# Singly-even self-dual codes with minimal shadow 

Stefka Bouyuklieva* and Wolfgang Willems<br>Faculty of Mathematics, University of Magdeburg 39016 Magdeburg, Germany


#### Abstract

In this note we investigate extremal singly-even self-dual codes with minimal shadow. For particular parameters we prove non-existence of such codes. By a result of Rains [11, the length of extremal singly-even self-dual codes is bounded. We give explicit bounds in case the shadow is minimal.


Index Terms: self-dual codes, singly-even codes, minimal shadow, bounds

## 1 Introduction

Let $C$ be a singly-even self-dual $\left[n, \frac{n}{2}, d\right]$ code and let $C_{0}$ be its doubly-even subcode. There are three cosets $C_{1}, C_{2}, C_{3}$ of $C_{0}$ such that $C_{0}^{\perp}=C_{0} \cup C_{1} \cup C_{2} \cup C_{3}$, where $C=C_{0} \cup C_{2}$. The set $S=C_{1} \cup C_{3}=C_{0}^{\perp} \backslash C$ is called the shadow of $C$. Shadows for self-dual codes were introduced by Conway and Sloane [5] in order to derive new upper bounds for the minimum weight of singly-even self-dual codes and to provide restrictions on their weight enumerators.

According to [10] the minimum weight $d$ of a self-dual code of length $n$ is bounded by $4[n / 24]+4$ for $n \not \equiv 22(\bmod 24)$ and by $4[n / 24]+6$ if $n \equiv 22(\bmod 24)$. We call a self-dual code meeting this bound extremal. Note that for some lengths, for instance length 34 , no extremal self-dual codes exist.

Some properties of the weight enumerator of $S$ are given in the following theorem.
Theorem 1 [5] Let $S(y)=\sum_{r=0}^{n} B_{r} y^{r}$ be the weight enumerator of $S$. Then

- $B_{r}=B_{n-r}$ for all $r$,
- $B_{r}=0$ unless $r \equiv n / 2(\bmod 4)$,
- $B_{0}=0$,

[^0]- $B_{r} \leq 1$ for $r<d / 2$,
- $B_{d / 2} \leq 2 n / d$,
- at most one $B_{r}$ is nonzero for $r<(d+4) / 2$.

Elkies studied in [6] the minimum weight $s$ (respectively the minimum norm) of the shadow of self-dual codes (respectively of unimodular lattices), especially in the cases where it attains a high value. Bachoc and Gaborit proposed to study the parameters $d$ and $s$ simultaneously [1. They proved that $2 d+s \leq \frac{n}{2}+4$, except in the case $n \equiv 22$ $(\bmod 24)$ where $2 d+s \leq \frac{n}{2}+8$. They called the codes attaining this bound $s$-extremal. In this note we study singly-even self-dual codes for which the minimum weight of the shadow has smallest possible value. possible.

Definition 1 We say that a self-dual code $C$ of length $24 m+8 l+2 r$ with $r=1,2,3$ and $l=0,1,2$ is a code with minimal shadow if $\operatorname{wt}(S)=r$. For $r=0, C$ is called of minimal shadow if $\operatorname{wt}(S)=4$.

Self-dual codes with minimal shadow are subject of two previous articles. The paper [3] is devoted to connections between self-dual codes of length $24 m+8 l+2$ with $\mathrm{wt}(S)=1$, combinatorial designs and secret sharing schemes. The structure of these codes are used to characterize access groups in a secret sharing scheme based on codes. There are two types of schemes which are proposed - with one-part secret and with two-part secret. Moreover, some of the considered codes support 1- and 2-designs. The performance of the extremal self-dual codes of length $24 m+8 l$ where $l=1,2$ have been studied in [2]. In particular, different types of codes with the same parameters are compared with regard to the decoding error probability. It turned out that for lengths $24 m+8$ singly-even codes with minimal shadow perform better than doubly-even codes. Thus from the point of view of data correction one is interested in singly-even codes with minimal shadow.

