Outage Exponent: A Unified Performance Metric for Parallel Fading Channels

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Abstract—The parallel fading channel, which consists of finite number of subchannels, is very important, because it can be used to formulate many practical communication systems. The outage probability, on the other hand, is widely used to analyze the relationship among the communication efficiency, reliability, SNR, and channel fading. To the best of our knowledge, the previous works only studied the asymptotic outage performance of the parallel fading channel which are only valid for a large number of subchannels or high SNRs. In this paper, a unified performance metric, which we shall refer to as the outage exponent, will be proposed. Our approach is mainly based on the large deviations theory and the Meijer's G-function. It is shown that the proposed outage exponent is not only an accurate estimation of the outage probability for any number of subchannels, any SNR, and any target transmission rate, but also provides an easy way to compute the outage capacity, finite-SNR diversity-multiplexing tradeoff, and SNR gain. The asymptotic performance metrics, such as the delay-limited capacity, ergodic capacity, and diversity-multiplexing tradeoff can be directly obtained by letting the number of subchannels or SNR tends to infinity. Similar to Gallager's error exponent, a reliable function for parallel fading channels, which illustrates a fundamental relationship between the transmission reliability and efficiency, can also be defined from the outage exponent. Therefore, the proposed outage exponent provides a complete and comprehensive performance measure for parallel fading channels.

Index Terms—Parallel fading channel, outage exponent, channel capacity, diversity-multiplexing tradeoff, large deviations theory, Meijer's *G*-function.

I. INTRODUCTION

THE parallel fading channel has a finite number of flat fading subchannels, where the channel gain of each subchannel only depends on its own fading statistics. In [1], this model is also referred to as the block-fading channel. The parallel fading channel is very important because many

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practical communication systems can be formulated into this model. The conventional narrow-band system, such as GSM, can be modeled as a parallel fading channel in the time domain [2]. The wide-band OFDM system, such as WiMAX and 3GPP LTE, is a parallel fading channel in the frequency domain [3]. By applying a singular value decomposition, the MIMO channel can also be formulated into a parallel fading channel in the space domain [4].

For the parallel fading channel, the ergodic capacity cannot be achieved because it only has a finite number of subchannels which yields a non-ergodic fading case [5]. Therefore, the outage probability, defined as the probability that the instantaneous channel capacity is smaller than a target transmission rate, becomes a fundamental performance metric for the non-ergodic parallel fading channel [2]. From the outage probability perspective, many important performance parameters, such as outage capacity, delay-limited capacity (or zero-outage capacity), ergodic capacity, diversity-multiplexing tradeoff, and finite-SNR diversity-multiplexing tradeoff, can be obtained directly. However, it is very difficult to accurately calculate the outage probability for the parallel fading channel except for two trivial cases, i.e., only one subchannel, and infinity number of subchannels. In [2], Ozarow, Shamai, and Wyner gave an integration formula for the outage probability of the parallel fading channel with two subchannels. This result is the first and can be considered as a milestone step. From then on, many works have been published on this topic. The results can be roughly divided into two categories: 1) the outage probability versus the number of subchannels; and 2) the outage performance versus the signal-to-noise ratio (SNR).

When considering the outage probability for any number of subchannels, Kaplan and Shamai provided an upper bound from the Chernoff bounding method in [1]. As the authors noticed, however, the upper bound is not tight. Another problem of this result lies in the fact that the target transmission rate should deviate from the ergodic capacity largely. Inspired by the idea in [6], some works also tried to estimate the error probability for the parallel fading channel from the theory of error exponent. The original result for the ergodic parallel channel can be found in [7]. Divsalar and Biglieri then proposed upper bounds on the error probability of coded systems over AWGN and fading channels in [8]. A similar work can also be found in [9]. Based on the second type of the Duman-Salehi (DS-2) bound in [10], Sason and Shamai proposed improved bounds on the decoding error probability of block codes over fully-interleaved fading channels [11]. They also evaluated the proposed bounds on turbo-like and

LDPC codes in [11]. By applying a similar approach, Wu, Xiang, and Ling proposed a new upper bound for blockfading channels in [12], which is tight for the channel with a large number of subchannels, i.e., the near-ergodic case. An excellent survey of this approach can be found in [13]. Because the theory of error exponent considers both the coding scheme and channel fading at the same time, the proposed bounds are often very complicated and not tight enough. Hence, it is not trivial to provide insights for the parallel fading channel clearly and directly from these results.

Another approach is to study the outage performance versus SNR, in which the ideal coding scheme is assumed to be used. An important result is to evaluate the diversity-multiplexing tradeoff for fading channels, where each point of the tradeoff curve is just the slope of the outage probability for a given multiplexing gain (or normalized target rate) as SNR tends to infinity. This concept was first proposed for MIMO channels [14], and the corresponding results for the parallel fading channel can be found in [4]. Since the diversity-multiplexing tradeoff is valid in the high SNR regime, the corresponding finite-SNR version for MIMO channels is independently proposed in [15] and [16], respectively. They have a similar definition and can converge to the diversity-multiplexing tradeoff when SNR tends to infinity. The finite-SNR diversity-multiplexing tradeoff can be used to estimate the additional SNR required to decrease the outage probability by a specified amount for a given multiplexing gain. This approach does not estimate the coefficient of the exponential function, which means it cannot be used to estimate the SNR gain for different coding schemes when SNR is not high enough.

In this paper, a unified performance metric for parallel fading channels, which we shall refer to as the outage exponent, will be proposed in order to analyze the relationship among the outage performance, the number of subchannels, and SNR at the same time. The proposed outage exponent has many advantages. It only focuses on the fading effect of the channel, and hence it is much tighter and simpler than the error exponent approach. The outage exponent also provides an accurate estimation of the outage probability for any number of subchannels, any SNR, and any target transmission rate. Similar to the error exponent, a reliable function for the parallel fading channel can then be defined to illustrate the fundamental relationship between the communication efficiency and reliability. From this reliable function, we will show that: 1) the outage probability will tend to zero as the number of subchannels tends to infinity, if and only if the average target rate is smaller than the ergodic capacity; and 2) the outage probability will tend to zero as the SNR tends to infinity for any average target rate lower than the capacity of additive white Gaussian noise (AWGN) channels. Furthermore, the outage capacity, finite-SNR diversitymultiplexing tradeoff, and SNR gain can also be obtained from the outage exponent. Then, the asymptotic performance metrics, such as the delay-limited capacity, ergodic capacity, and diversity-multiplexing tradeoff are just the limits of the previous results, and can be directly obtained by letting the number of subchannels or SNR tend to infinity. Therefore, the proposed outage exponent provides a complete performance framework for parallel fading channels. It also provides a powerful tool for analyzing and evaluating the performance of existing and upcoming communication systems.

In order to analyze the outage exponent for the parallel fading channel, we must consider two different cases. First of all, the outage exponent is analyzed for the case where the target rate is smaller than the ergodic capacity. Inspired by the successful application of large deviations theory on analyzing the bit-error probability for avalanche photodiode receivers [17], the latest results of large deviations theory in [18] and [19] are used to calculate tight upper and lower bounds on the outage probability, respectively. For the case where the target rate is close to and greater than the ergodic capacity, the Meijer's G-function and the method of integral around a contour in [20] are used to compute the upper and lower bounds. In order to achieve the above calculated outage exponent, the coding schemes which maximize the minimum product distance are also discussed. The proposed method combines the advantages of the rotated \mathbb{Z}^L -lattices code and permutation code [21], [22].

The rest of the paper is organized as follows. Section II presents the system model and the precise problem formulation. In Section III, the general definition and related properties of the outage exponent are presented. Section IV studies the outage exponent when the target rate is smaller than the ergodic capacity. The results of delay-limited capacity, ergodic capacity, and diversity-multiplexing tradeoff are also presented in this section. In Section V, the outage exponent and the reliable function are studied when the target rate is higher than the ergodic capacity. Section VI will illustrate the differences between the proposed outage exponent and the error exponent. Section VII studies some coding issues in the parallel fading channel. In Section VIII, numerical results are provided to verify the theoretical derivations. Finally, Section IX concludes the paper.

II. SYSTEM MODEL AND PROBLEM FORMULATION

A. Parallel Fading Channel Model

Consider a parallel fading channel with L subchannels, each of which undergoes independent flat Rayleigh fading. In narrowband systems, each subchannel may correspond to the duration of coherence time. In broadband systems, each subchannel corresponds to one coherence bandwidth in the slow fading scenario, or one coherence bandwidth in the duration of coherence time in the block-fading scenario. For convenience, we assume that each subchannel has a unit time duration and a unit bandwidth throughout this paper. In addition, we assume that the perfect channel state information (CSI) is only known at the receiver side.

Let h_l , $l = 1, \ldots, L$ denote the channel gain of the *l*th subchannel. Then, h_l is a random variable with the circularly symmetric Gaussian distribution $\mathcal{CN}(0, 1)$, and h_l is independent with $h_{l'}$ if $l \neq l'$. The row vector \boldsymbol{x}_l , $l = 1, 2, \ldots, L$ denotes the transmission symbols over the *l*th subchannel, while the row vector \boldsymbol{y}_l denotes the corresponding received symbols. The parallel fading channel can then be modeled by

$$Y = HX + W, \tag{1}$$

where

$$oldsymbol{Y} = egin{pmatrix} oldsymbol{y}_1 \ oldsymbol{y}_2 \ dots \ oldsymbol{y}_L \end{pmatrix}, \quad oldsymbol{X} = egin{pmatrix} oldsymbol{x}_1 \ oldsymbol{x}_2 \ dots \ oldsymbol{x}_L \end{pmatrix}, \quad oldsymbol{H} = ext{diag} \left(h_1, h_2, \dots, h_L
ight).$$

W is the white Gaussian noise matrix, where the elements of W are independent with the identical distribution of $\mathcal{CN}(0,1)$. Hence, the mutual information between the transmitter and the receiver, denoted by I(H), is then given by

$$I(\boldsymbol{H}) = \sum_{l=1}^{L} \ln\left(1 + \left|h_{l}\right|^{2} \gamma\right),$$

where γ is the received SNR. Throughout this paper, the natural logarithmic function $\ln(x)$ is used, and the unit of information is "nat".

B. Outage Formulation

The outage probability is an important concept in fading channels, which provides a way to characterize the performance of communication systems in non-ergodic fading scenarios. Clearly, the parallel fading channel is non-ergodic when L is finite. According to [5], the outage probability of the parallel fading channel is defined by

$$p_{\text{out}}(L,\gamma,R) = \Pr\left\{I\left(\boldsymbol{H}\right) < R\right\}$$
$$= \Pr\left\{\sum_{l=1}^{L} \ln\left(1 + |h_{l}|^{2}\gamma\right) < R\right\}$$
$$= \Pr\left\{\frac{1}{L}\sum_{l=1}^{L} \ln\left(1 + |h_{l}|^{2}\gamma\right) < \overline{R}\right\},$$
(2)

where R is the target transmission rate or coding rate, and $\overline{R} = \frac{R}{L}$ is the average rate on each subchannel. For convenience, p_{out} will be used in the following instead of $p_{\text{out}}(L, \gamma, R)$.

This definition characterizes the relationship among the outage probability p_{out} , the transmission rate R, the number of subchannels L, and the SNR γ . The outage probability and the transmission rate represent two key performance metrics for communication systems, i.e., the reliability and the efficiency, respectively. The number of subchannels determines the time interval and the bandwidth used by the transmission signal, i.e., the degree of freedom. At the same time, it also determines how many independent channel gains the transmission signal may undergo, i.e., the diversity order. The SNR represents the effective energy contained in the signal. Therefore, the outage formulation contains the fundamental elements which govern the transmission reliability and efficiency of non-ergodic fading channels.

Unfortunately, it is very difficult to derive the exact formula for the outage probability as defined in Eq. (2). By now, only some approximations have been proposed in previous works. In [1], from the Chernoff's bound, an upper bound of the outage probability is given by

$$p_{\text{out}} < \min_{\lambda \ge 0} \left\{ e^{\lambda \overline{R}} \left[\gamma^{-\frac{\lambda}{2L}} e^{\frac{1}{2\gamma}} W_{-\frac{\lambda}{2L}, \frac{L-\lambda}{2L}} \left(\gamma^{-1} \right) \right]^L \right\}, \quad (3)$$

where $W_{\nu,\mu}(z)$ is the Whittaker's function [20]. As the authors noticed in [1], however, the bound in Eq. (3) is not tight. In [4], a lower bound is given by

$$p_{\text{out}} > \left(\Pr\left\{ \ln\left(1 + |h_l|^2 \gamma\right) < \overline{R} \right\} \right)^L$$

$$= \left(1 - e^{-\frac{e\overline{R} - 1}{\gamma}} \right)^L.$$
(4)

However, Eq. (4) is only an approximation of the outage probability when SNR tends to infinity, which is not tight in realistic SNRs.

III. OUTAGE EXPONENT

As stated before, it is very difficult to compute the exact outage probability for the parallel fading channel directly. Likewise, it is also very tough to accurately analyze the decoding error probability for a given coding scheme. To overcome the difficulties, Gallager proposed a systematical approach to estimate the upper and lower bounds for the decoding error probability, which is often referred to as the *error exponent* [7]. Similarly, this paper tries to propose an *outage exponent* approach to calculate the exponentially tight upper and lower bounds for the outage probability in nonergodic fading channels.

