# Beam Selection Gain Versus Antenna Selection Gain 

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#### Abstract

We consider beam selection using a fixed beamforming network (FBN) at a base station with $M$ array antennas. In our setting, a Butler matrix is deployed at the RF stage to form $M$ beams, and then the best beam is selected for transmission. We provide the proofs of the key properties of the noncentral chi-square distribution and the following properties of the beam selection gain verifying that beam selection is superior to antenna selection in Rician channels with any $K$-factors. Furthermore, we find asymptotically tight stochastic bounds of the beam selection gain, which yield approximate closed form expressions of the expected selection gain and the ergodic capacity. Beam selection has the order of growth of the ergodic capacity $\Theta(\log (M))$ regardless of user location in contrast to $\Theta(\log (\log (M)))$ for antenna selection.


## I. Introduction

Deploying multiple antennas at a base station dramatically increases spectral efficiency. While multiple-input/multiple-output (MIMO) systems require multiple RF chains and elaborate signal processing units, Antenna selection has been an attractive solution for multiple antenna systems because only one RF chain is required to use the antenna with the highest signal-to-noise ratio (SNR).

With promise of higher spectral efficiency, we focus on beam selection instead of antenna selection using a FBN at a base station which deploys $M$ multiple linear equally spaced omnidirectional array antennas when each remote unit is equipped with an omnidirectional antenna. While the base station can adaptively steer beams to remote users using $M$ RF chains, we investigate the Butler matrix, a simple FBN at the RF stage producing orthogonal beams and requiring only one RF chain for the best beam to be selected for transmission [1]. The choice of the best beam can be achieved with partial channel state information (CSI) at the base station. The remote feeds back the index of the best beam to the base station for the forward link.

Although beam selection has been known to have no advantage over antenna selection in ideal Rayleigh fading channels, it has been established (using analysis and simulations) that beam selection can outperform antenna selection in correlated Rayleigh fading channels with limited angle spread [2]. For the case of Rician fading channels, there exist only limited analytical results of two very special cases of Rayleigh fading channels and deterministic channels except our own work in [3] while simulations and measurements have shown that beam selection using the Butler FBN outperforms antenna selection [4].

Motivated by this, we have analyzed the performance of beam selection using the Butler FBN for Rician fading channels with arbitrary $K$-factors and derived the exact distribution of the beam selection gain as a function of the azimuthal location of the remote user in our previous work [3], where some key properties

[^0]

Fig. 1. Beam selection system using the Butler FBN with $M$ linear equally spaced array antennas and beam pattern for $M=4$ and $d=\lambda_{c} / 2$.
of the noncentral chi-square distribution and the following properties of the beam selection gain have been presented without any proofs. Using these properties, we have compared the beam selection gain with the antenna selection gain for Rician fading channels and analytically proved that beam selection outperforms antenna selection.

In this paper, we provide the proofs omitted in [3], which verify our claim that beam selection is superior to antenna selection regardless of user location in Rician channels with any $K$-factors. Moreover, we find asymptotically tight stochastic bounds of the beam selection gain yielding approximate outage and the approximate expression for average performance. This approximation technique can be applied for most of average performance measures as shown for the expected selection gain and the ergodic capacity. Using these results, we obtain orders of growth of the expected selection gain and the ergodic capacity for beam selection, proved to be higher than those for antenna selection.

The remainder of this paper is organized as follows: In Section II we present our system model when the Butler FBN is used in the base station. In Section III) we analyze the beam selection gain using a statistical approach. In Section IV] we compare the gain of beam selection with that of antenna selection, and prove that beam selection outperforms antenna selection under any Rician channel transmission model. In Section V we find stochastic bounds of the beam selection gain and approximate closed form expressions of performance measures. Finally, we provide our conclusions in Section (VI)

## II. The System Model

We consider a base station endowed with $M \geq 2$ antennas (as depicted in Fig. (1) and remote units each endowed with one antenna. For the $m$-th port of the Butler matrix ( $m \in\{1, \ldots, M\}$ ), the SNR equals to $\rho \cdot \Gamma_{m}$ regardless of the direction of the communication link [3], where $\rho$ is the average SNR per port and $\Gamma_{m}$ denotes the gain of selecting the $m$-th port. This gain is given by

$$
\begin{equation*}
\Gamma_{m}=\left|\boldsymbol{b}_{m}^{T} \boldsymbol{h}\right|^{2} \tag{1}
\end{equation*}
$$

where the $M \times 1$ complex vector $\boldsymbol{h}=\left[h_{1}, \ldots, h_{M}\right]^{T}$ represents the flat fading channel gains for corresponding antennas normalized such that $\mathbb{E}\left[\left|h_{i}\right|^{2}\right]=1$ for $i=1,2, \cdots, M$, and the $1 \times M$ complex vector
$\boldsymbol{b}_{m}^{T}$ is the $m$-th row of the $M \times M$ Butler matrix given by

$$
\begin{equation*}
\boldsymbol{b}_{m}^{T}=\frac{1}{\sqrt{M}}\left[e^{j \frac{2 \pi}{M}\left(m-\frac{1}{2}\right)}, e^{j \frac{2 \pi}{M}\left(m-\frac{1}{2}\right) 2}, \ldots, e^{j \frac{2 \pi}{M}\left(m-\frac{1}{2}\right) M}\right] \tag{2}
\end{equation*}
$$

Then, the base station chooses the port with the highest SNR. To select the best beam for the forward link, the remote user only needs to feed back the index of the best beam to the base station (even when the channel is not reciprocal) and this is the only difference between reverse and forward link beam selection. From this point on, we will not distinguish reverse and forward link beam selection in this paper as they are analytically identical. The SNR is then given by $\rho \cdot \Gamma_{(M)}$, where the notation $z_{(m)}$ is used to denote the $m$-th smallest value from any set of finite samples $\left\{z_{1}, \ldots, z_{M}\right\}$, and thus $\Gamma_{(M)}=\max _{m \in\{1, \ldots, M\}} \Gamma_{m}$.

We define the beam selection gain as the ratio of the SNR of beam selection with a FBN to the average SNR of random antenna switching without a FBN, which is given by $\Gamma_{(M)}$.

## III. Beam Selection Gains in Fading Channels

It has been shown that beam selection outperforms antenna selection in ideal line-of-sight (LOS) channels, while beam selection performs as good as antenna selection in ideal non-line-of-sight (NLOS) channels [4]. We are interested in investigating the performance of beam selection under Rician channel models. This is the most frequently used realistic channel model in wireless communications. Under the Rician channel model, the normalized channel vector $\boldsymbol{h}$ can be modeled as multipath signals

$$
\begin{equation*}
\boldsymbol{h}=\sqrt{\frac{K}{K+1}} \boldsymbol{h}_{L}+\sqrt{\frac{1}{K+1}} \boldsymbol{h}_{N} \tag{3}
\end{equation*}
$$

The entries of complex vector $\boldsymbol{h}_{L}$ (which represents the normalized LOS component) are modeled to have unit power and fixed phase. The entries of the complex vector $\boldsymbol{h}_{N}$ (which represents the normalized NLOS component) are modeled by i.i.d. independent zero-mean circularly symmetric complex Gaussian random variables with unit variance. The parameter $K$ is referred to as the Rician $K$-factor, which represents the ratio of the LOS signal power to the NLOS signal power. The special cases of $K=\infty$ and $K=0$ represent ideal LOS (deterministic) and ideal NLOS (Rayleigh fading) channels, respectively.

## A. Deterministic Components

Consider the LOS component $\boldsymbol{h}_{L}$. Let $\theta$ denote the azimuthal angle of incident between a LOS signal and the line perpendicular to the linear equally spaced array antennas assuming two-dimensional geometry (horizontal plane) as shown in Fig. 1 Furthermore, assume that the distance between the base station and the mobile user is much larger than array antenna separation. Then for both reverse and forward link beam selection, $\boldsymbol{h}_{L}$ is given by

$$
\begin{array}{r}
\boldsymbol{h}_{L}=\exp (j \psi)\left[1, \exp \left(-j 2 \pi \frac{d}{\lambda_{c}} \sin \theta\right), \ldots\right. \\
\left.\exp \left(-j 2 \pi(M-1) \frac{d}{\lambda_{c}} \sin \theta\right)\right]^{T} \tag{4}
\end{array}
$$

where $\psi$ is an arbitrary phase shift of the signal from/to the first array antenna, $d$ is the distance between adjacent array antennas, and $\lambda_{c}$ is the carrier wavelength.

Let the SNR gain of the $m$-th beam in ideal LOS channels $(K=\infty)$ be denoted by

$$
\begin{align*}
\gamma_{m} & \triangleq\left|\boldsymbol{b}_{m}^{T} \boldsymbol{h}_{L}\right|^{2} \\
& = \begin{cases}M, & \text { if } \phi_{m}=2 \pi n, \quad n \in \mathbb{Z} \\
\frac{1}{M} \frac{\sin ^{2}\left(M \phi_{m} / 2\right)}{\sin ^{2}\left(\phi_{m} / 2\right)}, & \text { otherwise }\end{cases} \tag{5}
\end{align*}
$$

where

$$
\begin{equation*}
\phi_{m} \triangleq 2 \pi\left[\frac{1}{M}\left(m-\frac{1}{2}\right)-\frac{d}{\lambda_{c}} \sin \theta\right] . \tag{6}
\end{equation*}
$$

Since $\boldsymbol{h}_{L}$ is a function of $\theta, \gamma_{m}$ is also a function of $\theta$ and let us call a set of $M$ functions $\left\{\gamma_{m} \mid m=\right.$ $1, \ldots, M\}$ a beam pattern, which has the following properties:

$$
\begin{gather*}
\sum_{m=1}^{M} \gamma_{m}=M, \quad 0 \leq \gamma_{m} \leq M  \tag{7}\\
\gamma_{m}=M \quad \text { if and only if } \quad \phi_{m}=2 \pi \frac{n}{M}, \quad \frac{n}{M} \in \mathbb{Z} ;  \tag{8}\\
\gamma_{m}=0 \quad \text { if and only if } \quad \phi_{m}=2 \pi \frac{n}{M}, \frac{n}{M} \notin \mathbb{Z} \tag{9}
\end{gather*}
$$

where the azimuthal angle satisfying (8) is the beam direction. Let us define a lobe of a beam as a main lobe if the beam direction is inside that lobe. We assume

$$
\begin{equation*}
\frac{M-1}{2 M}<\frac{d}{\lambda_{c}}, \tag{10}
\end{equation*}
$$

for all $M$ beams to have at least one main lobe. We examine the beam pattern only from $\theta=0$ to the first beam direction given by

$$
\begin{equation*}
\theta=\nu \triangleq \arcsin \left(\frac{1}{2 M} \frac{\lambda_{c}}{d}\right) \tag{11}
\end{equation*}
$$

as discussed in [3].

