and $H$ designs are two possible generalizations of group divisible designs. Group divisible design is one of the most basic research objects in combinatorial design theory [6].

## $6.1 s$-fan designs

Hartman [28] first introduced the concept of $s$-fan designs in 1994. Let $s$ be a non-negative integer. An $s$-fan design is an $(s+3)$-tuple $\left(X, \mathcal{G}, \mathcal{B}_{1}, \mathcal{B}_{2}, \ldots, \mathcal{B}_{s}, \mathcal{T}\right)$ satisfying that $(X, \mathcal{G})$ is a 1-design, $\left(X, \mathcal{G} \cup \mathcal{B}_{i}\right)$ is a 2 -design for each $1 \leq i \leq s$ and $\left(X, \mathcal{G} \cup\left(\bigcup_{i=1}^{s} \mathcal{B}_{i}\right) \cup \mathcal{T}\right)$ is a 3-design. The elements of $\mathcal{G}$ and $\left(\bigcup_{i=1}^{s} \mathcal{B}_{i}\right) \cup \mathcal{T}$ are called groups and blocks, respectively.

For understanding the concept of $s$-fan designs, we first consider the case of $s=0$. A 0-fan design is a 3-tuple $(X, \mathcal{G}, \mathcal{T})$ satisfying that $(X, \mathcal{G})$ is a 1-design and $(X, \mathcal{G} \cup \mathcal{T})$ is a 3 -design.

Example 6.1 Take $X=I_{8}$ and $\mathcal{G}=\{\{0,2,4,6\},\{1,3,5,7\}\}$. Then $(X, \mathcal{G})$ is a 1-design. Let $\mathcal{T}$ consists of the following 12 blocks

| $\{0,1,2,3\}$, | $\{0,1,4,5\}$, | $\{0,1,6,7\}$, | $\{0,2,5,7\}$, | $\{0,3,4,7\}$, | $\{0,3,5,6\}$, |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\{1,2,4,7\}$, | $\{1,2,5,6\}$, | $\{1,3,4,6\}$, | $\{2,3,4,5\}$, | $\{2,3,6,7\}$, | $\{4,5,6,7\}$. |

It is readily checked that each 3-subset of $I_{8}$ is either contained in exactly one block of $\mathcal{T}$ or in exactly one group of $\mathcal{G}$, but not in both. Hence, $(X, \mathcal{G} \cup \mathcal{T})$ is a 3-design. This is an example of 0 -fan designs.

Next we give an example of 1-fan designs. A 1-fan design is a 4 -tuple $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ satisfying that $(X, \mathcal{G})$ is a 1-design, $(X, \mathcal{G} \cup \mathcal{B})$ is a 2-design and $(X, \mathcal{G} \cup \mathcal{B} \cup \mathcal{T})$ is a 3-design.

Example 6.2 Take $X=I_{9}$ and $\mathcal{G}=\{\{0,1,8\},\{2,3,6\},\{4,5,7\}\}$. Then $(X, \mathcal{G})$ is a 1 -design. Let $\mathcal{B}$ consists of the following 9 blocks

$$
\begin{array}{llllll}
\{2,4,8\}, & \{3,5,8\}, & \{0,2,7\}, & \{0,3,4\}, \quad\{1,3,7\}, \quad\{1,4,6\}, \\
\{1,2,5\}, & \{0,5,6\}, & \{6,7,8\} . & &
\end{array}
$$

It is readily checked that each 2 -subset of $I_{9}$ is either contained in exactly one block of $\mathcal{B}$ or in exactly one group of $\mathcal{G}$, but not in both. Hence, $(X, \mathcal{G} \cup \mathcal{B})$ is a 2 -design. Let $\mathcal{T}$ consists of the following 18 blocks

| $\{0,1,2,3\}$, | $\{0,1,4,5\}$, | $\{0,1,6,7\}$, | $\{0,2,4,6\}$, | $\{0,2,5,8\}$, | $\{0,3,5,7\}$, |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\{0,3,6,8\}$, | $\{0,4,7,8\}$, | $\{1,2,4,7\}$, | $\{1,2,6,8\}$, | $\{1,3,4,8\}$, | $\{1,3,5,6\}$, |
| $\{1,5,7,8\}$, | $\{2,3,4,5\}$, | $\{2,3,7,8\}$, | $\{2,5,6,7\}$, | $\{3,4,6,7\}$, | $\{4,5,6,8\}$. |

It is readily checked that $(X, \mathcal{G} \cup \mathcal{B} \cup \mathcal{T})$ is a 3-design. This is an example of 1-fan designs.
If there are $a_{i}$ groups of size $g_{i}$ in an $s$-fan design, $1 \leq i \leq m$, then the type of the $s$-fan design is defined to be $g_{1}^{a_{1}} g_{2}^{a_{2}} \cdots g_{m}^{a_{m}}$. Let $K_{1}, K_{2}, \ldots, K_{s}, K_{T}$ be sets of positive integers. If block sizes of $\mathcal{B}_{i}$ and $\mathcal{T}$ are from $K_{i}(1 \leq i \leq s)$ and $K_{T}$, respectively, then the $s$-fan design is denoted by $s$ - $\mathrm{FG}\left(3,\left(K_{1}, K_{2}, \ldots, K_{s}, K_{T}\right), \sum_{i=1}^{m} g_{i} a_{i}\right)$ of type $g_{1}^{a_{1}} g_{2}^{a_{2}} \cdots g_{m}^{a_{m}}$. Example 6.1 shows a 0 -FG $(3,(\emptyset, 4), 8)$ of type $4^{2}$. Example 6.2 presents a 1-FG $(3,(3,4), 9)$ of type $3^{3}$.

Lemma $6.3([18])$ The necessary conditions for the existence of a $0-F G\left(3,\left(\emptyset, K_{T}\right), g n\right)$ of type $g^{n}(n \geq 2)$ are
(1) $g^{2} n(n-1)(g n+g-3) \equiv 0(\bmod \alpha)$, where $\alpha=\operatorname{gcd}\left\{k(k-1)(k-2): k \in K_{T}\right\}$;
(2) $g(n-1)(g n+g-3) \equiv 0(\bmod \beta)$, where $\beta=\operatorname{gcd}\left\{(k-1)(k-2): k \in K_{T}\right\}$;
(3) if $g=1$, then $n \equiv 2(\bmod \gamma)$; if $g>1$, then $g n \equiv g \equiv 2(\bmod \gamma)$, where $\gamma=\operatorname{gcd}\left\{k-2: k \in K_{T}\right\}$.

Theorem 6.4 ([64]) There exists a $0-F G(3,(\emptyset, 4), g n)$ of type $g^{n}$ if and only if either $g=1$ and $n \equiv 2,4(\bmod 6)$, or $g$ is even and $g(n-1)(n-2) \equiv 0(\bmod 3)$.

### 6.1.1 The basic idea

Since an optimal 2-D $(u \times v, k, 2)$-OOC is equivalent to an optimal strictly $v$-cyclic 3 ( $u \times v, k, 1$ )-packing, we first consider how to construct a 3 -packing without the restriction of automorphism groups.

- Step 1: Start from a $0-\mathrm{FG}(3,(\emptyset, k), g n)$ of type $g^{n}(X, \mathcal{G}, \emptyset, \mathcal{T})$. By the definition of $s$-fan designs, $(X, \mathcal{G} \cup \mathcal{T})$ is a 3 -design. $(X, \mathcal{T})$ satisfies that each 3 -subset of $X$ not contained in some group of $\mathcal{G}$ occurs in exactly one block of $\mathcal{T}$, and each 3 -subset of $X$ contained in some group of $\mathcal{G}$ never occur in any block of $\mathcal{T}$.
- Step 2: If a $3-(g, k, 1)$-packing exists, then one can construct a 3 - $(g, k, 1)$-packing on the set $G$ for each $G \in \mathcal{G}$. Denote its block set by $\mathcal{A}_{G}$.
- Step 3: Let $\mathcal{A}=\cup_{G \in \mathcal{G}} \mathcal{A}_{G}$. It follows that each 3-subset of $X$ contained in some group of $\mathcal{G}$ occurs in at most one block of $\mathcal{A}$.
- Step 4: Let $\mathcal{C}=\mathcal{A} \cup \mathcal{T}$. We have that $(X, \mathcal{C})$ is a $3-(g n, k, 1)$-packing.

The main idea of the above construction is to fill in the groups of a 0 -fan design with a 3-packing. So this construction is termed as "Filling Construction". Furthermore, if one hope to obtain an optimal $3-(g n, k, 1)$-packing, it is necessary to input an optimal $3-(g, k, 1)$-packing. Note that the reverse is not always correct. Now our purpose is to construct strictly $v$-cyclic 3 - $(u \times v, k, 1)$-packings. We need to modify the above "Filling Construction" such that the initial 0 -fan design admits some special automorphisms.

An automorphism of an $s$-fan design $\left(X, \mathcal{G}, \mathcal{B}_{1}, \mathcal{B}_{2}, \ldots, \mathcal{B}_{s}, \mathcal{T}\right)$ is a permutation on $X$ leaving $\mathcal{G}, \mathcal{B}_{1}, \mathcal{B}_{2}, \ldots, \mathcal{B}_{s}, \mathcal{T}$ invariant, respectively. All automorphisms of an $s$-fan design form a group, called the full automorphism group of the s-fan design. Any subgroup of the full automorphism group is called an automorphism group of the $s$-fan design.

Let $G$ be an automorphism group of an $s$-fan design. All blocks of the $s$-fan design can be partitioned into some orbits under $G$. Choose any fixed block from each orbit and then call it a base block of this $s$-fan design. For any block $B$ of the $s$-fan design, the subgroup $\left\{\pi \in G: B^{\pi}=B\right\}$ is called the stabilizer of $B$ in $G$.

Example 6.5 Observe the 0-FG $(3,(\emptyset, 4), 8)$ of type $4^{2}$ from Example 6.1. Consider the permutation $\alpha=\left(\begin{array}{lll}0 & 1 & 2\end{array}\right)\left(\begin{array}{llll}4 & 5 & 6 & 7\end{array}\right)$ on $I_{8}$. It is easy to checked that $\alpha$ is an automorphism of this $0-F G$. All blocks are partitioned into 5 orbits under the action of $\alpha$. The 5 base blocks are $\{0,1,2,3\}^{*},\{4,5,6,7\}^{*},\{1,2,5,6\},\{0,1,6,7\}$ and $\{0,2,5,7\}$, where the stabilizer of each base block marked with $a^{*}$ is trivial, i.e., it contains only the identity permutation.

In the following we introduce two kinds of $s$-fan designs with special automorphism groups.

### 6.1.2 $h$-cyclic $s$-fan designs

Construct an $s$-fan design of type $\left(h g_{1}\right)^{a_{1}}\left(h g_{2}\right)^{a_{2}} \cdots\left(h g_{m}\right)^{a_{m}}$ on $\left(\bigcup_{i=1}^{m}\left(I_{a_{i}} \times I_{g_{i}}\right)\right) \times Z_{h}$ with the group set $\left\{\{x\} \times I_{g_{i}} \times Z_{h}: x \in I_{a_{i}}, 1 \leq i \leq m\right\}$. If this $s$-fan design admits an automorphism $\pi$ mapping $(x, y, j) \longmapsto(x, y, j+1)(\bmod (-,-, h)), x \in I_{a_{i}}, y \in I_{g_{i}}$ and $j \in Z_{h}$, then the $s$-fan design is said to be $h$-cyclic.

For each block $B$ of an $h$-cyclic $s$-fan design of type $\left(h g_{1}\right)^{a_{1}}\left(h g_{2}\right)^{a_{2}} \cdots\left(h g_{m}\right)^{a_{m}}$, if the stabilizer of $B$ in $Z_{h}$ is trivial, i.e., $\left\{\delta \in Z_{h}: B+\delta=B\right\}=\{0\}$, where $B+\delta=$ $\{(x, y, j+\delta):(x, y, j) \in B\}$, then the $s$-fan design is called strictly $h$-cyclic. A (strictly) $h$-cyclic $s$-fan design of type $h^{n}$ is often referred to as a (strictly) semi-cyclic s-fan design of type $h^{n}$ (cf. [18]).

The following construction is straightforward.
Construction 6.6 (Filling Construction-I) Suppose that the following exist:
(1) a strictly $h$-cyclic $0-F G\left(3,(\emptyset, k), \Sigma_{i=1}^{m} g_{i} a_{i} h\right)$ of type $\left(h g_{1}\right)^{a_{1}}\left(h g_{2}\right)^{a_{2}} \cdots\left(h g_{m}\right)^{a_{m}}$ with $b_{0}$ base blocks;
(2) a strictly h-cyclic $3-\left(g_{i} \times h, k, 1\right)$ packing with $b_{i}$ base blocks for each $1 \leq i \leq m$.

Then there exists a strictly h-cyclic $3-\left(\left(\sum_{i=1}^{m} g_{i} a_{i}\right) \times h, k, 1\right)$ packing with $b_{0}+\sum_{i=1}^{m} a_{i} b_{i}$ base blocks, which is a 2-D $\left(\left(\sum_{i=1}^{m} g_{i} a_{i}\right) \times h, k, 1\right)$-OOC.

Furthermore, if the given strictly h-cyclic 3-( $g_{i} \times h, k, 1$ ) packing is a strictly h-cyclic $S\left(3, k, g_{i} \times h\right)$ for each $1 \leq i \leq m$, then we obtain a strictly $h$-cyclic $S\left(3, k,\left(\sum_{i=1}^{m} g_{i} a_{i}\right) \times h\right)$, which is a perfect 2-D $\left(\left(\Sigma_{i=1}^{m} g_{i} a_{i}\right) \times h, k, 2\right)-O O C$.

Example 6.7 In this example, we construct an optimal 2-D $(4 \times 2,4,2)$-OOC.

- Step 1: First we construct a strictly 2-cyclic $0-F G(3,(\emptyset, 4), 8)$ of type $4^{2}$ on $I_{2} \times$ $I_{2} \times Z_{2}$ with the group set $\left\{\{x\} \times I_{2} \times Z_{2}: x \in I_{2}\right\}$. All the 6 base blocks are listed below.

$$
\begin{array}{lll}
\{(0,0,0),(0,0,1),(1,0,0),(1,1,0)\}, & \{(0,0,0),(1,0,0),(1,0,1),(0,1,0)\}, \\
\{(0,0,0),(1,0,0),(1,1,1),(0,1,1)\}, & \{(0,0,0),(1,0,1),(1,1,0),(0,1,1)\}, \\
\{(0,0,0),(1,1,0),(1,1,1),(0,1,0)\}, & \{(1,0,0),(1,1,0),(0,1,0),(0,1,1)\}
\end{array}
$$

- Step 2: Take an optimal strictly 2-cyclic $3-(2 \times 2,4,1)$ packing, which is trivial without base blocks.
- Step 3: Apply Construction 6.6 to obtain a strictly 2-cyclic 3-(4×2,4,1) packing with 6 base blocks, which achieves the upper bound in Theorem 5.7 and is an optimal $2-D(4 \times 2,4,2)$-OOC with 6 codewords. Note that $I_{2} \times I_{2} \times Z_{2} \cong I_{4} \times Z_{2}$. Hence $\Phi(4 \times 2,4,2)=J^{*}(4 \times 2)=6$.

Example 6.8 In this example, we construct an optimal 2-D $(4 \times 3,4,2)$-OOC.

- Step 1: First we construct a strictly 3-cyclic $0-F G(3,(\emptyset, 4), 12)$ of type $6^{2}$ on $I_{2} \times$ $I_{2} \times Z_{3}$ with the group set $\left\{\{x\} \times I_{2} \times Z_{3}: x \in I_{2}\right\}$. All the 15 base blocks are listed below:

$$
\begin{array}{ll}
\{(0,0,0),(0,0,1),(1,0,0),(1,0,1)\}, & \{(0,0,0),(0,0,1),(1,0,2),(1,1,0)\}, \\
\{(0,0,0),(0,0,1),(1,1,1),(1,1,2)\}, & \{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\}, \\
\{(0,0,0),(0,1,0),(1,0,1),(1,1,1)\}, & \{(0,0,0),(0,1,0),(1,0,2),(1,1,2)\}, \\
\{(0,0,0),(0,1,1),(1,0,0),(1,1,1)\}, & \{(0,0,0),(0,1,1),(1,0,1),(1,0,2)\}, \\
\{(0,0,0),(0,1,1),(1,1,0),(1,1,2)\}, & \{(0,0,0),(0,1,2),(1,0,0),(1,1,2)\}, \\
\{(0,0,0),(0,1,2),(1,0,1),(1,1,0)\}, & \{(0,0,0),(0,1,2),(1,0,2),(1,1,1)\}, \\
\{(0,1,0),(0,1,1),(1,0,0),(1,0,2)\}, & \{(0,1,0),(0,1,1),(1,0,1),(1,1,2)\},
\end{array}
$$

- Step 2: Construct an optimal strictly 3 -cyclic $3-(2 \times 3,4,1)$ packing on $\{x\} \times I_{2} \times Z_{3}$ for each $x \in I_{2}$, which has 1 base block and exists by Example 3.3. Then this step contributes 2 base blocks as follows

$$
\{(0,0,0),(0,1,0),(0,0,1),(0,1,1)\}, \quad\{(1,0,0),(1,1,0),(1,0,1),(1,1,1)\}
$$

- Step 3: Apply Construction 6.6 to obtain a strictly 3-cyclic $(4 \times 3,4,1)$ packing with 17 base blocks, which achieves the upper bound in Theorem 5.7 and is an optimal $2-D(4 \times 3,4,2)-O O C$ with 17 codewords. Note that $I_{2} \times I_{2} \times Z_{3} \cong I_{4} \times Z_{3}$. Hence $\Phi(4 \times 3,4,2)=J^{*}(4 \times 3)=17$.

Lemma 6.9 For any $v \equiv 1(\bmod 2)$, there exists a strictly $v$-cyclic $0-F G(3,(\emptyset, 4), 4 v)$ of type $(2 v)^{2}$.

Proof By Lemma 2.11 in [20], there is a semi-cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 4 v)$ of type $(2 v)^{2}$ on the point set $X=I_{2} \times Z_{2 v}$ and the group set $\mathcal{G}=\left\{\{x\} \times Z_{2 v}: x \in I_{2}\right\}$. Denote the family of its blocks by $\mathcal{T}$. For each $(x, i) \in X$, define a mapping

$$
\tau: \quad(x, i) \longmapsto(x, i-2\lfloor i / 2\rfloor,\lfloor i / 2\rfloor)
$$

Let $X^{\prime}=I_{2} \times I_{2} \times Z_{v}$ and $\mathcal{G}^{\prime}=\left\{\{x\} \times I_{2} \times Z_{v}: x \in I_{2}\right\}$. Let $\mathcal{T}^{\prime}=\bigcup_{T \in \mathcal{T}} \tau(T)$, where $\tau(T)=\{\tau(r): r \in T\}$. Since $v \equiv 1(\bmod 2)$, it is readily checked that $\left(X^{\prime}, \mathcal{G}^{\prime}, \emptyset, \mathcal{T}^{\prime}\right)$ is a strictly $v$-cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 4 v)$ of type $(2 v)^{2}$.

Remark 6.10 In Construction 6.6, even if the given strictly h-cyclic $3-\left(g_{i} \times h, k, 1\right)$ packing is optimal for each $1 \leq i \leq m$, the resulting strictly h-cyclic $3-\left(\left(\sum_{i=1}^{m} g_{i} a_{i}\right) \times\right.$ $h, k, 1)$ packing may not be optimal.

### 6.1.3 $(u, h)$-regular $s$-fan designs

Let $h$ divide $v$ and $H$ be a subgroup of order $h$ in $Z_{v}$, i.e., $H=\{0, v / h, \ldots,(h-1) v / h\}$. Let $H_{i}=H+i$ be a coset of $H$ in $Z_{v}, 0 \leq i<v / h$. Construct an $s$-fan design of type $(u h)^{v / h}$ on $I_{u} \times Z_{v}$ with the group set $\left\{I_{u} \times H_{i}: 0 \leq i<v / h\right\}$. If this $s$-fan design admits an automorphism $\pi$ mapping $(x, j) \longmapsto(x, j+1)(\bmod (-, v)), x \in I_{u}$ and $j \in Z_{v}$, then the $s$-fan design is said to be $(u, h)$-regular.

For each block $B$ of a $(u, h)$-regular $s$-fan design of type $(u h)^{v / h}$, if the stabilizer of $B$ in $Z_{v}$ is trivial, i.e., $\left\{\delta \in Z_{v}: B+\delta=B\right\}=\{0\}$, where $B+\delta=\{(x, j+\delta):(x, j) \in B\}$, then the $s$-fan design is called strictly $(u, h)$-regular.

Example 6.11 By Example 6.5, the 0-FG $(3,(\emptyset, 4), 8)$ of type $4^{2}$ from Example 6.1 admits an automorphism (0 123 ) (4567). Actually the reader may check that this 0$F G$ is isomorphic to a $(2,2)$-regular $0-F G$ under the mapping $\tau: v \rightarrow(\lfloor v / 4\rfloor, v(\bmod 4))$ from $I_{8}$ to $I_{2} \times Z_{4}$. But it is not strictly $(2,2)$-regular.

