

Quantum Error Correction beyond the Bounded Distance Decoding Limit

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Abstract—In this paper, we consider quantum error correction over depolarizing channels with non-binary low-density parity-check codes defined over Galois field of size 2^p . The proposed quantum error correcting codes are based on the binary quasi-cyclic CSS (Calderbank, Shor and Steane) codes. The resulting quantum codes outperform the best known quantum codes and surpass the performance limit of the bounded distance decoder. By increasing the size of the underlying Galois field, i.e., 2^p , the error floors are considerably improved.

Index Terms—LDPC code, non-binary LDPC codes, belief propagation, Galois field, iterative decoding, CSS codes, quantum error-correcting codes

I. INTRODUCTION

In 1963, Gallager invented low-density parity-check (LDPC) codes [1]. Due to the sparseness of the code representation, LDPC codes are efficiently decoded by the sum-product algorithm. By a powerful optimization method *density evolution* [2], developed by Richardson and Urbanke, messages of sum-product decoding can be statistically evaluated. The optimized LDPC codes exhibit error correcting performance very close to the Shannon limit [3]. Recently, LDPC codes have been generalized from a point of view of Galois fields, i.e. non-binary LDPC codes are proposed. Non-binary LDPC codes were invented by Gallager [1]. Davey and MacKay [4] found that non-binary LDPC codes can outperform binary ones.

Quantum LDPC codes, which are quantum error-correcting codes, have been developed in a similar manner to (classical) LDPC codes. By the discovery of CSS (Calderbank, Shor and Steane) codes [5], [6] and stabilizer codes [7], the notion of parity-check measurement, which is a generalized notion of parity-check matrix, is introduced to quantum information theory. In particular, a parity-check measurement for a CSS code is characterized by a pair of parity-check matrices which satisfy the following condition: the product of one of the pair and the transposed other is subjected to be a zero-matrix.

Quantum LDPC codes were first introduced by Postol in [8]. The above CSS constraint on the parity-check matrices makes the design of the quantum LDPC codes difficult. MacKay et al. proposed the *bicycle* codes [9] and Cayley graph based CSS codes [10]. In [11], Poulin et al. proposed serial turbo codes for the quantum error correction. These codes can be decoded by an efficient iterative decoder. To the best of the authors' knowledge, these codes [9], [10], [11] are the best known quantum error correcting codes among efficiently decodable quantum LDPC codes so far. In [12], Hagiwara and Imai proposed a construction method of CSS code pair that has

quasi-cyclic (QC) parity-check matrices with arbitrary regular even row weight $L \geq 4$ and column weight J such that $L/2 \geq J \geq 2$. However, the resulting codes do not outperform the codes proposed by MacKay et al. [9], [10].

Generally, LDPC CSS codes tend to have poor minimum distance. The minimum distance of an LDPC CSS code is upper-bounded by the row weight of the parity-check matrix. This is due to the dual and sparse constraint on the parity-check matrices. When the LDPC CSS codes are used with large code length, the poor minimum distance leads to high error floors. Therefore, it is desired to establish the construction method of quantum LDPC codes with large minimum distance. We should note that it is important to study quantum LDPC codes with large minimum distance which grows with code length [13] for constructing quantum LDPC codes with vanishing decoding error probability.

Non-binary LDPC codes are defined as codes over $\text{GF}(2^p)$ with $p > 2$. The parity-check matrices of non-binary LDPC codes are given as sparse matrices over $\text{GF}(2^p)$. In this paper, we investigate non-binary LDPC codes for quantum error correction. It is empirically known that the best classical non-binary LDPC codes have column weight $J = 2$ from a point of view of error-correcting performance [14]. Moreover, due to the sparse representation of non-binary parity-check matrices of column weight $J = 2$, the non-binary LDPC codes are efficiently decoded by FFT-based sum-product algorithm [15].

In this paper, we propose a construction method of a binary CSS code which can be viewed also as a pair of non-binary LDPC codes. More precisely, the proposed construction method produces a binary code pair (C, D) such that $C \supset D^\perp$, and C and D are also defined by non-binary sparse parity-check matrices over $\text{GF}(2^p)$ of column weight $J = 2$. This satisfies the constraint of CSS codes. To this end, we first construct $PJ \times PL$ binary QC parity-check matrix pair (\hat{H}_C, \hat{H}_D) with column weight $J = 2$ and row weight L such that $\hat{H}_C \hat{H}_D^T = 0$ by the method developed in [12]. Solving some linear equations on \mathbb{Z}_{2^p-1} , we get $PJ \times PL$ non-binary parity-check matrix pair (H_Γ, H_Δ) with column weight $J = 2$ and row weight L such that $H_\Gamma H_\Delta^T = 0$. It is known that a natural linear map from $\text{GF}(2^p)$ to $\text{GF}(2)^{p \times p}$ is given so that through this map, the non-binary LDPC matrix pair (H_Γ, H_Δ) can be viewed as a binary LDPC matrix pair (H_C, H_D) such that $H_C H_D^T = 0$. Numerical experiments show the resulting CSS codes outperform the best known quantum error correcting codes and surpass the performance limit of the bounded distance decoder. By increasing the size of the underlying Galois field, i.e., 2^p , the error floors are

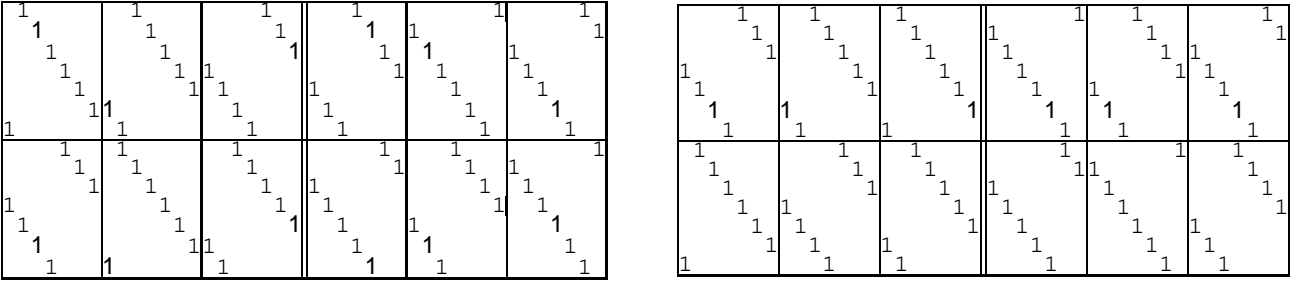


Fig. 1. An example of binary $(J = 2, L = 6, P = 7)$ -QC parity-check matrix pair (\hat{H}_C, \hat{H}_D) constructed by the method in Theorem 1 with $\sigma = 2$ and $\tau = 3$. It holds that $\hat{H}_C \hat{H}_D^T = 0$. For any row m' of \hat{H}_D

considerably improved.

