Singly-even self-dual codes with minimal shadow

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Abstract

In this note we investigate extremal singly-even self-dual codes with minimal shadow. For particular parameters we prove non-existence of such codes. By a result of Rains [11], the length of extremal singly-even self-dual codes is bounded. We give explicit bounds in case the shadow is minimal.

Index Terms: self-dual codes, singly-even codes, minimal shadow, bounds

1 Introduction

Let C be a singly-even self-dual $[n, \frac{n}{2}, d]$ code and let C_0 be its doubly-even subcode. There are three cosets C_1, C_2, C_3 of C_0 such that $C_0^{\perp} = C_0 \cup C_1 \cup C_2 \cup C_3$, where $C = C_0 \cup C_2$. The set $S = C_1 \cup C_3 = C_0^{\perp} \setminus C$ is called the shadow of C. Shadows for self-dual codes were introduced by Conway and Sloane [5] in order to derive new upper bounds for the minimum weight of singly-even self-dual codes and to provide restrictions on their weight enumerators.

According to [10] the minimum weight d of a self-dual code of length n is bounded by 4[n/24] + 4 for $n \not\equiv 22 \pmod{24}$ and by 4[n/24] + 6 if $n \equiv 22 \pmod{24}$. We call a self-dual code meeting this bound extremal. Note that for some lengths, for instance length 34, no extremal self-dual codes exist.

Some properties of the weight enumerator of S are given in the following theorem.

Theorem 1 [5] Let $S(y) = \sum_{r=0}^{n} B_r y^r$ be the weight enumerator of S. Then

- $B_r = B_{n-r}$ for all r,
- $B_r = 0$ unless $r \equiv n/2 \pmod{4}$,
- $B_0 = 0$,

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• $B_r \le 1$ for r < d/2,

•
$$B_{d/2} \leq 2n/d$$
,

• at most one B_r is nonzero for r < (d+4)/2.

Elkies studied in [6] the minimum weight s (respectively the minimum norm) of the shadow of self-dual codes (respectively of unimodular lattices), especially in the cases where it attains a high value. Bachoc and Gaborit proposed to study the parameters d and s simultaneously [1]. They proved that $2d + s \leq \frac{n}{2} + 4$, except in the case $n \equiv 22 \pmod{24}$ where $2d + s \leq \frac{n}{2} + 8$. They called the codes attaining this bound *s*-extremal. In this note we study singly-even self-dual codes for which the minimum weight of the shadow has smallest possible value. possible.

Definition 1 We say that a self-dual code C of length 24m + 8l + 2r with r = 1, 2, 3and l = 0, 1, 2 is a code with minimal shadow if wt(S) = r. For r = 0, C is called of minimal shadow if wt(S) = 4.

Self-dual codes with minimal shadow are subject of two previous articles. The paper [3] is devoted to connections between self-dual codes of length 24m + 8l + 2 with wt(S) = 1, combinatorial designs and secret sharing schemes. The structure of these codes are used to characterize access groups in a secret sharing scheme based on codes. There are two types of schemes which are proposed - with one-part secret and with two-part secret. Moreover, some of the considered codes support 1- and 2-designs. The performance of the extremal self-dual codes of length 24m + 8l where l = 1, 2 have been studied in [2]. In particular, different types of codes with the same parameters are compared with regard to the decoding error probability. It turned out that for lengths 24m + 8 singly-even codes with minimal shadow perform better than doubly-even codes. Thus from the point of view of data correction one is interested in singly-even codes with minimal shadow.

This article is organized as follows. In Section 2 we prove that extremal self-dual codes with minimal shadow of length 24m + 2t for t = 1, 2, 3, 5, 11 do not exist. Moreover, for t = 4, 6, 7 and 9, we obtain upper bounds for the length. We also prove that if extremal doubly-even self-dual codes of length n = 24m + 8 or 24m + 16 do not exist then extremal singly-even self-dual codes with minimal shadow do not exist for the same length. The only case for which we do not have a bound for the length is n = 24m + 20.

