# On the Labeling Problem of Permutation Group Codes under the Infinity Metric 

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#### Abstract

Codes over permutations under the infinity norm have been recently suggested as a coding scheme for correcting limited-magnitude errors in the rank modulation scheme. Given such a code, we show that a simple relabeling operation, which produces an isomorphic code, may drastically change the minimal distance of the code. Thus, we may choose a code structure for efficient encoding/decoding procedures, and then optimize the code's minimal distance via relabeling.

We formally define the relabeling problem, and show that all codes may be relabeled to get a minimal distance at most 2 . On the other hand, the decision problem of whether a code may be relabeled to distance 2 or more is shown to be NP-complete, and calculating the best achievable minimal distance after relabeling is proved hard to approximate.

Finally, we consider general bounds on the relabeling problem. We specifically show the optimal relabeling distance of cyclic groups. A general probabilistic bound is given, and then used to show both the $\operatorname{AGL}(p)$ group and the dihedral group on $p$ elements, may be relabeled to a minimal distance of $p-O(\sqrt{p \ln p})$.


## I. INTRODUCTION

FLASH memory is a prominent contender to address the increasing demand for dense storage devices. Initially, each flash-memory cell was able to store one bit of information. However, a multi-level technology is now common, in which each cell stores information by choosing one of $q \geqslant 2$ discrete levels. Hence, each cell can store $\log _{2} q$ bits.

Flash memories possess inherent problems one has to address in designing such storage device. The problems range from data reliability to costly write operations. Recently, the rank-modulation scheme was proposed [13] in order to address specifically these inherent problems. In this scheme, the information is stored in the permutation induced by the $n$ distinct charge levels being read from $n$ cells. Each cell has a rank which indicates its relative position when ordering the cells in descending charge level. The ranks of the $n$ cells induce a permutation of $\{1,2, \ldots, n\}$.

While this new scheme alleviates some of the problems associated with current flash technology, the flash-memory channel remains noisy and error correction must be employed to increase reliability. In a recent work [23], spike-error

[^0]correction for rank modulation was addressed. Such errors are characterized by a limited-magnitude change in charge level in the cells, which readily translates into a limited-magnitude change in the rank of, possibly, all cells in the stored permutation. These errors correspond to a bounded distance change in the induced permutation under the $\ell_{\infty}$-metric. We call codes protecting against such errors limited-magnitude rankmodulation codes, or LMRM-codes. Throughout the paper we will consider only LMRM-codes.

A similar error model for flash memory was considered not in the context of rank modulation in [5], while a different error-model (charge-constrained errors for rank modulation) was studied in [1], [14], [19]. Codes over permutations are also referred to as permutation arrays and have been studied in the past under different metrics [2], [3], [6], [7], [9], [11], [24]. Specifically, permutation arrays under the $\ell_{\infty}$-metric were considered in [17]. We also mention a generalization of the rank modulation scheme which uses partial permutations studied in [10], [21].

A code over permutations, being a subset of the symmetric group $S_{n}$, may happen to be a subgroup, in which case we call it a group code. Group theory offers a rich structure to be exploited when constructing and analyzing group codes, in an analogy to the case of linear codes over vector spaces. Hence, throughout this paper, we focus on LMRM group codes.

If $\mathcal{C}$ and $\mathcal{C}^{\prime}$ are conjugate subgroups of the symmetric group, then from a group-theoretic point of view, they are almost the same algebraic object, and they share many properties. However, from a coding point of view these two codes can possess vastly different minimal distance, which is one of the most important properties of a code. For example, consider the following two subgroups of $S_{n}, \mathcal{C}=\{\iota,(1, n)\}$ and $\mathcal{C}^{\prime}=$ $\{\iota,(1,2)\}$, where $\iota$ is the identity permutation and the rest of the permutations are given in a cycle notation. The subgroups $\mathcal{C}$ and $\mathcal{C}^{\prime}$ are conjugate but the minimal distance of $\mathcal{C}$ and $\mathcal{C}^{\prime}$ is $n-1$ and 1 respectively, which are the highest and the lowest possible minimal distances in the $\ell_{\infty}$-metric.

Hence, we conclude that the minimal distance of a code $\mathcal{C}$ depends crucially on the specific conjugate subgroup. Thus, while a certain group code might be chosen due to its grouptheoretic structure (perhaps allowing simple encoding or decoding), we may choose to use an isomorphic conjugate of the group, having the same group-theoretic structure, but with a higher minimal distance. We refer to the problem of finding the optimal minimal distance among all conjugate groups (sets) of a certain group (set) as the labeling problem.

Apart from introducing and motivating the labeling problem, we show that this algorithmic problem is hard. However, we
are able to show the existence of a labeling with high minimal distance for a variety of codes, based on the size of the code and the number of cycles in certain permutations derived from the code itself.

The rest of the paper is organized as follows. In Section II we define the notation, introduce the error model with the associated $\ell_{\infty}$-metric, as well as formally defining the labeling problem. We proceed in Section [III to introduce two algorithmic problems related to the labeling problem, and we show their hardness. In Section IV we give some labeling results on ordinary groups and we present our main result of the paper, which gives general labeling results for arbitrary codes based on a probabilistic argument. In addition with give a few corollaries by applying this result to some well-known groups. We conclude in Section $\square$ with a summary of the results and short concluding remarks.

## II. Definitions and Notations

For any $m, n \in \mathbb{N}, m \leqslant n$, let $[m, n]$ denote the set $\{m, m+1, \ldots, n\}$, where we also denote by $[n]$ the set $[1, n]$. Given any $n \in \mathbb{N}$ we denote by $S_{n}$ the set of all permutations over the set $[n]$.

We will mostly use the cycle notation for permutations $f \in S_{n}$, where $f=\left(f_{0}, f_{1}, \ldots, f_{k-1}\right)$ denotes the permutation mapping $f_{i} \mapsto f_{(i+1) \bmod k}$ for $i \in[0, k-1]$. We shall occasionally use the vector notation whereby a permutation $f=\left[f_{1}, f_{2}, \ldots, f_{n}\right] \in S_{n}$ denotes the mapping $i \mapsto f_{i}$, for all $i \in[n]$. Given two permutations $f, g \in S_{n}$, the product $f g$ is a permutation mapping $i \mapsto f(g(i))$ for all $i \in[n]$.

A code, $\mathcal{C}$ is a subset $\mathcal{C} \subseteq S_{n}$. Note that sometimes $\mathcal{C}$ will also be a subgroup of $S_{n}$, in which case we shall refer to $\mathcal{C}$ as a group code. For a code $\mathcal{C}$ and a permutation $f \in S_{n}$ we call the code $f \mathcal{C} f^{-1}=\left\{f c f^{-1}: c \in \mathcal{C}\right\}$ a conjugate of $\mathcal{C}$.

Consider $n$ flash memory cells which we name $1,2, \ldots, n$. The charge level of each cell is denoted by $c_{i} \in \mathbb{R}$ for all $i \in[n]$. In the rank-modulation scheme defined in [13], the information is stored by the permutation induced by the cells' charge levels in the following way: The induced permutation (in vector notation) is $\left[f_{1}, f_{2}, \ldots, f_{n}\right]$ iff $c_{f_{i}}>c_{f_{i+1}}$ for all $i \in[n-1]$.

Having stored a permutation in $n$ flash cells, a corrupted version of it may be read due to any of a variety of error sources (see [4]). To model a measure of the corruption in the stored permutations one can use any of the well-known metrics over $S_{n}$ (see [8]). Given a metric over $S_{n}$, defined by a distance function $d: S_{n} \times S_{n} \rightarrow \mathbb{N} \cup\{0\}$, an errorcorrecting code is a subset of $S_{n}$ with lower-bounded distance between distinct members.

In [14], the Kendall- $\tau$ metric was used, where the distance between two permutations is the number of adjacent transpositions required to transform one into the other. This metric is used when we can bound the total difference in charge levels.

