# Multiple Access Channel with Partial and Controlled Cribbing Encoders 

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#### Abstract

In this paper we consider a multiple access channel (MAC) with partial cribbing encoders. This means that each of two encoders obtains a deterministic function of the other encoder output with or without delay. The partial cribbing scheme is especially motivated by the additive noise Gaussian MAC since perfect cribbing results in the degenerated case of full cooperation between the encoders and requires an infinite entropy link. We derive a single letter characterization of the capacity of the MAC with partial cribbing for the cases of causal and strictly causal partial cribbing. Several numerical examples, such as quantized cribbing, are presented. We further consider and derive the capacity region where the cribbing depends on actions that are functions of the previous cribbed observations. In particular, we consider a scenario where the action is "to crib or not to crib" and show that a naive time-sharing strategy is not optimal.


Index Terms<br>Backward decoding, Block-Markov coding, Cribbing encoders, Cribbing with actions, Gaussian MAC, Quantized cribbing, Partial cribbing, Rate splitting, Superposition codes, "To crib or not to crib".

## I. Introduction

In his remarkable dissertation [1], Willems introduced a new problem of the multiple access channel (MAC) with cribbing encoders and derived its capacity region using a novel decoding technique called "backward decoding". Cribbing encoder refers to the case where the encoder knows perfectly the other output encoder, possibly with delay or lookahead. The work by Willems on MACs with cribbing encoders has been extended to the interference channel [2], and to state-dependent MAC [3]. However, for the Gaussian case, where the encoder output is of a continuous alphabet, the cribbing idea is not an interesting case [4] since it implies a full cooperation between the encoders regardless of the delay of the cribbing. This is due to the fact that in a single epoch time a noiseless continuous signal may transmit an infinite amount of information. Motivated by this fact, we introduce in this paper "partial cribbing", where one encoder only knows a quantized version, or, more generally, a deterministic function of the coded output of other encoder.

In this paper we consider two kinds of partial cribbing: causal and strictly-causal. Causal partial cribbing means that at time $i$ the encoder observes (and uses) the partial cribbing signal without delay, i.e., $Z_{i}$. Strictly-causal
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Fig. 1. Partial (deterministic-function) cribbing. Each encoder observes a deterministic function of the other encoder with or without delay. Encoder 1 observes the cribbing in a strictly causal way, i.e., with delay, and Encoder 2 observes the cribbing causally, i.e., without delay. The setting corresponds to Case B in this paper.
partial cribbing means that at time $i$ the encoder observes the partial cribbing with a delay, i.e., $Z_{i-1}$. We derive the capacity region for two different cases according to the causality or the strictly causality of the cribbing

Case A: The cribbing for both encoders is strictly-causal.
Case B: The cribbing for one encoder is causal and for the other encoder is strictly-causal.
The setting that is depicted in Fig. 1 is the case where one encoder has causal partial cribbing and the other strictly causal partial cribbing, namely Case B. To some extent, the partial cribbing problem is related to the semideterministic relay channel [5] which was solved using the partial decode and forward technique [6]. The partial cribbing setting has a similar structure to the semi-deterministic relay channel where Encoder 2 plays the role of relay and receives a deterministic function of the output of Encoder 1. However, the MAC with partial cribbing is different from the semi-deterministic relay in the sense that Encoder 2 has its own message to transmit in addition to its role of relaying information from Encoder 1. Another related problem is the semi-deterministic broadcast channel [7], where one of the receivers obtains a deterministic function of the input channel. In our problem Encoder 1 "is broadcasting" to Encoder 2 and to the decoder hence this part of the communication resembles the semideterministic broadcast channel. However, in our problem of partial cribbing only the decoder is actually required to decode the message error-free.

The coding scheme presented here for the partial cribbing uses the same techniques that were used for the perfect cribbing, i.e., block Markov coding, Shannon's strategies, super position coding, and backward decoding, and in addition to that, we use rate splitting in the code design. Rate splitting is needed since Encoder 2 can decode only part of the message transmitted by Encoder 1.

