# Euclidean and Hermitian Self-orthogonal Algebraic Geometry Codes and Their Application to Quantum Codes

Lingfei Jin and Chaoping Xing

Abstract—In the present paper, we show that if the dimension of an arbitrary algebraic geometry code over a finite field of even characters is slightly less than half of its length, then it is equivalent to an Euclidean self-orthogonal code. However, in the literatures, a strong contrition about existence of certain differential is required to obtain such a result. We also show a similar result on Hermitian self-orthogonal algebraic geometry codes. As a consequence, we can apply our result to quantum codes and obtain quantum codes with good asymptotic bounds.

Index Terms—Algebraic geometry codes, Euclidean selforthogonal, Hermitian self-orthogonal, Quantum codes

#### I. INTRODUCTION

Classical Euclidean self-orthogonal codes have been extensively studied due to their nice algebraic and combinatorial nature [17], [18]. Various constructions of classical Euclidean self-orthogonal codes have been studied through algebraic and combinatorial tools [6], [7], [12]. In recent years, this topic has become increasingly interesting due to application to quantum codes [1], [2], [5], [10], [11], [13]. For application to quantum codes, one is interested in not only classical Euclidean selforthogonal codes but also some other types of self-orthogonal codes such as Hermitian and simplectic self-orthogonal codes.

One good candidate for self-orthogonal codes is algebraic geometry codes. For instance, in [27], it is shown that algebraic geometry codes from a certain optimal tower are equivalent to Euclidean self-orthogonal codes. Unfortunate, this is not true for an arbitrary algebraic geometry codes in general. In fact, it requires a very strong condition in order that an algebraic geometry code is Euclidean self-orthogonal.

In this paper, we construct both Euclidean and Hermitian self-orthogonal codes through algebraic geometry codes. More precisely, we show that an arbitrary algebraic geometry code with dimension slightly less than half of its length over a finite field of characteristic 2 is Euclidean self-orthogonal. Furthermore, it is shown that an arbitrary algebraic geometry code with dimension slightly less than half of its length over a finite field of arbitrary characteristic is Hermitian selforthogonal when its tensor product is considered (see the details in ).

The paper is organized as follows.

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#### II. PRELIMINARY

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To construct self-orthogonal algebraic geometry codes, we need to recall some basic definition and results of algebraic curves and algebraic geometry codes. The reader may refer to [26], [28]

Let  $\mathcal{X}$  be a smooth, projective, absolutely irreducible curve of genus g defined over  $\mathbb{F}_q$ . We denote by  $\mathbb{F}_q(\mathcal{X})$  the function field of  $\mathcal{X}$ . An element of  $\mathbb{F}_q(\mathcal{X})$  is called a function. The normalized discrete valuation corresponding to a point P of  $\mathbb{F}_q(\mathcal{X})$  is written as  $\nu$ . A point P is said  $\mathbb{F}_q$ -rational if  $P^{\sigma} = P$ for all  $\sigma$  in the Galois group  $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ . Likewise, a divisor  $G = \sum_P m_p P$  is said  $\mathbb{F}_q$ -rational if  $G^{\sigma} = \sum_P m_P P^{\sigma} = G$ for all  $\sigma$  in the Galois group  $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ .

For an  $\mathbb{F}_q$ -rational divisor G, the Riemann-Roch space associated to G is

$$\mathcal{L}_{\mathbb{F}_q}(G) = \{ f \in \mathbb{F}_q(\mathcal{X}) : \operatorname{div}(f) + G \ge 0 \} \cup \{ 0 \}$$

Then  $\mathcal{L}_{\mathbb{F}_q}(G)$  is a finite-dimensional vector space over  $\mathbb{F}_q$  and we denote its dimension by  $\ell(G)$ . By the Riemann-Roch theorem we have

$$\ell(G) \ge \deg(G) + 1 - g$$

where the equality holds if  $\deg(G) \ge 2g - 1$ .