All computations have been carried out with Maple.

2 Extremal self-dual codes with minimal shadow

Let C be a singly-even self-dual code of length n = 24m + 8l + 2r where l = 0, 1, 2 and r = 0, 1, 2, 3. The weight enumerator of C and its shadow are given by [5]:

$$W(y) = \sum_{j=0}^{12m+4l+r} a_j y^{2j} = \sum_{i=0}^{3m+l} c_i (1+y^2)^{12m+4l+r-4i} (y^2(1-y^2)^2)^i$$
$$S(y) = \sum_{j=0}^{6m+2l} b_j y^{4j+r} = \sum_{i=0}^{3m+l} (-1)^i c_i 2^{12m+4l+r-6i} y^{12m+4l+r-4i} (1-y^4)^{2i}$$

Using these expressions we can write c_i as a linear combination of the a_j and as a linear combination of the b_j in the following way [10]:

$$c_i = \sum_{j=0}^{i} \alpha_{ij} a_j = \sum_{j=0}^{3m+l-i} \beta_{ij} b_j.$$
 (1)

Suppose C is an extremal singly-even self-dual code with minimal shadow, hence d = 4m + 4 and wt(S) = r if r = 1, 2, 3 and wt(S) = 4 if r = 0. Obviously in this case $a_0 = 1, a_1 = a_2 = \cdots = a_{2m+1} = 0$. According to Theorem 1, we have $b_0 = 1$ if r > 0 and $m \ge 1$, and $b_0 = 0, b_1 = 1$ if r = 0 and $m \ge 2$.

Moreover, if r > 0 and $m \ge 1$ then $b_1 = b_2 = \cdots = b_{m-1} = 0$. Otherwise S would contain a vector v of weight less than or equal to 4m - 4 + r, and if $u \in S$ is a vector of weight r then $u + v \in C$ with $wt(u + v) \le 4m + 2r - 4 \le 4m + 2$, a contradiction to the minimum distance of C. Similarly, if r = 0 and $m \ge 2$ then $b_2 = \cdots = b_{m-1} = 0$.

Remark 1 For extremal self-dual codes of length 24m + 8l + 2 we furthermore have $b_m = 0$. Otherwise S would contain a vector v of weight 4m + 1, and if $u \in S$ is the vector of weight 1 which exists since wt(S) = 1, then $u + v \in C$ with wt(u + v) $\leq 4m + 2$ contradicting the minimum distance of C.

If $m \ge 2$ we have by (1)

$$c_{2m+1} = \alpha_{2m+1,0} = \beta_{2m+1,\epsilon} + \sum_{j=m}^{m+l-1} \beta_{2m+1,j} b_j,$$
(2)

where $\epsilon = 1$ for r = 0 and $\epsilon = 0$ otherwise, since 3m+l-2m-1 = m+l-1. To evaluate this equation, which turns out to be crucial in the following, we need to consider the coefficients α_{i0} in details. In order to do this we denote by $\alpha_i(n)$ the coefficient α_{i0} if n is the length of the code. According to [10] we have

$$\alpha_i(n) = \alpha_{i0} = -\frac{n}{2i} [\text{coeff. of } y^{i-1} \text{ in } (1+y)^{-n/2-1+4i} (1-y)^{-2i}].$$
(3)

Let t = 4l + r and n = 24m + 8l + 2r = 24m + 2t. Then

$$\alpha_{2m+1}(n) = -\frac{12m+t}{2m+1} [\text{coeff. of } y^{2m} \text{ in } (1+y)^{-12m-t-1+8m+4}(1-y)^{-4m-2}]$$
$$= -\frac{12m+t}{2m+1} [\text{coeff. of } y^{2m} \text{ in } (1+y)^{-4m-t+3}(1-y)^{-4m-2}]$$