A. General Results

The general result on the outage exponent is given below in Theorem 1. For convenience, we let the symbol " \lesssim " denote the following relationship:

$$f(x) \lesssim g(x) \Leftrightarrow \begin{cases} f(x) \leq g(x);\\ \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1. \end{cases}$$
(5)

Similarly, we also define the symbol " \gtrsim ".

Theorem 1: For a parallel fading channel with L subchannels, the outage exponents are the exponentially tight upper and lower bounds of the outage probability

$$\begin{cases} p_{\text{out}} \lesssim p_{\text{ex}}^{\text{upper}} = \psi e^{-L \left[E_1(R,\gamma) + \frac{E_0(\gamma)}{L} + o(L) \right]}, \\ p_{\text{out}} \gtrsim p_{\text{ex}}^{\text{lower}} = \varphi e^{-L \left[E_1(R,\gamma) + \frac{E_0(\gamma)}{L} + o(L) \right]}; \end{cases}$$
(6)

where φ and ψ are constants or slowly varying functions with $\varphi \leq \psi$. $p_{\text{ex}}^{\text{upper}}$ and $p_{\text{ex}}^{\text{lower}}$ are referred to as the upper and lower outage exponents, respectively. The function $E_1(R, \gamma)$ is given by

$$E_{1}(R,\gamma) = \left(C_{\text{awgn}} - \overline{R}\right) E_{1,1}(\gamma) + E_{1,0}(\gamma), \quad (7)$$

where $E_1(R, \gamma) > 0$ if and only if $\overline{R} < \overline{C}$ with \overline{C} being the ergodic capacity of each subchannel, and $E_{1,1}(\gamma)$ and $E_{1,0}(\gamma)$ satisfy

$$\begin{cases} \lim_{\gamma \to \infty} \frac{C_{\text{awgn}} E_{1,1}(\gamma)}{\ln \gamma} = 1; \\ \lim_{\gamma \to \infty} \frac{E_{1,0}(\gamma)}{\ln \gamma} = 0. \end{cases}$$
(8)

The function $E_0(\gamma)$ satisfies

$$\lim_{\gamma \to \infty} \frac{E_0(\gamma)}{\ln \gamma} = 0.$$
(9)

Finally, the function o(L) tends to zero when $L \to \infty$.

Theorem 1 captures the intrinsic principles of the outage probability from two dimensions, namely, the number of subchannels L, and the SNR γ .

B. Relationship with Other Performance Metrics

The proposed outage exponent integrates many important performance metrics. As such, it gives a complete picture of the comprehensive performance for parallel fading channels. Based on Theorem 1, the relationship between the outage exponent and other performance metrics are discussed in this subsection.

1) Outage Capacity: For a given outage probability ε , the outage capacity is defined as the supremum of the transmission rate that satisfies $p_{\rm out} < \varepsilon$ in [5]. Therefore, the ε -outage capacity, denoted by C_{ε} , is given by

$$C_{\varepsilon} = \sup\left\{\overline{R} : p_{\text{out}} < \varepsilon\right\},\tag{10}$$

where $\sup A$ is the supremum of the set A. To obtain the outage capacity, an accurate estimation of the outage probability is needed. Note that this is given by the outage exponent in Theorem 1.

2) Delay-Limited Capacity (or Zero-Outage Capacity): In [5], the delay-limited capacity, denoted by $C_{\rm dl}$, which is also known as the zero-outage capacity, is the maximum transmission rate as the outage probability tends to zero when $L \rightarrow \infty$. Therefore, $C_{\rm dl}$ is defined as

$$C_{\rm dl} = \sup\left\{\overline{R} : \lim_{L \to \infty} p_{\rm out} = 0\right\},$$

Clearly, $C_{dl} = \lim_{\epsilon \to 0} \lim_{L \to \infty} C_{\epsilon}$. By applying the error exponent in Theorem I, we have

$$\lim_{L \to \infty} \frac{1}{L} \ln p_{\text{out}} = E_1(R, \gamma)$$

Hence, the delay-limited capacity can be obtained as

$$C_{\rm dl} = \sup\left\{\overline{R} : E_1\left(R,\gamma\right) > 0\right\},\tag{11}$$

3) Ergodic Capacity: In [5], the ergodic capacity for fading channels, denoted by \overline{C} , is defined as the statistical average of the channel capacity. According to the central limit theorem, we have

$$\overline{C} = \lim_{L \to \infty} \frac{I(\boldsymbol{H})}{L} = \mathsf{E}\left\{ \ln\left(1 + |h_l|^2 \gamma\right) \right\}.$$

From Theorem 1, if $\overline{R} < \overline{C}$, then $p_{\text{out}} \to 0$ as $L \to \infty$, which implies that $C_{\text{dl}} \ge \overline{C}$. On the other hand, if $E_1(R, \gamma) > 0$, we have $\overline{R} < \overline{C}$, which implies that $C_{\text{dl}} \le \overline{C}$. As a result, the delay limited capacity is equal to the ergodic capacity in parallel fading channels, that is

$$C_{\rm dl} = \overline{C} = \mathsf{E}\left\{\ln\left(1 + |h_l|^2 \gamma\right)\right\}.$$
 (12)

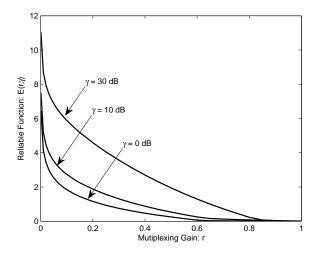


Fig. 1. Reliable function for the parallel fading channel with $\gamma = 0, 10, 30 \text{ dB}$, and L = 4.

4) Reliable Function: In [7], Gallager proposed an error exponent to reveal the fundamental principle that the decoding error probability exponentially varies with the coding length. The outage exponent, however, studies the exponentially decreasing of the outage probability as the number of fading subchannels increases. Therefore, the outage exponent reveals the fundamental principle that the outage probability varies with the ergodicity of the channel, i.e., from slow fading, block fading, to fast fading. Similar to the error exponent, the *reliable function* for the outage exponent can be defined as follows

$$E(R,\gamma) = E_1(R,\gamma) + \frac{E_0(\gamma)}{L} + o(L).$$
(13)

If we let

$$r = \frac{R}{C_{\text{awgn}}},$$

1

where r is referred to as the multiplexing gain, the reliable function can also be denoted by $E(r, \gamma)$. From the properties of $E(R, \gamma)$ in Theorem 1, we have the following results.

- For any fixed SNR, if \overline{R} is smaller than \overline{C} , the outage probability will tend to zero as $L \to \infty$.
- For any fixed L, if R is smaller than C_{awgn}, the outage probability will tend to zero as γ → ∞.

An example of the reliable function versus multiplexing gain is plotted in Fig. 1 for different SNRs. It can be seen that for any given r, the reliable function gets larger as the SNR increases. But all of the curves tend to zero as $r \rightarrow 1$, which is the same as the error exponent figures in [7].

5) Finite-SNR Diversity-Multiplexing Tradeoff: In realistic SNRs, the finite-SNR diversity-multiplexing tradeoff, proposed by [15], [16] for MIMO channels, is used to estimate the additional SNR required to decrease the outage probability by a specified amount for a given multiplexing gain. In [15], the finite-SNR diversity-multiplexing tradeoff is defined as

$$d_{\rm f}^*\left(r\right) = -\frac{\partial \ln p_{\rm out}}{\partial \ln \gamma}$$

which is derived by estimating the upper bound of the outage probability. From Theorem 1, the finite-SNR diversitymultiplexing tradeoff for parallel fading channels can be obtained by

$$d_{\rm f}^*\left(r\right) = L\gamma \frac{\partial E\left(r,\gamma\right)}{\partial\gamma},\tag{14}$$

where $d_{\rm f}^*(r)$ represents the reliable function in Eq. (13) from another perspective.

6) Diversity-Multiplexing Tradeoff: In [14], the diversitymultiplexing tradeoff is defined as the slope of the outage probability when SNR tends to infinity. This concept reveals the fundamental relationship between reliability and efficiency in the asymptotic case. From [4], the optimal diversitymultiplexing tradeoff for parallel fading channels is given by

$$d^{*}(r) = -\lim_{\gamma \to \infty} \frac{\ln p_{\text{out}}}{\ln \gamma} = L\left(1 - \frac{r}{L}\right).$$
(15)

This result was obtained by applying the lower bound in Eq. (4). From Theorem 1, $d^*(r)$ can be derived as follows

$$d^{*}(r) = \lim_{\gamma \to \infty} -\frac{\ln p_{\text{out}}}{\ln \gamma} = \lim_{\gamma \to \infty} -\frac{LE(R,\gamma)}{\ln \gamma}$$
$$= \lim_{\gamma \to \infty} \frac{L\left[E_{1}(R,\gamma) + \frac{E_{0}(\gamma)}{L} + o(L)\right]}{\ln \gamma}$$
$$= \lim_{\gamma \to \infty} \frac{L\left[\left(1 - \frac{r}{L}\right)E_{1,1}(\gamma)\ln(1+\gamma) + E_{1,0}(\gamma)\right]}{\ln \gamma}$$
$$= L\left(1 - \frac{r}{L}\right).$$

By applying the properties of the outage exponent, it is not difficult to verify that

$$d^{*}\left(r\right) = \lim_{\gamma \to \infty} d_{\mathrm{f}}^{*}\left(r\right) = \lim_{\gamma \to \infty} L\gamma \frac{\partial E\left(r,\gamma\right)}{\partial\gamma}.$$

7) SNR Gain (or Coding Gain): The SNR gain is the difference between the SNR values needed by two different coding schemes to achieve a given outage probability. This metric is very useful to evaluate coding schemes with the same diversity gain. In the high SNR regime, the coding gain is determined by the coefficient of the outage exponent.

IV. OUTAGE EXPONENT FOR $\overline{R} < \overline{C}$

The outage exponent for the parallel fading channel will be studied thoroughly in Sections IV and V. According to our study, the channel outage behavior is different for $\overline{R} < \overline{C}$ and $\overline{R} \geq \overline{C}$. Thus, the outage exponent will be analyzed in two different cases. For the $\overline{R} < \overline{C}$ case, our derivation is mainly based on the latest results in large deviations theory and Meijer's *G*-function [20], [23].

Before presenting the main results, we introduce some basic terminologies and notations for this section. Let $\{Y_n : n \in \mathbb{N}\}$ be a sequence of real-valued random variables with distribution $F_n(y)$. The logarithmic moment generating function of Y_n is given by

$$\Lambda_n\left(\xi\right) = \ln M_n\left(\xi\right) = \ln \mathsf{E}\left\{e^{\xi Y_n}\right\}.$$

The Legendre-Fenchel transform is then defined as

$$\Lambda_{n}^{*}(s) = \sup_{\xi \in \mathbb{R}} \left\{ \xi s - \Lambda_{n}(\xi) \right\}.$$

Define the Legendre duality as

$$\Xi(s) = \underset{\xi \in \mathbb{R}}{\operatorname{arg sup}} \left\{ \xi s - \Lambda_n(\xi) \right\}.$$

The limit of the logarithmic moment generating function is given by

$$\Lambda\left(\xi\right) = \lim_{n \to \infty} \frac{1}{n} \Lambda_n\left(\xi\right).$$

Finally, define the titled distribution of Y_n as

$$dF_{n}^{(\xi)}(y) = \frac{e^{\xi y} dF_{n}(y)}{M_{n}(\xi)},$$
(16)

and let $Y_{n}^{\left(\xi\right)}$ be a random variable having $F_{n}^{\left(\xi\right)}\left(y\right)$ as its distribution.

A. Upper Outage Exponent

The upper outage exponent for the parallel fading channel, defined as the exponentially tight upper bound of the outage probability, is given in the following theorem.

Theorem 2: For any R with $\overline{R} < \overline{C}$, the upper outage exponent $p_{\text{ex}}^{\text{upper}}$ for a parallel fading channel with L subchannels is given by

$$p_{\text{out}} \lesssim p_{\text{ex}}^{\text{upper}} = \frac{1}{\sqrt{2\pi}} e^{-L\left[E_1(R,\gamma) + \frac{\ln(\sigma \Xi(0))}{L} + \frac{\ln L}{2L}\right]},$$
 (17)

where

$$E_1(R,\gamma) = \left(C_{\text{awgn}} - \overline{R}\right) \Xi(0) - \Xi(0) \ln\left(1 + \frac{1}{\gamma}\right)$$

-
$$\ln\left(e^{\frac{1}{\gamma}}\Gamma\left(1 - \Xi(0), \gamma^{-1}\right)\right).$$
(18)

The parameter $\Xi(0)$ is the unique positive solution of the following equation

$$\overline{R} - \frac{1}{\Gamma(1-\xi,\gamma^{-1})} G_{2,3}^{3,0} \left(\frac{1}{\gamma} \middle| \begin{array}{c} 1,1\\0,0,1-\xi \end{array}\right) = 0, \quad (19)$$

while σ^2 is given by

$$\sigma^{2} = \frac{2}{\Gamma\left(1 - \Xi\left(0\right), \gamma^{-1}\right)} G_{3,4}^{4,0} \left(\frac{1}{\gamma} \left| \begin{array}{c} 1, 1, 1\\ 0, 0, 0, 1 - \Xi\left(0\right) \end{array} \right) - \left[\frac{1}{\Gamma\left(1 - \Xi\left(0\right), \gamma^{-1}\right)} G_{2,3}^{3,0} \left(\frac{1}{\gamma} \left| \begin{array}{c} 1, 1\\ 0, 0, 1 - \Xi\left(0\right) \end{array} \right) \right]_{(20)}^{2}.$$

In these equations, $\Gamma(z, \alpha)$ is the incomplete Gamma function, which is defined by

$$\Gamma(z,\alpha) = \int_{\alpha}^{\infty} e^{-t} t^{z-1} dt$$

and

$$G_{p,q}^{m,n}\left(z \begin{vmatrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{vmatrix}\right) = \frac{1}{2\pi i} \oint_{\mathcal{L}} \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds,$$

is the Meijer's G-function.