When $u=1$, a (strictly) (1,h)-regular $s$-fan design of type $h^{v / h}$ is often referred to as a (strictly) cyclic s-fan design of type $h^{v / h}$ (cf. [18]). We quote the following results for later use.

Lemma 6.12 ( [18])
(1) There exists a strictly cyclic $0-F G(3,(\emptyset, 4), 2 h)$ of type $h^{2}$ for any $h \equiv 0(\bmod 8)$.
(2) There exists a strictly cyclic $0-F G(3,(\emptyset, 4), 3 h)$ of type $h^{3}$ for any $h \equiv 0(\bmod 12)$.
(3) There exists a strictly cyclic $0-F G(3,(\emptyset, 4), 5 h)$ of type $h^{5}$ for any $h \equiv 0(\bmod 2)$.

Lemma $6.13([19])$ An $\operatorname{RoSQS}(v+1)$ for $v \equiv 1(\bmod 6)$ is equivalent to a strictly cyclic $1-F G(3,(3,4), v)$ of type $1^{v}$. An $\operatorname{RoSQS}(v+1)$ for $v \equiv 3(\bmod 6)$ is equivalent to a strictly cyclic 1-FG(3, $(3,4), v)$ of type $3^{v / 3}$.

Construction 6.14 (Filling Construction-II) Let $u h \geq k \geq 3$. Suppose that the following exist.
(1) a strictly $(u, h)$-regular 0-FG(3, $(\emptyset, k)$, uv) of type $(u h)^{v / h}$ with $b_{0}$ base blocks;
(2) a strictly $h$-cyclic 3- $(u \times h, k, 1)$ packing with $b_{1}$ base blocks.

Then there exists a strictly $v$-cyclic $3-(u \times v, k, 1)$ packing with $b_{0}+b_{1}$ base blocks.
Furthermore, if the given strictly $h$-cyclic $3-(u \times h, k, 1)$ packing is optimal with $J(u \times$ $h, k, 2)$ base blocks, then the derived strictly $v$-cyclic $3-(u \times v, k, 1)$ packing is also optimal with $J(u \times v, k, 2)$ base blocks, which is an optimal $2-D(u \times v, k, 2)$-OOC with $J(u \times v, k, 2)$ codewords.

Proof First we prove the first part of this construction.

- Step 1: Start from a strictly $(u, h)$-regular $0-\mathrm{FG}(3,(\emptyset, k), u v)$ of type $(u h)^{v / h}$. Denote the family of base blocks of this design by $\mathcal{F}$.
- Step 2: Let $\mathcal{E}$ be the family of base blocks of a strictly $h$-cyclic 3 - $(u \times h, k, 1)$ packing. For each $B=\left\{\left(x_{1}, j_{1}\right),\left(x_{2}, j_{2}\right), \ldots,\left(x_{k}, j_{k}\right)\right\} \in \mathcal{E}$ we take

$$
\frac{v}{h} B=\left\{\left(x_{1}, \frac{v}{h} j_{1}\right),\left(x_{2}, \frac{v}{h} j_{2}\right), \ldots,\left(x_{k}, \frac{v}{h} j_{k}\right)\right\} .
$$

- Step 3: Then $\mathcal{F} \cup\left\{\frac{v}{h} B: B \in \mathcal{E}\right\}$ forms the family of base blocks of the desired strictly $v$-cyclic ( $u \times v, k, 1$ ) packing.

For checking optimality of the required design in the second part, it suffices to show that

$$
\begin{align*}
& \frac{u((u v-1)(u v-2)-(u h-1)(u h-2))}{k(k-1)(k-2)}+\left\lfloor\frac{u}{k}\left\lfloor\frac{u h-1}{k-1}\left\lfloor\frac{u h-2}{k-2}\right\rfloor\right\rfloor\right\rfloor \\
& =\left\lfloor\frac{u}{k}\left\lfloor\frac{u v-1}{k-1}\left\lfloor\frac{u v-2}{k-2}\right\rfloor\right\rfloor\right\rfloor . \tag{1}
\end{align*}
$$

By Lemma 6.3 (3), since $u h>1$, the existence of a strictly ( $u, h$ )-regular 0-FG(3, ( $\emptyset, k)$, $u v)$ of type $(u h)^{v / h}$ implies that $u v-2 \equiv u h-2 \equiv 0(\bmod k-2)$. By Lemma 6.3 (2), one can verify that $(u v-1)(u v-2) \equiv(u h-1)(u h-2)(\bmod (k-1)(k-2))$. Let $(u v-1)(u v-2)=a_{1}(k-1)(k-2)+r$ and $(u h-1)(u h-2)=a_{2}(k-1)(k-2)+r$, where $0 \leq r<(k-1)(k-2)$. Thus for obtaining the equation (1), it suffices to prove that

$$
\begin{equation*}
\frac{u\left(a_{1}-a_{2}\right)}{k}+\left\lfloor\frac{u a_{2}}{k}\right\rfloor=\left\lfloor\frac{u a_{1}}{k}\right\rfloor . \tag{2}
\end{equation*}
$$

Note that $u\left(a_{1}-a_{2}\right) \equiv 0(\bmod k)$. Let $u a_{1}=b_{1} k+r_{1}$ and $u a_{2}=b_{2} k+r_{1}$, where $0 \leq r_{1}<k$. It is readily checked that the equation (2) holds.

Example 6.15 In this example, we construct an optimal $2-D(2 \times 4,4,2)$-OOC.

- Step 1: First we construct a strictly (2,2)-regular 0-FG(3, ( $\emptyset, 4), 8)$ of type $4^{2}$ on $I_{2} \times Z_{4}$ with the group set $\left\{I_{2} \times H_{i}: 0 \leq i \leq 1\right\}$, where $H_{0}=\{0,2\}$ is a subgroup of order 2 in $Z_{4}$ and $H_{1}=\{1,3\}$. All the 3 base blocks are listed below:

$$
\{(0,0),(0,1),(0,2),(1,1)\}, \quad\{(0,0),(0,1),(1,2),(1,3)\}, \quad\{(0,0),(1,0),(1,1),(1,3)\} .
$$

- Step 2: Take an optimal strictly 2-cyclic $3-(2 \times 2,4,1)$ packing, which is trivial without base blocks.
- Step 3: Apply Construction 6.14 to obtain a strictly 4-cyclic 3-( $2 \times 4,4,1$ ) packing with 3 base blocks, which achieves the upper bound in Theorem 5.7 and is an optimal $2-D(2 \times 4,4,2)$-OOC with 3 codewords. Hence $\Phi(2 \times 4,4,2)=J^{*}(2 \times 4)=3$.

Example 6.16 In this example, we construct an optimal 2-D $(2 \times 8,4,2)$-OOC.

- Step 1: First we construct a strictly (2,4)-regular $0-F G(3,(\emptyset, 4), 16)$ of type $8^{2}$ on $I_{2} \times Z_{8}$ with the group set $\left\{I_{2} \times H_{i}: 0 \leq i \leq 1\right\}$, where $H_{0}=\{0,2,4,6\}$ is a subgroup of order 4 in $Z_{8}$ and $H_{1}=\{1,3,5,7\}$. All the 14 base blocks are listed below:

$$
\begin{aligned}
& \{(0,0),(0,1),(0,2),(0,5)\}, \quad\{(0,0),(0,1),(0,3),(1,0)\}, \quad\{(0,0),(0,1),(0,6),(1,1)\}, \\
& \{(0,0),(0,1),(1,2),(1,3)\}, \quad\{(0,0),(0,1),(1,4),(1,5)\}, \quad\{(0,0),(0,1),(1,6),(1,7)\}, \\
& \{(0,0),(0,2),(1,1),(1,5)\}, \quad\{(0,0),(0,3),(1,1),(1,6)\}, \quad\{(0,0),(0,3),(1,2),(1,5)\}, \\
& \{(0,0),(0,3),(1,4),(1,7)\}, \quad\{(0,0),(0,4),(1,1),(1,7)\}, \quad\{(0,0),(1,0),(1,1),(1,3)\}, \\
& \{(0,0),(1,0),(1,5),(1,7)\}, \quad\{(1,0),(1,1),(1,2),(1,5)\} \text {. }
\end{aligned}
$$

- Step 2: Construct an optimal strictly 4-cyclic $3-(2 \times 4,4,1)$ packing with 3 base blocks, which exists by Example 6.15. Then this step contributes 3 base blocks as follows

$$
\{(0,0),(0,2),(0,4),(1,2)\}, \quad\{(0,0),(0,2),(1,4),(1,6)\}, \quad\{(0,0),(1,0),(1,2),(1,6)\} .
$$

- Step 3: Apply Construction 6.14 to obtain a strictly 8 -cyclic $(2 \times 8,4,1)$ packing with 17 base blocks, which achieves the upper bound in Theorem 5.7 and is an optimal $2-D(2 \times 8,4,2)$-OOC with 17 codewords. Hence $\Phi(2 \times 8,4,2)=J^{*}(2 \times 8)=17$.

The following result is simple but very useful.
Lemma 6.17 If there exists a strictly $(u, h)$-regular $0-F G(3,(\emptyset, k), u v)$ of type $(u h)^{v / h}$, then for any integer divisor $h_{1}$ of $h$, there exists a strictly $h_{1}-c y c l i c ~ 0-F G(3,(\emptyset, k), u v)$ of type $(u h)^{v / h}$.

Corollary 6.18 For any $v \not \equiv 2(\bmod 4)$, there exists a strictly v-cyclic 0-FG(3, ( $\emptyset, 4), 4 v)$ of type $(2 v)^{2}$.

Proof When $v \equiv 0(\bmod 4)$, by Lemma 6.12 there is a strictly cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 4 v)$ of type $(2 v)^{2}$. Apply Lemma 6.17 to obtain a strictly $v$-cyclic 0 -FG( $\left.3,(\emptyset, 4), 4 v\right)$ of type $(2 v)^{2}$. When $v \equiv 1(\bmod 2)$, the conclusion follows from Lemma 6.9.

Lemma 6.19 If there is a perfect $2-D(2 \times v, 4,2)-O O C$ with $v \equiv 1,5(\bmod 6)$, then there is a strictly $(2,1)$-regular 1-FG(3, $(2,4), 2 v)$ of type $2^{v}$.

Proof By Lemma 3.4, the necessary condition for the existence of a perfect 2-D $(2 \times$ $v, 4,2)$-OOC is $v \equiv 1,5(\bmod 6)$. Suppose that $(X, \mathcal{T})$ is a strictly $v$-cyclic $\operatorname{SQS}(2 \times v)$ with $X=I_{2} \times Z_{v}$, which is equivalent to a perfect 2-D $(2 \times v, 4,2)$-OOC. Let $\mathcal{G}=$ $\left\{I_{2} \times\{x\}: x \in Z_{v}\right\}$. Then $(X, \mathcal{G}, \emptyset, \mathcal{T})$ is a strictly $(2,1)$-regular $0-\mathrm{FG}(3,(\emptyset, 4), 2 v)$ of type $2^{v}$. Collect all 2 -subsets of $X$ from distinct groups of $\mathcal{G}$ into a set $\mathcal{B}$. Since $v$ is odd, $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ is a strictly $(2,1)$-regular $1-\mathrm{FG}(3,(2,4), 2 v)$ of type $2^{v}$.

## 6.2 $H$ designs

Mills first used the terminology of $H$ designs in [42]. Let $n, g, t$ be positive integers and $K$ be a set of positive integers. An $H$ design is a triple $(X, \mathcal{G}, \mathcal{B})$, where $\mathcal{G}$ is a partition of a set of points $X$ into $n$ subsets (called groups), each of cardinality $g$, and $\mathcal{B}$ is a collection of subsets of $X$ (called blocks), each of cardinality from $K$, such that each block intersects any given group in at most one point, and each $t$-subset of $X$ from $t$ distinct groups is contained in a unique block. Such a design is denoted by $H(n, g, K, t)$.

Example 6.20 Take $X=I_{8}$ and $\mathcal{G}=\{\{0,4\},\{1,5\},\{2,6\},\{3,7\}\}$. Let $\mathcal{B}$ consists of the following 8 blocks

$$
\begin{array}{llllll}
\{0,1,2,3\}, & \{4,5,6,7\}, & \{0,1,6,7\}, & \{2,3,4,5\}, & \{0,2,5,7\}, & \{1,3,4,6\}, \\
\{0,3,5,6\}, & \{1,2,4,7\} . & & &
\end{array}
$$

It is easy to see that each 3 -subset of $I_{8}$ from three distinct groups of $\mathcal{G}$ is contained in a unique block of $\mathcal{B}$. Then $(X, \mathcal{G}, \mathcal{B})$ is an $H(4,2,4,3)$.

Example 6.21 Let $\left(X, \mathcal{G}, \mathcal{B}_{1}, \mathcal{B}_{2}, \ldots, \mathcal{B}_{s}, \mathcal{T}\right)$ be an $s-F G\left(3,\left(K_{1}, K_{2}, \ldots, K_{s}, K_{T}\right), g n\right)$ of type $g^{n}$. Then for each $1 \leq i \leq s,\left(X, \mathcal{G}, \mathcal{B}_{i}\right)$ is an $H\left(n, g, K_{i}, 2\right)$ (called the $i$-th subdesign of the $s$-fan design).

Lemma 6.22 ( $[30,43])$ For any $n \geq 4, n \neq 5$, an $H(n, g, 4,3)$ exists if and only if $g n$ is even and $g(n-1)(n-2)$ is divisible by 3 . For $n=5$, an $H(5, g, 4,3)$ exists if $g$ is even, $g \neq 2$ and $g \not \equiv 10,26(\bmod 48)$.

An automorphism of an $H$ design $(X, \mathcal{G}, \mathcal{B})$ is a permutation on $X$ leaving $\mathcal{G}, \mathcal{B}$ invariant, respectively. All automorphisms of an $H$ design form a group, called the full automorphism group of the $H$ design. Any subgroup of the full automorphism group is called an automorphism group of the $H$ design.

Let $G$ be an automorphism group of an $H$ design. All blocks of the $H$ design can be partitioned into some orbits under $G$. Choose any fixed block from each orbit and then call it a base block of this $H$ design. For any block $B$ of the $H$ design, the subgroup $\left\{\pi \in G: B^{\pi}=B\right\}$ is called the stabilizer of $B$ in $G$.

Example 6.23 Observe the $H(4,2,4,3)$ from Example 6.20. Consider the permutation $(04)(15)(26)(37)$ on $I_{8}$. It is readily checked that $\alpha$ is an automorphism of this $H$ design. All blocks are partitioned into 4 orbits under the action of $\alpha$. The 4 base blocks are $\{0,1,2,3\},\{0,1,6,7\},\{0,2,5,7\},\{0,3,5,6\}$.

Construct an $H(n, l h, K, t)$ on $I_{n} \times I_{l} \times Z_{h}$ with the group set $\left\{\{x\} \times I_{l} \times Z_{h}: x \in I_{n}\right\}$. If this $H$ design admits an automorphism $\pi$ mapping $(x, y, j) \longmapsto(x, y, j+1)(\bmod$ $(-,-, h)), x \in I_{n}, y \in I_{l}$ and $j \in Z_{h}$, then the $H$ design is said to be $h$-cyclic. If the stabilizer of each block of an $h$-cyclic $H(n, l h, K, t)$ in $Z_{h}$ is trivial, i.e., for any block $B$, $\left\{\delta \in Z_{h}: B+\delta=B\right\}=\{0\}$, where $B+\delta=\{(x, y, j+\delta):(x, y, j) \in B\}$, then the $H$ design is called strictly $h$-cyclic. Note that one can verify that an $h$-cyclic $H$ design is always strictly $h$-cyclic.

Example 6.24 By Example 6.23, the H(4,2,4,3) from Example 6.20 admits an automorphism $(04)(15)(26)(37)$. Actually the reader may check that this $H$ design is isomorphic to a 2-cyclic $H(4,2,4,3)$ under the mapping $\tau: v \rightarrow(v(\bmod 4), 0,\lfloor v / 4\rfloor)$ from $I_{8}$ to $I_{4} \times I_{1} \times Z_{2}$.

When $l=1$, an $h$-cyclic $H(n, h, K, t)$ is often referred to as a semi-cyclic $H(n, h, K, t)$.
Lemma 6.25 ( [18]) For any $h \geq 1$, there exists a semi-cyclic $H(4, h, 4,3)$.

## 7 Weighting constructions

For applying Constructions 6.6 and 6.14 , we need some strictly $h$-cyclic 0 -FGs and strictly $(u, h)$-regular $s$-FGs. Construction 7.1 shows that if one has a strictly $h_{1}$-cyclic 1-FG of type $\left(g_{1} h_{1}\right)^{n}$, and gives each point of the 1-FG a weight $g_{2} h_{2}$, then a strictly $h_{1} h_{2}$-cyclic 0 -FG of type $\left(g_{1} g_{2} h_{1} h_{2}\right)^{n}$ can be obtained; Construction 7.3 shows that if one has a strictly $\left(g_{1}, h_{1}\right)$-regular 1-FG of type $\left(g_{1} h_{1}\right)^{n}$, and gives each point of the 1-FG a weight $g_{2} h_{2}$, then a strictly $\left(g_{1} g_{2}, h_{1} h_{2}\right)$-regular 0-FG of type $\left(g_{1} g_{2} h_{1} h_{2}\right)^{n}$ can be obtained. So Constructions 7.1 and 7.3 give an approach to find some infinite families of strictly $h$ cyclic 0 -FGs and strictly ( $u, h$ )-regular $s$-FGs. Then apply Constructions 6.6 and 6.14 to fill in the groups of these infinite families. We can obtain many optimal 2-D OOCs, which will be presented in Sections 8 and 9 .

Condition (3) in Constructions 7.1 and 7.3 implies that $h$-cyclic $H$ designs are important. Thus a recursive construction for $h$-cyclic $H$ designs is presented in Construction 7.5. The proofs of all constructions in this section are of design theory. Here we only focus on how these constructions work. The detailed proofs of Constructions 7.1 and 7.5 have been moved to Appendix III. The detailed proof of Construction 7.3 is omitted, which is similar to that of Construction 7.1.

Construction 7.1 (Weighting Construction-I) Let $K$ and $L_{i}$ for each $1 \leq i \leq s$ be all sets of positive integers greater than 1. Let $K_{T}$ and $L_{T}$ be both sets of positive integers greater than 2. Suppose that the following exist:
(1) a strictly $h_{1}$-cyclic 1-FG(3, $\left(K, K_{T}\right)$, $\left.n g_{1} h_{1}\right)$ of type $\left(g_{1} h_{1}\right)^{n}$ (called the master design);
(2) a strictly $h_{2}$-cyclic s-FG(3, $\left.\left(L_{1}, L_{2}, \ldots, L_{s}, L_{T}\right), k g_{2} h_{2}\right)$ of type $\left(g_{2} h_{2}\right)^{k}$ for each $k \in$ $K$;
(3) an $h_{2}$-cyclic $H\left(k, g_{2} h_{2}, L_{T}, 3\right)$ for each $k \in K_{T}$.

Then there exists a strictly $h_{1} h_{2}$-cyclic s-FG(3, $\left.\left(L_{1}, L_{2}, \ldots, L_{s}, L_{T}\right), n g_{1} g_{2} h_{1} h_{2}\right)$ of type $\left(g_{1} g_{2} h_{1} h_{2}\right)^{n}$.

Step 1: Start from

$$
\text { a strictly } h_{1} \text {-cyclic } 1-\mathrm{FG}\left(3,\left(K, K_{T}\right), n g_{1} h_{1}\right) \text { of type }\left(g_{1} h_{1}\right)^{n}(X, \mathcal{G}, \mathcal{B}, \mathcal{T})
$$

where $X=I_{n} \times I_{g_{1}} \times Z_{h_{1}}$ and $\mathcal{G}=\left\{\{x\} \times I_{g_{1}} \times Z_{h_{1}}: x \in I_{n}\right\}$.

- Denote the family of base blocks of this design by $\mathcal{F}=\mathcal{F}_{1} \cup \mathcal{F}_{2}$, where $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$ generate all the blocks of $\mathcal{B}$ and $\mathcal{T}$ respectively.
Step 2 (input): For any base block $B \in \mathcal{F}_{1}$, construct

$$
\text { a strictly } h_{2} \text {-cyclic } s \text {-FG }\left(3,\left(L_{1}, L_{2} \ldots, L_{s}, L_{T}\right),|B| g_{2} h_{2}\right) \text { of type }\left(g_{2} h_{2}\right)^{|B|}
$$

on $B \times I_{g_{2}} \times Z_{h_{2}}$ with the group set $\left\{\{x\} \times I_{g_{2}} \times Z_{h_{2}}: x \in B\right\}$.

- Denote the family of base blocks of the $j$-th subdesign $H\left(|B|, g_{2} h_{2}, L_{j}, 2\right)$ by $\mathcal{A}_{B}^{j}$ for $1 \leq j \leq s$. Denote the family of all the other base blocks by $\mathcal{D}_{B}$.
Step 3 (input): For any base block $B \in \mathcal{F}_{2}$, construct

$$
\text { an } h_{2} \text {-cyclic } H\left(|B|, g_{2} h_{2}, L_{T}, 3\right)
$$

on $B \times I_{g_{2}} \times Z_{h_{2}}$ with the group set $\left\{\{x\} \times I_{g_{2}} \times Z_{h_{2}}: x \in B\right\}$.