In this work we consider a different type of error - a limitedmagnitude spike error. Suppose a permutation $f \in S_{n}$ was stored by setting the charge levels of $n$ flash memory cells to $c_{1}, c_{2}, \ldots, c_{n}$. We say a single spike error of limited-magnitude $L$ has occurred in the $i$-th cell if the corrupted charge level, $c_{i}^{\prime}$,
obeys $\left|c_{i}-c_{i}^{\prime}\right| \leqslant L$. In general, we say spike errors of limitedmagnitude $L$ have occurred if the corrupted charge levels of all the cells, $c_{1}^{\prime}, c_{2}^{\prime}, \ldots, c_{n}^{\prime}$, obey

$$
\max _{i \in[n]}\left|c_{i}-c_{i}^{\prime}\right| \leqslant L
$$

Denote by $f^{\prime}$ the permutation induced by the cell charge levels $c_{1}^{\prime}, c_{2}^{\prime}, \ldots, c_{n}^{\prime}$ under the rank-modulation scheme. Under the plausible assumption that distinct charge levels are not arbitrarily close (due to resolution constraints and quantization at the reading mechanism), i.e., $\left|c_{i}-c_{j}\right| \geqslant \ell$ for some positive constant $\ell \in \mathbb{R}$ for all $i \neq j$, a spike error of limited-magnitude $L$ implies a constant $d \in \mathbb{N}$ such that

$$
\max _{i \in[n]}\left|f^{-1}(i)-f^{\prime-1}(i)\right|<d
$$

Loosely speaking, an error of limited magnitude cannot change the rank of the cell $i$ (which is simply $\left.f^{-1}(i)\right)$ by $d$ or more positions.

We therefore find it suitable to use the $\ell_{\infty}$-metric over $S_{n}$ defined by the distance function

$$
d_{\infty}(f, g)=\max _{i \in[n]}|f(i)-g(i)|
$$

for all $f, g \in S_{n}$. Since this will be the distance measure used throughout the paper, we will usually omit the $\infty$ subscript.

Definition 1. A limited-magnitude rank-modulation code (LMRM-code) with parameters $(n, M, d)$, is a subset $\mathcal{C} \subseteq S_{n}$ of cardinality $M$, such that $d_{\infty}(f, g) \geqslant d$ for all $f, g \in \mathcal{C}$, $f \neq g$. (We will sometimes omit the parameter M.)

We note that unlike the charge-constrained rank-modulation codes of [14], in which the codeword is stored in the permutation induced by the charge levels of the cells, here the codeword is stored in the inverse of the permutation.

Permutation codes under the $\ell_{\infty}$-metric have been studied before in [16], [23]. The size of spheres in this metric has been studied in [15], [20], and the size of optimal anticodes in [22].

For a code $\mathcal{C}$ we define its minimal distance and denote it by $d(\mathcal{C})$ as

$$
d(\mathcal{C})=\min _{\substack{f, g \in \mathcal{C} \\ f \neq g}} d(f, g)
$$

A labeling function is a permutation $l \in S_{n}$. A relabeling of a code $C$ by a labeling $l \in S_{n}$ is defined as the set $l \mathrm{Cl}^{-1}$. We say that the code $\mathcal{C}$ has minimal distance $d$ with a labeling function $l$ when

$$
d\left(l C l^{-1}\right)=d
$$

It is well known (see [8]) that the $\ell_{\infty}$-metric over $S_{n}$ is only right invariant and not left invariant, i.e., for any $f, g, h \in S_{n}$, $d(f, g)=d(f h, g h)$, and usually $d(f, g) \neq d(h f, h g)$, thus we would expect that in many cases $d(\mathcal{C}) \neq d\left(l \mathcal{C} l^{-1}\right)$. Therefore, the questions of which labeling permutation leads to the optimal minimal distance, and what is the optimal minimal distance, rise naturally in the context of error-correcting codes over permutations under the infinity metric. Note that $l$ is
called a labeling function because for a permutation in cycle notation $f=\left(a_{1}, \ldots, a_{k_{1}}\right) \ldots\left(a_{k_{j}+1}, \ldots, a_{n}\right)$ we get

$$
l f l^{-1}=\left(l\left(a_{1}\right), \ldots, l\left(a_{k_{1}}\right)\right) \ldots\left(l\left(a_{k_{j}+1}\right), \ldots, l\left(a_{n}\right)\right)
$$

The labeled permutation $l f l^{-1}$ has the same cycle structure as $f$ but the elements within each cycle are relabeled by $l$.

By virtue of the right invariance of the $\ell_{\infty}$-metric, we shall assume throughout the paper that any code $\mathcal{C} \subseteq S_{n}$ contains the identity permutation, since right cosets of $\mathcal{C}$ preserve the distances between codewords, and one of the cosets contains the identity. Furthermore,

$$
d(\mathcal{C})=\min _{g, h \in C, g \neq h} d\left(g h^{-1}, \iota\right)
$$

where $\iota$ is the identity element of $S_{n}$, and where the distance from the identity shall be called the weight of the permutation. This makes it easier to calculate the minimal distance of a group code since $g h^{-1}$ simply goes over all the codewords.

More specifically, we will explore the case where $\mathcal{C}$ is a subgroup of $S_{n}$ and ask which conjugate group of $\mathcal{C}$ has the largest minimal distance. We denote by $\mathcal{L}_{\min }(\mathcal{C})\left(\mathcal{L}_{\max }(\mathcal{C})\right)$ the minimal (maximal) achievable minimal distance among all the conjugates of a code $\mathcal{C}$.

## III. The Labeling Problem is hard to approximate

In this section we define two algorithmic problems regarding the labeling of codes, and show that they are hard to approximate. We shall begin by showing that for any code $\mathcal{C}, \mathcal{L}_{\text {min }}(\mathcal{C}) \leqslant 2$, which means that the minimal distance of a code depends crucially on its labeling. We then continue by showing the decision problem of whether $\mathcal{L}_{\max }(\mathcal{C}) \geqslant 2$ is NPcomplete, while finding out $\mathcal{L}_{\text {max }}(\mathcal{C})$ is hard to approximate.

Recall the conjugacy relation over $S_{n}$ : Two permutations $g, f \in S_{n}$ are said to be conjugate if there exists $h \in S_{n}$ such that $h g h^{-1}=f$. Conjugacy is an equivalence relation, and its equivalence classes are called conjugacy classes. Let $T=$ $\left\{C_{1}, C_{2}, \ldots, C_{k}\right\}$ be the set of conjugacy classes of $S_{n}$. It is known that two permutations have the same cycle structure if and only if they share the same conjugacy class. Denote by $B(\iota, r)$ the ball of radius $r$ centered at the identity,

$$
B(\iota, r)=\left\{f \in S_{n}: d(f, \iota) \leqslant r\right\}
$$

The following lemma will help us show that any code $\mathcal{C}$ has a "bad" labeling, i.e., a labeling with minimal distance 1 or 2.

Lemma 2. For any $n \in \mathbb{N}$ there is a permutation $f$ composed of a single $n$-cycle, i.e., $f=\left(a_{0}, a_{1}, \ldots, a_{n-1}\right) \in S_{n}$, such that $\left|a_{i}-a_{(i+1) \bmod n}\right| \leqslant 2$ for all $i \in[0, n-1]$.

Proof: The proof is by induction. For $n=1,2,3$ all $n$ cycles in $S_{n}$ satisfy the claim. We assume the claim holds for $n$, and prove it also holds for $n+1$. By the induction hypothesis there is $f=\left(a_{0}, a_{1}, \ldots, a_{n-1}\right) \in S_{n}$ that satisfies the claim. W.l.o.g., we can assume that $a_{n-1}=n-1$, $a_{0}=n$, and $a_{1}=n-2$, otherwise $f^{-1}$ would satisfy these conditions. Set $a_{n}=n+1$ and the permutation $f^{\prime}=$ $\left(a_{0}, a_{1}, \ldots, a_{n-1}, a_{n}\right) \in S_{n+1}$ satisfies the claim.