For t > 5 we obtain

$$\alpha_{2m+1}(n) = -\frac{12m+t}{2m+1} [\text{coeff. of } y^{2m} \text{ in } (1-y^2)^{-4m-t+3}(1-y)^{t-5}],$$

and if $t \leq 5$ then

$$\alpha_{2m+1}(n) = -\frac{12m+t}{2m+1} [\text{coeff. of } y^{2m} \text{ in } (1-y^2)^{-4m-2} (1+y)^{5-t}].$$

Since

$$(1-y^2)^{-a} = \sum_{0 \le j} {\binom{-a}{j}} (-1)^j y^{2j} = \sum_{0 \le j} {\binom{a+j-1}{j}} y^{2j} \text{ for } a > 0,$$

it follows in case $t \leq 5$ that

$$\alpha_{2m+1}(n) = -\frac{12m+t}{2m+1} [\text{coeff. of } y^{2m} \text{ in } (1+y)^{5-t} \sum_{j=0}^{m} \binom{4m+j+1}{j} y^{2j}]$$
$$= -\frac{12m+t}{2m+1} \sum_{s=0}^{\left\lfloor \frac{5-t}{2} \right\rfloor} \binom{5-t}{2s} \binom{5m+1-s}{m-s},$$

and in case t > 5 that

$$\alpha_{2m+1}(n) = -\frac{12m+t}{2m+1} [\text{coeff. of } y^{2m} \text{ in } (1-y)^{t-5} \sum_{j=0}^{m} \binom{4m+t+j-4}{j} y^{2j}]$$
$$= -\frac{12m+t}{2m+1} \sum_{s=0}^{\left\lfloor \frac{t-5}{2} \right\rfloor} \binom{t-5}{2s} \binom{5m+t-4-s}{m-s}.$$

For the different lengths n the values of $\alpha_{2m+1}(n)$ are listed in Table 1.

To evaluate equation (2) we also need β_{ij} which are known due to [10]. Here we have

$$\beta_{ij} = (-1)^i 2^{-n/2 + 6i} \frac{k - j}{i} \binom{k + i - j - 1}{k - i - j},\tag{4}$$

n	24m + 2	24m + 10	24m + 18	
α_{2m+1}	$-\frac{(12m+1)(56m+4)}{(2m+1)(m-1)}\binom{5m-1}{m-2}$	$-\frac{12m+5}{2m+1}\binom{5m+1}{m}$	$-\frac{12(7m+5)(4m+3)}{m(m-1)}\binom{5m+3}{m-2}$	
n	24m + 4	24m + 12	24m + 20	
α_{2m+1}	$-\frac{2(6m+1)(8m+1)}{m(2m+1)}\binom{5m}{m-1}$	$-6\binom{5m+2}{m}$	$-\frac{20(6m+5)(4m+3)}{m(m-1)}\binom{5m+4}{m-2}$	
n	24m + 6	24m + 14	24m + 22	
α_{2m+1}	$-\frac{3(4m+1)(6m+1)}{m(2m+1)}\binom{5m}{m-1}$	$-\frac{3(12m+7)}{m}\binom{5m+2}{m-1}$	$-\frac{6(12m+11)(6m+5)(8m+7)}{m(m-1)(m-2)}\binom{5m+4}{m-3}$	
n	24m + 8	24m + 16		
α_{2m+1}	$-\frac{4(3m+1)}{2m+1}\binom{5m+1}{m}$	$-\frac{16(3m+2)}{m}\binom{5m+3}{m-1}$		

Table 1: The values $\alpha_{2m+1}(n)$ for extremal self-dual codes

where $k = \lfloor n/8 \rfloor = 3m + l$. In particular,

$$\beta_{2m+1,j} = -2^{6-t} \frac{3m+l-j}{2m+1} {5m+l-j \choose m+l-1-j}$$
 and $\beta_{2m+1,m+l-1} = -2^{6-t}$.

Now we are prepared to prove:

Theorem 2 Extremal self-dual codes of lengths n = 24m + 2, 24m + 4, 24m + 6, 24m + 10 and 24m + 22 with minimal shadow do not exist.