The rest of this paper is organized as follows. Section II describes the construction method of a non-binary twisted LDPC parity-check matrix pair (H_G, H_Δ) of column weight $J = 2$. Section III describes the decoding algorithm of the binary twisted code pair (C, D) . Section IV demonstrates the decoding performance of the proposed codes.

II. CONSTRUCTION OF NON-BINARY MATRIX PAIR WITH COLUMN WEIGHT 2

In this section, we will construct a binary code pair (C, D) defined by orthogonal parity-check matrices H_C and H_D , where we call two matrices X and Y orthogonal if $XY^T = 0$. Let pN qubits be the code length. The binary codes C and D are designed in such a way that C and D can be represented also by non-binary sparse parity-check matrices over $\text{GF}(2^p)$ of column size N and column weight $J = 2$. To this end, we start with binary QC matrices and extend them to matrices over $\text{GF}(2^p)$. The following is the outline of construction procedure.

- A) Choose integers $J = 2, P$ and L such that $PL = N$. Construct a pair of orthogonal $PJ \times PL$ quasi-cyclic binary matrices \hat{H}_C and \hat{H}_D by following the procedure of [16].
- B) A pair of $PJ \times PL$ matrices H_Γ and H_Δ over $\text{GF}(2^p)$ are constructed by replacing non-zero entries of the matrices \hat{H}_C and \hat{H}_D with elements of $\text{GF}(2^p)$. The orthogonality condition $H_\Gamma H_\Delta^T = 0$ imposes a set of linear constraints on the non-zero entries of H_Γ and H_Δ , that can be solved by Gaussian elimination.
- C) The entries of H_Γ and H_Δ are mapped to elements of $\text{GF}(2)^{p \times p}$ that preserves addition and multiplication. These properties of the mapping imply that the resulting matrices H_C and H_D remain orthogonal.

The proposed code can be viewed as a concatenated code with an inner LDPC code and an outer code of rate 1 and length p . Indeed, the code obtained at step A) is a quantum LDPC code [16]. Steps B) and C) have the effect of replacing each qubit by p qubits while preserving the orthogonality property.

Each step in the procedure is explained in the following sub-sections.

A. Quasi-Cyclic Binary Construction

Let \hat{H}_C and \hat{H}_D be $PJ \times PL$ binary parity-check matrices defined as follows:

$$\begin{aligned} \hat{H}_C &:= (I(c_{j,\ell}))_{0 \leq j < J, 0 \leq \ell < L}, \\ \hat{H}_D &:= (I(d_{j,\ell}))_{0 \leq j < J, 0 \leq \ell < L}, \\ I(1) &:= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \in \{0, 1\}^{P \times P}, \\ I(c_{j,\ell}) &:= I(1)^{c_{j,\ell}}. \end{aligned}$$

We refer to such matrices as (J, L, P) -QC matrices.

Hagiwara and Imai proposed [12] the following method for constructing a QC parity-check matrix pair (\hat{H}_C, \hat{H}_D) . In the original paper [12], the construction method is more flexible about the row size of the matrices, i.e., \hat{H}_C and \hat{H}_D can have different row sizes. For simplicity, in this paper, we focus on \hat{H}_C and \hat{H}_D with the same row size JP .

Theorem 1 ([16, Theorem 5.2]). *Define $\mathbb{Z}_P^* := \{z \in \mathbb{Z}_P \mid \exists a \in \mathbb{Z}_P, za = 1\}$, and $\text{ord}(\sigma) := \min\{m > 0 \mid \sigma^m = 1\}$. For integers $P > 2, J, L, 0 \leq \sigma < P$ and $0 \leq \tau < P$ such that*

$$\sigma, \tau \in \mathbb{Z}_P^*, \quad (1)$$

$$L/2 = \text{ord}(\sigma), \quad (2)$$

$$1 \leq J \leq \text{ord}(\sigma),$$

$$\text{ord}(\sigma) \neq \#\mathbb{Z}_P^*,$$

$$1 - \sigma^j \in \mathbb{Z}_P^* \text{ for all } 1 \leq j < \text{ord}(\sigma), \quad (3)$$

$$\tau \neq 1, \sigma, \sigma^2, \dots, \sigma^{\text{ord}(\sigma)-1}, \quad (4)$$

let \hat{H}_C and \hat{H}_D be two (J, L, P) -QC binary matrices such that

$$\begin{aligned} \hat{H}_C &= (I(c_{j,\ell}))_{0 \leq j < J, 0 \leq \ell < L}, \\ \hat{H}_D &= (I(d_{j,\ell}))_{0 \leq j < J, 0 \leq \ell < L}, \\ c_{j,\ell} &:= \begin{cases} \sigma^{-j+\ell} & 0 \leq \ell < L/2 \\ \tau \sigma^{-j+\ell} & L/2 \leq \ell < L, \end{cases} \quad (5) \end{aligned}$$

$$d_{j,\ell} := \begin{cases} -\tau \sigma^{j-\ell} & 0 \leq \ell < L/2 \\ -\sigma^{j-\ell} & L/2 \leq \ell < L \end{cases} \quad (6)$$

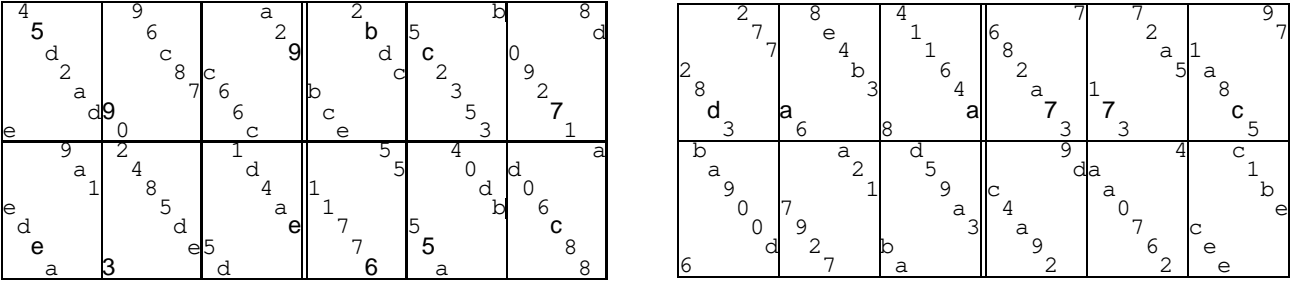


Fig. 2. An example of non-binary matrices $H_\Gamma = (\gamma_{m,n})_{0 \leq m < M, 0 \leq n < N}$ and $H_\Delta = (\delta_{m,n})_{0 \leq m < M, 0 \leq n < N}$ over $\text{GF}(2^4)$ such that $H_\Gamma H_\Delta^T = 0$ with $M = 14$ and $N = 42$. Each non-zero entry is represented as the hexadecimal number of $\log_\alpha(\gamma_{m,n})$, where α is a primitive element such that $\alpha^4 + \alpha + 1 = 0$. E.g., α^0 and α^{11} are represented as 0 and b, respectively.

then it holds that $\hat{H}_C \hat{H}_D^T = 0$ and there are no cycles of size 4 in the Tanner graph of \hat{H}_C and \hat{H}_D .