Corollary 3. Let $C$ be any conjugacy class of $S_{n}$, then

$$
B(\iota, 2) \cap C \neq \varnothing
$$

Proof: Every conjugacy class of $S_{n}$ is uniquely defined by the set of its cycles' lengths. Let $\left\{n_{1}, n_{2} \ldots, n_{k}\right\}$ be the cycles' lengths of the permutations in $C$, where $\sum_{i=1}^{k} n_{i}=n$. By Lemma 2 we conclude that there exists some $f \in C_{j}$ such that

$$
f=\left(a_{1}^{1}, a_{2}^{1}, \ldots, a_{n_{1}}^{1}\right)\left(a_{1}^{2}, a_{2}^{2}, \ldots, a_{n_{2}}^{2}\right) \ldots\left(a_{1}^{k}, a_{2}^{k}, \ldots, a_{n_{k}}^{k}\right)
$$

where for each $i$, the set $\left\{a_{j}^{i}\right\}_{j=1}^{n_{i}}=\left[1+\sum_{m=1}^{i-1} n_{m}, \sum_{m=1}^{i} n_{m}\right]$ and the cycle $\left(a_{1}^{i}, a_{2}^{i}, \ldots, a_{n_{i}}^{i}\right)$ satisfies Lemma 2 One can easily check that $d(f, \iota) \leqslant 2$, thus $f \in B(\iota, 2)$.

Now we are ready to prove that any code $\mathcal{C}$ has a "bad" labeling.
Theorem 4. For any code $\mathcal{C} \subseteq S_{n},|\mathcal{C}| \geqslant 2$, there exists a labeling of the elements such that the minimum distance is at most 2, i.e., there exists $l \in S_{n}$ such that $d\left(\mathcal{C l ^ { - 1 } )} \leqslant 2\right.$. Moreover, $\mathcal{C}$ has a labeling with minimal distance 1 if and only if the set $\left\{a b^{-1}: a, b \in \mathcal{C}\right\}$ contains an involution (a permutation of order 2).

Proof: Let $f \in \mathcal{C}, f \neq \iota$, be a permutation whose cycles' lengths are $\left\{n_{1}, n_{2} \ldots, n_{k}\right\}$ and where

$$
f=\left(a_{1}^{1}, a_{2}^{1}, \ldots, a_{n_{1}}^{1}\right)\left(a_{1}^{2}, a_{2}^{2}, \ldots, a_{n_{2}}^{2}\right) \ldots\left(a_{1}^{k}, a_{2}^{k}, \ldots, a_{n_{k}}^{k}\right)
$$

By Corollary 3 there exists $f^{\prime} \in B(\iota, 2)$ with the same cycle structure as $f$. Let $l \in S_{n}$ be the permutation that conjugates $f$ to $f^{\prime}$, i.e., $l f l^{-1}=f^{\prime}$. Therefore,

$$
d\left(l \mathcal{C l} l^{-1}\right) \leqslant d\left(l l l^{-1}, l f l^{-1}\right)=d\left(\iota, f^{\prime}\right) \leqslant 2
$$

We note that the only permutations of weight 1 are involutions in $S_{n}$, and that any involution in $S_{n}$ may be easily relabeled to be of weight 1 . Hence, $\mathcal{C}$ has a labeling with minimal distance 1 if and only if the set $\left\{a b^{-1}: a, b \in \mathcal{C}\right\}$ contains an involution.

After proving that the worst labeling satisfies $\mathcal{L}_{\text {min }}(\mathcal{C}) \leqslant 2$ for all $\mathcal{C} \subseteq S_{n}$, we turn to consider the best labeling. We show that the algorithmic decision problem of determining whether a certain code $\mathcal{C}$ has $\mathcal{L}_{\text {max }}(\mathcal{C})=1$ or $\mathcal{L}_{\max }(\mathcal{C}) \geqslant 2$ is NPcomplete.

## 2-DISTANCE PROBLEM:

- INPUT: A subset of permutations $\mathcal{C} \subseteq S_{n}$ given as a list of permutations, each given in vector notation.
- OUTPUT: The correct Yes or No answer to the question "Does $\mathcal{C}$ have a labeling that leads to a minimal distance at least 2 , i.e., is $\mathcal{L}_{\text {max }}(\mathcal{C}) \geqslant 2$ ? ".
We start with a few definitions. For a code $\mathcal{C} \subseteq S_{n}$, define its associated set of involutions as

$$
I(\mathcal{C})=\left\{g \in S_{n}: g^{2}=\iota, \quad g=a b^{-1} \neq \iota, a, b \in \mathcal{C}\right\}
$$

For any $g \in I(\mathcal{C})$ we define a set of edges, $E(g)$, in the complete graph on $n$ vertices, $K_{n}$, where the vertices are conveniently called $1,2, \ldots, n$, as

$$
E(g)=\left\{u v \in E\left(K_{n}\right): g(u)=v, u \neq v\right\}
$$

Recall that a Hamiltonian path in an undirected graph $G$ is a path which visits each vertex exactly once. The following theorem shows an equivalence between the property of a code having a labeling with minimal distance at least 2 and the existence of a certain Hamiltonian path in the complete graph $K_{n}$.
Theorem 5. Let $\mathcal{C} \subseteq S_{n}$ be a code, then $\mathcal{L}_{\max }(\mathcal{C}) \geqslant 2$ if and only if there exists a Hamiltonian path in $K_{n}$ which does not include all the edges $E(g)$, for any $g \in I(\mathcal{C})$.

Proof: Recall that $d(\mathcal{C})=\min _{f, h \in \mathcal{C}, f \neq h} d\left(f h^{-1}, \iota\right)$ and note that any permutation which contains a cycle of length 3 or more is at distance at least 2 from the identity. Hence, we only have to make sure the set of involutions, $I(\mathcal{C})$, has distance at least 2 from the identity.

If such a Hamiltonian path, $a_{1}, a_{2}, \ldots, a_{n}$, exists in $K_{n}$, then use this path as the labeling permutation and label the element $a_{i}$ as $i$, i.e., the labeling permutation $l \in S_{n}$ satisfies $l\left(a_{i}\right)=i$ for all $i \in[n]$. For any $g \in I(\mathcal{C})$ we know that there exists some $u v \in E(g)$ which does not belong to the Hamiltonian path in $K_{n}$, and therefore $|l(u)-l(v)| \geqslant 2$. From the definition of $E(g)$ we get that $g(u)=v$, and so $d\left(\lg l^{-1}, \iota\right) \geqslant 2$.

For the other direction, let $l \in S_{n}$ be a labeling such that $d\left(l \mathrm{Cl}^{-1}\right) \geqslant 2$. We now consider the Hamiltonian path $l^{-1}(1), l^{-1}(2), \ldots, l^{-1}(n)$ in $K_{n}$. By our choice of $l$, for any $g \in I(\mathcal{C})$ there exists $u, v \in[n]$ such that $g(u)=v$ and $|l(u)-l(v)| \geqslant 2$. Hence, the edge $u v$ does not belong to the constructed Hamiltonian path in $K_{n}$.

By the last theorem we conclude that any algorithm that finds a labeling of $\mathcal{C}$ with minimal distance at least 2 , actually finds a Hamiltonian path in $K_{n}$ which does not include all the edges $E(g)$, for any $g \in I(\mathcal{C})$. We are now able to show that the 2-DISTANCE problem is NP-complete.
Theorem 6. The 2-DISTANCE problem is NP-complete.
Proof: First, we show that 2-DISTANCE is in NP. For any given verifier, $l \in S_{n}$, which is a labeling function, we compute the distance between $\iota$ and all the elements of $I(\mathcal{C})$. Note that $|I(\mathcal{C})| \leqslant|\mathcal{C}|^{2}$ and constructing $I(\mathcal{C})$ may be easily done in polynomial time. Thus, the question can be verified in polynomial time.