Proof: See Appendix A.

Remark 1: The upper outage exponent proposed in Theorem 2 is also known as the second order saddle-point approximation for any given L and γ . The validity and accuracy of this approximation for finite L and γ is justified in [24].

By solving the equation $\varepsilon = p_{\text{out}} \lesssim p_{\text{ex}}^{\text{upper}}$, the following corollary can be obtained immediately.

Corollary 1: For a parallel fading channel with L subchannels, the ε -outage capacity is given by

$$C_{\varepsilon} \gtrsim \ln \frac{\gamma}{\left[\left(\varepsilon\sqrt{2\pi L}\sigma\Xi\left(0\right)\right)^{\frac{1}{L}}e^{\frac{1}{\gamma}}\Gamma\left(1-\Xi\left(0\right),\gamma^{-1}\right)\right]^{\frac{1}{\Xi\left(0\right)}}}.$$
(21)

According to Theorem 2, other two corollaries which illustrate the asymptotic performance of the parallel fading channel from two different dimensions can also be obtained directly.

Corollary 2: For a parallel fading channel with L subchannels, the delay-limited capacity is equal to the ergodic capacity when $L \rightarrow \infty$, i.e.,

$$C_{\rm dl} = \overline{C} = \mathsf{E}\left\{\ln\left(1 + |h_l|^2 \gamma\right)\right\}.$$
 (22)

Proof: From Theorem 2, if $\overline{R} < \overline{C}$, then $p_{\text{out}} \to 0$ as $L \to \infty$. Therefore, we have $C_{\text{dl}} \ge \overline{C}$.

On the other hand, we have $E_1(R, \gamma) = \Lambda^*(0)$. According to the definition of Legendre-Fenchel transform, $\Lambda^*(s)$ is nondeceasing for $s > \mathsf{E} \{Y_L\} = \overline{R} - \overline{C}$. Therefore, if $E_1(R, \gamma) >$ 0, we have $\overline{R} < \overline{C}$, i.e., $C_{dl} \le \overline{C}$. As a result, Eq. (22) holds.

Corollary 3: The diversity-multiplexing tradeoff for a parallel fading channel with L subchannels is given by

$$d^*\left(r\right) = L\left(1 - \frac{r}{L}\right).$$
(23)

Proof: According to the definition of diversitymultiplexing tradeoff and Theorem 2, we have

$$d^{*}(r) = \lim_{\gamma \to \infty} \frac{LE(R,\gamma)}{\ln \gamma}$$

From Eq. (13), it is not difficult to verify the following equations:

$$\lim_{\gamma \to \infty} \frac{C_{\text{awgn}} \Xi(0)}{\ln \gamma} = \lim_{\gamma \to \infty} \frac{\ln (1+\gamma) \Xi(0)}{\ln \gamma} = 1,$$
$$\lim_{\gamma \to \infty} \frac{\Xi(0) \ln \left(1 + \frac{1}{\gamma}\right)}{\ln \gamma} = 0,$$
$$\lim_{\gamma \to \infty} \frac{\ln \left(e^{\frac{1}{\gamma}} \Gamma \left(1 - \Xi(0), \gamma^{-1}\right)\right)}{\ln \gamma} = 0,$$
$$\lim_{\gamma \to \infty} \frac{\frac{\ln(\sigma \Xi(0))}{L} + \frac{\ln L}{2L}}{\ln \gamma} = 0.$$

Therefore, Eq. (15) holds.

B. Lower Outage Exponent

As stated before, the outage exponent also depends on the exponentially tight lower bound. According to the proof in Appendix A, we have

$$\varphi \lesssim \frac{p_{\text{out}}}{e^{-L\left[E_1(R,\gamma) + \frac{\ln(\sigma \Xi(0))}{L} + \frac{\ln L}{2L}\right]}} \lesssim \frac{1}{\sqrt{2\pi}}$$

Thus, we only need to give a good estimation on φ for the lower outage exponent. The results are summarized in the following theorem.

Theorem 3: For any R with $\overline{R} < \overline{C}$, the lower outage exponent $p_{\text{ex}}^{\text{lower}}$ for a parallel fading channel with L subchannels is given by

$$p_{\text{out}} \gtrsim p_{\text{ex}}^{\text{lower}} = \sup_{0 < \alpha < 1} \left\{ \left(1 - e^{-\Lambda_{\alpha}^{*}(\delta R)} - e^{-\Lambda_{\alpha}^{*}(R)} \right) e^{-LE_{1}^{\alpha}(R,\gamma)} \right\},$$
(24)

where

$$\delta = \frac{\alpha - e^{-\Lambda_{\alpha}^*(R)}}{1 - e^{-\Lambda_{\alpha}^*(R)}},\tag{25}$$

and

ł

$$\begin{cases} E_{1}^{\alpha}\left(R,\gamma\right) = \left(C_{\text{awgn}} - \delta\overline{R}\right) \Xi\left(\alpha R\right) - \Xi\left(\alpha R\right) \ln\left(1 + \frac{1}{\gamma}\right) \\ + \ln\left(e^{\frac{1}{\gamma}}\Gamma\left(1 + \Xi\left(\alpha R\right),\gamma^{-1}\right)\right); \\ \Lambda_{\alpha}^{*}\left(\delta R\right) = R\left(\alpha - \delta\right) \Xi\left(\alpha R\right) - \int_{\delta R}^{\alpha R} \Xi\left(t\right) dt; \\ \Lambda_{\alpha}^{*}\left(R\right) = \Lambda_{\alpha}^{*}\left(\delta R\right)|_{\delta=1}. \end{cases}$$

The parameter $\Xi(t)$ is the negative solution of the following equation

$$t - \frac{L}{\Gamma(1+\xi,\gamma^{-1})} G_{2,3}^{3,0}\left(\frac{1}{\gamma} \left| \begin{array}{c} 1,1\\0,0,1+\xi \end{array} \right) = 0.$$

Proof: See Appendix B.

From the proof of Theorem 3, it can be seen that $\Xi(R)$ here is equal to $-\Xi(0)$ in Theorem 2, which implies that $E_1^{\alpha}(R,\gamma)$ is the same as $E_1(R,\gamma)$. Therefore, the lower bound and the upper bound of the outage probability have the same exponent. The only difference lies in the coefficient of the main exponential function. A key step in the proof, i.e., Eq. (42), can also be replaced by applying the same technique in the proof of Theorem 2, which will result in a tighter bound.

V. OUTAGE EXPONENT FOR $\overline{R} \geq \overline{C}$

It is well known that if the coding rate is larger than the ergodic capacity, the decoding error probability of communication over ergodic channels is still greater than a constant when the coding length tends to infinity. For the parallel fading channel, the behavior of the outage probability when the target transmission rate is higher than the ergodic capacity is still unknown.

A. Upper and Lower Outage Exponents

If the large deviation theory is applied for $\overline{R} > \overline{C}$, the bounds will be obtained in the form of non outage probability, i.e., the dualities of Eqns. (17) and (24). These results will show that the outage probability tends to one when $L \to \infty$ in the R > C case. Furthermore, the large deviation bounds will be loose when \overline{R} is close to \overline{C} . Especially, the large deviation bound will be one, if $\overline{R} = \overline{C}$. Therefore, we need to find another method to study the behavior of the outage probability when SNR changes. The results are summarized in the following theorem.

Theorem 4: For any R with $\overline{R} \geq \overline{C}$, the outage probability of a parallel fading channel with L subchannels can be bounded by

$$\begin{cases} p_{\text{out}} \lesssim p_{\text{ex}}^{\text{upper}} = 1 - F_L\left(\frac{e^R - 1}{\gamma^L}\right), \\ p_{\text{out}} \gtrsim p_{\text{ex}}^{\text{lower}} = 1 - e^{\frac{L}{\gamma}} F_L\left(\frac{e^R}{\gamma^L}\right), \end{cases}$$
(26)

for the high SNR regime. In these equations, $F_L(z)$ is given by

$$F_L(z) = G_{0,L}^{L,0} \left(z \begin{vmatrix} - \\ 0, 1, \dots, 1 \end{vmatrix} \right).$$

In the low SNR regime, the outage probability can be approximated by

$$p_{\rm out} \approx 1 - \frac{\Gamma(L, r)}{(L-1)!},\tag{27}$$

where $r = \frac{R}{C_{\text{awgn}}}$ is the multiplexing gain. *Proof:* See Appendix C.

It can be seen that the low SNR approximation is irrelevant to γ . In the high SNR regime, the difference between the lower bound and the upper bound is only the coefficient of the Meijer's G-function. The coefficient in the lower bound is $e^{\frac{L}{\gamma}}$, while the coefficient in the upper bound is $e^{\frac{0}{\gamma}} = 1$. Since $e^{\frac{L}{\gamma}} \to 1$ as $\gamma \to \infty$, the lower and upper bounds will converge to the true value in the high SNR regime. However, in the low SNR regime, $e^{\frac{0}{\gamma}}$ is too small, while $e^{\frac{L}{\gamma}}$ is too large. Therefore, if we can modify the coefficient to a proper value, a more accurate approximation can then be obtained for the low SNR regime.

Proposition 1: For any R with $\overline{R} \geq \overline{C}$, the outage probability of a parallel fading channel with L subchannels can be approximated by

$$p_{\text{out}} \approx 1 - e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{e^R}{\gamma^L} \middle| \begin{array}{c} -\\ 0, 1, \dots, 1 \end{array} \right), \qquad (28)$$

in the high SNR regime, where $0 < \lambda < L$.

For the parallel fading channel with L subchannels, it is very difficult to estimate λ in theory. However, if we set the simulation value of the outage probability in the low SNR regime equal to 0 dB in (28), an estimation of λ can be obtained.

B. Reliable Function

The reliable function for the $\overline{R} \ge \overline{C}$ case will be analyzed in this subsection. Since the outage probability tends to one when $L \to \infty$ in this case, we will mainly focus on the behavior of the reliable function versus SNR. Before we study the reliable function, a basic lemma will be given first.

Lemma 1: Let f(x) be a strict concave function in $[0,\infty)$. If f(x) is differentiable in $(0, \infty)$, then for any $x \in (0, \infty)$, we have

$$f'(x) < \frac{f(x) - f(0)}{x}.$$
 (29)

Proof: Fix any $x \in (0, \infty)$. According to the Lagrange mean value theorem, there must be some $\xi \in (0, x)$ satisfying

$$f'(\xi) = \frac{f(x) - f(0)}{x}$$

Since f(x) is a strict concave function, then f''(x) < 0. Therefore, we have $f'(x) < f'(\xi)$, so that Eq. (29) holds.

With this lemma, the outage exponent can be constructed from the approximation of the outage probability in Eq. (28).

Theorem 5: For any R with $\overline{R} > \overline{C}$, the lower outage exponent for a parallel fading channel with L subchannels is given by

$$p_{\text{out}} \gtrsim p_{\text{ex}}^{\text{lower}} = e^{-L\left[\left(1-\frac{r}{L}\right)E_{1,1}(\gamma) + E_{1,0}(\gamma) + \frac{E_{0}(\gamma)}{L}\right]}, \quad (30)$$

where the functions $E_{1,1}(\gamma)$, $E_{1,0}(\gamma)$, and $E_0(\gamma)$ are given by

$$E_{1,1}(\gamma) = \frac{\frac{(1+\gamma)^{r-1}}{\gamma^{L-1}} e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{(1+\gamma)^r}{\gamma^L} \middle| \begin{array}{c} - \\ 0, 0, \dots, 0 \end{array} \right)}{1 - e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{(1+\gamma)^r}{\gamma^L} \middle| \begin{array}{c} - \\ 0, 1, \dots, 1 \end{array} \right)}, \quad (31)$$
$$E_{1,0}(\gamma) = \frac{1}{\gamma} E_{1,1}(\gamma), \quad (32)$$

and

$$E_{0}(\gamma) = \frac{-\frac{\lambda}{\gamma}e^{\frac{\lambda}{\gamma}}G_{0,L}^{L,0}\left(\frac{(1+\gamma)^{r}}{\gamma^{L}}\middle| \begin{array}{c} -\\ 0,1,\dots,1 \end{array}\right)}{L\left[1-e^{\frac{\lambda}{\gamma}}G_{0,L}^{L,0}\left(\frac{(1+\gamma)^{r}}{\gamma^{L}}\middle| \begin{array}{c} -\\ 0,1,\dots,1 \end{array}\right)\right]}.$$
 (33)

Proof: See Appendix D.