From Theorem 1, we obtain two $JP \times LP$ binary matrices \hat{H}_C and \hat{H}_D such that $\hat{H}_C \hat{H}_D^T = 0$ and the Tanner graphs of \hat{H}_C and \hat{H}_D are free of cycles of size 4. We give an example.

Example 2. With parameters $J = 2, L = 6, P = 7, \sigma = 2$ and $\tau = 3$, from Theorem 1, we are given a $JP \times LP$ binary matrix pair (\hat{H}_C, \hat{H}_D) such that $\hat{H}_C \hat{H}_D^T = 0$ as follows.

$$\hat{H}_C = \begin{pmatrix} I(1) & I(2) & I(4) & I(3) & I(6) & I(5) \\ I(4) & I(1) & I(2) & I(5) & I(3) & I(6) \end{pmatrix},$$

$$\hat{H}_D = \begin{pmatrix} I(4) & I(2) & I(1) & I(6) & I(3) & I(5) \\ I(1) & I(4) & I(2) & I(5) & I(6) & I(3) \end{pmatrix}.$$

The binary representation of these matrices are given in Fig. 1. It can be verified that there are no cycles of length 4 in Tanner graphs of \hat{H}_C and \hat{H}_D .

We observe a fundamental property of \hat{H}_C and \hat{H}_D as follows.

Observation 3. Let $m' := 5$. The m' -th row of \hat{H}_D has non-zero entries at the $n_0 = 2, n_2 = 7, n_4 = 20, n_1 = 25, n_5 = 29$ and $n_3 = 38$ -th columns. Denote the set of these indices by $N(m') := \{n_0, \dots, n_5\}$. Note that the index starts from 0. These non-zero entries are represented in thick font in Fig. 1. At each of these 6 columns in $N(m')$, \hat{H}_C has $J = 2$ non-zero entries at

$$(1, 2), (5, 7), (2, 20), (1, 25), (2, 29), (5, 38),$$

$$(12, 2), (13, 7), (11, 20), (13, 25), (12, 29), (11, 38).$$

Let these positions be denoted, respectively, by

$$(m_0, n_0), (m_2, n_2), (m_4, n_4), (m_0, n_1), (m_4, n_5), (m_2, n_3),$$

$$(m_5, n_0), (m_1, n_2), (m_3, n_4), (m_1, n_1), (m_5, n_5), (m_3, n_3).$$

We denote the set of these positions by $E(m')$. In the Tanner graph of \hat{H}_C , those non-zero entries from (m_0, n_0) to (m_5, n_0) consist of a cycle of size $2L = 12$. Along this cycle, the row index moves back and forth between the above and below of \hat{H}_C . Indeed, $0 \leq m_{2i} < P \leq m_{2i+1} < 2P$ for $i = 0, \dots, L/2 - 1$. On the other hand, the column index moves back and forth between the left and right of \hat{H}_C . Indeed, $0 \leq n_{2i} < LP/2 \leq n_{2i+1} < LP$ for $i = 0, \dots, L/2 - 1$.

We claim that this observation is general for any m' -th row of \hat{H}_C . The following lemma will be a key ingredient for constructing non-binary matrices H_Δ and H_Γ in Section II-B.

Lemma 4. Let \hat{H}_C, \hat{H}_D be the two $(2, L, P)$ -QC binary matrices dealt in Theorem 1. Let $N(m') := \{n_0, \dots, n_{L-1}\}$ be the support of the m' -th row of H_D . To be precise,

$$N(m') = \{n_0, \dots, n_{L-1}\} = \{0 \leq n < LP \mid \hat{d}_{m',n} \neq 0\}.$$

Let $E(m')$ be the set of non-zero entry positions in \hat{H}_C whose column is in $N(m')$. To be precise,

$$E(m') := \{(m, n) \mid \hat{c}_{m,n} \neq 0, n \in N(m')\}. \quad (7)$$

In this setting, in the Tanner graph of \hat{H}_C , for any $m' = 0, \dots, L-1$, the L variable nodes corresponding to the column index in $N(m')$ and the L adjacent check nodes form a cycle of length $2L$. In other words, there exist L distinct m_0, \dots, m_{L-1} and L distinct n_0, \dots, n_{L-1} , such that

$$\{(m_{2i-1}, n_{2i}), (m_{2i}, n_{2i}), (m_{2i}, n_{2i+1}), (m_{2i+1}, n_{2i+1}) \mid 0 \leq i < L/2\} = E(m'), \quad (8)$$

where we denote $m_{-1} := m_{L-1}$ and $m_{L+1} := m_1$.

Proof. Since $J = 2$, it follows that \hat{H}_C and \hat{H}_D comprise $2 \times L$ sub-matrices of size $P \times P$. For simplicity, we focus on the m' -th row chosen from the upper half rows of \hat{H}_D , i.e., $0 \leq m' < P$. The proof for $P \leq m' < 2P$ is essentially the same.