- Denote the family of base blocks of this design by $\mathcal{D}_{B}^{\prime}$.

Step 4 (mapping): Let

$$
\mathcal{A}_{j}=\bigcup_{B \in \mathcal{F}_{1}} \mathcal{A}_{B}^{j} \text { for } 1 \leq j \leq s, \quad \mathcal{D}=\left(\bigcup_{B \in \mathcal{F}_{1}} \mathcal{D}_{B}\right) \bigcup\left(\bigcup_{B \in \mathcal{F}_{2}} \mathcal{D}_{B}^{\prime}\right)
$$

For each $C \in\left(\bigcup_{1 \leq j \leq s} \mathcal{A}_{j}\right) \bigcup \mathcal{D}$ and each $(x, y, z, u, v) \in C$, define a mapping

$$
\tau: \quad(x, y, z, u, v) \longmapsto\left(x, y+u g_{1}, z+v h_{1}\right)
$$

Define $\tau(C)=\{\tau(c): c \in C\}$. Let

$$
\mathcal{A}_{j}^{*}=\bigcup_{C \in \mathcal{A}_{j}} \tau(C), 1 \leq j \leq s, \quad \mathcal{D}^{*}=\bigcup_{C \in \mathcal{D}} \tau(C)
$$

Step 5 (final): Take

$$
\mathcal{A}_{j}^{\prime}=\left\{A+\delta: A \in \mathcal{A}_{j}^{*}, \delta \in Z_{h_{1} h_{2}}\right\}, \quad \mathcal{D}^{\prime}=\left\{A+\delta: A \in \mathcal{D}^{*}, \delta \in Z_{h_{1} h_{2}}\right\}
$$

where $A+\delta=\left\{(x, y, z+\delta)\left(\bmod \left(-,-, h_{1} h_{2}\right)\right):(x, y, z) \in A\right\}$. Take

$$
X^{\prime}=I_{n} \times I_{g_{1} g_{2}} \times Z_{h_{1} h_{2}}, \quad \mathcal{G}^{\prime}=\left\{\{x\} \times I_{g_{1} g_{2}} \times Z_{h_{1} h_{2}}: x \in I_{n}\right\} .
$$

Then $\left(X^{\prime}, \mathcal{G}^{\prime}, \mathcal{A}_{1}{ }^{\prime}, \ldots, \mathcal{A}_{s}{ }^{\prime}, \mathcal{D}^{\prime}\right)$ is the required strictly $h_{1} h_{2}$-cyclic $s$ - $\mathrm{FG}\left(3,\left(L_{1}, L_{2}, \ldots, L_{s}, L_{T}\right)\right.$, $\left.n g_{1} g_{2} h_{1} h_{2}\right)$ of type $\left(g_{1} g_{2} h_{1} h_{2}\right)^{n}$.

Figure 2: Algorithm in Construction 7.1

The following example illustrates the algorithm presented in Figure 2.
Example 7.2 In this example, we construct an optimal $2-D(8 \times 2,4,2)-O O C$.

- Step 1: First construct a strictly 1-cyclic 1-FG(3, $(2,4), 4)$ of type $1^{4}(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ on $X=I_{4} \times I_{1} \times Z_{1}$ with the group set $\mathcal{G}=\left\{\{x\} \times I_{1} \times Z_{1}: x \in I_{4}\right\}$, which is trivial. Take

$$
\mathcal{F}_{1}=\{\{(i, 0,0),(j, 0,0)\}:\{i, j\} \in\{\{0,1\},\{0,2\},\{0,3\},\{1,2\},\{1,3\},\{2,3\}\}\}
$$

which generates 6 blocks of $\mathcal{B}$ under $Z_{1}$ such that $(X, \mathcal{G} \cup \mathcal{B})$ is a 2-design. Take

$$
\mathcal{F}_{2}=\{\{(0,0,0),(1,0,0),(2,0,0),(3,0,0)\}\}
$$

which generates the unique block of $\mathcal{T}$ such that $(X, \mathcal{G} \cup \mathcal{B} \cup \mathcal{T})$ is a 3-design.

- Step 2: For each $B=\{(i, 0,0),(j, 0,0)\} \in \mathcal{F}_{1}$, construct a strictly 2-cyclic 0 $F G(3,(\emptyset, 4), 8)$ of type $4^{2}$ on $B \times I_{2} \times Z_{2}$ with the group set $\left\{\{x\} \times I_{2} \times Z_{2}: x \in B\right\}$, which exists by Example 6.7. All the 6 base blocks of $\mathcal{D}_{B}$ are listed below.

$$
\begin{aligned}
& \{(i, 0,0,0,0),(i, 0,0,0,1),(j, 0,0,0,0),(j, 0,0,1,0)\}, \\
& \{(i, 0,0,0,0),(, 0,0,0,0,),(,, 0,0,0,1),(i, 0,0,1,0)\}, \\
& \{(i, 0,0,0,0),(j, 0,0,0,0),(j, 0,0,1,1),(i, 0,0,1,1)\}, \\
& \{(i, 0,0,0,0),(j, 0,0,0,1),(j, 0,0,1,0),(i, 0,0,1,1)\}, \\
& \{(i, 0,0,0,0),(j, 0,0,1,0),(j, 0,0,1,1),(i, 0,0,1,0)\}, \\
& \{(j, 0,0,0,0),(j, 0,0,1,0),(i, 0,0,1,0),(i, 0,0,1,1)\},
\end{aligned}
$$

- Step 3: For the unique $B \in \mathcal{F}_{2}$, construct a 2 -cyclic $H(4,4,4,3)$ on $B \times I_{2} \times Z_{2}$ with the group set $\left\{\{x\} \times I_{2} \times Z_{2}: x \in B\right\}$, which exists by Corollary 7.6. Denote the family of base blocks of this design by $\mathcal{D}_{B}^{\prime}$, and $\left|\mathcal{D}_{B}^{\prime}\right|=32$.
- Step 4: Let $\mathcal{D}=\left(\bigcup_{B \in \mathcal{F}_{1}} \mathcal{D}_{B}\right) \bigcup\left(\bigcup_{B \in \mathcal{F}_{2}} \mathcal{D}_{B}^{\prime}\right)$. For each $C \in \mathcal{D}$ and each $(x, y, z, u, v)$ $\in C$, define a mapping $\tau:(x, y, z, u, v) \longmapsto(x, y+u, z+v)$. Define $\tau(C)=\{\tau(c)$ : $c \in C\}$. Let $\mathcal{D}^{*}=\bigcup_{C \in \mathcal{D}} \tau(C)$. Then $\left|\mathcal{D}^{*}\right|=68$, which is just the number of base blocks in a strictly 2-cyclic 0-FG(3, ( $\emptyset, 4), 16)$ of type $4^{4}$.
- Step 5: Let $\mathcal{D}^{\prime}=\left\{A+\delta: A \in \mathcal{D}^{*}, \delta \in Z_{2}\right\}$, where $A+\delta=\{(x, y, z+\delta)(\bmod (-,-, 2))$ : $(x, y, z) \in A\}$. Take $X^{\prime}=I_{4} \times I_{2} \times Z_{2}$ and $\mathcal{G}^{\prime}=\left\{\{x\} \times I_{2} \times Z_{2}: x \in I_{4}\right\}$. Then $\left(X^{\prime}, \mathcal{G}^{\prime}, \emptyset, \mathcal{D}^{\prime}\right)$ is the required strictly 2-cyclic $0-F G(3,(\emptyset, 4), 16)$ of type $4^{4}$.
- Step 6: Apply Construction 6.6. Fill in the groups of the resulting strictly 2-cyclic $0-F G(3,(\emptyset, 4), 16)$ of type $4^{4}$ with a trivial optimal strictly 2 -cyclic $3-(2 \times 2,4,1)$ packing without base blocks. We have an optimal strictly 2 -cyclic $3-(8 \times 2,4,1)$ packing with 68 base blocks, which achieves the upper bound in Theorem 5.7 and is an optimal $2-D(8 \times 2,4,2)$-OOC. Hence $\Phi(8 \times 2,4,2)=J^{*}(8 \times 2)=68$.

Construction 7.3 (Weighting Construction-II) Let $K$ and $L_{i}$ for each $1 \leq i \leq s$ be all sets of positive integers greater than 1. Let $K_{T}$ and $L_{T}$ be both sets of positive integers greater than 2. Suppose that the following exist:
(1) a strictly $\left(g_{1}, h_{1}\right)$-regular $1-F G\left(3,\left(K, K_{T}\right), g_{1} h_{1} n\right)$ of type $\left(g_{1} h_{1}\right)^{n}$;
(2) a strictly $h_{2}$-cyclic s-FG(3, $\left.\left(L_{1}, L_{2}, \ldots, L_{s}, L_{T}\right), k g_{2} h_{2}\right)$ of type $\left(g_{2} h_{2}\right)^{k}$ for each $k \in K$;
(3) an $h_{2}$-cyclic $H\left(k, g_{2} h_{2}, L_{T}, 3\right)$ for each $k \in K_{T}$.

Then there exists a strictly $\left(g_{1} g_{2}, h_{1} h_{2}\right)$-regular $s$ - $F G\left(3,\left(L_{1}, L_{2}, \ldots, L_{s}, L_{T}\right), g_{1} g_{2} h_{1} h_{2} n\right)$ of type $\left(g_{1} g_{2} h_{1} h_{2}\right)^{n}$.

Step 1: Start from
a strictly $\left(g_{1}, h_{1}\right)$-regular 1-FG $\left(3,\left(K, K_{T}\right), g_{1} h_{1} n\right)$ of type $\left(g_{1} h_{1}\right)^{n}(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$,
on $X=I_{g_{1}} \times Z_{h_{1} n}$ with the group set $\mathcal{G}=\left\{I_{g_{1}} \times H_{i}: 0 \leq i<n\right\}$, where $H=\left\{0, n, \ldots,\left(h_{1}-1\right) n\right\}$ is a subgroup of order $h_{1}$ in $Z_{h_{1} n}$, and $H_{i}=H+i$ be a coset of $H$ in $Z_{h_{1} n}, 0 \leq i<n$.

- Denote the family of base blocks of this design by $\mathcal{F}=\mathcal{F}_{1} \cup \mathcal{F}_{2}$, where $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$ generate all the blocks of $\mathcal{B}$ and $\mathcal{T}$ respectively.
Step 2 (input): For any base block $B \in \mathcal{F}_{1}$, construct

$$
\text { a strictly } h_{2} \text {-cyclic } s \text {-FG }\left(3,\left(L_{1}, L_{2} \ldots, L_{s}, L_{T}\right),|B| g_{2} h_{2}\right) \text { of type }\left(g_{2} h_{2}\right)^{|B|}
$$

on $B \times I_{g_{2}} \times Z_{h_{2}}$ with the group set $\left\{\{x\} \times I_{g_{2}} \times Z_{h_{2}}: x \in B\right\}$.

- Denote the family of base blocks of the $j$-th subdesign $H\left(|B|, g_{2} h_{2}, L_{j}, 2\right)$ by $\mathcal{A}_{B}^{j}$ for $1 \leq j$ $\leq s$, and denote the family of all the other base blocks by $\mathcal{D}_{B}$.
Step 3 (input): For any base block $B \in \mathcal{F}_{2}$, construct

$$
\text { an } h_{2} \text {-cyclic } H\left(|B|, g_{2} h_{2}, L_{T}, 3\right)
$$

on $B \times I_{g_{2}} \times Z_{h_{2}}$ with the group set $\left\{\{x\} \times I_{g_{2}} \times Z_{h_{2}}: x \in B\right\}$.

- Denote the family of base blocks of this design by $\mathcal{D}_{B}^{\prime}$.

Step 4 (mapping): Let

$$
\mathcal{A}_{j}=\bigcup_{B \in \mathcal{F}_{1}} \mathcal{A}_{B}^{j} \text { for } 1 \leq j \leq s, \quad \mathcal{D}=\left(\bigcup_{B \in \mathcal{F}_{1}} \mathcal{D}_{B}\right) \bigcup\left(\bigcup_{B \in \mathcal{F}_{2}} \mathcal{D}_{B}^{\prime}\right) .
$$

For each $C \in\left(\bigcup_{1 \leq j \leq s} \mathcal{A}_{j}\right) \bigcup \mathcal{D}$ and each $(x, y, z, u) \in C$, define a mapping

$$
\tau: \quad(x, y, z, u) \longmapsto\left(x+z g_{1}, y+u h_{1} n\right) .
$$

Define $\tau(C)=\{\tau(c): c \in C\}$. Let

$$
\mathcal{A}_{j}^{*}=\bigcup_{C \in \mathcal{A}_{j}} \tau(C), 1 \leq j \leq s, \quad \mathcal{D}^{*}=\bigcup_{C \in \mathcal{D}} \tau(C) .
$$

Step 5 (final): Take

$$
\mathcal{A}_{j}^{\prime}=\left\{A+\delta: A \in \mathcal{A}_{j}^{*}, \delta \in Z_{h_{1} h_{2} n}\right\}, \quad \mathcal{D}^{\prime}=\left\{A+\delta: A \in \mathcal{D}^{*}, \delta \in Z_{h_{1} h_{2} n}\right\},
$$

where $A+\delta=\left\{(x, y+\delta)\left(\bmod \left(-, h_{1} h_{2} n\right)\right):(x, y) \in A\right\}$. Let $H^{\prime}=\left\{0, n, \ldots,\left(h_{1} h_{2}-1\right) n\right\}$ be a subgroup of order $h_{1} h_{2}$ in $Z_{h_{1} h_{2} n}$, and $H_{i}^{\prime}=H^{\prime}+i$ be a coset of $H^{\prime}$ in $Z_{h_{1} h_{2} n}, 0 \leq i<n$. Take

$$
X^{\prime}=I_{g_{1} g_{2}} \times Z_{h_{1} h_{2} n}, \quad \mathcal{G}^{\prime}=\left\{I_{g_{1} g_{2}} \times H_{i}^{\prime}: 0 \leq i<n\right\} .
$$

Then $\left(X^{\prime}, \mathcal{G}^{\prime}, \mathcal{A}_{1}{ }^{\prime}, \ldots, \mathcal{A}_{s}{ }^{\prime}, \mathcal{D}^{\prime}\right)$ is the required strictly $\left(g_{1} g_{2}, h_{1} h_{2}\right)$-regular $s$-FG $\left(3,\left(L_{1}, L_{2}, \ldots, L_{s}\right.\right.$, $\left.\left.L_{T}\right), g_{1} g_{2} h_{1} h_{2} n\right)$ of type $\left(g_{1} g_{2} h_{1} h_{2}\right)^{n}$.

Figure 3: Algorithm in Construction 7.3
The following example illustrates the algorithm presented in Figure 3.
Example 7.4 In this example, we construct an optimal $2-D(8 \times 4,4,2)-O O C$.

- Step 1: First we construct a strictly $(2,2)$-regular 1-FG(3, $(2,4), 8)$ of type $4^{2}$ as follows.
(1) Take a strictly $(2,2)$-regular $0-F G(3,(\emptyset, 4), 8)$ of type $4^{2}(X, \mathcal{G}, \emptyset, \mathcal{T})$ on $X=$ $I_{2} \times Z_{4}$ with the group set $\mathcal{G}=\left\{I_{2} \times H_{i}: 0 \leq i<2\right\}$, where $H_{0}=\{0,2\}$ is a subgroup of order 2 in $Z_{4}$ and $H_{1}=\{1,3\}$. It exists by Example 6.15. Denote the
family of base blocks of this design by $\mathcal{F}_{2}$. It follows that $\mathcal{F}_{2}$ generates all the blocks of $\mathcal{T}$ and $\left|\mathcal{F}_{2}\right|=3$.
(2) Collect all 2 -subsets from distinct groups of $\mathcal{G}$ into a set $\mathcal{B}$. Then $(X, \mathcal{G} \cup \mathcal{B})$ is a 2-design. Hence, $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ is a strictly (2,2)-regular 1-FG(3, (2, 4), 8) of type $4^{2}$. Take

$$
\mathcal{F}_{1}=\{\{(0,0),(0,1)\},\{(0,0),(1,1)\},\{(1,0),(1,1)\},\{(0,0),(1,3)\}\} .
$$

$\mathcal{F}_{1}$ generates all the blocks of $\mathcal{B}$.

- Step 2: For each $B \in \mathcal{F}_{1}$, construct a strictly 1-cyclic $0-F G(3,(\emptyset, 4), 8)$ of type $4^{2}$ on $B \times I_{4} \times Z_{1}$ with the group set $\left\{\{x\} \times I_{4} \times Z_{1}: x \in B\right\}$, which can be taken from Example 6.1. Denote the family of base blocks of this design by $\mathcal{D}_{B}$, and $\left|\mathcal{D}_{B}\right|=12$.
- Step 3: For each $B \in \mathcal{F}_{2}$, construct a 1 -cyclic $H(4,4,4,3)$ on $B \times I_{4} \times Z_{1}$ with the group set $\left\{\{x\} \times I_{4} \times Z_{1}: x \in B\right\}$, which exists by Corollary 7.6. Denote the family of base blocks of this design by $\mathcal{D}_{B}^{\prime}$, and $\left|\mathcal{D}_{B}^{\prime}\right|=64$.
- Step 4: Let $\mathcal{D}=\left(\bigcup_{B \in \mathcal{F}_{1}} \mathcal{D}_{B}\right) \bigcup\left(\bigcup_{B \in \mathcal{F}_{2}} \mathcal{D}_{B}^{\prime}\right)$. For each $C \in \mathcal{D}$ and each $(x, y, z, u) \in$ $C$, define a mapping $\tau:(x, y, z, u) \longmapsto(x+2 z, y+4 u)$. Define $\tau(C)=\{\tau(c): c \in$ $C\}$. Let $\mathcal{D}^{*}=\bigcup_{C \in \mathcal{D}} \tau(C)$. Then $\left|\mathcal{D}^{*}\right|=240$, which is just the number of base blocks in a strictly $(8,2)$-regular 0-FG(3, $(\emptyset, 4), 32)$ of type $16^{2}$.
- Step 5: Let $\mathcal{D}^{\prime}=\left\{A+\delta: A \in \mathcal{D}^{*}, \delta \in Z_{4}\right\}$, where $A+\delta=\{(x, y+\delta)(\bmod (-, 4))$ : $(x, y) \in A\}$. Let $H^{\prime}=\{0,2\}$ be a subgroup of order 2 in $Z_{4}$, and $H_{1}^{\prime}=\{1,3\}$. Take $X^{\prime}=I_{8} \times Z_{4}$ and $\mathcal{G}^{\prime}=\left\{I_{8} \times H_{i}^{\prime}: 0 \leq i<2\right\}$. Then $\left(X^{\prime}, \mathcal{G}^{\prime}, \emptyset, \mathcal{D}^{\prime}\right)$ is the required strictly $(8,2)$-regular $0-F G(3,(\emptyset, 4), 32)$ of type $16^{2}$.
- Step 6: Apply Construction 6.14. Fill in the groups of the resulting strictly $(8,2)-$ regular $0-F G(3,(\emptyset, 4), 32)$ of type $16^{2}$ with an optimal strictly 2 -cyclic $3-(8 \times 2,4,1)$ packing with 68 base blocks, which exists by Example 7.2. We have an optimal strictly 4 -cyclic $3-(8 \times 4,4,1)$ packing with 308 base blocks, which achieves the upper bound in Theorem 5.7 and is an optimal $2-D(8 \times 4,4,2)$-OOC with 308 codewords. Hence $\Phi(8 \times 4,4,2)=J^{*}(8 \times 4)=308$.

Step 1: Start from an $h_{1}$-cyclic $H\left(n, g_{1} h_{1}, K, t\right)(X, \mathcal{G}, \mathcal{B})$, where $X=I_{n} \times I_{g_{1}} \times Z_{h_{1}}$ and $\mathcal{G}=\left\{\{x\} \times I_{g_{1}} \times Z_{h_{1}}: x \in I_{n}\right\}$.

- Denote the family of base blocks of this design by $\mathcal{F}$.

Step 2 (input): For any base block $B \in \mathcal{F}$, construct an $h_{2}$-cyclic $H\left(|B|, g_{2} h_{2}, L, t\right)$ on $B \times I_{g_{2}} \times Z_{h_{2}}$ with the group set $\left\{\{x\} \times I_{g_{2}} \times Z_{h_{2}}: x \in B\right\}$.

- Denote the family of base blocks of this design by $\mathcal{D}_{B}$.