In Theorem 5, the formula of the outage exponent can only be expressed as a function of the multiplexing gain, because $\overline{R} \geq \overline{C}$ in this case. Consider the definition of the multiplexing gain and the condition $\overline{R} \geq \overline{C}$, the value of r must satisfy

$$\frac{r}{L} \ge \frac{\overline{C}}{C_{\text{awgn}}} = \frac{e^{\frac{1}{\gamma}} \Gamma\left(0, \gamma^{-1}\right)}{\ln\left(1 + \gamma\right)} = r_0\left(\gamma\right).$$

It can be shown that $r_0(\gamma)$ is a quasi-convex function, and

$$\lim_{\gamma \to 0} r_0(\gamma) = \lim_{\gamma \to \infty} r_0(\gamma) = 1.$$

Fig. 2 shows the curve of $r_0(\gamma)$. It can be seen that the minimum value of $r_0^*(\gamma) \approx 0.8331$ is achieved at $\gamma^* \approx 6.2442$ dB. Therefore, for any SNR, if the multiplexing gain is below this curve, the formulas in Theorems 2 and 3 give the tight bounds;

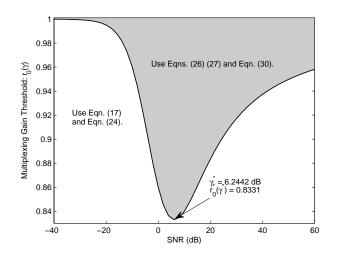


Fig. 2. The curve of $r_0(\gamma)$ versus γ .

whereas if the multiplexing gain is near and above the curve, the formulas in Theorems 4 and 5 give the tight bounds.

From Theorem 5, the finite-SNR diversity-multiplexing tradeoff can be directly obtained as described in the following.

Corollary 4: For a parallel fading channel with *L* subchannels, the finite-SNR diversity-multiplexing tradeoff is given by

$$d_{\rm f}^{*}(r,\gamma) = L \left[\left(1 - \frac{r}{L} \right) + \frac{1}{\gamma} \right] \frac{(1+\gamma)^{r-1}}{\gamma^{L-1}} \cdot \frac{e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{(1+\gamma)^{r}}{\gamma^{L}} \middle| \begin{array}{c} -\\ 0, 0, \dots, 0 \end{array} \right)}{1 - e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{(1+\gamma)^{r}}{\gamma^{L}} \middle| \begin{array}{c} -\\ 0, 1, \dots, 1 \end{array} \right)} \frac{\frac{\lambda}{\gamma} e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{(1+\gamma)^{r}}{\gamma^{L}} \middle| \begin{array}{c} -\\ 0, 1, \dots, 1 \end{array} \right)}{1 - e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{(1+\gamma)^{r}}{\gamma^{L}} \middle| \begin{array}{c} -\\ 0, 1, \dots, 1 \end{array} \right)} \cdot$$
(34)

From the finite-SNR diversity-multiplexing tradeoff, the diversity-multiplexing tradeoff can be obtained by letting $\gamma \rightarrow \infty$. The result is summarized in the following corollary.

Corollary 5: For a parallel fading channel with L subchannels, the diversity-multiplexing tradeoff is given by

$$d^{*}(r) = \lim_{\gamma \to \infty} d_{\mathrm{f}}^{*}(r,\gamma) = L\left(1 - \frac{r}{L}\right).$$
(35)

Proof: See Appendix E.

Remark 2: From the proof of Theorem 5 and Corollary 5, we can find that if r < L, the outage probability will tend to zero as SNR tends to infinity. Therefore, the diversity-multiplexing tradeoff is valid for any $\overline{R} < C_{\text{awgn}}$. As a result, for any fixed L, if only \overline{R} is smaller than C_{awgn} , the outage probability will reduce to zero as $\gamma \to \infty$.

VI. DIFFERENCES BETWEEN THE OUTAGE EXPONENT AND ERROR EXPONENT

In fading channels, the proposed outage exponent relates the communication reliability, efficiency, transmission rate, SNR,

and the number of fading subchannels. In ergodic channels, the relationship among the error probability, the transmission rate, channel noise, and the coding lengths is characterized by a commonly used approach, referred to as the error exponents. Recently, some works tried to generalize the error exponent theory to fading channels. Interested readers are referred to [13] and its references for a thorough survey on this topic. In this section, we will discuss the differences between our results and the theory of error exponents.

The classical results of the error exponent for parallel channels is summarized in the following lemma [7].

Lemma 2: Let $p_l(j|i), l = 1, ..., L$ be the transition probability of the *l*th subchannel, and $q(k_1, ..., k_L)$ be a probability assignment on the input vectors. If $q(k_1, ..., k_L) = \prod_{l=1}^{L} q_l(k_l)$, the error exponent for the parallel fading channel is then given by the exponentially tight upper bound of the error probability:

$$p_{\rm err} < e^{-N \sum_{l=1}^{L} E_r^l(\rho)}$$

where

$$E_{r}^{l}\left(\rho\right) = \max_{\boldsymbol{q}_{l}} \left\{ E_{o}^{l}\left(\rho, \boldsymbol{q}_{l}\right) - \rho \frac{\partial}{\partial \rho} E_{o}^{l}\left(\rho, \boldsymbol{q}_{l}\right) \right\}$$

and

$$R(\rho) = \sum_{l=1}^{L} R_l(\rho) = \sum_{l=1}^{L} \frac{\partial}{\partial \rho} E_o^l(\rho, \boldsymbol{q}_l).$$

The function $E_{\rho}^{l}(\rho, \boldsymbol{q}_{l})$ is given by

$$E_{o}^{l}(\rho, \boldsymbol{q}_{l}) = -\ln \sum_{j=0}^{J-1} \left(\sum_{k=0}^{K-1} q_{l}(k) p_{l}(j|k)^{\frac{1}{1+\rho}} \right)^{1+\rho}.$$

The parameter ρ is the magnitude of the slope of the $E_r^l(\rho)$ versus $R(\rho)$ curve.

From Lemma 2, if $p_l(i|j)$, l = 1, ..., L are the same for any subchannel, the error exponent will be reduced to

$$p_{\rm err} \le e^{-L\left(NE_r^\iota(\rho)\right)},\tag{36}$$

and

$$R(\rho) = LR_l(\rho) = L\frac{\partial}{\partial\rho}E_o^l(\rho, \boldsymbol{q}_l).$$
(37)

Therefore, if and only if $R_l < C_{\text{awgn}}$, p_{err} will tend to zero when $L \to \infty$ for a fixed N. It can be shown that $E_r^l(\rho)$ in Eq. (36) is just the error exponent for the *l*th subchannel.

By comparing Eqns. (36) (37) and Eq. (17), it can be found that the outage exponent and error exponent have a similar form. However, they are different. First of all, the outage exponent considers the non-ergodic parallel fading channel, i.e., each subchannel is associated with a random channel gain; whereas each subchannel in the error exponent is ergodic. The results that tried to generalize the error exponent method to fading channels, however, can only be applied to ergodic fading channels, e.g., the fully interleaved blockfading channel in [9], [11]. For the non-ergodic parallel fading channel, the transition probability of each subchannel requires knowledge of the channel, i.e., we need $p_l(j|i, h_l)$. Then, from

TABLE I PROPERTIES OF THE OUTAGE EXPONENT

	γ : Non-Asymptotic	γ : Asymptotic
L Non-Asymptotic	 Accurate estimation of p_{out}: Eqns. (6) (17) (24) (26). Outage capacity: Eqns. (10) (21). Reliable function: Eqns. (13) (18) (30). Finite-SNR diversity-multiplexing tradeoff: Eqns. (14) (34). SNR gain: Eqns. (6) (17) (24) (26). 	 diversity-multiplexing tradeoff: Eqns. (15) (23) (35). If R̄ < C_{awgn}, lim_{γ→∞} p_{out} = 0: Theorem 2.
L Asymptotic	 Delay-limited capacity: Eqns. (11) (22). Ergodic capacity: Eqns. (12) (22). If R̄ < C̄, lim_{L→∞} p_{out} = 0: Theorem 4. 	—

the perspective of error exponent, the outage probability is given by

$$p_{\text{out}} = \Pr\left\{\sup\left\{R\left(\rho\right) : \sum_{l=1}^{L} E_{r}^{l}\left(\rho, \boldsymbol{h}\right) > 0\right\} < R\right\}.$$
 (38)

Because $E_r^l(\rho, h)$ must be maximized over q_l for every sample of h, it is very difficult to get a closed-form formula for the outage probability. Some attempts can be found in [8], [12]. By comparing Eq. (2) and Eq. (38), it can be found that: 1) the outage exponent only focuses on the effect of channel fading; and 2) all of the details about the channel coding are dropped by assuming that an ideal coding scheme has been adopted. This is reasonable because the AWGN channel capacity can be approximately achieved by LDPC and Turbo codes. Actually, this is just the basic idea in [2].

Another important difference is that the error exponent for the parallel fading channel is L times the error exponents of each subchannel, as shown in Eq. (36). For the outage exponent, however, the reliable function defined in Eq. (13) is not the outage exponent for one subchannel, since the outage probability for each subchannel is given by

$$p_{\text{out}}^l = 1 - e^{-\frac{e^{\overline{R}} - 1}{\gamma}}.$$

Therefore, we can independently study the subchannels of an ergodic parallel fading channel. For the non-ergodic parallel fading channel, we have to view the channel as a whole so as to study its performance. To illustrate the differences more clearly, the obtained results in this paper for the outage exponent are briefly summarized in Table I.

VII. DISCUSSION ON CODING ISSUES FOR THE PARALLEL FADING CHANNEL

According to the above results, if the transmitter has perfect knowledge of CSI of each subchannel, the outage exponent can be achieved easily. As a matter of fact, the outage performance can be better, in the sense of coding gain, when the water-filling power allocation is applied. If the transmitter has no CSI, however, how to achieve the outage performance as good as possible is a key problem in this context. By recalling the definition of the outage probability given in Eq. (2), the optimal outage performance can be achieved if the coding scheme with rate R is capable of achieving the capacity of each subchannel without CSI at the transmitter. This section will discuss some basic issues in designing the best codes for the parallel fading channel.

It is known that the minimum distance between two codewords determines the decode error probability in the AWGN channel [25]. In fading channels, another key parameter is the minimum product distance of two codewords [4]. In brief, the minimum distance determines the capability of codes to combat noise; whereas the minimum product distance determines the capability to combat channel fading. Eq. (2) indicates that the outage probability is determined by channel fading. Therefore, the minimum product distance is the key metric to evaluate a coding scheme for the parallel fading channel.

In signal space, each codeword can be seen as a vector or point in the *L*-dimension space, i.e., *L*-dimension constellation. Moreover, the channel gains on *L* subchannels are orthogonal random variables, because they are independent with zero means. Therefore, the *l*th axis of the *L*-dimension constellation corresponds to the channel gain of the *l*th subchannel (i.e., h_l). Let $\mathbf{X}_A = (\mathbf{x}_{A1}, \mathbf{x}_{A2}, \dots, \mathbf{x}_{AL})^{\mathrm{T}}$ and $\mathbf{X}_B = (\mathbf{x}_{B1}, \mathbf{x}_{B2}, \dots, \mathbf{x}_{BL})^{\mathrm{T}}$ denote the codewords for the information *A* and *B*, respectively. The normalized product distance of \mathbf{X}_A and \mathbf{X}_B is then defined by

$$D_p = \frac{1}{\sqrt{\gamma}} \prod_{l=1}^{L} |\boldsymbol{x}_{Al} - \boldsymbol{x}_{Bl}|.$$
(39)

Therefore, the coding scheme must be designed to maximize the minimum value of D_p , denoted by D_p^{\min} , so as to optimize the outage performance. If $\mathbf{x}_{Al} = \mathbf{x}_{Bl}$ for some l, there must be a hyperplane which is orthogonal with the lth axis such that $\mathbf{x}_{Al} - \mathbf{x}_{Bl}$ is parallel with this hyperplane. Clearly, if all the channel gains on that hyperplane are very small, the receiver cannot distinguish \mathbf{X}_A from \mathbf{X}_B . In other words, we lose one dimension, i.e., the lth subchannel, to combat the channel fading. As a matter of fact, the number of terms that $\mathbf{x}_{Al} \neq \mathbf{x}_{Bl}, l = 1, 2, \dots, L$ in Eq. (39) is the diversity order achieved by this signal constellation [21].