This article is organized as follows. In Section 2 we prove that extremal self-dual codes with minimal shadow of length $24 m+2 t$ for $t=1,2,3,5,11$ do not exist. Moreover, for $t=4,6,7$ and 9 , we obtain upper bounds for the length. We also prove that if extremal doubly-even self-dual codes of length $n=24 m+8$ or $24 m+16$ do not exist then extremal singly-even self-dual codes with minimal shadow do not exist for the same length. The only case for which we do not have a bound for the length is $n=24 m+20$.

All computations have been carried out with Maple.

## 2 Extremal self-dual codes with minimal shadow

Let $C$ be a singly-even self-dual code of length $n=24 m+8 l+2 r$ where $l=0,1,2$ and $r=0,1,2,3$. The weight enumerator of $C$ and its shadow are given by [5]:

$$
\begin{gathered}
W(y)=\sum_{j=0}^{12 m+4 l+r} a_{j} y^{2 j}=\sum_{i=0}^{3 m+l} c_{i}\left(1+y^{2}\right)^{12 m+4 l+r-4 i}\left(y^{2}\left(1-y^{2}\right)^{2}\right)^{i} \\
S(y)=\sum_{j=0}^{6 m+2 l} b_{j} y^{4 j+r}=\sum_{i=0}^{3 m+l}(-1)^{i} c_{i} 2^{12 m+4 l+r-6 i} y^{12 m+4 l+r-4 i}\left(1-y^{4}\right)^{2 i}
\end{gathered}
$$

Using these expressions we can write $c_{i}$ as a linear combination of the $a_{j}$ and as a linear combination of the $b_{j}$ in the following way [10]:

$$
\begin{equation*}
c_{i}=\sum_{j=0}^{i} \alpha_{i j} a_{j}=\sum_{j=0}^{3 m+l-i} \beta_{i j} b_{j} . \tag{1}
\end{equation*}
$$

Suppose $C$ is an extremal singly-even self-dual code with minimal shadow, hence $d=4 m+4$ and $\operatorname{wt}(S)=r$ if $r=1,2,3$ and $\operatorname{wt}(S)=4$ if $r=0$. Obviously in this case $a_{0}=1, a_{1}=a_{2}=\cdots=a_{2 m+1}=0$. According to Theorem 1, we have $b_{0}=1$ if $r>0$ and $m \geq 1$, and $b_{0}=0, b_{1}=1$ if $r=0$ and $m \geq 2$.

Moreover, if $r>0$ and $m \geq 1$ then $b_{1}=b_{2}=\cdots=b_{m-1}=0$. Otherwise $S$ would contain a vector $v$ of weight less than or equal to $4 m-4+r$, and if $u \in S$ is a vector of weight $r$ then $u+v \in C$ with $\operatorname{wt}(u+v) \leq 4 m+2 r-4 \leq 4 m+2$, a contradiction to the minimum distance of $C$. Similarly, if $r=0$ and $m \geq 2$ then $b_{2}=\cdots=b_{m-1}=0$.

Remark 1 For extremal self-dual codes of length $24 m+8 l+2$ we furthermore have $b_{m}=0$. Otherwise $S$ would contain a vector $v$ of weight $4 m+1$, and if $u \in S$ is the vector of weight 1 which exists since $\operatorname{wt}(S)=1$, then $u+v \in C$ with $\mathrm{wt}(u+v) \leq 4 m+2$ contradicting the minimum distance of $C$.

If $m \geq 2$ we have by (1)

$$
\begin{equation*}
c_{2 m+1}=\alpha_{2 m+1,0}=\beta_{2 m+1, \epsilon}+\sum_{j=m}^{m+l-1} \beta_{2 m+1, j} b_{j}, \tag{2}
\end{equation*}
$$

where $\epsilon=1$ for $r=0$ and $\epsilon=0$ otherwise, since $3 m+l-2 m-1=m+l-1$. To evaluate this equation, which turns out to be crucial in the following, we need to consider the coefficients $\alpha_{i 0}$ in details. In order to do this we denote by $\alpha_{i}(n)$ the coefficient $\alpha_{i 0}$ if $n$ is the length of the code. According to [10] we have

$$
\begin{equation*}
\alpha_{i}(n)=\alpha_{i 0}=-\frac{n}{2 i}\left[\text { coeff. of } y^{i-1} \text { in }(1+y)^{-n / 2-1+4 i}(1-y)^{-2 i}\right] . \tag{3}
\end{equation*}
$$