In order to verify the completeness we shall reduce the HAMILTONIAN-PATH problem (see [12]) to our problem. Let $G(V, E)$ be a graph on $n$ vertices (given as an $n \times n$ adjacency matrix) in which we want to decide whether a Hamiltonian path exists. Define the code

$$
\mathcal{C}=\{(u, v): u v \notin E\} \cup\{\iota\} \subseteq S_{n}
$$

where $(u, v)$ is the permutation that fixes everything in place except commuting the elements $u$ and $v$. Obviously, we can construct $\mathcal{C}$ from $G$ in polynomial time. We then run the 2 DISTANCE algorithm on $\mathcal{C}$ and return its answer.

We observe that

$$
\begin{aligned}
I(\mathcal{C})= & \{(u, v)(k, l):(u, v),(k, l) \in \mathcal{C},\{u, v\} \cap\{k, l\}=\varnothing\} \\
& \cup \mathcal{C} \backslash\{\iota\}
\end{aligned}
$$

If $a_{1}, a_{2}, \ldots, a_{n}$ is a Hamiltonian path in $G$, then it is also a Hamiltonian path in $K_{n}$ not containing all of $E(g)$, for any
$g \in I(\mathcal{C})$. This is true because $E(g)$ only contains edges that are not in $E$.

For the other direction, if there is a Hamiltonian path in $K_{n}$ which does not include all the edges of $E(g)$ for any $g \in I(\mathcal{C})$, then, in particular, this path does not include all of $E(g), g \in \mathcal{C}$, $g \neq \iota$. Since for any such $g=(u, v) \in \mathcal{C}, E(g)=\{u v\}$, and $u v \notin E$, this path is also a Hamiltonian path in $G$.

We now define a harder algorithmic question and deduce by Theorem 6 that this problem is hard to approximate.

## OPTIMAL-DISTANCE PROBLEM:

- INPUT: A subset of permutations $\mathcal{C} \subseteq S_{n}$ given in vector notation.
- OUTPUT: The integer $\mathcal{L}_{\max }(\mathcal{C})$.

For a constant $\epsilon>1$ we say the problem may be $\epsilon$ approximated if there exists an efficient algorithm that for any input $\mathcal{C}$ computes $f(\mathcal{C})$ which satisfies

$$
\frac{1}{\epsilon} \mathcal{L}_{\max }(\mathcal{C}) \leqslant f(\mathcal{C}) \leqslant \epsilon \mathcal{L}_{\max }(\mathcal{C})
$$

Corollary 7. For any constant $1<\epsilon<2$, the OPTIMALDISTANCE problem cannot be $\epsilon$-approximated unless $P=$ NP.

Proof: Assume there exists an efficient algorithm computing $f(\mathcal{C}) \in \mathbb{N}$ which is an $\epsilon$-approximation of $\mathcal{L}_{\max }(\mathcal{C})$. If $\mathcal{L}_{\max }(\mathcal{C})=1$ then $f(\mathcal{C})<2$ and so $f(\mathcal{C}) \leqslant 1$. If, however, $\mathcal{L}_{\max }(\mathcal{C}) \geqslant 2$, then $f(\mathcal{C})>1$. Thus, given such an efficient algorithm exists, we can decide whether $\mathcal{L}_{\max }(\mathcal{C}) \geqslant 2$, i.e., efficiently solve the 2-DISTANCE problem. By Theorem6we know that the 2-DISTANCE problem is NP-complete, and so $P=N P$.

## IV. Constructions and Bounds

In the previous section we have shown that the 2 DISTANCE and OPTIMAL-DISTANCE problems are hard. We are therefore motivated to focus on solving and bounding the latter problem for specific families of codes, and in particular, codes that form a subgroup of the symmetric group $S_{n}$. The rich structure offered by such codes makes them easier to analyze, in much the same way as linear codes in vector space. Furthermore, knowing good labelings for certain groups is of great interest since one can use them as building blocks when constructing larger codes (see for example the direct and semi-direct product constructions in [23]).

## A. Optimal Labeling for Cyclic Groups

The most simple basic groups one can think of are cyclic groups. Recall that for a cyclic group $G$ there is an element $g \in G$ such that $G$ is generated by the powers of $g$, i.e., $G=$ $\left\{g^{k}: k \in \mathbb{N}\right\}$. We also recall that a group $G$ acting on $[n]$ is said to be transitive if for every $a, b \in[n]$ there exists $g \in G$ such that $g(a)=b$. The following theorem gives an exact optimal labeling for transitive cyclic groups over the set $[n]$.
Theorem 8. Let $\mathcal{C} \subseteq S_{n}$ be a transitive cyclic group over the set $[n]$, then the optimal minimal distance for $\mathcal{C}$ is

$$
\mathcal{L}_{\max }(\mathcal{C})=n-\left\lceil\frac{\sqrt{4 n-3}-1}{2}\right\rceil
$$

Proof: Let $f=\left(a_{1}, a_{2}, \ldots, a_{n}\right) \in \mathcal{C}$ be a generato 1 of $\mathcal{C}$, and let $d$ be an achievable minimal distance, i.e., there is a labeling $l$ such that $d\left(l \mathcal{C l}{ }^{-1}\right)=d$. Denote $\mathcal{C}^{\prime}=l \mathcal{C l}{ }^{-1}$, then $f^{\prime}=l f l^{-1}=\left(l\left(a_{1}\right), l\left(a_{2}\right), \ldots, l\left(a_{n}\right)\right)$ is a generator of $\mathcal{C}^{\prime}$. Define

$$
B=\{(x, y) \in[n] \times[n]:|x-y| \geqslant d\} .
$$

From the minimal distance of $\mathcal{C}^{\prime}$ we know that for any $g \in \mathcal{C}^{\prime}$, $g \neq \iota, d(g, \iota) \geqslant d$. Hence, there is at least one pair $(x, y) \in B$ such that $g(x)=y$. On the other hand, $\mathcal{C}$ is cyclic and transitive and so is $\mathcal{C}^{\prime}$, so for any pair $(x, y) \in B$ there is exactly one $g \in \mathcal{C}^{\prime}$ such that $g(x)=y$. It follows that

$$
\left|\mathcal{C}^{\prime} \backslash\{\iota\}\right|=n-1 \leqslant|B|=(n-d)(n-d+1)
$$

Solving the inequality and remembering that $d$ is an integer, we get

$$
d \leqslant n-\left\lceil\frac{\sqrt{4 n-3}-1}{2}\right\rceil
$$

In order to show the upper bound is achievable, conveniently denote $k=\lceil(\sqrt{4 n-3}-1) / 2\rceil$ and define the sets

$$
A_{1}=[1, k], \quad A_{2}=[k+1, n-k], \quad A_{3}=[n-k+1, n] .
$$

We define the following labeling $l \in S_{n}$,

1) First set $l\left(a_{i}\right)=i$ for all $i \in A_{1}$.
2) Then set $l\left(a_{(n+1-i)(2 k-n+i) / 2+1}\right)=i$ for all $i \in A_{3}$.
3) Finally set $l\left(a_{j}\right)=i$ for all $i \in A_{2}$, where $j$ is chosen arbitrarily from the left-over indices.
We will show that for any $s \in[n-1], d\left(f^{s}, \iota\right) \geqslant n-k$. Note that it is enough to show the claim for $s \leqslant\lceil n / 2\rceil$ since if $s>\lceil n / 2\rceil$ then by the right invariant property $d\left(f^{s}, \iota\right)=$ $d\left(\iota, f^{-s}\right)=d\left(\iota, f^{n-s}\right)$.