First, we will clarify the support $N(m')$ of the m' -th row of \hat{H}_D . From (6), the upper half of \hat{H}_D can be written by L sub-matrices as follows.

$$(I(-\tau\sigma^{-0}) \dots I(-\tau\sigma^{-(L/2-1)}) \parallel I(-\sigma^{-L/2}) \dots I(-\sigma^{-(L-1)})).$$

For each sub-matrix, the m' -th row has the only one non-zero entry. For $0 \leq i < L/2$, let n_{2i} be the column index of such a non-zero entry in the i -th sub-matrix. Similarly, let n_{2i+1} be the column index of such a non-zero entry in the $([-i]_{L/2} + L/2)$ -th sub-matrix. Obviously, $N(m') = \{n_0, \dots, n_{L-1}\}$ and n_0, \dots, n_{L-1} are distinct. Note that the n_3 -th, \dots , n_{L-3} -th and n_1 -th columns are in the $L/2$ right half $(L-1)$ -th, \dots , $(L/2+1)$ -th and $(L/2)$ -th sub-matrices of \hat{H}_D , respectively. The m' -th row of $I(x)$ has the only non-zero entry at the $(x+m')$ -th column. From this, it follows that we can rewrite

$$n_{2i} = [-\tau\sigma^{-i} + m']_P + iP, \quad (9)$$

$$n_{2i+1} = [-\sigma^{[-i]_{L/2}} + m']_P + ([-i]_{L/2} + L/2)P, \quad (10)$$

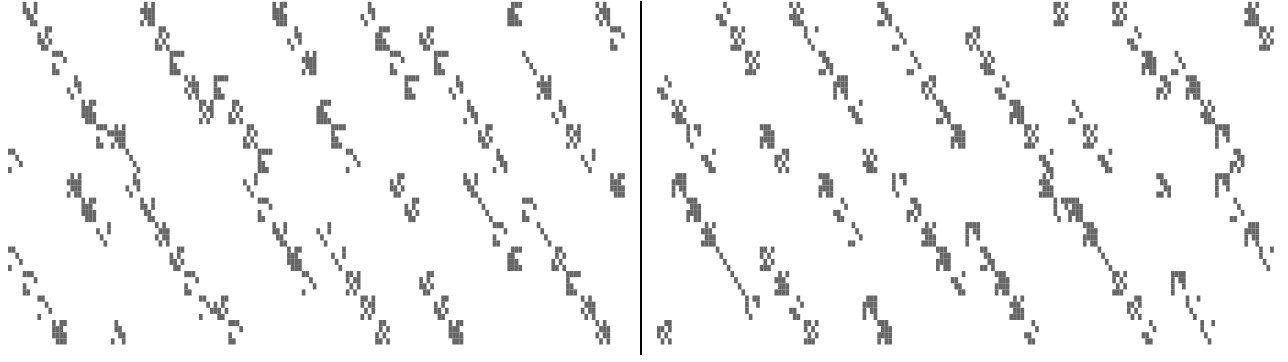


Fig. 3. An example of binary $pM \times pN$ matrices H_C and H_D such that $H_C H_D^T = 0$ with $p = 4$, $M = 14$ and $N = 42$. Non-zero entries are represented in black. The codes have many cycles of size 4 as binary codes. On the other hand, the codes have no cycles of size 4 as non-binary codes.

where we define $[x]_t \in \mathbb{Z}$ for $x \in \mathbb{Z}$ as $0 \leq [x]_t < t$ such that $[x]_t = x \pmod{t}$.

Secondly, we will prove (8) by clarifying the structure of $E(m')$. Define

$$\begin{aligned} m_{2i} &:= [-\sigma^i - \tau\sigma^{-i} + m']_P, \\ m_{2i-1} &:= [-\sigma^{i-1} - \tau\sigma^{-i} + m']_P + P. \end{aligned}$$

We claim that $(m_{2i-1}, n_{2i}), (m_{2i}, n_{2i}) \in E(m')$ for $0 \leq i < L/2$. Consider the non-zero entry in the (j, i) -th sub-matrix for $j = 0, 1$, and $0 \leq i < L/2$. Let the position be denoted by (m, n_{2i}) . From (5) and $i < L/2$, it follows that the m -th row of \hat{H}_C has non-zero entries at the $([\sigma^{-j+i} + m]_P + iP)$ -th column. Therefore, it follows from (9) that $(m, n_{2i}) \in E(m')$ if and only if

$$[\sigma^{-j+i} + m]_P + iP = [-\tau\sigma^{-i} + m']_P + iP,$$

for $j = 0, 1$. Thus, we conclude that $m = m_{2i-1}$ and $m = m_{2i}$ for $j = 0, 1$, respectively, which proves the claim. In a similar manner, it can be shown that $(m_{2i}, n_{2i+1}), (m_{2i+1}, n_{2i+1}) \in E(m')$ for $0 \leq i < L/2$, where we denoted $m_{L+1} := m_1$. These prove (8).

Finally, we show that the $2L$ non-zero entries in $E(m')$ consist of a cycle of length $2L$. To this end, it is suffice to show that m_0, m_2, \dots, m_{L-2} are distinct, and m_1, m_3, \dots, m_{L-1} are distinct. Assume $m_{2i} = m_{2i'}$ for $i \neq i'$. Then we have $\sigma^i + \tau\sigma^{-i} + m' = \sigma^{i'} + \tau\sigma^{-i'} + m' \pmod{P}$. Some calculations reveal that $(1 - \sigma^{i'-i})(\sigma^i - \tau\sigma^{-i'}) = 0 \pmod{P}$. From (3) it follows that $\sigma^i - \tau\sigma^{-i'} = 0 \pmod{P}$, which contradicts with (4). Hence, we conclude $i = i'$. In the same manner, we can show m_1, m_3, \dots, m_{L-1} are distinct. \square

B. Non-binary matrix pair construction

Define $M := JP$ and $N := LP$. So far, we obtain orthogonal matrices $\hat{H}_C = (\hat{c}_{m,n})_{0 \leq m < M, 0 \leq n < N}$ and $\hat{H}_D = (\hat{d}_{m,n})_{0 \leq m < M, 0 \leq n < N}$ whose Tanner graphs are free of cycles of size 4. In this sub-section, we will construct orthogonal non-binary $M \times N$ matrices $H_\Gamma = (\gamma_{m,n})_{0 \leq m < M, 0 \leq n < N}$ and $H_\Delta = (\delta_{m,n})_{0 \leq m < M, 0 \leq n < N}$ over $\text{GF}(2^p)$ such that $\gamma_{m,n} \neq 0$ iff $\hat{c}_{m,n} \neq 0$ and $\delta_{m,n} \neq 0$ iff $\hat{d}_{m,n} \neq 0$. Obviously, the Tanner graphs of H_Γ and H_Δ are free of cycles of size 4. We will determine the non-zero entries of H_Γ and H_Δ such that $H_\Gamma H_\Delta^T = 0$ in the rest of this sub-section.