We can also consider the tensor product of  $\mathcal{L}_{\mathbb{F}_q}(G)$  with  $\mathbb{F}_{q^2}$ , denoted by  $\mathcal{L}_{\mathbb{F}_{q^2}}(G)$ , i.e.,

$$\mathcal{L}_{\mathbb{F}_{q^2}}(G) = \mathcal{L}_{\mathbb{F}_q}(G) \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2} = \{ f \in \mathbb{F}_{q^2}(\mathcal{X}) : \operatorname{div}(f) + G \ge 0 \} \cup \{ 0 \}$$

Then  $\mathcal{L}_{\mathbb{F}_{q^2}}(G)$  is a vector space over  $\mathbb{F}_{q^2}$  of dimension  $\ell(G)$ . Let  $P_1, \ldots, P_n$  be pairwise distinct  $\mathbb{F}_q$ -rational points of  $\mathcal{X}$  and  $D = P_1 + \cdots + P_n$ . Choose an  $\mathbb{F}_q$ -rational divisor G in  $\mathcal{X}$  such that  $\operatorname{supp}(G) \bigcap \operatorname{supp}(D) = \emptyset$ , and a vector  $\mathbf{v} = (v_1, \ldots, v_n)$  such that  $v_i \in (\mathbb{F}_q)^*, (i = 1, \ldots, n)$ . Then  $\nu_{P_i}(f) \geq 0$  for all  $1 \leq i \leq n$  and any  $f \in \mathcal{L}_{\mathbb{F}_q}(G)$ .

Consider the map

$$\Psi: \mathcal{L}(G) \to \mathbb{F}_q^n, \quad f \mapsto (v_1 f(P_1), \dots, v_n f(P_n)).$$

Obviously the image of the  $\Psi$  is a subspace of  $\mathbb{F}_q^n$ . The image of  $\Psi$  is denoted as  $C_{\mathcal{L}}(D, G)$  which is called an algebraicgeometry code(or AG code for short). If deg(G) < n, then  $\Psi$  is an embedding and we have dim $(C_{\mathcal{L}}(D,G)) = \ell(G)$ . By Riemann-Roch theorem we can estimate the parameters of an AG code (see [26]).

Proposition 2.1:  $C_{\mathcal{L}}(D, G, \mathbf{v})$  is an [n, k, d]- linear code over  $\mathbb{F}_q$  with parameters

$$k = \ell(G) - \ell(G - D), \quad d \ge n - \deg(G).$$

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(a) If G satisfies  $g \leq \deg(G) < n$ , then

$$k = \ell(G) \ge \deg(G) - g + 1, \quad d \ge n - \deg(G).$$

- (b) If additionally 2g−2 < deg(G) < n, then k = deg(G) − g + 1.
- *Remark* 2.2: (i) The proposition above implies that  $k + d \ge n + 1 g$ . Compared with the Singleton bound, we can know all the AG codes in the above are MDS codes while in rational function field.
- (ii) Note that C<sub>L</sub>(D, G, 1) is the ordinal algebraic geometry code defined by Goppa, where 1 denotes the all-one vector (1,..., 1).

Now we discuss the Euclidean dual of the algebraic code  $C_{\mathcal{L}}(D,G;\mathbf{v})$ .

For two vectors  $\mathbf{a} = (a_1, \dots, a_n)$ ,  $\mathbf{b} = (b_1, \dots, b_n)$  in  $\mathbb{F}_q^n$ , Euclidean inner product is defined by  $\langle \mathbf{a}, \mathbf{b} \rangle_E = \sum_{i=1}^n a_i b_i$ . For a linear code C over  $\mathbb{F}_q$ , the *Euclidean dual* of C is defined by

$$C^{\perp_E} := \{ \mathbf{v} \in \mathbb{F}_q^n : < \mathbf{v}, \mathbf{c} >= 0 \ \forall \ \mathbf{c} \in C \}.$$

Let  $\Omega$  denote the differential space of  $\mathbb{F}_q(\mathcal{X})$ . For an  $\mathbb{F}_q$ -rational divisor G, we define

$$\Omega(G) = \{ w \in \Omega : \operatorname{div}(w) \ge G \}$$

and denote the dimension of  $\Omega(G)$  by i(G). Then one has the following relationship

$$i(G) = \ell(K - G),$$

where K is a canonical divisor.