Let $s \in[\lceil n / 2\rceil]$, and note that

$$
\begin{aligned}
\sum_{i=1}^{k} i & =\frac{1}{2}\left\lceil\frac{\sqrt{4 n-3}-1}{2}\right\rceil\left\lceil\frac{\sqrt{4 n-3}+1}{2}\right\rceil \\
& \geqslant \frac{1}{2} \cdot \frac{\sqrt{4 n-3}-1}{2} \cdot \frac{\sqrt{4 n-3}+1}{2} \\
& =\frac{4 n-4}{8} \\
& =\frac{n-1}{2}
\end{aligned}
$$

However, since $\sum_{i=1}^{k} i$ is an integer we get that

$$
\sum_{i=1}^{k} i \geqslant\left\lceil\frac{n-1}{2}\right\rceil=\left\lfloor\frac{n}{2}\right\rfloor
$$

Thus, let $m \in[k]$ be the smallest integer such that

$$
\sum_{j=0}^{m-1}(k-j)=\frac{m(2 k-m+1)}{2} \geqslant s
$$

Hence

$$
\begin{equation*}
\frac{m(2 k-m+1)}{2}-s+1 \leqslant k-m+1 \tag{1}
\end{equation*}
$$

[^1]From labeling rule 2 we get that

$$
a_{\frac{m(2 k-m+1)}{2}+1}=n-m+1,
$$

and from labeling rule 1

$$
a_{\frac{m(2 k-m+1)}{2}-s+1}=\frac{m(2 k-m+1)}{2}-s+1
$$

and so

$$
\begin{align*}
& d\left(f^{s}, \iota\right)= \max _{i \in[n]}\left|f^{s}(i)-i\right| \\
& \geqslant \left\lvert\, f^{s}\left(\frac{m(2 k-m+1)}{2}-s+1\right)\right. \\
& \left.\quad-\left(\frac{m(2 k-m+1)}{2}-s+1\right) \right\rvert\, \\
& \geqslant\left|f^{s}\left(a_{\frac{m(2 k-m+1)}{2}-s+1}\right)-(k-m+1)\right|  \tag{2}\\
&=\left|a_{\frac{m(2 k-m+1)}{2}+1}-(k-m+1)\right| \\
&=|n-m+1-(k-m+1)| \\
&= n-k
\end{align*}
$$

where (2) follows from (1).
Since the labeling of indices in $A_{2}$ is arbitrary, we actually have $(n-2 k)$ ! different good labelings resulting from the theorem.

Example 9. Applying Theorem 8 for the case $n=10$ we get that $k=3$, and the optimal minimal distance is $\mathcal{L}_{\max }(\mathcal{C})=$ $n-k=10-3=7$. Moreover, such a labeling is $a_{1}=1$, $a_{2}=2, a_{3}=3, a_{4}=10, a_{6}=9, a_{7}=8$, and one of the cycles that generates the cyclic group of minimal distance 7 is

$$
(1,2,3,10,4,9,8,5,6,7)
$$

## B. The Neighboring-Sets Method

In this section we present a general method we call the neighboring-sets method. With this method, lower and upper bounds on $\mathcal{L}_{\max }(\mathcal{C})$ may be obtained provided certain neighboring sets of indices exist. We shall first describe the general method, and then apply it, using further probabilistic arguments, to show strong bounds on $\mathcal{L}_{\max }(\operatorname{AGL}(p))$ where $\operatorname{AGL}(p)$ is the affine general linear group of order $p$, as well as $\mathcal{L}_{\max }\left(D_{n}\right)$, where $D_{n}$ is the dihedral group of order $n$.

We start by recalling the definitions of $D_{n}$ and $\operatorname{AGL}(p)$ and dispensing with small parameters, for which we can give exact bounds.

Definition 10. For $n \in \mathbb{N}$, the dihedral group of order $n$, denoted $D_{n}$ is the group generated by the two permutations

$$
D_{n}=\langle(1,2, \ldots, n),(1, n)(2, n-1) \ldots(\lfloor n / 2\rfloor,\lceil n / 2\rceil)\rangle .
$$

We refer to the labeling of $D_{n}$ described in the definition above as the natural labeling of $D_{n}$.
Definition 11. Let $p \in \mathbb{N}$ be a prime, then $\operatorname{AGL}(p)$ is defined by the subgroup of permutations that acts on the set $[0, p-1]$ and is generated by the permutations $f(x)=x+1$ and $g(x)=$
ax, where all calculations are over $\operatorname{GF}(p)$ and $a$ is a primitive element in $\mathrm{GF}(p)$.

Throughout we shall consider only $\operatorname{AGL}(p)$ for $p \geqslant 3$. Like before, we refer to the natural labeling of $\operatorname{AGL}(p)$ as the labeling derived from the permutations $f$ and $g$ described above. For example, the natural labeling of AGL(5) is the group generated by the permutations (in cycle notation) $f=$ $(0,1,2,3,4)$ and $g=(1,2,4,3)$. The following theorem gives us the minimal distance of the natural labeling of $\operatorname{AGL}(p)$.

Theorem 12. For any prime $p \geqslant 3$, $\operatorname{AGL}(p)$ with the natural labeling has minimal distance $(p-1) / 2$.

Proof: Because $\operatorname{AGL}(p)$ is a group and the metric is right invariant it suffices to check only the distances from the identity permutation. Let $\sigma_{b}$ be the permutation $\sigma_{b}: x \mapsto x+b$ for some $b \in[1, p-1]$. If $b \geqslant(p-1) / 2$ then $\left|\sigma_{b}(0)-0\right| \geqslant$ $(p-1) / 2$. Otherwise, $\left|\sigma_{b}(p-1)-(p-1)\right| \geqslant(p-1) / 2$. Thus, in any case, $d\left(\sigma_{b}, l\right) \geqslant(p-1) / 2$.

Let $\tau \in \operatorname{AGL}(p)$ be an arbitrary permutation of the kind $\tau(x)=a x+b$ where $a \neq 1$. Both of the permutations $\sigma_{(p-1) / 2}$ and $\tau$ represent lines in the affine plane with different slopes, and so there exists $x_{0} \in[0, p-1]$ such that $\tau\left(x_{0}\right)=\sigma_{(p-1) / 2}\left(x_{0}\right)$. Hence, $\left|\tau\left(x_{0}\right)-x_{0}\right| \geqslant(p-1) / 2$ and then $d(\tau, \iota) \geqslant(p-1) / 2$, which concludes the proof.

The next theorem shows that the natural labeling is optimal for any prime $p<8$.

Theorem 13. For any prime $3 \leqslant p<8$,

$$
\mathcal{L}_{\max }(\operatorname{AGL}(p))=\frac{p-1}{2}
$$

Proof: Let $I$ be the set of involutions of $\operatorname{AGL}(p)$. It is easy to verify that any permutation $g \in I$ is of the form $g(x)=$ $-x+b$ for some $b \in \mathrm{GF}(p)$, and so $|I|=p$. We note also that for any $x_{1}, x_{2} \in \mathrm{GF}(p)$ there is exactly one involution $g \in I$ such that $g\left(x_{1}\right)=x_{2}$ (finding $g$ is by solving the equation $\left.x_{2}=-x_{1}+b\right)$.

Assume that we have a labeling of $\operatorname{AGL}(p)$ with minimal distance more than the natural minimal distance. In particular, with this labeling every involution has minimal distance at least $(p+1) / 2$ from the identity permutation. Let

$$
B=\left\{\{x, y\}: x, y \in \operatorname{GF}(p),|x-y| \geqslant \frac{p+1}{2}\right\}
$$

Now, for any $g \in I$ there is at least one unordered pair $\{x, y\} \in B$ such that $g(x)=y$. It follows that

$$
|B|=\frac{p^{2}-1}{8} \geqslant|I|=p
$$

Solving the inequality we get $p \geqslant 4+\sqrt{17}>8$.
We can get a very similar result (which we omit) regarding the distance of the natural labeling of the dihedral group $D_{n}$, showing it to be approximately $n / 2$.

It is tempting to assume that for large $p$ and $n$ we can get labelings for $\operatorname{AGL}(p)$ and $D_{n}$ with normalized distance tending to 1 , by virtue of their size alone: $\left|D_{n}\right|=2 n$ and $|\operatorname{AGL}(p)|=p(p-1)$, both vanishing in comparison to the
size of $S_{n}$ and $S_{p}$, respectively. However, a simple example of a code

$$
\mathcal{C}=\{\iota\} \cup\left\{l(1,2) l^{-1}: l \in S_{n}\right\} \subseteq S_{n}
$$

dispels this thought since $|\mathcal{C}|=n(n-1) / 2+1, d(\mathcal{C})=1$, and for any $l \in S_{n}$ we have $l \mathcal{C l} l^{-1}=\mathcal{C}$, so relabeling does not change the code's distance. Thus, we turn to describe the neighboring-sets method which will attain better results for $\operatorname{AGL}(p)$ and $D_{n}$.