To the best of our knowledge, there are two main approaches to maximize the minimum product distance. The first class is the rotated \mathbb{Z}^L -lattices code, which is based on the algebraic number theory and lattices theory. The basic idea is to construct the *L*-dimension constellation with the size 2^R based on \mathbb{Z}^L -lattices. Then, the constellation will be rotated to an appropriate angle such that D_p^{\min} is maximized. According to the lattices theory, D_p^{\min} can be calculated theoretically. One can refer to [21], [26] and its references for detailed discussions on this coding scheme. Another advantage of the rotated \mathbb{Z}^L -lattices code lies in the fact that it can reduce the peakto-average power ratio (PAPR) [27]. As a matter of fact, the single carrier frequency division multiple access (SC-FDMA) can be seen as a special case of rotated \mathbb{Z}^L -lattices codes, where the rotated angle is determined by the discrete Fourier transform (DFT) matrix. This rotation can increase D_n^{\min} or the distance between two points with zero product distance. Therefore, compared to uncoded orthogonal frequency division multiple access (OFDMA) systems, SC-FDMA has a better performance in form of decoding error probability and PAPR [28]. Another approach is the permutation code which was proposed to achieve the optimal diversity-multiplexing tradeoff [4], [22]. The basic idea is to use the constellation of size 2^R on a complex plane for each subchannel. The points in the constellation for each subchannel are permuted, so that the product distance is maximized. The permutation operation on each subchannel can be seen as the product of the original information times a specific matrix, which is referred to as the universal decodable matrix (UDM). Based on the Pascal triangle, the universal decodable matrices can be constructed directly for every subchannel [29]. From the perspective of rotated \mathbb{Z}^{L} -lattices code, the permutation code can be seen as a way to construct a L-dimension constellation with size 2^{LR} . The permutation rules imply that we should choose 2^{R} points from 2^{LR} , so that the product distance is maximized. It is not trivial to evaluate the decoding error probability of the permutation code, because the analytic formula for D_n^{\min} has not been obtained. According to the construction process of the permutation code, the PAPR performance could be worse than the rotated \mathbb{Z}^L -lattices code for the same average power. Another important difference is that the constellation size of each subchannel is $2^{\overline{R}} = 2^{R/L}$ for the rotated \mathbb{Z}^L -lattices code, while it is 2^R for the permutation code.

It is possible to combine the advantages of the rotated \mathbb{Z}^{L} lattices code and permutation code. First, we can construct a L-dimension constellation with the size of M ($2^R \leq M$). Then, the constellation can be rotated to an appropriate angle to maximize D_p^{\min} . After that, 2^R points from M can be chosen to further maximize D_n^{\min} . In order to find the optimal coding scheme, we consider the asymptotic case that $M \to \infty$. Therefore, the L-dimension constellation becomes a continuos L-dimension hypersphere whose radius is determined by the peak power of the coding scheme. Then, the optimal coding is to find the coordinates for 2^R points which maximize D_n^{\min} under the constraint of average power. This is an interesting idea that formulates the optimal coding design problem, which is often seen as a complicated discrete optimization problem, into an optimization problem on continuos variables. However, the general approach, which is out of the scope of this paper, still needs to be investigated in depth.

VIII. NUMERICAL RESULTS

In this section, some numerical results will be presented to verify the theoretical derivation in the previous sections. According to the proof of the proposed theorems, the accuracy of the bounds will be tight as the number of subchannels increases. Hence, we only need to compare the results for small L. Since \overline{C} is a function of SNR, the average multiplexing gain (or normalized target rate) $\overline{r} = \frac{r}{L} = \frac{\overline{R}}{C_{awgn}}$ is used to guarantee $\overline{R} < \overline{C}$ or $\overline{R} \ge \overline{C}$. The first set of simulations is to verify the accuracy of the proposed upper and lower bounds. In the simulation, we first generate the samples of the parallel fading channel, and then compare the instantaneous channel capacity with the target transmission rate, which yields the simulation results for the outage probability. Fig. 3 compares the simulation results and the theoretical results of outage exponent for the $\overline{R} < \overline{C}$ case. In Fig. 3, the number of subchannels is L = 4, and the average multiplexing gain is $\overline{r} = 0.1$. The outage exponent is computed through Eq. (17). It can be seen that the proposed outage exponent is nearly identical with the simulation results in the full SNR range. The Kaplan-Shamai's upper bound in Eq. (3) and the Tse-Viswanath's approximation in Eq. (4) have the accurate slope but not the intercept in this case.

For Fig. 4, we compare the simulation results and the outage exponent for the $\overline{R} \ge \overline{C}$ case. The proposed upper and lower bounds are calculated through Eq. (26). It can be seen that the proposed upper and lower bounds will converge to the simulation values as SNR increases. Because the proposed lower bound is only valid in the high SNR regime, the point at 0 dB of the curve is lacking. Hence, the modified lower bound in Proposition 1 has been proposed to overcome this shortcoming. In Fig. 4, it can be seen that the modified lower bound is an accurate estimation for the outage probability. Clearly, the upper bound in Eq. (3) cannot provide an accurate estimation of the outage probability. The approximation in Eq. (4) does not have a good accuracy either, since it can only estimate the slope of the outage probability in high SNRs for large average multiplexing gains.

Fig. 5 shows the simulation results of the outage probability and the proposed bounds from $-20 \,\mathrm{dB}$ to $20 \,\mathrm{dB}$. It can be seen that the low SNR approximation in Eq. (27), which is ploted from $-20 \,\mathrm{dB}$ to $-5 \,\mathrm{dB}$, is very tight as expected. From 0 to 20 dB, the proposed approximation in Proposition 1 has been used, which is also very tight. The Kaplan-Shamai's upper bound and Tse-Viswanath's approximation are also plotted for comparison. In the Kaplan-Shamai's upper bound, i.e., Eq. (3), the large deviation principle is applied. Therefore, the slope is accurate when $\overline{R} < \overline{C}$ or $\overline{R} > \overline{C}$, and is not accurate enough when \overline{R} is close to \overline{C} . According to Fig. 2, when SNR is between 0 dB and 10 dB, $\overline{r} = 0.8$ is very close to the threshold curve r_0 . In this range, Kaplan-Shamai's upper bound converges to 1. When SNR is smaller than 0 dB or larger than 10 dB, the difference between $\overline{r} = 0.8$ and the threshold curve in Fig. 2 is considerably large. Therefore, the Kaplan-Shamai's upper bound characterizes the accurate slope in this SNR range.

The second set of simulations illustrate the behavior of the outage probability as the number of subchannels increases. Figs. 6 and 7 show the simulated outage probability and the theoretical results for $\overline{R} < \overline{C}$ and $\overline{R} \ge \overline{C}$, respectively. The number of subchannels L is set to be 2, 4, and 6, respectively. As expected, the proposed bounds are accurate even for L = 2. From Fig. 6, it can be found that the slopes of the outage probability for L = 2, 4, 6 do not vary much with one another. The performance gain are mainly comes from the SNR gain. This phenomenon can be explained from Eq. (17). Clearly, for

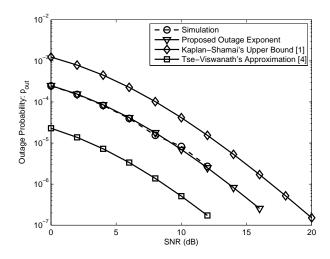


Fig. 3. Outage probability comparison for L = 4 and $\overline{r} = 0.1$.

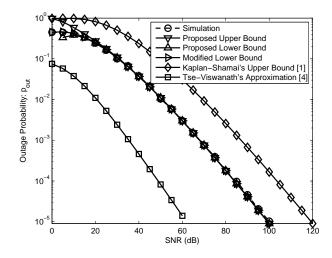


Fig. 4. Outage probability comparison for L = 4 and $\overline{r} = 0.8$.

a fixed L, we have

$$p_{\text{out}} \lesssim \frac{1}{\sqrt{2\pi L}\sigma \Xi(0)} e^{-LE_1(R,\gamma)}$$

Therefore, varying L results in a great change in the coefficient of the exponential function. However, as shown in Fig. 7, the performance gains are mainly determined by the diversity gain for the $\overline{R} \geq \overline{C}$ case. This phenomenon occurs because the coefficient in Eq. (30) is one for any L. This results indicate that the power allocation is very important for the small average multiplexing gain case, but may not result in a significant performance gain in the large average multiplexing gain case.

The outage capacity defined as the maximum transmission rate which guarantee the outage probability is smaller than a target value. Fig. 8 shows the comparison between the simulated outage capacity and theoretical results. The normalized outage capacity in the figure is just the maximum average multiplexing gain which can be achieved for a given outage probability. In the simulation, the binary search based ordinal

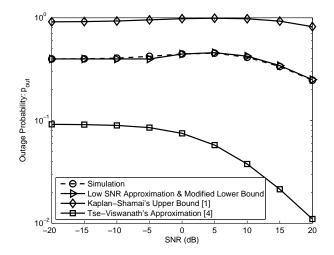


Fig. 5. Outage probability comparison for L = 4 and $\overline{r} = 0.8$ in the low SNR regime.

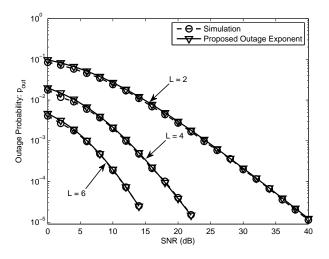


Fig. 6. Outage probability comparison for L = 2, 4, 6, and $\overline{r} = 0.3$.

optimization method is used to calculate C_{ε} [30]. First, fix a desired outage probability p_{out} , and let $C_{norm} = C_{norm}^{max} = 2$ and $C_{\text{norm}}^{\min} = 0$. If we set $R = C_{\text{norm}} \cdot C_{\text{awgn}}$ and give a SNR value, the simulated outage probability \hat{p}_{out} can then be obtained. If $|\hat{p}_{out} - p_{out}| < \delta$, C_{norm} is seen as the correct value. However, if $|\hat{p}_{out} - p_{out}| \ge \delta$, C_{norm}^{max} and C_{norm}^{min} will be changed with the following rules:

- 1) if $\hat{p}_{out} p_{out} \ge \delta$, set $C_{norm}^{max} = C_{norm}$ and go to 3);
- 2) if $\hat{p}_{out} p_{out} \le -\delta$, set $C_{norm}^{\min} = C_{norm}$; 3) set $C_{norm} = \frac{1}{2} \left(C_{norm}^{\max} + C_{norm}^{\min} \right)$.

After the correct C_{norm} is obtained, the SNR will be changed to a new value. The normalized ε -outage capacity at any SNR value can then be obtained. In our simulations, δ is set to $\frac{1}{1000}p_{\text{out}}$. The corresponding theoretical results are computed through Eq. (21), since $r_0(\gamma) \to 1$ as $\gamma \to \infty$. From Fig. 8, it can be seen that the theoretical results are very accurate for $p_{\text{out}} = 0.001$ and $p_{\text{out}} = 0.01$. For $p_{\text{out}} = 0.1$, the two curves have a small gap in the low SNR regime, because $r_0(\gamma)$ achieves the minimum value at $\gamma^* \approx 6.2442 \, \text{dB}$.

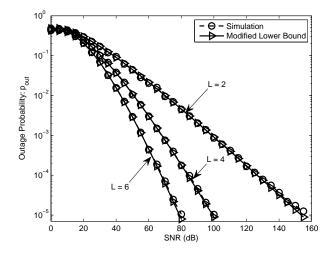


Fig. 7. Outage probability comparison for L = 2, 4, 6, and $\overline{r} = 0.8$.

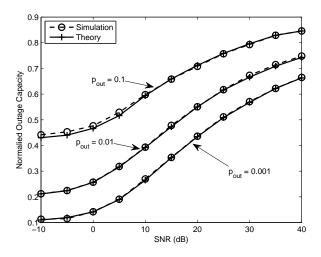


Fig. 8. Outage capacity comparison for $p_{out} = 0.001, 0.01, 0.1, and L = 4$.

The last set of simulations is to illustrate the diversity gain versus SNR for a given average multiplexing gain. For $\overline{R} < \overline{C}$, as shown in Fig. 9, the average multiplexing gain is set to $\overline{r} = 0.7$. The simulated diversity gain is obtained by differentiating the outage probability. The theoretical results is computed by Eq. (17). The asymptotic diversity gain is also plotted for comparison. It can be seen that the theoretical and simulation curves have a small gap when SNR is smaller than 20 dB. This is because $\overline{r} = 0.7$ is close to the minimum value of $r_0(\gamma)$ in that SNR range, which yields a loose large deviation bound. Fig. 10 shows the diversity at a given average multiplexing gain for $\overline{r} = 0.8$. In contrast to Fig. 9, the theoretical curve in this figure is calculated by Eq. (34). Although $\overline{r} = 0.8$ is still smaller than the minimum value of $r_0(\gamma)$, the proposed bound is nearly identical with the simulation results. Therefore, the proposed bound for $R \geq C$ is also very tight for \overline{r} is of the middle values. From these figures, the finite-SNR diversity gain approaches to the asymptotic value as SNR increases. Therefore, for high SNRs, such as more than 40 dB, the asymptotic diversity-

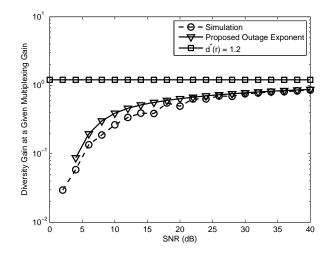


Fig. 9. Diversity gain at a given average multiplexing gain for L = 4 and $\overline{r} = 0.7$.