We define the code  $C_{\Omega}(D, G, \mathbf{v})$  as

$$C_{\Omega}(D,G,\mathbf{v}) = \{ (v_1 \operatorname{res}_{P_1}(w), \dots, v_n \operatorname{res}_{P_n}(w)) : w \in \Omega(G-D) \},\$$

where  $\operatorname{res}_{P_i}(w)$  stands for the residue of w at  $P_i$ .

 $C_{\Omega}(D, G, \mathbf{v})$  is an  $[n, i(G - D) - i(G), \geq \deg G - (2g - 2)]$  linear code over  $\mathbb{F}_q$ . Furthermore,  $C_{\Omega}(D, G, \mathbf{v}^{-1})$  is the Euclidean dual of  $C_{\mathcal{L}}(D, G, \mathbf{v})$ , where  $\mathbf{v}^{-1}$  denotes the vector  $(v_1^{-1}, \ldots, v_n^{-1})$ .

## III. SELF-ORTHOGONAL ALGEBRAIC GEOMETRY CODES

In this section, we first show existence of a ceratin vector in the Euclidean dual code  $C_{\Omega}(D, G, \mathbf{1})$  of  $C_{\mathcal{L}}(D, G, \mathbf{1})$ . Based on this result, we are able to show that any algebraic geometry codes are equivalent to Euclidean and Hamming self-orthogonal codes.

# A. A result on algebraic geometry codes

Proposition 3.1: The code  $C_{\Omega}(D, 2G, 1)$  contains at least a codeword of Hamming weight n if

$$\deg(G) < \frac{1}{2} \left( n - 1 - n \log_q \left( 1 + \frac{2}{q} \right) \right)$$

*Proof:* Let *m* denote the degree of *G*. The number of codewords with Hamming weight *n* in  $C_{\Omega}(D, 2G, \mathbf{v})$  is the size of

$$\Omega(2G - \sum_{j=1}^{n} P_j) \setminus \bigcup_{i=1}^{n} \Omega(2G - \sum_{j=1}^{n} P_j + P_i).$$

We denote this set by A and denote  $\Omega(2G - \sum_{j=1}^{n} P_j + P_i)$  by  $A_i$ . Thus, it's sufficient to prove A is not an empty set. By the inclusion-exclusion principle, we have

$$\begin{aligned} |A| &= \left| \Omega(2G - \sum_{j=1}^{n} P_j) \right| - \sum_{i=1}^{n} |A_i| + \sum_{h,k} |A_h \cap A_k| \\ &+ \dots + (-1)^{n-2m-2g+2} \sum \left| \bigcap_{j=1}^{n-2m+2g-2} A_{i_j} \right| \\ &= q^{n-2m+g-1} - \binom{n}{1} q^{n-2m+g-2} + \binom{n}{2} q^{n-2m+g-3} \\ &+ \dots + (-1)^{n-2m-1} \binom{n}{n-2m-1} q^g \\ &+ \sum_{k=n-2m}^{n-2m+2g-2} (-1)^k \sum_{i_1,\dots,i_k} \left| \bigcap_{j=1}^k A_{i_j} \right| \\ &= q^{n-2m+g-1} \left( 1 - \frac{1}{q} \right)^n + c, \end{aligned}$$

where

$$c = \sum_{k=n-2m}^{n-2m+2g-2} (-1)^k \sum_{i_1,\dots,i_k} \left| \bigcap_{j=1}^k A_{i_j} \right|$$
$$- \sum_{k=n-2m}^n (-1)^k \binom{n}{k} q^{n-2m+g-1-k}$$
$$\geq - \sum_{i_1,\dots,i_{n-2m-1}} \left| \bigcap_{j=1}^{n-2m-1} A_{i_j} \right|$$
$$- \sum_{k=n-2m}^n \binom{n}{k} q^{n-2m+g-1-k}$$
$$= -q^g \sum_{k=n-2m-1}^n \binom{n}{k} q^{n-2m-1-k}$$
$$\geq -q^g \left(1 + \frac{1}{q}\right)^n.$$

The desired result follows from the condition.