Recently, several problems on "action" in information theory have been considered in [8]-[11]. In these problems the side information is not freely available but depends on an action that has a cost. The solution of partial cribbing allows us to consider the case where the cribbing is action dependent. Namely, there is an action that is a function of
the previously cribbed observations and this action determines the current cribbing function. This kind of questions may be raised in cognitive communication systems where sensing other users signals is a resource with a cost. In particular we show through a simple example where the action is "to crib or not to crib" that a time-sharing action is not necessarily optimal.

The remainder of the paper is organized as follows: In Section we introduce the setting of MAC with partial cribbing and state the capacity region for strictly-causal (Case A), as well as mixed causal and strictly-causal (Case B). In sections III and IV respectively we provide the converse and achievability proofs of the capacity region for each case of partial cribbing. In Section $\square$ we consider the case where a common message, known to the encoders, needs to be transmitted to the decoder in addition to the private messages. We show that no additional auxiliary random variable is needed to characterize the capacity region since the partial cribbing is utilized via generating a common message between the users. In Section VI we consider the case where one of the encoders has no message to send; hence it becomes a special case of the semi-deterministic relay channel with and without delay. We show that indeed the region obtained via partial cribbing and the region obtained via a semi-deterministic relay channel coincide. In Section VII we consider a Gaussian MAC with quantized cribbing. We provide a simple achievable scheme and show numerically that even with a few bit quantizer we obtain an achievable region that is very close to the perfect cribbing capacity region. In Section VIII we consider a scenario where a limited-resource action controls the cribbing. In particular, we investigate an example where the action is "to crib or not to crib" and solve it analytically. In Section IX we conclude the paper and suggest some research directions that have not been yet solved such as noncausal partial cribbing, noisy cribbing and a few action related problems.

## II. Problem definition and Main Result

The MAC setting consists of two transmitters (encoders) and one receiver (decoder). Each transmitter $l \in\{1,2\}$ chooses an index $m_{l}$ uniformly from the set $\left\{1, \ldots, 2^{n R_{l}}\right\}$ and independently of the other transmitter. The input to the channel from encoder $l \in\{1,2\}$ is denoted by $\left\{X_{l, 1}, X_{l, 2}, X_{l, 3}, \ldots\right\}$. Encoder 1 and Encoder 2 obtain a deterministic function of the output of Encoder 2 and 1, respectively, of the form $Z_{2, i}=g_{2}\left(X_{2, i}\right)$, and $Z_{1, i}=g_{1}\left(X_{1, i}\right)$. The output of the channel is denoted by $\left\{Y_{1}, Y_{2}, Y_{3}, \ldots\right\}$. The channel is characterized by a conditional probability $P\left(y_{i} \mid x_{1, i}, x_{2, i}\right)$. The channel probability does not depend on the time index $i$ and is memoryless, i.e.,

$$
\begin{equation*}
P\left(y_{i} \mid x_{1}^{i}, x_{2}^{i}, y^{i-1}\right)=P\left(y_{i} \mid x_{1, i}, x_{2, i}\right), \tag{1}
\end{equation*}
$$

where the superscripts denote sequences in the following way: $x_{l}^{i}=\left(x_{l, 1}, x_{l, 2}, \ldots, x_{l, i}\right), l \in\{1,2\}$. Since the settings in this paper do not include feedback from the receiver to the transmitters, i.e., $P\left(x_{1, i}, x_{2, i} \mid x_{1}^{i-1}, x_{2}^{i-1}, y^{i-1}\right)=$ $P\left(x_{1, i}, x_{2, i} \mid x_{1}^{i-1}, x_{2}^{i-1}\right)$, Equation (1) implies that

$$
\begin{equation*}
P\left(y_{i} \mid x_{1}^{n}, x_{2}^{n}, y^{i-1}\right)=P\left(y_{i} \mid x_{1, i}, x_{2, i}\right) \tag{2}
\end{equation*}
$$

Definition 1: A $\left(2^{n R_{1}}, 2^{n R_{2}}, n\right)$ code with partial cribbing, as shown in Fig. 1 consists at time $i$ of an encoding function at Encoder 1