## B. Probabilistic Analysis

Now, let us consider the statistical channel model including NLOS components. The cumulative distribution function (cdf) of $\Gamma_{m}$ is given by [3]

$$
\begin{align*}
F_{m}(x) & \triangleq \operatorname{Pr}\left\{\Gamma_{m} \leq x\right\} \\
& =\left.F_{\chi^{\prime 2}}(2(K+1) x \mid n, \delta)\right|_{n=2, \delta=2 K \gamma_{m}} \\
& =\left.\mathbb{E}\left[F_{\chi^{2}}\left(2(K+1) x \mid n+2 P_{\delta / 2}\right)\right]\right|_{n=2, \delta=2 K \gamma_{m}}, \tag{12}
\end{align*}
$$

where $F_{\chi^{\prime 2}}(x \mid n, \delta)$ is the noncentral chi-square cdf with $n$ degrees of freedom and the noncentrality parameter $\delta, P_{\delta / 2}$ is a Poisson random variable with mean $\delta / 2$, and $F_{\chi^{2}}(x \mid q)$ is the chi-square cdf with $q$ degrees of freedom, given by

$$
\begin{equation*}
F_{\chi^{2}}(x \mid q)=1-e^{-x / 2} \sum_{k=0}^{q / 2-1} \frac{(x / 2)^{k}}{k!}=e^{-x / 2} \sum_{k=q / 2}^{\infty} \frac{(x / 2)^{k}}{k!} \tag{13}
\end{equation*}
$$

if $q$ is an even number as in (12) where $q=n+\left.2 P_{\delta / 2}\right|_{n=2}$. Note that given $K$, evaluating $\gamma_{m}$ is enough to know the distribution of the SNR gain $\Gamma_{m}$. The beam selection gain $\Gamma_{(M)}$ is given by

$$
\begin{equation*}
F_{(M)}(x) \triangleq \operatorname{Pr}\left\{\Gamma_{(M)} \leq x\right\}=\prod_{m=1}^{M} F_{m}(x) \tag{14}
\end{equation*}
$$

and thus for $x>0$,

$$
\begin{equation*}
\log F_{(M)}(x)=\sum_{m=1}^{M} \log F_{m}(x) . \tag{15}
\end{equation*}
$$

We have the following useful key theorem on the noncentral chi-square distribution, whose proof can be found in the Appendix.

Theorem 1: The logarithm of the noncentral chi-square cdf with two degrees of freedom

$$
\begin{equation*}
\log F_{\chi^{\prime 2}}(x \mid 2, \delta) \tag{16}
\end{equation*}
$$

is a strictly decreasing and strictly concave function of the noncentrality parameter $\delta \geq 0$ for any given $x>0$ assuming that the base of logarithm is greater than one.

Now, we are ready to show the following theorem, where stochastic order relations are introduced in [5] Ch. 9].

Theorem 2: For any given $x>0, F_{(M)}(x)$, the cdf of the beam selection gain $\Gamma_{(M)}$, is a strictly decreasing function of $\theta$ from zero to the first beam direction $\nu=\arcsin \left(\frac{1}{2 M} \frac{\lambda_{c}}{d}\right)$. Therefore, in this interval, $\Gamma_{(M)}$ is stochastically increasing, stochastically smallest at $\theta=0$, and stochastically largest at $\theta=\nu$.

Proof: This proof is given in the Appendix.
The corollary below follows naturally from Theorem 2
Corollary 3: For $\theta \in[-\pi / 2, \pi / 2]$ and any integer $|m| \leq \frac{M}{\lambda_{c} / d}, \Gamma_{(M)}$ is stochastically increasing as $\theta$ increases if

$$
\begin{align*}
\theta \in & {\left[\arcsin \left(\frac{m}{M} \frac{\lambda_{c}}{d}\right)\right.} \\
& \left.\arcsin \left(\min \left\{\frac{m+1 / 2}{M} \frac{\lambda_{c}}{d}, 1\right\}\right)\right], \tag{17}
\end{align*}
$$

and stochastically decreasing as $\theta$ increases if

$$
\begin{gather*}
\theta \in\left[\arcsin \left(\max \left\{\frac{m-1 / 2}{M} \frac{\lambda_{c}}{d},-1\right\}\right),\right. \\
\left.\arcsin \left(\frac{m}{M} \frac{\lambda_{c}}{d}\right)\right] \tag{18}
\end{gather*}
$$

It is exactly opposite for the other half of the horizontal plane, $\theta \in[\pi / 2,3 \pi / 2]$. Therefore, $\Gamma_{(M)}$ with $\theta=0$ and $\theta=\nu$ are achievable stochastic lower and upper bounds, respectively for $\Gamma_{(M)}$ with an arbitrary $\theta$.

Corollary 3 tells us that the expected performance measures over $\Gamma_{(M)}$ with $\theta=0$ and $\theta=\nu$ can serve as lower and upper bounds, respectively, for the averages of any performance measures which are increasing functions of SNR, e.g., the channel capacity. They can also serve as upper and lower bounds, respectively, for the averages of any performance measures which are decreasing functions of SNR, e.g., the bit error rate (BER), applying the result in [5] pp. 405-406].

## IV. Beam Selection Versus Antenna Selection

Let us consider the antenna selection gain under the same scenario used for beam selection case except the fact that the Butler FBN will not be deployed for antenna selection. When the $m$-th antenna is selected among $M$ antennas in the base station, the SNR is given by $\rho \cdot H_{m}$, where $H_{m} \triangleq\left|h_{m}\right|^{2}$. Assuming that the antenna with the highest SNR is always selected, the antenna selection gain is defined as the ratio of the SNR of antenna selection to the average SNR of random antenna switching, which can be expressed by $H_{(M)}$. For any $m$, the cdf of $H_{m}$ becomes

$$
\begin{equation*}
G(x) \triangleq \operatorname{Pr}\left\{H_{m} \leq x\right\}=F_{\chi^{\prime 2}}(2(K+1) x \mid 2,2 K) . \tag{19}
\end{equation*}
$$

Therefore, the cdf of $H_{(M)}$ is given by

$$
\begin{equation*}
G_{(M)}(x) \triangleq \operatorname{Pr}\left\{H_{(M)} \leq x\right\}=G^{M}(x) . \tag{20}
\end{equation*}
$$

With the proofs of previous theorems, we can confirm that the following lemma holds.
Lemma 1: For the same Rician $K$-factor, beam selection always outperforms antenna selection, i.e., the beam selection gain $\Gamma_{(M)}$ is stochastically larger than the antenna selection gain $H_{(M)}$.

Proof: Applying the concavity result in Theorem 1 and Jensen's inequality gives us

$$
\begin{align*}
& \log G_{(M)}(x)=M \log F_{\chi^{\prime 2}}(2(K+1) x \mid 2,2 K) \\
& \geq \sum_{m=1}^{M} \log F_{\chi^{\prime 2}}\left(2(K+1) x \mid 2,2 K \gamma_{m}\right) \\
& =\log F_{(M)}(x), \tag{21}
\end{align*}
$$

for any given $x>0$.

## V. Asymptotic Selection Gains

It has been shown that the beam selection gain is stochastically upper and lower bounded by $\Gamma_{(M)}$ with $\theta$ of zero and the first beam direction $\nu=\arcsin \left(\frac{1}{2 M} \frac{\lambda_{c}}{d}\right)$, respectively. Our interest in this section is to see how these two extremes change as the number of antennas $M$ increases and then obtain the asymptotic selection gain for an arbitrary location of the remote user. Furthermore, these analytical results can be applied to study the outage and the ergodic capacity of beam selection systems. For this purpose, consider the SNR gain $\Gamma_{m}(\theta)$ and its $\operatorname{cdf} F_{m}(\cdot \mid \theta)$ as functions of the azimuthal angle $\theta$.

## A. Bounds and Approximations

First, we can obtain the stochastic lower bound for the beam selection gain of the user at the beam direction $\Gamma_{(M)}(\nu)$ given by

$$
\begin{align*}
& F_{(M)}(x \mid \nu)=\prod_{m=1}^{M} F_{\chi^{\prime 2}}\left(2(K+1) x \mid 2,2 K \gamma_{m}(\nu)\right) \\
& \quad=F_{\chi^{\prime 2}}(2(K+1) x \mid 2,2 K M) \cdot F_{\chi^{2}}^{M-1}(2(K+1) x \mid 2) \\
& \quad=Q_{M}(x) W^{M-1}(x) \\
& \quad \leq Q_{M}(x) \tag{22}
\end{align*}
$$

where $Q$ and $W$ are defined by

$$
\begin{align*}
Q_{\gamma}(x) & \triangleq F_{\chi^{\prime 2}}(2(K+1) x \mid 2,2 K \gamma),  \tag{23}\\
W(x) & \triangleq F_{\chi^{2}}(2(K+1) x \mid 2) . \tag{24}
\end{align*}
$$

Fig. 2 shows $F_{(M)}(x \mid \nu)$ and its stochastic lower bound $Q_{M}$. It can be seen that the lower bound $Q_{M}$ approaches to the cdf $F_{(M)}(x \mid \nu)$ as $M$ increases, which will be proved.