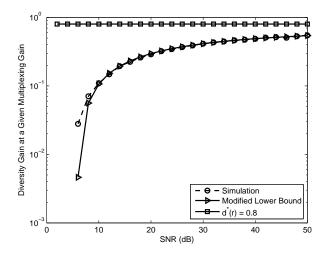


Fig. 10. Diversity gain at a given average multiplexing gain for L = 4 and $\overline{r} = 0.8$.

multiplexing tradeoff can be used to evaluate the performance instead of the corresponding finite-SNR diversity-multiplexing tradeoff.

Remark 3: Throughout this section, the numerical evaluation of the theoretical results need to compute the Meijer's G-function. However, the numerical computation problem of Meijer's G-function has not been fully solved. MATLAB R2010a does not have this function at all, while the realizations in Mathematica 7.0 and Maple 14 have severe bugs in some cases. From a large number of experiments, for instance, Mathematica 7.0 will give wrong results if the input parameters are decimal and not fractional. The designers for Mathematica, who wrote the program of Meijer's G-function, also pointed out the design problems of this function, especially the logarithmic cases [31]. Therefore, a software package of Meijer's G-function is developed for MATLAB in [32]. In the simple poles case, the algorithm can go straightforwardly to the generalized hyper-geometric functions. Meanwhile, the algorithm can also deal with some cases which Maple 14

cannot handle. For the super complicated and potentially buggy cases, i.e., the logarithmic cases, it is very difficulty to design a general algorithm. Aiming at this problem, the accurate formulas to evaluate the residues of the integration have been derived, and can be used to numerically evaluate Meijer's *G*-function with arbitrary precision.

IX. CONCLUSIONS

This paper focused on the parallel fading channel and proposed a unified performance metric, referred to as the outage exponent. The outage exponent is defined as the exponentially tight upper and lower bounds of the outage probability for any number of subchannels, any SNR, and any target transmission rate. Based on the latest results in large deviations theory, Meijer's G-function, and the method of integral around a contour, the outage exponent is obtained for both $\overline{R} < \overline{C}$ and $\overline{R} \geq \overline{C}$ cases. From the accurate estimation of the outage probability, the reliable function, outage capacity, finite-SNR diversity-multiplexing tradeoff, SNR gain, and also the asymptotic performance metrics, including the delay-limited capacity, ergodic capacity, and diversity-multiplexing tradeoff have been calculated. In order to achieve the proposed outage exponent, the coding schemes which maximize the minimum product distance have also been discussed. Therefore, it can be concluded that the proposed outage exponent framework provides a powerful tool for analyzing and evaluating the performance of existing and upcoming communication systems.

ACKNOWLEDGMENT

The authors would like to thank Professor John S. Sadowsky for insightful discussion on large deviations theory.

APPENDIX A Proof of Theorem 2

Define a sequence of random variables $\{X_l : l = 1, ..., L\}$ by letting

$$X_l = \overline{R} - \ln\left(1 + |h_l|^2 \gamma\right).$$

We further let $Y_L = \sum_{l=1}^{L} X_l$, then the outage probability defined in Eq. (2) is equivalent to

$$p_{\text{out}} = \Pr\left\{\frac{1}{L}\sum_{l=1}^{L}X_l > 0\right\} = \Pr\left\{\frac{1}{L}Y_L > 0\right\}.$$

Because the elements of $\{h_l : l = 1, ..., L\}$ are independent with the identical distribution of CN(0, 1), the logarithmic moment generating function of Y_L is given by

$$\begin{split} \Lambda_L\left(\xi\right) &= \ln\mathsf{E}\left\{e^{\xi Y_L}\right\} = \ln\left(e^{\xi\overline{R}}\mathsf{E}\left\{e^{-\xi\ln\left(1+|h_l|^2\gamma\right)}\right\}\right)^L\\ &= L\left[\xi\overline{R} + \ln\left(\int_0^\infty e^{-\xi x}\frac{1}{\gamma}e^{-\frac{e^x-1}{\gamma}+x}dx\right)\right]\\ &= L\left[\xi\overline{R} + \ln\left(e^{\frac{1}{\gamma}}\frac{1}{\gamma}\int_1^\infty t^{-\xi}e^{-\frac{t}{\gamma}}dt\right)\right]\\ &= L\left[\left(\overline{R} - \ln\gamma\right)\xi + \ln\left(e^{\frac{1}{\gamma}}\Gamma\left(1-\xi,\gamma^{-1}\right)\right)\right]. \end{split}$$

Therefore, we have

$$\Lambda \left(\xi \right) = \lim_{L \to \infty} \frac{1}{L} \Lambda_L \left(\xi \right)$$
$$= \left(\overline{R} - \ln \gamma \right) \xi + \ln \left(e^{\frac{1}{\gamma}} \Gamma \left(1 - \xi, \gamma^{-1} \right) \right)$$

Notice that $\overline{R} < \overline{C}$, i.e., $\mathsf{E} \{Y_L\} < 0$, the Legendre-Fenchel transform of $\Lambda(\xi)$ at s = 0 is then given by

$$\Lambda^{*}(0) = \sup_{\xi \in \mathbb{R}} \{-\Lambda(\xi)\}$$

$$= -\inf_{\xi \ge 0} \left\{ \left(\overline{R} - \ln\gamma\right)\xi + \ln\left(e^{\frac{1}{\gamma}}\Gamma\left(1 - \xi, \gamma^{-1}\right)\right) \right\}$$

$$= \left(\ln\gamma - \overline{R}\right)\Xi(0) - \ln\left(e^{\frac{1}{\gamma}}\Gamma\left(1 - \Xi(0), \gamma^{-1}\right)\right) \quad (40)$$

$$= \left(C_{\text{awgn}} - \overline{R}\right)\Xi(0) - \Xi(0)\ln\left(1 + \frac{1}{\gamma}\right)$$

$$- \ln\left(e^{\frac{1}{\gamma}}\Gamma\left(1 - \Xi(0), \gamma^{-1}\right)\right)$$

Since $\Lambda(\xi)$ is convex and differentiable, $\xi = \Xi(0)$ is the solution of the following equation:

$$\frac{\partial}{\partial\xi}\Lambda\left(\xi\right) = 0.$$

By computing the derivative, we have

$$\frac{\partial}{\partial\xi}\Lambda\left(\xi\right) = \overline{R} - \frac{1}{\Gamma\left(1-\xi,\gamma^{-1}\right)} G_{2,3}^{3,0}\left(\frac{1}{\gamma} \left| \begin{array}{c} 1,1\\0,0,1-\xi \end{array} \right).$$

Therefore, $\Xi(0)$ is the solution of Eq. (19). According to the results in [18], σ^2 is given by

$$\begin{split} \sigma^{2} &= \frac{\partial^{2}}{\partial \xi^{2}} \Lambda\left(\xi\right) \bigg|_{\xi=\Xi(0)} \\ &= \frac{2}{\Gamma\left(1-\Xi\left(0\right),\gamma^{-1}\right)} G_{3,4}^{4,0} \left(\frac{1}{\gamma} \left| \begin{array}{c} 1,1,1\\ 0,0,0,1-\Xi\left(0\right) \right) \right) - \\ &\left[\frac{1}{\Gamma\left(1-\Xi\left(0\right),\gamma^{-1}\right)} G_{2,3}^{3,0} \left(\frac{1}{\gamma} \left| \begin{array}{c} 1,1\\ 0,0,1-\Xi\left(0\right) \right) \right]^{2}. \end{split}$$

Clearly, we have

$$\Lambda_{L}^{*}(0) = \sup_{\xi \ge 0} \left\{ -\Lambda_{L}(\xi) \right\} = L\Lambda^{*}(0).$$

According to the proof of Cramér's theorem in Chapter 2.2.1 of [23], the following inequality holds for any $L \in \mathbb{N}$:

$$p_{\text{out}} \le \psi e^{-L\Lambda^*(0)} = p_{\text{ex}}^{\text{upper}}.$$

where ψ is a slowly varying function. As a matter of fact, from Remark (c) on Cramér's theorem in Chapter 2.2.1 of [23], a loose estimation of ψ is 2. Let $\mathcal{F} = \{Y_L : \frac{1}{L}Y_L \ge 0\}$ and $\mathcal{G} = \{Y_L : \frac{1}{L}Y_L > 0\}$ with $\mathcal{G} \subseteq \mathcal{F}$, then from Cramér's theorem, we have

$$-\Lambda^{*}(0) \leq \liminf_{L \to \infty} \frac{1}{L} \ln \Pr \left\{ \mathcal{G} \right\}$$
$$\leq \limsup_{L \to \infty} \frac{1}{L} \ln \Pr \left\{ \mathcal{F} \right\} \leq -\Lambda^{*}(0),$$

which implies that

$$\lim_{L \to \infty} \frac{1}{L} \ln p_{\text{out}} = -\Lambda^* \left(0 \right).$$

Therefore, $\Lambda^*(0)$ is a good rate function in the sense of large deviation, and there must be a slowly varying function φ with $\varphi < \psi$ satisfying

$$p_{\rm ex}^{\rm lower} = \varphi e^{-L\Lambda^*(0)} \le p_{\rm out}.$$

The above arguments can also be verified by the derivations and results in two important literatures on probability inequalities and large deviation results for sums of independent random variables [33], [34].

In order to obtain a tight upper bound, the only needed work is to estimate ψ accurately. Because the elements of $\{X_l : l = 1, \ldots, L\}$ are independent with identical distribution, the sequence of random variables $\{Y_L^{(\Xi(0))} : L \in \mathbb{N}\}$ with the titled distribution defined in Eq. (16) obeys the central limit theorem. Thus, the results in [18] can be used to give an accurate estimation of ψ as

$$\psi = \frac{1}{\sqrt{2\pi L}\sigma\Xi\left(0\right)},$$

for any $L \in \mathbb{N}$, with

$$\lim_{L \to \infty} \frac{1}{L} \ln \frac{p_{\text{out}}}{\psi} = -\Lambda^* \left(0 \right)$$

Therefore, we have

$$\lim_{L \to \infty} \frac{1}{p_{\text{out}}} \left(\frac{1}{\sqrt{2\pi L} \sigma \Xi(0)} e^{-L\Lambda^*(0)} \right) = \lim_{L \to \infty} \frac{p_{\text{ex}}^{\text{upper}}}{p_{\text{out}}} = 1.$$

According to the definition of limit, for any given $\varepsilon > 0$, there is a number $L^* \in \mathbb{N}$ such that

$$\frac{p_{\rm ex}^{\rm upper}}{p_{\rm out}} - 1 < \varepsilon,$$

holds for any $L \ge L^*$. Therefore, for any $L \in \mathbb{N}$, the outage probability p_{out} is upper bounded by $p_{\text{ex}}^{\text{upper}}$, and the accuracy of $p_{\text{ex}}^{\text{upper}}$ increases as L increases for any given γ .

To sum up, we have

$$p_{\text{out}} \lesssim p_{\text{ex}}^{\text{upper}} = \frac{1}{\sqrt{2\pi L}\sigma\Xi(0)} e^{-L\Lambda^*(0)}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-L\left[\Lambda^*(0) + \frac{\ln(\sigma\Xi(0))}{L} + \frac{\ln L}{2L}\right]},$$
(41)

in the sense of L and γ , where the symbol " \lesssim " is defined in Eq. (5).

APPENDIX B Proof of Theorem 3

Define a random variable Y by letting

$$Y = \sum_{l=1}^{L} \ln\left(1 + |h_l|^2 \gamma\right),$$

whose moment generating function is given by

$$M\left(\xi\right) = \left[e^{\frac{1}{\gamma}}\gamma^{\xi}\Gamma\left(1+\xi,\gamma^{-1}\right)\right]^{L}$$

Clearly, we have $p_{out} = \Pr{\{Y < R\}}$.