Case A, B, $\quad f_{1, i}:\left\{1, \ldots, 2^{n R_{1}}\right\} \times \mathcal{Z}_{2}^{i-1} \mapsto X_{1, i}$,
and an encoding function at Encoder 2 that changes according to the following case settings

$$
\begin{array}{ll}
\text { Case A } & f_{2, i}:\left\{1, \ldots, 2^{n R_{2}}\right\} \times \mathcal{Z}_{1}^{i-1} \mapsto X_{1, i}, \\
\text { Case B } & f_{2, i}:\left\{1, \ldots, 2^{n R_{2}}\right\} \times \mathcal{Z}_{1}^{i} \mapsto X_{1, i}, \tag{4}
\end{array}
$$

and a decoding function,

$$
\begin{equation*}
g: \mathcal{Y}^{n} \mapsto\left\{1, \ldots, 2^{n R_{1}}\right\} \times\left\{1, \ldots, 2^{n R_{2}}\right\} \tag{5}
\end{equation*}
$$

The average probability of error for $\left(2^{n R_{1}}, 2^{n R_{2}}, n\right)$ code is defined as

$$
\begin{equation*}
P_{e}^{(n)}=\frac{1}{2^{n\left(R_{1}+R_{2}\right)}} \sum_{m_{1}, m_{2}} \operatorname{Pr}\left\{g\left(Y^{n}\right) \neq\left(m_{1}, m_{2}\right) \mid\left(m_{1}, m_{2}\right) \text { sent }\right\} \tag{6}
\end{equation*}
$$

A rate $\left(R_{1}, R_{2}\right)$ is said to be achievable for the encoder with partial cribbing if there exists a sequence of $\left(2^{n R_{1}}, 2^{n R_{2}}, n\right)$ codes with $P_{e}^{(n)} \rightarrow 0$. The capacity region of the MAC is the closure of all achievable rates. The following theorem describes the capacity region of MAC with partial cribbing for two different cases of causality.

Let us define the following regions $\mathcal{R}_{A}, \mathcal{R}_{B}$, which are contained in $\mathbb{R}_{+}^{2}$, namely, contained in the set of nonnegative two dimensional real numbers.

$$
\mathcal{R}_{A}=\left\{\begin{array}{l}
R_{1} \leq H\left(Z_{1} \mid U\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right)  \tag{7}\\
R_{2} \leq H\left(Z_{2} \mid U\right)+I\left(X_{2} ; Y \mid X_{1}, Z_{2}, U\right) \\
R_{1}+R_{2} \leq I\left(X_{1}, X_{2} ; Y \mid U, Z_{1}, Z_{2}\right)+H\left(Z_{1}, Z_{2} \mid U\right), \\
R_{1}+R_{2} \leq I\left(X_{1}, X_{2} ; Y\right), \text { for } \\
P(u) P\left(x_{1}, z_{1} \mid u\right) P\left(x_{2}, z_{2} \mid u\right) P\left(y \mid x_{1}, x_{2}\right) .
\end{array}\right\}
$$

The region $\mathcal{R}_{B}$ is defined with the same set of inequalities as in (7), but the joint distribution is of the form

$$
\begin{equation*}
P(u) P\left(x_{1}, z_{1} \mid u\right) P\left(x_{2}, z_{2} \mid z_{1}, u\right) P\left(y \mid x_{1}, x_{2}\right) \tag{8}
\end{equation*}
$$

Theorem 1 (Capacity region): The capacity regions of the MAC with strictly-causal (Case A), mixed causal and strictly-causal (Case B) partial cribbing as described in Def. 1 are $\mathcal{R}_{A}, \mathcal{R}_{B}$, respectively.

Lemma 2: To exhaust $\mathcal{R}_{A}$ and $\mathcal{R}_{B}$ it is enough to restrict the alphabet of $U$, to satisfy

$$
\begin{equation*}
\left.|\mathcal{U}| \leq \min \left(|\mathcal{Y}|+3,\left|\mathcal{X}_{1}\right|\left|\mathcal{X}_{2}\right|+2\right)\right) \tag{9}
\end{equation*}
$$

The proof of Theorem 1 and Lemma 2 is given in the next section.