## B. Euclidean Self-orthogonal AG Codes

In this subsection, we restrict ourselves to finite fields  $\mathbb{F}_q$ of even characteristic. A linear code *C* is called *Euclidean self-orthogonal* if  $\langle \mathbf{u}, \mathbf{v} \rangle_E = 0$  for all  $\mathbf{u}, \mathbf{v} \in C$ . It is clear that the dimension of an Euclidean self-orthogonal code is at most the half of its length.

Theorem 3.2:  $C_{\mathcal{L}}(D, G, \mathbf{1})$  is equivalent to an Euclidean self-orthogonal code if

$$\deg(G) < \frac{1}{2}\left(n - 1 - n\log_q\left(1 + \frac{2}{q}\right)\right)$$

**Proof:** From Proposition 3.1, there exists a codeword  $\mathbf{u} = (u_1, \ldots, u_n)$  of Hamming weight n in  $C_{\mathcal{L}}(D, 2G, \mathbf{1})^{\perp_E} = C_{\Omega}(D, 2G, \mathbf{1})$ . Since  $v_i$  are elements in  $\mathbb{F}_q^*$  and q is a power of 2, there exist  $v_i \in \mathbb{F}_q^*$  such that  $v_i^2 = u_i$  for  $i = 1, \ldots, n$ . For any two codewords  $(v_1 f(P_1), \ldots, v_n f(P_n))$ 

and  $(v_1h(P_1), \ldots, v_nh(P_n))$  in  $C_{\mathcal{L}}(D, G, \mathbf{v})$  for some  $f, h \in \mathcal{L}_{\mathbb{F}_q}(G)$ , their Euclidean inner product is

$$\sum_{i=1}^{n} v_i^2 f(P_i) h(P_i) = \sum_{i=1}^{n} u_i(fh)(P_i) = 0.$$

Therefore,  $C_{\mathcal{L}}(D, G, \mathbf{v})$  is Euclidean self-orthogonal and our result follows.

Remark 3.3: In deg(G) > 2g - 2, then the dimension of  $C_{\mathcal{L}}(D, G, \mathbf{1})$  is deg(G) - g + 1. Hence, from Theorem 3.2, an algebraic geometry code is equivalent to an Euclidean self-orthogonal code if its dimension is at most  $\frac{1}{2}\left(n-1-n\log_q\left(1+\frac{2}{q}\right)\right)-g$ .

The following example shows that the condition that  $\mathbb{F}_q$  has even characteristic is necessary.

*Example 3.4:* We consider the algebraic code  $C_{\mathcal{L}}(D, G, \mathbf{1})$  over  $\mathbb{F}_5$  from the rational function field

$$\{(f(0), f(1), f(2), f(3)): f \in \mathbb{F}_5[x], \deg(f) \le 1\},\$$

where the divisors D and G are clear from the above context. It is in fact a generalized Reed-Solomon code (see Section IV). It is easy to see that its equivalent code  $C(D, G, \mathbf{v})$  is Euclidean self-orthogonal if and only if  $(v_1^2, v_2^2, v_3^2, v_4^2)$  is a nonzero solution of

$$\left(\begin{array}{rrrrr} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 4 \end{array}\right) \mathbf{x} = \mathbf{0}$$

On the other hand, all possible nonzero solutions of above equation are  $\lambda(2, 4, 1, 3)$  for some nonzero  $\lambda$ . However, 2 and 3 are non-square elements in  $\mathbb{F}_5$ , while 4, 1 are square elements in  $\mathbb{F}_5$ . This implies that  $(v_1^2, v_2^2, v_3^2, v_4^2)$  can not be a nonzero solution, i.e.,  $C(D, G, \mathbf{v})$  is not Euclidean self-orthogonal.

## C. Hermitian self-orthogonal AG codes

To study Hermitian self-orthogonal codes, we have to consider codes over  $\mathbb{F}_{q^2}$ .