For $H_\Gamma H_\Delta^T = 0$, it is required that the m' -th row of H_Δ is in the null-space of H_Γ for each $0 \leq m' < JP$. From Lemma 4, this is equivalent to

$$\begin{bmatrix} \gamma_{m_0, n_0} & \gamma_{m_0, n_1} & & & \\ & \ddots & \ddots & & \\ & & \gamma_{m_{L-2}, n_{L-2}} & \gamma_{m_{L-2}, n_{L-1}} & \\ \gamma_{m_{L-1}, n_0} & & & \gamma_{m_{L-1}, n_{L-1}} & \end{bmatrix} \begin{bmatrix} \delta_{m', n_0} \\ \vdots \\ \delta_{m', n_{L-1}} \end{bmatrix} = 0, \quad (11)$$

for each $0 \leq m' < JP$. In order to find the non-zero entries of H_Γ and H_Δ , this equation needs to have non-trivial solutions, i.e., the determinant of the left matrix, denoted by $\Gamma_{m'}$, in (11) is 0:

$$\det(\Gamma_{m'}) = \gamma_{m_0, n_0} \cdots \gamma_{m_{L-1}, n_{L-1}} - \gamma_{m_0, n_1} \cdots \gamma_{m_{L-1}, n_0} = 0. \quad (12)$$

Divide $E(m')$ in (8) into two parts as in the proof of Lemma 4:

$$\begin{aligned} E(m') &= E_1(m') \cup E_2(m'), \\ E_1(m') &:= \{(m_0, n_0), (m_1, n_1), \dots, (m_{L-1}, n_{L-1})\}, \\ E_2(m') &:= \{(m_0, n_1), (m_1, n_2), \dots, (m_{L-1}, n_0)\}. \end{aligned}$$

Then (12) can be transformed to

$$\prod_{(m,n) \in E_1(m')} \gamma_{m,n} \prod_{(m,n) \in E_2(m')} \gamma_{m,n}^{-1} = 1. \quad (13)$$

For $\alpha^x \in \text{GF}(2^p)$, define $\log_\alpha(\alpha^x) := x \pmod{2^p - 1}$. Then \log_α is well-defined. The equation above is equivalent to the following linear equation over \mathbb{Z}_{2^p-1} .

$$\sum_{(m,n) \in E_1(m')} \log_\alpha \gamma_{m,n} - \sum_{(m,n) \in E_2(m')} \log_\alpha \gamma_{m,n} = 0. \quad (14)$$

Thus, we have JP linear equations over \mathbb{Z}_{2^p-1} for $m' = 0, \dots, JP-1$. Solving these linear equations by the Gaussian elimination, we get the candidate solution space of the non-zero entries of H_Γ such that (14) holds for $m' = 0, \dots, JP-1$. Picking non-zero entries of H_Γ randomly from the candidate solution space and solving (11), we obtain non-zero entries of H_Δ . We give an example.

Example 5. Using \hat{H}_C and \hat{H}_D given in Example 2, we get an $M \times N$ non-binary matrix pair (H_Γ, H_Δ) over $\text{GF}(2^p)$ such

that $H_\Gamma H_\Delta^\top = 0$ with $M = JP = 14$ and $N = LP = 42$. The resulting (H_Γ, H_Δ) is depicted in Fig. 2.

This construction can be viewed as picking (H_Γ, H_Δ) randomly from $\{(H_\Gamma, H_\Delta) \mid H_\Gamma H_\Delta^\top = 0\}$, where H_Γ and H_Δ are constrained to have non-zero entries at the same positions as \hat{H}_C and \hat{H}_D , respectively. Since \hat{H}_C and \hat{H}_D is equivalent with some column permutation [12], the construction has symmetry for H_Γ and H_Δ . This symmetry leads to almost the same decoding performance which will be observed by computer experiments in Section IV.

C. Binary Quasi-Cyclic CSS LDPC Codes

So far, we obtained $M \times N$ sparse non-binary $\text{GF}(2^p)$ parity-check matrices H_Δ and H_Γ , where $N := PL$ and $M := PJ$. It is known that non-binary codes have the binary representation of their parity-check matrices. In this section, we show that two parity-check matrices H_Γ and H_Δ over $\text{GF}(2^p)$ such that $H_\Gamma H_\Delta^\top = 0$ can be represented by two binary matrices H_C and H_D such that $H_C H_D^\top = 0$.

Let $\text{GF}(2^p)$ has a primitive element α with its primitive polynomial $\pi(x) = \sum_{i=0}^{p-1} \pi_i x^i + x^p$. It is known [17] that the following map A from $\text{GF}(2^p)$ to $\text{GF}(2)^{p \times p}$ is bijective and its image is isomorphic to $\text{GF}(2^p)$ as a field by sum and multiple as matrices.

$$\begin{aligned} \text{GF}(2^p) \ni \alpha^i &\mapsto A(\alpha^i) := A(\alpha)^i \in \text{GF}(2)^{p \times p}, \\ A(0) &= 0, \\ A(\alpha) &:= \begin{bmatrix} 0 & 0 & 0 & 0 & \pi_0 \\ 1 & 0 & 0 & 0 & \pi_1 \\ 0 & 1 & 0 & 0 & \pi_2 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 1 & \pi_{p-1} \end{bmatrix}. \end{aligned}$$

Moreover, it holds that

$$\begin{aligned} A(\alpha^i) \underline{v}(\alpha^j) &= \underline{v}(\alpha^{i+j}), \\ \text{where } \alpha^i &= \sum_{j=0}^{p-1} a_j \alpha^j \in \text{GF}(2^p), \\ \text{and } \underline{v}(\alpha^i) &:= (a_0, \dots, a_{p-1})^\top \in \text{GF}(2)^p. \end{aligned}$$

Furthermore, with an abuse of notation we define $A(\underline{v}(\alpha^j)) := \underline{v}(\alpha^j)$.

Fact 6. Let H_Γ and H_Δ be matrices over $\text{GF}(2^p)^{M \times N}$ and let H_C and H_D be two matrices over $\text{GF}(2)^{pM \times pN}$ such that

$$\begin{aligned} H_\Gamma &= (\gamma_{m,n})_{0 \leq m < M, 0 \leq n < N}, \\ H_\Delta &= (\delta_{m,n})_{0 \leq m < M, 0 \leq n < N}, \\ H_C &= (A(\gamma_{m,n}))_{0 \leq m < M, 0 \leq n < N}, \\ H_D &= (A^\top(\delta_{m,n}))_{0 \leq m < M, 0 \leq n < N}. \end{aligned}$$

Then, it holds that if $H_\Gamma H_\Delta^\top = 0$, then $H_C H_D^\top = 0$.