Proof. According to [10] any extremal self-dual code of length 24m + 22 has minimum distance 4m + 6 and the minimum weight of its shadow is 4m + 7. Thus the shadow is not minimal since a minimal shadow must have minimum weight 3. (There is a misprint in [10] where it is stated that the minimum weight of the shadow is 4m + 6. But actually the weights in this shadow are of type 4j + 3).

In the other four cases we have

$$c_{2m+1} = \alpha_{2m+1,0} = \beta_{2m+1,0} \tag{5}$$

by (2). In case n = 24m + 10 we use the fact that $b_m = 0$, according to Remark 1. Simplifying equation (5) according to Table 1 we obtain

$$48m^{2} + 26m + 1 = 0, \quad \text{if } n = 24m + 2$$

$$24m^{2} + 14m + 1 = 0, \quad \text{if } n = 24m + 4$$

$$48m^{2} + 30m + 3 = 0, \quad \text{if } n = 24m + 6$$

$$6m + 3 = 0, \quad \text{if } n = 24m + 10$$

Since all these equations have no solutions $m \ge 0$ extremal self-dual codes with minimal shadow do not exist for $n \equiv 2, 4, 6, 10 \mod 24$.

Remark 2 So far no extremal self-dual codes of length 24m + 2t are known for t = 1, 2, 3, 5. According to [8] extremal self-dual codes of length 24m + 2r do not exist for r = 1, 2, 3 and $m = 1, 2, \ldots, 6, 8, \ldots, 12, 16, \ldots, 22$. Thus if there is (for instance) a self-dual [170, 85, 32] code it will not have minimal shadow, by Theorem 2.

The next result is a crucial observation in order to prove explicit bounds for the existence of extremal singly-even self-dual codes.

Theorem 3 Extremal singly-even self-dual codes with minimal shadow of lengths n = 24m+8, 24m+12, 24m+14 and 24m+18 have uniquely determined weight enumerators.

Proof. For m = 0 and m = 1 see Remark 3 and the examples at the end of the paper. Now let $m \ge 2$.

In case n = 24m + 12 or n = 24m + 14 we have

$$c_i = \alpha_{i0} = \beta_{i0} + \sum_{j=m}^{3m+1-i} \beta_{ij} b_j \text{ for } i \le 2m+1 \text{ and}$$

 $c_i = \alpha_{i0} + \sum_{j=2m+2}^{i} \alpha_{ij} a_j = \beta_{i0} \text{ for } i > 2m+1.$

Therefore $c_i = \alpha_{i0}$ for i = 0, 1, ..., 2m + 1 and $c_i = \beta_{i0}$ for i = 2m + 2, ..., 3m + 1.

In the case n = 24m + 8 we have $b_0 = 0$, $b_1 = 1$ and $b_2 = \cdots = b_{m-1} = 0$. Hence $c_i = \alpha_{i0}$ for $i = 0, 1, \ldots, 2m + 1$ and $c_i = \beta_{i1}$ for $i = 2m + 2, \ldots, 3m + 1$.

Similarly, if n = 24m + 18 we obtain $c_i = \alpha_{i0}$ for $i = 0, 1, \ldots, 2m + 1$ and $c_i = \beta_{i0}$ for $i = 2m + 2, \ldots, 3m + 2$. In both cases the weight enumerator can be computed as above.

By (3) and (4), the values of c_i can be calculated and they depend only on the length n. Thus the weight enumerators are unique in all cases.