Definition 14. Let $\mathcal{C} \subseteq S_{n}$ be any set of permutations acting on $[n]$. Two disjoint subsets $A, B \subseteq[n]$ are called $\mathcal{C}$-neighboring sets if for any $f \in \mathcal{C}, f \neq \iota$, the following holds

$$
(f(A) \cap B) \cup(f(B) \cap A) \neq \varnothing
$$

We define $O(\mathcal{C})$ to be the smallest integer $O(\mathcal{C})=|A|+|B|$, where $A$ and $B$ are $\mathcal{C}$-neighboring sets. If there are no such sets then we define $O(\mathcal{C})=\infty$.

First we show that if $\mathcal{C}$ is a group then, $O(\mathcal{C})$ is closely related to its optimal minimal distance.
Theorem 15. Let $\mathcal{C} \subseteq S_{n}$ be a group that acts on $[n]$ with $O(\mathcal{C})<\infty$, then

$$
n-O(\mathcal{C})+1 \leqslant \mathcal{L}_{\max }(\mathcal{C})
$$

Moreover, if $\mathcal{L}_{\max }(\mathcal{C}) \geqslant \frac{n}{2}$ then also

$$
\mathcal{L}_{\max }(\mathcal{C}) \leqslant n-\frac{O(\mathcal{C})}{2}
$$

Proof: Since $O(\mathcal{C})<\infty$ there exist $\mathcal{C}$-neighboring sets $A, B \subseteq[n]$ such that $|A|+|B|=O(\mathcal{C})$. Let the labeling function $l \in S_{n}$ be such that $l(A)=[1,|A|]$, and $l(B)=$ $[n-|B|+1, n]$. It is trivial to check that $\mathcal{C} l^{-1}$ has minimal distance $n-O(\mathcal{C})+1 \leqslant d(\mathcal{C})$.

For the other inequality, assume that the labeling $l$ of $\mathcal{C}$ gives the optimal minimal distance, $d\left(l \mathcal{C l}^{-1}\right)=\mathcal{L}_{\max }(\mathcal{C}) \geqslant \frac{n}{2}$. It follows that $n-\mathcal{L}_{\max }(\mathcal{C})<\mathcal{L}_{\max }(\mathcal{C})+1$, so $A=[1, n-$ $\left.\mathcal{L}_{\text {max }}(\mathcal{C})\right]$, and $B=\left[\mathcal{L}_{\text {max }}(\mathcal{C})+1, n\right]$, are two disjoint sets. We will show that $A$ and $B$ are $\mathcal{C}$-neighboring sets.

For any $n-\mathcal{L}_{\max }(\mathcal{C})<i<\mathcal{L}_{\max }(\mathcal{C})+1$, if such $i$ exists at all, and for any $f \in \mathcal{C} l^{-1}, f \neq \iota$, we have $|f(i)-i|<\mathcal{L}_{\max }(\mathcal{C})$. However, $d(f, \iota) \geqslant \mathcal{L}_{\max }(\mathcal{C})$ and so necessarily $(f(A) \cap B) \cup(f(B) \cap A) \neq \varnothing$. Thus, $A$ and $B$ are $\mathcal{C}$-neighboring sets. Hence, $O(\mathcal{C}) \leqslant 2\left(n-\mathcal{L}_{\max }(\mathcal{C})\right)$, and the result follows.

It is pointed out in the definition that some groups $\mathcal{C} \subseteq S_{n}$ might have $O(\mathcal{C})=\infty$, e.g., $O\left(S_{n}\right)=\infty$. The following theorem shows that for any prime $p>5, O(\operatorname{AGL}(p))$ is finite while also showing a lower bound.

Theorem 16. If $p=3,5$, then $O(\operatorname{AGL}(p))=\infty$. For any prime $p \geqslant 7$,

$$
O(\operatorname{AGL}(p)) \geqslant \max \{\sqrt{2(p-1)}, 6\}
$$

For primes $p \geqslant 37$ we also have

$$
O(\operatorname{AGL}(p)) \leqslant p
$$

Proof: We first start with the lower bounds. It is well known that $\operatorname{AGL}(p)$ is 2 -transitive, i.e., for any $(a, b),(c, d) \in[0, p-1]^{2}, a \neq b, c \neq d$, there exists $f \in \operatorname{AGL}(p)$ such that $f((a, b))=(c, d)$. If $O(\operatorname{AGL}(p)) \leqslant 5$ and $A$ and $B$ are $\operatorname{AGL}(p)$-neighboring sets then, w.l.o.g., we can assume that $|A| \leqslant 2$. Hence there exists $f \in \operatorname{AGL}(p)$, $f \neq \iota$, such that $f(A)=A$ which contradicts the fact that $A$ and $B$ are $\operatorname{AGL}(p)$-neighboring sets. As a consequence we also get that $O(\operatorname{AGL}(3))=O(\operatorname{AGL}(5))=\infty$.

The second lower bound is based on a counting argument. $\operatorname{AGL}(p)$ contains a permutation $f$ composed of one cycle of length $p$. For any $i \in[p-1]$ there exists at least one $(k, m) \in(A \times B) \cup(B \times A)$ such that $f^{i}(k)=m$. On the other hand, for any $(k, m) \in(A \times B) \cup(B \times A)$ there exists only one $i \in[p-1]$ such that $f^{i}(k)=m$. Thus,

$$
\begin{equation*}
p-1 \leqslant|(A \times B) \cup(B \times A)|=2|A| \cdot|B| \tag{3}
\end{equation*}
$$

and the result follows because the minimum of $O(\operatorname{AGL}(p))=$ $|A|+|B|$ given by (3) is $\sqrt{2(p-1)}$.

For the upper bound we will show that there are $\operatorname{AGL}(p)$ neighboring sets $A, B \subseteq[0, p-1]$ of sizes $(p-1) / 2$ and $(p+1) / 2$, respectively, and thus $O(\operatorname{AGL}(p)) \leqslant p$. We note that $A$ and $B$ of the appropriate sizes are neighboring sets if and only if $f(A) \neq A$ for all $f \neq \iota$. We shall therefore try to bound the number of such "bad" subsets $A$. Assume $A \subseteq[0, p-1],|A|=\frac{p-1}{2}$, and $f \in \operatorname{AGL}(p), f \neq \iota$. Then $f(A)=A$ iff $A$ is a union of cycles of $f$. We define a polynomial which is related to the cycle-index polynomial of $f$ as

$$
Z_{f}(x)=\prod_{i}\left(1+x^{i}\right)^{a_{i}(f)}
$$

where $a_{i}(f)$ is the number of cycles of $f$ of length $i$. It follows that the number of "bad" sets $A$ for $f$ is the coefficient of $x^{(p-1) / 2}$ in $Z_{f}(x)$. Summing over all permutations $f \in \operatorname{AGL}(p)$ except the identity permutation will upper bound the number of such "bad" sets in AGL $(p)$.