Let F(y) denote the distribution of Y, define the titled distribution for Y as

$$dF^{\left(\xi\right)}\left(y\right) = \frac{e^{\xi y}dF\left(y\right)}{M\left(\xi\right)}$$

For convenience, we let $F_{\alpha}(y) = F^{(\Xi(\alpha R))}(y)$, and use O_{α} to denote the operation O under the titled distribution $F_{\alpha}(y)$. Let Z be a random variable with the distribution of $F_{\alpha}(y)$, then for any α

$$\mathsf{E}_{\alpha}\left\{Z\right\} = \frac{1}{M\left(\Xi\left(\alpha R\right)\right)} \int_{-\infty}^{+\infty} y e^{\xi y} dF\left(y\right)$$
$$= \left.\frac{\partial}{\partial \xi} \ln M\left(\xi\right)\right|_{\xi=\Xi\left(\alpha R\right)} = \alpha R.$$

Therefore, for any $0 < \delta < \alpha < 1$, we have

$$F(R) = M(\Xi(\alpha R)) \int_{0}^{R} e^{-t\Xi(\alpha R)} dF_{\alpha}(t)$$

$$\geq M(\Xi(\alpha R)) \int_{\delta R}^{R} e^{-t\Xi(\alpha R)} dF_{\alpha}(t)$$

$$\geq M(\Xi(\alpha R)) e^{-\delta R M(\Xi(\alpha R))} [F_{\alpha}(R) - F_{\alpha}(\delta R)].$$

Since $\delta R < \mathsf{E}_{\alpha} \{Z\} < R$, by applying Cramér's theorem [23], we have

$$\begin{cases} F_{\alpha}\left(\delta R\right) \leq e^{-\Lambda_{\alpha}\left(\delta R\right)};\\ F_{\alpha}\left(R\right) \leq 1 - e^{-\Lambda_{\alpha}\left(R\right)}. \end{cases}$$
(42)

Therefore, we have

$$\begin{split} F\left(R\right) &\geq M\left(\Xi\left(\alpha R\right)\right)e^{-\delta R \Xi\left(\alpha R\right)}\left[F_{\alpha}\left(R\right) - F_{\alpha}\left(\delta R\right)\right] \\ &= e^{\ln\left(M\left(\Xi\left(\alpha R\right)\right)\right) - \delta R \Xi\left(\alpha R\right)}\left[F_{\alpha}\left(R\right) - F_{\alpha}\left(\delta R\right)\right] \\ &\geq \left(1 - e^{-\Lambda_{\alpha}\left(R\right)} - e^{-\Lambda_{\alpha}\left(\delta R\right)}\right)e^{-\Lambda^{*}\left(\alpha R\right) + \Xi\left(\alpha R\right)R\left(\alpha - \delta\right)}. \end{split}$$

Clearly, $\Lambda^{*}(R)$ is given by

$$\Lambda^{*}(R) = L \left[-\left(C_{\text{awgn}} - \overline{R}\right) \Xi(R) + \Xi(R) \ln\left(1 + \frac{1}{\gamma}\right) - \ln\left(e^{\frac{1}{\gamma}}\Gamma\left(1 + \Xi(R), \gamma^{-1}\right)\right) \right],$$

and $\Xi(R)$ is the solution of

$$R - \frac{L}{\Gamma(1+\xi,\gamma^{-1})} G_{2,3}^{3,0} \left(\frac{1}{\gamma} \begin{vmatrix} 1,1\\0,0,1+\xi \end{vmatrix}\right) = 0.$$

Therefore, we have

$$\Lambda^{*}(\alpha R) - \Xi(\alpha R) R(\alpha - \delta) = LE_{1}^{\alpha}(R, \gamma).$$

In the following, $\Lambda_{\alpha}(R)$ and $\Lambda_{\alpha}(\delta R)$ will be obtained from $\Xi(R)$. Notice that

$$\frac{\partial}{\partial R}M\left(\Xi\left(R\right)\right) = RM\left(\Xi\left(R\right)\right)\Xi'\left(R\right).$$

By computing the integration of both sides for the above formula, we have

$$\int_{R}^{\mathsf{E}(X)} \frac{1}{M(\Xi(t))} \frac{\partial}{\partial R} M(\Xi(t)) \, dt = \int_{R}^{\mathsf{E}(X)} t\Xi'(t) \, dt.$$

Since $\Xi(\mathsf{E}(X)) = 0$, the integration equation will become

$$\ln M\left(\Xi\left(R\right)\right) = R\Xi\left(R\right) + \int_{R}^{\mathsf{E}(X)} \Xi\left(t\right) dt.$$

According to the definition of Legendre-Fenchel transform, we have

$$\Lambda^* \left(R \right) = -\int_R^{\mathsf{E}\{Y\}} \Xi\left(t \right) dt. \tag{43}$$

The moment generating function for Z is given by

$$\mathsf{E}_{\alpha}\left\{e^{\xi Z}\right\} = \frac{M\left(\xi + \Xi\left(\alpha R\right)\right)}{M\left(\Xi\left(\alpha R\right)\right)}.$$

Then, $\Xi_{\alpha}(s)$ satisfies

$$z = \frac{\partial}{\partial \xi} \frac{M\left(\xi + \Xi\left(\alpha R\right)\right)}{M\left(\Xi\left(\alpha R\right)\right)} \bigg|_{\xi = \Xi_{\alpha}\left(z\right)}$$
$$= \frac{\partial}{\partial \theta} \ln M\left(\theta\right) \bigg|_{\theta = \Xi_{\alpha}(z) + \Xi\left(\alpha R\right)}.$$

Therefore, we have

$$\Xi_{\alpha}(z) = \Xi(z) - \Xi(\alpha R)$$

According to Eq. (43), we have

$$\Lambda_{\alpha}^{*}(\delta R) = -\int_{\delta R}^{\mathsf{E}_{\alpha}\{\mathbf{Z}\}} \Xi_{\alpha}(t) dt$$
$$= R(\alpha - \delta) \Xi(\alpha R) - \int_{\delta R}^{\alpha R} \Xi(t) dt$$

and $\Lambda_{\alpha}^{*}(R) = \Lambda_{\alpha}^{*}(\delta R)|_{\delta=1}$. Next, we will prove there exists α and δ such that $1 - e^{-\Lambda_{\alpha}(R)} - e^{-\Lambda_{\alpha}(\delta R)} > 0$ for any $\overline{R} < \overline{C}$. According to the monotonicity of $\Xi_{\alpha}(R)$, we have

$$\frac{\partial}{\partial \delta} \Lambda_{\alpha}^{*} \left(\delta R \right) = R \left(\Xi \left(\delta R \right) - \Xi \left(\alpha R \right) \right) < 0,$$

and

$$\frac{\partial^2}{\partial \delta^2} \Lambda^*_{\alpha} \left(\delta R \right) = R^2 \Xi' \left(\delta R \right) > 0.$$

Therefore, $\Lambda^*_{\alpha}(\delta R)$ achieves the maximum value at $\delta = 0$. According to the result in [23], we have

$$\inf \left\{ z : z \in \mathbb{R}, F_{\alpha}(z) > 0 \right\} = 0,$$

and $F_{\alpha}(z) = e^{-\Lambda_{\alpha}^{*}(0)}$. On the other hand,

$$\lim_{R \to 0^+} F_{\alpha}\left(R\right) = \frac{1}{M\left(\Xi\left(\alpha R\right)\right)} \lim_{R \to 0^+} \int_0^R e^{\Xi\left(\alpha R\right)t} dF\left(t\right) = 0.$$

Therefore, we have $\lim_{\delta \to 0} \Lambda^*_{\alpha}(\delta R) = \infty$. Furthermore,

$$\frac{\partial}{\partial \alpha} \Lambda_{\alpha}^{*}(R) = -R^{2} (1 - \alpha) \Xi'(\alpha R) \le 0$$

Consider the result that $\Lambda_{\alpha}^{*}(R)|_{\alpha=1} = 0$, there exists $\epsilon > 0$ and $0 < \alpha^* < 1$ such that $\Lambda^*_{\alpha^*}(\hat{R}) = \epsilon$. Then, δ^* can be chosen as follows:

$$\Lambda_{\alpha^*}^* \left(\delta^* R \right) > -\ln\left(1 - e^{-\epsilon} \right).$$

Therefore, there is α^* and δ^* satisfying

$$1 - e^{-\Lambda_{\alpha^*}(R)} - e^{-\Lambda_{\alpha^*}(\delta^* R)} > 0.$$

Finally, we determine the relationship between α and δ . By letting

$$\frac{\partial}{\partial \alpha} \left(1 - e^{-\Lambda_{\alpha}(R)} - e^{-\Lambda_{\alpha}(\delta R)} \right) e^{-\Lambda^*(\alpha R) + \Xi(\alpha R)R(\alpha - \delta)} = 0,$$

Eq. (25) can then be obtained. As a matter of fact, the main method of this proof is based on [19].

APPENDIX C **PROOF OF THEOREM 4**

Define a function $F_L(z)$ as

$$F_L(z) = 1 - \int_{\mathcal{D}(z,0)} e^{-\sum_{l=1}^L x_l} d\boldsymbol{x},$$

where the integration domain is defined by

$$\mathcal{D}(z,\eta) = \left\{ \boldsymbol{x} \left| \prod_{l=1}^{L} x_l < z, \boldsymbol{x} > \eta \right. \right\}.$$

Consider the tight upper bound in the high SNR regime, it is easy to verify that

$$p_{\text{out}} = \Pr\left\{\sum_{l=1}^{L} \ln\left(1 + |h_l|^2 \gamma\right) < R\right\}$$
$$\approx \Pr\left\{\prod_{l=1}^{L} |h_l|^2 < \frac{e^R - 1}{\gamma^L}\right\}.$$

and

$$\begin{aligned} p_{\text{out}} &= \Pr\left\{\sum_{l=1}^{L} \ln\left(1 + |h_l|^2 \gamma\right) < R\right\} \\ &\leq \Pr\left\{\ln\left(1 + \gamma^L \prod_{l=1}^{L} |h_l|^2\right) < R\right\} \\ &= \int_{\mathcal{D}\left(\frac{e^R - 1}{\gamma^L}, 0\right)} e^{-\sum_{l=1}^{L} x_l} d\boldsymbol{x} \\ &= 1 - F_L\left(\frac{e^R - 1}{\gamma^L}\right). \end{aligned}$$

Therefore, we have

$$p_{\text{out}} \lesssim 1 - F_L\left(\frac{e^R - 1}{\gamma^L}\right)$$

holds in the high SNR regime.

Before deriving the tight lower bound in the high SNR regime, we first rewrite the outage probability as

$$p_{\text{out}} = \Pr\left\{\sum_{l=1}^{L} \ln\left(1 + |h_l|^2 \gamma\right) < R\right\}$$
$$= \Pr\left\{\prod_{l=1}^{L} \left(\frac{1}{\gamma} + |h_l|^2\right) < \frac{e^R}{\gamma^L}\right\}$$
$$= e^{\frac{L}{\gamma}} \int_{\mathcal{D}\left(\frac{e^R}{\gamma^L}, \frac{1}{\gamma}\right)} e^{-\sum_{l=1}^{L} x_l} d\boldsymbol{x}.$$

Next, the principle of mathematical induction will be used to show that $p_{\text{out}} \ge 1 - e^{\frac{L}{\gamma}} F_L \left(e^R \gamma^{-L} \right)$. For L = 1, it is easy to verify that

$$p_{\text{out}} = 1 - e^{-\frac{e^R - 1}{\gamma}} = 1 - e^{\frac{1}{\gamma}} F_1\left(\frac{e^R}{\gamma}\right)$$

Assume the proposition holds for $L = k, k \ge 1$. Then, for the situation of L = k + 1, the lower bound of p_{out} is calculated as follows:

$$\begin{split} p_{\text{out}} &= e^{\frac{k+1}{\gamma}} \int_{\mathcal{D}\left(\frac{e^R}{\gamma^{k+1}}, \frac{1}{\gamma}\right)} e^{-\sum_{l=1}^{k+1} x_l} d\boldsymbol{x} \\ &= e^{\frac{1}{\gamma}} \int_{\frac{1}{\gamma}}^{\infty} e^{-x_1} \left[\int_{\mathcal{D}\left(\frac{e^R}{\gamma^{k+1}x_1}, \frac{1}{\gamma}\right)} e^{-\sum_{l=1}^{k} x_l} d\boldsymbol{x}' \right] dx_1 \\ &\geq e^{\frac{1}{\gamma}} \int_{\frac{1}{\gamma}}^{\infty} e^{-x_1} \left[1 - e^{\frac{k}{\gamma}} F_k\left(\frac{e^R}{\gamma^{k+1}x_1}\right) \right] dx_1 \\ &= 1 - e^{\frac{k+1}{\gamma}} \int_{0}^{\infty} e^{-x_1} F_k\left(\frac{e^R}{\gamma^{k+1}x_1}\right) dx_1 \\ &= 1 - e^{\frac{k+1}{\gamma}} F_{k+1}\left(\frac{e^R}{\gamma^{k+1}}\right), \end{split}$$

where $x' = [x_2, \ldots, x_{k+1}]$. According to the principle of mathematical induction, the inequality

$$p_{\text{out}} \gtrsim 1 - e^{\frac{L}{\gamma}} F_L\left(\frac{e^R}{\gamma^L}\right)$$

holds for any $L \in \mathbb{N}$.