In [15], Zhang obtained upper bounds for the lengths of the extremal binary doublyeven codes. He proved that extremal doubly-even codes of length n = 24m + 8l do not exist if $m \ge 154$ (for l = 0), $m \ge 159$ (for l = 1) and $m \ge 164$ (for l = 2). For extremal singly-even codes there is also a bound due to Rains [11]. Unfortunately, he only states the existence of a bound. In the next corollary we give explicit bounds for extremal singly-even self-dual codes with minimal shadow for lengths congruent 8, 12, 14 and 18 mod 24. In the proof we need the value of $c_{2m} = \alpha_{2m,0}$. According to [10] we have

$$\begin{aligned} \alpha_{2m}(n) &= -\frac{24m+2t}{4m} [\text{coeff. of } y^{2m-1} \text{ in } (1+y)^{-4m-t-1}(1-y)^{-4m}] \\ &= -\frac{12m+t}{2m} [\text{coeff. of } y^{2m-1} \text{ in } (1-y)^{t+1}(1-y^2)^{-4m-t-1}] \\ &= -\frac{12m+t}{2m} [\text{coeff. of } y^{2m-1} \text{ in } (1-y)^{t+1} \sum_{j=0}^m \binom{4m+t+j}{j} y^{2j}] \\ &= \frac{12m+t}{2m} \sum_{s=1}^{\lfloor \frac{t+2}{2} \rfloor} \binom{t+1}{2s-1} \binom{5m+t-s}{m-s} \end{aligned}$$

where t = 4l + r and n = 24m + 8l + 2r = 24m + 2t. The values for $\alpha_{2m}(n)$ are listed in Table 2.

n	$\alpha_{2m}(n)$
24m + 8	$\frac{8(4m+1)(11m+3)(3m+1)}{m(m-1)(m-2)} \binom{5m+1}{m-3}$
24m + 12	$\frac{24(116m^2 + 79m + 15)(1 + 2m)^2}{m(m-1)(m-2)(m-3)} \binom{5m+2}{m-4}$
24m + 14	$\frac{24(1+2m)(12m+7)(28m^2+22m+5)}{m(m-1)(m-2)(m-3)} \binom{5m+3}{m-4}$
24m + 16	$\frac{16(3m+2)(2m+1)(1216m^3+1956m^2+1073m+210)}{m(m-1)(m-2)(m-3)(m-4)} \binom{5m+3}{m-5}$
24m + 18	$\frac{120(2m+1)(4m+3)(176m^3+308m^2+189m+42)}{m(m-1)(m-2)(m-3)(m-4)} \binom{5m+4}{m-5}$
24m + 20	$\frac{16(6m+5)(2m+1)(4m+3)(1592m^3+3280m^2+2363m+630)}{m(m-1)(m-2)(m-3)(m-4)(m-5)} \binom{5m+4}{m-6}$

Furthermore, $\beta_{2m,j} = 2^{-t} \frac{3m+l-j}{2m} {5m+l-1-j \choose m+l-j}$. Hence $\beta_{2m,m+l} = 2^{-t}$ and $\beta_{2m,m+l-1} = 2^{1-t}(2m+1)$.

Corollary 4 There are no extremal singly-even self-dual codes of length n with minimal shadow if

- (i) n = 24m + 8 and $m \ge 53$,
- (*ii*) n = 24m + 12 and $m \ge 142$,

- (iii) n = 24m + 14 and $m \ge 146$,
- (iv) n = 24m + 18 and $m \ge 157$.

Proof. Using the equation

$$c_i = \alpha_{i0} = \beta_{i\epsilon} + \sum_{j=m}^{3m+l-i} \beta_{ij} b_j \quad \text{for } i \le 2m+1,$$

where $\epsilon = 1$ if n = 24m + 8 and $\epsilon = 0$ in the other cases, we see that

$$b_{m+l-1} = -2^{t-6} (\alpha_{2m+1,0} - \beta_{2m+1,\epsilon}).$$

The values of b_m for n = 24m + 8, 24m + 12 and 24m + 14 are given in Table 3.