Now, consider the beam selection gain of the user exactly between beams $\Gamma_{(M)}(0)$ and its cdf given by

$$
\begin{equation*}
F_{(M)}(x \mid 0)=\prod_{m=1}^{M} F_{\chi^{\prime 2}}\left(2(K+1) x \mid 2,2 K \gamma_{m}(0)\right) \tag{25}
\end{equation*}
$$



Fig. 2. Distributions of the beam selection gain $\Gamma_{(M)}(\nu)$ and its stochastic lower bound for $K=$ $-10,0,10 \mathrm{~dB}$, where $d=\lambda_{c} / 2$ is assumed.

Let us choose a vector $\boldsymbol{u}=\left[u_{(1)}, \ldots, u_{(M)}\right]$ which majorizes the beam pattern $\left\{\gamma_{m}(0) \mid m=1, \ldots, M\right\}$ as

$$
\begin{align*}
& \gamma_{1}(0)=\gamma_{M}(0)=u_{(M)}=u_{(M-1)}=\frac{1}{M \sin ^{2}(\pi / 2 M)} \triangleq a_{M} \\
& >u_{(M-2)}=u_{(M-3)}=\frac{M}{2}-\frac{1}{M \sin ^{2}(\pi / 2 M)} \triangleq b_{M}>\gamma_{2}(0)=\gamma_{M-1}(0) \\
& >\gamma_{3}(0), \ldots, \gamma_{M-2}(0)>u_{(M-4)}=\ldots=u_{(1)}=0, \tag{26}
\end{align*}
$$

where majorization is introduced in [6, p. 45].

Notation: For any two real-valued sequences $c_{M}$ and $d_{M}$, we define

$$
\begin{array}{ll}
c_{M} \approx d_{M} & \text { if and only if } \quad \lim _{M \rightarrow \infty}\left|c_{M}-d_{M}\right|=0 \\
c_{M} \sim d_{M} & \text { if and only if } \quad \lim _{M \rightarrow \infty} c_{M} / d_{M}=1 \\
c_{M}=\Theta\left(d_{M}\right) & \text { if and only if } \quad 0<\lim _{M \rightarrow \infty} c_{M} / d_{M}<\infty
\end{array}
$$

Using this notation, we can see

$$
\begin{align*}
a_{M} & \sim \frac{4}{\pi^{2}} M=(0.4053 \ldots) \times M  \tag{27}\\
b_{M} & \sim\left(\frac{1}{2}-\frac{4}{\pi^{2}}\right) M=(0.0947 \ldots) \times M \tag{28}
\end{align*}
$$

Applying Hardy-Littlewood-Pólyas theorem in [6, pp. 88-91] and the strict concavity of (16) to (25) yields the stochastic upper bound

$$
\begin{align*}
& F_{(M)}(x \mid 0) \geq \prod_{m=1}^{M} F_{\chi^{\prime 2}}\left(2(K+1) x \mid 2,2 K u_{(m)}\right) \\
& \quad=F_{\chi^{\prime 2}}^{2}\left(2(K+1) x \mid 2,2 K a_{M}\right) \cdot F_{\chi^{\prime 2}}^{2}\left(2(K+1) x \mid 2,2 K b_{M}\right) \cdot F_{\chi^{2}}^{M-4}(2(K+1) x \mid 2) \\
& \quad=Q_{a_{M}}^{2}(x) \cdot Q_{b_{M}}^{2}(x) \cdot W^{M-4}(x) \tag{29}
\end{align*}
$$

Thus, we have the stochastic lower and upper bound for $F_{(M)}(x \mid 0)$ given by

$$
\begin{equation*}
Q_{a_{M}}^{2}(x) \geq F_{(M)}(x \mid 0) \geq Q_{a_{M}}^{2}(x) \cdot Q_{b_{M}}^{2}(x) \cdot W^{M-4}(x) \tag{30}
\end{equation*}
$$

Fig. 3 shows $F_{(M)}(x \mid 0)$ and its stochastic lower bound $Q_{a_{M}}^{2}$ and upper bound $Q_{a_{M}}^{2} Q_{b_{M}}^{2} W^{M-4}$. We also observe that as the lower and upper bounds are merged into each other, so does $F_{(M)}(x \mid 0)$ as $M$ increases.

The following theorem verifies that the stochastic lower bounds in (22) and (30) are indeed asymptotically tight.

Theorem 4: For $K>0$ and $p \in[0,1)$,

$$
\begin{equation*}
F_{(M)}^{-1}(p \mid \nu) \approx Q_{M}^{-1}(p) \tag{31}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{(M)}^{-1}(p \mid 0) \approx Q_{a_{M}}^{-1}(\sqrt{p}) \tag{32}
\end{equation*}
$$

as $M$ increases.
Proof: This proof is given in the Appendix.
We also have the following theorem useful for average performance evaluation, whose proof can be found in the Appendix.

Theorem 5: Let $h$ be any differentiable function defined on $[0, \infty)$ such that $h^{\prime}$ is bounded. If $h$ is integrable with respect to $Q_{M}$, then

$$
\begin{equation*}
\int_{0}^{\infty} h(x) d F_{(M)}(x \mid \nu) \approx \int_{0}^{\infty} h(x) d Q_{M}(x) \tag{33}
\end{equation*}
$$

as $M$ increases. If $h$ is integrable with respect to $Q_{a_{M}}^{2}$, then

$$
\begin{equation*}
\int_{0}^{\infty} h(x) d F_{(M)}(x \mid 0) \approx \int_{0}^{\infty} h(x) d Q_{a_{M}}^{2}(x) \tag{34}
\end{equation*}
$$

as $M$ increases.


Fig. 3. Distributions of the beam selection gain $\Gamma_{(M)}(0)$ and its stochastic lower and upper bounds for $K=-10,0,10 \mathrm{~dB}$, where $d=\lambda_{c} / 2$ is assumed.

Theorems 4 and 5 in this subsection demonstrate that for large $M$, the distributions of the beam selection gain of the user at the beam direction $\Gamma_{(M)}(\nu)$ and the beam selection gain of the user exactly between beams $\Gamma_{(M)}(0)$ can be well approximated by $Q_{M}(x)$ and $Q_{a_{M}}^{2}(x)$, respectively, which are the noncentral chi-square distribution and its square. These are useful as their closed-form expressions are complicated and thus not insightful.

## B. Performance Analysis

It can be seen that outage probabilities with $\theta=0$ and $\theta=\nu$ for a given rate $C_{0}$ can be approximated by

$$
\begin{align*}
P_{\text {out }}\left(C_{0}\right) & \triangleq \operatorname{Pr}\left\{\log _{2}\left(1+\rho \Gamma_{(M)}(\theta)\right) \leq C_{0}\right\} \\
& \approx \begin{cases}Q_{M}\left(\frac{2^{C_{0}-1}}{\rho}\right), & \text { if } \theta=\nu, \\
Q_{a_{M}}^{2}\left(\frac{2^{C_{0}}-1}{\rho}\right), & \text { if } \theta=0,\end{cases} \tag{35}
\end{align*}
$$

for large $M$. Furthermore, Theorem 4 can be used to approximate outage capacities with $\theta=0$ and $\theta=\nu$ as

$$
\begin{align*}
C_{\text {out }}\left(P_{0}\right) & \triangleq P_{\text {out }}^{-1}\left(P_{0}\right)=\log _{2}\left[1+\rho F_{(M)}^{-1}\left(P_{0} \mid \theta\right)\right] \\
& \approx \begin{cases}\log _{2}\left[1+\rho Q_{M}^{-1}\left(P_{0}\right)\right], & \text { if } \theta=\nu, \\
\log _{2}\left[1+\rho Q_{a_{M}}^{-1}\left(\sqrt{P_{0}}\right)\right], & \text { if } \theta=0\end{cases} \tag{36}
\end{align*}
$$

for large $M$.
Let us apply Theorem 5 to the mean selection gain $\mathbb{E}\left[\Gamma_{(M)}\right]$ by taking $h(x)=x$. The expected beam selection gain for $\theta=\nu$ is given by

$$
\begin{equation*}
\mathbb{E}\left[\Gamma_{(M)}(\nu)\right] \approx \int_{0}^{\infty} x d Q_{M}(x)=\frac{K M+1}{K+1}=\Theta(M) \tag{37}
\end{equation*}
$$

The expected beam selection gain for $\theta=0$ is given by

$$
\begin{equation*}
\mathbb{E}\left[\Gamma_{(M)}(0)\right] \approx \int_{0}^{\infty} x d Q_{a_{M}}^{2}(x) \tag{38}
\end{equation*}
$$

as $M$ increases. Although it seems difficult to solve the integration in (38), we can obtain upper and lower bounds using an inequality in [7] p. 62] because $Q_{a_{M}}^{2}$ is the cdf of the maximum of two samples from $Q_{a_{M}}$, whose mean and variance are $\left(K a_{M}+1\right) /(K+1)$ and $\left(2 K a_{M}+1\right) /(K+1)^{2}$, respectively. These bounds are given by

$$
\begin{equation*}
\frac{K a_{M}+1}{K+1} \leq \int_{0}^{\infty} x d Q_{a_{M}}^{2}(x) \leq \frac{K a_{M}+1}{K+1}+\frac{1}{\sqrt{3}} \frac{\sqrt{2 K a_{M}+1}}{K+1} \tag{39}
\end{equation*}
$$

which yields

$$
\begin{equation*}
\mathbb{E}\left[\Gamma_{(M)}(0)\right] \approx \int_{0}^{\infty} x d Q_{a_{M}}^{2}(x) \sim \frac{K a_{M}+1}{K+1}=\Theta(M) \tag{40}
\end{equation*}
$$

Hence, $\mathbb{E}\left[\Gamma_{(M)}\right]=\Theta(M)$ regardless of user location, which is faster than $\Theta(\log M)$ for antenna selection [8].