For two vectors  $\mathbf{a} = (a_1, \ldots, a_n)$ ,  $\mathbf{b} = (b_1, \ldots, b_n)$  in  $\mathbb{F}_{q^2}$ , we define Hermitian inner product by  $\langle \mathbf{a}, \mathbf{b} \rangle_{H} = \sum_{i=1}^{n} a_i b_i^q$ . For an  $\mathbb{F}_q$  linear code C in  $\mathbb{F}_{q^2}$ , the *Hermitian dual*  $C^{\perp_H}$  of an  $\mathbb{F}_q$ -linear code  $C \subseteq \mathbb{F}_{q^2}^n$  consists of vectors in  $\mathbb{F}_{q^2}$  that are orthogonal with all the codewords in C with respect to Hermitian inner product defined above. It follows immediately that  $C^{\perp_H} = (C^q)^{\perp}$ , where  $C^q = \{(c_1^q, \ldots, c_n^q) : (c_1, \ldots, c_n) \in C\}$ . This implies that the Hermitian dual  $C^{\perp_H}$  of C is  $(C^q)^{\perp_E}$ .

Let  $\mathcal{X}$  be an algebraic curve in  $\mathbb{F}_q$ , let  $P_1, \ldots, P_n$  be pairwise distinct  $\mathbb{F}_q$ -rational points and let G be an  $\mathbb{F}_q$ -rational divisor such that  $\operatorname{supp}(G) \cap \{P_1, \ldots, P_n\} = \emptyset$ . Define a code over  $\mathbb{F}_{q^2}$ 

$$C_{\mathcal{L}}(D,G,\mathbf{v};\mathbb{F}_{q^2}) := \{(v_1f(P_1),\ldots,v_nf(P_n)): f \in \mathcal{L}_{\mathbb{F}_{q^2}}(G)\}$$

Then  $C_{\mathcal{L}}(D, G, \mathbf{1}; \mathbb{F}_{q^2})$  is an  $[n, \ell(G), d \ge n - \deg(G)]$ -linear code over  $\mathbb{F}_{q^2}$  if  $\deg(G) < n$ . In fact,  $C_{\mathcal{L}}(D, G, \mathbf{1}; \mathbb{F}_{q^2})$  is the tensor product  $C_{\mathcal{L}}(D, G, \mathbf{1}; \mathbb{F}_{q^2}) \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$ .

Theorem 3.5:  $C_{\mathcal{L}}(D, G, \mathbf{1}; \mathbb{F}_{q^2})$  is equivalent to an Hermitian self-orthogonal code if

$$\deg(G) < \frac{1}{2}\left(n - 1 - n\log_q\left(1 + \frac{2}{q}\right)\right).$$

*Proof:* From Proposition 3.1, there exists a codeword  $\mathbf{u} = (u_1, \ldots, u_n)$  of Hamming weight n in  $C_{\mathcal{L}}(D, 2G, \mathbf{1})^{\perp_E} = C_{\Omega}(D, 2G, \mathbf{1})$ . Since  $v_i$  are elements in  $\mathbb{F}_q^*$ , there exist  $v_i \in \mathbb{F}_{q^2}^*$  such that  $v_i^{q+1} = u_i$  for  $i = 1, \ldots, n$ . Moreover,  $\mathbf{u}$  is also Euclidean orthogonal to  $C_{\mathcal{L}}(D, 2G, \mathbf{1}; \mathbb{F}_{q^2})$  as  $C_{\mathcal{L}}(D, G, \mathbf{1}; \mathbb{F}_{q^2})$  has a basis from  $C_{\mathcal{L}}(D, 2G, \mathbf{1})$ 

Consider two codewords  $(v_1f(P_1), \ldots, v_nf(P_n))$  and  $(v_1h(P_1), \ldots, v_nh(P_n))$  in  $C_{\mathcal{L}}(D, G, \mathbf{v})$  for some  $f, h \in \mathcal{L}_{q^2}(G)$ . Then  $fh^{\sigma}$  is an element of  $\mathcal{L}_{q^2}(G)$ , where  $\sigma$  is the Frobenius in the Galois group  $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ . their Hermitian inner product is

$$\sum_{i=1}^{n} v_i^{q+1} f(P_i)(h(P_i))^q = \sum_{i=1}^{n} u_i(fh^{\sigma})(P_i^{\sigma}) = \sum_{i=1}^{n} u_i(fh^{\sigma})(P_i) = 0.$$

Therefore,  $C_{\mathcal{L}}(D, G, \mathbf{v}; \mathbb{F}_{q^2})$  is Hermitian self-orthogonal and our result follows.