Let $t=4 l+r$ and $n=24 m+8 l+2 r=24 m+2 t$. Then

$$
\begin{aligned}
\alpha_{2 m+1}(n) & =-\frac{12 m+t}{2 m+1}\left[\text { coeff. of } y^{2 m} \text { in }(1+y)^{-12 m-t-1+8 m+4}(1-y)^{-4 m-2}\right] \\
& =-\frac{12 m+t}{2 m+1}\left[\text { coeff. of } y^{2 m} \text { in }(1+y)^{-4 m-t+3}(1-y)^{-4 m-2}\right]
\end{aligned}
$$

For $t>5$ we obtain

$$
\alpha_{2 m+1}(n)=-\frac{12 m+t}{2 m+1}\left[\text { coeff. of } y^{2 m} \text { in }\left(1-y^{2}\right)^{-4 m-t+3}(1-y)^{t-5}\right],
$$

and if $t \leq 5$ then

$$
\alpha_{2 m+1}(n)=-\frac{12 m+t}{2 m+1}\left[\text { coeff. of } y^{2 m} \text { in }\left(1-y^{2}\right)^{-4 m-2}(1+y)^{5-t}\right] .
$$

Since

$$
\left(1-y^{2}\right)^{-a}=\sum_{0 \leq j}\binom{-a}{j}(-1)^{j} y^{2 j}=\sum_{0 \leq j}\binom{a+j-1}{j} y^{2 j} \text { for } a>0,
$$

it follows in case $t \leq 5$ that

$$
\begin{aligned}
\alpha_{2 m+1}(n) & =-\frac{12 m+t}{2 m+1}\left[\text { coeff. of } y^{2 m} \text { in }(1+y)^{5-t} \sum_{j=0}^{m}\binom{4 m+j+1}{j} y^{2 j}\right] \\
& =-\frac{12 m+t}{2 m+1} \sum_{s=0}^{\left[\frac{5-t}{2}\right]}\binom{5-t}{2 s}\binom{5 m+1-s}{m-s}
\end{aligned}
$$

and in case $t>5$ that

$$
\begin{aligned}
\alpha_{2 m+1}(n) & =-\frac{12 m+t}{2 m+1}\left[\text { coeff. of } y^{2 m} \text { in }(1-y)^{t-5} \sum_{j=0}^{m}\binom{4 m+t+j-4}{j} y^{2 j}\right] \\
& =-\frac{12 m+t}{2 m+1} \sum_{s=0}^{\left[\frac{t-5}{2}\right]}\binom{t-5}{2 s}\binom{5 m+t-4-s}{m-s}
\end{aligned}
$$

For the different lengths $n$ the values of $\alpha_{2 m+1}(n)$ are listed in Table 1 .
To evaluate equation (2) we also need $\beta_{i j}$ which are known due to [10]. Here we have

$$
\begin{equation*}
\beta_{i j}=(-1)^{i} 2^{-n / 2+6 i} \frac{k-j}{i}\binom{k+i-j-1}{k-i-j}, \tag{4}
\end{equation*}
$$

