Distributed Source Coding of Correlated Gaussian Remote Sources

Yasutada Oohama

***Abstract*—We consider the distributed source coding system for L correlated Gaussian remote sources $X_i, i=1,2,\cdots,L$, where $X_i, i=1,2,\cdots,L$ are L correlated Gaussian random variables. We deal with the case where each of L distributed encoders can not directly observe X_i but its noisy version $Y_i = X_i + N_i$. Here $N_i, i=1,2,\cdots,L$ are independent additive L Gaussian noises also independent of $X_i, i=1,2,\cdots,L$. On this coding system the determination problem of the rate distortion region remains open. In this paper, we derive explicit outer and inner bounds of the rate distortion region. We further find an explicit sufficient condition for those two bounds to match. We also study the sum rate part of the rate distortion region when the correlation has some symmetrical property and derive a new lower bound of the sum rate part. We derive a sufficient condition for this lower bound to be tight. The derived sufficient condition depends only on the correlation property of the sources and their observations.**

***Index Terms*—Multiterminal source coding, Gaussian, rate-distortion region, CEO problem.**

I. Introduction

In multi-user source networks distributed coding of correlated information sources is a form of communication system which is significant from both theoretical and practical point of view. The first fundamental theory in those coding systems was established by Slepian and Wolf [1]. They considered a distributed source coding system of two correlated information sources. Those two sources are separately encoded and sent to a single destination, where the decoder reconstruct the original sources. In this system, Slepian and Wolf [1] determined the admissible rate region, the set that consists of a pair of transmission rates for which two sources can be decoded with an arbitrary small error probability.

In the above distributed source coding system we can consider the case where the source outputs should be reconstructed with average distortions smaller than prescribed levels. Such a situation suggests the multiterminal rate-distortion theory.

The rate distortion theory for the distributed source coding system formulated by Slepian and Wolf has been studied by [2]-[9]. Recently, Wagner *et al.* [10] have given a complete solution in the case of Gaussian information sources and mean squared distortion.

As a practical situation of the distributed source coding system, we can consider a case where the separate encoders can not directly observe the original source outputs but can observe their noisy versions. This situation was first studied by Yamamoto and Ito [11]. Subsequently, a similar distributed source coding system was studied by Flynn and R. M. Gray [12].

In this paper we consider the distributed source coding system for L correlated Gaussian remote sources $X_i, i = 1,2,\cdots,L$, where $X_i, i=1,2,\cdots,L$ are L correlated Gaussian random variables. We deal with the case where each of L distributed encoders can not directly observe X_i but its noisy version $Y_i = X_i + N_i$. Here $N_i, i = 1,2,\cdots,L$ are independent additive L Gaussian noises also independent of $X_i, i=1,2,\cdots,L$. In the above setup $Y_i, i=1,2,\cdots,L$ can be regarded as correlated Gaussian observations of $X_i, i = 1,2,\cdots,L$, respectively. This coding system can also be considered as a vector version of the Gaussian CEO problem investigated by [13], [14], and [15], where $X_i, i=1,2,\cdots,L$ are identical.

The above distributed source coding system was first posed and investigated by Pandya *et al.* [16]. They derived upper and lower bounds of the sum rate part of the rate distortion region. Oohama [17], [18] derived explicit outer and inner bounds of the rate distortion region. Wagner *et al.* [10] determined the rate distortion region in the case of $L=2$.

In [18], Oohama also derived a sufficient condition for his outer bound to coincide with the inner bound. Subsequently, Oohama [19] derived a matching condition which is simple and stronger than that of Oohama [18].

In this paper, we derive a new sufficient condition with respect to the source correlation and the distortion under which the inner and outer bounds match. We show that if the distortion is smaller than a threshold value which is a function of the source correlation, the inner and outer bounds match and find an explicit form of this threshold value. This sufficient condition is a significant improvement of the condition derived by Oohama [19]. We also investigate the sum rate part of rate distortion region. The optimal sum rate part of the outer bound derived by Oohama [18] serves as a lower bound of the sum rate part of the rate distortion region. When the covariance matrix Σ_{X^L} of the remote source $X^L = (X_1, X_2, \cdots, X_L)$ have a certain symmetrical property and the noise variances of $N_i, i=1,2,\cdots,L$ have an identical variance denoted by σ^2, we derive a new lower bound of the sum rate part. We further derive a sufficient condition for this lower bound to be tight. The derived sufficient condition depends only on Σ_{X^L} and σ^2. From this matching condition we can see that an explicit form of the sum rate part of the rate distortion region can be found when the noise variance σ^2 is relatively high compared with the eigen values of Σ_{X^L}.

In Oohama [17], [18], details of derivations of the inner and outer bound were omitted. In this paper we also present the details of derivation of those two bounds.

Manuscript received xxx, 20XX; revised xxx, 20XX.

Y. Oohama is with the Department of Information Science and Intelligent Systems, University of Tokushima, 2-1 Minami Josanjima-Cho, Tokushima 770-8506, Japan.

The rest of this paper is organized as follows. In Section II, we present problem formulations and state the previous works on those problems. In Section III, we give our main result. We first derive explicit inner and outer bounds of the rate distortion region. Next we presented an explicit sufficient condition for the outer bound to coincide with the inner bound. In Section IV, we explicitly compute the matching condition for two examples of Gaussian sources. In Sections V and VI we give the proofs of the results. Finally, in Section VII, we conclude the paper.

II. Problem Statement and Previous Results

A. Formal Statement of Problem

In this subsection we present a formal statement of problem. Throughout this paper all logarithms are taken to the base natural. Let $\Lambda=\{1,2,\cdots,L\}$ and let $X_i, i\in\Lambda$ be correlated zero mean Gaussian random variables taking values in the real lines $\mathcal{X}_i^n$. We write a L dimensional random vector as $X^L=(X_1,X_2,\cdots,X_L)$ and use similar notation of other random variables. We denote the covariance matrix of X^L by Σ_{X^L}. Let $\{(X_{1,t}, X_{2,t},\cdots,X_{L,t})\}_{t=1}^{\infty}$ be a stationary memoryless multiple Gaussian source. For each $t=1,2,\cdots$, $(X_{1,t},X_{2,t},\cdots, X_{L,t})$ obeys the same distribution as $(X_1, X_2,\cdots,X_L)$. Let a random vector consisting of n independent copies of the random variable X_i be denoted by $\boldsymbol{X}_i=X_{i,1}X_{i,2}\cdots X_{i,n}$. Furthermore, let $\boldsymbol{X}^L$ denote the random vector $(\boldsymbol{X}_1,\boldsymbol{X}_2,\cdots,\boldsymbol{X}_L)$.

We consider the separate coding system for L correlated sources, where L encoders can only access noisy version Y_i of X_i for $i=1,2,\cdots,L$, that is,

$$Y_i=X_i+N_i, i\in\Lambda \tag{1}$$

where N_i, $i\in\Lambda$ are zero mean independent Gaussian random variables with variance $\sigma_{N_i}^2$. We assume that X^L and N^L are independent. The separate coding system for L correlated Gaussian remote sources is shown in Fig. 1. For each $i\in\Lambda$, the noisy version $\boldsymbol{Y}_i$ of $\boldsymbol{X}_i$ is separately encoded to $\varphi_i(\boldsymbol{Y}_i)$. The L encoded data $\varphi_i(\boldsymbol{Y}_i)$, $i\in\Lambda$ are sent to the information processing center, where the decoder observes them and outputs the estimation $(\hat{\boldsymbol{X}}_1,\hat{\boldsymbol{X}}_2,\cdots,\hat{\boldsymbol{X}}_L)$ of $(\boldsymbol{X}_1,\boldsymbol{X}_2,\cdots,\boldsymbol{X}_L)$ by using the decoder function $\psi=(\psi_1,\psi_2,\cdots,\psi_L)$.

The encoder functions $\varphi_i\,, i\in\Lambda$ are defined by

$$\varphi_i:\mathcal{X}_i^n\to\mathcal{M}_i=\{1,2,\cdots,M_i\} \tag{2}$$

and satisfy rate constraints

$$\frac{1}{n}\log M_i\leq R_i+\delta \tag{3}$$

where δ is an arbitrary prescribed positive number. The decoder function $\psi=(\psi_1,\psi_2,\cdots,\psi_L)$ is defined by

$$\psi_i:\mathcal{M}_1\times\cdots\times\mathcal{M}_L\to\hat{\mathcal{X}}_i^n\,, i=1,2,\cdots,L, \tag{4}$$

where $\hat{\mathcal{X}}_i$ is the real line in which a reconstructed random variable of X_i takes values. Denote by $\mathcal{F}_\delta^{(n)}(R_1,R_2,\cdots,R_L)$ the set that consists of all the $(L+1)$ tuple of encoder and decoder functions $(\varphi_1,\varphi_2,\cdots,\varphi_L,\psi)$ satisfying (2)-(4). For $\boldsymbol{X}^L=(\boldsymbol{X}_1,\boldsymbol{X}_2,\cdots,\boldsymbol{X}_L)$ and its estimation

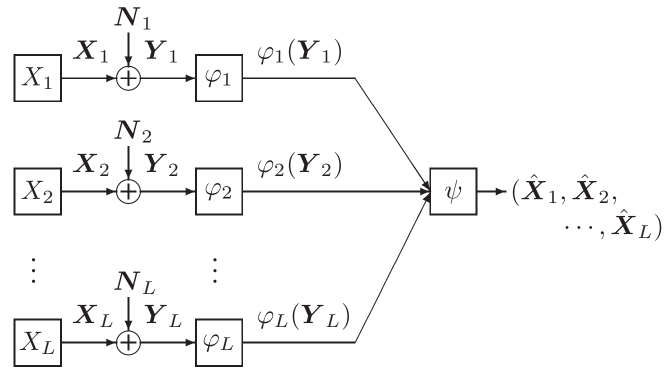

Fig. 1. Separate coding system for L correlated Gaussian observations

$$\begin{aligned}\hat{\boldsymbol{X}}^L&=(\hat{\boldsymbol{X}}_1,\hat{\boldsymbol{X}}_2,\cdots,\hat{\boldsymbol{X}}_L)\\&\triangleq(\psi_1(\varphi_1(\boldsymbol{Y}_1)),\psi_2(\varphi_2(\boldsymbol{Y}_2)),\cdots,\psi_L(\varphi_L(\boldsymbol{Y}_L)),\end{aligned}$$

set

$$\begin{aligned}d_{ii}&\triangleq\mathrm{E}||\boldsymbol{X}_i-\hat{\boldsymbol{X}}_i||^2\,,\\d_{ij}&\triangleq\mathrm{E}\langle\boldsymbol{X}_i-\hat{\boldsymbol{X}}_i,\boldsymbol{X}_j-\hat{\boldsymbol{X}}_j\rangle\,,1\leq i\neq j\leq L.\end{aligned}$$

where $||\boldsymbol{a}||$ stands for the Euclid norm of n dimensional vector $\boldsymbol{a}$ and $\langle\boldsymbol{a},\boldsymbol{b}\rangle$ stands for the inner product between $\boldsymbol{a}$ and $\boldsymbol{b}$. Let $\Sigma_{\boldsymbol{X}^L-\hat{\boldsymbol{X}}^L}$ be a covariance matrix with d_{ij} in its (i,j) element.

In this communication system we can consider two distortion criterions. For each distortion criterion we define the determination problem of the rate distortion region. Those two problems are shown below.

Problem 1. Vector Distortion Criterion: Fix positive vector $D^L=(D_1,D_2,\cdots,D_L)$. For a given D^L, the rate vector $(R_1,R_2,\cdots,R_L)$ is *admissible* if for any positive $\delta>0$ and any n with $n\geq n_0(\delta)$, there exists $(\varphi_1,\varphi_2,\cdots,\varphi_L,\psi)\in\mathcal{F}_\delta^{(n)}(R_1,R_2\cdots,R_L)$ such that

$$\left[\frac{1}{n}\Sigma_{\boldsymbol{X}^L-\hat{\boldsymbol{X}}^L}\right]_{ii}\leq D_i+\delta\,,$$

where $[A]_{ii}$ stands for the (i,j) entry of the matrix A. Let $\mathcal{R}_L(D^L)$ denote the set of all the admissible rate vector. On a form of $\mathcal{R}_L(D^L)$, we have a particular interest in its sum rate part. To examine this quantity, define

$$R_{\mathrm{sum},L}(D^L)\triangleq\min_{(R_1,R_2,\cdots,R_L)\in\mathcal{R}_L(D^L)}\left\{\sum_{i=1}^L R_i\right\}.$$

To determine $R_{\mathrm{sum},L}(D^L)$ in an explicit form is also of our interest.

Problem 2. Sum Distortion Criterion: Fix positive D. For a given positive D, the rate vector $(R_1,R_2,\cdots,R_L)$ is *admissible* if for any positive $\delta>0$ and any n with $n\geq n_0(\delta)$, there exists $(\varphi_1,\varphi_2,\cdots,\varphi_L,\psi)\in\mathcal{F}_\delta^{(n)}(R_1,R_2\cdots,R_L)$ such that

$$\mathrm{tr}\left[\frac{1}{n}\Sigma_{\boldsymbol{X}^L-\hat{\boldsymbol{X}}^L}\right]\leq D+\delta\,,$$

Let $\mathcal{R}_L(D)$ denote the set of all the admissible rate vector. To examine the sum rate part of $\mathcal{R}_L(D)$, define

$$R_{\mathrm{sum},L}(D) \stackrel{\triangle}{=} \min_{(R_1,R_2,\cdots,R_L)\in\mathcal{R}_L(D)} \left\{ \sum_{i=1}^{L} R_i \right\} .$$

We can easily show that we have the following relation between $\mathcal{R}_L(D)$ and $\mathcal{R}_L^{(\mathrm{in})}(D^L)$:

$$\mathcal{R}_L(D) = \bigcup_{\sum_{i=1}^{L} D_i \le D} \mathcal{R}_L(D^L) . \tag{5}$$

In this paper our argument is concentrated on the study of Problem 2. It is well known that when $D \ge \mathrm{tr}[\Sigma_{X^L}]$, $R_1 = R_2 = \cdots = R_L = 0$ is admissible. In this case, we have

$$\mathcal{R}_L(D) = \{(R_1, \cdots, R_L) : R_i \ge 0, i \in \Lambda\} .$$

In the subsequent arguments we focus on our arguments in the case of $D < \mathrm{tr}[\Sigma_{X^L}]$.

B. Previous Results

In this subsection we state previous results on the determination problem of $\mathcal{R}_L(D)$. We first state a previous result on an inner bound of $\mathcal{R}_L(D)$ and $\mathcal{R}_L(D^L)$. Let $U_i, i \in \Lambda$ be random variables taking values in real lines $\mathcal{U}_i$. For any subset $S \subseteq \Lambda$, we introduce the notation $U_S \stackrel{\triangle}{=} (U_i)_{i\in S}$. In particular, $U_\Lambda = U^L = (U_1, U_2, \cdots, U_L)$. Similar notations are used for other random variables. Define

$$\begin{aligned}
\mathcal{G}(D^L) \stackrel{\triangle}{=} \{U^L :\ & U^L \text{ is a Gaussian} \\
& \text{random vector that satisfies} \\
& U_S \to Y_S \to X^L \to Y_{S^c} \to U_{S^c} , \\
& U^L \to Y^L \to X^L \\
& \text{for any } S \subseteq \Lambda \text{ and} \\
& \mathrm{E}\left[X_i - \tilde{\psi}_i(U^L)\right]^2 \le D_i \\
& \text{for some linear mapping} \\
& \tilde{\psi}_i : \mathcal{U}^L \to \hat{\mathcal{X}}_i, i \in \Lambda . \}
\end{aligned}$$

and set

$$\begin{aligned}
\hat{\mathcal{R}}_L^{(\mathrm{in})}(D^L) \stackrel{\triangle}{=} \mathrm{conv}\{R^L :\ & \text{There exists } U^L \in \mathcal{G}(D^L) \\
& \text{such that} \\
& \sum_{i\in S} R_i \ge I(U_S; Y_S | U_{S^c}) \\
& \text{for any } S \subseteq \Lambda . \} , \\
\hat{\mathcal{R}}_L^{(\mathrm{in})}(D) \stackrel{\triangle}{=} \mathrm{conv}\{R^L :\ & \text{There exist } D^L \text{ and} \\
& U^L \in \mathcal{G}(D^L) \text{ such that} \\
& \sum_{i\in S} R_i \ge I(U_S; Y_S | U_{S^c}) \\
& \text{for any } S \subseteq \Lambda \text{ and} \\
& \sum_{i=1}^{L} D_i \le D . \} ,
\end{aligned}$$

where conv$\{A\}$ denotes a convex hull of the set A. We can easily show that we have the following relation between $\hat{\mathcal{R}}_L^{(\mathrm{in})}(D)$ and $\hat{\mathcal{R}}_L^{(\mathrm{in})}(D^L)$:

$$\hat{\mathcal{R}}_L^{(\mathrm{in})}(D) = \bigcup_{\sum_{i=1}^{L} D_i \le D} \hat{\mathcal{R}}_L^{(\mathrm{in})}(D^L) . \tag{6}$$

Then, we have the following result.

Theorem 1 (Berger [4] and Tung [5]):

$$\hat{\mathcal{R}}_L^{(\mathrm{in})}(D) \subseteq \mathcal{R}_L(D) , \hat{\mathcal{R}}_L^{(\mathrm{in})}(D^L) \subseteq \mathcal{R}_L(D^L) .$$

The inner bound $\hat{\mathcal{R}}_L^{(\mathrm{in})}(D^L)$ is well known as the inner bound of Berger [4] and Tung [5]. The inner bound $\hat{\mathcal{R}}_L^{(\mathrm{in})}(D)$ can be regarded as a variant of their inner bound.

The source coding problem considered in this paper was first posed and investigated by Pandya *et al.*[16]. They dealt with the case that $Y^L = X^L A + N^L$, where A is $L \times L$ a positive definite attenuation matrix. When A is an identity matrix, the problem studied by Pandya *et al.* is the same as the problem considered here. They derived upper and lower bounds of $R_{\mathrm{sum,L}}(D)$.

Recently, Wagner *et al.* [10] have determined $\mathcal{R}_2(D_1, D_2)$. Their result is as follows.

Theorem 2 (Wagner et al. [10]): For any positive D_1 and D_2, we have

$$\mathcal{R}_2(D_1, D_2) = \hat{\mathcal{R}}_2^{(\mathrm{in})}(D_1, D_2) .$$

From the above theorem, (5) and (6), we immediatly obtain the following corollary.

Corollary 1 (Wagner et al. [10]): For any positive D, we have

$$\mathcal{R}_2(D) = \hat{\mathcal{R}}_2^{(\mathrm{in})}(D) .$$

According to Wagner *et al.* [10], the results of Oohama [9], [14], and [15] play an essential role in deriving the above result. The determination problems of $\mathcal{R}_L(D^L)$ and $\mathcal{R}_L(D)$ for $L \ge 3$ still remains to be solved. Their method for the proof depends heavily on the specific property of $L = 2$. It is hard to generalize it to the case of $L \ge 3$.

III. Main Results

In this section we state our results on $\mathcal{R}_L(D)$ and $R_{\mathrm{sum},L}(D)$.