Next, we will show that $F_L(z)$ is the Meijer's *G*-function as shown in the theorem. The proof is also based on the principle of mathematical induction. For L = 1,

$$F_1(z) = 1 - \int_0^z e^{-x} dx = e^{-z} = G_{0,1}^{1,0} \left(z \begin{vmatrix} - \\ 0 \end{vmatrix} \right),$$

where the last equation follows that

$$G_{0,1}^{1,0}\left(z \middle| \begin{matrix} -\\ 0 \end{matrix}\right) = \frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma\left(-s\right) z^{s} ds$$
$$= \sum_{n=0}^{\infty} \frac{(-z)^{n}}{n!} = e^{-z}.$$

For L = 2, we have

$$F_{2}(z) = 1 - \iint_{\mathcal{D}(z,0)} e^{-x_{1}-x_{2}} dx_{1} dx_{2}$$

$$= 1 - \int_{0}^{\infty} e^{-x_{1}} \left(\int_{0}^{\frac{z}{x_{1}}} e^{-x_{2}} dx_{2} \right) dx_{1}$$

$$= 1 - \int_{0}^{\infty} e^{-x_{1}} \left(1 - e^{-\frac{z}{x_{1}}} \right) dx_{1}$$

$$= \int_{0}^{\infty} e^{-x_{1}-\frac{z}{x_{1}}} dx_{1}$$

$$= 2\sqrt{z}K_{1} \left(2\sqrt{z} \right) = G_{0,2}^{2,0} \left(z \middle| \begin{array}{c} - \\ 0, 1 \end{array} \right),$$

where $K_{\nu}(z)$ is the modified Bessel function of the second kind, and the last equation follows Eq. (3) in Chapter 9.34 of [20]

$$G_{0,2}^{2,0}\left(\frac{z^2}{4} \left| \frac{-}{\frac{\mu - \nu}{2}, \frac{\mu + \nu}{2}} \right) = 2\left(\frac{z}{2}\right)^{\mu} K_{\nu}(z).$$

Assume the proposition holds for $L = k, k \ge 1$. Then, for L = k + 1, $F_{k+1}(z)$ is calculated as

$$F_{k+1}(z) = 1 - \int_{\mathcal{D}(z,0)} e^{-\sum_{l=1}^{k+1} x_l} d\mathbf{x}$$

= $1 - \int_0^\infty e^{-x_1} \left[\int_{\mathcal{D}\left(\frac{z}{x_1},0\right)} e^{-\sum_{l=1}^k x_l} d\mathbf{x}' \right] dx_1$
= $\int_0^\infty e^{-x_1} G_{0,k}^{k,0} \left(\frac{z}{x_1} \middle| \begin{array}{c} -\\ 0,1,\dots,1 \end{array} \right) dx_1$
= $\frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma(-s) \Gamma^k (1-s) z^s ds$
= $G_{0,k+1}^{k+1,0} \left(z \middle| \begin{array}{c} -\\ 0,1,\dots,1 \end{array} \right).$

It should be noted that the path \mathcal{L} runs from $-\infty$ to $+\infty$ in such a way that the poles of the functions $\Gamma(-s)$ and $\Gamma(1-s)$ lie to the right of \mathcal{L} . Therefore, the path \mathcal{L} can be chosen as $\Re(\mathcal{L}) = -\frac{1}{2}$, then $\Re(1-s) = \frac{3}{2}$ for any s along this path. Under this condition, $\Gamma(1-s)$ can be expanded as $\int_0^\infty x^{-s} e^{-x} dx$. According to the principle of mathematical induction, the equation

$$F_L(z) = G_{0,L}^{L,0} \left(z \begin{vmatrix} - \\ 0, 1, \dots, 1 \end{vmatrix} \right)$$

holds for any $L \in \mathbb{N}$. As a matter of fact, this proof can also be carried out by applying the Mellin's transform to the product of independent random variables [35].

In the low SNR regime, let $R = rC_{awgn}$, the approximation is given by

$$p_{\text{out}} = \Pr\left\{\sum_{l=1}^{L} \ln\left(1 + |h_l|^2 \gamma\right) < R\right\}$$
$$\approx \Pr\left\{\ln\left(1 + \gamma \sum_{l=1}^{L} |h_l|^2\right) < r\ln\left(1 + \gamma\right)\right\}$$
$$= \Pr\left\{\sum_{l=1}^{L} |h_l|^2 < \frac{(1 + \gamma)^r - 1}{\gamma}\right\}$$
$$= 1 - \frac{\Gamma\left(L, r\right)}{(L - 1)!},$$

where the property that the sum of exponential distributed random varibles is an Erlang distributed random varible has been used.

APPENDIX D Proof of Theorem 5

Define a function $f(\gamma)$ as follows:

$$f(\gamma) = \ln \left(1 - e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{\left(1+\gamma\right)^r}{\gamma^L} \middle| \begin{array}{c} -\\ 0,1,\dots,1 \end{array} \right) \right).$$

According to Theorem 4 and Proposition 1, we have

$$p_{\rm out} \approx e^{f(\gamma)}$$

For any $x \in (0, 1)$, it is easy to verify that

$$\begin{cases} \frac{\partial}{\partial x} \ln \left(1 - x \right) = \frac{-1}{1 - x} < 0;\\ \frac{\partial^2}{\partial x^2} \ln \left(1 - x \right) = \frac{-1}{\left(1 - x \right)^2} < 0. \end{cases}$$

According to Lemma 1, the outage probability can be bounded Since $e^{\frac{\lambda}{\gamma}} \to 1$ and $\frac{1+\gamma}{\gamma} \to 1$ as $\gamma \to \infty$, according to the rule of L'Hospital, we have

$$p_{\text{out}} \approx e^{f(\gamma)} \ge e^{\gamma \frac{\partial}{\partial \gamma} f(\gamma)}.$$

According to the definition of Meijer's G-function, we have

$$\begin{split} & \frac{\partial}{\partial z} G_{0,L}^{L,0} \left(z \middle| \begin{array}{c} - \\ 0, 1, \dots, 1 \end{array} \right) \\ = & \frac{\partial}{\partial z} \frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma \left(-s \right) \Gamma^{L-1} \left(1 - s \right) z^{s} ds \\ = & \frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma \left(-s \right) \Gamma^{L-1} \left(1 - s \right) s z^{s-1} ds \\ = & - \frac{1}{2\pi i} \oint_{\mathcal{L}} \left(-1 - t \right) \Gamma \left(-1 - t \right) \Gamma^{L-1} \left(-t \right) z^{t} dt \\ = & - \frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma^{L} \left(-t \right) z^{t} dt \\ = & - G_{0,L}^{L,0} \left(z \middle| \begin{array}{c} - \\ 0, 0, \dots, 0 \end{array} \right). \end{split}$$

Then, for $\gamma \frac{\partial}{\partial \gamma} f\left(\gamma\right)$ we have

$$\begin{split} \gamma \frac{\partial}{\partial \gamma} f\left(\gamma\right) &= -L \left[\left(1 - \frac{r}{L}\right) + \frac{1}{\gamma} \right] \frac{\left(1 + \gamma\right)^{r-1}}{\gamma^{L-1}} \cdot \\ & \frac{e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{\left(1 + \gamma\right)^r}{\gamma^L} \right| \begin{array}{c} - \\ 0, 0, \dots, 0 \end{array} \right)}{1 - e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{\left(1 + \gamma\right)^r}{\gamma^L} \right| \begin{array}{c} - \\ 0, 1, \dots, 1 \end{array} \right)} + \\ & \frac{\frac{\lambda}{\gamma} e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{\left(1 + \gamma\right)^r}{\gamma^L} \right| \begin{array}{c} - \\ 0, 1, \dots, 1 \end{array} \right)}{1 - e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{\left(1 + \gamma\right)^r}{\gamma^L} \right| \begin{array}{c} - \\ 0, 1, \dots, 1 \end{array} \right)} \\ & \frac{1 - e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{\left(1 + \gamma\right)^r}{\gamma^L} \right| \begin{array}{c} - \\ 0, 1, \dots, 1 \end{array} \right)} \\ & \frac{1 - e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{\left(1 + \gamma\right)^r}{\gamma^L} \right| \begin{array}{c} - \\ 0, 1, \dots, 1 \end{array} \right)} \\ \end{split}$$

Therefore, Eq. (30) holds.

APPENDIX E **PROOF OF THEOREM 5**

From the proof of Theorem 4, we have

$$\lim_{\gamma \to \infty} \frac{\frac{\lambda}{\gamma} e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{(1+\gamma)^r}{\gamma^L} \middle| \begin{array}{c} -\\ 0,1,\dots,1 \end{array} \right)}{1 - e^{\frac{\lambda}{\gamma}} G_{0,L}^{L,0} \left(\frac{(1+\gamma)^r}{\gamma^L} \middle| \begin{array}{c} -\\ 0,1,\dots,1 \end{array} \right)} = 0.$$

$$\begin{split} &\lim_{\gamma \to \infty} d_{\mathrm{f}}^{*}\left(r,\gamma\right) \\ = &d^{*}\left(r\right) \lim_{z \to 0} \frac{zG_{0,L}^{L,0}\left(z \middle| \begin{array}{c} - \\ 0,0,\dots,0 \end{array}\right)}{1 - G_{0,L}^{L,0}\left(z \middle| \begin{array}{c} - \\ 0,1,\dots,1 \end{array}\right)} \\ = &d^{*}\left(r\right) \lim_{z \to 0} \left[1 + \frac{z\frac{\partial}{\partial z}G_{0,L}^{L,0}\left(z \middle| \begin{array}{c} - \\ 0,0,\dots,0 \end{array}\right)}{G_{0,L}^{L,0}\left(z \middle| \begin{array}{c} - \\ 0,0,\dots,0 \end{array}\right)} \\ = &d^{*}\left(r\right) + d^{*}\left(r\right) \lim_{z \to 0} \frac{G_{0,L}^{L,0}\left(z \middle| \begin{array}{c} - \\ 0,0,\dots,1 \end{array}\right)}{G_{0,L}^{L,0}\left(z \middle| \begin{array}{c} - \\ 0,0,\dots,1 \end{array}\right)}, \end{split}$$

where the last equality follows that

$$\begin{aligned} z \frac{\partial}{\partial z} G_{0,L}^{L,0} \left(z \middle| \begin{array}{c} - \\ 0, 0, \dots, 0 \end{array} \right) \\ = & z \frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma^{L} \left(-s \right) s z^{s-1} ds \\ = & \frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma^{L-1} \left(-s \right) \Gamma \left(1 - s \right) z^{s} ds \\ = & G_{0,L}^{L,0} \left(z \middle| \begin{array}{c} - \\ 0, 0, \dots, 1 \end{array} \right). \end{aligned}$$

By repeating this process, we have

$$\lim_{z \to 0} \frac{G_{0,L}^{L,0}\left(z \middle| \begin{array}{c} - \\ 0,0,\dots,1 \end{array}\right)}{G_{0,L}^{L,0}\left(z \middle| \begin{array}{c} - \\ 0,0,\dots,0 \end{array}\right)} \\ = \lim_{z \to 0} \frac{G_{0,L}^{L,0}\left(z \middle| \begin{array}{c} - \\ 1,1,\dots,1 \end{array}\right)}{G_{0,L}^{L,0}\left(z \middle| \begin{array}{c} - \\ 0,1,\dots,1 \end{array}\right)}.$$

In the following, the principle of mathematical induction will be used to show the above limit is zero. For L = 1, we have

$$\lim_{z \to 0} \frac{G_{0,1}^{1,0} \left(z \begin{vmatrix} - \\ 1 \end{matrix} \right)}{G_{0,1}^{1,0} \left(z \begin{vmatrix} - \\ 0 \end{matrix} \right)} = \lim_{z \to 0} \frac{\frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma \left(1 - s \right) z^{s} ds}{e^{-z}}$$
$$= \frac{\lim_{z \to 0} z \sum_{n=0}^{\infty} \frac{(-z)^{n}}{n!}}{\lim_{z \to 0} e^{-z}}$$
$$= \frac{0}{1} = 0.$$

In the above derivation, we used the property that the poles of $\Gamma(1-s)$ are $s = n \in \mathbb{N}$, and the corresponding residuals are $\frac{(-1)^n}{n!}$. Suppose the proposition holds for L = k, when

$$\begin{split} & L = k + 1 \text{ we have} \\ & \lim_{z \to 0} \frac{G_{0,k+1}^{k+1,0} \left(z \middle| \begin{array}{c} - \\ 1,1,\dots,1 \end{array} \right)}{G_{0,k+1}^{k+1,0} \left(z \middle| \begin{array}{c} - \\ 0,1,\dots,1 \end{array} \right)} \\ & = \lim_{z \to 0} \frac{\frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma^{k+1} \left(1 - s \right) z^{s} ds}{\frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma \left(-s \right) \Gamma^{k} \left(1 - s \right) z^{s} ds} \\ & = \lim_{z \to 0} \frac{\frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma \left(-s \right) \Gamma^{k-1} \left(1 - s \right) z^{s} \int_{0}^{\infty} t^{-s} e^{-t} dt ds}{\frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma \left(-s \right) \Gamma^{k-1} \left(1 - s \right) z^{s} \int_{0}^{\infty} t^{-s} e^{-t} dt ds} \\ & = \lim_{z \to 0} \frac{\int_{0}^{\infty} e^{-t} \frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma^{k} \left(1 - s \right) \frac{z}{t} \int_{0}^{s} ds dt}{\int_{0}^{\infty} e^{-t} \frac{1}{2\pi i} \oint_{\mathcal{L}} \Gamma \left(-s \right) \Gamma^{k-1} \left(1 - s \right) \left(\frac{z}{t} \right)^{s} ds dt} \\ & = \frac{\int_{0}^{\infty} e^{-t} \lim_{z \to 0} G_{0,k}^{k,0} \left(\frac{z}{t} \middle| \begin{array}{c} - \\ 1,1,\dots,1 \end{array} \right) dt}{\int_{0}^{\infty} e^{-t} \lim_{z \to 0} G_{0,k}^{k,0} \left(\frac{z}{t} \middle| \begin{array}{c} - \\ 0,1,\dots,1 \end{array} \right) dt} = \frac{0}{1} = 0. \end{split}$$

The integration path \mathcal{L} is $\Re(\mathcal{L}) = -\frac{1}{2}$. According to the principle of mathematical induction, the limit is zero for any $L \in \mathbb{N}$.

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