Lemma 2: Let $\rho>0$ denote SNR. As $M$ increases, the ergodic capacity of the user at the beam direction $(\theta=\nu)$ is given by

$$
\begin{equation*}
\mathbb{E}\left[\log _{2}\left(1+\rho \Gamma_{(M)}(\nu)\right)\right] \approx \log _{2}\left(1+\rho \frac{K M+1}{K+1}\right) \tag{41}
\end{equation*}
$$

and the ergodic capacity of the user exactly between beams $(\theta=0)$ is given by

$$
\begin{equation*}
\mathbb{E}\left[\log _{2}\left(1+\rho \Gamma_{(M)}(0)\right)\right] \approx \log _{2}\left(1+\rho \frac{K a_{M}+1}{K+1}\right) \tag{42}
\end{equation*}
$$

Proof: This proof is given in the Appendix.


Fig. 4. Ergodic capacity versus $M$ for $K=0 \mathrm{~dB}$ at $\rho=5 \mathrm{~dB}$, where $d=\lambda_{c} / 2$ is assumed.

This lemma also yields the order of growth of the ergodic capacity $\mathbb{E}\left[\log _{2}\left(1+\rho \Gamma_{(M)}\right)\right] \approx \Theta(\log (M))$ regardless of user location, which is faster than $\Theta(\log (\log (M)))$ for antenna selection [8]. Fig. 4 shows the ergodic capacity and its approximations in (41) and (42) for SNR $\rho=5 \mathrm{~dB}$. We see that the approximations approach the numerically integrated exact values as $M$ increases.

## VI. Conclusion

We considered beam selection using the Butler FBN at the base station with multiple linear equally spaced omnidirectional array antennas. Completing the analysis of the beam selection gain, we provided the proofs of the key properties verifying that beam selection is superior to antenna selection in Rician channels with any $K$-factors. We also found asymptotically tight stochastic bounds of the beam selection gain and approximate closed form expressions of the expected selection gain and the ergodic capacity. Using these results, it was shown that beam selection has higher order of growth of the ergodic capacity than antenna selection. Graphical results were provided demonstrating the underlying gains and supporting our approximations.

## APPENDIX

Proof of Theorem [7. Without loss of generality, assume the natural logarithm. For any given $x>0$, (16) can be expressed as

$$
\begin{align*}
\log F_{\chi^{\prime 2}}(x \mid 2, \delta) & =\log \left[\sum_{i=0}^{\infty} \frac{e^{-\delta / 2}(\delta / 2)^{i}}{i!} \alpha_{i}\right] \\
& =-\frac{\delta}{2}+\log \left[\sum_{i=0}^{\infty} \frac{(\delta / 2)^{i}}{i!} \alpha_{i}\right], \tag{43}
\end{align*}
$$

where $\alpha_{i}$ is defined as

$$
\begin{equation*}
\alpha_{i} \triangleq F_{\chi^{2}}(x \mid 2+2 i)=e^{-x / 2} \sum_{k=i+1}^{\infty} \frac{(x / 2)^{k}}{k!} \tag{44}
\end{equation*}
$$

from (13). Differentiating (43) gives us

$$
\begin{gather*}
\frac{\partial}{\partial \delta} \log F_{\chi^{\prime 2}}(x \mid 2, \delta)=-\frac{1}{2}+\frac{1}{2} \cdot \frac{\sum_{i=0}^{\infty} \frac{(\delta / 2)^{i}}{i!} \alpha_{i+1}}{\sum_{i=0}^{\infty} \frac{(\delta / 2)^{i}}{i!} \alpha_{i}} \\
=\frac{\sum_{i=0}^{\infty} \frac{(\delta / 2)^{i}}{i!}\left(\alpha_{i+1}-\alpha_{i}\right)}{2 \sum_{i=0}^{\infty} \frac{(\delta / 2)^{i}}{i!} \alpha_{i}}<0 \tag{45}
\end{gather*}
$$

for $\delta>0$ because $\alpha_{i+1}<\alpha_{i}$ from (44), and thus (16) is a strictly decreasing function of $\delta \geq 0$.
Now, prove that (16) is a strictly concave function of $\delta \geq 0$. The second derivative of (43) is given by

$$
\begin{align*}
& \frac{\partial^{2}}{\partial \delta^{2}} \log F_{\chi^{\prime 2}}(x \mid 2, \delta) \\
= & \frac{\left(\sum_{i=0}^{\infty} \frac{(\delta / 2)^{i}}{i!} \alpha_{i}\right)\left(\sum_{i=0}^{\infty} \frac{(\delta / 2)^{i}}{i!} \alpha_{i+2}\right)-\left(\sum_{i=0}^{\infty} \frac{(\delta / 2)^{i}}{i!} \alpha_{i+1}\right)^{2}}{4\left(\sum_{i=0}^{\infty} \frac{(\delta / 2)^{i}}{i!} \alpha_{i}\right)^{2}} \tag{46}
\end{align*}
$$

the $i$-th order term of whose numerator can be simplified as

$$
\begin{equation*}
\frac{(\delta / 2)^{i}}{i!}\left(\alpha_{0} \alpha_{i+2}-\alpha_{1} \alpha_{i+1}\right) . \tag{47}
\end{equation*}
$$

Let us show that (46) is negative by proving that (47) is negative for $\delta>0$. Consider $\alpha_{i-1} / \alpha_{i}$, which is an increasing function of $i$ because

$$
\begin{align*}
\frac{\alpha_{i-1}}{\alpha_{i}}-1 & =\frac{\frac{(x / 2)^{i}}{i!}}{\sum_{k=i+1}^{\infty} \frac{(x / 2)^{k}}{k!}} \\
& =\frac{1}{\sum_{k=1}^{\infty} \frac{(x / 2)^{k}}{(i+k)!/ i!}} \tag{48}
\end{align*}
$$

and $(i+k)!/ i$ ! increases as $i$ increases for any positive integer $k$. Therefore,

$$
\begin{equation*}
\frac{\alpha_{0}}{\alpha_{1}}<\frac{\alpha_{1}}{\alpha_{2}}<\ldots<\frac{\alpha_{i+1}}{\alpha_{i+2}}<\ldots \tag{49}
\end{equation*}
$$

which yields the strict concavity of (16).
Proof of Theorem [2• Define

$$
\begin{equation*}
\beta \triangleq 2 \pi \frac{d}{\lambda_{c}} \sin \theta \tag{50}
\end{equation*}
$$

Under the condition (10), $\beta$ is an increasing and continuous function of $\theta$ and has the range $\left[0, \frac{\pi}{M}\right]$. Therefore, we only need to show that $F_{(m)}$ is a strictly decreasing function of $\beta$ in the domain $\left[0, \frac{\pi}{M}\right]$. From (6),

$$
\begin{equation*}
\phi_{m}(\beta) \triangleq \frac{2 \pi}{M}\left(m-\frac{1}{2}\right)-\beta=\phi_{m}(0)-\beta \tag{51}
\end{equation*}
$$

and by defining

$$
\eta(\phi)= \begin{cases}M & \text { if } \phi=2 \pi n, \quad n \in \mathbb{N}  \tag{52}\\ \frac{1}{M} \frac{\sin ^{2}(M \phi / 2)}{\sin ^{2}(\phi / 2)} & \text { otherwise },\end{cases}
$$

we can represent

$$
\begin{equation*}
\gamma_{m}(\beta)=\eta\left(\phi_{m}(\beta)\right) . \tag{53}
\end{equation*}
$$

Note that $\eta(\phi)$ is a periodic function with period $2 \pi$, symmetric with respect to the axis $\phi=\pi n$, and the value of $\eta$ at $\phi=2 \pi n$ makes $\eta(\phi)$ a continuous function of $\phi$.