A. Definition of Functions and their Properties

In this subsection we define several functions which are necessary to describe our results and present their properties. For $r_i \ge 0, i \in \Lambda$, let $N_i(r_i)$, $i \in \Lambda$ be L independent Gaussian random variables with mean 0 and variance $\sigma_{N_i}^2/(1 - \mathrm{e}^{-2r_i})$. Let $\Sigma_{N^L(r^L)}$ be a covariance matrix for the random vector $N^L(r^L)$. For any subset $S \subseteq \Lambda$, we set $r_S \stackrel{\triangle}{=} (r_i)_{i\in S}$. In particular, $r_\Lambda = r^L = (r_1, r_2, \cdots, r_L)$. Fix nonnegative vector

r^L. Let $\alpha_i = \alpha_i(r^L), i \in \Lambda$ be L eigen values of the matrix $\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1}$. For $S \subseteq \Lambda$, and $\theta > 0$, define

$$\Sigma_{N^L(r_{S^c})}^{-1} \triangleq \Sigma_{N^L(r^L)}^{-1}\Big|_{r_S=\mathbf{0}} ,$$

$$\underline{J}_S(\theta, r_S|r_{S^c}) \triangleq \frac{1}{2}\log^+\left[\frac{\prod_{i\in S} \mathrm{e}^{2r_i}}{\theta\left|\Sigma_{X^L}^{-1} + \Sigma_{N^L(r_{S^c})}^{-1}\right|}\right],$$

$$J_S\left(r_S|r_{S^c}\right) \triangleq \frac{1}{2}\log\left[\frac{\left|\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1}\right|\left\{\prod_{i\in S} \mathrm{e}^{2r_i}\right\}}{\left|\Sigma_{X^L}^{-1} + \Sigma_{N^L(r_{S^c})}^{-1}\right|}\right],$$

where $S^c = \Lambda - S$ and $\log^+ x \triangleq \max\{\log x, 0\}$. Let $\mathcal{B}_L(D)$ be the set of all nonnegative vectors r^L that satisfy

$$\mathrm{tr}\left[\left(\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1}\right)^{-1}\right] \leq D . \tag{7}$$

Let $\partial\mathcal{B}_L(D)$ be the boundary of $\mathcal{B}_L(D)$, that is, the set of all nonnegative vectors r^L that satisfy

$$\mathrm{tr}\left[\left(\Sigma_{X^L}^{-1} + \Sigma_{N(r^L)}^{-1}\right)^{-1}\right] = D .$$

Let ξ be nonnegative number that satisfy

$$\sum_{i=1}^{L}\left\{[\xi - \alpha_i^{-1}]^+ + \alpha_i^{-1}\right\} = D.$$

Define

$$\theta(D, r^L) \triangleq \prod_{i=1}^{L}\left\{[\xi - \alpha_i^{-1}]^+ + \alpha_i^{-1}\right\} .$$

We can show that for $S \subseteq \Lambda$, $\underline{J}_S(\theta(D, r^L), r_S|r_{S^c})$ and $J_S(r_S|r_{S^c})$ satisfy the following two properties.

Property 1:

a) If $r^L \in \mathcal{B}_L(D)$, then, for any $S \subseteq \Lambda$,

$$\underline{J}_S(\theta(D, r^L), r_S|r_{S^c}) \leq J_S(r_S|r_{S^c}) .$$

The equality holds when $r^L \in \partial\mathcal{B}_L(D)$.

b) Suppose that $r^L \in \mathcal{B}_L(D)$. If $r^L\big|_{r_S=\mathbf{0}}$ still belongs to $\mathcal{B}_L(D)$, then,

$$\underline{J}_S(\theta(D, r^L), r_S|r_{S^c})\big|_{r_S=\mathbf{0}} = J_S(r_S|r_{S^c})|_{r_S=\mathbf{0}} = 0 .$$

Property 2: Fix $r^L \in \mathcal{B}_L(D)$. For $S \subseteq \Lambda$, set

$$f_S = f_S(r_S|r_{S^c}) \triangleq \underline{J}_S(\theta(D, r^L), r_S|r_{S^c}) .$$

By definition it is obvious that $f_S, S \subseteq \Lambda$ are nonnegative. We can show that $f \triangleq \{f_S\}_{S\subseteq\Lambda}$ satisfies the followings:

a) $f_\emptyset = 0$.
b) $f_A \leq f_B$ for $A \subseteq B \subseteq \Lambda$.
c) $f_A + f_B \leq f_{A\cap B} + f_{A\cup B}$.

In general (Λ, f) is called a *co-polymatroid* if the nonnegative function f on 2^Λ satisfies the above three properties. Similarly, we set

$$\tilde{f}_S = \tilde{f}_S(r_S|r_{S^c}) \triangleq J_S(r_S|r_{S^c}) , \quad \tilde{f} = \left\{\tilde{f}_S\right\}_{S\subseteq\Lambda} .$$

Then, $(\Lambda, \tilde{f})$ also has the same three properties as those of (Λ, f) and becomes a co-polymatroid.

B. Results

In this subsection we present our results on $\mathcal{R}_L(D)$. To describe our result on inner and outer bounds of $\mathcal{R}_L(D)$, set

$$\mathcal{R}_L^{(\mathrm{out})}(D, r^L) \triangleq \Big\{R^L : \sum_{i\in S} R_i \geq \underline{J}_S\left(\theta(D, r^L), r_S|r_{S^c}\right) \text{ for any } S \subseteq \Lambda . \Big\} ,$$

$$\mathcal{R}_L^{(\mathrm{out})}(D) \triangleq \bigcup_{r^L\in\mathcal{B}_L(D)} \mathcal{R}_L^{(\mathrm{out})}(D, r^L) ,$$

$$\mathcal{R}_L^{(\mathrm{in})}(r^L) \triangleq \Big\{R^L : \sum_{i\in S} R_i \geq J_S\left(r_S|r_{S^c}\right) \text{ for any } S \subseteq \Lambda . \Big\} ,$$

$$\mathcal{R}_L^{(\mathrm{in})}(D) \triangleq \mathrm{conv}\left\{\bigcup_{r^L\in\mathcal{B}_L(D)} \mathcal{R}_L^{(\mathrm{in})}(r^L)\right\} .$$

Our main result is as follows.

Theorem 3:

$$\mathcal{R}_L^{(\mathrm{in})}(D) \subseteq \hat{\mathcal{R}}_L^{(\mathrm{in})}(D) \subseteq \mathcal{R}_L(D) \subseteq \mathcal{R}_L^{(\mathrm{out})}(D) .$$

Proof of this theorem will be given in Section V.

An essential gap between $\mathcal{R}_L^{(\mathrm{out})}(D)$ and $\mathcal{R}_L^{(\mathrm{in})}(D)$ is the difference between $\underline{J}_S(\theta(D, r^L), r_S|r_{S^c})$ in the definition of $\mathcal{R}_L^{(\mathrm{out})}(D)$ and $J_S\left(r_S| \; r_{S^c}\right)$ in the definition of $\mathcal{R}_L^{(\mathrm{in})}(D)$. By Property 1 part a) and the definitions of $\mathcal{R}_L^{(\mathrm{out})}(D, r^L)$ and $\mathcal{R}_L^{(\mathrm{in})}(r^L)$, if $r^L \in \partial\mathcal{B}_L(D)$, then,

$$\mathcal{R}_L^{(\mathrm{out})}(D, r^L) = \mathcal{R}_L^{(\mathrm{in})}(r^L) ,$$

which suggests a possibility that in some nontrivial cases $\mathcal{R}_L^{(\mathrm{out})}(D)$ and $\mathcal{R}_L^{(\mathrm{in})}(D)$ match. For $L \geq 3$, we present a sufficient condition for $\mathcal{R}_L^{(\mathrm{out})}(D) \subseteq \mathcal{R}_L^{(\mathrm{in})}(D)$. We consider the following condition on $\theta(D, r^L)$.

Condition: For any $i \in \Lambda$, $\mathrm{e}^{-2r_i}\theta(D, r^L)$ is a monotone decreasing function of $r_i \geq 0$.

We call this condition the MD condition. The following is a key lemma to derive the matching condition.

Lemma 1: If $\theta(D, r^L)$ satisfies the MD condition on $\mathcal{B}_L(D)$, then,

$$\mathcal{R}_L^{(\mathrm{in})}(D) = \hat{\mathcal{R}}_L^{(\mathrm{in})}(D) = \mathcal{R}_L(D) = \mathcal{R}_L^{(\mathrm{out})}(D).$$

Proof of this lemma will be given in Section VI. Based on Lemma 1, we derive a sufficient condition for $\theta(D, r^L)$ to satisfy the MD condition.

Let $a_{ii}, i = 1, 2, \cdots, L$ be (i, i)-element of $\Sigma_{X^L}^{-1}$ and set $c_i \triangleq \frac{1}{\sigma_{N_i}^2}$. Let $\alpha_{\min} = \alpha_{\min}(r^L)$ and $\alpha_{\max} = \alpha_{\max}(r^L)$ be the minimum and maximum eigen values of $\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1}$, respectively. The following is a key lemma to derive a sufficient condition for the MD condition to hold.

Lemma 2: If $\alpha_{\min}(r^L)$ and $\alpha_{\max}(r^L)$ satisfy

$$\frac{1}{\alpha_{\min}(r^L)} - \frac{1}{\alpha_{\max}(r^L)} \leq \frac{1}{a_{ii} + c_i} , \quad \text{for } i \in \Lambda$$

on $\mathcal{B}_L(D)$, then, $\theta(D, r^L)$ satisfies the MD condition on $\mathcal{B}_L(D)$.

Set

$$\mathcal{C} \triangleq \{(D, \Sigma_{X^L}, \Sigma_{N^L}) : r^L \in \mathcal{B}_L(D) \text{ for some nonnegative } r^L.\}.$$

When $r^L \geq s^L$, we have

$$\begin{aligned} &\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1} \succeq \Sigma_{X^L}^{-1} + \Sigma_{N^L(s^L)}^{-1}, \\ \Rightarrow &\left(\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1}\right)^{-1} \preceq \left(\Sigma_{X^L}^{-1} + \Sigma_{N^L(s^L)}^{-1}\right)^{-1}, \end{aligned} \quad (8)$$

where $B \succeq A$ stands for that $B - A$ is positive semi-definite. The equation (8) implies that $\mathrm{tr}\left[\left(\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1}\right)^{-1}\right]$ is a monotone decreasing function of r^L. Hence, we have

$$\mathcal{C} = \left\{(D, \Sigma_{X^L}, \Sigma_{N^L}) : D > \mathrm{tr}\left[\left(\Sigma_{X^L}^{-1} + \Sigma_{N^L}^{-1}\right)^{-1}\right]\right\}.$$

From Lemmas 1, 2 and an elementary computation we obtain the following.

Theorem 4: Let $\alpha_{\max}^*$ be the maximum eigen value of $\Sigma_{X^L}^{-1} + \Sigma_{N^L}^{-1}$. If

$$\mathrm{tr}\left[\left(\Sigma_{X^L}^{-1} + \Sigma_{N^L}^{-1}\right)^{-1}\right] < D \leq \tfrac{L+1}{\alpha_{\max}^*},$$

then,

$$\mathcal{R}_L^{(\mathrm{in})}(D) = \hat{\mathcal{R}}_L^{(\mathrm{in})}(D) = \mathcal{R}_L(D) = \mathcal{R}_L^{(\mathrm{out})}(D).$$

In particular,

$$\begin{aligned} &R_{\mathrm{sum},L}(D) \\ &= \min_{r^L \in \mathcal{B}_L(D)} \left\{ \sum_{i=1}^{L} r_i + \frac{1}{2} \log \frac{\left|\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1}\right|}{\left|\Sigma_{X^L}^{-1}\right|} \right\}. \end{aligned} \quad (9)$$

Proofs of Lemma 2 and Theorem 4 will be stated in Section VI. From Theorem 4, we can see that we have several nontrivial cases where $\mathcal{R}_L^{(\mathrm{in})}(D)$ and $\mathcal{R}_L^{(\mathrm{out})}(D)$ match. In Oohama [19], the author derived the sufficient matching condition $D \leq \frac{L+\frac{1}{L-1}}{\alpha_{\max}^*}$ on upper bound of D. Thus the matching condition presented here provides a significant improvement of that of Oohama [19] for large L.

We further examine an explicit characterization of $R_{\mathrm{sum},L}(D)$ when the source has a certain symmetrical property. Let

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & L \\ \tau(1) & \tau(2) & \cdots & \tau(i) & \cdots & \tau(L) \end{pmatrix}$$

be a cyclic shift on Λ, that is,

$$\tau(1) = 2, \tau(2) = 3, \cdots, \tau(L-1) = L, \tau(L) = 1.$$

Let $p_{X_\Lambda}(x_\Lambda) = p_{X_1 X_2 \cdots X_L}(x_1, x_2, \cdots, x_L)$ be a probability density function of X^L. The source X^L is said to be cyclic shift invariant if we have

$$\begin{aligned} p_{X_\Lambda}(x_{\tau(\Lambda)}) &= p_{X_1 X_2 \cdots X_L}(x_2, x_3, \cdots, x_L, x_1) \\ &= p_{X_1 X_2 \cdots X_L}(x_1, x_2, \cdots, x_{L-1}, x_L) \end{aligned}$$

for any $(x_1, x_2, \cdots, x_L) \in \mathcal{X}^L$. In the following argument we assume that X^L satisfies the cyclic shift invariant property. We further assume that $N_i, i \in \Lambda$ are independent identically distributed (i.i.d.) Gaussian random variables with mean 0 and variance σ^2. Then, the observation $Y^L = X^L + N^L$ also satisfies the cyclic shift invariant property.

Fix $r > 0$, let $N_i(r)$, $i \in \Lambda$ be L i.i.d. Gaussian random variables with mean 0 and variance $\sigma^2/(1-\mathrm{e}^{-2r})$. Let $\Sigma_{N^L(r)}$ be a covariance matrix for the random vector $N^L(r)$. Let $\lambda_i, i \in \Lambda$ be L eigen values of the matrix Σ_{X^L} and let $\beta_i = \beta_i(r), i \in \Lambda$ be L eigen values of the matrix $\Sigma_{X^L}^{-1} + \Sigma_{N^L(r)}^{-1}$. Using the eigen values of Σ_{X^L}, $\beta_i(r), i \in \Lambda$ can be written as

$$\beta_i(r) = \frac{1}{\lambda_i} + \frac{1}{\sigma^2}(1 - \mathrm{e}^{-2r}).$$

Let ξ be a nonnegative number that satisfies $\sum_{i=1}^{L}\{[\xi - \beta_i^{-1}]^+ + \beta_i^{-1}\} = D$. Define

$$\theta(D, r) \triangleq \prod_{i=1}^{L}\left\{[\xi - \beta_i^{-1}]^+ + \beta_i^{-1}\right\},$$

$$\underline{J}(\theta(D, r), r) \triangleq \frac{1}{2}\log\left[\frac{\mathrm{e}^{2Lr}\,|\Sigma_{X^L}|}{\theta(D, r)}\right],$$

and set

$$\phi(r) \triangleq \mathrm{tr}\left[\left(\Sigma_{X^L}^{-1} + \Sigma_{N^L(r)}^{-1}\right)^{-1}\right] = \sum_{i=1}^{L}\frac{1}{\beta_i(r)}.$$

Since $\phi(r)$ is a monotone decreasing function of r, there exists a unique r such that $\phi(r) = D$, we denote it by $r^*(D)$. Note that

$$\underbrace{(r, r, \cdots, r)}_{L} \in \mathcal{B}_L(D) \Leftrightarrow \phi(r) \leq D \Leftrightarrow r \geq r^*(D),$$

$$\theta(D, r^*) = \left|\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^*)}^{-1}\right|^{-1}.$$

Set

$$R_{\mathrm{sum},L}^{(\mathrm{l})}(D) \triangleq \min_{r \geq r^*(D)} \underline{J}(\theta(D, r), r).$$

Then, we have the following.

Theorem 5: Assume that the source X^L and its noisy version $Y^L = X^L + N^L$ are cyclic shift invariant. Then, we have

$$R_{\mathrm{sum},L}(D) \geq R_{\mathrm{sum},L}^{(\mathrm{l})}(D).$$

Proof of this theorem will be stated in Section V.

Next, we examine a sufficient condition for $R_{\mathrm{sum},L}^{(\mathrm{l})}(D)$ to coincide with $R_{\mathrm{sum},L}(D)$. It is obvious from the definition of $\underline{J}(\theta(D, r), r)$ that when $\mathrm{e}^{-2Lr}\theta(D, r)$ is a monotone decreasing function of $r \in [r^*(D), +\infty)$, we have $R_{\mathrm{sum},L}^{(\mathrm{l})}(D) = R_{\mathrm{sum},L}(D)$.

Lemma 3: Let a be an identical diagonal element of $\Sigma_{X^L}^{-1}$. Set $c \triangleq \frac{1}{\sigma^2}$. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the minimum and maximum eigen values of Σ_{X^L}, respectively. Let the minimum and maximum eigen values of $\Sigma_{X^L}^{-1} + \Sigma_{N^L(r)}^{-1}$ be denoted by $\beta_{\min} = \beta_{\min}(r)$ and $\beta_{\max} = \beta_{\max}(r)$, respectively. Those are given by

$$\beta_{\min}(r) = \frac{1}{\lambda_{\max}} + \frac{1}{\sigma^2}(1 - \mathrm{e}^{-2r}),$$

$$\beta_{\max}(r) = \frac{1}{\lambda_{\min}} + \frac{1}{\sigma^2}(1 - \mathrm{e}^{-2r}).$$

If $\beta_{\min}(r)$ and $\beta_{\max}(r)$ satisfy

$$\frac{1}{\beta_{\min}(r)} - \frac{1}{\beta_{\max}(r)} \leq \frac{L\sigma^2 \mathrm{e}^{2r}}{L-1} \cdot \frac{\beta_{\min}(r)}{\beta_{\max}(r)}$$

for $r \geq r^*(D)$, then, $\mathrm{e}^{-2Lr}\theta(D,r)$ is a monotone decreasing function of $r \in [r^*(D), \infty)$.

From Lemma 3 and an elementary computation we obtain the following.

Theorem 6: Assume that X^L and $Y^L = X^L + N^L$ are cyclic shift invariant. If

$$\sigma^2 \geq \frac{L-1}{L} \cdot \frac{\lambda_{\max}}{\lambda_{\min}} (\lambda_{\max} - \lambda_{\min}) \,, \tag{10}$$

then, $R^{(1)}_{\mathrm{sum},L}(D) = R_{\mathrm{sum},L}(D)$. Furthermore, the curve $R = R_{\mathrm{sum},L}(D)$ has the following parametric form:

$$\left.\begin{array}{l} R = \frac{1}{2} \log \left[|\Sigma_{X^L}| \mathrm{e}^{2Lr} \prod_{i=1}^{L} \beta_i(r) \right] \,, \\ D = \sum_{i=1}^{L} \frac{1}{\beta_i(r)} \,. \end{array}\right\}$$

Proofs of Lemma 3 and Theorem 6 will be stated in Section VI. Note that the condition (10) depends only on the correlation property of X^L and N^L. From Theorem 6 we can see that for (X^N, N^N) satisfying the cyclic shift invariant property the determination problem of $R_{\mathrm{sum,L}}(D)$ is solved if the identical varaince σ^2 of $N_i, i \in \Lambda$ is relatively high compared with the eigen values of Σ_{X^L}.