Proof: Let $(H_C H_D^\top)_{m,n}$ be the (m, n) -th $p \times p$ binary sub-matrix of $H_C H_D^\top$, and let $(H_\Gamma H_\Delta^\top)_{m,n}$ be the (m, n) -th

entry of $H_\Gamma H_\Delta^\top$. Then, for any $0 \leq m < M$ and $0 \leq n < N$,

$$\begin{aligned} (H_C H_D^\top)_{m,n} &= \sum_{k=0}^{N-1} A(\gamma_{m,k}) A(\delta_{k,n}) \\ &= \sum_{k=0}^{N-1} A(\gamma_{m,k} \delta_{k,n}) \\ &= A(\sum_{k=0}^{N-1} \gamma_{m,k} \delta_{k,n}) \\ &= A((H_\Gamma H_\Delta^\top)_{m,n}) = A(0) = 0. \end{aligned}$$

■

Example 7. Using H_Γ and H_Δ given in Example 5, we get a $pM \times pN$ binary matrix pair (H_C, H_D) such that $H_C H_D^\top = 0$ with $p = 4$, $pM = pJP = 56$ and $pN = pLP = 168$. The resulting (H_C, H_D) is depicted in Fig. 3.

III. DECODING ALGORITHM

In this section, we describe the decoding algorithm for the CSS code pair (C, D) constructed by the proposed method in Section II and II-C. The decoding algorithm is based on the decoding algorithm of classical non-binary LDPC codes [15]. The input of the decoding algorithm is the syndrome. We assume the depolarizing channels [9, Section V] with depolarizing probability $2f_m/3$, where f_m can be viewed as the marginal probability for X and Z errors.

Let $M \times N$ be the size of the non-binary parity-check matrix H_Γ over $\text{GF}(2^p)$. The code length is pN qubits. We deal with a p -bit sequence as a non-binary symbol which is simply referred to as symbol. Moreover, we deal with the symbol interchangeably as a symbol in $\text{GF}(2^p)$.

Note that the channel is the normal depolarizing channel. We assume the decoder knows the depolarizing probability $3f_m/2$. For each row $m = 1, \dots, M$ in H_Γ , let N_m be the set of the non-zero entry indices in the m -th row. To be precise, $N_m := \{n \mid \gamma_{m,n} \neq 0\}$. The decoder is given the syndrome symbols $\underline{s}_m \in \text{GF}(2)^p$ for $m = 1, \dots, M$. To be precise, the decoder does not know the flipped qubits but their syndromes:

$$\underline{s}_m = \sum_{n \in N_m} A(\gamma_{m,n}) \underline{y}_n, \quad (15)$$

where A is the isomorphism defined in Section II-C and $\underline{y}_n \in \text{GF}(2)^p$ is a p -bit sequence corresponding to the n -th p -qubit sequence of flipped pN qubits.

For simplicity, we concentrate on the decoding algorithm for C , since the decoding algorithm for D is given by replacing Γ with Δ , and $A(\cdot)$ with $A^\top(\cdot)$ in the following algorithm.

The decoding algorithm of C

initialization:

For each column $n = 1, \dots, N$ in H_Γ , let M_n be the set of the non-zero entry indices in the n -th column. To be precise, $M_n := \{m \mid \gamma_{m,n} \neq 0\}$. For each column n in H_Γ for $n = 1, \dots, N$, calculate the initial probability $p_n^{(0)}(\underline{e})$ as follows.

$$p_n^{(0)}(\underline{e}) = \Pr(\underline{e}_n = \underline{e} \mid \underline{Y}_n = \underline{0}) = f_m^{W_H(\underline{e})} (1 - f_m)^{p - W_H(\underline{e})}$$

for $\underline{e} \in \text{GF}(2)^p$, where f_m is the flip probability of the channel and $W_H(\underline{e})$ is the Hamming weight of \underline{e} . For each column $n = 1, \dots, N$ in H_Γ , copy the initial message $p_{nm}^{(0)} = p_n^{(0)} \in [0, 1]^{2^p}$ for $m \in M_n$. Set the iteration round as $\ell := 0$.

horizontal step:

Each row m has L incoming messages $p_{nm}^{(\ell)}$ for $v \in N_m$. The m -th row sends the following message $q_{mn}^{(\ell+1)} \in [0, 1]^{2^p}$ to each column $n \in N_m$.

$$\tilde{p}_{nm}^{(\ell)}(\underline{e}) = p_{nm}^{(\ell)}(A(\gamma_{nm}^{-1})\underline{e}) \text{ for } \underline{e} \in \text{GF}(2)^p, \quad (16)$$

$$\tilde{q}_{mn}^{(\ell+1)} = \mathbf{1}_{\underline{s}_m} \bigotimes_{n' \in N_m \setminus \{n\}} \tilde{p}_{n'm}^{(\ell)},$$

$$q_{mn}^{(\ell+1)}(\underline{e}) = \tilde{q}_{mn}^{(\ell+1)}(A(\gamma_{nm})\underline{e}) \text{ for } \underline{e} \in \text{GF}(2)^p. \quad (17)$$

where $\mathbf{1}_{\underline{s}_m}$ is a probability on $\text{GF}(2)^p$ such that $\mathbf{1}_{\underline{s}_m}(\underline{e}) = 1$ for $\underline{e} = \underline{s}_m$ and 0 otherwise, and $q_1 \otimes q_2 \in [0, 1]^{2^{2p}}$ is a convolution of $q_1 \in [0, 1]^{2^p}$ and $q_2 \in [0, 1]^{2^p}$. To be precise,

$$(q_1 \otimes q_2)(\underline{e}) = \sum_{\substack{\underline{f}, \underline{g} \in \text{GF}(2)^p \\ \underline{e} = \underline{f} + \underline{g}}} q_1(\underline{f})q_2(\underline{g}) \text{ for } \underline{e} \in \text{GF}(2)^p.$$

The convolutions are efficiently calculated via FFT and IFFT [18], [19]. Increment the iteration round as $\ell := \ell + 1$.

vertical step:

Each column $n = 1, \dots, N$ in H_Δ has $J = 2$ non-zero entries. Let M_n be the set of the column indices of the non-zero entry. The message $p_{nm}^{(\ell)} \in [0, 1]^{2^p}$ sent from n to $m \in M_n$ is given by

$$p_{nm}^{(\ell)}(\underline{e}) = \xi q_n^{(0)}(\underline{e}) \prod_{m' \in M_n \setminus \{m\}} q_{m'n}^{(\ell)}(\underline{e}) \text{ for } \underline{e} \in \text{GF}(2)^p,$$

where ξ is the normalization factor so that $\sum_{\underline{e} \in \text{GF}(2)^p} p_{nm}^{(\ell)}(\underline{e}) = 1$.

tentative decision:

For each $n = 1, \dots, N$, the tentatively estimated v -th transmitted symbol is given as

$$\hat{\underline{e}}_n^{(\ell)} = \underset{\underline{e} \in \text{GF}(2)^p}{\text{argmax}} p_n^{(0)}(\underline{e}) \prod_{m \in M_n} q_{mn}^{(\ell)}(\underline{e}).$$

If $(\hat{\underline{e}}_0, \dots, \hat{\underline{e}}_N)$ has the same syndrome as $(\underline{s}_1, \dots, \underline{s}_M)$ which is defined in (15), in other words, for $m = 1, \dots, LP$,

$$\sum_{n \in N_m} A(\gamma_{mn})\hat{\underline{e}}_n^{(\ell)} = \hat{\underline{s}}_m \in \text{GF}(2)^p$$

for all $c = 1, \dots, M$, the decoder outputs $(\hat{\underline{e}}_0, \dots, \hat{\underline{e}}_N)$ as the estimated error. Otherwise, repeat the latter 3 decoding steps. If the iteration round ℓ reaches a pre-determined number, the decoder outputs FAIL.