The group $\operatorname{AGL}(p)$ is a disjoint union (except for the identity) of $p$ groups which are: the cyclic group of order $p$ generated by $(0,1, \ldots, p-1)$, and $p-1$ cyclic groups generated by a permutation of the form $\left(a_{0}, a_{1}, \ldots, a_{p-2}\right)\left(a_{p-1}\right)$. Since, in a cyclic group of order $\ell$, for each $i \mid \ell$ there are $\phi(i)$ elements of order $i$, where $\phi$ is Euler's totient function, we can define the polynomial $Z_{\mathrm{AGL}(p)}(x)$ and readily verify that

$$
\begin{aligned}
& \mathrm{Z}_{\mathrm{AGL}(p)}(x) \triangleq \sum_{f \in \operatorname{AGL}(p), f \neq \iota} \mathrm{Z}_{f}(x)= \\
& \quad=(p-1)\left(1+x^{p}\right)+\sum_{\substack{i \mid p-1 \\
i>1}} p \phi(i)(1+x)\left(1+x^{i}\right)^{\frac{p-1}{i}}
\end{aligned}
$$

We shall now upper-bound the coefficient $a_{(p-1) / 2}$ of $x^{(p-1) / 2}$ in $\mathrm{Z}_{\mathrm{AGL}(p)}$,

$$
a_{\frac{p-1}{2}}=\sum_{\substack{2 i \mid p-1 \\ i>1}} p \phi(i)\binom{\frac{p-1}{i}}{\frac{p-1}{2 i}} \leqslant \frac{p^{3}}{\sqrt{\frac{\pi(p-1)}{4}}} \cdot 2^{\frac{p-1}{2}}
$$

where the upper bound is derived by upper bounding $\phi(i) \leqslant p$, upper bounding the central binomial coefficient using [18], and taking at most $p$ summands.

On the other hand, the number of subsets of $[0, p-1]$ of size $(p-1) / 2$ is exactly $\binom{p}{(p-1) / 2}$. One can easily verify that

$$
\binom{p}{(p-1) / 2}>\frac{p^{3}}{\sqrt{\frac{\pi(p-1)}{4}}} \cdot 2^{\frac{p-1}{2}}
$$

for all primes $p \geqslant 37$. Thus, there are sets $A$ such that $f(A) \neq$ $A$, as required.

Example 17. Let $p=7$. By Theorem 16 we have the lower bound $O(\operatorname{AGL}(7)) \geqslant 6$, and indeed the sets $A=\{0,1,2\}$, $B=\{4,5,6\}$ are AGL(7)-neighboring sets. Furthermore, by Theorem 15 we get that $7-O(\mathrm{AGL}(7))+1=2 \leqslant$ $\mathcal{L}_{\max }(\mathrm{AGL}(7))$. However, by Theorem 13 we know that $\mathcal{L}_{\max }(\operatorname{AGL}(7))=3$.

The following theorem is our main result of this section. It gives a generic labeling result for a code $\mathcal{C}$ over the set $[n]$ based solely on the size of the code and the number of cycles in the set of permutations $\left\{g h^{-1}: g, h \in \mathcal{C}\right\}$.

Theorem 18. Let $\mathcal{C} \subseteq S_{n}$ be a code. If there exist $p, t \in \mathbb{R}$, $0<p<\frac{1}{2}$, and $t>0$, such that

$$
\begin{equation*}
e^{-\frac{2 t^{2}}{n}}+e^{-n p^{2} /(1-p)} \sum_{\substack{f=g h^{-1} \\ g, h \in \mathcal{C}, g \neq h}} e^{c(f) p^{2} /(1-p)}<1 \tag{4}
\end{equation*}
$$

where $c(f)$ is the number of cycles in the permutation $f$, then there exists a labeling $l \in S_{n}$ such that

$$
\mathcal{L}_{\max }(\mathcal{C}) \geqslant d\left(l C l^{-1}\right) \geqslant n+1-\lfloor 2 p n+t\rfloor
$$

Proof: We use a probabilistic argument to show such a labeling exists. We partition the set $[n]$ into three disjoint sets, $A, B$, and $C$, according the probabilities $P(i \in A)=p$, $P(i \in B)=p$, and $P(i \in C)=1-2 p$, where elements are placed independently.

Assume first that $f \in S_{n}$ is a single cycle, i.e., $f=$ $\left(a_{0}, a_{1}, \ldots, a_{k-1}\right)$. We define the events

$$
D_{i}(f)=\left\{a_{i} \in A \text { and } a_{i+1} \in B \text { or } a_{i} \in B \text { and } a_{i+1} \in A\right\}
$$

for each $i \in[0, k-1]$, and where the indices are taken modulo $k$. Where it is clear from context, we shall write $D_{i}$ for short. We also define the event $D_{f}$ to be that $A$ and $B$ are $\{f\}$ neighboring sets.

We would like to evaluate the probability that $A$ and $B$ are not $\{f\}$-neighboring sets, i.e., the probability $P\left(\overline{D_{f}}\right)=$ $P\left(\cap_{i=0}^{k-1} \overline{D_{i}}\right)$. It is easy to calculate that

$$
P\left(\overline{D_{i}}\right)=1-2 p^{2}
$$

Furthermore, for all $i \in[0, k-1]$ we denote

$$
p_{i}=P\left(\overline{D_{i}} \mid \overline{D_{0}}, \ldots, \overline{D_{i-1}}\right)
$$

We find the following recursion, for all $i \in[0, k-3]$ :

$$
\begin{aligned}
p_{i+1}= & P\left(\overline{D_{i+1}} \mid \overline{D_{0}}, \ldots, \overline{D_{i}}\right) \\
= & P\left(a_{i+1} \in C \mid \overline{D_{0}}, \ldots, \overline{D_{i}}\right) \\
& \cdot P\left(\overline{D_{i+1}} \mid \overline{D_{0}}, \ldots, \overline{D_{i}}, a_{i+1} \in C\right) \\
& +P\left(a_{i+1} \notin C \mid \overline{D_{0}}, \ldots, \overline{D_{i}}\right) \\
& \cdot P\left(\overline{D_{i+1}} \mid \overline{D_{0}}, \ldots, \overline{D_{i}}, a_{i+1} \notin C\right) \\
= & P\left(a_{i+1} \in C \mid \overline{D_{0}}, \ldots, \overline{D_{i}}\right) \\
& +P\left(a_{i+1} \notin C \mid \overline{D_{0}}, \ldots, \overline{D_{i}}\right) \cdot(1-p) .
\end{aligned}
$$

In addition,

$$
\begin{aligned}
P\left(a_{i+1} \in C \mid \overline{D_{0}}, \ldots, \overline{D_{i}}\right)= & \frac{P\left(a_{i+1} \in C \mid \overline{D_{0}}, \ldots, \overline{D_{i-1}}\right)}{P\left(\overline{D_{i}} \mid \overline{D_{0}}, \ldots, \overline{D_{i-1}}\right)} \\
& \cdot P\left(\overline{D_{i}} \mid \overline{D_{0}}, \ldots, \overline{D_{i-1}}, a_{i+1} \in C\right) \\
= & \frac{1-2 p}{p_{i}} .
\end{aligned}
$$

It follows that for all $i \in[0, k-3]$,

$$
\begin{aligned}
p_{0} & =1-2 p^{2} \\
p_{i+1} & =1-p+p \cdot \frac{1-2 p}{p_{i}}
\end{aligned}
$$

It is easily seen that for all $i \in[0, k-2], p_{i} \geqslant 1-p$, and so for all $i \in[0, k-3]$,

$$
p_{i+1}=1-p+p \cdot \frac{1-2 p}{p_{i}} \leqslant 1-\frac{p^{2}}{1-p}
$$

Furthermore, since $0<p<\frac{1}{2}$,

$$
p_{0}=1-2 p^{2} \leqslant 1-\frac{p^{2}}{1-p}
$$

Combining the above, we get that

$$
\begin{aligned}
P\left(\overline{D_{f}}\right) & =P\left(\cap_{i=0}^{k-1} \overline{D_{i}}\right) \\
& =\prod_{i=0}^{k-1} P\left(\overline{D_{i}} \mid \cap_{j=0}^{i-1} \overline{D_{j}}\right) \\
& \leqslant \prod_{i=0}^{k-2} p_{i} \leqslant\left(1-\frac{p^{2}}{1-p}\right)^{k-1} \\
& \leqslant e^{-(k-1) p^{2} /(1-p)}
\end{aligned}
$$

since $1-x \leqslant e^{-x}$ for all $x \in \mathbb{R}$.
Let $g \in S_{n}$ be a general permutation, with cycles' lengths $l_{1}, l_{2}, \ldots, l_{k}$, and $\sum_{i=1}^{k} l_{i}=n$, then the probability that $A$ and $B$ are not $\{g\}$-neighboring sets is,

$$
P\left(\overline{D_{g}}\right) \leqslant \prod_{i=1}^{k} e^{-\left(l_{i}-1\right) p^{2} /(1-p)}=e^{-(n-k) p^{2} /(1-p)}
$$