IV. Computation of Matching Conditions

In this section we explicitly compute the matching condition for some class of Gaussian information sources. Define

$$u_i \stackrel{\triangle}{=} a_{ii} + c_i(1 - \mathrm{e}^{-2r_i}) \,, i \in \Lambda \,. \tag{11}$$

From (11), we have

$$2r_i = \log \frac{c_i}{a_{ii} + c_i - u_i} \,.$$

By the above transformation we regard $\theta(D, r^L)$ and $\Sigma^{-1}_{X^L} + \Sigma^{-1}_{N^L(r^L)}$ as functions of u^L, that is, $\theta(D, r^L) = \theta(D, u^L)$ and

$$\Sigma^{-1}_{X^L} + \Sigma^{-1}_{N^L(r^L)} = \Sigma^{-1}_{X^L} + \Sigma^{-1}_{N^L(u^L)} \,.$$

We consider the case where Σ_{X^L} have identical diagonal and nondiagonal elements, that is,

$$\begin{aligned} \mathrm{Var}[X_i] &= \sigma^2_{X_i} = 1 \,, \text{ for } i \in \Lambda, \\ \mathrm{Cov}[X_i, X_j] &= \rho \sigma_{X_i} \sigma_{X_j} = \rho \text{ for } i, j \in \Lambda, i \neq j. \end{aligned}$$

In this identical variance case, (i,j) elements a_{ij} of $\Sigma^{-1}_{X^L}$ is given by

$$a_{ij} = \begin{cases} \frac{1+(L-2)\rho}{(1-\rho)(1+(L-1)\rho)} & \text{if } i = j \,, \\ \frac{-\rho}{(1-\rho)(1+(L-1)\rho)} & \text{if } i \neq j \,. \end{cases}$$

For simplicity of notations we set $a \stackrel{\triangle}{=} a_{ii}, b \stackrel{\triangle}{=} -a_{ij}$. We first derive an explicit form of the set $\mathcal{B}_L(D)$. To this end we use the following formula

$$\begin{vmatrix} z_1 & \delta & \ldots & \delta \\ \delta & z_2 & \ldots & \delta \\ \vdots & \vdots & \ddots & \vdots \\ \delta & \delta & \ldots & z_L \end{vmatrix} = \left\{ \prod_{i=1}^{L} (z_i - \delta) \right\} \left\{ 1 + \delta \sum_{i=1}^{L} \frac{1}{z_i - \delta} \right\} . \tag{12}$$

Using (12), the condition

$$\mathrm{tr} \left[\left(\Sigma^{-1}_{X^L} + \Sigma^{-1}_{N^L(u^L)} \right)^{-1} \right] \leq D \tag{13}$$

is explicitly given by the following:

$$\sum_{i \neq j} \frac{b^2}{(u_i + b)(u_j + b)} - (1 + Db) \sum_{i=1}^{L} \frac{b}{u_i + b} + Db \geq 0 \,. \tag{14}$$

Set

$$\kappa_1 \stackrel{\triangle}{=} \frac{1}{2} \cdot \frac{1 + Db}{L - 1} \,, \kappa_2 \stackrel{\triangle}{=} \frac{L}{4(L-1)} (1 + Db)^2 - Db \,.$$

Then, the above condition is rewritten as

$$\sum_{i \neq j} \left(\kappa_1 - \tfrac{b}{u_i + b} \right) \left(\kappa_1 - \tfrac{b}{u_j + b} \right) \geq \kappa_2 \,. \tag{15}$$

From (15), we can see that the region $\mathcal{C}$ is given by the set of all (a, b, c^L, D) satisfying

$$\sum_{i \neq j} \left(\kappa_1 - \tfrac{b}{a + b + c_i} \right) \left(\kappa_1 - \tfrac{b}{a + b + c_j} \right) \geq \kappa_2 \,. \tag{16}$$

The above condition is equivalent to

$$\sum_{i \neq j} \frac{b^2}{(a + b + c_i)(a + b + c_j)} - (1 + Db) \sum_{i=1}^{L} \frac{b}{a + b + c_i} + Db \geq 0 \,. \tag{17}$$

Solving (17) with respect to D, we obtain

$$D \geq \frac{\sum_{i=1}^{L} \frac{1}{a+b+c_i} - \sum_{i \neq j} \frac{b}{(a+b+c_i)(a+b+c_j)}}{1 - \sum_{i=1}^{L} \frac{b}{a+b+c_i}} \,. \tag{18}$$

From Theorem 4, we obtain the following corollary.

Corollary 2: If D satisfy

$$\frac{\sum_{i=1}^{L} \frac{1}{a+b+c_i} - \sum_{i \neq j} \frac{b}{(a+b+c_i)(a+b+c_j)}}{1 - \sum_{i=1}^{L} \frac{b}{a+b+c_i}} \leq D \leq \frac{L+1}{\alpha^*_{\max}},$$

then

$$\mathcal{R}^{(\mathrm{in})}_L(D) = \mathcal{R}_L(D) = \mathcal{R}^{(\mathrm{out})}_L(D).$$

Next we derive a more explicit sufficient condition. Set

$$c_{\min} \stackrel{\triangle}{=} \min_{1\le i\le L} c_i, \quad c_{\max} \stackrel{\triangle}{=} \max_{1\le i\le L} c_i\,.$$

Then, the condition

$$L(L-1)\left(\kappa_1 - \tfrac{b}{a+b+c_{\min}}\right)^2 > \kappa_2 \tag{19}$$

is a sufficient condition for $(a,b,c^L,D)\in\mathcal{C}$. The above condition is equivalent to

$$D \ge \tfrac{L}{a+b+c_{\min}}\cdot\left(1+\tfrac{b}{a+b+c_{\min}-Lb}\right)\,. \tag{20}$$

On the other hand, the maximum eigen value of $\Sigma_{X^L}^{-1}$ $+\Sigma_{N^L(u^L)}^{-1}$ satisfies

$$\alpha_{\max}^* \le \max_{1\le j\le L}\{u_j+b\} \le a+b+c_{\max}\,. \tag{21}$$

Properties on bounds of the eigen values of $\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}$ including the property stated in (21) and their proofs are given in Appendix C. From (20), (21), and Corollary 2, we obtain the following theorem.

Theorem 7: If $(a,b,c_{\min},c_{\max},D)$ satisfies

$$\tfrac{L}{a+b+c_{\min}}\cdot\left(1+\tfrac{b}{a+b+c_{\min}-Lb}\right)\le D\le \tfrac{L+1}{a+b+c_{\max}} \tag{22}$$

then,

$$\mathcal{R}_L^{(\mathrm{in})}(D)=\mathcal{R}_L(D)=\mathcal{R}_L^{(\mathrm{out})}(D).$$

In particular,

$$\begin{aligned}&R_{\mathrm{sum},L}(D)\\ &=\min_{r^L\in\mathcal{B}_L(D)}\left\{\sum_{i=1}^L r_i+\frac{1}{2}\log\frac{\left|\Sigma_{X^L}^{-1}+\Sigma_{N^L(r^L)}^{-1}\right|}{\left|\Sigma_{X^L}^{-1}\right|}\right\}\,.\end{aligned} \tag{23}$$

It can be seen from (22) that the matching condition holds for sufficiently small b and $c_{\max}$. This implies that the determination problem of $\mathcal{R}_L(D)$ is solved if the correlation of X^L is relatively small and the noise variance of N^L is relatively large.

Now we derive an explicit form of $R_{\mathrm{sum},L}(D)$ in the case where $c=c_{\min}=c_{\max}$. In this case, we have

$$\begin{aligned}\mathcal{C}=\{(a,b,c,D):&\\ &D\ge\tfrac{L}{a+b+c}\cdot\left(1+\tfrac{b}{a+b+c-Lb}\right)\}\,.\end{aligned}$$

Set

$$L_1 \stackrel{\triangle}{=} \frac{L}{2}\left[1+Db+\sqrt{(1-Db)^2+\tfrac{4Db}{L}}\right]\,. \tag{24}$$

Solving the minimization problem in the right member of (23), we have the following.

Theorem 8: If (a,b,c,D) satisfies

$$\tfrac{L}{a+b+c}\cdot\left(1+\tfrac{b}{a+b+c-Lb}\right)\le D\le\tfrac{L}{a+b+c}\left(1+\tfrac{1}{L}\right) \tag{25}$$

then,

$$\begin{aligned}R_{\mathrm{sum},L}(D)&=\frac{L}{2}\log\left(\frac{(1-\rho)L_1c}{D(a+b+c)-L_1}\right)\\ &\quad+\frac{1}{2}\log\left\{\frac{1+(L-1)\rho}{1-\rho}\left(1-\frac{LDb}{L_1}\right)\right\}\,.\end{aligned}$$

Proof of this theorem is given in Appendix A.

Next, we consider another example where the source and its noisy observation are cyclic shift invariant. Let $L=4$ and

$$\Sigma_{X^4}=\begin{bmatrix}1&\rho&0&\rho\\ \rho&1&\rho&0\\ 0&\rho&1&\rho\\ \rho&0&\rho&1\end{bmatrix},\ |\rho|<\frac{1}{2},\ \Sigma_{N^4}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\,. \tag{26}$$

In this case, we have

$$\begin{aligned}&|\Sigma_{X^4}|=1-4\rho^2, a=\frac{1-2\rho^2}{1-4\rho^2},\\ &\lambda_1=1-2\rho,\lambda_2=\lambda_3=1,\lambda_4=1+2\rho\,.\end{aligned}$$

Four eigen valules $\beta_i(r), i=1,2,3,4$ are given by

$$\left.\begin{aligned}&\beta_1(r)=1-2\rho+\tfrac{1}{\sigma^2}(1-\mathrm{e}^{-2r})\,,\\ &\beta_2(r)=\beta_3(r)=1+\tfrac{1}{\sigma^2}(1-\mathrm{e}^{-2r})\,,\\ &\beta_4(r)=1+2\rho+\tfrac{1}{\sigma^2}(1-\mathrm{e}^{-2r})\,.\end{aligned}\right\} \tag{27}$$

The matching condition is

$$\sigma^2\ge 3|\rho|\frac{1+2|\rho|}{1-2|\rho|}\,.$$

Summerizing the above argument, we obtain the following.

Theorem 9: We consider the case where $L=4$, Σ_{X^4} and Σ_{N^4} are given by (26). If

$$\sigma^2\ge 3|\rho|\frac{1+2|\rho|}{1-2|\rho|},$$

then, the rate distortion curve $R=R_{\mathrm{sum},4}(D)$ has the following parametric form:

$$\left.\begin{aligned}&R=\frac{1}{2}\log\left[(1-4\rho^2)\mathrm{e}^{8r}\prod_{i=1}^4\beta_i(r)\right]\,,\\ &D=\sum_{i=1}^4\frac{1}{\beta_i(r)}\,,\end{aligned}\right\}$$

where $\beta_i(r), i=1,2,3,4$ are definded by (27).

From this theorem we can see that for the above example of (X^4,N^4) satisfying the cyclic shift invariant property the determination problem of $R_{\mathrm{sum},4}(D)$ is solved if the identical varaince σ^2 is relatively high or correlation coefficient ρ is relatively small.

The determination problem of $R_{\mathrm{sum},L}(D)$ was first investigated by Pandya *et al.* [16]. They derived upper and lower bound of $R_{\mathrm{sum},L}(D)$. Pandya *et al.* [16] also numerically compared those two bounds to show that the gap between them is relatively small for some examples. In this paper we have determined $R_{\mathrm{sum},L}(D)$ for some nontrivial case of Gaussian sources.

V. Derivation of Outer and Inner Bounds

In this section we state the proofs of Theorems 3 and 5 stated in Section III.

A. Derivation of the Outer Bound

In this subsection we prove the inclusion $\mathcal{R}_L(D) \subseteq \mathcal{R}_L^{(\mathrm{out})}(D)$ stated in Theorem 3. We use the following two well known lemmas to prove this inclusion.

Lemma 4 (Water Filling Lemma): Let $a_i, i = 1,2,\cdots,L$ be L positive numbers. The maximum of $\prod_{i=1}^{L} \xi_i$ subject to $\sum_{i=1}^{L} \xi_i \leq D$ and $\xi_i \geq a_i, i = 1,2,\cdots,L$ is given by

$$\prod_{i=1}^{L} \left\{ [\xi - a_i]^+ + a_i \right\} ,$$

where ξ is determined by $\sum_{i=1}^{L} \{[\xi - a_i]^+ + a_i\} = D$.

Lemma 5: For any n dimensional random vector $\boldsymbol{U}_i, i = 1,2$, we have

$$\frac{1}{n} h(\boldsymbol{U}_1|\boldsymbol{U}_2) \leq \frac{1}{2}\log\left[(2\pi\mathrm{e}) \cdot \frac{1}{n}\mathrm{E}||\boldsymbol{U}_1 - \boldsymbol{U}_2||^2\right] , \quad (28)$$

where $h(\cdot)$ stands for the differential entropy.

Next, we state an important lemma which is a mathematical core of the converse coding theorem. For $i = 1,2,\cdots,L$, set

$$W_i \stackrel{\triangle}{=} \varphi_i(\boldsymbol{Y}_i), r_i^{(n)} \stackrel{\triangle}{=} \frac{1}{n} I(\boldsymbol{Y}_i; W_i|\boldsymbol{X}_i) . \quad (29)$$

For $S \subseteq \Lambda$, let Q_S be a unitary matrix which transforms X_S into $Z_S = X_S Q_S$. For $\boldsymbol{X}_S = (X_{S,1},\ X_{S,2},\ \cdots, X_{S,n})$, we set

$$\boldsymbol{Z}_S = \boldsymbol{X}_S Q_S = (X_{S,1}Q_S, X_{S,2}Q_S, \cdots, X_{S,n}Q_S) .$$

Then, we have the following lemma.

Lemma 6: For any $S \subseteq \Lambda$, we have

$$\begin{aligned} & h\left(\boldsymbol{Z}_i |\, \boldsymbol{Z}_{S-\{i\}} W_S\right) \\ & \geq \frac{n}{2}\log\left\{(2\pi\mathrm{e})\left[Q_S^{-1}\left(\Sigma_{X_S}^{-1} + \Sigma_{N_S(r_S^{(n)})}^{-1}\right)Q_S\right]_{ii}^{-1}\right\} , \end{aligned}$$

where $[C]_{ij}$ stands for the (i,j) element of the matrix C.

Proof of this lemma will be stated in Appendix B. This lemma provides a strong result on outer bound of the rate distortion region. From Lemma 6, we obtain the following corollary.

Corollary 3: For any $S \subseteq \Lambda$, we have

$$I(\boldsymbol{X}_S; W_S) \leq \frac{n}{2}\log\left|I + \Sigma_{X_S}\Sigma_{N_S(r_S^{(n)})}^{-1}\right| . \quad (30)$$

Proof: We choose unitary matrix Q_S so that

$$Q_S^{-1}\left(\Sigma_{X_S}^{-1} + \Sigma_{N_S(r_S^{(n)})}^{-1}\right)Q_S$$

becomes the following diagonal matrix:

$$Q_S^{-1}\left(\Sigma_{X_S}^{-1} + \Sigma_{N_S(r_S^{(n)})}^{-1}\right)Q_S = \begin{bmatrix} \nu_1 & & & \text{\Large 0} \\ & \nu_2 & & \\ & & \ddots & \\ \text{\Large 0} & & & \nu_{|S|} \end{bmatrix} . \quad (31)$$

Then, we have the following chain of inequalities:

$$\begin{aligned} & I(\boldsymbol{X}_S; W_S) \\ & \stackrel{\text{(a)}}{=} h\left(\boldsymbol{X}_S\right) - h\left(\boldsymbol{Z}_S|W_S\right) \\ & \leq h\left(\boldsymbol{X}_S\right) - \sum_{i=1}^{|S|} h\left(\boldsymbol{Z}_i|\boldsymbol{Z}_{S-\{i\}}W_S\right) \\ & \stackrel{\text{(b)}}{\leq} \frac{n}{2}\log\left[(2\pi\mathrm{e})^{|S|}\left|\Sigma_{X_S}\right|\right] \\ & \quad + \sum_{i=1}^{|S|}\frac{n}{2}\log\left\{\frac{1}{2\pi\mathrm{e}}\left[Q_S^{-1}\left(\Sigma_{X_S}^{-1} + \Sigma_{N_S(r_S^{(n)})}^{-1}\right)Q_S\right]_{ii}\right\} \\ & \stackrel{\text{(c)}}{=} \frac{n}{2}\log\left|\Sigma_{X_S}\right| + \sum_{i=1}^{|S|}\frac{n}{2}\log\left[\nu_i\right] \\ & = \frac{n}{2}\log\left|\Sigma_{X_S}\right| + \frac{n}{2}\log\left|\Sigma_{X_S}^{-1} + \Sigma_{N_S(r_S^{(n)})}^{-1}\right| \\ & = \frac{n}{2}\log\left|I + \Sigma_{X_S}\Sigma_{N_S(r_S^{(n)})}^{-1}\right| . \end{aligned} \quad (32)$$

Step (a) follows from the rotation invariance of the (conditional) differential entropy. Step (b) follows from Lemma 6. Step (c) follows from (31). ■

Using Lemmas 4-6, Corollary 3 and a standard argument on the proof of converse coding theorems, we can prove $\mathcal{R}_L(D) \subseteq \mathcal{R}_L^{(\mathrm{out})}(D)$.

Proof of $\mathcal{R}_L(D) \subseteq \mathcal{R}_L^{(\mathrm{out})}(D)$: Assume that $(R_1, R_2, \cdots, R_L) \in \mathcal{R}_L(D)$. Then, for any $\delta > 0$ and any n with $n \geq n_0(\delta)$, there exists $(\varphi_1, \varphi_2, \cdots,\ \varphi_L,\ \psi) \in \mathcal{F}_\delta^{(n)}(R_1, R_2 \cdots, R_L)$ such that

$$\sum_{i=1}^{L}\mathrm{E}||\boldsymbol{X}_i - \hat{\boldsymbol{X}}_i||^2 \leq D + \delta .$$

We set $\boldsymbol{Z}_\Lambda \stackrel{\triangle}{=} \boldsymbol{X}_\Lambda Q$, $\hat{\boldsymbol{Z}}_\Lambda \stackrel{\triangle}{=} \hat{\boldsymbol{X}}_\Lambda Q$. Furthermore, for $i \in \Lambda$, set

$$\xi_i^{(n)} \stackrel{\triangle}{=} \frac{1}{n}\mathrm{E}||\boldsymbol{Z}_i - \hat{\boldsymbol{Z}}_i||^2 .$$

By rotation invariance of the squared norm, we have

$$\begin{aligned} \sum_{i=1}^{L}\xi_i^{(n)} &= \sum_{i=1}^{L}\frac{1}{n}\mathrm{E}||\boldsymbol{Z}_i - \hat{\boldsymbol{Z}}_i||^2 \\ &= \sum_{i=1}^{L}\frac{1}{n}\mathrm{E}||\boldsymbol{X}_i - \hat{\boldsymbol{X}}_i||^2 \leq D + \delta . \end{aligned} \quad (33)$$

By Lemmas 5 and 6, for $i = 1,2,\cdots,L$, we have

$$\begin{aligned} & \frac{n}{2}\log\left[(2\pi\mathrm{e})\xi_i^{(n)}\right] \geq h(\boldsymbol{Z}_i - \hat{\boldsymbol{Z}}_i) \geq h(\boldsymbol{Z}_i|\hat{\boldsymbol{Z}}_i) \\ & \geq h(\boldsymbol{Z}_i|W_\Lambda) \geq h(\boldsymbol{Z}_i|\boldsymbol{Z}_{\Lambda-\{i\}}W_\Lambda) \\ & \geq \frac{n}{2}\log\left\{(2\pi\mathrm{e})\left[Q^{-1}\left(\Sigma_{X_\Lambda}^{-1} + \Sigma_{N_\Lambda(r_\Lambda^{(n)})}^{-1}\right)Q\right]_{ii}^{-1}\right\} , \end{aligned}$$

from which we have

$$\begin{aligned} \xi_i^{(n)} \geq & \left[Q^{-1}\left(\Sigma_{X_\Lambda}^{-1} + \Sigma_{N_\Lambda(r_\Lambda^{(n)})}^{-1}\right)Q\right]_{ii}^{-1} \\ & \text{for } i \in \Lambda . \end{aligned} \quad (34)$$