Note that, in this algorithm, the correlations between X errors and Z errors are neglected. In [9, Section VI, C] MacKay et al. used the knowledge about the channel properties for decoding, which improved the decoding performance. The most complex part of the decoding is the horizontal step, which requires $O(Nq \log(q))$ multiplications and additions when calculated via FFT, where $q = 2^p$.

IV. NUMERICAL RESULT

In this section, we demonstrate the proposed CSS code pair decoded by the algorithm described in the previous section. The proposed CSS code pair (C, D) is constructed as follows. First, by Theorem 1, construct $JP \times LP$ binary matrices \hat{H}_C

and \hat{H}_D with $J = 2, L$, and P . Secondly, by the scheme described in Section II, construct $JP \times LP$ non-binary matrices H_Γ and H_Δ over $\text{GF}(2^p)$. Finally, by the scheme described in Section II-C, we have $pJP \times pLP$ binary matrices H_C and H_D . Thus, we obtain C and D are defined by the parity-check matrices H_C and H_D , respectively. Note that C and D can not only be viewed as binary codes defined by H_C and H_D but also be viewed as non-binary codes defined by H_Γ and H_Δ . The code length of the proposed CSS code is given as $n = pLP$ qubits or equivalently LP symbols. The quantum rate R_Q of the proposed CSS code is given as

$$R_Q = 1 - 2J/L.$$

Fig. 4 shows the block error probability of the constituent codes C and D of the proposed CSS code pair (C, D) over the depolarizing channel with marginal flip probability f_m of X and Z errors. Parameter are chosen $J = 2, L = 6, 8$ and 14 for $R_Q = 1/3, 1/2$ and $5/7$, respectively. The depolarizing probability is given by $3f_m/2$. The correlations between X errors and Z errors are neglected. Due to the symmetry of construction of C and D , the block error probability of the constituent codes C and D are almost the same, hence we plot the block error probability of either C or D . It is observed that for fixed $q = 2^p$ and R_Q , the codes with larger code length tend to have higher error floors. This is due to the fact that the proposed codes have poor minimum distance which is upper-bounded by pL . The error floors can be improved by using larger p , i.e., larger field $\text{GF}(2^p)$, which leads to the requirements of more complex decoding computations $O(Nq \log(q))$, where $q = 2^p$.

Fig. 5 compares the proposed quantum codes with the best quantum codes so far. The horizontal axis is the flip probability at which the block error probability of one of the constituent classical code is 0.5×10^{-4} . The vertical axis is the quantum rate R_Q of quantum codes. Since the proposed CSS codes have constituent classical codes C and D of the same classical rate $R_C = 1 - J/L$, the quantum rate R_Q is given as $R_Q = 2R_C - 1 = 1 - 2J/L$. It can be seen that the proposed codes outperform the state-of-the-art codes. In fact, the proposed codes surpass the BDD curve which is the limit of the bounded distance decoder, while the other codes fall inside the BDD curve.

V. CONCLUSION

We proposed a novel construction method of CSS codes. The resulting CSS codes can be viewed as non-binary LDPC codes over $\text{GF}(2^p)$. Due to the sparse representation of the parity-check matrices, the proposed codes are efficiently decoded. The simulation results over the depolarizing channels show that the proposed codes outperform the other quantum error correcting codes which exhibited the best decoding performance so far. The error floors are lowered by increasing the size of the underlying Galois field, i.e., 2^p .

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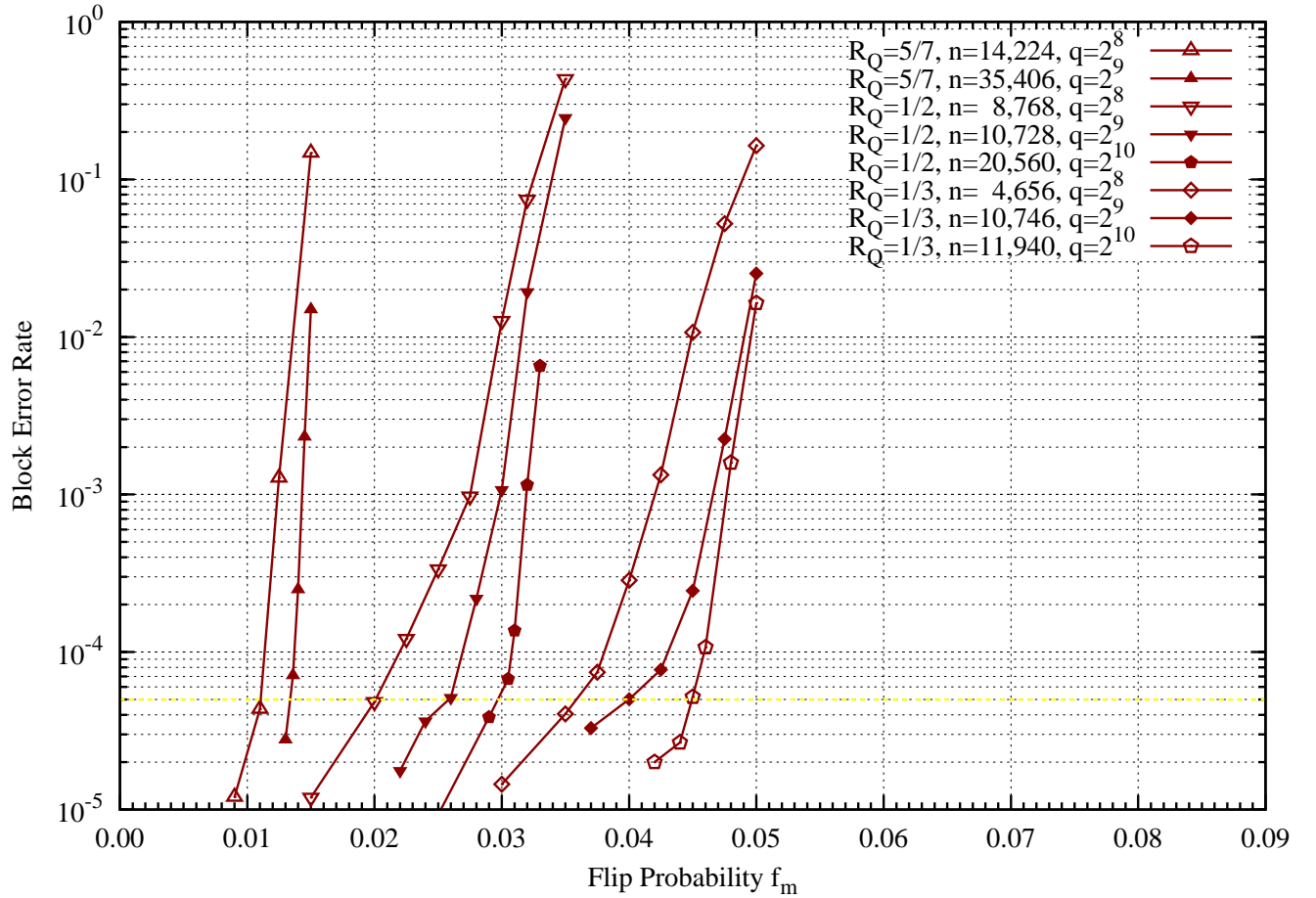