Step 3 (mapping): Let $\mathcal{D}=\bigcup_{B \in \mathcal{F}} \mathcal{D}_{B}$. For each $C \in \mathcal{D}$ and each $(x, y, z, u, v) \in C$, define a mapping

$$
\tau: \quad(x, y, z, u, v) \longmapsto\left(x, y+u g_{1}, z+v h_{1}\right)
$$

Define $\tau(C)=\{\tau(c): c \in C\}$. Let $\mathcal{D}^{*}=\bigcup_{C \in \mathcal{D}} \tau(C)$.
Step 4 (final): Take

$$
\mathcal{D}^{\prime}=\left\{D+\delta: D \in \mathcal{D}^{*}, \delta \in Z_{h_{1} h_{2}}\right\}
$$

where $D+\delta=\left\{(x, y, z+\delta)\left(\bmod \left(-,-, h_{1} h_{2}\right)\right):(x, y, z) \in D\right\}$. Take

$$
X^{\prime}=I_{n} \times I_{g_{1} g_{2}} \times Z_{h_{1} h_{2}}, \quad \mathcal{G}^{\prime}=\left\{\{x\} \times I_{g_{1} g_{2}} \times Z_{h_{1} h_{2}}: x \in I_{n}\right\} .
$$

Then $\left(X^{\prime}, \mathcal{G}^{\prime}, \mathcal{D}^{\prime}\right)$ is the required $h_{1} h_{2}$-cyclic $H\left(n, g_{1} g_{2} h_{1} h_{2}, L, t\right)$.
Figure 4: Algorithm in Construction 7.5

Construction 7.5 (Weighting Construction-III) Suppose that the following exist:
(1) an $h_{1}$-cyclic $H\left(n, g_{1} h_{1}, K, t\right)$;
(2) an $h_{2}$-cyclic $H\left(k, g_{2} h_{2}, L, t\right)$ for each $k \in K$.

Then there exists an $h_{1} h_{2}$-cyclic $H\left(n, g_{1} g_{2} h_{1} h_{2}, L, t\right)$.
Corollary 7.6 For any $h \geq 1$ and $n \geq 4, n \neq 5$, if $g n$ is even and $g(n-1)(n-2)$ is divisible by 3 , then there is an $h$-cyclic $H(n, g h, 4,3)$. For any $h \geq 1$ and $n=5$, an $h$-cyclic $H(5, g h, 4,3)$ exists if $g$ is even, $g \neq 2$ and $g \not \equiv 10,26(\bmod 48)$.

Proof By Lemma 6.25, for any $h \geq 1$, there exists a semi-cyclic $H(4, h, 4,3)$ (i.e., an $h$-cyclic $H(4, h, 4,3))$. Apply Construction 7.5 with $h_{1}=g_{2}=1, g_{1}=g$ and $h_{2}=h$. Combine the results of Lemma 6.22 to complete the proof.

## 8 Small orders of optimal 2-D ( $u \times v, 4,2$ )-OOCs

In this section, we obtain some small orders of optimal 2-D OOCs. Some of them are obtained by computer search, and some of them are obtained by applying filling constructions in Section 6.

Lemma 8.1 There exists an optimal 2-D $(u \times v, 4,2)$-OOC with $J^{*}(u \times v)$ codewords for each $(u, v) \in\{(3,3),(2,6),(3,4),(6,2),(7,2),(2,11)\}$.

Proof We here give a construction of a $3-(u v, 4,1)$-packing on $I_{u v}$. Let $\alpha=(01 \cdots v-$ 1) $(v v+1 \cdots 2 v-1) \cdots((u-1) v \cdots u v-1)$ be a permutation on $I_{u v}$, which consists of $u$ cycles of length $v$. Let $G$ be the group generated by $\alpha$. Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of $G$. Obviously this design is isomorphic to a strictly $v$-cyclic 3 - $(u \times v, 4,1)$-packing, which achieves the upper bound in Theorem 5.7, and is an optimal 2-D ( $u \times v, 4,2$ )-OOC.

| $(u, v)=(3,3):$ | $\{0,1,3,4\}$ | \{0, 1, 5, 6\} | $\{0,1,7,8\}$ | $\{0,3,6,8\}$ | $\{0,4,5,7\}$ | $\{3,4,7,8\}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(u, v)=(2,6):$ | $\{0,1,2,6\}$ | $\{0,1,3,8\}$ | $\{0,1,4,7\}$ | $\{0,1,9,10\}$ | $\{0,2,8,10\}$ | \{0,6, 7,9$\}$ |
|  | $\{0,6,10,11\}$ | $\{0,7,8,11\}$ |  |  |  |  |
| $(u, v)=(3,4):$ | $\{0,2,4,5\}$ | $\{0,1,2,8\}$ | $\{0,1,4,9\}$ | $\{0,1,5,7\}$ | $\{0,1,6,10\}$ | $\{0,4,7,10\}$ |
|  | $\{0,4,8,11\}$ | $\{0,5,6,11\}$ | $\{0,5,8,10\}$ | $\{0,6,8,9\}$ | $\{0,7,9,11\}$ | \{4, 5, 6, 9\} |
| $(u, v)=(6,2):$ | $\{0,1,2,4\}$ | $\{0,1,6,8\}$ | $\{0,2,5,9\}$ | $\{0,2,3,6\}$ | $\{0,2,7,8\}$ | $\{0,2,10,11\}$ |
|  | $\{0,3,4,7\}$ | $\{0,3,8,9\}$ | $\{0,4,5,6\}$ | $\{0,4,8,10\}$ | $\{0,4,9,11\}$ | $\{0,5,7,10\}$ |
|  | $\{0,5,8,11\}$ | $\{0,6,7,11\}$ | $\{0,6,9,10\}$ | $\{2,3,4,9\}$ | $\{2,4,5,10\}$ | $\{2,4,6,7\}$ |
|  | $\{2,4,8,11\}$ | $\{2,5,7,11\}$ | $\{2,6,8,10\}$ | $\{2,6,9,11\}$ | $\{2,7,9,10\}$ | \{4, 6, 8, 9\} |
|  | $\{4,7,10,11\}$ |  |  |  |  |  |
| $(u, v)=(7,2):$ | $\{0,1,2,4\}$ | $\{0,1,6,8\}$ | $\{0,1,10,12\}$ | $\{0,2,3,7\}$ | $\{0,2,5,10\}$ | $\{0,2,6,13\}$ |
|  | $\{0,2,8,11\}$ | $\{0,2,9,12\}$ | $\{0,3,4,6\}$ | $\{0,3,8,10\}$ | $\{0,3,9,11\}$ | $\{0,3,12,13\}$ |
|  | $\{0,4,5,13\}$ | $\{0,4,7,10\}$ | $\{0,4,8,9\}$ | $\{0,4,11,12\}$ | $\{0,5,6,9\}$ | $\{0,5,7,11\}$ |
|  | $\{0,5,8,12\}$ | $\{0,6,7,12\}$ | $\{0,6,10,11\}$ | $\{0,7,8,13\}$ | $\{0,9,10,13\}$ | $\{2,3,4,9\}$ |
|  | $\{2,3,10,12\}$ | $\{2,4,5,6\}$ | $\{2,4,7,12\}$ | $\{2,4,8,13\}$ | $\{2,4,10,11\}$ | $\{2,5,9,13\}$ |
|  | $\{2,5,11,12\}$ | $\{2,6,7,11\}$ | $\{2,6,8,12\}$ | $\{2,6,9,10\}$ | $\{2,7,8,9\}$ | $\{2,7,10,13\}$ |
|  | $\{4,5,8,10\}$ | $\{4,6,7,9\}$ | $\{4,6,8,11\}$ | $\{4,6,12,13\}$ | $\{4,7,11,13\}$ | $\{4,9,10,12\}$ |
|  | $\{6,8,10,13\}$ | $\{6,9,11,13\}$ |  |  |  |  |
| $(u, v)=(2,11):$ | $\{0,1,2,4\}$ | $\{0,1,5,7\}$ | $\{0,1,6,9\}$ | $\{0,1,8,11\}$ | $\{0,1,12,13\}$ | $\{0,1,14,15\}$ |
|  | $\{0,1,16,17\}$ | $\{0,1,18,19\}$ | $\{0,1,20,21\}$ | $\{0,2,5,11\}$ | $\{0,2,12,14\}$ | $\{0,2,13,15\}$ |
|  | $\{0,2,16,19\}$ | $\{0,2,17,21\}$ | $\{0,2,18,20\}$ | $\{0,3,7,12\}$ | $\{0,3,11,15\}$ | $\{0,3,13,16\}$ |
|  | $\{0,3,17,19\}$ | $\{0,3,18,21\}$ | $\{0,4,11,18\}$ | $\{0,4,12,15\}$ | $\{0,4,13,17\}$ | $\{0,4,19,21\}$ |
|  | $\{0,5,12,17\}$ | $\{0,5,13,18\}$ | $\{0,5,14,19\}$ | $\{0,5,15,20\}$ | $\{0,5,16,21\}$ | $\{0,11,14,21\}$ |
|  | $\{0,11,17,20\}$ | $\{0,12,16,20\}$ | $\{11,12,13,17\}$ | $\{11,12,14,19\}$ | $\{11,12,18,20\}$ |  |

Lemma 8.2 There exists a strictly $(2,6)$-regular $0-F G(3,(\emptyset, 4), 24)$ of type $12^{2}$.
Proof We here give a construction of a $0-\mathrm{FG}(3,(\emptyset, 4), 24)$ of type $12^{2}$ on $I_{24}$ with the group set $\{\{2 i+j: 0 \leq i \leq 11\}: 0 \leq j \leq 1\}$. Let $\alpha=\left(\begin{array}{llll}0 & 1 & \cdots & 11\end{array}\right)(1213 \cdots 23)$ be a permutation on $I_{24}$, which consists of 2 cycles of length 12 . Let $G$ be the group generated by $\alpha$. Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of $G$. Obviously this design is isomorphic to a strictly $(2,6)$-regular $0-\mathrm{FG}(3,(\emptyset, 4), 24)$ of type $12^{2}$.

| $\{0,1,2,5\}$ | $\{0,1,3,8\}$ | $\{0,1,6,9\}$ | $\{0,1,7,12\}$ | $\{0,1,10,13\}$ | $\{0,1,14,15\}\{0,1,16,17\}$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\{0,1,18,19\}$ | $\{0,1,20,21\}$ | $\{0,1,22,23\}$ | $\{0,2,9,13\}$ | $\{0,2,17,19\}$ | $\{0,2,21,23\}\{0,3,12,19\}$ |
| $\{0,3,13,18\}$ | $\{0,3,14,21\}$ | $\{0,3,17,22\}$ | $\{0,3,20,23\}$ | $\{0,4,13,19\}$ | $\{0,4,15,23\}\{0,4,17,21\}$ |
| $\{0,5,12,21\}$ | $\{0,5,13,20\}$ | $\{0,5,14,23\}$ | $\{0,5,15,18\}$ | $\{0,5,19,22\}$ | $\{0,6,13,21\}\{0,12,13,15\}$ |
| $\{0,12,17,23\}$ | $\{0,13,16,23\}$ | $\{12,13,14,21\}$ | $\{12,13,16,19\}$ | $\{12,13,17,22\}$ |  |

Lemma 8.3 There exists an optimal $2-D(2 \times 12,4,2)$-OOC with $J^{*}(2 \times 12)=41$ codewords.

Proof Start from a strictly $(2,6)$-regular $0-\mathrm{FG}(3,(\emptyset, 4), 24)$ of type $12^{2}$, which exists by Lemma 8.2. Apply Construction 6.14 with an optimal strictly 6 -cyclic $3-(2 \times 6,4,1)$ packing from Lemma 8.1 to obtain a strictly 12 -cyclic $3-(2 \times 12,4,1)$ packing with 41 base blocks, which achieves the upper bound in Theorem 5.7, and is an optimal 2-D ( $2 \times 12,4,2$ )-OOC with 41 codewords.

Lemma 8.4 There exists a strictly 2-cyclic 0-FG(3, ( $\emptyset, 4), 24)$ of type $12^{2}$.
Proof We here give a construction of a $0-\mathrm{FG}(3,(\emptyset, 4), 24)$ of type $12^{2}$ on $I_{24}$ with the group set $\{\{0,1, \ldots, 11\}+i: i \in\{0,12\}\}$. Let $\alpha=(01)(23) \cdots$ (22 23) be a permutation on $I_{24}$ and $G$ be the group generated by $\alpha$. Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of $G$. Obviously this design is isomorphic to a strictly 2 -cyclic 0 -FG( $3,(\emptyset, 4), 24$ ) of type $12^{2}$.
$\{0,1,12,14\}$
$\{0,2,20,21\}$
$\{0,1,16,18\}$
$\{0,2,22,23\}$
$\{0,3,21,22\}$
$\{0,4,12,16\}$
$\{0,5,12,17\}$

Lemma 8.5 There exists an optimal $2-D(12 \times 2,4,2)-O O C$ with $J^{*}(12 \times 2)=248$ codewords.

Proof Start from a strictly 2 -cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 24)$ of type $12^{2}$, which exists by Lemma 8.4. Applying Construction 6.6 with an optimal strictly 2 -cyclic $3-(6 \times 2,4,1)$ packing from Lemma 8.1, we have a strictly 2 -cyclic 3 - $(12 \times 2,4,1)$ packing with 248 base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D ( $12 \times 2,4,2$ )-OOC with 248 codewords exists.

Lemma 8.6 There exists a strictly (2,3)-regular $0-F G(3,(\emptyset, 4), 30)$ of type $6^{5}$.
Proof We here give a construction of a $0-\mathrm{FG}(3,(\emptyset, 4), 30)$ of type $6^{5}$ on $I_{30}$ with the group set $\{\{5 i+j: 0 \leq i \leq 5\}: 0 \leq j \leq 4\}$. Let $\alpha=(01 \cdots 14)(1516 \cdots 29)$ be a permutation on $I_{30}$, which consists of 2 cycles of length 15 . Let $G$ be the group generated by $\alpha$. Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of $G$. Obviously this design is isomorphic to a strictly $(2,3)$-regular $0-\mathrm{FG}(3,(\emptyset, 4), 30)$ of type $6^{5}$.

| $\{0,1,2,4\}$ | $\{0,1,5,6\}$ | $\{0,1,7,9\}$ | $\{0,1,8,12\}$ | $\{0,1,13,15\}$ | $\{0,1,16,17\}$ | $\{0,1,18,19\}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\{0,1,20,21\}$ | $\{0,1,22,23\}$ | $\{0,1,24,25\}$ | $\{0,1,26,27\}$ | $\{0,1,28,29\}$ | $\{0,2,5,12\}$ | $\{0,2,6,11\}$ |
| $\{0,2,7,15\}$ | $\{0,2,10,16\}$ | $\{0,2,18,20\}$ | $\{0,2,19,21\}$ | $\{0,2,22,24\}$ | $\{0,2,23,25\}$ | $\{0,2,26,28\}$ |
| $\{0,2,27,29\}$ | $\{0,3,6,15\}$ | $\{0,3,7,16\}$ | $\{0,3,9,19\}$ | $\{0,3,18,21\}$ | $\{0,3,20,23\}$ | $\{0,3,22,26\}$ |
| $\{0,3,24,29\}$ | $\{0,3,25,28\}$ | $\{0,4,15,19\}$ | $\{0,4,16,20\}$ | $\{0,4,17,24\}$ | $\{0,4,18,22\}$ | $\{0,4,21,27\}$ |
| $\{0,4,23,26\}$ | $\{0,4,25,29\}$ | $\{0,5,16,22\}$ | $\{0,5,17,27\}$ | $\{0,5,18,23\}$ | $\{0,5,19,24\}$ | $\{0,5,26,29\}$ |
| $\{0,6,17,22\}$ | $\{0,6,18,24\}$ | $\{0,6,19,28\}$ | $\{0,6,20,26\}$ | $\{0,6,21,29\}$ | $\{0,6,23,27\}$ | $\{0,7,17,25\}$ |
| $\{0,7,18,26\}$ | $\{0,7,19,27\}$ | $\{0,7,20,24\}$ | $\{0,7,22,29\}$ | $\{0,7,23,28\}$ | $\{0,15,17,29\}$ | $\{0,15,21,28\}$ |
| $\{0,15,24,27\}$ | $\{0,16,19,25\}$ | $\{0,16,23,29\}$ | $\{0,16,24,28\}$ | $\{15,16,17,28\}$ | $\{15,16,19,23\}\{15,16,20,22\}$ |  |
| $\{15,16,21,25\}$ | $\{15,16,24,26\}\{15,17,20,27\}$ |  |  |  |  |  |

Lemma 8.7 There exists an optimal $2-D(2 \times 15,4,2)$-OOC with $J^{*}(2 \times 15)=67$ codewords.

Proof Start from a strictly $(2,3)$-regular $0-\mathrm{FG}(3,(\emptyset, 4), 30)$ of type $6^{5}$, which exists by Lemma 8.6. Apply Construction 6.14 with an optimal strictly 3 -cyclic 3 - $(2 \times 3,4,1)$ packing from Example 3.3 to obtain a strictly 15 -cyclic 3 - $(2 \times 15,4,1)$ packing with 67 base blocks, which achieves the upper bound in Theorem 5.7, and is an optimal 2-D ( $2 \times 15,4,2$ )-OOC with 67 codewords.

Lemma 8.8 There exists a strictly (3,2)-regular 0-FG(3, ( $\emptyset, 4), 30)$ of type $6^{5}$.
Proof We here give a construction of a $0-\mathrm{FG}(3,(\emptyset, 4), 30)$ of type $6^{5}$ on $I_{30}$ with the group set $\{\{5 i+j: 0 \leq i \leq 5\}: 0 \leq j \leq 4\}$. Let $\alpha=(01 \cdots 9)(1011 \cdots 19)(2021 \cdots 29)$ and $\beta=(01020)(11121) \cdots(91929)$ be two permutations on $I_{30}$ and $G$ be the group generated by $\alpha$ and $\beta$. Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of $G$. Obviously this design is isomorphic to a $(3,2)$-regular $0-\mathrm{FG}(3,(\emptyset, 4), 30)$ of type $6^{5}$.

| $\{0,1,2,4\}$ | $\{0,1,5,7\}$ | $\{0,1,6,10\}$ | $\{0,1,8,11\}$ | $\{0,1,12,13\}$ | $\{0,1,14,15\}$ | $\{0,1,16,17\}$ | $\{0,1,18,20\}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\{0,1,19,22\}$ | $\{0,1,21,23\}$ | $\{0,2,7,26\}$ | $\{0,2,11,22\}$ | $\{0,2,12,18\}$ | $\{0,2,14,20\}$ | $\{0,2,15,17\}$ | $\{0,2,16,21\}$ |
| $\{0,2,23,29\}$ | $\{0,2,24,28\}$ | $\{0,3,6,24\}$ | $\{0,3,10,14\}$ | $\{0,3,11,18\}$ | $\{0,3,12,27\}$ | $\{0,3,15,23\}$ | $\{0,3,16,20\}$ |
| $\{0,3,17,28\}$ | $\{0,3,19,26\}$ | $\{0,4,10,21\}$ | $\{0,4,12,26\}$ | $\{0,4,15,19\}$ | $\{0,4,17,27\}$ | $\{0,5,11,27\}$ | $\{0,5,12,21\}$ |

$\{0,5,13,28\}$

Lemma 8.9 There exists an optimal $2-D(3 \times 10,4,2)-O O C$ with $J^{*}(3 \times 10)=100$ codewords.

Proof Start from a strictly $(3,2)$-regular $0-\mathrm{FG}(3,(\emptyset, 4), 30)$ of type $6^{5}$, which exists by Lemma 8.8. Apply Construction 6.14 with an optimal strictly 2 -cyclic $3-(3 \times 2,4,1)$ packing from Example 3.3 to obtain a strictly 10 -cyclic $3-(3 \times 10,4,1)$ packing with 100 base blocks, which achieves the upper bound in Theorem 5.7, and is an optimal 2-D ( $3 \times 10,4,2$ )-OOC with 100 codewords.

Lemma 8.10 Let $n \equiv 18(\bmod 24)$. If there is an optimal $1-D(n, 4,2)-O O C$, which achieves the Johnson bound $J(1 \times n, 4,2)=\left\lfloor\frac{1}{4}\left\lfloor\frac{n-1}{3}\left\lfloor\frac{n-2}{2}\right\rfloor\right\rfloor\right\rfloor$, then for any integer factorization $n=u v$, there is an optimal $2-D(u \times v, 4,2)$-OOC with $J^{*}(u \times v)=\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{n-1}{3}\left\lfloor\frac{n-2}{2}\right\rfloor\right\rfloor-\right.\right.$ 1)」codewords.

Proof By Corollary 2.2, if there exists an optimal 1-D (uv, 4, 2)-OOC with $J(1 \times u v, 4,2)$ codewords, then there exists a 2 -D $(u \times v, 4,2)$-OOC with $u J(1 \times u v, 4,2)$ codewords. It is readily checked that $u J(1 \times u v, 4,2)=u\left(u^{2} v^{2}-3 u v-6\right) / 24=\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor=$ $J^{*}(u \times v)$. This number achieves the upper bound in Theorem 5.7. This completes the proof.

Note that when $n \equiv 18(\bmod 24), J(1 \times n, 4,2)=\left\lfloor\frac{1}{4}\left\lfloor\frac{n-1}{3}\left\lfloor\frac{n-2}{2}\right\rfloor\right\rfloor\right\rfloor=\left\lfloor\frac{1}{4}\left(\left\lfloor\frac{n-1}{3}\left\lfloor\frac{n-2}{2}\right\rfloor\right\rfloor-\right.\right.$ 1) $\rfloor$. Hence no confusion occurs in Lemma 8.10. By Lemma 2.6(3), there is an optimal 1-D ( $n, 4,2$ )-OOC with $J(1 \times n, 4,2)$ codewords for each $n \in\{18,42,90\}$. Then we have

Corollary 8.11 Let $n \in\{18,42,90\}$. For any integer factorization $n=n_{1} n_{2}$, there is an optimal 2-D $\left(n_{1} \times n_{2}, 4,2\right)$-OOC with $J^{*}\left(n_{1} \times n_{2}\right)$ codewords.