First, prove that for $\beta \in\left[0, \frac{\pi}{M}\right]$ the beam pattern $\left\{\gamma_{m}\right\}$ can be sorted in nonincreasing order as follows:

$$
\begin{equation*}
\gamma_{1}(\beta) \geq \gamma_{M}(\beta) \geq \gamma_{2}(\beta) \geq \gamma_{M-1}(\beta) \geq \ldots \geq \gamma_{\left\lfloor\frac{M}{2}\right\rfloor+1}(\beta) \tag{54}
\end{equation*}
$$

where $\lfloor\cdot\rfloor$ is a floor function. It can be easily shown that

$$
\begin{equation*}
\gamma_{M+1-m}(\beta)=\eta\left(\phi_{m}(-\beta)\right)=\gamma_{m}(-\beta) . \tag{55}
\end{equation*}
$$

We get the following equivalent inequalities of (54)

$$
\begin{equation*}
\eta\left(\phi_{1}(\beta)\right) \geq \eta\left(\phi_{1}(-\beta)\right) \geq \ldots \geq \eta\left(\phi_{\left\lfloor\frac{M+1}{2}\right\rfloor}\left((-1)^{M-1} \beta\right)\right) . \tag{56}
\end{equation*}
$$

We can see that

$$
\begin{equation*}
\eta\left(\phi_{m}( \pm \beta)\right)=\frac{1}{M} \frac{\sin ^{2}\left(\frac{M}{2} \phi_{m}( \pm \beta)\right)}{\sin ^{2}\left(\frac{1}{2} \phi_{m}( \pm \beta)\right)}=\frac{1}{M} \frac{\cos ^{2}\left(\frac{M}{2} \beta\right)}{\sin ^{2}\left(\frac{1}{2} \phi_{m}( \pm \beta)\right)} \tag{57}
\end{equation*}
$$

and

$$
\begin{equation*}
0 \leq \phi_{1}(\beta) \leq \phi_{1}(-\beta) \leq \ldots \leq \phi_{\left\lfloor\frac{M+1}{2}\right\rfloor}\left((-1)^{M-1} \beta\right) \leq \pi \tag{58}
\end{equation*}
$$

which yields (56) because in (57), the numerator $\sin ^{2}\left(\frac{M}{2} \phi\right)$ has the same value at $\phi=\phi_{m}( \pm \beta)$ for any fixed $\beta$ and all $m$, and the denominator $\sin ^{2}\left(\frac{1}{2} \phi\right)$ is increasing function of $\phi \in[0, \pi]$. Define the nondecreasingly sorted vector $\gamma$ from $\left\{\gamma_{m}\right\}$ given by

$$
\begin{align*}
\gamma & \triangleq\left[\gamma_{(1)}, \gamma_{(2)}, \ldots, \gamma_{(M)}\right] \\
& =\left[\gamma_{\left\lfloor\frac{M}{2}\right\rfloor+1}, \ldots, \gamma_{M-1}, \gamma_{2}, \gamma_{M}, \gamma_{1}\right] \tag{59}
\end{align*}
$$

for $\beta \in\left[0, \frac{\pi}{M}\right]$. Let us show that $\gamma\left(\beta_{2}\right)$ strictly majorizes $\gamma\left(\beta_{1}\right)$ for $0 \leq \beta_{1}<\beta_{2} \leq \frac{\pi}{M}$, which means

$$
\begin{equation*}
\sum_{i=1}^{M} \gamma_{(i)}\left(\beta_{1}\right)=\sum_{i=1}^{M} \gamma_{(i)}\left(\beta_{2}\right) \tag{60}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{i=1}^{m} \gamma_{(i)}\left(\beta_{1}\right)>\sum_{i=1}^{m} \gamma_{(i)}\left(\beta_{2}\right) \tag{61}
\end{equation*}
$$

for all $m \in\{1, \ldots, M-1\}$. We already have (60) from (7), and thus it suffices to prove (61). Under the assumption that (61) is proved, using Hardy-Littlewood-Pólyas theorem in [6, pp. 88-91] based on the strict concavity of (16) proved in Theorem 1 gives us

$$
\begin{align*}
& \log F_{(M)}\left(x \mid K, \beta_{1}\right)=\sum_{m=1}^{M} \log F_{\chi^{\prime 2}}\left(2(K+1) x \mid 2,2 K \gamma_{m}\left(\beta_{1}\right)\right) \\
& \quad>\sum_{m=1}^{M} \log F_{\chi^{\prime 2}}\left(2(K+1) x \mid 2,2 K \gamma_{m}\left(\beta_{2}\right)\right)=\log F_{(M)}\left(x \mid K, \beta_{2}\right) \tag{62}
\end{align*}
$$

which basically shows that $F_{(m)}$ is a strictly decreasing function of $\beta$.
Let us prove that $\gamma_{1}(\beta)$ and $\gamma_{M}(\beta)$ are strictly increasing and strictly decreasing respectively. For $\phi \neq 2 \pi n$, it can be shown that

$$
\begin{equation*}
\eta^{\prime}(\phi)=\frac{1}{M} \frac{\sin ^{2}\left(\frac{M}{2} \phi\right)}{\sin ^{2}\left(\frac{1}{2} \phi\right)}\left[M \cot \left(\frac{M}{2} \phi\right)-\cot \left(\frac{1}{2} \phi\right)\right] . \tag{63}
\end{equation*}
$$

We can show $\eta^{\prime}(\phi)$ is negative for $0<\phi<\frac{2 \pi}{M}$ because by the Taylor series expansion,

$$
\begin{align*}
& M \cot \left(\frac{M}{2} \phi\right)-\cot \left(\frac{1}{2} \phi\right) \\
& \quad=M\left[\frac{2}{M \phi}-\sum_{i=1}^{\infty} \frac{2^{2 i}\left|B_{2 i}\right|}{(2 i)!}\left(\frac{M}{2} \phi\right)^{2 i-1}\right]-\left[\frac{2}{\phi}-\sum_{i=1}^{\infty} \frac{2^{2 i}\left|B_{2 i}\right|}{(2 i)!}\left(\frac{1}{2} \phi\right)^{2 i-1}\right] \\
& \quad=-\sum_{i=1}^{\infty} \frac{2^{2 i}\left|B_{2 i}\right|}{(2 i)!}\left(M^{2 i}-1\right)\left(\frac{\phi}{2}\right)^{2 i-1}<0 \tag{64}
\end{align*}
$$

where $B_{i}$ is the $i$-th Bernoulli number. Therefore, $\eta(\phi)$ is strictly decreasing in $\left[0, \frac{2 \pi}{M}\right]$, and thus $\eta(\phi)$ is strictly increasing in $\left[2 \pi \frac{M-1}{M}, 2 \pi\right]$ by the symmetry. Since

$$
\begin{equation*}
\gamma_{1}(\beta)=\eta\left(\frac{\pi}{M}-\beta\right) \tag{65}
\end{equation*}
$$

and

$$
\begin{equation*}
\gamma_{M}(\beta)=\eta\left(2 \pi \frac{M-1 / 2}{M}-\beta\right), \tag{66}
\end{equation*}
$$

we have proved our claim.
Now, consider the case when $M \geq 3$ and $m=2, \ldots, M-1$. We can see that if $m<(M+1) / 2$, $\gamma_{M+1-m}(\beta)$ is strictly decreasing because the numerator and the denominator in (52) are strictly decreasing and strictly increasing respectively as functions of $\beta$. Moreover, we can show the fact that $\gamma_{m}(\beta)+\gamma_{M+1-m}(\beta)$ is strictly decreasing, which can lead to the consequence that $\gamma_{\frac{M+1}{2}}(\beta)$ is strictly decreasing for odd $M$ and thus $\gamma_{M+1-m}(\beta)$ is strictly decreasing for $m=(M+1) / 2$ as well. It suffices to prove that

$$
\begin{equation*}
\gamma_{m}^{\prime}(\beta)+\gamma_{M+1-m}^{\prime}(\beta)<0 \tag{67}
\end{equation*}
$$

for $\beta \in\left(0, \frac{\pi}{M}\right)$. From (55),

$$
\begin{align*}
\gamma_{m}(\beta)+\gamma_{M+1-m}(\beta) & =\frac{1}{M} \frac{\sin ^{2}\left(M \phi_{m}(\beta) / 2\right)}{\sin ^{2}\left(\phi_{m}(\beta) / 2\right)}+\frac{1}{M} \frac{\sin ^{2}\left(M \phi_{m}(-\beta) / 2\right)}{\sin ^{2}\left(\phi_{m}(-\beta) / 2\right)} \\
& =\frac{1}{M} \cos ^{2}\left(\frac{M}{2} \beta\right)\left[\csc ^{2}\left(\phi_{m}(\beta) / 2\right)+\csc ^{2}\left(\phi_{m}(-\beta) / 2\right)\right] \tag{68}
\end{align*}
$$

because it can be shown that

$$
\begin{equation*}
\sin ^{2}\left(M \phi_{m}(\beta) / 2\right)=\sin ^{2}\left(M \phi_{m}(-\beta) / 2\right)=\cos ^{2}\left(\frac{M}{2} \beta\right) . \tag{69}
\end{equation*}
$$

By defining

$$
\begin{gather*}
f(\beta) \triangleq \cos ^{2}\left(\frac{M}{2} \beta\right)  \tag{70}\\
g_{1}(\beta) \triangleq \csc ^{2}\left(\phi_{m}(\beta) / 2\right), \quad g_{2}(\beta) \triangleq \csc ^{2}\left(\phi_{m}(-\beta) / 2\right), \tag{71}
\end{gather*}
$$

we have the expression

$$
\begin{equation*}
\gamma_{m}^{\prime}+\gamma_{M+1-m}^{\prime}=\frac{1}{M} f\left(g_{1}+g_{2}\right)\left(\frac{f^{\prime}}{f}+\frac{g_{1}^{\prime}+g_{2}^{\prime}}{g_{1}+g_{2}}\right) \tag{72}
\end{equation*}
$$

Since $f>0$ and $g_{1}+g_{2}>0$ for $\beta \in\left(0, \frac{\pi}{M}\right)$, we only need to show

$$
\begin{equation*}
h \triangleq \frac{f^{\prime}}{f}+\frac{g_{1}^{\prime}+g_{2}^{\prime}}{g_{1}+g_{2}}<0 . \tag{73}
\end{equation*}
$$

Simple derivations give us

$$
\begin{gather*}
\frac{f^{\prime}}{f}=-M \tan \left(\frac{M}{2} \beta\right)  \tag{74}\\
g_{1}^{\prime}=\frac{d}{d \beta}\left[\csc ^{2}\left(\frac{\phi_{m}(\beta)}{2}\right)\right]=\csc ^{2}\left(\frac{\phi_{m}(\beta)}{2}\right) \cot \left(\frac{\phi_{m}(\beta)}{2}\right) \tag{75}
\end{gather*}
$$

and similarly

$$
\begin{equation*}
g_{2}^{\prime}=-\csc ^{2}\left(\frac{\phi_{m}(-\beta)}{2}\right) \cot \left(\frac{\phi_{m}(-\beta)}{2}\right) . \tag{76}
\end{equation*}
$$