Now we proceed to the derivation of the outer bound. We first observe that

$$W_S \to \boldsymbol{X}_S \to \boldsymbol{X}_{S^{\rm c}} \to W_{S^{\rm c}} \tag{35}$$

hold for any subset S of Λ. For any subset $S \subseteq \Lambda$, we obtain the following chain of inequalities:

$$\begin{aligned}
&\sum_{i\in S} n(R_i+\delta) \geq \sum_{i\in S} \log M_i \\
&\geq \sum_{i\in S} H(W_i) \geq H(W_S|W_{S^{\rm c}}) \\
&= I(\boldsymbol{X}_\Lambda; W_S|W_{S^{\rm c}}) + H(W_S|W_{S^{\rm c}}\boldsymbol{X}_\Lambda) \\
&= I(\boldsymbol{X}_\Lambda; W_S|W_{S^{\rm c}}) + \sum_{i\in S} H(W_i|\boldsymbol{X}_\Lambda) \\
&= I(\boldsymbol{X}_\Lambda; W_S|W_{S^{\rm c}}) + \sum_{i\in S} H(W_i|\boldsymbol{X}_i) \\
&\overset{\rm (a)}{=} I(\boldsymbol{X}_\Lambda; W_S|W_{S^{\rm c}}) + n\sum_{i\in S} r_i^{(n)}.
\end{aligned} \tag{36}$$

Step (a) follows from (35). We estimate a lower bound of $I(\boldsymbol{X}_\Lambda; W_S|W_{S^{\rm c}})$. Observe that

$$\begin{aligned}
I(\boldsymbol{X}_\Lambda; W_S|W_{S^{\rm c}}) &= I(\boldsymbol{X}_\Lambda; W_\Lambda) - I(\boldsymbol{X}_\Lambda; W_{S^{\rm c}}) \\
&= I(\boldsymbol{X}_\Lambda; W_\Lambda) - I(\boldsymbol{X}_{S^{\rm c}}; W_{S^{\rm c}}).
\end{aligned} \tag{37}$$

Since an upper bound of $I(\boldsymbol{X}_{S^{\rm c}}; W_{S^{\rm c}})$ is derived by Corollary 3, it suffices to estimate a lower bound of $I(\boldsymbol{X}_\Lambda; W_\Lambda)$. On a lower bound of this quantity we have the following chain of inequalities:

$$\begin{aligned}
&I(\boldsymbol{X}_\Lambda; W_\Lambda) \\
&= h(\boldsymbol{X}_\Lambda) - h(\boldsymbol{X}_\Lambda|W_\Lambda) \overset{\rm (a)}{=} h(\boldsymbol{X}_\Lambda) - h(\boldsymbol{Z}_\Lambda|W_\Lambda) \\
&= h(\boldsymbol{X}_\Lambda) - \sum_{i=1}^{L} h(\boldsymbol{Z}_i|\boldsymbol{Z}^{i-1}W_\Lambda) \\
&\geq h(\boldsymbol{X}_\Lambda) - \sum_{i=1}^{L} h(\boldsymbol{Z}_i|\hat{\boldsymbol{Z}}_i) \\
&\overset{\rm (b)}{\geq} \frac{n}{2}\log\left[(2\pi{\rm e})^L|\Sigma_{X_\Lambda}|\right] - \sum_{i=1}^{L}\frac{n}{2}\log\left[(2\pi{\rm e})\xi_i^{(n)}\right] \\
&= \frac{n}{2}\log|\Sigma_{X_\Lambda}| - \frac{n}{2}\log\left[\prod_{i=1}^{L}\xi_i^{(n)}\right].
\end{aligned} \tag{38}$$

Step (a) follows from the rotation invariance of the differential entropy. Step (b) follows from Lemma 5. Combining (37), (38) and Corollary 3, we have

$$\begin{aligned}
&I(\boldsymbol{X}_\Lambda; W_S|W_{S^{\rm c}}) + n\sum_{i\in S} r_i^{(n)} \\
&\geq \frac{n}{2}\log\left[\frac{\prod_{i\in S}{\rm e}^{2r_i^{(n)}}|\Sigma_{X_\Lambda}|}{\left|I+\Sigma_{X_{S^{\rm c}}}\Sigma^{-1}_{N_{S^{\rm c}}(r^{(n)}_{S^{\rm c}})}\right|\prod_{i=1}^{L}\xi_i^{(n)}}\right] \\
&= \frac{n}{2}\log\left[\frac{\prod_{i\in S}{\rm e}^{2r_i^{(n)}}|\Sigma_{X_\Lambda}|}{\left|I+\Sigma_{X_{\Lambda}}\Sigma^{-1}_{N_{\Lambda}(r^{(n)}_{S^{\rm c}})}\right|\prod_{i=1}^{L}\xi_i^{(n)}}\right] \\
&= \frac{n}{2}\log\left[\frac{\prod_{i\in S}{\rm e}^{2r_i^{(n)}}}{\left|\Sigma^{-1}_{X_\Lambda}+\Sigma^{-1}_{N_{\Lambda}(r^{(n)}_{S^{\rm c}})}\right|\prod_{i=1}^{L}\xi_i^{(n)}}\right].
\end{aligned}$$

Note here that

$$I(\boldsymbol{X}_\Lambda; W_S|W_{S^{\rm c}}) + n\sum_{i\in S} r_i^{(n)}$$

is nonnegative. Hence, we have

$$\begin{aligned}
&I(\boldsymbol{X}_\Lambda; W_S|W_{S^{\rm c}}) + n\sum_{i\in S} r_i^{(n)} \\
&\geq n\underline{J}_S\left(\prod_{i=1}^{L}\xi_i^{(n)}, r_S^{(n)}\middle| r_{S^{\rm c}}^{(n)}\right).
\end{aligned} \tag{39}$$

Combining (36) and (39), we obtain

$$\sum_{i\in S}(R_i+\delta) \geq \underline{J}_S\left(\prod_{i=1}^{L}\xi_i^{(n)}, r_S^{(n)}\middle| r_{S^{\rm c}}^{(n)}\right). \tag{40}$$

for $S \subseteq \Lambda$. For $i \in \Lambda$, set

$$\begin{aligned}
r_i &\overset{\triangle}{=} \limsup_{n\to\infty} r_i^{(n)} = \limsup_{n\to\infty}\frac{1}{n}I(\boldsymbol{Y}_i; W_i|\boldsymbol{X}_i), \\
\xi_i &\overset{\triangle}{=} \limsup_{n\to\infty} \xi_i^{(n)} = \limsup_{n\to\infty}\frac{1}{n}{\rm E}||\boldsymbol{Z}_i - \hat{\boldsymbol{Z}}_i||^2.
\end{aligned}$$

Then, by letting $n \to \infty$ in (33), (34), and (40), we obtain

$$\left.\begin{aligned}
&\sum_{i=1}^{L}\xi_i \leq D+\delta, \\
&\xi_i \geq \left[Q^{-1}\left(\Sigma^{-1}_{X_\Lambda}+\Sigma^{-1}_{N_\Lambda(r_\Lambda)}\right)Q\right]^{-1}_{ii}, \\
&\quad \text{for } i \in \Lambda, \\
&\sum_{i\in S}(R_i+\delta) \geq \underline{J}_S\left(\textstyle\prod_{i=1}^{L}\xi_i, r_S\middle| r_{S^{\rm c}}\right) \\
&\quad \text{for } S \subseteq \Lambda.
\end{aligned}\right\} \tag{41}$$

Since δ can be made arbitrary small, we obtain

$$\left.\begin{aligned}
&\sum_{i=1}^{L}\xi_i \leq D, \\
&\xi_i \geq \left[Q^{-1}\left(\Sigma^{-1}_{X_\Lambda}+\Sigma^{-1}_{N_\Lambda(r_\Lambda)}\right)Q\right]^{-1}_{ii}, \\
&\quad \text{for } i \in \Lambda, \\
&\sum_{i\in S}R_i \geq \underline{J}_S\left(\textstyle\prod_{i=1}^{L}\xi_i, r_S\middle| r_{S^{\rm c}}\right) \\
&\quad \text{for } S \subseteq \Lambda.
\end{aligned}\right\} \tag{42}$$

Here we choose unitary matrix Q so that $Q^{-1}(\Sigma^{-1}_{X_\Lambda} + \Sigma^{-1}_{N_\Lambda(r_\Lambda)})Q$ becomes the following diagonal matrix:

$$Q^{-1}\left(\Sigma^{-1}_{X_\Lambda}+\Sigma^{-1}_{N_\Lambda(r_\Lambda)}\right)Q = \begin{bmatrix}\alpha_1 & & & \mathbf{0}\\ & \alpha_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \alpha_L\end{bmatrix}. \tag{43}$$

From the second inequality of (42), we have

$$\xi_i \geq \alpha_i^{-1} = \alpha_i^{-1}(r_\Lambda), \quad i = 1, 2, \cdots, L, \tag{44}$$

which together with the first inequality of (42) yields that

$$\sum_{i=1}^{L} \alpha_i^{-1}(r_\Lambda) = \mathrm{tr}\left[\left(\Sigma_{X_\Lambda}^{-1} + \Sigma_{N_\Lambda(r_\Lambda)}^{-1}\right)^{-1}\right] \le \sum_{i=1}^{L} \xi_i \le D\,. \tag{45}$$

On the other hand, by the first inequality of (42), (44), and Lemma 4, we have

$$\prod_{i=1}^{L} \xi_i \le \theta(D, r_\Lambda)\,, \tag{46}$$

which together with the third inequality of (42) yields that

$$\sum_{i\in S} R_i \ge \underline{J}_S(\theta(D,r_\Lambda), r_S | r_{S^c}) \text{ for } S \subseteq \Lambda\,. \tag{47}$$

(45) and (47) imply that $\mathcal{R}_L(D) \subseteq \mathcal{R}_L^{(\mathrm{out})}(D)$. ■

Proof of $R_{\mathrm{sum},L}(D) \ge R_{\mathrm{sum},L}^{(\mathrm{l})}(D)$: Assume that $(R_1, R_2, \cdots, R_L) \in \mathcal{R}_L(D)$. Then, for any $\delta > 0$ and any n with $n \ge n_0(\delta)$, there exists $(\varphi_1, \varphi_2, \cdots, \varphi_L, \psi) \in \mathcal{F}_\delta^{(n)}(R_1, R_2 \cdots, R_L)$ such that

$$\sum_{i=1}^{L} \mathrm{E}||\boldsymbol{X}_i - \hat{\boldsymbol{X}}_i||^2 \le D + \delta\,.$$

For each $l = 0, 1, \cdots, L-1$, we use $(\varphi_{\tau^l 1}, \varphi_{\tau^l(2)}, \cdots, \varphi_{\tau^l(L)})$ for the encoding of $(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_L)$. For $i \in \Lambda$ and for $l = 0, 1, \cdots, L-1$, set

$$W_{l,i} \triangleq \varphi_{\tau^l(i)}(\boldsymbol{Y}_i), \quad \hat{\boldsymbol{X}}_{l,i} \triangleq \psi_{\tau^l(i)}(\varphi_{\tau^l(i)}(\boldsymbol{Y}_1)),$$
$$r_{l,i}^{(n)} \triangleq \frac{1}{n} I(\boldsymbol{Y}_i; W_{l,i} | \boldsymbol{X}_i).$$

In particular,

$$r_{0,i}^{(n)} = r_i^{(n)} = \frac{1}{n} I(\boldsymbol{Y}_i; W_i | \boldsymbol{X}_i), \quad \text{for } i \in \Lambda.$$

Furthermore, set

$$r_{\tau^l(\Lambda)}^{(n)} \triangleq (r_{l,1}^{(n)}, r_{l,2}^{(n)}, \cdots, r_{l,L}^{(n)})\,, \text{ for } l = 0, 1, \cdots, L-1\,,$$
$$r^{(n)} \triangleq \frac{1}{L} \sum_{i=1}^{L} r_i^{(n)}\,.$$

By the cyclic shift invariant property of the source X^L and its noisy observation $Y^L = X^L + N^L$, we have

$$\sum_{i=1}^{L} \mathrm{E}||\boldsymbol{X}_i - \hat{\boldsymbol{X}}_{l,i}||^2 \le D + \delta \quad \text{for } 0 \le l \le L-1\,, \tag{48}$$

$$\frac{1}{L}\sum_{l=0}^{L-1} r_{l,i}^{(n)} = \frac{1}{L}\sum_{l=0}^{L-1} r_{\tau^l(i)}^{(n)} = \frac{1}{L}\sum_{j=1}^{L} r_j^{(n)} = r^{(n)} \quad \text{for } 1 \le i \le L\,. \tag{49}$$

We choose $L \times L$ unitary matrix $Q = [q_{ij}]$ so that

$$Q^{-1}\Sigma_{X_\Lambda}^{-1} Q = \begin{bmatrix} \frac{1}{\lambda_1} & & & \text{\huge 0} \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ \text{\huge 0} & & & \frac{1}{\lambda_L} \end{bmatrix}. \tag{50}$$

Then, we have

$$\begin{aligned} & Q^{-1}\left(\Sigma_{X_\Lambda}^{-1} + \Sigma_{N_\Lambda(r^{(n)})}^{-1}\right) Q \\ &= \begin{bmatrix} \frac{1}{\lambda_1} & & & \text{\huge 0} \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ \text{\huge 0} & & & \frac{1}{\lambda_L} \end{bmatrix} + \frac{1-\mathrm{e}^{-2r^{(n)}}}{\sigma^2} \begin{bmatrix} 1 & & & \text{\huge 0} \\ & 1 & & \\ & & \ddots & \\ \text{\huge 0} & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} \beta_1 & & & \text{\huge 0} \\ & \beta_2 & & \\ & & \ddots & \\ \text{\huge 0} & & & \beta_L \end{bmatrix}. \end{aligned}$$

We set $\boldsymbol{Z}_\Lambda \triangleq \boldsymbol{X}_\Lambda Q$, $\hat{\boldsymbol{Z}}_{\tau^l(\Lambda)} \triangleq \hat{\boldsymbol{Z}}_\Lambda Q$. Furthermore, set

$$\xi_{l,i}^{(n)} \triangleq \frac{1}{n}\mathrm{E}||\boldsymbol{Z}_i - \hat{\boldsymbol{Z}}_{l,i}||^2\,, \quad \bar{\xi}_i^{(n)} \triangleq \frac{1}{L}\sum_{l=0}^{L-1} \xi_{l,i}^{(n)}\,.$$

By the rotation invariance of the squared norm and (48), we have

$$\begin{aligned} \sum_{i=1}^{L} \bar{\xi}_i^{(n)} &= \sum_{i=1}^{L} \frac{1}{L}\sum_{l=0}^{L-1} \frac{1}{n}\mathrm{E}||\boldsymbol{Z}_i - \hat{\boldsymbol{Z}}_{l,i}||^2 \\ &= \frac{1}{L}\sum_{l=0}^{L-1}\sum_{i=1}^{L} \frac{1}{n}\mathrm{E}||\boldsymbol{X}_i - \hat{\boldsymbol{X}}_{l,i}||^2 \le D + \delta\,. \end{aligned} \tag{51}$$

On the other hand, for $i \in \Lambda$, we have the following chain of inequalities:

$$\begin{aligned} &\frac{n}{2}\log\left[(2\pi\mathrm{e})\bar{\xi}_i^{(n)}\right] = \frac{n}{2}\log\left[(2\pi\mathrm{e})\frac{1}{L}\sum_{l=0}^{L-1}\xi_{l,i}^{(n)}\right] \\ &\overset{\text{(a)}}{\ge} \frac{1}{L}\sum_{l=0}^{L-1}\frac{n}{2}\log\left[(2\pi\mathrm{e})\xi_{l,i}^{(n)}\right] \overset{\text{(b)}}{\ge} \frac{1}{L}\sum_{l=0}^{L-1} h(\boldsymbol{Z}_i|\hat{\boldsymbol{Z}}_{l,i}) \qquad (52) \\ &\ge \frac{1}{L}\sum_{l=0}^{L-1} h(\boldsymbol{Z}_i|\boldsymbol{Z}_{\Lambda-\{i\}}W_{\tau^l(\Lambda)}) \\ &\overset{\text{(c)}}{\ge} \frac{1}{L}\sum_{l=0}^{L-1}\frac{n}{2}\log\left\{(2\pi\mathrm{e})\left[Q^{-1}\left(\Sigma_{X_\Lambda}^{-1} + \Sigma_{N_\Lambda(r_\Lambda^{(n)})}^{-1}\right)Q\right]_{ii}^{-1}\right\} \\ &\overset{\text{(d)}}{=} \frac{1}{L}\sum_{l=0}^{L-1}\frac{n}{2}\log\left\{(2\pi\mathrm{e})\left[\frac{1}{\lambda_i} + \sum_{j=1}^{L} q_{ji}^2 \cdot \frac{1-\mathrm{e}^{-2r_{l,j}^{(n)}}}{\sigma^2}\right]^{-1}\right\} \\ &\overset{\text{(e)}}{\ge} \frac{n}{2}\log\left\{(2\pi\mathrm{e})\left[\frac{1}{\lambda_i} + \frac{1}{L}\sum_{l=0}^{L-1}\sum_{j=1}^{L} q_{ji}^2 \cdot \frac{1-\mathrm{e}^{-2r_{l,j}^{(n)}}}{\sigma^2}\right]^{-1}\right\}. \end{aligned} \tag{53}$$

Step (a) follows from the concavity of $\log t$. Step (b) follows from Lemma 5. Step (c) follows from Lemma 6. Step (d) follows from (50). Step (e) follows from the convexity of

$-\log t$. From (53), we have

$$
\begin{aligned}
\bar{\xi}_i^{(n)} &\geq \left[\frac{1}{\lambda_i}+\frac{1}{L}\sum_{l=0}^{L-1}\sum_{j=1}^{L} q_{ji}^2\cdot\frac{1-\mathrm{e}^{-2r_{l,j}^{(n)}}}{\sigma^2}\right]^{-1}\\
&\overset{\text{(a)}}{\geq} \left[\frac{1}{\lambda_i}+\sum_{j=1}^{L} q_{ji}^2\cdot\frac{1-\mathrm{e}^{-2\frac{1}{L}\sum_{l=0}^{L-1} r_{l,j}^{(n)}}}{\sigma^2}\right]^{-1}\\
&= \left[\frac{1}{\lambda_i}+\frac{1-\mathrm{e}^{-2r^{(n)}}}{\sigma^2}\right]^{-1}\\
&= \beta_i^{-1}(r^{(n)}),\quad \text{for } i\in\Lambda\,.
\end{aligned}\tag{54}
$$

Step (a) follows from the concavity of $1-\mathrm{e}^{-2t}$. On the other hand, by (51) and (54), we have

$$
\phi(r^{(n)})=\sum_{i=1}^{L}\beta_i^{-1}(r^{(n)})\leq\sum_{i=1}^{L}\bar{\xi}_i^{(n)}\leq D+\delta\,. \tag{55}
$$

Now we proceed to an evaluation of lower bound of the sum rate. In a manner quite similar to the derivation of (36) in the proof of $\mathcal{R}_L(D)\subseteq\mathcal{R}_L^{(\mathrm{out})}(D)$, we have

$$
\begin{aligned}
&\sum_{i\in\Lambda} n(R_{\tau^l(i)}+\delta)\\
&\geq I(\boldsymbol{X}_\Lambda;W_{\tau^l(\Lambda)})+n\sum_{i\in\Lambda} r_{l,i}^{(n)}\quad \text{for } 0\leq l\leq L-1.
\end{aligned}\tag{56}
$$

From (56), we have

$$
\begin{aligned}
\sum_{i\in\Lambda} n(R_i+\delta) &= \frac{1}{L}\sum_{l=0}^{L-1}\sum_{i\in\Lambda} n(R_{\tau^l(i)}+\delta)\\
&\geq \frac{1}{L}\sum_{l=0}^{L-1} I(\boldsymbol{X}_\Lambda;W_{\tau^l(\Lambda)})+nLr^{(n)}\,.
\end{aligned}\tag{57}
$$