Fig. 4. The block error probability of the constituent codes C and D of the proposed CSS code pair (C, D) over the depolarizing channel with marginal flip probability f_m of X and Z errors. These codes are defined over $\text{GF}(q)$ for $q = 2^8, 2^9, 2^{10}$ and have quantum rate $R_Q = 1/3, 1/2, 5/7$. The code length is n qubits.

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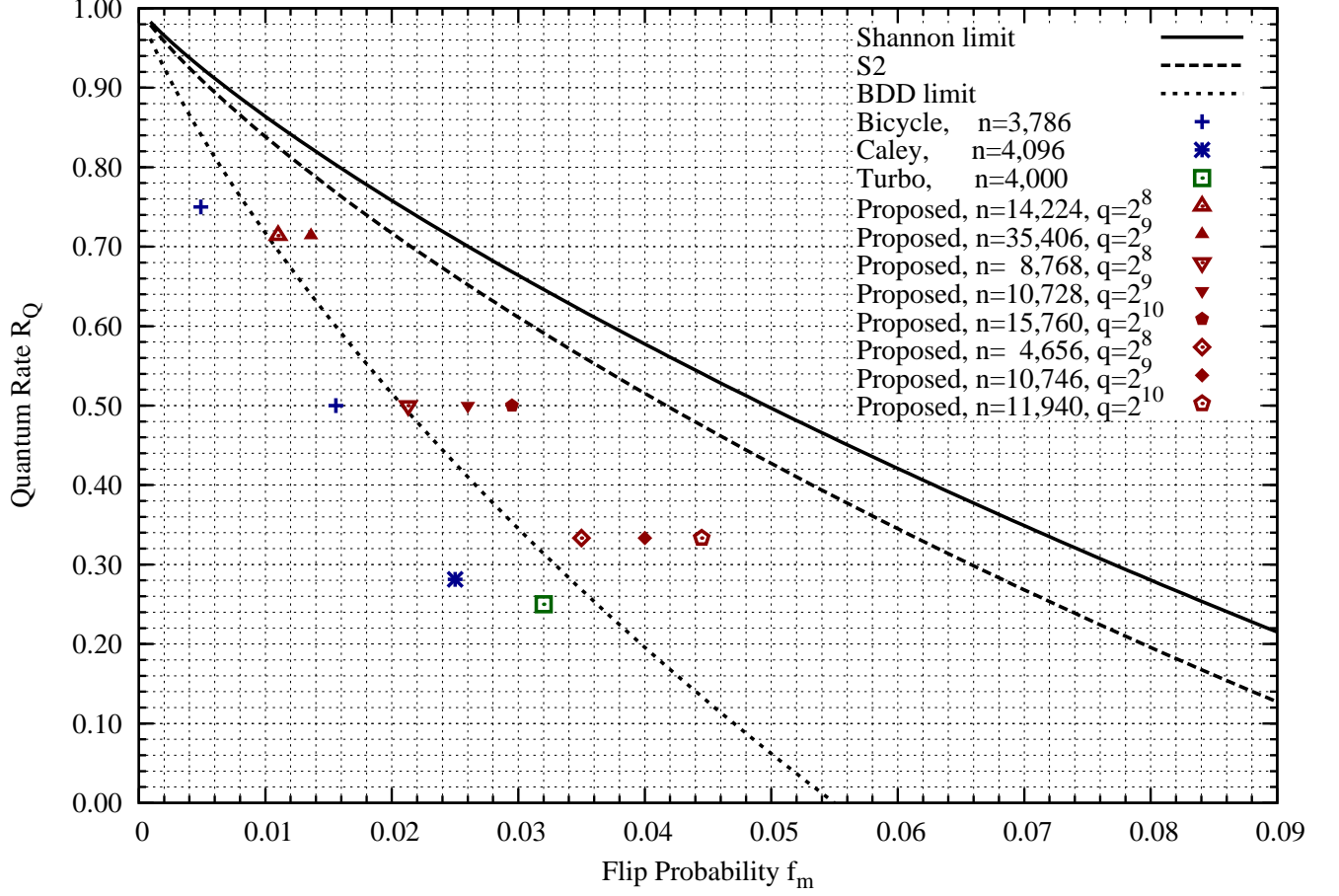


Fig. 5. The performance of the proposed CSS code pair (C, D) compared with the best CSS codes so far from [9], [10] and [11] over the depolarizing channel with marginal flip probability f_m of X and Z errors. Each point is plotted at which the block probabilities of both two constituent codes are 5×10^{-5} . The block probability of the entire CSS code is $1 - (1 - 5 \times 10^{-5})^2 \sim 10^{-4}$. The Shannon limit of the depolarizing channel: $R_Q = 1 - h(3f_m/2) - 3f_m/2 \log_2(3)$, where $h(\cdot)$ is the binary entropy function. The curve labelled S2 is the achievable quantum rate if the correlations between X errors and Z errors are neglected: $R_Q = 1 - 2h(f_m)$. The curve labeled BDD is the performance limit when the bounded distance decoder is employed and the correlations between X errors and Z errors are neglected: $R_Q = 1 - 2h(2f_m)$. The code length is n qubits. The proposed codes are defined over $GF(q)$.