We get

$$
\begin{equation*}
h(\beta)=-M \tan \left(\frac{M}{2} \beta\right)+\frac{\csc ^{2}\left(\frac{\phi_{m}(\beta)}{2}\right) \cot \left(\frac{\phi_{m}(\beta)}{2}\right)-\csc ^{2}\left(\frac{\phi_{m}(-\beta)}{2}\right) \cot \left(\frac{\phi_{m}(-\beta)}{2}\right)}{\csc ^{2}\left(\frac{\phi_{m}(\beta)}{2}\right)+\csc ^{2}\left(\frac{\phi_{m}(-\beta)}{2}\right)} . \tag{77}
\end{equation*}
$$

Because $0<\phi_{m}(\beta) / 2, \phi_{m}(-\beta) / 2<\pi$, applying the mean value theorem yields

$$
\begin{equation*}
\frac{\csc ^{2}\left(\frac{\phi_{m}(\beta)}{2}\right) \cot \left(\frac{\phi_{m}(\beta)}{2}\right)-\csc ^{2}\left(\frac{\phi_{m}(-\beta)}{2}\right) \cot \left(\frac{\phi_{m}(-\beta)}{2}\right)}{-\beta}=-2 \csc ^{2} \psi \cot ^{2} \psi-\csc ^{4} \psi \tag{78}
\end{equation*}
$$

for some $\psi \in\left(\phi_{m}(\beta) / 2, \phi_{m}(-\beta) / 2\right)$. Then,

$$
\begin{equation*}
h(\beta)=-M \tan \left(\frac{M}{2} \beta\right)+\frac{\beta \csc ^{4} \psi\left(2 \cos ^{2} \psi+1\right)}{\csc ^{2}\left(\frac{\phi_{m}(\beta)}{2}\right)+\csc ^{2}\left(\frac{\phi_{m}(-\beta)}{2}\right)} . \tag{79}
\end{equation*}
$$

We can see that for $\psi \in\left[\phi_{m}(\beta) / 2, \phi_{m}(-\beta) / 2\right], \csc ^{4} \psi\left(2 \cos ^{2} \psi+1\right)$ has maximum at either $\psi=\phi_{m}(\beta) / 2$ or $\psi=\phi_{m}(-\beta) / 2$ and let it be denoted by $\psi_{b}$. We are ready to show the following series of inequalities

$$
\begin{align*}
h(\beta) & <-M \tan \left(\frac{M}{2} \beta\right)+\frac{\beta \csc ^{4} \psi_{b}\left(2 \cos ^{2} \psi_{b}+1\right)}{\csc ^{2} \psi_{b}+1} \\
& <-\frac{M^{2}}{2} \beta+\beta \csc ^{2} \psi_{b}\left(-2+\frac{10}{3-\cos 2 \psi_{b}}\right) \\
& <-\frac{M^{2}}{2} \beta+3 \beta \csc ^{2} \psi_{b} \\
& <-\frac{M^{2}}{2} \beta+3 \beta \frac{1}{\sin ^{2} \frac{\pi}{M}} \\
& <\frac{M^{2}}{2} \beta\left[-1+\frac{6}{\pi^{2}\left\{1-\frac{1}{6}\left(\frac{\pi}{M}\right)^{2}\right\}^{2}}\right] \\
& <0 \tag{80}
\end{align*}
$$

where the last inequality holds as $M \geq 3$. This proves (73), and thus (67) follows.
It is clear that $\sum_{i=1}^{m} \gamma_{(i)}(\beta)$ is strictly decreasing for all $m \in\{1, \ldots, M-1\}$ because

$$
\begin{equation*}
\gamma_{M+1-m}(\beta) \tag{81}
\end{equation*}
$$

and

$$
\begin{equation*}
\gamma_{m}(\beta)+\gamma_{M+1-m}(\beta) \tag{82}
\end{equation*}
$$

are strictly decreasing for $m=1, \ldots,\left\lfloor\frac{M+1}{2}\right\rfloor$, which we has been proved above, and $\sum_{i=1}^{m} \gamma_{(i)}(\beta)$ becomes either the sum of (82) for multiple $m$ or the sum of (81) and (82) for multiple $m$. The validity of (62) completes our proof.

Proof of Theorem (4) All functions in (31) and (32) take the value 0 if and only if $x=0$. Thus, we can assume $p \in(0,1)$. To show (31), define $x_{1} \triangleq Q_{M}^{-1}(p)$ and $x_{2} \triangleq F_{(M)}^{-1}(p \mid \nu)$ and this yields

$$
\begin{equation*}
Q_{M}\left(x_{1}\right)=Q_{M}\left(x_{2}\right) W^{M-1}\left(x_{2}\right)=p . \tag{83}
\end{equation*}
$$

Let us introduce a new variable $x_{3}$ to obtain upper bound for $x_{2}$ given by

$$
\begin{equation*}
x_{3} \triangleq Q_{M}^{-1}\left(\frac{p}{W^{M-1}\left(x_{1}\right)}\right) \geq x_{2} \geq x_{1} . \tag{84}
\end{equation*}
$$

Let us show $x_{3} \approx x_{1}$, and then $x_{2} \approx x_{1}$ in (31) follows immediately. The value of $x_{1}$ can be computed using Sankaran's approximation in [9], where it has been suggested that for a random variable $X$ with the cdf $F_{\chi^{\prime 2}}(x \mid n, \delta)$, $\{X-(n-1) / 2\}^{1 / 2}-\{\delta+(n-1) / 2\}^{1 / 2}$ is approximately zero mean Gaussian with unit variance and this approximation improves if either $n$ or $\delta$ increases. Thus as $M$ increases,

$$
\begin{align*}
x_{1} & \approx \frac{1}{2(K+1)}\left[\frac{1}{2}+\left\{\left(2 K M+\frac{1}{2}\right)^{\frac{1}{2}}+\Phi^{-1}(p)\right\}^{2}\right] \\
& \sim \frac{K}{K+1} M \tag{85}
\end{align*}
$$

where $\Phi^{-1}$ is the inverse function of the Gaussian cdf given by

$$
\begin{equation*}
\Phi(x)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{x} e^{-\frac{t^{2}}{2}} d t \tag{86}
\end{equation*}
$$

Let us use the notations $\mu_{F}$ and $\sigma_{F}^{2}$ to denote the mean and variance of distribution $F$, respectively. Then, it can be shown that

$$
\begin{equation*}
\mu_{W^{M-1}} \approx \frac{q_{M-1}+\zeta}{2(K+1)}, \quad \sigma_{W^{M-1}}^{2} \approx \frac{\pi^{2} / 6}{\{2(K+1)\}^{2}} \tag{87}
\end{equation*}
$$

where $q_{M-1} \triangleq W^{-1}(1-1 /(M-1)) \approx \ln (M-1)$ and $\zeta \triangleq 0.5772 \ldots$ (Euler's constant) [8]. For $x_{1}>\mu_{W^{M-1}}$ (this is true for all $M>C$ for some $C$ ), applying one-sided Chebyshev's inequality in [10, p. 152] yields

$$
\begin{equation*}
1-W^{M-1}\left(x_{1}\right) \leq \frac{1}{1+\left(x_{1}-\mu_{W^{M-1}}\right)^{2} / \sigma_{W^{M-1}}^{2}} \tag{88}
\end{equation*}
$$

and thus

$$
\begin{equation*}
1 \leq \frac{1}{W^{M-1}\left(x_{1}\right)} \leq 1+\epsilon_{M} \tag{89}
\end{equation*}
$$

where

$$
\begin{equation*}
0<\epsilon_{M} \triangleq \frac{\sigma_{W^{M-1}}^{2}}{\left(x_{1}-\mu_{W^{M-1}}\right)^{2}} \sim \frac{\pi^{2}}{24} \frac{1}{K^{2} M^{2}} \tag{90}
\end{equation*}
$$

by (85) and (87) as $M$ increases. We have

$$
\begin{align*}
x_{3}- & x_{1} \\
\leq & Q_{M}^{-1}\left(p\left(1+\epsilon_{M}\right)\right)-Q_{M}^{-1}(p) \\
\approx & \frac{1}{K+1}\left(2 K M+\frac{1}{2}\right)^{\frac{1}{2}}\left\{\Phi^{-1}\left(p\left(1+\epsilon_{M}\right)\right)-\Phi^{-1}(p)\right\} \\
& +\frac{1}{2(K+1)}\left[\left\{\Phi^{-1}\left(p\left(1+\epsilon_{M}\right)\right)\right\}^{2}-\left\{\Phi^{-1}(p)\right\}^{2}\right] \\
\approx & \frac{1}{K+1}\left(2 K M+\frac{1}{2}\right)^{\frac{1}{2}} \epsilon_{M} p \frac{1}{\Phi^{\prime}\left(\Phi^{-1}(p(1+\varepsilon))\right)}, \quad \varepsilon \in\left(1, \epsilon_{M}\right) \\
\approx & 0, \tag{91}
\end{align*}
$$

as $M$ increases, because $(2 K M+1 / 2)^{1 / 2} \cdot \epsilon_{M} \approx 0$ from (90). Hence, (31) is proved. Now, (32) can be shown similarly. For any $p \in(0,1)$, let us define $x_{4} \triangleq Q_{a_{M}}^{-1}(\sqrt{p})$ and $x_{5}$ as

$$
\begin{equation*}
Q_{a_{M}}^{2}\left(x_{5}\right) \cdot Q_{b_{M}}^{2}\left(x_{5}\right) \cdot W^{M-4}\left(x_{5}\right)=p \tag{92}
\end{equation*}
$$