We estimate a lower bound of the first quantity in the right members of (57). On this quantity we have the following chain of inequalities:

$$
\begin{aligned}
&\frac{1}{L}\sum_{l=0}^{L-1} I(\boldsymbol{X}_\Lambda;W_{\tau^l(\Lambda)})\\
&= h(\boldsymbol{X}_\Lambda)-\frac{1}{L}\sum_{l=0}^{L-1} h(\boldsymbol{X}_\Lambda|W_{\tau^l(\Lambda)})\\
&= h(\boldsymbol{X}_\Lambda)-\frac{1}{L}\sum_{l=0}^{L-1} h(\boldsymbol{Z}_\Lambda|W_{\tau^l(\Lambda)})\\
&= h(\boldsymbol{X}_\Lambda)-\frac{1}{L}\sum_{l=0}^{L-1}\sum_{i=1}^{L} h(\boldsymbol{Z}_i|\boldsymbol{Z}^{i-1}W_{\tau^l(\Lambda)})\\
&\geq h(\boldsymbol{X}_\Lambda)-\sum_{i=1}^{L}\frac{1}{L}\sum_{l=0}^{L-1} h(\boldsymbol{Z}_i|\hat{\boldsymbol{Z}}_{l,i})\\
&\overset{\text{(a)}}{\geq} \frac{n}{2}\log\left[(2\pi\mathrm{e})^L|\Sigma_{X_\Lambda}|\right]-\sum_{i=1}^{L}\frac{n}{2}\log\left[(2\pi\mathrm{e})\bar{\xi}_i^{(n)}\right]\\
&= \frac{n}{2}\log|\Sigma_{X_\Lambda}|-\frac{n}{2}\log\left[\prod_{i=1}^{L}\bar{\xi}_i^{(n)}\right]\,.
\end{aligned}\tag{58}
$$

Step (a) follows from (52). Combining (57) and (58), we obtain

$$
\sum_{i\in\Lambda}(R_i+\delta)\geq\underline{J}\left(\prod_{i=1}^{L}\bar{\xi}_i^{(n)},r^{(n)}\right)\,. \tag{59}
$$

Set

$$
\begin{aligned}
r &\triangleq \limsup_{n\to\infty} r^{(n)}=\limsup_{n\to\infty}\frac{1}{L}\sum_{i=1}^{L}\frac{1}{n}I(\boldsymbol{Y}_i;W_i|\boldsymbol{X}_i),\\
\bar{\xi}_i &\triangleq \limsup_{n\to\infty}\bar{\xi}_i^{(n)}=\limsup_{n\to\infty}\frac{1}{L}\sum_{l=0}^{L-1}\frac{1}{n}\mathrm{E}||\boldsymbol{Z}_i-\hat{\boldsymbol{Z}}_{l,i}||^2.
\end{aligned}
$$

By letting $n\to\infty$ in (54), (55), and (59), we obtain

$$
\left.
\begin{aligned}
\bar{\xi}_i\geq\beta_i^{-1}(r)\ \text{for } i\in\Lambda\,,\\
\phi(r)=\sum_{i=1}^{L}\beta_i^{-1}(r)\leq\sum_{i=1}^{L}\bar{\xi}_i\leq D+\delta\,,\\
\sum_{i\in\Lambda}(R_i+\delta)\geq\underline{J}\left(\prod_{i=1}^{L}\bar{\xi}_i,r\right)\,.
\end{aligned}
\right\} \tag{60}
$$

Since δ can be made arbitrary small, we have

$$
\left.
\begin{aligned}
\bar{\xi}_i\geq\beta_i^{-1}(r)\ \text{for } i\in\Lambda\,,\\
\phi(r)=\sum_{i=1}^{L}\beta_i^{-1}(r)\leq\sum_{i=1}^{L}\bar{\xi}_i\leq D\,,\\
\sum_{i\in\Lambda}R_i\geq\underline{J}\left(\prod_{i=1}^{L}\bar{\xi}_i,r\right)\,.
\end{aligned}
\right\} \tag{61}
$$

From the first and second inequality of (61) and Lemma 4, we have

$$
\prod_{i=1}^{L}\bar{\xi}_i\leq\theta(D,r)\,.
$$

Hence, we have

$$
\sum_{i\in\Lambda}R_i\geq\underline{J}(\theta(D,r),r)\ \text{and}\ \phi(r)\leq D\,,
$$

which imply that $R_{\mathrm{sum},L}(D)\geq R_{\mathrm{sum},L}^{(\mathrm{l})}(D)$. ■

B. Derivation of the Inner Bound

In this subsection we prove $\mathcal{R}_L^{(\mathrm{in})}(D)\subseteq\mathcal{R}_L(D)$ stated in Theorem 3.

Proof of $\mathcal{R}_L^{(\mathrm{in})}(D)\subseteq\mathcal{R}_L(D)$: Since $\hat{\mathcal{R}}_L^{(\mathrm{in})}(D)\subseteq\mathcal{R}_L(D)$ is proved by Theorem 1, it suffices to show $\mathcal{R}_L^{(\mathrm{in})}(D)\subseteq\hat{\mathcal{R}}_L^{(\mathrm{in})}(D)$ to prove $\mathcal{R}_L^{(\mathrm{in})}(D)\subseteq\mathcal{R}_L(D)$. We assume that $R^L\in\mathcal{R}_L^{(\mathrm{in})}(D)$. Then, there exists nonnegative vector r^L such that

$$
\mathrm{tr}\left[\left(\Sigma_{X^L}^{-1}+\Sigma_{N^L(r^L)}^{-1}\right)^{-1}\right]\leq D \tag{62}
$$

and

$$
\sum_{i\in S}R_i\geq K(r_S|r_{S^c})\ \text{for any } S\subseteq\Lambda\,. \tag{63}
$$

Let $V_i, i\in\Lambda$ be L independent Gaussian random variables with mean 0 and variance $\sigma_{V_i}^2$. Define Gaussian random variables $U_i, i \in \Lambda$ by $U_i = X_i + N_i + V_i$. By definition it is obvious that

$$\left.\begin{array}{l} U^L \to Y^L \to X^L \\ U_S \to Y_S \to X^L \to Y_{S^c} \to U_{S^c} \\ \text{ for any } S \subseteq \Lambda\,. \end{array}\right\} \qquad (64)$$

For given $r_i \geq 0, i \in \Lambda$ and $D > 0$, choose $\sigma_{V_i}^2$ so that $\sigma_{V_i}^2 = \sigma_{N_i}^2/(\mathrm{e}^{2r_i} - 1)$ when $r_i > 0$. When $r_i = 0$, we choose U_i so that U_i take the constant value zero. Then, the covariance matrix of $N^L + V^L$ becomes $\Sigma_{N^L(r^L)}$. Choose covariance matrix Σ_D so that

$$\mathrm{tr}[\Sigma_D] = D\,, \quad \Sigma_D \succeq (\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1})^{-1}.$$

Since (62), the above choice of Σ_D is possible. Define the linear function $\tilde{\psi}$ of U^L by

$$\tilde{\psi}\left(U^L\right) = U^L \Sigma_{N^L(r^L)}^{-1} (\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1})^{-1}\,.$$

Set $\hat{X}^L = \tilde{\psi}\left(U^L\right)$ and

$$\begin{array}{l} d_{ii} \stackrel{\triangle}{=} \mathrm{E}||X_i - \hat{X}_i||^2\,, \\ d_{ij} \stackrel{\triangle}{=} \mathrm{E}\left(X_i - \hat{X}_i\right)\left(X_j - \hat{X}_j\right), 1 \leq i \neq j \leq L. \end{array}$$

Let $\Sigma_{X^L - \hat{X}^L}$ be a covariance matrix with d_{ij} in its (i,j) element. Then, by simple computations we can show that

$$\Sigma_{X^L - \hat{X}^L} = (\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1})^{-1} \preceq \Sigma_D \qquad (65)$$

and that for any $S \subseteq \Lambda$,

$$J_S(r_S|r_{S^c}) = I(Y_S; U_S|U_{S^c})\,. \qquad (66)$$

From (62) and (65), we have

$$\begin{aligned} & ||X^L - \tilde{\psi}\left(U^L\right)||^2 = ||X^L - \hat{X}^L||^2 \\ & = \mathrm{tr}\left[\left(\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1}\right)^{-1}\right] \leq \mathrm{tr}\left[\Sigma_D\right] = D\,. \end{aligned} \qquad (67)$$

From (64) and (67), we have $U^L \in \mathcal{G}(D)$. Then, from (66)

$$\mathcal{R}_L^{(\mathrm{in})}(D) \subseteq \hat{\mathcal{R}}_L^{(\mathrm{in})}(D)\,,$$

completing the proof. ∎

VI. Proofs of the Results on Matching Conditions

In this section we prove Lemmas 1-3 and Theorems 4 and 6 stated in Section III.

A. *Proof of Lemma 1*

In this subsection we prove Lemma 1. We first present a preliminary observation on $\mathcal{R}_L^{(\mathrm{out})}(D)$. For $r^L \in \mathcal{B}_L(D)$, we examine a form of the region

$$\mathcal{R}_L^{(\mathrm{out})}(D, r^L) = \Big\{R^L : \sum_{i \in S} R_i \geq \underline{J}_S\left(\theta(D, r^L), r_S|r_{S^c}\right) \text{ for any } S \subseteq \Lambda\,.\Big\}\,.$$

Let (Λ, f), $f = \{f_S(r_S|r_{S^c})\}_{S \subseteq \Lambda}$ be a co-polymatroid defined in Property 2. Using (Λ, f), $\mathcal{R}_L^{(\mathrm{out})}(D, r^L)$ is expressed as

$$\mathcal{R}_L^{(\mathrm{out})}(D, r^L) = \Big\{R^L : \sum_{i \in S} R_i \geq f_S\left(r_S|r_{S^c}\right) \text{ for any } S \subseteq \Lambda\,.\Big\}\,.$$

The set $\mathcal{R}_L^{(\mathrm{out})}(D, r^L)$ forms a kind of polytope which is called a *co-polymatroidal polytope* in the terminology of matroid theory. It is well known as a property of this kind of polytope that the polytope $\mathcal{R}_L^{(\mathrm{out})}(D, r^L)$ consists of $L!$ end-points whose components are given by

$$\left.\begin{array}{l} R_{\pi(i)} \\ = f_{\{\pi(i),\cdots,\pi(L)\}}(r_{\{\pi(i),\cdots,\pi(L)\}}|r_{\{\pi(1),\cdots,\pi(i-1)\}}) \\ \quad - f_{\{\pi(i+1),\cdots,\pi(L)\}}(r_{\{\pi(i+1),\cdots,\pi(L)\}}|r_{\{\pi(1),\cdots,\pi(i)\}}) \\ \qquad \text{for } i = 1, 2, \cdots, L-1\,, \\ R_{\pi(L)} = f_{\{\pi(L)\}}(r_{\pi(L)}|r_{\{\pi(1),\cdots,\pi(L-1)\}})\,, \end{array}\right\} \qquad (68)$$

where π is an arbitrary permutation on Λ, that is

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & L \\ \pi(1) & \pi(2) & \cdots & \pi(i) & \cdots & \pi(L) \end{pmatrix}.$$

For $l = 1, 2, \cdots, L$, set

$$\begin{aligned} \mathcal{B}_{\pi,l}(D) &\stackrel{\triangle}{=} \{r^L : r^L \in \mathcal{B}_L(D) \text{ and } \\ &\qquad r_{\pi(i)} = 0 \text{ for } i = l+1, \cdots, L\}\,, \\ \partial\mathcal{B}_{\pi,l}(D) &\stackrel{\triangle}{=} \{r^L : r^L \in \partial\mathcal{B}_L(D) \text{ and } \\ &\qquad r_{\pi(i)} = 0 \text{ for } i = l+1, \cdots, L\}\,. \end{aligned}$$

In particular, when π is the identity map, we omit π to write $\mathcal{B}_l(D)$ and $\partial\mathcal{B}_l(D)$. By Property 1, when $r^L \in \mathcal{B}_{\pi,l}(D)$, the end-point given by (68) becomes

$$\left.\begin{array}{l} R_{\pi(i)} \\ = f_{\{\pi(i),\cdots,\pi(l)\}}(r_{\{\pi(i),\cdots,\pi(l)\}}|r_{\{\pi(1),\cdots,\pi(i-1)\}}) \\ \quad - f_{\{\pi(i+1),\cdots,\pi(l)\}}(r_{\{\pi(i+1),\cdots,\pi(l)\}}|r_{\{\pi(1),\cdots,\pi(i)\}}) \\ \qquad \text{for } i = 1, 2, \cdots, l-1\,, \\ R_{\pi(l)} = f_{\{\pi(l)\}}(r_{\pi(l)}|r_{\{\pi(1),\cdots,\pi(l-1)\}})\,, \\ R_{\pi(i)} = 0, \text{ for } i = l+1, \cdots, L\,. \end{array}\right\} \qquad (69)$$

Proof of Lemma 1: Fix $r^L \in \mathcal{B}_L(D)$ arbitrary. Let R^L be a nonnegative rate vector such that L components of R^L satisfy (68). To prove Lemma 1, it suffices to show that this nonnegative vector belongs to $\mathcal{R}_L^{(\mathrm{in})}(D)$. For $l = 1, 2, \cdots, L$, we prove the claim that under the MD condition, if $r^L \in \mathcal{B}_{\pi,l}(D)$, then, the rate vector R^L satisfying (69) belongs to $\mathcal{R}_L^{(\mathrm{in})}(D)$. We prove this claim by induction with respect to l. When $l = 1$, from (69), we have

$$\left.\begin{array}{l} R_{\pi(1)} = f_{\{\pi(1)\}}(r_{\pi(1)})\,, \\ R_{\pi(i)} = 0, \text{ for } i = 2, \cdots, L\,. \end{array}\right\} \qquad (70)$$

The function $f_{\{\pi(1)\}}(r_{\pi(1)})$ is computed as

$$\begin{aligned} & f_{\{\pi(1)\}}(r_{\pi(1)}) \\ & = \underline{J}_{\{\pi(1)\}}\left(\theta(D, r^L), r_{\pi(1)}|r_{\{\pi(1)\}^c}\right)\Big|_{r_{\{\pi(1)\}^c} = \mathbf{0}} \\ & = \frac{1}{2}\log^+\left[\frac{\mathrm{e}^{2r_{\pi(1)}}}{|\Sigma_{X^L}^{-1}|\,\theta(D, r^L)|_{r_{\{\pi(1)\}^c} = \mathbf{0}}}\right]. \end{aligned} \qquad (71)$$

Since $r^L \in \mathcal{B}_{\pi,l}(D)$, we can decrease $r_{\pi(1)}$ keeping $r^L \in \mathcal{B}_{\pi,1}(D)$ so that it arrives at $r_{\pi(1)}^* = 0$ or a positive $r_{\pi(1)}^*$ satisfying

$$(r_{\pi(1)}^*, r_{\{\pi(1)\}^c}) = (r_{\pi(1)}^*, \underbrace{0, \cdots, 0}_{L-1}) \in \partial\mathcal{B}_{\pi,1}(D)\,. \qquad (72)$$

Let $(R^*_{\pi(1)},\ \cdots, R^*_{\pi(L)})$ be a rate vector corresponding to $(r^*_{\pi(1)},\ r_{\{\pi(1)\}^{\rm c}})$. If $r^*_{\pi(1)}=0$, we have $r^L=\mathbf{0}\in\mathcal{B}_L(D)$. Then, we have

$$\mathrm{tr}\left[\left(\Sigma_{X^L}^{-1}+\Sigma_{N^L(r^L)}^{-1}\right)^{-1}\right]=\mathrm{tr}\left[\Sigma_{X^L}\right]\le D\,.$$

This contradicts the first assumption of $D<\ \mathrm{tr}\left[\Sigma_{X^L}\right]$. Therefore, $r^*_{\pi(1)}$ must be positive. Then, from (72), we have

$$(R^*_{\pi(1)},\cdots,R^*_{\pi(L)})=(R^*_{\pi(1)},\underbrace{0,\cdots,0}_{L-1})\in\mathcal{R}_L^{(\rm in)}(D)\,.$$

By (71) and the MD condition, $f_{\{\pi(1)\}}(r_{\pi(1)})$ is a monotone increasing function of $r_{\pi(1)}$. Then, we have $R_{\pi(1)}\ge R^*_{\pi(1)}$. Hence, we have

$$(R_{\pi(1)},\cdots,R_{\pi(L)})=(R_{\pi(1)},\underbrace{0,\cdots,0}_{L-1})\in\mathcal{R}_L^{(\rm in)}(D)\,.$$

Thus, the claim holds for $l=1$. We assume that the claim holds for $l-1$. Since $\mathrm{tr}\left[(\Sigma_{X^L}^{-1}+\Sigma_{N^L(r^L)}^{-1})^{-1}\right]$ is a monotone increasing function of $r_{\pi(l)}$ on $\mathcal{B}_{\pi,l}(D)$, we can decrease $r_{\pi(l)}$ keeping $r^L\in\mathcal{B}_{\pi,l}(D)$ so that it arrives at $r^*_{\pi(l)}=0$ or a positive $r^*_{\pi(l)}$ satisfying

$$(r^*_{\pi(l)},r_{\{\pi(l)\}^{\rm c}})\in\partial\mathcal{B}_{\pi,l}(D)\,. \tag{73}$$

Let $(R^*_{\pi(1)},\ \cdots, R^*_{\pi(L)})$ be a rate vector corresponding to $(r^*_{\pi(l)},\ r_{\{\pi(l)\}^{\rm c}})$. By Property 2 part b) and the MD condition, the l functions

$$\begin{array}{l}f_{\{\pi(i),\cdots,\pi(l)\}}(r_{\{\pi(i),\cdots,\pi(l)\}}|r_{\{\pi(1),\cdots,\pi(i-1)\}})\\ -f_{\{\pi(i+1),\cdots,\pi(l)\}}(r_{\{\pi(i+1),\cdots,\pi(l)\}}|r_{\{\pi(1),\cdots,\pi(i)\}})\\ \quad \text{for } i=1,2,\cdots,l-1\,,\\ f_{\{\pi(l)\}}(r_{\pi(l)}|r_{\{\pi(1),\cdots,\pi(l-1)\}})\end{array}$$

appearing in the right members of (69) are monotone increasing functions of $r_{\pi(l)}$. Then, from (69), we have

$$\left.\begin{array}{l}R_{\pi(i)}\ge R^*_{\pi(i)} \text{ for } i=1,2,\cdots,l\,,\\ R_{\pi(i)}=R^*_{\pi(i)}=0 \text{ for } i=l+1,\cdots,L\,.\end{array}\right\} \tag{74}$$

When $r^*_{\pi(l)}=0$, we have $(r^*_{\pi(l)},r_{\{\pi(l)\}^{\rm c}})\in\mathcal{B}_{\pi,l-1}(D)$. Then, by induction hypothesis we have

$$(R^*_{\pi(1)},\cdots,R^*_{\pi(L)})\in\mathcal{R}_L^{(\rm in)}(D)\,.$$

When $r^*_{\pi(l)}>0$, from (73), we have

$$(R^*_{\pi(1)},\cdots,R^*_{\pi(L)})\in\mathcal{R}_L^{(\rm in)}(D)\,.$$

Hence, by (74), we have

$$\begin{array}{l}(R_{\pi(1)},\cdots,R_{\pi(L)})\\ =(R_{\pi(1)},\cdots,R_{\pi(l)},\underbrace{0,\cdots,0}_{L-l})\in\mathcal{R}_L^{(\rm in)}(D)\,.\end{array}$$

Thus, the claim is proved. ∎

B. *Proofs of Lemmas 2 and 3 and Theorems 4 and 6*

In this subsection we prove Lemmas 2 and 3 and Theorems 4 and 6.