Table 1: The values $\alpha_{2 m+1}(n)$ for extremal self-dual codes

| $n$ | $24 m+2$ | $24 m+10$ | $24 m+18$ |
| :---: | :---: | :---: | :---: |
| $\alpha_{2 m+1}$ | $-\frac{(12 m+1)(56 m+4)}{(2 m+1)(m-1)}\binom{5 m-1}{m-2}$ | $-\frac{12 m+5}{2 m+1}\binom{5 m+1}{m}$ | $-\frac{12(7 m+5)(4 m+3)}{m(m-1)}\binom{5 m+3}{m-2}$ |
| $n$ | $24 m+4$ | $24 m+12$ | $24 m+20$ |
| $\alpha_{2 m+1}$ | $-\frac{2(6 m+1)(8 m+1)}{m(2 m+1)}\binom{5 m}{m-1}$ | $-6\binom{5 m+2}{m}$ | $-\frac{20(6 m+5)(4 m+3)}{m(m-1)}\binom{5 m+4}{m-2}$ |
| $n$ | $24 m+6$ | $24 m+14$ | $24 m+22$ |
| $\alpha_{2 m+1}$ | $-\frac{3(4 m+1)(6 m+1)}{m(2 m+1)}\binom{5 m}{m-1}$ | $-\frac{3(12 m+7)}{m}\binom{5 m+2}{m-1}$ | $-\frac{6(12 m+11)(6 m+5)(8 m+7)}{m m+4}\binom{5 m+1}{m-3}$ |
| $n$ | $-\frac{4(3 m+1)}{2 m+1}\binom{5 m+1}{m}$ | $-\frac{16(3 m+2)}{m}\binom{5 m+3}{m-1}$ |  |
| $\alpha_{2 m+1}$ |  |  |  |

where $k=\lfloor n / 8\rfloor=3 m+l$. In particular,

$$
\beta_{2 m+1, j}=-2^{6-t} \frac{3 m+l-j}{2 m+1}\binom{5 m+l-j}{m+l-1-j} \quad \text { and } \quad \beta_{2 m+1, m+l-1}=-2^{6-t} .
$$

Now we are prepared to prove:
Theorem 2 Extremal self-dual codes of lengths $n=24 m+2,24 m+4,24 m+6$, $24 m+10$ and $24 m+22$ with minimal shadow do not exist.

Proof. According to [10] any extremal self-dual code of length $24 m+22$ has minimum distance $4 m+6$ and the minimum weight of its shadow is $4 m+7$. Thus the shadow is not minimal since a minimal shadow must have minimum weight 3 . (There is a misprint in [10] where it is stated that the minimum weight of the shadow is $4 m+6$. But actually the weights in this shadow are of type $4 j+3$ ).

In the other four cases we have

$$
\begin{equation*}
c_{2 m+1}=\alpha_{2 m+1,0}=\beta_{2 m+1,0} \tag{5}
\end{equation*}
$$

by (21). In case $n=24 m+10$ we use the fact that $b_{m}=0$, according to Remark 1 .
Simplifying equation (5) according to Table 1 we obtain

$$
\begin{aligned}
48 m^{2}+26 m+1=0, & \text { if } n=24 m+2 \\
24 m^{2}+14 m+1=0, & \text { if } n=24 m+4 \\
48 m^{2}+30 m+3=0, & \text { if } n=24 m+6 \\
6 m+3=0, & \text { if } n=24 m+10
\end{aligned}
$$

Since all these equations have no solutions $m \geq 0$ extremal self-dual codes with minimal shadow do not exist for $n \equiv 2,4,6,10 \bmod 24$.

Remark 2 So far no extremal self-dual codes of length $24 m+2 t$ are known for $t=$ $1,2,3,5$. According to [8] extremal self-dual codes of length $24 m+2 r$ do not exist for $r=1,2,3$ and $m=1,2, \ldots, 6,8, \ldots, 12,16, \ldots, 22$. Thus if there is (for instance) a self-dual $[170,85,32]$ code it will not have minimal shadow, by Theorem 2 .

The next result is a crucial observation in order to prove explicit bounds for the existence of extremal singly-even self-dual codes.

Theorem 3 Extremal singly-even self-dual codes with minimal shadow of lengths $n=$ $24 m+8,24 m+12,24 m+14$ and $24 m+18$ have uniquely determined weight enumerators.

Proof. For $m=0$ and $m=1$ see Remark 3 and the examples at the end of the paper. Now let $m \geq 2$.