Table 3: The parameter b_m for extremal self-dual codes of length n

n	24m	2 + 8	24m	+ 12	24m + 14	
b_m	$\frac{6m+1}{m}$	$\begin{pmatrix} 5m\\m-1 \end{pmatrix}$	$\frac{12m+5}{2m+1}$	$\binom{5m+1}{m}$	$\frac{168m^2 + 164m + 39}{(2m+1)(4m+3)} \binom{5m+3}{m}$	1

If n = 24m + 18 we have

$$b_m = 0$$
 and $b_{m+1} = \frac{(24m+17)(17m+10)}{(2m+1)(4m+5)} {5m+2 \choose m+1}.$

In the first three cases we compute

$$b_{m+1} = \frac{\alpha_{2m,0} - \beta_{2m,\epsilon} - \beta_{2m,m} b_m}{\beta_{2m,m+1}}.$$

If n = 24m + 8 we obtain

$$b_{m+1} = \frac{16(6m+1)(-4m^3 + 209m^2 + 141m + 24)}{5m(m+1)(4m+3)} \binom{5m+1}{m-1}$$

In case $m \ge 53$ the polynomial $-4m^3 + 209m^2 + 141m + 24$ takes negative values, hence $b_{m+1} < 0$, a contradiction.

For 24m + 12 we have

$$b_{m+1} = \frac{2(12m+5)(-32m^4+4496m^3+4242m^2+1257m+117)}{(5m+1)(4m+3)(4m+5)(2m+3)} \binom{5m+2}{m+1}$$

If $m \ge 142$ the polynomial $-32m^4 + 4496m^3 + 4242m^2 + 1257m + 117$ takes negative values, hence $b_{m+1} < 0$, a contradiction.

For 24m + 14 the calculations lead to

$$b_{m+1} = \frac{2(-5376m^6 + 772352m^5 + 1663728m^4 + 1386448m^3 + 557970m^2 + 107643m + 7875)}{(4m+3)(4m+5)(2m+3)(4m+7)(5m+1)} \binom{5m+2}{m+1}$$

which is negative if $m \ge 146$.

In the last case we have to compute

$$b_{m+2} = \frac{\alpha_{2m,0} - \beta_{2m,0} - \beta_{2m,m+1}b_{m+1}}{\beta_{2m,m+2}}$$

The computations yield

$$b_{m+2} = \frac{2(24m+17)(-544m^5+83696m^4+184210m^3+149089m^2+52809m+6930)}{(4m+5)(2m+3)(4m+7)(4m+9)(5m+2)} \binom{5m+3}{m+2}$$

which is negative for $m \ge 157$.

Proposition 5 If there are no extremal doubly-even self-dual codes of length n = 24m + 8 or 24m + 16 then there are no extremal singly-even self-dual codes of length n with minimal shadow.

Proof. We shall prove the contraposition. Let C be a singly-even self-dual [n = 24m + 8l, 12m + 4l, 4m + 4] code and suppose that the coset C_1 contains the vector u of weight 4. If $v \in C_3$ then $u + v \in C_2$ and hence wt $(u + v) \ge 4m + 6$. It follows that

$$wt(v) \ge 4m + 6 - 4 + 2wt(u * v) \ge 4m + 4,$$

since C_1 is not orthogonal to C_3 , which means that $u * v \equiv 1 \pmod{2}$ for $u \in C_1, v \in C_3$ (see [4]). Thus wt $(C_3) \ge 4m + 4$. Therefore $C_0 \cup C_3$ is an extremal doubly-even code with parameters [24m + 8l, 12m + 4l, 4m + 4].

Corollary 6 There are no extremal singly-even self-dual codes with minimal shadow of length n = 24m + 16 for $m \ge 164$.

Proof. This follows immediately from the Zhang bound [15] for doubly-even codes in connection with Proposition 5. \Box

Summarizing the results in Theorem 2, Corollary 4 and Corollary 6 we have proved either the non-existence or an explicit bound for the length n of an extremal singly-even self-dual code unless $n \equiv 20 \pmod{24}$. To find an explicit bound for n = 24m + 20seems to be difficult since the weight enumerator is not unique in this case. **Remark 3** Extremal singly-even self-dual codes of length 24m + 8 are constructed only for m = 1, i.e. n = 32. There are exactly three inequivalent singly-even self-dual [32, 16, 8] codes. Yorgov proved that there are no extremal singly-even self-dual codes with minimal shadow of length 24m + 8 in the case m is even and $\binom{5m}{m}$ is odd [14].