## III. CONVERSE

Here we provide the converse proof of Theorem 1 for the two cases, A and B.
Converse proof of Case A: Assume that we have a $\left(2^{n R_{1}}, 2^{n R_{2}}, n\right)$ code as in Definition 1 Case A. We will show the existence of a joint distribution $P(u) P\left(z_{1} \mid u\right) P\left(z_{2} \mid u\right) P\left(x_{1} \mid z_{1}, u\right) P\left(x_{2} \mid z_{2}, u\right) P\left(y \mid x_{1}, x_{2}\right)$ that satisfies the inequalities of (7) within some $\epsilon_{n}$, where $\epsilon_{n}$ goes to zero as $n \rightarrow \infty$. Consider

$$
n\left(R_{1}+R_{2}\right)=H\left(M_{1}, M_{2}\right)
$$

$$
\begin{align*}
& =H\left(M_{1}, M_{2}\right)+H\left(M_{1}, M_{2} \mid Y^{n}\right)-H\left(M_{1}, M_{2} \mid Y^{n}\right) \\
& \stackrel{(a)}{=} I\left(M_{1}, M_{2} ; Y^{n}\right)+n \epsilon_{n} \\
& \stackrel{(b)}{=} I\left(X_{1}^{n}, X_{2}^{n} ; Y^{n}\right)+n \epsilon_{n} \\
& = \\
& \sum_{i=1}^{n} I\left(X_{1}^{n}, X_{2}^{n} ; Y_{i} \mid Y^{i-1}\right)+n \epsilon_{n}  \tag{10}\\
& \stackrel{(c)}{\leq} \sum_{i=1}^{n} I\left(X_{1, i}, X_{2, i} ; Y_{i}\right)+n \epsilon_{n}
\end{align*}
$$

where (a) follows from Fano's inequality, (b) from the fact that $\left(X_{1}^{n}, X_{2}^{n}\right)$ is a deterministic function of $\left(M_{1}, M_{2}\right)$ and the Markov chain $Y^{n}-\left(X_{1}^{n}, X_{2}^{n}\right)-\left(M_{1}, M_{2}\right)$ and (c) from the Markov chain $Y_{i}-\left(X_{1, i}, X_{2, i}\right)-\left(X_{1}^{n}, X_{2}^{n}, Y^{i-1}\right)$. Now consider,

$$
\begin{align*}
n\left(R_{1}+R_{2}\right) & =H\left(M_{1}, M_{2}\right) \\
& \stackrel{(a)}{=} H\left(M_{1}, M_{2}, Z_{1}^{n}, Z_{2}^{n}\right) \\
& =H\left(Z_{1}^{n}, Z_{2}^{n}\right)+H\left(M_{1}, M_{2} \mid Z_{1}^{n}, Z_{2}^{n}\right) \\
& \stackrel{(b)}{=} H\left(Z_{1}^{n}, Z_{2}^{n}\right)+I\left(M_{1}, M_{2} ; Y^{n} \mid Z_{1}^{n}, Z_{2}^{n}\right)+n \epsilon_{n} \\
& =H\left(Z_{1}^{n}, Z_{2}^{n}\right)+I\left(X_{1}^{n}, X_{2}^{n} ; Y^{n} \mid Z_{1}^{n}, Z_{2}^{n}\right)+n \epsilon_{n} \\
& =\sum_{i=1}^{n} H\left(Z_{1, i}, Z_{2, i} \mid Z_{1}^{i-1}, Z_{2}^{i-1}\right)+I\left(X_{1}^{n}, X_{2}^{n} ; Y_{i} \mid Y^{i-1}, Z_{1}^{n}, Z_{2}^{n}\right)+n \epsilon_{n} \\
& \leq \sum_{i=1}^{n} H\left(Z_{1, i}, Z_{2, i} \mid Z_{1}^{i-1}, Z_{2}^{i-1}\right)+I\left(X_{1, i}, X_{2, n} ; Y_{i} \mid Z_{1}^{i}, Z_{2}^{i}\right)+n \epsilon_{n} \\
& \stackrel{(c)}{=} \sum_{i=1}^{n} H\left(Z_{1, i}, Z_{2, i} \mid U_{i}\right)+I\left(X_{1, i}, X_{2, n} ; Y_{i} \mid Z_{1, i}, Z_{2, i}, U_{i}\right)+n \epsilon_{n} \tag{11}
\end{align*}
$$

where (a) follows from the fact that $\left(Z_{1}^{n}, Z_{2}^{n}\right)$ are a deterministic function of $\left(M_{1}, M_{2}\right)$, (b) from Fano's inequality, and (c) from the following definition of a random variable

$$
\begin{equation*}
U_{i} \triangleq\left(Z_{1}^{i-1}, Z_{2}^{i-1}\right) \tag{12}
\end{equation*}
$$