*Remark 3.6:* To show Euclidean self-orthogonality of  $C_{\mathcal{L}}(D, G, \mathbf{1})$ , we requires that  $\mathbb{F}_q$  has even characteristics. However, we do not need this condition for Hermitian elforthogonality of  $C_{\mathcal{L}}(D, G, \mathbf{1}; \mathbb{F}_{q^2})$ .

## IV. EXAMPLES

In this section, we illustrate our result by considering algebraic codes from projective line and elliptic curves.

# A. Self-orthogonal generalized Reed-Solomn codes

Let's recall some basic results of generalized Reed-Solomon codes ( $\mathcal{G}RS$  codes for short) first. Let  $\mathbb{F}_q$  be a finite field of qelements, choose n distinct elements  $\alpha_1, \ldots, \alpha_n$  of  $\mathbb{F}_q$ , and nnonzero elements  $v_1, \ldots, v_n$  of  $\mathbb{F}_q$ . Denote  $\mathbf{a} = (\alpha_1, \ldots, \alpha_n)$ and  $\mathbf{v} = (v_1, \ldots, v_n)$ .

Let  $P_i$  be the only zero of  $x - \alpha_i$  and let  $\infty$  be the only pole of x. Put  $D = \sum_{i=1}^{n} P_i$  and  $G = (k-1)\infty$ for some k between 1 and n. Then we denote our algebraic geometry codes  $C_{\mathcal{L}}(D, G, \mathbf{v})$  and  $C_{\mathcal{L}}(D, G, \mathbf{1}; \mathbb{F}_{q^2})$  by  $\mathcal{G}RS_k(\mathbf{a}, \mathbf{v})$  and  $\mathcal{G}RS_k(\mathbf{a}, \mathbf{v}, \mathbb{F}_{q^2})$ , respectively. First of all, the Euclidean dual code of  $C_{\mathcal{L}}(D, 2G, \mathbf{v}) = \mathcal{G}RS_{2k-1}(\mathbf{a}, \mathbf{v})$ is  $\mathcal{G}RS_{n-2k+1}(\mathbf{a}, \mathbf{v}')$ , where  $\mathbf{v}'$  is a nonzero solution of the following system

$$\begin{pmatrix} v_{1} & v_{2} & \dots & v_{n} \\ v_{1}\alpha_{1} & v_{2}\alpha_{2} & \dots & v_{n}\alpha_{n} \\ v_{1}\alpha_{1}^{2} & v_{2}\alpha_{2}^{2} & \cdots & v_{n}\alpha_{n}^{2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ v_{1}\alpha_{1}^{n-2} & v_{2}\alpha_{2}^{n-2} & \cdots & v_{n}\alpha_{n}^{n-2} \end{pmatrix} \mathbf{x}^{T} = \mathbf{0}. \quad (IV.1)$$

}.Note that the solution space of the above system has dimension 1 and every nonzero solution has all coordinates not equal to zero. It is clear that  $\mathcal{G}RS_{n-2k+1}(\mathbf{a}, \mathbf{v}')$  has a codeword  $f(P_1), \ldots, f(P_n)$ ) of weight *n* for an irreducible polynomial *f* of degree 2 as long as  $n-2k \geq 2$ . Therefore, a generalized

Reed-Solomon code  $\mathcal{G}RS_k(\mathbf{a}, \mathbf{1})$  is equivalent to an Euclidean self-orthogonal code if  $k \leq n - 1$ .

With the same arguments, we can show that a generalized Reed-Solomon code  $\mathcal{GRS}_k(\mathbf{a}, \mathbf{1}, \mathbb{F}_{q^2})$  is equivalent to a Hermitian self-orthogonal code if  $k \leq n - 1$ .

## B. Codes over elliptic curves

The AG codes in the example of  $\mathcal{G}RS$  codes are associated with the projective line whose genus is 0. In this subsection, we consider another example of AG code basing on elliptic curves.