From (30), defining $x_{6}$ yields

$$
\begin{equation*}
x_{6} \triangleq Q_{a_{M}}^{-1}\left(\frac{\sqrt{p}}{Q_{b_{M}}\left(x_{4}\right) W^{\frac{M-4}{2}}\left(x_{4}\right)}\right) \geq x_{5} \geq F_{(M)}^{-1}(p \mid 0) \geq x_{4} . \tag{93}
\end{equation*}
$$

Assuming $x_{6} \approx x_{4}$, we have $F_{(M)}^{-1}(p \mid 0) \approx x_{4}$ in (32). Now, as $M$ increases, it can be shown that

$$
\begin{equation*}
x_{4} \sim \frac{K}{K+1} a_{M} \tag{94}
\end{equation*}
$$

as above. Note that $Q_{b_{M}} W^{\frac{M-4}{2}}$ is the distribution of the maximum of two independent random variables following $Q_{b_{M}}$ and $W^{\frac{M_{1-4}}{2}}$. It can be easily proved that

$$
\begin{align*}
\mu_{Q_{b_{M} W} \frac{M-4}{2}} & \leq \mu_{Q_{b_{M}}}+\mu_{W}{ }^{\frac{M-4}{2}} \\
& \approx \frac{K b_{M}+1}{K+1}+\frac{q_{\frac{M-4}{2}}+\zeta}{2(K+1)} \\
& \sim \frac{K}{K+1} b_{M} \tag{95}
\end{align*}
$$

and

$$
\begin{align*}
\sigma_{Q_{b_{M}} W^{\frac{M-4}{2}}}^{2} & \leq \sigma_{Q_{b_{M}}}^{2}+\sigma_{W^{\frac{M-4}{2}}}^{2}+\left(\mu_{W^{\frac{M-4}{2}}}\right)^{2} \\
& \approx \frac{K b_{M}+1}{(K+1)^{2}}+\frac{\pi^{2} / 6}{\{2(K+1)\}^{2}}+\left(\frac{q^{\frac{M-4}{2}}+\zeta}{2(K+1)}\right)^{2} \\
& \sim \frac{K}{(K+1)^{2}} b_{M} \tag{96}
\end{align*}
$$

Once again using one-sided Chebyshev's inequality,

$$
\begin{equation*}
1 \leq \frac{1}{Q_{b_{M}}\left(x_{4}\right) W^{\frac{M-4}{2}}\left(x_{4}\right)} \leq 1+\epsilon_{M}^{\prime}, \tag{97}
\end{equation*}
$$

where

$$
\begin{align*}
0<\epsilon_{M}^{\prime} & \triangleq \frac{\sigma_{Q_{b_{M}} W^{\frac{M-4}{2}}}^{2}}{\left(x_{4}-\mu_{Q_{b_{M}} W^{\frac{M-4}{2}}}\right)^{2}} \\
& \sim \frac{1}{K} \frac{1 / 2-4 / \pi^{2}}{\left(8 / \pi^{2}-1 / 2\right)^{2}} \frac{1}{M} \\
& =(0.9820 \ldots) \times \frac{1}{K M}, \tag{98}
\end{align*}
$$

As $M$ increases, this implies $\left(2 K a_{M}+1 / 2\right)^{1 / 2} \cdot \epsilon_{M}^{\prime} \approx 0$, which leads us $x_{6} \approx x_{4}$ as in (91).
Proof of Theorem 5; Let us show (33), first. Let $X_{F}$ denote a random variable following any distribution $F$. Obviously, $\Gamma_{(M)}(\nu)$ is stochastically larger than $X_{Q_{M}}$ from (22). Using the idea of coupling [5] Sec. 9.2], define

$$
\begin{equation*}
\Gamma_{(M)}^{*}(\nu) \triangleq F_{(M)}^{-1}\left(Q_{M}\left(X_{Q_{M}}\right) \mid \nu\right) . \tag{99}
\end{equation*}
$$

Then, $\Gamma_{(M)}(\nu)$ and $\Gamma_{(M)}^{*}(\nu)$ share the same distribution but $\Gamma_{(M)}^{*}(\nu) \geq X_{Q_{M}}$ with probability 1 . By the mean value theorem, we have

$$
\begin{equation*}
h\left(\Gamma_{(M)}^{*}(\nu)\right)-h\left(X_{Q_{M}}\right)=h^{\prime}(\varepsilon)\left[\Gamma_{(M)}^{*}(\nu)-X_{Q_{M}}\right] \tag{100}
\end{equation*}
$$

for some $\varepsilon \in\left(X_{Q_{M}}, \Gamma_{(M)}^{*}(\nu)\right)$. Using this,

$$
\begin{align*}
\mid \mathbb{E} & {\left[h\left(\Gamma_{(M)}(\nu)\right)-h\left(X_{Q_{M}}\right)\right] \mid } \\
& =\left|\mathbb{E}\left[h\left(\Gamma_{(M)}^{*}(\nu)\right)-h\left(X_{Q_{M}}\right)\right]\right| \\
& \leq \mathbb{E}\left[\left|h\left(\Gamma_{(M)}^{*}(\nu)\right)-h\left(X_{Q_{M}}\right)\right|\right] \\
& \leq C \cdot \mathbb{E}\left[\left|\Gamma_{(M)}^{*}(\nu)-X_{Q_{M}}\right|\right] \\
& =C \cdot\left|\mathbb{E}\left[\Gamma_{(M)}(\nu)-X_{Q_{M}}\right]\right|, \tag{101}
\end{align*}
$$

where $\left|h^{\prime}\right|$ is bounded by $C$. Now, let us show $\mathbb{E}\left[\Gamma_{(M)}(\nu)\right] \approx \mathbb{E}\left[X_{Q_{M}}\right]$. For any $x \geq 0$,

$$
\begin{align*}
Q_{M}(x) & \geq F_{(M)}(x \mid \nu)=Q_{M}(x) W^{M-1}(x) \\
& \geq\left[Q_{M}(x)+W^{M-1}(x)-1\right]^{+}, \tag{102}
\end{align*}
$$

where $[\cdot]^{+}$is defined as

$$
[y]^{+} \triangleq \begin{cases}y & \text { if } y \geq 0  \tag{103}\\ 0 & \text { if } y<0\end{cases}
$$

As $Q_{M}+W^{M-1}-1$ is an increasing and continuous function of $[0, \infty)$ onto $[-1,1)$, there exists only one $\alpha \geq 0$ such that

$$
\begin{equation*}
Q_{M}(\alpha)+W^{M-1}(\alpha)-1=0 \tag{104}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
0 & \leq \mathbb{E}\left[\Gamma_{(M)}(\nu)-X_{Q_{M}}\right] \\
& =\int_{0}^{\infty}\left[\left(1-F_{(M)}(x \mid \nu)\right)-\left(1-Q_{M}(x)\right)\right] \mathrm{d} x \\
& \leq \int_{0}^{\infty}\left[Q_{M}(x)-\left[Q_{M}(x)+W^{M-1}(x)-1\right]^{+}\right] \mathrm{d} x \\
& =\int_{0}^{\alpha} Q_{M}(x) \mathrm{d} x+\int_{\alpha}^{\infty}\left[1-W^{M-1}(x)\right] \mathrm{d} x \\
& \leq \int_{0}^{\beta} Q_{M}(x) \mathrm{d} x+\int_{\beta}^{\infty}\left[1-W^{M-1}(x)\right] \mathrm{d} x \tag{105}
\end{align*}
$$

for any $\beta \geq 0$ as (105) can be minimized by choosing $\beta=\alpha$. Let us obtain the upper bound for the first term of (105) using the Marcum Q-function defined and bounded as

$$
\begin{align*}
\Psi(a, b) & \triangleq \int_{b}^{\infty} x e^{\left(x^{2}+a^{2}\right) / 2} I_{0}(a x) \mathrm{d} x \\
& \geq 1-\frac{a}{a-b} \exp \left(-\frac{1}{2}(a-b)^{2}\right) \text { if } a>b \tag{106}
\end{align*}
$$

where $I_{0}(x)$ is the modified Bessel function of the first kind with order zero [11]. Using the connection between the Rice distribution and the noncentral chi-square distribution with two degrees of freedom, it can be shown that

$$
\begin{align*}
Q_{M}(x) & =F_{\chi^{\prime 2}}(2(K+1) x \mid 2,2 K M) \\
& =1-\Psi(\sqrt{2 K M}, \sqrt{2(K+1) x}) . \tag{107}
\end{align*}
$$

From (106) and 107), the first term in (105) is bounded as

$$
\begin{align*}
& \int_{0}^{\beta} Q_{M}(x) \mathrm{d} x \leq \beta Q_{M}(\beta) \\
& \quad \leq \frac{\beta}{1-\sqrt{(K+1) \beta /(K M)}} \exp \left[-K M\left(1-\sqrt{\frac{(K+1) \beta}{K M}}\right)^{2}\right] \tag{108}
\end{align*}
$$

for $\beta<K M /(K+1)$. If we take $\beta$ such that

$$
\begin{equation*}
\lim _{M \rightarrow \infty} \frac{\beta}{M}<\frac{K}{K+1}, \tag{109}
\end{equation*}
$$

then (108) goes to zero. Consider the second term of (105). Note that $W$ is the exponential distribution, which has an increasing failure rate (IFR) [12, Sec. 3.2]. From the chains of implication in [12], p. 159], $W$ is a new better than used (NBU) distribution, which is closed under the formation of coherent systems including parallel systems, and thus the distribution $W^{M-1}$ is a new better than used in expectation (NBUE) as well as NBU. Using the bound for NBUE in [12, p. 187], the second term in (105) is bounded as

$$
\begin{equation*}
\int_{\beta}^{\infty}\left[1-W^{M-1}(x)\right] \mathrm{d} x \leq \mu_{W^{M-1}} e^{-\beta / \mu_{W} M-1} . \tag{110}
\end{equation*}
$$