Lemma 8.12 There exists an optimal 2-D $(u \times v, 4,2)-O O C$ with $J^{*}(u \times v)$ codewords for each $(u, v) \in\{(5,4),(7,4),(6,5)\}$.

Proof Apply Theorem 2.1 with some known optimal 2-D $\left(u_{1} \times v_{1}, 4,2\right)$-OOCs. One can have all the required optimal 2-D $(u \times v, 4,2)$-OOCs. For illustrating the details, we give the following table.

| $\left(u_{1}, v_{1}\right)$ | Source | number of codewords | $\Rightarrow$ | $(u, v)$ | number of codewords | $J^{*}(u \times v)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(1,20)$ | Lemma 2.6 | 14 |  | $(5,4)$ | 70 | 70 |
| $(1,28)$ | Lemma 2.6 | 29 |  | $(7,4)$ | 203 | 203 |
| $(2,15)$ | Lemma 8.7 | 67 |  | $(6,5)$ | 201 | 201 |

## 9 Infinite families of optimal 2-D $(u \times v, 4,2)-\mathrm{OOCs}$

In this section, on one hand we shall give some infinite families of optimal 2-D ( $u \times v, 4,2$ )OOCs, which will be presented as Theorems. On the other hand, although we can not complete the existence of optimal 2-D $(u \times v, 4,2)$-OOCs, we hope to present some possible approaches to complete it, which will be presented as Propositions.

Lemma 9.1 There exists an optimal 2-D $(u \times 2,4,2)$-OOC with $J^{*}(u \times 2)$ codewords for any $u \equiv 2,4(\bmod 6)$.

Proof Let $n=u / 2$. Then $n \equiv 1,2(\bmod 3)$. When $n=1$, an optimal 2-D $(2 \times 2,4,2)$ OOC is trivial without base blocks. In the following consider $n \geq 2$. First we shall show that there is a strictly 2 -cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 4 n)$ of type $4^{n}$ for any $n \equiv 1,2(\bmod 3)$ and $n \geq 2$. When $n=2$, a strictly 2 -cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 8)$ of type $4^{2}$ exists by Example 6.7. When $n \equiv 1,2(\bmod 3), n \geq 4$ and $n \neq 5$, start from a $1-\mathrm{FG}(3,(2, n), n)$ of type $1^{n}$, which contains one block of size $n$ and all 2 -subsets of $n$ points. Apply Construction 7.1 with $h_{1}=1$ and $h_{2}=2$ to obtain a strictly 2 -cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 4 n)$ of type $4^{n}$, where the needed 2 -cyclic $H(n, 4,4,3)$ is from Corollary 7.6. When $n=5$, there is a strictly cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 20)$ of type $4^{5}$ from Lemma 6.12. By Lemma 6.17, it implies a strictly 2 -cyclic 0 -FG $(3,(\emptyset, 4), 20)$ of type $4^{5}$.

Next applying Construction 6.6 with an optimal strictly 2 -cyclic $3-(2 \times 2,4,1)$ packing, which is trivial without base blocks, we have a strictly 2 -cyclic $3-(2 n \times 2,4,1)$ packing, which contains $\left\lfloor\frac{2 n}{4}\left(\left\lfloor\frac{4 n-1}{3}\left\lfloor\frac{4 n-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor=n(n-1)(4 n+1) / 3$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(2 n \times 2,4,2)$-OOC with $J^{*}(2 n \times 2)$ codewords exists. It is an optimal 2-D $(u \times 2,4,2)$-OOC with $J^{*}(u \times 2)$ codewords.

Proposition 9.2 Let $v \equiv 1(\bmod 2)$ or $v \equiv 0(\bmod 4)$. Suppose that there is an optimal $2-D(2 \times v, 4,2)$-OOC with $J^{*}(2 \times v)$ codewords. Then there is an optimal $2-D(u \times v, 4,2)-$ OOC with $J^{*}(u \times v)$ codewords for any $u \equiv 2,4(\bmod 6)$. Especially when $v \equiv 1,5(\bmod$ $6)$, the resulting optimal $2-D(u \times v, 4,2)$-OOC is perfect.
Proof Let $n=u / 2$. Then $n \equiv 1,2(\bmod 3)$. When $n=1$, the conclusion follows from the assumption. In the following consider $n \geq 2$. First we shall show that there is a strictly $v$-cyclic 0 -FG $(3,(\emptyset, 4), 2 v n)$ of type $(2 v)^{n}$ for any $n \equiv 1,2(\bmod 3)$ and $n \geq 2$. When $n=2$, a strictly $v$-cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 4 v)$ of type $(2 v)^{2}$ is from Corollary 6.18. When $n \equiv 1,2(\bmod 3), n \geq 4$ and $n \neq 5$, start from a $1-\mathrm{FG}(3,(2, n), n)$ of type $1^{n}$, which contains a block of size $n$ and all 2 -subsets of $n$ points. Apply Construction 7.1 with $h_{1}=1$ and $h_{2}=v$ to obtain a strictly $v$-cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 2 v n)$ of type $(2 v)^{n}$, where the needed $v$-cyclic $H(n, 2 v, 4,3)$ is from Corollary 7.6. When $n=5$, there is a strictly cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 10 v)$ of type $(2 v)^{5}$ from Lemma 6.12. By Lemma 6.17, it implies a strictly $v$-cyclic 0 -FG $(3,(\emptyset, 4), 10 v)$ of type $(2 v)^{5}$.

Next apply Construction 6.6 with an optimal strictly $v$-cyclic $3-(2 \times v, 4,1)$ packing with $J^{*}(2 \times v)$ base blocks, which exists by assumption. Note that by Theorem 5.7,

$$
J^{*}(2 \times v)= \begin{cases}\left\lfloor\frac{2}{4}\left\lfloor\frac{2 v-1}{3}\left\lfloor\frac{2 v-2}{2}\right\rfloor\right\rfloor\right\rfloor, & \text { if } v \equiv 1,5(\bmod 6), \\ \left\lfloor\frac{2}{4}\left(\left\lfloor\frac{2 v-1}{3}\left\lfloor\frac{2 v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor, & \text { if } v \equiv 3(\bmod 6) \text { or } v \equiv 4,8(\bmod 12), \\ \left\lfloor\frac{2}{4}\left(\left\lfloor\frac{2 v-1}{3}\left\lfloor\frac{2 v-2}{2}\right\rfloor\right\rfloor-2\right)\right\rfloor, & \text { if } v \equiv 0(\bmod 12) .\end{cases}
$$

Then we have a strictly $v$-cyclic 3 - $(2 n \times v, 4,1)$ packing, which contains

$$
\begin{cases}n(n v-1)(2 n v-1) / 6=\left\lfloor\frac{2 n}{4}\left\lfloor\frac{2 n v-1}{3}\left\lfloor\frac{2 n v-2}{2}\right\rfloor\right\rfloor\right\rfloor, & \text { if } v \equiv 1,5(\bmod 6), \\ n\left(2 n^{2} v^{2}-3 n v-3\right) / 6=\left\lfloor\frac{2 n}{4}\left(\left\lfloor\frac{2 n v-1}{3}\left\lfloor\frac{2 n v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor, & \text { if } v \equiv 3(\bmod 6), \\ n(n v-2)(2 n v+1) / 6=\left\lfloor\frac{2 n}{4}\left(\left\lfloor\frac{2 n v-1}{3}\left\lfloor\frac{2 n v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor, & \text { if } v \equiv 4,8(\bmod 12), \\ n\left(2 n^{2} v^{2}-3 n v-6\right) / 6=\left\lfloor\frac{2 n}{4}\left(\left\lfloor\frac{2 n v-1}{3}\left\lfloor\frac{2 n v-2}{2}\right\rfloor\right\rfloor-2\right)\right\rfloor, & \text { if } v \equiv 0(\bmod 12),\end{cases}
$$

base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal $2-\mathrm{D}(2 n \times v, 4,2)$-OOC with $J^{*}(2 n \times v)$ codewords exists. It is an optimal $2-\mathrm{D}(u \times v, 4,2)$ OOC with $J^{*}(u \times v)$ codewords. Especially by Lemma 3.4 , when $v \equiv 1,5(\bmod 6)$, the resulting optimal $2-\mathrm{D}(u \times v, 4,2)$-OOC is perfect.

Lemma 9.3 There is an optimal 2-D $\left(2 \times 2^{n}, 4,2\right)$-OOC with $J^{*}\left(2 \times 2^{n}\right)$ codewords for any positive integer $n$.

Proof When $n=1$, an optimal 2-D $(2 \times 2,4,2)$-OOC is trivial without codewords. When $n=2$, the conclusion follows from Example 6.15. When $n \geq 3$, by Lemma 6.12 there exists a strictly cyclic $0-\mathrm{FG}\left(3,(\emptyset, 4), 2^{n+1}\right)$ of type $\left(2^{n}\right)^{2}$, denoted by $(X, \mathcal{G}, \emptyset, \mathcal{T})$, which is also a strictly $\left(1,2^{n}\right)$-regular $0-\mathrm{FG}\left(3,(\emptyset, 4), 2^{n+1}\right)$ of type $\left(2^{n}\right)^{2}$. Collect all 2-subsets from distinct groups of $\mathcal{G}$ into a set $\mathcal{B}$. Then $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ is a strictly $\left(1,2^{n}\right)$-regular 1-FG $\left(3,(2,4), 2^{n+1}\right)$ of type $\left(2^{n}\right)^{2}$. Start from this 1-FG and apply Construction 7.3 with $h_{1}=2^{n}$ and $h_{2}=1$ to obtain a strictly $\left(2,2^{n}\right)$-regular $0-\mathrm{FG}\left(3,(\emptyset, 4), 2^{n+2}\right)$ of type $\left(2^{n+1}\right)^{2}$, where the needed strictly 1-cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 4)$ of type $2^{2}$ is from Theorem 6.4, and the needed 1-cyclic $H(4,2,4,3)$ is from Corollary 7.6. Now use induction on $n$. When $n=3$, there is an optimal 2-D $\left(2 \times 2^{3}, 4,2\right)$-OOC with $J^{*}\left(2 \times 2^{3}\right)$ codewords from Example 6.16. Assume that an optimal 2-D $\left(2 \times 2^{n}, 4,2\right)$-OOC with $J^{*}\left(2 \times 2^{n}\right)$ codewords exists for some $n \geq 3$. Then start from a strictly $\left(2,2^{n}\right)$-regular $0-\mathrm{FG}\left(3,(\emptyset, 4), 2^{n+2}\right)$ of type $\left(2^{n+1}\right)^{2}$, and apply Construction 6.14 with an optimal 2-D $\left(2 \times 2^{n}, 4,2\right)$-OOC with $J^{*}\left(2 \times 2^{n}\right)$ codewords to obtain an 2 -D $\left(2 \times 2^{n+1}, 4,2\right)$-OOC, which contains $\left(2^{n}-1\right)\left(2^{n+2}+\right.$ $1) / 3=\left\lfloor\frac{2}{4}\left(\left\lfloor\frac{2^{n+2}-1}{3}\left\lfloor\frac{2^{n+2}-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$ codewords. This number achieves the upper bound in Theorem 5.7 (note that for any integer $\left.n \geq 2,2^{n} \equiv 4,8(\bmod 12)\right)$. Thus an optimal 2 -D $\left(2 \times 2^{n+1}, 4,2\right)$-OOC with $J^{*}\left(2 \times 2^{n+1}\right)$ codewords exists.

Combining the results of Proposition 9.2 and Lemmas 9.1, 9.3, we have
Theorem 9.4 There is an optimal 2-D (u×2n,4,2)-OOC with $J^{*}\left(u \times 2^{n}\right)$ codewords for any $u \equiv 2,4(\bmod 6)$ and any positive integer $n$.

Proposition 9.2 can only deal with the case of $u \equiv 2,4(\bmod 6)$ and $v \not \equiv 2(\bmod 4)$. When $v \equiv 2(\bmod 4)$, we have the following proposition.

Proposition 9.5 Let $u \equiv 2,4(\bmod 6)$ and $v \equiv 2(\bmod 4)$. Suppose that there is an optimal 2-D $(u / 2 \times 2 v, 4,2)-O O C$ with $J^{*}(u / 2 \times 2 v)$ codewords. Then there is an optimal $2-D(u \times v, 4,2)-O O C$ with $J^{*}(u \times v)$ codewords.

Proof By Theorem 2.1, if there exists an optimal 2-D $(u / 2 \times 2 v, 4,2)$-OOC with $J^{*}(u / 2 \times$ $2 v)$ codewords, then there exits a 2 - $\mathrm{D}(u \times v, 4,2)$-OOC with $2 J^{*}(u / 2 \times 2 v)$ codewords. Note that by Theorem 5.7,

$$
J^{*}(u / 2 \times 2 v)= \begin{cases}\left\lfloor\frac{u}{8}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor, & \text { if } v \equiv 2,10(\bmod 12) \\ \left\lfloor\frac{u}{8}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-2\right)\right\rfloor, & \text { if } v \equiv 6(\bmod 12)\end{cases}
$$

It is readily checked that $2 J^{*}(u / 2 \times 2 v)=$

$$
\begin{cases}u(u v+1)(u v-4) / 24=\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor, & \text { if } v \equiv 2,10(\bmod 12) \\ u\left(u^{2} v^{2}-3 u v-12\right) / 24=\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-2\right)\right\rfloor, & \text { if } v \equiv 6(\bmod 12)\end{cases}
$$

This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(u \times v, 4,2)$ OOC with $J^{*}(u \times v)$ codewords exists.

The following proposition shows another approach to obtain some optimal 2-D (u× $v, 4,2)$-OOCs with $v \equiv 2(\bmod 4)$.

Proposition 9.6 If there is a perfect $2-D(2 \times v, 4,2)-O O C$ with $v \equiv 1,5(\bmod 6)$, then there is an optimal 2-D $(u \times 2 v, 4,2)$-OOC for any $u \equiv 8,16(\bmod 24)$ with $J^{*}(u \times 2 v)$ codewords.

Proof Let $n=u / 8$. Then $n \equiv 1,2(\bmod 3)$. There is a strictly cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 16 n)$ of type $(8 n)^{2}$, which exists by Lemma 6.12 . By Lemma 6.17 , it implies a strictly 2-cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 16 n)$ of type $(8 n)^{2}$. By Lemma 6.19, if there is a perfect 2-D $(2 \times v, 4,2)$ OOC with $v \equiv 1,5(\bmod 6)$, then there is a strictly $(2,1)$-regular $1-\mathrm{FG}(3,(2,4), 2 v)$ of type $2^{v}$. Start from this 1-FG and apply Construction 7.3 with $h_{1}=1$ and $h_{2}=2$ to obtain a strictly $(8 n, 2)$-regular $0-\mathrm{FG}(3,(\emptyset, 4), 16 n v)$ of type $(16 n)^{v}$, where the needed 2-cyclic $H(4,8 n, 4,3)$ is from Corollary 7.6. Applying Construction 6.14 with an optimal strictly 2 -cyclic 3 - $(8 n \times 2,4,1)$ packing with $J^{*}(8 n \times 2)=\left\lfloor\frac{8 n}{4}\left(\left\lfloor\frac{16 n-1}{3}\left\lfloor\frac{16 n-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$ base blocks, which exists by Lemma 9.1, we have a strictly $2 v$-cyclic 3 -( $8 n \times 2 v, 4,1$ ) packing, which contains $4 n(4 n v-1)(16 n v+1) / 3=\left\lfloor\frac{8 n}{4}\left(\left\lfloor\frac{16 n v-1}{3}\left\lfloor\frac{16 n v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(8 n \times 2 v, 4,2)$ OOC with $J^{*}(8 n \times 2 v)$ codewords exists. It is an optimal 2-D $(u \times 2 v, 4,2)$-OOC with $J^{*}(u \times 2 v)$ codewords.

Theorem 9.7 Let $p \equiv 7(\bmod 12)$ be a prime or $p \in\{37,61,73,109,157,181,229$, $277\}$. There exists an optimal $(u \times 2 p, 4,2)$-OOC with $J^{*}(u \times 2 p)$ codewords for any $u \equiv 8,16(\bmod 24)$.

Proof Start from a perfect 2-D $(2 \times p, 4,2)$-OOC, which exists by Theorem 4.5. Apply Proposition 9.6 to complete the proof.

Lemma 9.8 Let $v \equiv 1(\bmod 2)$ or $v \equiv 0(\bmod 12)$. Suppose that there is an optimal $2-D$ $(12 \times v, 4,2)-O O C$ with $J^{*}(12 \times v)$ codewords. Then there is an optimal 2-D $(u \times v, 4,2)$ OOC with $J^{*}(u \times v)$ codewords for any $u \equiv 0(\bmod 12)$.

Proof Let $n=u / 12$. When $n=1$, the conclusion follows from the assumption. When $n \geq 2$, by Theorem 6.4 , there exists a $0-\mathrm{FG}(3,(\emptyset, 4), 6 n)$ of type $6^{n}(X, \mathcal{G}, \emptyset, \mathcal{T})$. Collect all 2 -subsets from distinct groups of $\mathcal{G}$ into a set $\mathcal{B}$. Then $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ is a 1 - $\operatorname{FG}(3,(2,4), 6 n)$ of type $6^{n}$. Apply Construction 7.1 with $h_{1}=1$ and $h_{2}=v$ to obtain a strictly $v$-cyclic 0 $\mathrm{FG}(3,(\emptyset, 4), 12 n v)$ of type $(12 v)^{n}$, where the needed strictly $v$-cyclic 0 -FG $(3,(\emptyset, 4), 4 v)$ of type $(2 v)^{2}$ is from Corollary 6.18, and the needed $v$-cyclic $H(4,2 v, 4,3)$ is from Corollary 7.6. Apply Construction 6.6 with an optimal strictly $v$-cyclic $3-(12 \times v, 4,1)$ packing with $J^{*}(12 \times v)$ base blocks, which exists by assumption. Note that by Theorem 5.7,

$$
J^{*}(12 \times v)= \begin{cases}\left\lfloor\frac{12}{4}\left(\left\lfloor\frac{12 v-1}{3}\left\lfloor\frac{12 v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor, & \text { if } v \equiv 1(\bmod 2), \\ \left\lfloor\frac{12}{4}\left(\left\lfloor\frac{12 v-1}{3}\left\lfloor\frac{12 v-2}{2}\right\rfloor\right\rfloor-2\right)\right\rfloor, & \text { if } v \equiv 0(\bmod 12) .\end{cases}
$$

Then we have a strictly $v$-cyclic $3-(12 n \times v, 4,1)$ packing, which contains

$$
\begin{cases}3 n\left(24 n^{2} v^{2}-6 n v-1\right)=\left\lfloor\frac{12 n}{4}\left(\left\lfloor\frac{12 n v-1}{3}\left\lfloor\frac{12 n v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor, & \text { if } v \equiv 1(\bmod 2), \\ 6 n\left(12 n^{2} v^{2}-3 n v-1\right)=\left\lfloor\frac{12 n}{4}\left(\left\lfloor\frac{12 n v-1}{3}\left\lfloor\frac{12 n v-2}{2}\right\rfloor\right\rfloor-2\right)\right\rfloor, & \text { if } v \equiv 0(\bmod 12),\end{cases}
$$

base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(12 n \times v, 4,2)$-OOC with $J^{*}(12 n \times v)$ codewords exists. It is an optimal 2-D $(u \times v, 4,2)$ OOC with $J^{*}(u \times v)$ codewords.

The use of Lemma 9.8 depends on the existence of optimal 2-D $(12 \times v, 4,2)$-OOCs with $J^{*}(12 \times v)$ codewords. The following lemma shows an approach to obtain some optimal 2-D ( $12 \times v, 4,2$ )-OOCs from perfect 2-D $(2 \times v, 4,2)$-OOCs.

Lemma 9.9 Suppose that there is a perfect $2-D(2 \times v, 4,2)$-OOC with $v \equiv 1,5(\bmod$ $6)$. Then there is an optimal $2-D(12 \times v, 4,2)-O O C$ with $J^{*}(12 \times v)$ codewords.

Proof By Lemma 6.19, if there is a perfect 2-D $(2 \times v, 4,2)$-OOC with $v \equiv 1,5(\bmod 6)$, then there is a strictly $(2,1)$-regular 1-FG $(3,(2,4), 2 v)$ of type $2^{v}$. Start from this 1-FG and apply Construction 7.3 with $h_{1}=1$ and $h_{2}=1$ to obtain a strictly ( 12,1 )-regular $0-\mathrm{FG}(3,(\emptyset, 4), 12 v)$ of type $12^{v}$, where the needed strictly 1-cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 12)$ of type $6^{2}$ is from Theorem 6.4, and the needed 1-cyclic $H(4,6,4,3)$ is from Corollary 7.6. Applying Construction 6.14 with an optimal strictly 1 -cyclic $3-(12 \times 1,4,1)$ packing with 51 base blocks from Theorem 3.2, we have a strictly $v$-cyclic $3-(12 \times v, 4,1)$ packing, which contains $72 v^{2}-18 v-3=\left\lfloor\frac{12}{4}\left(\left\lfloor\frac{12 v-1}{3}\left\lfloor\frac{12 v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(12 \times v, 4,2)$-OOC with $J^{*}(12 \times v)$ codewords exists.