Let  $\mathcal{X}$  be an elliptic curve over  $\mathbb{F}_q$  and let  $\mathcal{P}$  denote the set of  $\mathbb{F}_q$ -rational points on  $\mathcal{X}$ . Choose an  $\mathbb{F}_q$ -rational divisor G such that  $\operatorname{supp}(G) \cap \mathcal{P} = \emptyset$ . For  $2 \operatorname{deg}(G) +$  $2 \leq n < |\mathcal{P}|$ , we choose a closed point Q of degree  $n-2 \operatorname{deg}(G)$  and  $n \mathbb{F}_q$ -rational points  $P_1, \ldots, P_n$  such that  $\sum_{i=1}^n P_1 - 2G + Q$  is equivalent to a canonical divisor K(note that this can be always done). Let  $\operatorname{div}(w) = K$  for a differential w and  $\operatorname{div}(x) = K - (\sum_{i=1}^n P_1 - 2G + Q)$ . Then the differential xw belongs to  $\Omega(2G - D = \sum_{i=1}^n P_i)$ . Moreover,  $(\operatorname{res}_{P_1}(xw), \ldots, \operatorname{res}_{P_n}(xw))$  is a codeword of  $C_{\Omega}(D, 2G, \mathbf{1})$  with Hamming weight n. This implies that  $C_{\mathcal{L}}(D, G, \mathbf{1})$  is equivalent to an Euclidean self-orthogonal code and  $C_{\mathcal{L}}(D, G, \mathbf{1}, \mathbb{F}_{q^2})$  is equivalent to a Hermitian selforthogonal code.

### V. APPLICATION TO QUANTUM CODES

The main purpose of this section is to apply our selforthogonal codes to construction of quantum codes and derive an asymptotic bound.

Let us first introduce some notations and results on quantum codes. Let  $\mathbb{C}$  be the field of complex numbers. For an positive integer n, denote  $V_n = (\mathbb{C}^{q^n})^{\otimes n} = \mathbb{C}^{q^n}$ . Any  $K \ge 1$  dimensional subspace Q of  $V_n$  is called a q-ary quantum code with length n, dimension  $K \ge 1$ . Then Q is a  $((n, K, d))_q$  code or  $[[n, k, d]]_q$  code if Q can detect d-1 errors and correct  $\lfloor \frac{d-1}{2} \rfloor$  where  $k = \log_q K$ . Similar as the classical code, for any  $[[n, k, d]]_q$  quantum code, the quantum singleton bound tells us  $n \ge k + 2d - 2$ . Q is called a quantum MDS code if it achieves the quantum singleton bound. In order to use our results to construct quantum code, we need to introduce two lemmas for connection.

Lemma 5.1: (see [13])There is an q-ary  $[[n, n - 2k, d^{\perp}]]$ quantum code whenever there exists a q-ary classical Euclidean self-orthogonal [n, k]-linear code with dual distance  $d^{\perp}$ .

*Lemma 5.2:* (see [2]) There is an q-ary  $[[n, n - 2k, d^{\perp}]]$ quantum code whenever there exists a q-ary classical Hermitian self-orthogonal [n, k]-linear code with dual distance  $d^{\perp}$ .

Now, using the theorems in the previous sections, we can derive several classes of quantum codes immediately.

Theorem 5.3: For finite field  $\mathbb{F}_q$  and  $1 \leq n \leq q+1, k \leq n-1$ , there exists a q-ary [[n, n-2k, k+1]]-quantum MDS code.

Theorem 5.4: For finite field  $\mathbb{F}_q$ ,  $2m + 2 \le n \le q + 1 + \lfloor 2\sqrt{q} \rfloor$ , there exists a q-ary [[n, n - 2m, m]]-quantum code.

*Proof:* Applying the result in GRS codes to Theorem 5.1 yields the desired results.

Theorem 5.5: For finite field  $\mathbb{F}_q$ ,  $2m + 2 \le n < q + 1 + \lfloor 2\sqrt{q} \rfloor$ , there exists a q-ary [[n, n - 2m, m]]-quantum code.

*Proof:* Applying the result of elliptic curves to Theorem 5.2 gives the desired results.