Let $S=|A|+|B|=X_{1}+X_{2}+\cdots+X_{n}$, where $X_{i}$ is the indicator random variable for the event $a_{i} \in A \cup B$. By the
union bound

$$
\begin{aligned}
& P\left(\begin{array}{l}
\bigcup_{\substack{f=g h^{-1} \\
g, h \in C, g \neq h}} \overline{D_{f}} \cup\{S \geqslant E(S)+t\} \\
\leqslant P(S \geqslant E(S)+t)+\sum_{\substack{f=g h^{-1} \\
g, h \in \mathcal{C}, g \neq h}} P\left(\overline{D_{f}}\right)
\end{array}\right. \\
& \leqslant e^{-\frac{2 t^{2}}{n}}+e^{-n p^{2} /(1-p)} \sum_{\substack{f=g h^{-1} \\
g, h \in \mathcal{C}, g \neq h}} e^{c(f) p^{2} /(1-p)}
\end{aligned}
$$

$$
<1
$$

where $P(S \geqslant E(S)+t)$ was upper-bounded using Hoeffding's inequality.

Therefore, with positive probability neither of these events occur, i.e., there is a labeling for $\mathcal{C}$ such that for any $h, g \in \mathcal{C}$, $h \neq g, A$ and $B$ are $\left\{g h^{-1}\right\}$-neighboring sets and $S=|A|+$ $|B| \leqslant E(S)+t=2 p n+t$, and the result follows.
Note that when $\mathcal{C}$ forms a subgroup of $S_{n}$ then the summation in equation (4) is done only over the elements of $\mathcal{C} \backslash\{\iota\}$. Theorem 18 easily gives us achievable-labeling results for any subgroup of $S_{n}$ only by knowing the number of cycles in each of its elements.
We say that $a \in[n]$ is a fixed point of a permutation $f \in S_{n}$ if $f(a)=a$. The minimal degree of a subgroup $\mathcal{C} \subseteq S_{n}$ is the minimum number of non-fixed points among the nonidentity permutations in $\mathcal{C}$. The following corollary connects the minimal degree of a group and an achievable distance by applying Theorem 18 .

Corollary 19. Let $\mathcal{C}$ be a subgroup of $S_{n}$ with minimal degree $d$, such that there exist $t>0,0<p<\frac{1}{2}$, satisfying

$$
e^{-\frac{2 t^{2}}{n}}+|\mathcal{C}| e^{-\frac{d p^{2}}{2(1-p)}}<1
$$

then $\mathcal{C}$ has a labeling $l \in S_{n}$ with

$$
d\left(l \mathcal{C} l^{-1}\right) \geqslant n+1-\lfloor 2 p n+t\rfloor .
$$

Proof: If $\mathcal{C}$ has minimal degree $d$, then the number of cycles of any $g \in \mathcal{C}, g \neq l$, is at most $n-\frac{d}{2}$ and the claim follows by Theorem 18

We now proceed to show strong bounds on $\mathcal{L}_{\max }(\operatorname{AGL}(p))$ and $\mathcal{L}_{\max }\left(D_{n}\right)$.

Theorem 20. For $q$, a large enough prime,
$q-O(\sqrt{q \ln q}) \leqslant \mathcal{L}_{\max }(\operatorname{AGL}(q)) \leqslant q-\left\lceil\frac{\sqrt{4 q-3}-1}{2}\right\rceil$.
Proof: For the upper bound we simply note that a transitive cyclic group of order $q$ is a subgroup of $\operatorname{AGL}(q)$, and then use Theorem 8 For the lower bound we recall that AGL $(q)$ is sharply 2-transitive, hence, its minimal degree is $q-1$. By Corollary 19

$$
e^{-\frac{2 t^{2}}{q}}+|\operatorname{AGL}(q)| e^{-\frac{(q-1) p^{2}}{2(1-p)}} \leqslant e^{-\frac{2 t^{2}}{q}}+q^{2} e^{-\frac{(q-1) p^{2}}{2}}
$$

For $t=\sqrt{q \ln (q+1)}$ and $p=\sqrt{\frac{4 \ln (q+1)}{q-1}}$, we get

$$
e^{-\frac{2 t^{2}}{q}}+q^{2} e^{-\frac{(q-1) p^{2}}{2}}=\frac{1}{(q+1)^{2}}+\frac{q^{2}}{(q+1)^{2}}<1
$$

We note that for $q$ large enough, $p<\frac{1}{2}$. It follows that

$$
\begin{aligned}
\mathcal{L}_{\max }(\operatorname{AGL}(q)) & \geqslant q+1-\lfloor 2 q p+t\rfloor \\
& \geqslant q-2 q \sqrt{\frac{4 \ln (q+1)}{q-1}}-\sqrt{q \ln (q+1)} \\
& =q-O(\sqrt{q \ln q}) .
\end{aligned}
$$

Theorem 21. For the dihedral group, $D_{n}, n \geqslant 37$,

$$
n-O(\sqrt{n \ln n}) \leqslant \mathcal{L}_{\max }\left(D_{n}\right) \leqslant n-\left\lceil\frac{\sqrt{4 n-3}-1}{2}\right\rceil
$$

Proof: For the upper bound, again we note that a transitive cyclic group of order $n$ is a subgroup of $D_{n}$ and then use Theorem 8, For the lower bound, we know that $\left|D_{n}\right|=2 n$, and that $D_{n}$ has minimal degree $d \geqslant n-2$ (it is $n-2$ for even $n$, and $n-1$ for odd $n$ ). We use Corollary 19 with

$$
t=\sqrt{\frac{n \ln (2 n+2)}{2}} \quad p=\sqrt{\frac{\ln (2 n+2)}{n / 2-1}}
$$

and get

$$
\begin{aligned}
e^{-\frac{2 t^{2}}{n}}+\left|D_{n}\right| e^{-\frac{d p^{2}}{2(1-p)}} & \leqslant e^{-\frac{2 t^{2}}{n}}+2 n e^{-\frac{(n-2) p^{2}}{2}} \\
& =\frac{1}{2 n+2}+\frac{2 n}{2 n+2}<1
\end{aligned}
$$

It is easy to verify that $p<\frac{1}{2}$ for all $n \geqslant 37$. Thus,

$$
\begin{aligned}
\mathcal{L}_{\max }\left(D_{n}\right) & \geqslant n+1-\lfloor 2 p n+t\rfloor \\
& \geqslant n-2 n \sqrt{\frac{\ln (2 n+2)}{n / 2-1}}-\sqrt{\frac{n \ln (2 n+2)}{2}} \\
& =n-O(\sqrt{n \ln n}) .
\end{aligned}
$$

## V. Summary

In this work we examined the relabeling of permutation codes under the infinity metric. While relabeling preserves the code structure, producing an isomorphic code, it may drastically reduce or increase the relabeled code's minimal distance.

We formally defined the relabeling problem and showed that all codes may be relabeled to get a minimal distance of at most 2 . Deciding whether one can relabel a given code to achieve minimal distance 2 or more was shown to be an NPcomplete problem. In addition, calculating the best minimal distance achievable after relabeling was shown to be hard to approximate.

We then turned to bounding the best achievable minimal distance after relabeling for certain groups, and in particular, cyclic groups, dihedral groups, and affine general linear groups. For cyclic groups, an exact solution and relabeling was
shown. For the other two families of groups, a probabilistic method was used to give a general bound which turned out to provide strong bounds on the relabeling distance.

Finding out how the best achievable minimal distance after relabeling depends on certain group properties, and finding its exact value for other well-known groups, is still an open problem.

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[^1]:    ${ }^{1} \mathrm{~A}$ single-cycle generator must exist since $\mathcal{C}$ is transitive.