Furthermore, consider

$$
\begin{aligned}
n R_{1} & =H\left(M_{1}\right) \\
& \stackrel{(a)}{=} H\left(M_{1} \mid M_{2}\right) \\
& \stackrel{(b)}{=} H\left(M_{1}, Z_{1}^{n} \mid M_{2}\right) \\
& =H\left(Z_{1}^{n} \mid M_{2}\right)+H\left(M_{1} \mid Z_{1}^{n}, M_{2}\right) \\
& =H\left(Z_{1}^{n} \mid M_{2}\right)+H\left(M_{1} \mid M_{2}, Z_{1}^{n}\right)+H\left(M_{1} \mid Y^{n}, M_{2}, Z_{1}^{n}\right)-H\left(M_{1} \mid Y^{n}, M_{2}, Z_{1}^{n}\right) \\
& \stackrel{(c)}{=} \sum_{i=1}^{n} H\left(Z_{1, i} \mid Z_{1}^{i-1}, M_{2}\right)+I\left(Y_{i} ; M_{1} \mid Y^{i-1}, M_{2}, Z_{1}^{n}\right)+n \epsilon_{n}
\end{aligned}
$$

$$
\begin{align*}
& \stackrel{(d)}{=} \sum_{i=1}^{n} H\left(Z_{1, i} \mid Z_{1}^{i-1}, Z_{2}^{i-1}, M_{2}\right)+I\left(Y_{i} ; M_{1}, X_{1, i} \mid Y^{i-1}, M_{2}, X_{2, i}, Z_{1}^{n}, Z_{2}^{n}\right)+n \epsilon_{n} \\
& \stackrel{(e)}{=} \sum_{i=1}^{n} H\left(Z_{1, i} \mid Z_{1}^{i-1}, Z_{2}^{i-1}\right)+I\left(Y_{i} ; X_{1, i} \mid X_{2, i}, Z_{1}^{i}, Z_{2}^{i}\right)+n \epsilon_{n} \\
& =\sum_{i=1}^{n} H\left(Z_{1, i} \mid U_{i}\right)+I\left(Y_{i} ; X_{1, i} \mid X_{2, i}, U_{i}, Z_{1, i}\right)+n \epsilon_{n} \tag{13}
\end{align*}
$$

where (a) follows from the fact that the messages $M_{1}$ and $M_{2}$ are independent of each other, (b) follows from the fact that $Z_{1}^{n}$ is a deterministic function of $\left(M_{1}, M_{2}\right)$, (c) follows from Fano's inequality, and (d) from the fact that $X_{1, i}$ is a deterministic functions of $\left(M_{1}, Z_{2}^{i-1}\right)$ and $X_{2, i}$ is a deterministic function of $\left(M_{2}, Z_{1}^{i-1}\right)$. Step (e) follows from the Markov chain $Y_{i}-\left(X_{1, i}, X_{2, i}, Z^{n}\right)-\left(M_{1}, M_{2}, Y^{i-1}\right)$ and the fact that conditioning reduces entropy. Similarly to (13) we obtain

$$
\begin{equation*}
n R_{2} \leq \sum_{i=1}^{n} H\left(Z_{2, i} \mid U_{i}\right)+I\left(Y_{i} ; X_{2, i} \mid X_{1, i}, U_{i}, Z_{2, i}\right)+n \epsilon_{n} \tag{14}
\end{equation*}
$$