We first observe that using the eigen values $\alpha_k=\alpha_k(u^L)$, $k\in\Lambda$ of $\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}$, the condition

$$\mathrm{tr}\left[\left(\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}\right)^{-1}\right]\le D$$

is rewritten as

$$\sum_{i=1}^{L}\frac{1}{\alpha_i(u^L)}\le D\,. \tag{75}$$

Next, we present a lemma necessary to prove Lemma 2.

Lemma 7: For the eigen values $\alpha_k=\alpha_k(u^L),k\in\Lambda$ of $\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}$ and for $u_i,i\in\Lambda$, we have the followings:

$$\alpha_{\min}\le u_i\le\alpha_{\max}\,,\ \frac{\partial\alpha_k}{\partial u_i}\ge 0,\ \text{for } k\in\Lambda,\ \sum_{k=1}^{L}\frac{\partial\alpha_k}{\partial u_i}=1\,.$$

Proof of this lemma needs some analytical arguments on the eigen values of positive semidefinite Hermitian matrix. Detail of the proof will be given in Appendix C.

Proof of Lemma 2: Let S be a set of integers that satisfies $\alpha_i^{-1}\ge\xi$ in the definition of $\theta(D,u^L)$. Then, $\theta(D,u^L)$ is computed as

$$\theta(D,u^L)=\tfrac{1}{(L-|S|)^{L-|S|}}\left(\prod_{k\in S}\frac{1}{\alpha_k}\right)\left(D-\sum_{k\in S}\frac{1}{\alpha_k}\right)^{L-|S|}\,.$$

Fix $i\in\Lambda$ arbitrary. For simplicity of notation we set $A_i\triangleq(a_{ii}+c_i)$ and set

$$\Psi\triangleq\log\frac{Dc_i}{A_i-u_i}-\log\theta(D,u^L)\,.$$

Computing the partial derivative of Ψ by u_i, we obtain

$$\frac{\partial\Psi}{\partial u_i}=\sum_{k\in S}\left(\frac{\partial\alpha_k}{\partial u_i}\right)\left[\frac{1}{\alpha_k}-\frac{L-|S|}{D-\sum_{k\in S}\frac{1}{\alpha_k}}\frac{1}{\alpha_k^2}\right]+\frac{1}{A_i-u_i}\,. \tag{76}$$

From Lemma 7 and (76), we obtain

$$\frac{\partial\Psi}{\partial u_i}\ge\sum_{k\in S}\left(\frac{\partial\alpha_k}{\partial u_i}\right)\left[\frac{1}{\alpha_k}-\frac{L-|S|}{D-\sum_{k\in S}\frac{1}{\alpha_k}}\frac{1}{\alpha_k^2}+\frac{1}{A_i-\alpha_{\min}}\right]\,.$$

To examine signs of contents of the above summation we set

$$\begin{array}{l}\Phi_k\triangleq\left\{D-\sum_{k\in S}\frac{1}{\alpha_k}-\frac{L-|S|}{\alpha_k}\right\}(A_i-\alpha_{\min})\\ \quad+\alpha_k\left(D-\sum_{k\in S}\frac{1}{\alpha_k}\right)\,.\end{array}$$

If $|S| = L$, $\Phi_k \geq 0, k \in \Lambda$ is obvious. We hereafter assume $|S| \leq L-1$. Computing Φ_k, we obtain

$$\begin{aligned}
\Phi_k &= A_i \left(D - \sum_{k \in S} \frac{1}{\alpha_k} \right) - \frac{L-|S|}{\alpha_k} \cdot (A_i - \alpha_{\min}) \\
&\quad +(\alpha_k - \alpha_{\min}) \left(D - \sum_{k \in S} \frac{1}{\alpha_k} \right) \\
&\geq A_i \left(D - \sum_{k \in S} \frac{1}{\alpha_k} \right) - \frac{L-|S|}{\alpha_k} \cdot (A_i - \alpha_{\min}) \\
&\overset{\text{(a)}}{\geq} A_i \sum_{k \in \Lambda - S} \frac{1}{\alpha_k} - \frac{L-|S|}{\alpha_k} \cdot (A_i - \alpha_{\min}) \\
&\geq A_i \cdot \frac{L-|S|}{\alpha_{\max}} - \frac{L-|S|}{\alpha_{\min}} \cdot (A_i - \alpha_{\min}) \\
&= A_i (L - |S|) \left(\frac{1}{\alpha_{\max}} - \frac{1}{\alpha_{\min}} + \frac{1}{A_i} \right) . \qquad (77)
\end{aligned}$$

Step (a) follows from the inequality (75), that is,

$$D - \sum_{k=1}^{L} \frac{1}{\alpha_k(r^L)} \geq 0 \,.$$

From (77), we can see that if

$$\frac{1}{\alpha_{\min}(r^L)} - \frac{1}{\alpha_{\max}(r^L)} \leq \frac{1}{A_i} \text{ for } i \in \Lambda,$$

then, $\Phi_k \geq 0$ for $k \in S$. ■

Proof of Theorem 4: By (75), we have

$$\begin{aligned}
\frac{1}{\alpha_{\min}(r^L)} &\leq D - \frac{L-1}{\alpha_{\max}(r^L)} \\
&= \frac{1}{\alpha_{\max}(r^L)} + D - \frac{L}{\alpha_{\max}(r^L)} \,.
\end{aligned}$$

Hence, if

$$D - \frac{L}{\alpha_{\max}(r^L)} \leq \frac{1}{a_{ii} + c_i} \,,$$

or equivalent to

$$\left(D - \frac{1}{a_{ii} + c_i} \right) \alpha_{\max}(r^L) \leq L \qquad (78)$$

holds for $r^L \in \mathcal{B}_L(D)$ and $i \in \Lambda$, the condition on $\alpha_{\min}$ and $\alpha_{\max}$ in Lemma 2 holds. By Lemma 7, we have

$$\alpha_{\max}(r^L) \leq \alpha^*_{\max} \text{ for } r^L \in \mathcal{B}_L(D). \qquad (79)$$

It can be seen from (78) and (79) that

$$\left(D - \frac{1}{a_{ii} + c_i} \right) \alpha^*_{\max} \leq L \text{ for } i \in \Lambda \,. \qquad (80)$$

is a sufficient condition for (78) to hold. By Lemma 7, we have

$$a_{ii} + c_i \leq \alpha^*_{\max} \text{ for } i \in \Lambda. \qquad (81)$$

From (80) and (81), we have

$$\left(D - \frac{1}{a_{ii} + c_i} \right) \alpha^*_{\max} \leq D\alpha^*_{\max} - 1 \,.$$

Thus, if we have $D\alpha^*_{\max} - 1 \leq L$ or equivalent to $D \leq (L+1)/\alpha^*_{\max}$, we have (80). ■

Proof of Lemma 3: Let S be a set of integers that satisfies $\beta_i^{-1} \geq \xi$ in the definition of $\theta(D, r)$. Then $\theta(D, r)$ is computed as

$$\theta(D,r) = \tfrac{1}{(L-|S|)^{L-|S|}} \left(\prod_{k \in S} \frac{1}{\beta_k} \right) \left(D - \sum_{k \in S} \frac{1}{\beta_k} \right)^{L-|S|} .$$

Fix $i \in \Lambda$ arbitrary and set

$$\Psi \triangleq 2Lr - \log \theta(D, r) \,.$$

Computing the derivative of Ψ by r, we obtain

$$\begin{aligned}
\frac{\mathrm{d}\Psi}{\mathrm{d}r} &= \frac{2}{\sigma^2 \mathrm{e}^{2r}} \sum_{k \in S} \left[\frac{1}{\beta_k} - \frac{L-|S|}{D - \sum_{k \in S} \frac{1}{\beta_k}} \frac{1}{\beta_k^2} \right] + 2L \\
&= \frac{2}{\sigma^2 \mathrm{e}^{2r}} \sum_{k \in S} \left[\frac{1}{\beta_k} - \frac{L-|S|}{D - \sum_{k \in S} \frac{1}{\beta_k}} \frac{1}{\beta_k^2} + \sigma^2 \mathrm{e}^{2r} \cdot \frac{L}{|S|} \right] .
\end{aligned}$$

To examine signs of contents of the above summation we set

$$\begin{aligned}
\Phi_k &\triangleq D - \sum_{k \in S} \frac{1}{\beta_k} - \frac{L-|S|}{\beta_k} \\
&\quad + \sigma^2 \mathrm{e}^{2r} \frac{L}{|S|} \beta_k \left(D - \sum_{k \in S} \frac{1}{\beta_k} \right) .
\end{aligned}$$

If $|S| = L$, $\Phi_k \geq 0, k \in \Lambda$ is obvious. We hereafter assume $|S| \leq L-1$. Computing Φ_k, we obtain

$$\begin{aligned}
&\Phi_k \\
&\overset{\text{(a)}}{\geq} \sum_{k \in \Lambda - S} \frac{1}{\beta_k} - \frac{L-|S|}{\beta_k} + \sigma^2 \mathrm{e}^{2r} \frac{L}{|S|} \beta_k \sum_{k \in \Lambda - S} \frac{1}{\beta_k} \\
&\geq \frac{L-|S|}{\beta_{\max}} - \frac{L-|S|}{\beta_{\min}} + \sigma^2 \mathrm{e}^{2r} \frac{L}{|S|} (L - |S|) \frac{\beta_{\min}}{\beta_{\max}} \\
&= (L-|S|) \left[\frac{1}{\beta_{\max}} - \frac{1}{\beta_{\min}} + \sigma^2 \mathrm{e}^{2r} \frac{L}{|S|} \cdot \frac{\beta_{\min}}{\beta_{\max}} \right] . \qquad (82)
\end{aligned}$$

Step (a) follows from

$$D - \sum_{k=1}^{L} \frac{1}{\beta_k} \geq 0 \Leftrightarrow D - \sum_{k \in S} \frac{1}{\beta_k} \geq \sum_{k \in \Lambda - S} \frac{1}{\beta_k} \,.$$

From (82), we can see that if

$$\frac{1}{\beta_{\min}} - \frac{1}{\beta_{\max}} \leq \sigma^2 \mathrm{e}^{2r} \frac{L}{|S|} \cdot \frac{\beta_{\min}}{\beta_{\max}} \,, \qquad (83)$$

then, $\Phi_k \geq 0$ for $k \in S$. Since $|S| \leq L-1$,

$$\frac{1}{\beta_{\min}(r)} - \frac{1}{\beta_{\max}(r)} \leq \sigma^2 \mathrm{e}^{2r} \frac{L}{L-1} \cdot \frac{\beta_{\min}(r)}{\beta_{\max}(r)}$$

is a sufficient condition for (83) to hold. ■

Proof of Theorem 6: Computing $\beta_{\min}^{-1} - \beta_{\max}^{-1}$, we have

$$\begin{aligned}
&\frac{1}{\beta_{\min}(r)} - \frac{1}{\beta_{\max}(r)} \\
&= \frac{\lambda_{\max} - \lambda_{\min}}{\left\{ 1 + \frac{\lambda_{\max}}{\sigma^2} (1 - \mathrm{e}^{-2r}) \right\} \left\{ 1 + \frac{\lambda_{\min}}{\sigma^2} (1 - \mathrm{e}^{-2r}) \right\}} \\
&\leq \lambda_{\max} - \lambda_{\min} \,.
\end{aligned}$$

On the other hand

$$\mathrm{e}^{2r}\frac{\beta_{\min}(r)}{\beta_{\max}(r)} = \mathrm{e}^{2r}\frac{1+\frac{\lambda_{\max}}{\sigma^2}(1-\mathrm{e}^{-2r})}{1+\frac{\lambda_{\min}}{\sigma^2}(1-\mathrm{e}^{-2r})}\cdot\frac{\lambda_{\min}}{\lambda_{\max}} \geq \frac{\lambda_{\min}}{\lambda_{\max}}\,.$$

Hence, if

$$\lambda_{\max}-\lambda_{\min} \leq \sigma^2\frac{L}{L-1}\cdot\frac{\lambda_{\min}}{\lambda_{\max}}\,,$$

or equivalent to

$$\sigma^2 \geq \frac{L-1}{L}\cdot\frac{\lambda_{\max}}{\lambda_{\min}}(\lambda_{\max}-\lambda_{\min})\,,$$

we have

$$\frac{1}{\beta_{\min}(r)}-\frac{1}{\beta_{\max}(r)} \leq \sigma^2\mathrm{e}^{2r}\frac{L}{L-1}\cdot\frac{\beta_{\min}(r)}{\beta_{\max}(r)}$$

for $r \geq 0$, completing the proof. ∎

VII. Conclusion

We have considered the distributed source coding of correlated Gaussian observation and given a partial solution to this problem by deriving explicit outer bound of the rate distortion region. Furthermore, we established a sufficient condition under which this outer bound is tight.

In this paper our arguments have been concentrated on Problem 2, the determination problem of $\mathcal{R}_L(D)$. On Problem 1, the determination problem of $\mathcal{R}_L(D^L)$, the techniques we have used to derive the outer bound of $\mathcal{R}_L(D)$ are not sufficient to derive an outer bound of $\mathcal{R}_L(D^L)$.

In [20], we introduced a unified approach to deal with Problems 1 and 2 and derived outer bounds of the rate distortion regions on those two problems. For Problem 1, the outer bound of [20] has a form of positive semi definite programming. For Problem 2, the outer bound of [20] is the same as that of this paper. Recently, we have obtained some extentions of the results of Oohama [20]. Details of those results are to be presented in a future paper.

Appendix

A. *Proof of Theorem 8.*

In this appendix we prove Theorem 8.

Proof of Theorem 8: We first observe that

$$\sum_{i=1}^{L} r_i + \frac{1}{2}\log\frac{\left|\Sigma_{X^L}^{-1}+\Sigma_{N^L(r^L)}^{-1}\right|}{\left|\Sigma_{X^L}^{-1}\right|} = \sum_{i=1}^{L} r_i + \frac{1}{2}\log\left|\Sigma_{X^L}^{-1}+\Sigma_{N^L(r^L)}^{-1}\right| + \frac{1}{2}\log\left|\Sigma_{X^L}\right|\,, \tag{84}$$

$$\left|\Sigma_{X^L}\right| = (1-\rho)^L\left\{1+\frac{\rho L}{1-\rho}\right\}\,, \tag{85}$$

$$\left|\Sigma_{X^L}^{-1}+\Sigma_{N^L(r^L)}^{-1}\right| = \left(1-\sum_{i=1}^{L}\frac{b}{u_i+b}\right)\prod_{i=1}^{L}(u_i+b)\,. \tag{86}$$

Set

$$v_i \stackrel{\triangle}{=} \frac{1}{u_i+b} = \{a+b+c(1-\mathrm{e}^{-2r_i})\}^{-1}\,.$$

Then, we have

$$\left.\begin{aligned} u_i &= v_i^{-1}-b\,,\\ r_i &= \frac{1}{2}\log\frac{c}{a+b+c-v_i^{-1}}\,.\end{aligned}\right\} \tag{87}$$

From (14) in Section IV and (87), we can see that the condition $r^L \in \mathcal{B}_L(D)$ is equivalent to

$$\begin{aligned} & b\sum_{i\neq j} v_i v_j - (1+Db)\sum_{i=1}^{L} v_i + D \geq 0\\ \Leftrightarrow\ & b\left(\sum_{i=1}^{L} v_i\right)^2 - b\sum_{i=1}^{L} v_i^2\\ & -(1+Db)\sum_{i=1}^{L} v_i + D \geq 0\,. \end{aligned} \tag{88}$$

From (86) and (87), we have

$$\begin{aligned} & \sum_{i=1}^{L} r_i + \frac{1}{2}\log\left|\Sigma_{X^L}^{-1}+\Sigma_{N^L(r^L)}^{-1}\right|\\ &= \sum_{i=1}^{L}\frac{1}{2}\log\frac{c}{(a+b+c)v_i-1}\\ &\quad +\frac{1}{2}\log\left(1-b\sum_{i=1}^{L} v_i\right)\\ &\stackrel{\rm (a)}{\geq} \frac{L}{2}\log\frac{c}{(a+b+c)\frac{1}{L}\sum_{i=1}^{L} v_i - 1}\\ &\quad +\frac{1}{2}\log\left(1-b\sum_{i=1}^{L} v_i\right)\,. \end{aligned} \tag{89}$$

Step (a) follows from the convexity of $-\log t$. Here, we set

$$\gamma \stackrel{\triangle}{=} \left\{\frac{1}{L}\sum_{i=1}^{L} v_i\right\}^{-1}\,.$$

Then, from (89), we have

$$\begin{aligned} & \sum_{i=1}^{L} r_i + \frac{1}{2}\log\left|\Sigma_{X^L}^{-1}+\Sigma_{N^L(r^L)}^{-1}\right|\\ &\geq \frac{L}{2}\log\frac{(Dc)\gamma}{D(a+b+c)-\gamma} + \frac{1}{2}\log\left(1-(Db)\frac{L}{\gamma}\right)\,. \end{aligned} \tag{90}$$

Since

$$\sum_{i} v_i^2 \geq L\cdot\left(\frac{1}{L}\sum_{i=1}^{L} v_i\right)^2 = L\gamma^{-2}$$

and (88), we obtain

$$\begin{aligned} & bL(L-1)\gamma^{-2} - (1+Db)L\gamma^{-1} + D \geq 0\\ \Leftrightarrow\ & \left(\frac{D\gamma}{L}\right)^2 - (1+Db)\left(\frac{D\gamma}{L}\right) + Db\left(1-\frac{1}{L}\right) \geq 0 \end{aligned} \tag{91}$$

Since $v_i \le b^{-1}$ for $i \in \Lambda$, γ must be $\gamma \ge Lb$. Solving (91) under this constraint, we obtain

$$D\gamma \ge \frac{L}{2}\left[1 + Db + \sqrt{(1-Db)^2 + \tfrac{4Db}{L}}\right] = L_1 \,. \tag{92}$$

Combining (84), (85), (90), and (92), we have

$$\begin{aligned}
&\sum_{i=1}^{L} r_i + \frac{1}{2}\log\frac{\left|\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1}\right|}{\left|\Sigma_{X^L}^{-1}\right|} \\
&\ge \frac{L}{2}\log\left(\frac{(1-\rho)L_1 c}{D(a+b+c)-L_1}\right) \\
&\quad + \frac{1}{2}\log\left\{\frac{1+(L-1)\rho}{1-\rho}\left(1-\frac{LDb}{L_1}\right)\right\}.
\end{aligned}$$

The equality holds

$$r_i = \frac{1}{2}\log\frac{Dc}{D(a+b+c)-L_1}\,,\ \text{for } i \in \Lambda\,,$$

completing the proof. ∎

B. Proof of Lemma 6

In this appendix we prove Lemma 6. Without loss of generality we may assume that $S = \{1, 2, \cdots, s\}$. We write unitary matrix Q_S as $Q_S = [q_{ij}]$, where q_{ij} stands for the (i,j) element of Q_S. The unitary matrix Q_S transforms X_S into $Z_S = X_S Q_S$. The following lemma states an important property on the distribution of Gaussian random vector Z_S. This lemma is a basis of the proof of Lemma 6.

Lemma 8: For any $i \in S$, we have the following.

$$Z_i = -\frac{1}{g_{ii}}\sum_{j\ne i}\nu_{ij}Z_j + \frac{1}{g_{ii}}\sum_{j=1}^{s}\frac{q_{ji}}{\sigma_{N_j}^2}Y_j + \hat{N}_i\,, \tag{93}$$

where

$$g_{ii} = \left[Q_S^{-1}\Sigma_{X_S}^{-1}Q_S\right]_{ii} + \sum_{j=1}^{s}\frac{q_{ji}^2}{\sigma_{N_j}^2}\,, \tag{94}$$

ν_{ij}, $j \in S - \{i\}$ are suitable constants and $\hat{N}_i$ is a zero mean Gaussian random variables with variance $\frac{1}{g_{ii}}$. For each $i \in S$, $\hat{N}_i$ is independent of $Z_j, j \in S-\{i\}$ and $Y_j, j \in S$.