Note

$$
\begin{equation*}
\mu_{W^{M-1}} \approx \frac{\ln (M-1)+\zeta}{2(K+1)} \tag{111}
\end{equation*}
$$

from (87), and thus we can find a sequence $\beta$ such that (110) converges to zero as $M$ increases while $\lim _{M \rightarrow \infty} \beta / M<K /(K+1)$, e.g., $\beta=0.5 K \sqrt{M} /(K+1)$. We now prove (34). It can be seen that

$$
\begin{align*}
Q_{a_{M}}^{2}(x) & \geq F_{(M)}(x \mid 0), Q_{a_{M}}^{2}(x) Q_{b_{M}}^{2}(x) \\
& \geq Q_{a_{M}}^{2}(x) Q_{b_{M}}^{2}(x) W^{M-4}(x) . \tag{112}
\end{align*}
$$

By the similar reasoning as above, it needs to be proved that

$$
\begin{equation*}
\mathbb{E}\left[\Gamma_{(M)}(0)\right] \approx \mathbb{E}\left[X_{Q_{a_{M}}^{2}}\right] \tag{113}
\end{equation*}
$$

as $M$ increases. We can easily show $\mathbb{E}\left[X_{Q_{a_{M}}^{2} Q_{b_{M}}^{2} W^{M-4}}\right] \approx \mathbb{E}\left[X_{Q_{a_{M}}^{2} Q_{b_{M}}^{2}}\right]$ as above. Assuming

$$
\begin{equation*}
\mathbb{E}\left[X_{Q_{a_{M}}^{2} Q_{b_{M}}^{2}}\right] \approx \mathbb{E}\left[X_{Q_{a_{M}}^{2}}\right] \tag{114}
\end{equation*}
$$

yields $\mathbb{E}\left[X_{Q_{a_{M}}^{2} Q_{b_{M}}^{2} W^{M-4}}\right] \approx \mathbb{E}\left[X_{Q_{a_{M}}^{2}}\right]$, and thus (113) follows. Hence, we will show (114) to complete this proof. For this, we need to find a sequence $\beta \geq 0$ such that

$$
\begin{equation*}
\int_{0}^{\beta} Q_{a_{M}}^{2}(x) \mathrm{d} x+\int_{\beta}^{\infty}\left[1-Q_{b_{M}}^{2}(x)\right] \mathrm{d} x \rightarrow 0 \tag{115}
\end{equation*}
$$

as $M$ increases. To make the first term of (115) diminish, $\beta$ can be chosen as

$$
\begin{equation*}
\lim _{M \rightarrow \infty} \frac{\beta}{a_{M}}<\frac{K}{K+1} . \tag{116}
\end{equation*}
$$

As $Q_{b_{M}}$ is the noncentral chi-square distribution with two degrees of freedom, $Q_{b_{M}}$ is IFR [13], and thus $Q_{b_{M}}^{2}$ is NBUE as above. From the definition of NBUE in [12, p. 159], the second term of (115] is upper bounded as

$$
\begin{align*}
\int_{\beta}^{\infty}\left[1-Q_{b_{M}}^{2}(x)\right] \mathrm{d} x & \leq \mu_{Q_{b_{M}}^{2}}\left[1-Q_{b_{M}}^{2}(\beta)\right] \\
& \leq 4 \mu_{Q_{b_{M}}}\left[1-Q_{b_{M}}(\beta)\right] \tag{117}
\end{align*}
$$

For $a<b$, Marcum Q-function is upper bounded as [11]

$$
\begin{equation*}
\Psi(a, b) \leq \frac{b}{b-a} \exp \left(-\frac{1}{2}(b-a)^{2}\right) \tag{118}
\end{equation*}
$$

Then, 117) can be further bounded as

$$
\begin{align*}
& \int_{\beta}^{\infty}\left[1-Q_{b_{M}}^{2}(x)\right] \mathrm{d} x \\
& \quad \leq 4 \mu_{Q_{b_{M}}} \Psi\left(\sqrt{2 K b_{M}}, \sqrt{2(K+1) \beta}\right) \\
& \quad \leq 4 \frac{K b_{M}+1}{K+1} \frac{1}{1-1 / \sqrt{(K+1) \beta /\left(K b_{M}\right)}} \exp \left[-K b_{M}\left(\sqrt{\frac{(K+1) \beta}{K b_{M}}}-1\right)^{2}\right] \tag{119}
\end{align*}
$$

which goes to zero if we take $\beta$ such that

$$
\begin{equation*}
\frac{K}{K+1}<\lim _{M \rightarrow \infty} \frac{\beta}{b_{M}}<\infty \tag{120}
\end{equation*}
$$

From the growth rates of $a_{M}$ and $b_{M}$ in (27) and (28), $\beta$ can be selected such that (116) and (120) are satisfied simultaneously, e.g., $\beta=0.25 \cdot K M /(K+1)$, which proves (114) and (34) consequently.

Proof of Lemma 2. Obviously, $\log _{2}(1+\rho x)$ is integrable with respect to $Q_{M}$ and $Q_{a_{M}}^{2}$ as

$$
\begin{equation*}
\mathbb{E}\left[\log _{2}\left(1+\rho X_{Q_{M}}\right)\right] \leq \log _{2}\left(1+\rho \mu_{Q_{M}}\right) \tag{121}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{E}\left[\log _{2}\left(1+\rho X_{Q_{a_{M}}^{2}}\right)\right] \leq \log _{2}\left(1+\rho \mu_{Q_{a_{M}}^{2}}\right) \tag{122}
\end{equation*}
$$

by Jensen's inequality. From these and Theorem 5, we have

$$
\begin{equation*}
\mathbb{E}\left[\log _{2}\left(1+\rho \Gamma_{(M)}(\nu)\right)\right] \approx \mathbb{E}\left[\log _{2}\left(1+\rho X_{Q_{M}}\right)\right] \tag{123}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{E}\left[\log _{2}\left(1+\rho \Gamma_{(M)}(0)\right)\right] \approx \mathbb{E}\left[\log _{2}\left(1+\rho X_{Q_{a_{M}}^{2}}\right)\right] \tag{124}
\end{equation*}
$$

as $M$ increases. Then, we need to show that

$$
\begin{equation*}
\mathbb{E}\left[\log _{2}\left(1+\rho X_{Q_{M}}\right)\right] \approx \log _{2}\left(1+\rho \mu_{Q_{M}}\right) \tag{125}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{E}\left[\log _{2}\left(1+\rho X_{Q_{a_{M}}^{2}}\right)\right] \approx \log _{2}\left(1+\rho \mu_{Q_{a_{M}}^{2}}\right) \tag{126}
\end{equation*}
$$

as $M$ increases. Assuming that these are true, (41) and (42) follow naturally from (37) and (39). We will now prove 125. By Chebyshev's inequality, for any given $\varepsilon>0$, we have

$$
\begin{align*}
\mathbb{E} & {\left[\log _{2}\left(1+\rho X_{Q_{M}}\right)\right] } \\
& =\int_{0}^{\infty} \log _{2}(1+\rho x) \mathrm{d} Q_{M}(x) \\
& \geq\left[\log _{2}\left(1+\rho \mu_{Q_{M}}\right)-\frac{\varepsilon}{2}\right]\left[1-\operatorname{Pr}\left\{\log _{2}\left(\frac{1+\rho X_{Q_{M}}}{1+\rho \mu_{Q_{M}}}\right) \leq-\frac{\varepsilon}{2}\right\}\right] \\
& =\left[\log _{2}\left(1+\rho \mu_{Q_{M}}\right)-\frac{\varepsilon}{2}\right]\left[1-\operatorname{Pr}\left\{\frac{X_{M}-\mu_{Q_{M}}}{\sigma_{Q_{M}}} \leq-\frac{\mu_{Q_{M}}+1 / \rho}{\sigma_{Q_{M}}}\left(1-2^{-\varepsilon / 2}\right)\right\}\right] \\
& \geq\left[\log _{2}\left(1+\rho \mu_{Q_{M}}\right)-\frac{\varepsilon}{2}\right]\left[1-\frac{\sigma_{Q_{M}}}{\left(1-2^{-\varepsilon / 2}\right)\left(\mu_{Q_{M}}+1 / \rho\right)}\right] \\
& \geq \log _{2}\left(1+\rho \mu_{Q_{M}}\right)-\frac{\varepsilon}{2}-\log _{2}\left(1+\rho \mu_{Q_{M}}\right) \frac{\sigma_{Q_{M}}}{\left(1-2^{-\varepsilon / 2}\right)\left(\mu_{Q_{M}}+1 / \rho\right)} \\
& \geq \log _{2}\left(1+\rho \mu_{Q_{M}}\right)-\varepsilon \tag{127}
\end{align*}
$$

for large enough $M$ because $\mu_{Q_{M}}=(K M+1) /(K+1)$ given in 37) and $\sigma_{M}=\sqrt{K M+1} /(K+1)$, which proves (125). Moreover, (126) can be shown similarly as $\mu_{Q_{a_{M}}^{2}} \geq \mu_{Q_{a_{M}}}=\left(K a_{M}+1\right) /(K+1)$ and $\sigma_{Q_{a_{M}}^{2}} \leq \sqrt{2} \sigma_{Q_{a_{M}}}=\sqrt{2\left(2 K a_{M}+1\right)} /(K+1)$ by the variance bound in [7] p. 69]

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