Now let us verify that the three Markov chains $Z_{1, i}-U_{i}-Z_{2, i}, X_{1, i}-\left(U_{i}, Z_{1, i}\right)-\left(X_{2, i}\right)$, and $X_{2, i}-\left(U_{i}, Z_{2, i}\right)-\left(X_{1, i}\right)$ hold. The first Markov chain is due to the Markov $\left(M_{1}, Z_{2}^{i-1}\right)-\left(Z_{1}^{i-1}, Z_{2}^{i-1}\right)-\left(M_{2}, Z_{1}^{i-1}\right)$ or equivalently $M_{1}-$ $\left(Z_{1}^{i-1}, Z_{2}^{i-1}\right)-M_{2}$ and the second Markov chain is due to the Markov chain $\left(M_{1}, Z_{2}^{i-1}\right)-\left(Z_{1}^{i}, Z_{2}^{i-1}\right)-\left(M_{2}, Z_{1}^{i-1}\right)$ or equivalently $M_{1}-\left(Z_{1}^{i}, Z_{2}^{i-1}\right)-M_{2}$. The Markov chain follows from the joint distribution $P\left(m_{1}, m_{2}, z_{1}^{n}, z_{2}^{n}\right)=$ $P\left(m_{1}\right) P\left(m_{2}\right) \prod_{i=1}^{n} P\left(z_{1, i} \mid z_{2}^{i-1}, m_{1}\right) \prod_{i=1}^{n} P\left(z_{2, i} \mid z_{1}^{i-1}, m_{2}\right)$ and the observation that

$$
\begin{align*}
P\left(m_{1} \mid z_{1}^{n}, z_{2}^{n}, m_{2}\right) & =\frac{P\left(m_{1}\right) P\left(m_{2}\right) \prod_{i=1}^{n} P\left(z_{1, i} \mid z_{2}^{i-1}, m_{1}\right) \prod_{i=1}^{n} P\left(z_{2, i} \mid z_{1}^{i-1}, m_{2}\right)}{\left(P\left(m_{2}\right) \prod_{i=1}^{n} P\left(z_{2, i} \mid z_{1}^{i-1}, m_{2}\right)\right) \sum_{m_{1}} P\left(m_{1}\right) \prod_{i=1}^{n} P\left(z_{1, i} \mid z_{2}^{i-1}, m_{1}\right)} \\
& =\frac{\prod_{i=1}^{n} P\left(z_{1, i} \mid z_{2}^{i-1}, m_{1}\right)}{\sum_{m_{1}} P\left(m_{1}\right) \prod_{i=1}^{n} P\left(z_{1, i} \mid z_{2}^{i-1}, m_{1}\right)} \tag{15}
\end{align*}
$$

does not depend on $m_{2}$. The third Markov chain is an exchange between the indexes 1 and 2 , namely, $M_{1}, X_{1, i}, Z_{1, i}$ is exchanged with $M_{2}, X_{2, i}, Z_{2, i}$, respectively. Finally, let $Q$ be a random variable independent of $\left(X_{1}^{n}, X_{2}^{n}, Y^{n}\right)$, and uniformly distributed over the set $\{1,2,3, . ., n\}$. We define the random variables $U \triangleq\left(Q, U_{Q}\right)$ and obtain that the region given in (7) is an outer bound to any achievable rate.

Once Case A is proved, Case B follows straightforwardly using the following modification.
Converse proof for Case B: We repeat the same converse as for Case A, except that in the final step we need to show the Markov chain $X_{2, i}-\left(U_{i}, Z_{1, i}, Z_{2, i}\right)-X_{1, i}$ rather than $X_{2, i}-\left(U_{i}, Z_{2, i}\right)-X_{1, i}$ as in Case A. Since for case B the Markov chain $\left(M_{2}, Z_{1}^{i}\right)-\left(Z_{1}^{i}, Z_{2}^{i}\right)-M_{1}$ holds it follows that $X_{2, i}-\left(M_{2}, Z_{1}^{i}\right)-\left(U_{i}, Z_{i}\right)-\left(M_{1}, Z_{2}^{i-1}\right)-X_{1, i}$ holds too.

Now we prove Lemma 2 which allows us to bound the cardinality of the auxiliary random variable $U$ without decreasing the rate regions $\mathcal{R}_{A}, \mathcal{R}_{B}$.