In case $n=24 m+12$ or $n=24 m+14$ we have

$$
\begin{gathered}
c_{i}=\alpha_{i 0}=\beta_{i 0}+\sum_{j=m}^{3 m+1-i} \beta_{i j} b_{j} \quad \text { for } i \leq 2 m+1 \quad \text { and } \\
c_{i}=\alpha_{i 0}+\sum_{j=2 m+2}^{i} \alpha_{i j} a_{j}=\beta_{i 0} \quad \text { for } i>2 m+1
\end{gathered}
$$

Therefore $c_{i}=\alpha_{i 0}$ for $i=0,1, \ldots, 2 m+1$ and $c_{i}=\beta_{i 0}$ for $i=2 m+2, \ldots, 3 m+1$.
In the case $n=24 m+8$ we have $b_{0}=0, b_{1}=1$ and $b_{2}=\cdots=b_{m-1}=0$. Hence $c_{i}=\alpha_{i 0}$ for $i=0,1, \ldots, 2 m+1$ and $c_{i}=\beta_{i 1}$ for $i=2 m+2, \ldots, 3 m+1$.

Similarly, if $n=24 m+18$ we obtain $c_{i}=\alpha_{i 0}$ for $i=0,1, \ldots, 2 m+1$ and $c_{i}=\beta_{i 0}$ for $i=2 m+2, \ldots, 3 m+2$. In both cases the weight enumerator can be computed as above.

By (3) and (4), the values of $c_{i}$ can be calculated and they depend only on the length $n$. Thus the weight enumerators are unique in all cases.

In [15, Zhang obtained upper bounds for the lengths of the extremal binary doublyeven codes. He proved that extremal doubly-even codes of length $n=24 m+8 l$ do not exist if $m \geq 154$ (for $l=0$ ), $m \geq 159$ (for $l=1$ ) and $m \geq 164$ (for $l=2$ ). For extremal singly-even codes there is also a bound due to Rains [11. Unfortunately, he only states the existence of a bound. In the next corollary we give explicit bounds for extremal singly-even self-dual codes with minimal shadow for lengths congruent $8,12,14$ and 18 $\bmod 24$.

In the proof we need the value of $c_{2 m}=\alpha_{2 m, 0}$. According to 10 we have

$$
\begin{aligned}
\alpha_{2 m}(n) & =-\frac{24 m+2 t}{4 m}\left[\text { coeff. of } y^{2 m-1} \text { in }(1+y)^{-4 m-t-1}(1-y)^{-4 m}\right] \\
& =-\frac{12 m+t}{2 m}\left[\text { coeff. of } y^{2 m-1} \text { in }(1-y)^{t+1}\left(1-y^{2}\right)^{-4 m-t-1}\right] \\
& =-\frac{12 m+t}{2 m}\left[\text { coeff. of } y^{2 m-1} \text { in }(1-y)^{t+1} \sum_{j=0}^{m}\binom{4 m+t+j}{j} y^{2 j}\right] \\
& =\frac{12 m+t}{2 m} \sum_{s=1}^{\left[\frac{t+2}{2}\right]}\binom{t+1}{2 s-1}\binom{5 m+t-s}{m-s}
\end{aligned}
$$

where $t=4 l+r$ and $n=24 m+8 l+2 r=24 m+2 t$. The values for $\alpha_{2 m}(n)$ are listed in Table 2

Table 2: The values $\alpha_{2 m}(n)$ for an extremal self-dual $\left[n=24 m+2 t, \frac{n}{2}, 4 m+4\right]$ code

| $n$ | $\alpha_{2 m}(n)$ |
| :---: | :---: |
| $24 m+8$ | $\frac{8(4 m+1)(11 m+3)(3 m+1)}{m(m-1)(m-2)}\binom{5 m+1}{m-3}$ |
| $24 m+12$ | $\frac{24\left(116 m^{2}+79 m+15\right)(1+2 m)^{2}}{m(m-1)(m-2)(m-3)}\binom{5 m+2}{m-4}$ |
| $24 m+14$ | $\frac{24(1+2 m)(12 m+7)\left(28 m^{2}+22 m+5\right)}{m(m-1)(m-2)(m-3)}\binom{5 m+3}{m-4}$ |
| $24 m+16$ | $\frac{16(3 m+2)(2 m+1)\left(1216 m^{3}+1956 m^{2}+1073 m+210\right)}{m(m-1)(m-2)(m-3)(m-4)}\binom{5 m+3}{m-5}$ |
| $24 m+18$ | $\frac{120(2 m+1)(4 m+3)\left(176 m^{3}+308 m^{2}+189 m+42\right)}{m(m-1)(m-2)(m-3)(m-4)}\binom{5 m+4}{m-5}$ |
| $24 m+20$ | $\frac{16(6 m+5)(2 m+1)(4 m+3)\left(1592 m^{3}+3280 m^{2}+2363 m+630\right)}{m(m-1)(m-2)(m-3)(m-4)(m-5)}\binom{5 m+4}{m-6}$ |