Proof: Without loss of generality we may assume $i = 1$. Let $\Sigma_{X_S Y_S}$ be a covariance matrix on the pair of the Gaussian random vectors X_S and Y_S. Since $Y_S = X_S + N_S$, we have

$$\Sigma_{X_S Y_S} = \begin{bmatrix}\Sigma_{X_S} & \Sigma_{X_S}\\ \Sigma_{X_S} & \Sigma_{X_S} + \Sigma_{N_S}\end{bmatrix}.$$

Since $Z_S = X_S Q_S$, we have

$$\Sigma_{Z_S Y_S} = \begin{bmatrix}Q_S^{-1}\Sigma_{X_S}Q_S & Q_S^{-1}\Sigma_{X_S}\\ \Sigma_{X_S}Q_S & \Sigma_{X_S}+\Sigma_{N_S}\end{bmatrix}.$$

The density function $p_{Z_S Y_S}(z_S, y_S)$ of (Z_S, Y_S) is given by

$$p_{Z_S Y_S}(z_S, y_S) = \frac{1}{(2\pi\mathrm{e})^s\left|\Sigma_{Z_S Y_S}\right|^{\frac{1}{2}}}\mathrm{e}^{-\frac{1}{2}[z_S y_S]\Sigma_{Z_S Y_S}^{-1}\begin{bmatrix}{}^{\mathrm t}z_S\\ {}^{\mathrm t}y_S\end{bmatrix}},$$

where $\Sigma_{Z_S Y_S}^{-1}$ has the following form:

$$\Sigma_{Z_S Y_S}^{-1} = \begin{bmatrix}Q_S^{-1}(\Sigma_{X_S}^{-1}+\Sigma_{N_S}^{-1})Q_S & -Q_S^{-1}\Sigma_{N_S}^{-1}\\ -\Sigma_{N_S}^{-1}Q_S & \Sigma_{N_S}^{-1}\end{bmatrix}.$$

Set

$$\left.\begin{aligned}
\nu_{ij} &\stackrel{\triangle}{=} \left[Q_S^{-1}(\Sigma_{X_S}^{-1}+\Sigma_{N_S}^{-1})Q_S\right]_{ij}\\
&= \left[Q_S^{-1}\Sigma_{X_S}^{-1}Q_S\right]_{ij} + \sum_{k=1}^{s}\frac{q_{ki}q_{kj}}{\sigma_{N_k}^2}\,,\\
\beta_{ij} &\stackrel{\triangle}{=} -\left[Q_S^{-1}\Sigma_{N_S}^{-1}\right]_{ij} = -\frac{q_{ji}}{\sigma_{N_j}^2}\,.
\end{aligned}\right\} \tag{95}$$

Now, we consider the following partition of $\Sigma_{Z_S Y_S}^{-1}$:

$$\begin{aligned}
\Sigma_{Z_S Y_S}^{-1} &= \begin{bmatrix}Q_S^{-1}(\Sigma_{X_S}^{-1}+\Sigma_{N_S}^{-1})Q_S & -Q_S^{-1}\Sigma_{N_S}^{-1}\\ -\Sigma_{N_S}^{-1}Q_S & \Sigma_{N_S}^{-1}\end{bmatrix}\\
&= \left[\begin{array}{c|c} g_{11} & g_{12}\\ \hline {}^{\mathrm t}g_{12} & G_{22}\end{array}\right],
\end{aligned}$$

where g_{11}, g_{12}, and G_{22} are scalar, $2s-1$ dimensional vector, and $(2s-1)\times(2s-1)$ matrix, respectively. It is obvious from the above partition of $\Sigma_{Z_S Y_S}^{-1}$ that we have

$$\left.\begin{aligned}
g_{11} &= \nu_{11} = \left[Q_S^{-1}\Sigma_{X_S}^{-1}Q_S\right]_{11} + \sum_{k=1}^{s}\frac{q_{k1}^2}{\sigma_{N_k}^2}\,,\\
g_{12} &= [\nu_{12}\cdots\nu_{1s}\beta_{11}\beta_{12}\cdots\beta_{1s}]\,.
\end{aligned}\right\} \tag{96}$$

It is well known that $\Sigma_{Z_S Y_S}^{-1}$ has the following expression:

$$\begin{aligned}
\Sigma_{Z_S Y_S}^{-1} &= \left[\begin{array}{c|c} g_{11} & g_{12}\\ \hline {}^{\mathrm t}g_{12} & G_{22}\end{array}\right]\\
&= \left[\begin{array}{c|c} 1 & 0_{12}\\ \hline \frac{1}{g_{11}}{}^{\mathrm t}g_{12} & I_{L-1}\end{array}\right]\left[\begin{array}{c|c} g_{11} & 0_{12}\\ \hline {}^{\mathrm t}0_{12} & G_{22}-\frac{1}{g_{11}}{}^{\mathrm t}g_{12}g_{12}\end{array}\right]\\
&\qquad\times\left[\begin{array}{c|c} 1 & \frac{1}{g_{11}}g_{12}\\ \hline {}^{\mathrm t}0_{12} & I_{L-1}\end{array}\right].
\end{aligned}$$

Set

$$\left.\begin{aligned}
\hat{n}_1 &\stackrel{\triangle}{=} [z_1|z_{S-\{1\}}y_S]\left[\begin{array}{c}1\\ \hline \frac{1}{g_{11}}{}^{\mathrm t}g_{12}\end{array}\right]\\
&= z_1 + \tfrac{1}{g_{11}}\left[z_{S-\{1\}}y_S\right]{}^{\mathrm t}g_{12}\,.
\end{aligned}\right\} \tag{97}$$

Then, we have

$$\begin{aligned}
&[z_S y_S]\Sigma_{Z_S Y_S}\begin{bmatrix}{}^{\mathrm t}z_S\\ {}^{\mathrm t}y_S\end{bmatrix}\\
&= [z_1|z_{S-\{1\}}y_S]\left[\begin{array}{c|c} g_{11} & g_{12}\\ \hline {}^{\mathrm t}g_{12} & G_{22}\end{array}\right]\left[\begin{array}{c} z_1\\ \hline {}^{\mathrm t}z_{S-\{1\}}\\ {}^{\mathrm t}y_S\end{array}\right]\\
&= [\hat{n}_1|z_{S-\{1\}}y_S]\left[\begin{array}{c|c} g_{11} & 0_{12}\\ \hline {}^{\mathrm t}0_{12} & G_{22}-\frac{1}{g_{11}}{}^{\mathrm t}g_{12}g_{12}\end{array}\right]\left[\begin{array}{c}\hat{n}_1\\ \hline {}^{\mathrm t}z_{S-\{1\}}\\ {}^{\mathrm t}y_S\end{array}\right].
\end{aligned} \tag{98}$$

From (95)-(97), we have

$$\begin{aligned}
\hat{n}_1 &= z_1 + \frac{1}{g_{11}}\sum_{j=2}^{s}\nu_{1j}z_j + \frac{1}{g_{11}}\sum_{j=1}^{s}\beta_{1j}y_j\\
&= z_1 + \frac{1}{g_{11}}\sum_{j=2}^{s}\nu_{1j}z_j - \frac{1}{g_{11}}\sum_{j=1}^{s}\frac{q_{j1}}{\sigma_{N_j}^2}y_j\,.
\end{aligned} \tag{99}$$

It can be seen from (98) and (99) that the random variable $\hat{N}_1$ defined by

$$\hat{N}_1 \stackrel{\triangle}{=} Z_1 + \frac{1}{g_{11}}\sum_{j=2}^{s}\nu_{1j}Z_j - \frac{1}{g_{11}}\sum_{j=1}^{s}\frac{q_{j1}}{\sigma_{N_j}^2}Y_j$$

is a zero mean Gaussian random variable with variance $\frac{1}{g_{11}}$ and is independent of $Z_{S-\{1\}}$ and Y_S. This completes the proof of Lemma 8. ∎

The followings are two variants of the entropy power inequality.

Lemma 9: Let $\boldsymbol{U}_i, i = 1,2,3$ be n dimensional random vectors with densities and let T be a random variable taking values in a finite set. We assume that $\boldsymbol{U}_3$ is independent of $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, and T. Then, we have

$$\tfrac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\boldsymbol{U}_2+\boldsymbol{U}_3|\boldsymbol{U}_1T)} \geq \tfrac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\boldsymbol{U}_2|\boldsymbol{U}_1T)} + \tfrac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\boldsymbol{U}_3)}\,.$$

Lemma 10: Let $\boldsymbol{U}_i$, $i = 1,2,3$ be n random vectors with densities. Let T_1, T_2 be random variables taking values in finite sets. We assume that those five random variables form a Markov chain $(T_1, \boldsymbol{U}_1) \to \boldsymbol{U}_3 \to (T_2, \boldsymbol{U}_2)$ in this order. Then, we have

$$\begin{aligned}&\tfrac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\boldsymbol{U}_1+\boldsymbol{U}_2|\boldsymbol{U}_3T_1T_2)}\\ &\geq \tfrac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\boldsymbol{U}_1|\boldsymbol{U}_3T_1)} + \tfrac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\boldsymbol{U}_2|\boldsymbol{U}_3T_2)}\,.\end{aligned}$$

Proof of Lemma 6: By Lemma 8, we have

$$\boldsymbol{Z}_i = -\frac{1}{g_{ii}}\sum_{j\neq i}\nu_{ij}\boldsymbol{Z}_j + \frac{1}{g_{ii}}\sum_{j=1}^{s}\frac{q_{ji}}{\sigma_{N_j}^2}\boldsymbol{Y}_j + \hat{\boldsymbol{N}}_i\,, \tag{100}$$

where $\hat{\boldsymbol{N}}_i$ is a vector of n independent copies of zero mean Gaussian random variables with variance $\frac{1}{g_{ii}}$. For each $i \in S$, $\hat{\boldsymbol{N}}_i$ is independent of $\boldsymbol{Z}_j, j \in S - \{i\}$ and $\boldsymbol{Y}_j, j \in S$. Set

$$h^{(n)} \triangleq \frac{1}{n}h(\boldsymbol{Z}_i|\boldsymbol{Z}_{S-\{i\}}, W_S)\,.$$

Furthermore, for $k \in \Lambda$, define

$$S_k \triangleq \{k, k+1, \cdots, s\}\,, \Psi_k = \Psi_k(\boldsymbol{Y}_{S_k}) \triangleq \sum_{j=k}^{s}\frac{q_{ji}}{\sigma_{N_j}^2}\boldsymbol{Y}_j\,.$$

Applying Lemma 9 to (100), we have

$$\frac{\mathrm{e}^{2h^{(n)}}}{2\pi\mathrm{e}} \geq \frac{1}{(g_{ii})^2}\frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{Z}_{S-\{i\}},W_S)} + \frac{1}{g_{ii}}\,. \tag{101}$$

On the quantity $h(\Psi_1|\boldsymbol{z}_{S-\{i\}}, W_S)$ in the right member of (101), we have the following chain of equalities:

$$\begin{aligned}&h(\Psi_1|\boldsymbol{Z}_{S-\{i\}}, W_S)\\ &= I(\Psi_1;\boldsymbol{X}_S|\boldsymbol{Z}_{S-\{i\}}, W_S) + h(\Psi_1|\boldsymbol{X}_S, \boldsymbol{Z}_{S-\{i\}}, W_S)\\ &\overset{\text{(a)}}{=} I(\Psi_1;\boldsymbol{Z}_S|\boldsymbol{Z}_{S-\{i\}}, W_S) + h(\Psi_1|\boldsymbol{X}_S, W_S)\\ &= I(\Psi_1;\boldsymbol{Z}_i|\boldsymbol{Z}_{S-\{i\}}, W_S) + h(\Psi_1|\boldsymbol{X}_S, W_S)\\ &= h(\boldsymbol{Z}_i|\boldsymbol{Z}_{S-\{i\}}, W_S) - h(\boldsymbol{Z}_i|\Psi_1, \boldsymbol{Z}_{S-\{i\}}, W_S)\\ &\quad + h(\Psi_1|\boldsymbol{X}_S, W_S)\\ &\overset{\text{(b)}}{=} nh^{(n)} - h(\boldsymbol{Z}_i|\Psi_1, \boldsymbol{Z}_{S-\{i\}}) + h(\Psi_1|\boldsymbol{X}_S, W_S)\\ &= nh^{(n)} - \frac{n}{2}\log\left[2\pi\mathrm{e}(g_{ii})^{-1}\right] + h(\Psi_1|\boldsymbol{X}_S, W_S)\,.\end{aligned} \tag{102}$$

Step (a) follows from that $\boldsymbol{Z}_S$ can be obtained from $\boldsymbol{X}_S$ by the invertible matrix Q. Step (b) follows from the Markov chain

$$\boldsymbol{Z}_i \to (\Psi_1, \boldsymbol{Z}_{S-\{i\}}) \to \boldsymbol{Y}_S \to W_S.$$

From (102), we have

$$\frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{Z}_{S-\{i\}},W_S)} = \frac{\mathrm{e}^{2h^{(n)}}}{2\pi\mathrm{e}}g_{ii}\cdot\frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{X}_S,W_S)}. \tag{103}$$

Substituting (103) into (101), we obtain

$$\frac{\mathrm{e}^{2h^{(n)}}}{2\pi\mathrm{e}} \geq \frac{\mathrm{e}^{2h^{(n)}}}{2\pi\mathrm{e}}\frac{1}{g_{ii}}\cdot\frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{X}_S,W_S)} + \frac{1}{g_{ii}}\,. \tag{104}$$

Solving (104) with respect to $\frac{\mathrm{e}^{2h^{(n)}}}{2\pi\mathrm{e}}$, we obtain

$$\frac{\mathrm{e}^{2h^{(n)}}}{2\pi\mathrm{e}} \geq \left[g_{ii} - \frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{X}_S,W_S)}\right]^{-1}\,. \tag{105}$$

Next, we evaluate a lower bound of $\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{X}_S,W_S)}$. Note that for $j = 1, 2, \cdots, s-1$ we have the following Markov chain:

$$\left(W_{S_{j+1}}, \Psi_{j+1}(\boldsymbol{Y}_{S_{j+1}})\right) \to \boldsymbol{X}_S \to \left(W_j, \tfrac{q_{ji}}{\sigma_{N_j}^2}\boldsymbol{Y}_j\right)\,. \tag{106}$$

Based on (106), we apply Lemma 10 to $\frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_j|\boldsymbol{X}_S,W_S)}$ for $j = 1, 2, \cdots, s-1$. Then, for $j = 1, 2, \cdots, s-1$, we have the following chains of inequalities :

$$\begin{aligned}&\frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_j|\boldsymbol{X}_S,W_S)}\\ &= \frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h\left(\Psi_{j+1}+\frac{q_{ji}}{\sigma_{N_1}^2}\boldsymbol{Y}_j\middle|\boldsymbol{X}_S,W_{S_{j+1}},W_j\right)}\\ &\geq \frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h\left(\Psi_{j+1}|\boldsymbol{X}_S,W_{S_{j+1}}\right)} + \frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h\left(\frac{q_{ji}}{\sigma_{N_j}^2}\boldsymbol{Y}_j\middle|\boldsymbol{X}_S,W_j\right)}\\ &= \frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h\left(\Psi_{j+1}|\boldsymbol{X}_S,W_{S_{j+1}}\right)} + q_{ji}^2\frac{\mathrm{e}^{-2r_j^{(n)}}}{\sigma_{N_j}^2}\,.\end{aligned} \tag{107}$$

Using (107) iteratively for $j = 1, 2, \cdots, s-1$, we have

$$\frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{X}_S,W_S)} \geq \sum_{j=1}^{s}q_{ji}^2\frac{\mathrm{e}^{-2r_j^{(n)}}}{\sigma_{N_j}^2}\,. \tag{108}$$

Combining (94), (105), and (108), we have

$$\begin{aligned}\frac{\mathrm{e}^{2h^{(n)}}}{2\pi\mathrm{e}} &\geq \left\{\left[Q_S^{-1}\Sigma_{X_S}^{-1}Q_S\right]_{ii} + \sum_{j=1}^{s}q_{ji}^2\frac{1-\mathrm{e}^{-2r_j^{(n)}}}{\sigma_{N_j}^2}\right\}^{-1}\\ &= \left[Q_S^{-1}(\Sigma_{X_S}^{-1} + \Sigma_{N_S(r_S^{(n)})}^{-1})Q_S\right]_{ii}^{-1}\,,\end{aligned} \tag{109}$$

completing the proof. ∎

C. Eigen Values of $\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1}$

In this appendix we prove some properties on eigen values of $\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1}$. Using those properties, we prove Lemma 7.

We first consider the case treated in section IV, where $\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1}$ has the identical value b of non diagonal elements. Using (12), we can show that $\alpha_i, i = 1, 2, \cdots, L$ are L solutions to the following eigen value equation:

$$\left(1 - \sum_{i=1}^{L}\frac{b}{u_i + b - \alpha}\right)\prod_{i=1}^{L}(u_i + b - \alpha) = 0\,. \tag{110}$$

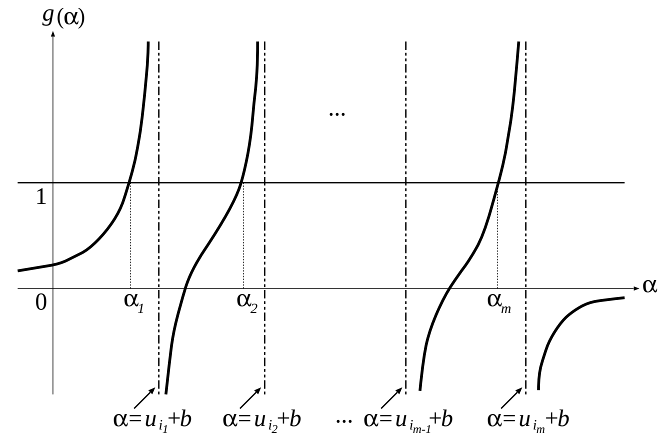

Fig. 1. Shape of $g(\alpha)$.

Let m be the number of distinct values of u_1,u_2,$\cdots$,u_L and let $u_{i_1} < u_{i_2} < \cdots < u_{i_m}$ be the ordered list of those values. For each $j = 1, 2, \cdots, m$, set $\mathcal{L}_j \stackrel{\triangle}{=} \{l : u_l = u_{i_j}\}$ and $l_j \stackrel{\triangle}{=} |\mathcal{L}_j|$. Then, the eigen value equation (110) becomes

$$\left(1 - \sum_{j=1}^{m} \frac{bl_j}{u_{i_j} + b - \alpha}\right) \prod_{j=1}^{m} (u_{i_j} + b - \alpha)^{l_j} = 0\,. \tag{111}$$

From (111), we obtain the following proposition.

Proposition 1: Eigen values of $\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1}$ satisfies the following two properties.

a) The matrix $\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1}$ has m positive eigen values, which are the m distinct solutions of the nonlinear scalar equation

$$1 = g(\alpha) \stackrel{\triangle}{=} \sum_{j=1}^{m} \frac{bl_j}{u_{i_j} + b - \alpha}\,. \tag{112}$$

Let $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ be the ordered list of solutions of (112). Then, we have

$$\begin{aligned} &0 < \alpha_1 < u_{i_1} + b < \alpha_2 < u_{i_2} + b < \cdots \\ &\quad < \alpha_m < u_{i_m} + b\,. \end{aligned} \tag{113}$$

The multiplicity of those eigen values is 1.

b) When $l_j \geq 2$, the matrix $\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1}$ has the eigen value $u_{i_j} + b$ with the multiplicity $l_j - 1$.