Lemma 9.10 Let $v \equiv 3(\bmod 6)$ or $v \equiv 0(\bmod 12)$. Suppose that there is an optimal $2-D(2 \times v, 4,2)$-OOC with $J^{*}(2 \times v)$ codewords. Then there is an optimal $2-D(12 \times v, 4,2)$ OOC with $J^{*}(12 \times v)$ codewords.

Proof By Proposition 9.2, if there is an optimal 2-D $(2 \times v, 4,2)$-OOC with $J^{*}(2 \times v)$ codewords for $v \equiv 3(\bmod 6)$ or $v \equiv 0(\bmod 12)$, then there is an optimal 2-D $(4 \times v, 4,2)$ OOC with $J^{*}(4 \times v)$ codewords. Note that by Theorem 5.7,

$$
J^{*}(4 \times v)= \begin{cases}\left\lfloor\frac{4}{4}\left(\left\lfloor\frac{4 v-1}{3}\left\lfloor\frac{4 v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor, & \text { if } v \equiv 3(\bmod 6), \\ \left\lfloor\frac{4}{4}\left(\left\lfloor\frac{4 v-1}{3}\left\lfloor\frac{4 v-2}{2}\right\rfloor\right\rfloor-2\right)\right\rfloor, & \text { if } v \equiv 0(\bmod 12) .\end{cases}
$$

By Lemma 6.12, there is a strictly cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 12 v)$ of type $(4 v)^{3}$. By Lemma 6.17, it implies a strictly $v$-cyclic 0 -FG $(3,(\emptyset, 4), 12 v)$ of type $(4 v)^{3}$. Start from this strictly $v$-cyclic 0 -FG and apply Construction 6.6 with an optimal strictly $v$-cyclic 3 - $(4 \times v, 4,1)$ packing with $J^{*}(4 \times v)$ base blocks, which is equivalent to an optimal 2-D $(4 \times v, 4,2)$ OOC with $J^{*}(4 \times v)$ codewords, to obtain a strictly $v$-cyclic $3-(12 \times v, 4,1)$ packing, which contains

$$
\begin{cases}72 v^{2}-18 v-3=\left\lfloor\frac{12}{4}\left(\left\lfloor\frac{12 v-1}{3}\left\lfloor\frac{12 v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor, & \text { if } v \equiv 3(\bmod 6), \\ 72 v^{2}-18 v-6=\left\lfloor\frac{12}{4}\left(\left\lfloor\frac{12 v-1}{3}\left\lfloor\frac{12 v-2}{2}\right\rfloor\right\rfloor-2\right)\right\rfloor, & \text { if } v \equiv 0(\bmod 12),\end{cases}
$$

base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2 -D $(12 \times v, 4,2)$-OOC with $J^{*}(12 \times v)$ codewords exists.

Combining the results of Lemmas 9.8, 9.9 and 9.10, we have the following proposition.
Proposition 9.11 Let $v \equiv 1(\bmod 2)$ or $v \equiv 0(\bmod 12)$. Suppose that there is an optimal 2-D $(2 \times v, 4,2)$-OOC with $J^{*}(2 \times v)$ codewords. Then there is an optimal 2-D $(u \times v, 4,2)$-OOC with $J^{*}(u \times v)$ codewords for any $u \equiv 0(\bmod 12)$.

Lemma 9.12 There exists a strictly 3-cyclic $0-F G(3,(\emptyset, 4), 18)$ of type $6^{3}$.
Proof We here give a construction of a $0-\mathrm{FG}(3,(\emptyset, 4), 18)$ of type $6^{3}$ on $I_{18}$ with the group set $\{\{0,1,2,3,4,5\}+i: i \in\{0,6,12\}\}$. Let $\alpha=(012)(345)(678)(91011)(121314)(15$ 16 17) be a permutation on $I_{18}$ and $G$ be the group generated by $\alpha$. Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of $G$. Obviously this design is isomorphic to a strictly 3 -cyclic 0 -FG $(3,(\emptyset, 4), 18)$ of type $6^{3}$.

| $\{0,1,6,7\}$ | $\{0,1,8,11\}$ | $\{0,1,9,10\}$ | $\{0,1,12,13\}$ | $\{0,1,14,15\}$ | $\{0,1,16,17\}$ | $\{0,3,6,9\}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\{0,3,7,8\}$ | $\{0,3,10,11\}$ | $\{0,3,12,15\}$ | $\{0,3,13,16\}$ | $\{0,3,14,17\}$ | $\{0,4,6,12\}$ | $\{0,4,7,14\}$ |
| $\{0,4,8,16\}$ | $\{0,4,9,13\}$ | $\{0,4,10,15\}$ | $\{0,4,11,17\}$ | $\{0,5,6,16\}$ | $\{0,5,7,12\}$ | $\{0,5,8,17\}$ |
| $\{0,5,9,14\}$ | $\{0,5,10,13\}$ | $\{0,5,11,15\}$ | $\{0,6,10,17\}$ | $\{0,6,11,14\}$ | $\{0,6,13,15\}$ | $\{0,7,9,16\}$ |
| $\{0,7,11,13\}$ | $\{0,7,15,17\}$ | $\{0,8,9,15\}$ | $\{0,8,10,12\}$ | $\{0,8,13,14\}$ | $\{0,9,12,17\}$ | $\{0,10,14,16\}$ |
| $\{0,11,12,16\}$ | $\{3,4,6,7\}$ | $\{3,4,8,11\}$ | $\{3,4,9,10\}$ | $\{3,4,12,17\}$ | $\{3,4,13,14\}$ | $\{3,4,15,16\}$ |
| $\{3,6,10,14\}$ | $\{3,6,11,17\}$ | $\{3,6,12,16\}$ | $\{3,7,9,13\}$ | $\{3,7,11,16\}$ | $\{3,7,12,14\}$ | $\{3,8,9,12\}$ |
| $\{3,8,10,15\}$ | $\{3,8,16,17\}$ | $\{3,9,14,15\}$ | $\{3,10,13,17\}$ | $\{3,11,13,15\}$ | $\{6,7,12,17\}$ | $\{6,7,13,16\}$ |
| $\{6,7,14,15\}$ | $\{6,9,12,13\}$ | $\{6,9,14,17\}$ | $\{6,9,15,16\}$ | $\{9,10,12,14\}$ | $\{9,10,13,16\}$ | $\{9,10,15,17\}$ |

Lemma 9.13 There exists an optimal 2-D $(u \times 3,4,2)$-OOC with $J^{*}(u \times 3)$ codewords for any $u \equiv 6(\bmod 12)$.

Proof Let $n=(u+2) / 2$. Then $n \equiv 4(\bmod 6)$. There is an $\operatorname{SQS}(n)$ [25]. Delete one point to obtain a 1-FG $(3,(3,4), n-1)$ of type $1^{n-1}$. Start from this 1-FG and apply Construction 7.1 with $h_{1}=1$ and $h_{2}=3$ to obtain a strictly 3 -cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 6(n-1))$ of type $6^{n-1}$, where the needed strictly 3 -cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 18)$ of type $6^{3}$ is from Lemma 9.12, and the needed 3 -cyclic $H(4,6,4,3)$ is from Corollary 7.6. Applying Construction 6.6 with an optimal strictly 3 -cyclic $3-(2 \times 3,4,1)$ packing with $J^{*}(2 \times 3)=1$ base block from Example 3.3, we have a strictly 3 -cyclic $3-(2(n-1) \times 3,4,1)$ packing, which contains $(n-1)\left(6 n^{2}-15 n+8\right) / 2$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(2(n-1) \times 3,4,2)$-OOC with $J^{*}(2(n-1) \times 3)$ codewords exists. It is an optimal 2-D $(u \times 3,4,2)$-OOC with $J^{*}(u \times 3)$ codewords.

Theorem 9.14 There exists an optimal 2-D $(u \times 3,4,2)$-OOC with $J^{*}(u \times 3)$ codewords for any $u \equiv 0(\bmod 2)$.

Proof When $u \equiv 2,4(\bmod 6)$, apply Proposition 9.2 with an optimal 2-D $(2 \times 3,4,2)$ OOC with $J^{*}(2 \times 3)=1$ base block from Example 3.3 to obtain an optimal 2-D ( $u \times$ $3,4,2)$-OOC with $J^{*}(u \times 3)$ codewords. When $u \equiv 0(\bmod 12)$, apply Proposition 9.11 with an optimal 2-D $(2 \times 3,4,2)$-OOC with $J^{*}(2 \times 3)=1$ base block from Example 3.3 to obtain an optimal 2-D $(u \times 3,4,2)$-OOC with $J^{*}(u \times 3)$ codewords. When $u \equiv 6(\bmod$ 12), the conclusion follows from Lemma 9.13.

Proposition 9.15 If there is an RoSQS(v+1), then there is an optimal 2-D $(u \times v, 4,2)$ OOC with $J^{*}(u \times v)$ codewords for any $u \equiv 0(\bmod 6)$.

Proof By Lemma 6.13 , when $v \equiv 1(\bmod 6)$, an $\operatorname{RoSQS}(v+1)$ is equivalent to a strictly cyclic $1-F G(3,(3,4), v)$ of type $1^{v}$, which is also a strictly $(1,1)$-regular $1-\mathrm{FG}(3,(3,4), v)$ of type $1^{v}$. Start from this 1-FG and apply Construction 7.3 with $h_{1}=1$ and $h_{2}=1$ to obtain a strictly $(u, 1)$-regular $0-\mathrm{FG}(3,(\emptyset, 4), u v)$ of type $u^{v}$ for any $u \equiv 0(\bmod 6)$, where the needed strictly 1-cyclic $0-\mathrm{FG}(3,(\emptyset, 4), 3 u)$ of type $u^{3}$ exists from Theorem 6.4, and the needed 1-cyclic $H(4, u, 4,3)$ is from Corollary 7.6. Applying Construction 6.14 with an optimal strictly 1-cyclic 3 - $(u \times 1,4,1)$ packing with $J^{*}(u \times 1)$ base blocks from Theorem 3.2, we have a strictly $v$-cyclic 3 - $(u \times v, 4,1)$ packing, which contains $u\left(u^{2} v^{2}-3 u v-6\right) / 24=\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(u \times v, 4,2)$-OOC with $J^{*}(u \times v)$ codewords exists.

By Lemma 6.13 , when $v \equiv 3(\bmod 6)$, an $\operatorname{RoSQS}(v+1)$ is equivalent to a strictly cyclic $1-F G(3,(3,4), v)$ of type $3^{v / 3}$, which is also a strictly $(1,3)$-regular $1-\mathrm{FG}(3,(3,4), v)$ of type $3^{v / 3}$. Start from this 1-FG and apply Construction 7.3 with $h_{1}=3$ and $h_{2}=1$ to obtain a strictly $(u, 3)$-regular $0-\mathrm{FG}(3,(\emptyset, 4), u v)$ of type $(3 u)^{v / 3}$ for any $u \equiv 0(\bmod$ 6). Applying Construction 6.14 with an optimal strictly 3 -cyclic 3 - $(u \times 3,4,1)$ packing with $J^{*}(u \times 3)$ base blocks from Theorem 9.14 , we have a strictly $v$-cyclic 3 - $(u \times v, 4,1)$ packing, which contains $u\left(u^{2} v^{2}-3 u v-6\right) / 24=\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2-D $(u \times v, 4,2)$-OOC with $J^{*}(u \times v)$ codewords exists.

Combining Theorem 4.4 and Proposition 9.15 , many infinite families of optimal 2-D $(u \times v, 4,2)$-OOCs with $J^{*}(u \times v)$ codewords will be obtained. As an example, we have

Theorem 9.16 Let $p \equiv 7(\bmod 12)$ be a prime or $p \in\{37,61,73,109,157,181,229$, $277\}$. There exist a perfect $2-D(u \times p, 4,2)-O O C$ for any $u \equiv 2,4(\bmod 6)$, and an optimal 2-D $(u \times p, 4,2)-O O C$ with $J^{*}(u \times p)$ codewords for any $u \equiv 0(\bmod 6)$.

Proof When $u \equiv 2,4(\bmod 6)$, start from a perfect $2-\mathrm{D}(2 \times p, 4,2)$-OOC, which exists by Theorem 4.5, and apply Proposition 9.2 to obtain a perfect $2-\mathrm{D}(u \times p, 4,2)$-OOC. When $u \equiv 0(\bmod 6)$, start from an $\operatorname{RoSQS}(p+1)$, which exists by Theorem 4.4, and apply Proposition 9.15 to obtain an optimal 2-D $(u \times p, 4,2)$-OOC with $J^{*}(u \times p)$ codewords.

Lemma 9.17 If there is a perfect $2-D(2 \times v, 4,2)-O O C$ with $v \equiv 1,5(\bmod 6)$, then there is an optimal $2-D(12 \times 2 v, 4,2)-O O C$ with $J^{*}(12 \times 2 v)$ codewords.

Proof By Lemma 6.19 , if there is a perfect $2-\mathrm{D}(2 \times v, 4,2)$-OOC with $v \equiv 1,5(\bmod 6)$, then there is a strictly $(2,1)$-regular $1-\mathrm{FG}(3,(2,4), 2 v)$ of type $2^{v}$. Start from this 1-FG and apply Construction 7.3 with $h_{1}=1$ and $h_{2}=2$ to obtain a strictly (12,2)-regular $0-\mathrm{FG}(3,(\emptyset, 4), 24 v)$ of type $24^{v}$, where the needed strictly 2 -cyclic 0 -FG $(3,(\emptyset, 4), 24)$ of type $12^{2}$ is from Lemma 8.4, and the needed 2-cyclic $H(4,12,4,3)$ is from Corollary 7.6. Applying Construction 6.14 with an optimal strictly 2 -cyclic 3 - $(12 \times 2,4,1)$ packing with $J^{*}(12 \times 2)=248$ base blocks from Lemma 8.5 , we have a strictly $2 v$-cyclic 3 $(12 \times 2 v, 4,1)$ packing, which contains $4\left(72 v^{2}-9 v-1\right)=\left\lfloor\frac{12}{4}\left(\left\lfloor\frac{24 v-1}{3}\left\lfloor\frac{24 v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor-1$ base blocks. This number achieves the upper bound in Theorem 5.7. Thus an optimal 2 -D $(12 \times 2 v, 4,2)$-OOC with $J^{*}(12 \times 2 v)$ codewords exists.

Theorem 9.18 Let $p \equiv 7(\bmod 12)$ be a prime or $p \in\{37,61,73,109,157,181,229$, $277\}$. There exists an optimal $(12 \times 2 p, 4,2)$-OOC with $J^{*}(12 \times 2 p)$ codewords.

Proof Start from a perfect 2-D $(2 \times p, 4,2)$-OOC, which exists by Theorem 4.5. Apply Lemma 9.17 to complete the proof.

Table II
Small orders of optimal 2-D $(u \times v, 4,2)$-OOCs with $\Phi(u \times v, 4,2)=J^{*}(u \times v)$
codewords for $6 \leq u v \leq 34$

| $u v$ | $u \times v$ | $\Phi(u \times v, 4,2)$ | Source | $u \times v$ | $\Phi(u \times v, 4,2)$ | Source |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | $2 \times 3$ | 1 | Example 3.3 | $3 \times 2$ | 1 | Example 3.3 |
| 8 | $2 \times 4$ | 3 | Example 6.15 | $4 \times 2$ | 6 | Example 6.7 |
| 9 | $3 \times 3$ | 6 | Lemma 8.1 |  |  |  |
| 10 | $2 \times 5$ | 6 | Theorem 2.7(2) | $5 \times 2$ | 15 | Theorem 2.7(2) |
| 12 | $2 \times 6$ | 8 | Lemma 8.1 | $3 \times 4$ | 12 | Lemma 8.1 |
|  | $4 \times 3$ | 17 | Example 6.8 | $6 \times 2$ | 25 | Lemma 8.1 |
| 14 | $2 \times 7$ | 13 | Theorem 4.5 | $7 \times 2$ | 44 | Lemma 8.1 |
| 15 | $3 \times 5$ | 21 | Theorem 2.7(2) | $5 \times 3$ | 35 | Theorem 2.7(2) |
| 16 | $2 \times 8$ | 17 | Example 6.16 | $4 \times 4$ | 34 | Theorem 9.4 |
|  | $8 \times 2$ | 68 | Lemma 9.1 |  |  |  |
| 18 | $2 \times 9$ | 22 | Corollary 8.11 | $3 \times 6$ | 33 | Corollary 8.11 |
|  | $6 \times 3$ | 66 | Corollary 8.11 | $9 \times 2$ | 99 | Corollary 8.11 |
| 20 | $2 \times 10$ | 28 | Theorem 2.7(4) | $4 \times 5$ | 57 | Proposition 9.2 |
|  | $5 \times 4$ | 70 | Lemma 8.12 | $10 \times 2$ | 140 | Lemma 9.1 |
| 21 | $3 \times 7$ | 45 | Theorem 2.7(2) | $7 \times 3$ | 105 | Theorem 2.7(2) |
| 22 | $2 \times 11$ | 35 | Lemma 8.1 | $11 \times 2$ |  | $? ?$ |
| 24 | $2 \times 12$ | 41 | Lemma 8.3 | $3 \times 8$ |  | $? ?$ |
|  | $4 \times 6$ | 82 | Proposition 9.5 | $6 \times 4$ |  | $? ?$ |
|  | $8 \times 3$ | 166 | Theorem 9.14 | $12 \times 2$ | 248 | Lemma 8.5 |
| 25 | $5 \times 5$ | 110 | Theorem 2.7(2) |  |  |  |
| 26 | $2 \times 13$ | 50 | Theorem 2.7(2) | $13 \times 2$ | 325 | Theorem 2.7(2) |
| 27 | $3 \times 9$ | 78 | Theorem 2.7(2) | $9 \times 3$ | 234 | Theorem 2.7(2) |
| 28 | $2 \times 14$ | 58 | Theorem 2.7(4) | $4 \times 7$ | 117 | Proposition 9.2 |
|  | $7 \times 4$ | 203 | Lemma 8.12 | $14 \times 2$ | 406 | Lemma 9.1 |
| 30 | $2 \times 15$ | 67 | Lemma 8.7 | $3 \times 10$ | 100 | Lemma 8.9 |
|  | $5 \times 6$ |  | $? ?$ | $6 \times 5$ | 201 | Lemma 8.12 |
|  | $10 \times 3$ | 335 | Theorem 9.14 | $15 \times 2$ |  | $? ?$ |
| 32 | $2 \times 16$ | 77 | Theorem 9.4 | $4 \times 8$ | 154 | Theorem 9.4 |
|  | $8 \times 4$ | 308 | Theorem 9.4 | $16 \times 2$ | 616 | Lemma 9.1 |
| 33 | $3 \times 11$ | 120 | Theorem 2.7(2) | $11 \times 3$ | 440 | Theorem 2.7(2) |
| 34 | $2 \times 17$ | 88 | Theorem 2.7(2) | $17 \times 2$ | 748 | Theorem 2.7(2) |

Finally we summarize the existence of small orders of optimal 2-D $(u \times v, 4,2)$-OOCs with $J^{*}(u \times v)$ codewords as follows.

Theorem 9.19 There exists an optimal 2-D $(u \times v, 4,2)$-OOC with $J^{*}(u \times v)$ codewords for each $(u, v)$ satisfying $6 \leq u v \leq 34$ and $(u, v) \notin\{(1,9),(1,12),(1,13),(11,2),(23,1)$, $(3,8),(6,4),(5,6),(15,2)\}$. When $(u, v) \in\{(1,9),(1,12),(1,13)\}$, there exists an optimal $2-D(u \times v, 4,2)-O O C$ with $J(1 \times u v, 4,2)-1$ codewords.

Proof By Theorem 3.2, there exists an optimal 2-D $(u \times 1,4,2)$-OOC with $J^{*}(u \times 1)$ codewords for each $6 \leq u \leq 34$ and $u \neq 23$. By Lemma 2.6(3), there exists an optimal 2 -D $(1 \times v, 4,2)$-OOC with $J^{*}(1 \times v)$ codewords for each $7 \leq v \leq 34$ and $v \notin\{9,12,13\}$. Note that $J^{*}(1 \times v)=J(1 \times v, 4,2)$ when $v \not \equiv 0(\bmod 24)$, and $J^{*}(1 \times v)=J(1 \times v, 4,2)-1$ when $v \equiv 0(\bmod 24)$. When $v \in\{9,12,13\}$, by Lemma 2.6(4), there exists an optimal 2 -D $(1 \times v, 4,2)$-OOC with $J(1 \times v, 4,2)-1$ codewords. An optimal 2-D $(1 \times 6,4,2)$-OOC is trivial without codewords. For all other cases of $(u, v)$ such that there is an optimal

2-D $(u \times v, 4,2)$-OOC with $J^{*}(u \times v)$ codewords, we show the sources in Table II, where the question marks "??" indicates the orders for each of which the existence of an optimal 2 -D $(u \times v, 4,2)$-OOC with $J^{*}(u \times v)$ codewords is still open.