Examples. Extremal singly-even self-dual codes of lengths 24m + 12, 24m + 14 and 24m + 18:

m = 0: There are unique extremal singly-even codes of lengths 12, 14 and 16, and they have minimal shadows. There are two inequivalent self-dual [18,9,4] codes, but only one of them is a code with minimal shadow (see [5]).

m = 1: Extremal self-dual codes of lengths 36, 38 and 42 with minimal shadow are constructed. Only for the length 36 there is a complete classification [9]. There are 16 inequivalent self-dual [36, 18, 8] codes with minimal shadow and their weight enumerator is $W = 1 + 225y^8 + 2016y^{10} + 9555y^{12} + \cdots$ (see [7]).

m = 2: There exists a doubly circulant code with parameters [60, 30, 12] and shadow of minimum weight 2, denoted by D13 in [5]. The first examples for extremal self-dual codes with minimal shadow of lengths 62 and 66 are constructed in [12] and [13], respectively.

Finally, we would like to mention that similar to the case of extremal doubly-even self-dual codes there is a large gap between the bounds for extremal singly-even self-dual codes and what we really can construct.

References

- C. Bachoc and P. Gaborit, Designs and self-dual codes with long shadows, J. Combin. Theory Ser. A, 105 (2004), 15–34.
- [2] S. Bouyuklieva, A. Malevich and W. Willems, On the performance of binary extremal self-dual codes, Advances in Mathematics of Communications 5 (2011), 267–274.
- [3] S. Bouyuklieva and Z. Varbanov, Some connections between self-dual codes, combinatorial designs and secret sharing schemes, Advances in Mathematics of Communications 5 (2011), 191–198.
- [4] R. Brualdi and V. Pless, Weight Enumerators of Self-Dual Codes, *IEEE Trans. Inform. Theory* 37 (1991), 1222–1225.
- [5] J.H.Conway and N.J.A.Sloane, A new upper bound on the minimal distance of self-dual codes, *IEEE Trans. Inform. Theory*, 36 (1990), 1319–1333.

- [6] N. Elkies, Lattices and codes with longshadows, Math. Res. Lett. 2 (5) (1995), 643-651.
- [7] C.A. Melchor and P. Gaborit, On the classification of extremal [36, 18, 8] binary self-dual codes, *IEEE Trans. Inform. Theory*, 54 (2008), 4743–4750.
- [8] S. Han and J.B. Lee, Nonexistence of some extremal self-dual codes, J. Korean Math. Soc. 43 (2006), No. 6, 1357-1369.
- [9] W.C. Huffman, On the classification and enumeration of self-dual codes, *Finite Fields Appl.* 11 (2005), 451–490.
- [10] E.M. Rains, Shadow bounds for self-dual codes, IEEE Trans. Inform. Theory 44 (1998), 134–139.
- [11] E.M. Rains, New asymptotic bounds for self-dual codes and lattices, *IEEE Trans.* Inform. Theory 49 (2003), 1261–1274.
- [12] R. Russeva and N. Yankov, On binary self-dual codes of lengths 60, 62, 64 and 66 having an automorphism of order 9, *Designs, Codes and Cryptography* 45 (2007), 335-346.
- [13] H.P. Tsai, Extremal self-dual codes of length 66 and 68, IEEE Trans. Inform. Theory 45 (1999), 2129-2133.
- [14] V. Yorgov, On the minimal weight of some singly-even codes, *IEEE Transactions* on Information Theory 45 (1999), 2539-2541.
- [15] S. Zhang, On the nonexistence of extremal self-dual codes, *Discrete Appl. Math.* 91 (1999), 277-286.