Furthermore, $\beta_{2 m, j}=2^{-t} \frac{3 m+l-j}{2 m}\binom{5 m+l-1-j}{m+l-j}$. Hence $\beta_{2 m, m+l}=2^{-t}$ and $\beta_{2 m, m+l-1}=2^{1-t}(2 m+1)$.

Corollary 4 There are no extremal singly-even self-dual codes of length $n$ with minimal shadow if
(i) $n=24 m+8$ and $m \geq 53$,
(ii) $n=24 m+12$ and $m \geq 142$,
(iii) $n=24 m+14$ and $m \geq 146$,
(iv) $n=24 m+18$ and $m \geq 157$.

Proof. Using the equation

$$
c_{i}=\alpha_{i 0}=\beta_{i \epsilon}+\sum_{j=m}^{3 m+l-i} \beta_{i j} b_{j} \quad \text { for } i \leq 2 m+1,
$$

where $\epsilon=1$ if $n=24 m+8$ and $\epsilon=0$ in the other cases, we see that

$$
b_{m+l-1}=-2^{t-6}\left(\alpha_{2 m+1,0}-\beta_{2 m+1, \epsilon}\right)
$$

The values of $b_{m}$ for $n=24 m+8,24 m+12$ and $24 m+14$ are given in Table 3.

Table 3: The parameter $b_{m}$ for extremal self-dual codes of length $n$

| $n$ | $24 m+8$ | $24 m+12$ | $24 m+14$ |
| :---: | :---: | :---: | :---: |
| $b_{m}$ | $\frac{6 m+1}{m}\binom{5 m}{m-1}$ | $\frac{12 m+5}{2 m+1}\binom{5 m+1}{m}$ | $\frac{168 m^{2}+164 m+39}{(2 m+1)(4 m+3)}\binom{5 m+1}{m}$ |

If $n=24 m+18$ we have

$$
b_{m}=0 \quad \text { and } \quad b_{m+1}=\frac{(24 m+17)(17 m+10)}{(2 m+1)(4 m+5)}\binom{5 m+2}{m+1} .
$$

In the first three cases we compute

$$
b_{m+1}=\frac{\alpha_{2 m, 0}-\beta_{2 m, \epsilon}-\beta_{2 m, m} b_{m}}{\beta_{2 m, m+1}}
$$

If $n=24 m+8$ we obtain

$$
b_{m+1}=\frac{16(6 m+1)\left(-4 m^{3}+209 m^{2}+141 m+24\right)}{5 m(m+1)(4 m+3)}\binom{5 m+1}{m-1}
$$

In case $m \geq 53$ the polynomial $-4 m^{3}+209 m^{2}+141 m+24$ takes negative values, hence $b_{m+1}<0$, a contradiction.

For $24 m+12$ we have

$$
b_{m+1}=\frac{2(12 m+5)\left(-32 m^{4}+4496 m^{3}+4242 m^{2}+1257 m+117\right)}{(5 m+1)(4 m+3)(4 m+5)(2 m+3)}\binom{5 m+2}{m+1}
$$

If $m \geq 142$ the polynomial $-32 m^{4}+4496 m^{3}+4242 m^{2}+1257 m+117$ takes negative values, hence $b_{m+1}<0$, a contradiction.