## 10 Conclusion

In this paper, we gave some combinatorial constructions for optimal 2-D ( $u \times v, k, 2$ )OOCs. As applications, many infinite families of optimal 2-D $(u \times v, 4,2)$-OOCs are obtained. We summarize all infinite families obtained in this paper in Table III. Although we can not complete the existence of optimal 2-D ( $u \times v, 4,2$ )-OOCs, we hope to present some possible approaches to reduce the existence problem. We summarize these approaches in Table IV.

Table III
New infinite families of optimal 2-D $(u \times v, 4,2)$-OOCs with $J^{*}(u \times v)$ codewords in this paper

| Parameters | Conditions | Source |
| :---: | :---: | :---: |
| $\left(n_{1} \times n_{2}, 4,2\right)$ | $\begin{gathered} n_{1} n_{2}=u v, \\ u \in\left\{4^{n}-1: n \geq 1\right\} \cup\{1,27,33,39,51,87,123,183\} \text { and } \\ v \in S=\{p \equiv 7(\bmod 12): p \text { is a prime }\} \cup \\ \left\{2^{n}-1: \text { odd integer } n \geq 1\right\} \cup\{25,37,61,73,109,157,181,229,277\}, \\ \text { or } v \text { is a product of integers from } S \\ \hline \end{gathered}$ | Theorem 2.7(1) |
| $(2 \times 2 n, 4,2)$ | $n=p_{1}^{r_{1}} p_{2}^{r_{2}} \cdots p_{s}^{r_{s}}$ $p_{i}=13$ or $p_{i} \equiv 5(\bmod 12)$ is a prime and $p_{i}<1500000$, $r_{i} \geq 1$ for $1 \leq i \leq s$ | Theorem 2.7(3) |
| $\left(u \times 2^{n}, 4,2\right)$ | $u \equiv 2,4(\bmod 6)$ and $n \geq 1$ | Theorem 9.4 |
| $(u \times 2 p, 4,2)$ | $\begin{gathered} u \equiv 8,16(\bmod 24) \text { or } u=12 \\ p \equiv 7(\bmod 12) \text { a prime or } p \in\{37,61,73,109,157,181,229,277\} \\ \hline \end{gathered}$ | Theorem 9.7 <br> Theorem 9.18 |
| ( $u \times p, 4,2)$ | $\begin{gathered} u \equiv 0(\bmod 2) \\ p \equiv 7(\bmod 12) \text { a prime or } p \in\{3,37,61,73,109,157,181,229,277\} \\ \hline \end{gathered}$ | Theorem 9.14 <br> Theorem 9.16 |

Table IV
Possible approaches to construct optimal 2-D $(u \times v, 4,2)$-OOCs

| Input | $\Rightarrow$ | Output | Source |
| :---: | :---: | :---: | :---: |
| $\begin{gathered} \text { optimal 1-D }(n, 4,2) \text {-OOC } \\ \text { with } J(1 \times n, 4,2) \text { codewords, } \\ n \equiv 1,3(\bmod 6) \text { or } n \equiv 2,10(\bmod 24) \end{gathered}$ |  | optimal 2-D ( $u \times v, 4,2$ )-OOC with $J(u \times v, 4,2)$ codewords, for any integer factorization $n=u v$ | Corollary 2.5(1) |
| $\begin{aligned} & \text { optimal } 2 \text { - } \mathrm{D}(2 \times v, 4,2) \text {-OOC } \\ & \text { with } J^{*}(2 \times v) \text { codewords, } \\ & v \equiv 1(\bmod 2) \text { or } v \equiv 0(\bmod 4) \end{aligned}$ |  | $\begin{gathered} \hline \text { optimal 2-D }(u \times v, 4,2) \text {-OOC } \\ \text { with } J^{*}(u \times v) \text { codewords, } \\ u \equiv 2,4(\bmod 6) \end{gathered}$ | Proposition 9.2 |
| $\begin{gathered} \text { optimal } 2 \text {-D }(u / 2 \times 2 v, 4,2) \text {-OOC } \\ \text { with } J^{*}(u / 2 \times 2 v) \text { codewords, } \\ u \equiv 2,4(\bmod 6) \text { and } v \equiv 2(\bmod 4) \end{gathered}$ |  | optimal 2-D $(u \times v, 4,2)$-OOC with $J^{*}(u \times v)$ codewords | Proposition 9.5 |
| $\begin{gathered} \text { perfect } 2 \text {-D }(2 \times v, 4,2) \text {-OOC } \\ v \equiv 1,5(\bmod 6) \end{gathered}$ |  | $\begin{gathered} \text { optimal 2-D }(u \times 2 v, 4,2) \text {-OOC } \\ \text { with } J^{*}(u \times 2 v) \text { codewords, } \\ u \equiv 8,16(\bmod 24) \\ \hline \end{gathered}$ | Proposition 9.6 |
| $\begin{aligned} & \text { optimal } 2 \text {-D }(2 \times v, 4,2) \text {-OOC } \\ & \text { with } J^{*}(2 \times v) \text { codewords, } \\ & v \equiv 1(\bmod 2) \text { or } v \equiv 0(\bmod 12) \end{aligned}$ |  | $\begin{gathered} \hline \text { optimal 2-D }(u \times v, 4,2) \text {-OOC } \\ \text { with } J^{*}(u \times v) \text { codewords, } \\ u \equiv 0(\bmod 12) \\ \hline \end{gathered}$ | Proposition 9.11 |
| $\begin{gathered} \hline \operatorname{RoSQS}(v+1) \\ v \equiv 1,3(\bmod 6) \end{gathered}$ |  | $\begin{gathered} \hline \text { optimal 2-D }(u \times v, 4,2) \text {-OOC } \\ \text { with } J^{*}(u \times v) \text { codewords, } \\ u \equiv 0(\bmod 6) \\ \hline \end{gathered}$ | Proposition 9.15 |

By Theorem 5.7, we see that in many cases the Johnson bound can not be achieved. A natural question is whether the bounds established in Theorem 5.7 is good enough to make each optimal 2-D $(u \times v, 4,2)$-OOC achieve it. Although many infinite families are given to achieve the upper bound in Theorem 5.7 , we still tend to think it not true. For example we conjecture that when $u \equiv 0(\bmod 6)$ and $v \equiv 2,4(\bmod 6)$, the upper bound is $\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor-\left\lfloor\frac{u}{12}\right\rfloor$. If the conjecture is correct, the condition in Lemma 9.8 can be relaxed to $v \not \equiv 2(\bmod 4)$, which implies that the condition in Proposition 9.11 can also be relaxed to $v \not \equiv 2(\bmod 4)$.

Another question is to find more constructions for optimal 2-D $(2 \times v, 4,2)$-OOCs, which are very useful by Propositions 9.2, 9.6 and 9.11. In 1991 Phelps [50] constructed a class of 2 -chromatic $\operatorname{SQS}(22)$ using cyclic large sets of $2-(11,3,1)$ packings. It seems that Phelps's method can be generalized to construct some strictly $v$-cyclic $\operatorname{SQS}(2 \times v) \mathrm{s}$ for $v \equiv 1,5(\bmod 6)$, which are also perfect $2-\mathrm{D}(2 \times v, 4,2)$-OOCs. The interested reader may refer to the paper [50].

## Appendix I

Proof of Construction 4.2: For checking the correctness of the algorithm shown in Figure 1, first count the number of base blocks in $\mathcal{A}$. It is clear that $\left|\mathcal{A}_{1} \cup \mathcal{A}_{1}^{\prime}\right|=$ $(p-1)(p-3) / 12,\left|\mathcal{A}_{2} \cup \mathcal{A}_{2}^{\prime}\right|=(p-1) / 3,\left|\mathcal{A}_{3}\right| \leq 3 \times(p-1) / 6 \times(p-1) / 2=(p-1)^{2} / 4$. Thus $|\mathcal{A}| \leq(p-1)(2 p-1) / 6$.

Since the number $(p-1)(2 p-1) / 6$ is the right number of base blocks in a strictly $p$-cyclic $S Q S(2 \times p)$, in the following it suffices to show that each triple of $I_{2} \times Z_{p}$ appears in at least one block of the resulting design. (1) Each triple of $\{0\} \times Z_{p}$ appears in one block of $\mathcal{A}_{1} \cup \mathcal{A}_{2}$ and their cyclic shifts. (2) Each triple of $\{1\} \times Z_{p}$ appears in one block of $\mathcal{A}_{1}^{\prime} \cup \mathcal{A}_{2}^{\prime}$ and their cyclic shifts. (3) Each triple of the form $\left\{x_{0}, y_{0}, z_{1}\right\}$ appears in one block of $\mathcal{A}_{2} \cup \mathcal{A}_{3}$ and their cyclic shifts. (4) Each triple of the form $\left\{x_{1}, y_{1}, z_{0}\right\}$ appears in one block of $\mathcal{A}_{2}^{\prime} \cup \mathcal{A}_{3}$ and their cyclic shifts.

## Appendix II

Lemma 5.2 Let $u \equiv 0(\bmod 12)$ and $v \equiv 2,4(\bmod 6)$. Then $\Phi(u \times v, 4,2) \leq$ $\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor-1$.

Proof First we shall show that $\Phi(u \times 2,4,2) \leq\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{2 u-1}{3}\left\lfloor\frac{2 u-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor-1$. By Lemma 5.1, $\Phi(u \times 2,4,2) \leq\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{2 u-1}{3}\left\lfloor\frac{2 u-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$. Suppose that $\Phi(u \times 2,4,2)=\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{2 u-1}{3}\left\lfloor\frac{2 u-2}{2}\right\rfloor\right\rfloor-\right.\right.$ 1) $\rfloor$. Then there were a strictly 2 -cyclic 3 - $(u \times 2,4,1)$-packing with $\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{2 u-1}{3}\left\lfloor\frac{2 u-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor$ base blocks. Let $\mathcal{L}$ be the leave of the strictly 2 -cyclic $3-(u \times 2,4,1)$-packing. Count the number of 3 -subsets in the leave $\mathcal{L}$. It is $\binom{2 u}{3}-\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{2 u-1}{3}\left\lfloor\frac{2 u-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor \cdot 2 \cdot 4=8 u / 3$.

For each $a \in I_{u}$ and each $i \in Z_{2}$, consider the number $n$ of 3 -subsets containing the point $(a, i)$ in the leave $\mathcal{L}$. Delete one point from a strictly 2 -cyclic 3 - $(u \times 2,4,1)$-packing to obtain a $2-(2 u-1,3,1)$-packing, which contains at most $\lfloor(2 u-1)(2 u-2) / 6\rfloor-1$ blocks when $2 u \equiv 0(\bmod 6)[26]$. Since each 3 -subset of $I_{u} \times Z_{2}$ occurs in at most one block, we have $n \geq\binom{ 2 u-1}{2}-3(\lfloor(2 u-1)(2 u-2) / 6\rfloor-1)=4$, which implies that $|\mathcal{L}| \geq 4 \cdot 2 u / 3$. Due to $|\mathcal{L}|=8 u / 3, n$ must be equal to 4 . Note that the above conclusion holds for each $a \in I_{u}$ and each $i \in Z_{2}$.

For each $a \in I_{u}$, consider the number $m$ of the base blocks containing the two points $(a, 0),(a, 1)$. Since each 3 -subset of $I_{u} \times Z_{2}$ occurs in at most one block and each
base block containing the two points $(a, 0),(a, 1)$ generates exactly two different blocks containing the same two points, the number $m$ is at most $\lfloor(2 u-2) / 4\rfloor=(2 u-4) / 4$. Thus there are at least two 3 -subsets containing the two points $(a, 0),(a, 1)$ in the leave, denoted by $\left\{(a, 0),(a, 1),\left(b_{a}, 0\right)\right\}$ and $\left\{(a, 0),(a, 1),\left(b_{a}, 1\right)\right\}$, where $b_{a} \in I_{u}$ and $b_{a} \neq a$. Note that the above conclusion holds for each $a \in I_{u}$. We have that $\mathcal{L} \supset$ $\left\{\left\{(a, 0),(a, 1),\left(b_{a}, 0\right)\right\},\left\{(a, 0),(a, 1),\left(b_{a}, 1\right)\right\}: a \in I_{u}\right\}$.

Given any $a \in I_{u}$, consider the number $r$ of the blocks containing the two points $(a, 0),\left(b_{a}, 0\right)$. Since each 3 -subset of $I_{u} \times Z_{2}$ occurs in at most one block, the number $r$ is at most $\lfloor(2 u-3) / 2\rfloor=(2 u-4) / 2$. Thus there is at least another one 3 -subset in the leave containing the two points $(a, 0),\left(b_{a}, 0\right)$. Assume that $\left\{(a, 0),\left(b_{a}, 0\right),(x, k)\right\} \in \mathcal{L}$, where $(x, k) \neq(a, 1)$. Similarly, consider the blocks containing the two points $(a, 0)$, $\left(b_{a}, 1\right)$ and assume that $\left\{(a, 0),\left(b_{a}, 1\right),(y, l)\right\} \in \mathcal{L}$, where $(y, l) \neq(a, 1)$.

If $(x, k) \neq\left(b_{a}, 1\right)$, then $\left\{(a, 0),\left(b_{a}, 0\right),(x, k)\right\} \neq\left\{(a, 0),\left(b_{a}, 1\right),(y, l)\right\}$. Since the number of 3 -subsets containing the point $(a, 0)$ in the leave is exactly four, they must be $\left\{(a, 0),(a, 1),\left(b_{a}, 0\right)\right\},\left\{(a, 0),(a, 1),\left(b_{a}, 1\right)\right\},\left\{(a, 0),\left(b_{a}, 0\right),(x, k)\right\},\left\{(a, 0),\left(b_{a}, 1\right),(y, l)\right\}$. Note that $(x, k) \neq\left(b_{a}, 0\right)$. Consider the number $s$ of the blocks containing the two points $(a, 0),(x, k)$. Since each 3 -subset of $I_{u} \times Z_{2}$ occurs in at most one block, the number $s$ is at most $\lfloor(2 u-3) / 2\rfloor=(2 u-4) / 2$. Thus there is at least another one 3 -subset in the leave containing the two points $(a, 0),(x, k)$. It implies that $(y, l)=(x, k)$. Due to $\left\{(a, 0),\left(b_{a}, 1\right),(y, l)\right\} \in \mathcal{L}$, i.e., $\left\{(a, 0),\left(b_{a}, 1\right),(x, k)\right\} \in \mathcal{L}$, under the action of $Z_{2}$ we have $\left\{(a, 1),\left(b_{a}, 0\right),(x, k+1)\right\} \in \mathcal{L}$. It implies that there are at least five 3 -subsets containing the point $\left(b_{a}, 0\right)$, i.e., $\left\{(a, 0),(a, 1),\left(b_{a}, 0\right)\right\},\left\{(a, 0),\left(b_{a}, 0\right),(x, k)\right\}$, $\left\{(a, 1),\left(b_{a}, 0\right),(x, k+1)\right\},\left\{\left(b_{a}, 0\right),\left(b_{a}, 1\right),\left(b_{b_{a}}, 0\right)\right\},\left\{\left(b_{a}, 0\right),\left(b_{a}, 1\right),\left(b_{b_{a}}, 1\right)\right\}$. A contradiction.

$$
\text { If }(x, k)=\left(b_{a}, 1\right) \text { and }(y, l) \neq\left(b_{a}, 0\right) \text {, then }\left\{(a, 0),\left(b_{a}, 0\right),(x, k)\right\} \neq\left\{(a, 0),\left(b_{a}, 1\right),(y, l)\right\} .
$$

Note that $(y, l) \neq\left(b_{a}, 1\right)$, and hence $(y, l) \neq(x, k)$. Since the number of 3 -subsets containing the point $(a, 0)$ in the leave is exactly four, they must be $\left\{(a, 0),(a, 1),\left(b_{a}, 0\right)\right\}$, $\left\{(a, 0),(a, 1),\left(b_{a}, 1\right)\right\},\left\{(a, 0),\left(b_{a}, 0\right),(x, k)\right\},\left\{(a, 0),\left(b_{a}, 1\right),(y, l)\right\}$. It implies that there is only one 3 -subset containing the two points $(a, 0)$ and $(y, l)$ in the leave. Similar arguments to those in the paragraph 4 of this proof, it is impossible.

If $(x, k)=\left(b_{a}, 1\right)$ and $(y, l)=\left(b_{a}, 0\right)$, then $\left\{(a, 0),\left(b_{a}, 0\right),(x, k)\right\}=\left\{(a, 0),\left(b_{a}, 1\right),(y, l)\right\}$. Since the number of 3 -subsets containing the point $(a, 0)$ in the leave is exactly four, three of them must be $\left\{(a, 0),(a, 1),\left(b_{a}, 0\right)\right\},\left\{(a, 0),(a, 1),\left(b_{a}, 1\right)\right\}$ and $\left\{(a, 0),\left(b_{a}, 0\right),\left(b_{a}, 1\right)\right\}$. Assume that the 4th 3 -subset containing the point $(a, 0)$ is $\{(a, 0),(z, i),(w, j)\}$. Similar arguments to those in the paragraph 4 of this proof, we have that the number of 3 -subsets containing the points $(a, 0),(z, i)$ in the leave must be even. A contradiction. Hence $\Phi(u \times 2,4,2) \leq\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{2 u-1}{3}\left\lfloor\frac{2 u-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor-1$.

Next consider the number of $\Phi(u \times v, 4,2)$. If there is an optimal 2-D $(u \times v, 4,2)$-OOC with $\Phi(u \times v, 4,2)$ codewords, then by Theorem 2.1, for integer factorization $v=2 v_{1}$, there exits a 2-D $\left(u v_{1} \times 2,4,2\right)$-OOC with $v_{1} \Phi(u \times v, 4,2)$ codewords. Since $u v_{1} \equiv 0$ ( $\bmod 12$ ), we have $v_{1} \Phi(u \times v, 4,2) \leq\left\lfloor\frac{u v_{1}}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor-1$. It is readily checked that $\Phi(u \times v, 4,2) \leq\left\lfloor\frac{1}{v_{1}}\left(\left\lfloor\frac{u v_{1}}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor-1\right)\right\rfloor=u\left(u^{2} v^{2}-3 u v-6\right) / 24-1=$ $\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-1\right)\right\rfloor-1$.

Lemma 5.3 Let $u v \equiv 0(\bmod 12)$ and $v \equiv 0(\bmod 6)$. Then $\Phi(u \times v, 4,2) \leq\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\right.\right.\right.$ $\left.\left.\left.\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-2\right)\right\rfloor$.

Proof Let $\mathcal{L}$ be the leave of a strictly $v$-cyclic 3 - $(u \times v, 4,1)$-packing. Let $\mathcal{L}_{1}=$ $\left\{\{(a, i),(a, v / 3+i),(a, 2 v / 3+i)\}: a \in I_{u}, 0 \leq i<v / 3\right\}$. Since each orbit of 3 -subsets in $\mathcal{L}_{1}$ is of length $v / 3$ under the action of $Z_{v}$, each 3 -subset in $\mathcal{L}_{1}$ must be contained in $\mathcal{L}$, i.e., $\mathcal{L}_{1} \subset \mathcal{L}$.

For each $a \in I_{u}$ and each $i \in Z_{v}$, consider the number $n$ of the blocks containing the two points $(a, i),(a, v / 3+i)$. Since each 3 -subset of $I_{u} \times Z_{v}$ occurs in at most one block and $\{(a, i),(a, v / 3+i),(a, 2 v / 3+i)\} \in \mathcal{L}$, the number $n$ is at most $\lfloor(u v-3) / 2\rfloor=(u v-4) / 2$. Thus there is at least another one 3 -subset in the leave containing the two points ( $a, i$ ), $(a, v / 3+i)$, denoted by $\left\{(a, i),(a, v / 3+i),\left(b_{a, i}, j_{a, i}\right)\right\}$, where $\left(b_{a, i}, j_{a, i}\right) \neq(a, 2 v / 3+i)$. Note that the above conclusion holds for each $a \in I_{u}$ and each $i \in Z_{v}$. Thus we have that $\mathcal{L}_{2}=\left\{\left\{(a, i),(a, v / 3+i),\left(b_{a, i}, j_{a, i}\right)\right\}: a \in I_{u}, i \in Z_{v}\right\} \subset \mathcal{L} \backslash \mathcal{L}_{1}$.

For each $a \in I_{u}$ and each $0 \leq i<v / 2$, consider the number $m$ of the base blocks containing the two points $(a, i),(a, v / 2+i)$. Since each 3 -subset of $I_{u} \times Z_{v}$ occurs in at most one block and each base block containing the two points $(a, i),(a, v / 2+i)$ generates exactly two different blocks containing the same two points, the number $m$ is at most $\lfloor(u v-2) / 4\rfloor=(u v-4) / 4$. Thus there are at least two 3 -subsets containing the two points $(a, i),(a, v / 2+i)$ in the leave, denoted by $\left\{(a, i),(a, v / 2+i),\left(c_{a, i}, k_{a, i}\right)\right\}$ and $\left\{(a, i),(a, v / 2+i),\left(c_{a, i}, v / 2+k_{a, i}\right)\right\}$. Note that the above conclusion holds for each $a \in I_{u}$ and each $0 \leq i<v / 2$. Thus we have that $\mathcal{L}_{3}=\{\{(a, i),(a, v / 2+$ $\left.\left.i),\left(c_{a, i}, k_{a, i}\right)\right\},\left\{(a, i),(a, v / 2+i),\left(c_{a, i}, v / 2+k_{a, i}\right)\right\}: a \in I_{u}, 0 \leq i<v / 2\right\} \subset \mathcal{L} \backslash \mathcal{L}_{1}$. For convenience assume that $\mathcal{L}_{3}=\left\{\left\{(a, i),(a, v / 2+i),\left(c_{a, i}, l_{a, i}\right)\right\}: a \in I_{u}, i \in Z_{v}\right\}$, where $l_{a, i}=k_{a, i}$ when $0 \leq i<v / 2$, and $l_{a, i}=v / 2+k_{a, i}$ when $v / 2 \leq i<v$.