Now, we introduce some results on quantum codes and their asymptotic bounds. For a q-ary quantum code Q, we denote by n(Q), K(Q), and d(Q) the length, the dimension, and the minimum distance of Q, respectively. Let  $U_q^Q$  be the set of ordered pairs  $(\delta, R) \in \mathbb{R}^2$  for which there exists a family  $\{Q_i\}_{i=1}^{\infty}$  of q-ary codes with  $n(Q_i) \to \infty$  and

$$\delta = \lim_{i \to \infty} \frac{d(Q_i)}{n(Q_i)}, \quad R = \lim_{i \to \infty} \frac{\log_q K(Q_i)}{n(Q_i)}$$

where  $\log_q$  denotes the logarithm to the base q. One of the central asymptotic problems for quantum codes is to determine the domain  $U_q^Q$ . As in classical coding, it is a hard problem to determine  $U_q^Q$  completely. Instead, we are satisfied with some bounds on  $U_q^Q$ .

A very good existence lower bound for *p*-ary quantum codes was introduced by Ashikhmin and Knill [?]. It is called the quantum Gilbert-Varshamov bound. As in classical coding theory, the quantum Gilbert-Varshamov bound is a benchmark for the function  $\alpha_q^Q(\delta)$ .

For  $0 < \delta < 1$ , define the q-ary entropy function

$$H_q(\delta) := \delta \log_q(q-1) - \delta \log_q \delta - (1-\delta) \log_q (1-\delta),$$

and put

$$R_{GV}(q,\delta) := 1 - \delta \log_q(q+1) - H_q(\delta).$$

Then the Gilbert-Varshamov bound says that

$$\alpha_q^Q(\delta) \ge R_{GV}(q,\delta) \quad \text{for all } \delta \in (0,\frac{1}{2}).$$
 (V.1)

Later on, a bound from algebraic geometry codes was derived in [8], [9], [19] and this algebraic geometry bound improves the Gilbert-Vrahsamov bound for large q as in the classical case. To introduce the asymptotic algebraic geometry bound, we need some further notations.

For any prime power q and any integer  $g \ge 0$ , put

$$N_q(g) := \max N(\mathcal{X}),$$

where the maximum is extended over all curves  $\mathcal{X}/\mathbb{F}_q$  with  $g(\mathcal{X}) = g$ .

We also define the following asymptotic quantity

$$A(q) := \limsup_{g \to \infty} \frac{N_q(g)}{g}$$

We know from [28] that  $A(q) = \sqrt{q} - 1$  if q is a square.

The algebraic geometry bound [9] says that for a prime power q, one has

$$\alpha_q^Q(\delta) \ge 1 - 2\delta - \frac{2}{A(q)}.\tag{V.2}$$

In the following part, we prove the bound (V.2) for  $\delta$  in the range  $(0, 1/2 - 2/A(q) - \log_q(1 + 2/q))$  using our result introduced in the previous sections.

*Proof of the bound (V.2)* 

*Proof:* Let  $\{\mathcal{X}/\mathbb{F}_q\}$  be a family of curves such that  $g(\mathcal{X}) \to \infty$  and  $N(\mathcal{X})/g(\mathcal{X}) \to A(q)$ .

For  $0 < \delta < 1/2 - 2/A(q) - \log_q(1 + 2/q)$ , define two families of integers  $\{n = N(\mathcal{X})\}_{\mathcal{X}}$  and  $\{m = \lfloor \delta N(\mathcal{X}) \rfloor + 2g\}_{\mathcal{X}}$ . Then  $n/g(\mathcal{X}) \to A(q)$  and  $(m - 2g)/n \to \delta$ .

For each curve, let  $P_1, \ldots, P_n$  be  $n \mathbb{F}_q$ -rational points and choose a divisor G of degree m such that  $\operatorname{supp}(G) \cap \{P_1, \ldots, P_n\} = \emptyset$ .

By Proposition 3.5, from each curve  $\mathcal{X}$  with sufficiently large genus in the family we have a Hermitian self-orthogonal code over  $\mathbb{F}_{q^2}$  with parameters [n, n - 2(m - g + 1)] and dual distance at least m - 2g + 2. By Lemma 5.1, we obtain a qary quantum  $((n, q^{n-2(m-g+1)}, m-2g+2))$  code. The desire bound follows.

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