Proof: We first prove the part a). From (111), we can see that every solution of the equation $1 = g(\alpha)$ is an eigen value of $\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1}$. Since

$$g'(\alpha) = \sum_{j=1}^{m} \frac{bl_j}{(u_{i_j} + b - \alpha)^2} > 0\,,$$

$g(\alpha)$ is differentiable and monotone increasing in each of the m open intervals $(-\infty, u_{i_1} + b)$, $(u_{i_1} + b, u_{i_2} + b)$, $\cdots$, $(u_{i_{m-1}} + b, u_{i_m} + b)$. Since $g(\alpha)$ is unbounded in each of these intervals, it has positive and negative values there, and thus $1 = g(\alpha)$ has a unique solution in each of these m disjoint intervals. In particular, since

$$\left|\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1}\right| = (1 - g(0)) \prod_{i=1}^{L} (u_i + b) > 0,$$

we have $0 < g(0) < 1$. This implies that $1 = g(\alpha)$ has a unique solution in the interval $(0, u_{i_1} + b)$. Furthermore, since

$$\lim_{\alpha \downarrow u_{i_m} + b} g(\alpha) = -\infty\,, \quad \lim_{\alpha \to +\infty} g(\alpha) = 0\,,$$

there is no eigen value in the open interval $(u_{i_m} + b, +\infty)$. Summarizing the above arguments, we obtain (113). For convenience we show the shape of $g(\alpha)$ in Fig. 1. The part b) is obvious from (111). ■

Next, we consider the case where X^L is a general covariance matrix. Set

$$\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1} = \left[\begin{array}{c|c} u_1 & b_{12} \\ \hline {}^{\mathrm{t}}b_{12} & B_{22} \end{array}\right].$$

Let η_1, η_2, $\cdots$, η_{L-1} be $L-1$ eigen values of B_{22}. Since B_{22} is positive definite, those $L-1$ eigen values are positive. Let p be the number of distinct eigen values of B_{22} and let $\eta_{k_1} < \eta_{k_2} < \cdots < \eta_{k_p}$ be the ordered list of eigen values of B_{22}. For each $j = 1, 2, \cdots, p$, set $\mathcal{T}_j \stackrel{\triangle}{=} \{l : \eta_l = \eta_{k_j}\}$ and $t_j \stackrel{\triangle}{=} |\mathcal{T}_j|$. For each $j = 1, 2, \cdots, p$, the quantity t_j is the multiplicity of the eigen value η_{k_j}. Choose the $(L-1) \times (L-1)$ unitary matrix Q_{22} so that

$${}^{\mathrm{t}}Q_{22} B_{22} Q_{22} = Q_{22}^{-1} B_{22} Q_{22} = \begin{bmatrix} \eta_1 & & & \mathbf{0} \\ & \eta_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \eta_{L-1} \end{bmatrix}$$

and set

$$\tilde{b}_{12} = [\tilde{b}_1 \tilde{b}_2 \cdots \tilde{b}_{L-1}] \stackrel{\triangle}{=} b_{12} Q_{22}\,.$$

Then, we have the following lemma.

Lemma 11:

$$\begin{aligned} &\left|\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1} - \alpha I_L\right| \\ &= (u_1 - \alpha) \prod_{l=1}^{L-1} (\eta_l - \alpha) - \sum_{l=1}^{L-1} \tilde{b}_j^2 \prod_{j \neq l} (\eta_j - \alpha) \\ &= \left(u_1 - \alpha - \sum_{l=1}^{L-1} \frac{\tilde{b}_j^2}{\eta_l - \alpha}\right) \prod_{l=1}^{L-1} (\eta_l - \alpha)\,. \end{aligned}$$

Proof: Set

$$Q \stackrel{\triangle}{=} \left[\begin{array}{c|c} 1 & 0_{12} \\ \hline {}^{\mathrm{t}}0_{12} & Q_{22} \end{array}\right].$$

Then, we have

$$\begin{aligned} &Q^{-1}(\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1} - \alpha I_L) Q \\ &= {}^{\mathrm{t}}Q(\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1} - \alpha I_L) Q \\ &= \left[\begin{array}{c|c} 1 & 0_{12} \\ \hline {}^{\mathrm{t}}0_{12} & {}^{\mathrm{t}}Q_{22} \end{array}\right] \left[\begin{array}{c|c} u_1 - \alpha & b_{12} \\ \hline {}^{\mathrm{t}}b_{12} & B_{22} - \alpha I_{L-1} \end{array}\right] \left[\begin{array}{c|c} 1 & 0_{12} \\ \hline {}^{\mathrm{t}}0_{12} & Q_{22} \end{array}\right] \\ &= \left[\begin{array}{c|c} u_1 - \alpha & b_{12} Q_{22} \\ \hline {}^{\mathrm{t}}Q_{22} {}^{\mathrm{t}}b_{12} & {}^{\mathrm{t}}Q_{22}(B_{22} - \alpha I_{L-1}) Q_{22} \end{array}\right] \\ &= \left[\begin{array}{c|cccc} u_1 - \alpha & & \tilde{b}_{12} & & \\ \hline & \eta_1 - \alpha & & & \mathbf{0} \\ {}^{\mathrm{t}}\tilde{b}_{12} & & \eta_2 - \alpha & & \\ & & & \ddots & \\ & \mathbf{0} & & & \eta_{L-1} - \alpha \end{array}\right]. \end{aligned} \tag{114}$$

By (114), we have

$$
\begin{aligned}
&\left|\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}-\alpha I_L\right| \\
&=\left|Q^{-1}(\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}-\alpha I_L)Q\right| \\
&=\left|\begin{array}{c|cccc} u_1-\alpha & & \tilde{b}_{12} & & \\ \hline & \eta_1-\alpha & & & \mathbf{0} \\ {}^{\mathrm{t}}\tilde{b}_{12} & & \eta_2-\alpha & & \\ & & & \ddots & \\ & \mathbf{0} & & & \eta_{L-1}-\alpha \end{array}\right| \\
&=(u_1-\alpha)\prod_{l=1}^{L-1}(\eta_l-\alpha)-\sum_{l=1}^{L-1}\tilde{b}_j^2\prod_{j\neq l}(\eta_j-\alpha) \\
&=\left(u_1-\alpha-\sum_{l=1}^{L-1}\frac{\tilde{b}_j^2}{\eta_l-\alpha}\right)\prod_{l=1}^{L-1}(\eta_l-\alpha)\,,
\end{aligned}
$$

completing the proof. ■

From Lemma 11, we obtain the following proposition. The first two parts in this proposition are known results (cf. [21]).

Proposition 2: Set $\epsilon_j \stackrel{\triangle}{=} \sum_{l\in\mathcal{T}_j}\tilde{b}_l^2$ and

$$
\begin{aligned}
\mathcal{C}_1 &\stackrel{\triangle}{=} \{j:1\le j\le p,\epsilon_j>0\}\,,\\
\mathcal{C}_2 &\stackrel{\triangle}{=} \{j:1\le j\le p,\epsilon_j=0\}\,.
\end{aligned}
$$

Then, eigen values of $\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}$ satisfies the following three properties.

a) Set $w=|\mathcal{C}_1|$. Let $j_1<j_2<\cdots<j_w$ be the ordered list of $\mathcal{C}_1$. For $i=1,2,\cdots,w$, set $k_{j_i}\stackrel{\triangle}{=}\tilde{k}_i$. Then, the matrix $\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}$ has $(w+1)$ eigen values, which are the $(w+1)$ distinct solutions of the nonlinear scalar equation

$$
u_1=\tilde{g}(\alpha)\stackrel{\triangle}{=}\alpha-\sum_{j\in C_1}\frac{\epsilon_j}{\alpha-\eta_{k_j}}=\alpha-\sum_{i=1}^{w}\frac{\epsilon_{j_i}}{\alpha-\eta_{\tilde{k}_i}}. \quad (115)
$$

Let $\mathcal{E}_0$ be the set of solutions of (115) and let $\alpha_1<\alpha_2<\cdots<\alpha_{w+1}$ be its ordered list. Then, we have

$$
\begin{aligned}
0<\alpha_1<\eta_{\tilde{k}_1}<\alpha_2<\eta_{\tilde{k}_2}<\cdots&\\
<\alpha_w<\eta_{\tilde{k}_w}<\alpha_{w+1}\,,& \quad (116)\\
\alpha_1<u_1<\alpha_{w+1}\,.& \quad (117)
\end{aligned}
$$

b) Set

$$
\mathcal{E}_1\stackrel{\triangle}{=}\{\eta_{k_j}:t_j\ge 2,j\in\mathcal{C}_1\}\,,\mathcal{E}_2\stackrel{\triangle}{=}\{\eta_{k_j}:j\in\mathcal{C}_2\}\,.
$$

By the above definition and (116), we have $\mathcal{E}_0\cap\mathcal{E}_1=\mathcal{E}_1\cap\mathcal{E}_2=\emptyset$. The set of all distinct eigen values of $\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}$ is given by $\mathcal{E}_0\cup\mathcal{E}_1\cup\mathcal{E}_2$. For each $\eta_{k_j}\in\mathcal{E}_1$, the multiplicity of η_{k_j} becomes t_j-1. For each $\eta_{k_j}\in\mathcal{E}_2\cap(\mathcal{E}_0)^{\mathrm{c}}$, the multiplicity of η_{k_j} remains t_j. For each $\eta_{k_j}\in\mathcal{E}_2\cap\mathcal{E}_0$, the multiplicity of η_{k_j} becomes t_j+1. The multiplicity of $\alpha\in\mathcal{E}_0\cap(\mathcal{E}_2)^{\mathrm{c}}$ is 1.

c) Every eigen value of $\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}$ is a monotone increasing function of u_1.

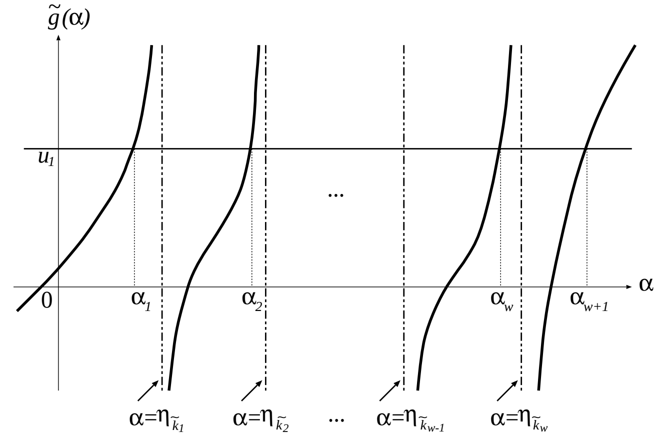

Fig. 2. Shape of $\tilde{g}(\alpha)$.

Proof: By Lemma 11, the eigen value equation of $\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}$ is

$$
\begin{aligned}
&\left(u_1-\alpha-\sum_{l=1}^{L-1}\frac{\tilde{b}_l^2}{\eta_l-\alpha}\right)\prod_{j=1}^{L-1}(\eta_j-\alpha)\\
&=\left(u_1-\alpha-\sum_{j=1}^{p}\frac{\epsilon_j}{\eta_{k_j}-\alpha}\right)\prod_{j=1}^{p}(\eta_{k_j}-\alpha)^{t_j}\\
&=\left(u_1-\alpha-\sum_{j\in\mathcal{C}_1}\frac{\epsilon_j}{\eta_{k_j}-\alpha}\right)\\
&\quad\times\left\{\prod_{j\in\mathcal{C}_1}(\eta_{k_j}-\alpha)^{t_j}\right\}\left\{\prod_{j\in\mathcal{C}_2}(\eta_{k_j}-\alpha)^{t_j}\right\}=0\,. \quad (118)
\end{aligned}
$$

We first prove the part a). From (118), we can see that every solution of the equation $u_1=g(\alpha)$ is an eigen value of $\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}$. Since

$$
\tilde{g}'(\alpha)=1+\sum_{i=1}^{w}\frac{\epsilon_{j_k}}{(\alpha-\eta_{\tilde{k}_i})^2}>0\,,
$$

$\tilde{g}(\alpha)$ is differentiable and monotone increasing in each of the $(w+1)$ open intervals $(-\infty,\eta_{\tilde{k}_1})$, $(\eta_{\tilde{k}_1},\eta_{\tilde{k}_2})$, $\cdots$, $(\eta_{\tilde{k}_w},\infty)$. Since $\tilde{g}(\alpha)$ is unbounded in each of these intervals, it has positive and negative values there, and thus $u_1=\tilde{g}(\alpha)$ has a unique solution in each of these $(w+1)$ disjoint intervals. In particular, since

$$
\left|\Sigma_{X^L}^{-1}+\Sigma_{N^L(u^L)}^{-1}\right|=(u_1-\tilde{g}(0))\prod_{j=1}^{L-1}\eta_j>0,
$$

we have $0<\tilde{g}(0)<u_1$. This implies that $u_1=\tilde{g}(\alpha)$ has a unique solution in the interval $(0,\eta_{\tilde{k}_1})$. Hence, (116) is proved. It remains to prove (117). Since $u_1=\tilde{g}(\alpha_1)=\tilde{g}(\alpha_{w+1})$, we have

$$
\left.\begin{aligned}
u_1-\alpha_1&=\sum_{i=1}^{w}\frac{\epsilon_{j_i}}{\eta_{\tilde{k}_i}-\alpha_1}\stackrel{\text{(a)}}{>}0\,,\\
u_1-\alpha_{w+1}&=\sum_{i=1}^{w}\frac{\epsilon_{j_i}}{\eta_{\tilde{k}_i}-\alpha_{w+1}}\stackrel{\text{(b)}}{<}0\,.
\end{aligned}\right\}
$$

Steps (a) and (b) follow from (116). For convenience, the shape of $\tilde{g}(\alpha)$ is shown in Fig. 2. Thus, the proof of the part a) is completed. The part b) is obvious from (118). Finally, we show the part c). Taking the derivative of (115) with respect to u_1, we obtain

$$1 = \tilde{g}'(\alpha)\frac{\mathrm{d}\alpha}{\mathrm{d}u_1} = \left(1 + \sum_{i=1}^{w} \frac{\epsilon_{j_i}}{(\alpha - \eta_{\tilde{k}_i})^2}\right)\frac{\mathrm{d}\alpha}{\mathrm{d}u_1}\,,$$

from which we obtain

$$\frac{\mathrm{d}\alpha}{\mathrm{d}u_1} = \left(1 + \sum_{i=1}^{w} \frac{\epsilon_{j_i}}{(\alpha - \eta_{\tilde{k}_i})^2}\right)^{-1} > 0\,.$$

Hence, every eigen value belonging to $\mathcal{E}_0$ is monotone increasing function of u_1. If the eigen value does not belong to $\mathcal{E}_0$, it does not depend on u_1. Thus, the part c) is proved. ■

Proof of Lemma 7: It suffices to prove the claim of Lemma 7 for $i = 1$, that is,

$$\alpha_{\max} \geq u_1 \geq \alpha_{\min}\,, \tag{119}$$

$$\frac{\partial \alpha_k}{\partial u_1} \geq 0, \text{ for } k \in \Lambda, \tag{120}$$

$$\sum_{k=1}^{L} \frac{\partial \alpha_k}{\partial u_1} = 1\,. \tag{121}$$

Inequalities (119) and (120) follow from Proposition 2 parts a) and c), respectively. It remains to prove (121). Since for any matrix its trace is equal to the sum of its eigen values, we have

$$\sum_{k=1}^{L} \alpha_k = \sum_{k=1}^{L} u_k\,. \tag{122}$$

Taking partial derivative of both sides of (122) with respect to u_1, we obtain (121). ■

References

[1] D. Slepian and J. K. Wolf, "Noiseless coding of correlated information sources," *IEEE Trans. Inform. Theory*, vol. IT-19, pp. 471-480, July 1973.

[2] A. D. Wyner and J. Ziv, "The rate-distortion function for source coding with side information at the decoder," *IEEE Trans. Inform. Theory*, vol. IT-22, pp. 1-10, Jan. 1976.

[3] A. D. Wyner, "The rate-distortion function for source coding with side information at the decoder-II: General sources," *Inform. Contr.*, vol. 38, pp. 60-80, July 1978.

[4] T. Berger, "Multiterminal source coding," in *the Information Theory Approach to Communications* (CISM Courses and Lectures, no. 229), G. Longo, Ed. Vienna and New York : Springer-Verlag, 1978, pp. 171-231.

[5] S. Y. Tung, "Multiterminal source coding," Ph.D. dissertation, School of Electrical Engineering, Cornell University, Ithaca, NY, May 1978.

[6] T. Berger, K. B. Houswright, J. K. Omura, S. Tung, and J. Wolfowitz, "An upper bound on the rate distortion function for source coding with partial side information at the decoder," *IEEE Trans. Inform. Theory*, vol. IT-25, pp. 664-666, Nov. 1979.

[7] A. H. Kaspi and T. Berger, "Rate-distortion for correlated sources with partially separated encoders," *IEEE Trans. Inform. Theory*, vol. IT-28, pp. 828-840, Nov. 1982.

[8] T. Berger and R. W. Yeung, "Multiterminal source encoding with one distortion criterion," *IEEE Trans. Inform. Theory*, vol. IT-35, pp. 228-236, Mar. 1989.

[9] Y. Oohama, "Gaussian multiterminal source coding," *IEEE Trans. Inform. Theory*, vol. 43, pp. 1912-1923, Nov. 1997.

[10] A. B. Wagner, S. Tavildar, and P. Viswanath "Rate region of the quadratic Gaussian two-encoder source-coding problem," *IEEE Trans. Inform. Theory*, vol. 54, pp. 1938-1961, May 2008.

[11] H. Yamamoto and K. Itoh, "Source coding theory for multiterminal communication systems with a remote source", *Trans. of the IECE of Japan*, vol. E63, no.10, pp. 700-706, Oct. 1980.

[12] T. J. Flynn and R. M. Gray, "Encoding of correlated observations," *IEEE Trans. Inform. Theory*, vol. IT-33, no. 6, pp. 773-787, Nov. 1987.

[13] H. Viswanathan and T. Berger, "The quadratic Gaussian CEO problem," *IEEE Trans. Inform. Theory*, vol. 43, no. 5, pp. 1549-1559, Sept. 1997.

[14] Y. Oohama, "The rate-distortion function for the quadratic Gaussian CEO problem," *IEEE Trans. Inform. Theory*, vol. 44, no. 3, pp. 1057-1070, May 1998.

[15] Y. Oohama, "Rate-distortion theory for Gaussian multiterminal source coding systems with several side Informations at the decoder," *IEEE Trans. Inform. Theory*, vol. 51, no. 7, pp. 2577-2593, July 2005.

[16] A. Pandya, A. Kansal, G. Pottie and M. Srivastava, "Fidelity and resource sensitive data gathering," *Proceedings of the 42nd Allerton Conference*, Allerton, IL, June 2004.

[17] Y. Oohama, "Rate distortion region for separate coding of correlated Gaussian remote observations," *Proceedings of the 43rd Allerton Conference*, Allerton, IL, pp. 2237-2246, Sept. 2005.

[18] Y. Oohama, "Separate source coding of correlated Gaussian remote sources," *Proceedings of Information Theory & Applications Inaugural Workshop*, UCSD, CA, Feb. 6-10, 2006.

[19] Y. Oohama, "Rate distortion region for distributed source coding of correlated Gaussian remote sources," *Proceedings of the IEEE International Symposium on Information Theory*, Toronto, Canada, July 6-11, pp. 41-45, 2008.

[20] Y. Oohama, "Distributed source coding of correlated Gaussian observations," *Proceedings of the 2008 International Symposium on Information Theory and its Applications,*, Auckland, New Zealand, December 7-10, pp. 1441-1446, 2008.

[21] A. Dembo, "Bounds on the extreme eigen values of positive-definite Toepliz matrices," *IEEE Trans. Inform. Theory*, vol. 34, No. 2, pp. 352-355, March 1988.