If $\mathcal{L}_{2} \cap \mathcal{L}_{3}=\emptyset$, then $|\mathcal{L}| \geq 7 u v / 3$. If $\mathcal{L}_{2} \cap \mathcal{L}_{3} \neq \emptyset$, assume that $\left\{\left(x, i_{1}\right),(x, v / 3+\right.$ $\left.\left.i_{1}\right),\left(b_{x, i_{1}}, j_{x, i_{1}}\right)\right\}=\left\{\left(x, i_{2}\right),\left(x, v / 2+i_{2}\right),\left(c_{x, i_{2}}, l_{x, i_{2}}\right)\right\}$ for some $x \in I_{u}$ and some $i_{1}, i_{2} \in Z_{v}$. If $\left(x, i_{1}\right) \neq\left(c_{x, i_{2}}, l_{x, i_{2}}\right)$, we have $\left(b_{x, i_{1}}, j_{x, i_{1}}\right)=\left(x, v / 2+i_{1}\right)$. If $\left(x, i_{1}\right)=\left(c_{x, i_{2}}, l_{x, i_{2}}\right)$, we have $\left(b_{x, i_{1}}, j_{x, i_{1}}\right)=\left(x, 5 v / 6+i_{1}\right)$. Thus each 3 -subset in $\mathcal{L}_{2} \cap \mathcal{L}_{3}$ is of the form $\left\{\left(x, i_{1}\right),\left(x, v / 3+i_{1}\right),\left(x, v / 2+i_{1}\right)\right\}$ or $\left\{\left(x, i_{1}\right),\left(x, v / 3+i_{1}\right),\left(x, 5 v / 6+i_{1}\right)\right\}$. Let $\mathcal{L}_{2} \cap \mathcal{L}_{3}=$ $\left\{\{(x, i),(x, v / 3+i),(x, v / 2+i)\}: x \in A, i \in Z_{v}\right\} \cup\{\{(x, i),(x, v / 3+i),(x, 5 v / 6+i)\}:$ $\left.x \in B, i \in Z_{v}\right\}=\left\{\{(x, i),(x, v / 3+i),(x, v / 2+i)\}: x \in A, i \in Z_{v}\right\} \cup\{\{(x, v / 2+$ $\left.i),(x, 5 v / 6+i),(x, v / 3+i)\}: x \in B, i \in Z_{v}\right\}$, where $A, B \subset I_{u}$ and $A \cap B=\emptyset$. Then $\mathcal{L}_{2} \backslash\left(\mathcal{L}_{2} \cap \mathcal{L}_{3}\right)=\left\{\left\{(a, i),(a, v / 3+i),\left(b_{a, i}, j_{a, i}\right)\right\}: a \in I_{u} \backslash(A \cup B), i \in Z_{v}\right\}$ and $\mathcal{L}_{3} \backslash\left(\mathcal{L}_{2} \cap \mathcal{L}_{3}\right)=\left\{\left\{(a, i),(a, v / 2+i),\left(c_{a, i}, l_{a, i}\right)\right\}: a \in I_{u} \backslash(A \cup B), i \in Z_{v}\right\}$.

Let $\{(x, v / 3+i),(x, v / 2+i)\} \subset T \in \mathcal{L}_{2} \cap \mathcal{L}_{3}$. Consider the number of the blocks containing the two points $(x, v / 3+i),(x, v / 2+i)$. It is at most $\lfloor(u v-3) / 2\rfloor=(u v-$ 4)/2. Thus there is at least another one 3 -subset containing the two points $(x, v / 3+i)$, $(x, v / 2+i)$ in $\mathcal{L} \backslash\left(\mathcal{L}_{2} \cap \mathcal{L}_{3}\right)$, denoted by $\left\{(x, v / 3+i),(x, v / 2+i),\left(d_{x, i}, r_{x, i}\right)\right\}$, where $\left(d_{x, i}, r_{x, i}\right) \neq(x, i)$ if $x \in A$, and $\left(d_{x, i}, r_{x, i}\right) \neq(x, 5 v / 6+i)$ if $x \in B$. Let $\mathcal{L}_{4}=\{\{(x, v / 3+$ $\left.\left.i),(x, v / 2+i),\left(d_{x, i}, r_{x, i}\right)\right\}:\{(x, v / 3+i),(x, v / 2+i)\} \subset T \in \mathcal{L}_{2} \cap \mathcal{L}_{3}\right\}$. Then $\mathcal{L}_{4} \subset \mathcal{L}$ and $\mathcal{L}_{4} \cap\left(\mathcal{L}_{2} \cup \mathcal{L}_{3}\right)=\emptyset$. Since $\left|\mathcal{L}_{4}\right|=\left|\mathcal{L}_{2} \cap \mathcal{L}_{3}\right|$, we have $|\mathcal{L}| \geq\left|\mathcal{L}_{1}\right|+\left|\mathcal{L}_{2}\right|+\left|\mathcal{L}_{3} \backslash\left(\mathcal{L}_{2} \cap \mathcal{L}_{3}\right)\right|+\left|\mathcal{L}_{4}\right|=$ $7 u v / 3$.

Thus there are at least $7 u v / 33$-subsets in the leave. It implies that $\Phi(u \times v, 4,2) \leq$ $\left\lfloor\left(\binom{u v}{3}-\frac{7}{3} u v\right) /(4 v)\right\rfloor=\left\lfloor\frac{1}{24} u\left(u^{2} v^{2}-3 u v-12\right)\right\rfloor$. It is readily checked that $\left\lfloor\frac{u}{4}\left(\left\lfloor\frac{u v-1}{3}\left\lfloor\frac{u v-2}{2}\right\rfloor\right\rfloor-\right.\right.$ 2) $\rfloor=\left\lfloor\frac{1}{24} u\left(u^{2} v^{2}-3 u v-12\right)\right\rfloor$.

## Appendix III

Proof of Construction 7.1: For checking the correctness of the algorithm shown in Figure 2, it suffices to show that: (1) the resulting design is strictly $h_{1} h_{2}$-cyclic; (2) any

3-subset $S$ of $X^{\prime}$ satisfying that $\left|S \cap G^{\prime}\right|<3$ for each $G^{\prime} \in \mathcal{G}^{\prime}$ is contained in a unique block of the resulting design; (3) any 2-subset $R$ of $X^{\prime}$ satisfying that $\left|R \cap G^{\prime}\right|<2$ for each $G^{\prime} \in \mathcal{G}^{\prime}$ is contained in a unique block of $\mathcal{A}_{i}^{\prime}$ for each $1 \leq i \leq s$.

For convenience assume that $\mathcal{A}_{B}=\bigcup_{j=1}^{s} \mathcal{A}_{B}^{j}$ for each $B \in \mathcal{F}_{1}$.
(1) Suppose that $A=\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right): 1 \leq l \leq r\right\}$ is a base block of the resulting design, where $x_{l} \in I_{n}, y_{l} \in I_{g_{1}}, u_{l} \in I_{g_{2}}, z_{l} \in Z_{h_{1}}, v_{l} \in Z_{h_{2}}$. We need to show that the stabilizer of $A$ is trivial, i.e., $A+\delta=A$ if and only if $\delta \equiv 0\left(\bmod h_{1} h_{2}\right)$. The sufficiency follows immediately, so we consider the necessity. Assume that $\delta=\delta_{1}+\delta_{2} h_{1}$, $\delta_{1} \in Z_{h_{1}}, \delta_{2} \in Z_{h_{2}}$. If $A+\delta=A$, we have
$\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right): 1 \leq l \leq r\right\}=\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+\delta_{1}+\left(v_{l}+\delta_{2}\right) h_{1}\right): 1 \leq l \leq r\right\}$,
where the arithmetic is modulo $\left(-,-, h_{1} h_{2}\right)$. It follows that

$$
\left\{\left(x_{l}, y_{l}, z_{l}\right): 1 \leq l \leq r\right\}=\left\{\left(x_{l}, y_{l}, z_{l}+\delta_{1}\right): 1 \leq l \leq r\right\},
$$

where the arithmetic is modulo $\left(-,-, h_{1}\right)$. Let $U=\left\{\left(x_{l}, y_{l}, z_{l}\right): 1 \leq l \leq r\right\}$.
If $A \in \mathcal{A}_{j}^{\prime}, 1 \leq j \leq s$, then $|U|=r \geq 2$. Since the subdesign $(X, \mathcal{G}, \mathcal{B})$ of the master design 1-FG $\left(3,\left(K, K_{T}\right), n g_{1} h_{1}\right)$ of type $\left(g_{1} h_{1}\right)^{n}(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ is strictly $h_{1}$-cyclic and it requires that any 2 -subset of $X$ which intersects each group of $\mathcal{G}$ in at most one point occurs in exactly one block, we have $\delta_{1}=0$.

If $A \in \mathcal{D}^{\prime}$, without loss of generality assume that $A \in \mathcal{D}^{*}$. If $A=\tau(C)$ for some $C \in \bigcup_{B \in \mathcal{F}_{2}} \mathcal{D}_{B}^{\prime}$, then $|U|=r \geq 3$. Since the master design 1-FG(3, $\left.\left(K, K_{T}\right), n g_{1} h_{1}\right)$ of type $\left(g_{1} h_{1}\right)^{n}$ is strictly $h_{1}$-cyclic and it requires that any 3 -subset of $X$ which intersects each group of $\mathcal{G}$ in at most two points occurs in exactly one block, we have $\delta_{1}=0$. If $A=\tau(C)$ for some $C \in \bigcup_{B \in \mathcal{F}_{1}} \mathcal{D}_{B}$, then $|U| \geq 2$. Note that in this case $U$ may be a multiset, i.e., $|U|$ may be not equal to $r$. By similar arguments to those in the case of $A \in \mathcal{A}_{j}^{\prime}$, we have $\delta_{1}=0$.

Hence,

$$
\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right): 1 \leq l \leq r\right\}=\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+\left(v_{l}+\delta_{2}\right) h_{1}\right): 1 \leq l \leq r\right\}
$$

where the arithmetic is modulo $\left(-,-, h_{1} h_{2}\right)$. It follows that

$$
\left\{\left(x_{l}, y_{l}, z_{l}, u_{l}, v_{l}\right): 1 \leq l \leq r\right\}=\left\{\left(x_{l}, y_{l}, z_{l}, u_{l}, v_{l}+\delta_{2}\right): 1 \leq l \leq r\right\},
$$

where the arithmetic is modulo $\left(-,-,-,-, h_{2}\right)$. Since the input designs are all strictly $h_{2}$-cyclic, we have $\delta_{2}=0$. Thus the resulting design is strictly $h_{1} h_{2}$-cyclic.
(2) Take any triple $S=\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right): 1 \leq l \leq 3\right\} \subset X^{\prime}$, where $x_{l} \in I_{n}$, $y_{l} \in I_{g_{1}}, u_{l} \in I_{g_{2}}, z_{l} \in Z_{h_{1}}, v_{l} \in Z_{h_{2}}$ and $x_{1}, x_{2}, x_{3}$ are not equal at the same time.

Case 1. Suppose that $x_{1}, x_{2}, x_{3}$ are pairwise distinct. Then there exist a unique base block $F$ in $\mathcal{F}$ and a unique element $\delta_{1} \in Z_{h_{1}}$, such that $\left\{\left(x_{l}, y_{l}, z_{l}^{*}\right): 1 \leq l \leq 3\right\} \subseteq F$ and $\left(x_{l}, y_{l}, z_{l}^{*}\right)+\delta_{1}=\left(x_{l}, y_{l}, z_{l}^{*}+\delta_{1}\right)=\left(x_{l}, y_{l}, z_{l}\right), 1 \leq l \leq 3$, where the arithmetic is modulo $\left(-,-, h_{1}\right)$. It follows that $\left(x_{l}, y_{l}, z_{l}\right)-\delta_{1}=\left(x_{l}, y_{l}, z_{l}^{*}\right)$.

If $F \in \mathcal{F}_{1}$, then there exist a unique base block $B \in \mathcal{A}_{F} \bigcup \mathcal{D}_{F}$ and a unique element $\delta_{2} \in Z_{h_{2}}$, such that $\left\{\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}^{*}\right): 1 \leq l \leq 3\right\} \subseteq B$ and $\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}^{*}\right)+\delta_{2}=$ $\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}^{*}+\delta_{2}\right)=\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}\right), 1 \leq l \leq 3$, where the arithmetic is modulo $\left(-,-,-,-, h_{2}\right)$. By the mapping $\tau$, we have that $\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}^{*}+\left(v_{l}^{*}+\delta_{2}\right) h_{1}\right)=\left(x_{l}, y_{l}+\right.$ $\left.u_{l} g_{1}, z_{l}^{*}+v_{l} h_{1}\right)=\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}-\delta_{1}+v_{l} h_{1}\right)$, where the arithmetic is modulo $\left(-,-, h_{1} h_{2}\right)$.

Let $\delta=\delta_{1}+\delta_{2} h_{1}$. It follows that $\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}^{*}+v_{l}^{*} h_{1}+\delta\right)=\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right)$. By (1) the resulting design is strictly $h_{1} h_{2}$-cyclic, so $\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right): 1 \leq l \leq 3\right\}$ is contained in the unique block $\tau(B)+\delta$, which is generated by $\tau(B)$. Similar arguments hold for $F \in \mathcal{F}_{2}$, where $B \in \mathcal{D}_{F}^{\prime}$.

Case 2. Suppose that $x_{1}=x_{2}, x_{1} \neq x_{3}$, and $\left(y_{1}, z_{1}\right) \neq\left(y_{2}, z_{2}\right)$. Then there exist a unique base block $F$ in $\mathcal{F}_{2}$ and a unique element $\delta_{1} \in Z_{h_{1}}$, such that $\left\{\left(x_{l}, y_{l}, z_{l}^{*}\right): 1 \leq\right.$ $l \leq 3\} \subseteq F$ and $\left(x_{l}, y_{l}, z_{l}^{*}+\delta_{1}\right)=\left(x_{l}, y_{l}, z_{l}\right), 1 \leq l \leq 3$, where the arithmetic is modulo $\left(-,-, h_{1}\right)$. There exist a unique base block $B \in \mathcal{D}_{F}^{\prime}$ and a unique element $\delta_{2} \in Z_{h_{2}}$, such that $\left\{\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}^{*}\right): 1 \leq l \leq 3\right\} \subseteq B$ and $\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}^{*}+\delta_{2}\right)=\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}\right)$, $1 \leq l \leq 3$, where the arithmetic is modulo $\left(-,-,-,-, h_{2}\right)$. By similar arguments to those in Case 1, $\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right): 1 \leq l \leq 3\right\}$ is contained in the unique block $\tau(B)+\delta$, where $\delta=\delta_{1}+\delta_{2} h_{1}$.

Case 3. Suppose that $\left(x_{1}, y_{1}, z_{1}\right)=\left(x_{2}, y_{2}, z_{2}\right),\left(u_{1}, v_{1}\right) \neq\left(u_{2}, v_{2}\right)$ and $x_{1} \neq x_{3}$. Then there exist a unique base block $F$ in $\mathcal{F}_{1}$ and a unique element $\delta_{1} \in Z_{h_{1}}$, such that $\left\{\left(x_{l}, y_{l}, z_{l}^{*}\right): 1 \leq l \leq 3\right\} \subseteq F$ and $\left(x_{l}, y_{l}, z_{l}^{*}+\delta_{1}\right)=\left(x_{l}, y_{l}, z_{l}\right), 1 \leq l \leq 3$, where the arithmetic is modulo $\left(-,-, h_{1}\right)$. There exist a unique base block $B \in \mathcal{D}_{F}$ and a unique element $\delta_{2} \in Z_{h_{2}}$, such that $\left\{\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}^{*}\right): 1 \leq l \leq 3\right\} \subseteq B$ and $\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}^{*}+\right.$ $\left.\delta_{2}\right)=\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}\right), 1 \leq l \leq 3$, where the arithmetic is modulo $\left(-,-,-,-, h_{2}\right)$. By similar arguments to those in Case $1,\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right): 1 \leq l \leq 3\right\}$ is contained in the unique block $\tau(B)+\delta$, where $\delta=\delta_{1}+\delta_{2} h_{1}$.
(3) Take any 2-subset $R=\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right): 1 \leq l \leq 2\right\} \subset X^{\prime}$, where $x_{l} \in I_{n}$, $y_{l} \in I_{g_{1}}, u_{l} \in I_{g_{2}}, z_{l} \in Z_{h_{1}}, v_{l} \in Z_{h_{2}}$ and $x_{1} \neq x_{2}$. Then there exist a unique base block $F$ in $\mathcal{F}_{1}$ and a unique element $\delta_{1} \in Z_{h_{1}}$, such that $\left\{\left(x_{l}, y_{l}, z_{l}^{*}\right): 1 \leq l \leq 2\right\} \subseteq F$ and $\left(x_{l}, y_{l}, z_{l}^{*}+\delta_{1}\right)=\left(x_{l}, y_{l}, z_{l}\right), 1 \leq l \leq 2$, where the arithmetic is modulo $\left(-,-, h_{1}\right)$.

Given any $1 \leq j \leq s$. There exist a unique base block $B$ in $\mathcal{A}_{j}$ and a unique element $\delta_{2} \in Z_{h_{2}}$, such that $\left\{\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}^{*}\right): 1 \leq l \leq 2\right\} \subseteq B$ and $\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}^{*}+\delta_{2}\right)=$ $\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}\right), 1 \leq l \leq 2$, where the arithmetic is modulo $\left(-,-,-,-, h_{2}\right)$. By the mapping $\tau$, we have that $\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}^{*}+\left(v_{l}^{*}+\delta_{2}\right) h_{1}\right)=\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}^{*}+v_{l} h_{1}\right)=$ $\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}-\delta_{1}+v_{l} h_{1}\right)$, where the arithmetic is modulo $\left(-,-, h_{1} h_{2}\right)$. Let $\delta=\delta_{1}+\delta_{2} h_{1}$. It follows that $\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}^{*}+v_{l}^{*} h_{1}+\delta\right)=\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right)$. By (1) the resulting design is strictly $h_{1} h_{2}$-cyclic, so $\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right): 1 \leq l \leq 2\right\}$ is contained in the unique block $\tau(B)+\delta$, which is generated by $\tau(B)$.

Proof of Construction 7.5: For checking the correctness of the algorithm shown in Figure 4, take any $t$-subset $T=\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right): 1 \leq l \leq t\right\} \subset X^{\prime}$, where $x_{l} \in I_{n}$, $y_{l} \in I_{g_{1}}, u_{l} \in I_{g_{2}}, z_{l} \in Z_{h_{1}}, v_{l} \in Z_{h_{2}}$ and $\left|\left\{x_{l}: 1 \leq l \leq t\right\}\right|=t$. Then there exist a unique base block $F$ in $\mathcal{F}$ and a unique element $\delta_{1} \in Z_{h_{1}}$, such that $\left\{\left(x_{l}, y_{l}, z_{l}^{*}\right): 1 \leq l \leq t\right\} \subseteq F$ and $\left(x_{l}, y_{l}, z_{l}^{*}+\delta_{1}\right)=\left(x_{l}, y_{l}, z_{l}\right), 1 \leq l \leq t$, where the arithmetic is modulo $\left(-,-, h_{1}\right)$. There exist a unique base block $B$ in $\mathcal{D}_{F}$ and a unique element $\delta_{2} \in Z_{h_{2}}$, such that $\left\{\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}^{*}\right): 1 \leq l \leq t\right\} \subseteq B$ and $\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}^{*}+\delta_{2}\right)=\left(x_{l}, y_{l}, z_{l}^{*}, u_{l}, v_{l}\right), 1 \leq$ $l \leq t$, where the arithmetic is modulo $\left(-,-,-,-, h_{2}\right)$. By the mapping $\tau$, we have that $\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}^{*}+\left(v_{l}^{*}+\delta_{2}\right) h_{1}\right)=\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}^{*}+v_{l} h_{1}\right)=\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}-\delta_{1}+v_{l} h_{1}\right)$, where the arithmetic is modulo $\left(-,-, h_{1} h_{2}\right)$. Let $\delta=\delta_{1}+\delta_{2} h_{1}$. It follows that ( $x_{l}, y_{l}+$ $\left.u_{l} g_{1}, z_{l}^{*}+v_{l}^{*} h_{1}+\delta\right)=\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right)$. Thus $\left\{\left(x_{l}, y_{l}+u_{l} g_{1}, z_{l}+v_{l} h_{1}\right): 1 \leq l \leq t\right\}$ is contained in the unique block $\tau(B)+\delta$, which is generated by $\tau(B)$.

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