For $24 m+14$ the calculations lead to
$b_{m+1}=\frac{2\left(-5376 m^{6}+772352 m^{5}+1663728 m^{4}+1386448 m^{3}+557970 m^{2}+107643 m+7875\right)}{(4 m+3)(4 m+5)(2 m+3)(4 m+7)(5 m+1)}\binom{5 m+2}{m+1}$
which is negative if $m \geq 146$.
In the last case we have to compute

$$
b_{m+2}=\frac{\alpha_{2 m, 0}-\beta_{2 m, 0}-\beta_{2 m, m+1} b_{m+1}}{\beta_{2 m, m+2}} .
$$

The computations yield
$b_{m+2}=\frac{2(24 m+17)\left(-544 m^{5}+83696 m^{4}+184210 m^{3}+149089 m^{2}+52809 m+6930\right)}{(4 m+5)(2 m+3)(4 m+7)(4 m+9)(5 m+2)}\binom{5 m+3}{m+2}$
which is negative for $m \geq 157$.

Proposition 5 If there are no extremal doubly-even self-dual codes of length $n=24 m+$ 8 or $24 m+16$ then there are no extremal singly-even self-dual codes of length $n$ with minimal shadow.

Proof. We shall prove the contraposition. Let $C$ be a singly-even self-dual $[n=24 m+$ $8 l, 12 m+4 l, 4 m+4]$ code and suppose that the coset $C_{1}$ contains the vector $u$ of weight 4. If $v \in C_{3}$ then $u+v \in C_{2}$ and hence $\operatorname{wt}(u+v) \geq 4 m+6$. It follows that

$$
\mathrm{wt}(v) \geq 4 m+6-4+2 \mathrm{wt}(u * v) \geq 4 m+4
$$

since $C_{1}$ is not orthogonal to $C_{3}$, which means that $u * v \equiv 1(\bmod 2)$ for $u \in C_{1}, v \in C_{3}$ (see [4). Thus wt $\left(C_{3}\right) \geq 4 m+4$. Therefore $C_{0} \cup C_{3}$ is an extremal doubly-even code with parameters $[24 m+8 l, 12 m+4 l, 4 m+4]$.

Corollary 6 There are no extremal singly-even self-dual codes with minimal shadow of length $n=24 m+16$ for $m \geq 164$.

Proof. This follows immediately from the Zhang bound [15] for doubly-even codes in connection with Proposition 5.

Summarizing the results in Theorem 2, Corollary 4 and Corollary 6 we have proved either the non-existence or an explicit bound for the length $n$ of an extremal singly-even self-dual code unless $n \equiv 20(\bmod 24)$. To find an explicit bound for $n=24 m+20$ seems to be difficult since the weight enumerator is not unique in this case.

Remark 3 Extremal singly-even self-dual codes of length $24 m+8$ are constructed only for $m=1$, i.e. $n=32$. There are exactly three inequivalent singly-even self-dual $[32,16,8]$ codes. Yorgov proved that there are no extremal singly-even self-dual codes with minimal shadow of length $24 m+8$ in the case $m$ is even and $\binom{5 m}{m}$ is odd [14].

Examples. Extremal singly-even self-dual codes of lengths $24 m+12,24 m+14$ and $24 m+18$ :
$m=0$ : There are unique extremal singly-even codes of lengths 12,14 and 16 , and they have minimal shadows. There are two inequivalent self-dual $[18,9,4]$ codes, but only one of them is a code with minimal shadow (see [5).
$m=1$ : Extremal self-dual codes of lengths 36,38 and 42 with minimal shadow are constructed. Only for the length 36 there is a complete classification [9. There are 16 inequivalent self-dual $[36,18,8]$ codes with minimal shadow and their weight enumerator is $W=1+225 y^{8}+2016 y^{10}+9555 y^{12}+\cdots$ (see [7]).
$m=2$ : There exists a doubly circulant code with parameters [60,30,12] and shadow of minimum weight 2 , denoted by $D 13$ in [5]. The first examples for extremal selfdual codes with minimal shadow of lengths 62 and 66 are constructed in [12] and [13], respectively.

Finally, we would like to mention that similar to the case of extremal doubly-even self-dual codes there is a large gap between the bounds for extremal singly-even self-dual codes and what we really can construct.

## References

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[^0]:    *Supported by the Humboldt Foundation. On leave from Faculty of Mathematics and Informatics, Veliko Tarnovo University, 5000 Veliko Tarnovo, Bulgaria