Proof of Lemma 2. We invoke the support lemma [12] pp. 310]. The external random variable $U$ must have $|\mathcal{Y}|-1$ letters to preserve $P(y)$ plus four more to preserve the expressions $H\left(Z_{1} \mid U\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right), H\left(Z_{2} \mid U\right)+$ $I\left(X_{2} ; Y \mid X_{1}, Z_{2}, U\right), I\left(X_{1}, X_{2} ; Y \mid U, Z_{1}, Z_{2}\right)+H\left(Z_{1}, Z_{2} \mid U\right)$, and $H\left(Y \mid X_{1}, X_{2}, U,\right)$. Alternatively, the external random variable $U$ must have $\left|\mathcal{X}_{1}\right|\left|\mathcal{X}_{2}\right|-1$ letters to preserve $P\left(x_{1}, x_{2}\right)$ and three more to preserve the expressions
$H\left(Z_{1} \mid U\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right), H\left(Z_{2} \mid U\right)+I\left(X_{2} ; Y \mid X_{1}, Z_{2}, U\right), I\left(X_{1}, X_{2} ; Y \mid U, Z_{1}, Z_{2}\right)+H\left(Z_{1}, Z_{2} \mid U\right)$. Hence the cardinality of $U$ may be bounded by $\left.\min \left(|\mathcal{Y}|+3,\left|\mathcal{X}_{1}\right|\left|\mathcal{X}_{2}\right|+2\right)\right)$.

## IV. Achievability proof of Theorem 1

In this section we provide the achievability proof of Theorem 1 for the two cases, A and B. Throughout the achievability proofs in the paper we use the definition of a strong typical set. The set $T_{\epsilon}^{(n)}(X, Y, Z)$ of $\epsilon$-typical $n$-sequences is defined by $\left\{\left(x^{n}, y^{n}, z^{n}\right): \left.\frac{1}{n} N\left(x, y, z \mid x^{n}, y^{n}, z^{n}\right)-p(x, y, z) \right\rvert\, \leq \epsilon p(x, y, z) \forall(x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}\right\}$, where $N\left(x, y, z \mid x^{n}, y^{n}, z^{n}\right)$ is the number of appearances of $(x, y, z)$ in the $n$-sequnce $\left(x^{n}, y^{n}, z^{n}\right)$. Additionally, we will use the following well-known lemma [12]-[15],

Lemma 3 (Joint typicality lemma): Consider a joint distribution $P_{X, Y, Z}$ and suppose $\left(x^{n}, y^{n}\right) \in T_{\epsilon}^{(n)}(X, Y)$. Let $\tilde{Z}^{n}$ be distributed according to $\prod_{i=1}^{n} P_{Z \mid X}\left(\tilde{z}_{i} \mid x_{i}\right)$. Then,

$$
\begin{equation*}
\operatorname{Pr}\left\{\left(x^{n}, y^{n}, \tilde{Z}^{n}\right) \in T_{\epsilon}^{(n)}(X, Y, Z)\right\} \leq 2^{-n(I(Y ; Z \mid X)-\delta(\epsilon))} \tag{16}
\end{equation*}
$$

where $\lim _{\epsilon \rightarrow 0} \delta(\epsilon)=0$.
For the achievability proof, we use the rate-splitting coding technique in addition to the techniques used by Willems [16], i.e., block Markov coding, super-position coding, Shannon's strategies and backward decoding. The rate splitting technique introduces additional rate variables which are redundant and we eliminate them using the Fourier-Motzkin elimination.

Achievability Proof of Case A: Let us split rate $R_{1}$ into two rates $R_{1}^{\prime}$ and $R_{1}^{\prime \prime}$ such that $R_{1}=R_{1}^{\prime}+R_{1}^{\prime \prime}$ and similarly $R_{2}$ into $R_{2}^{\prime}$ and $R_{2}^{\prime \prime}$ such that $R_{2}=R_{2}^{\prime}+R_{2}^{\prime \prime}$. Let $m_{1}^{\prime} \in\left[1, \ldots, 2^{n R_{1}^{\prime}}\right], m_{1}^{\prime \prime} \in\left[1, \ldots, 2^{n R_{1}^{\prime \prime}}\right], m_{2}^{\prime} \in\left[1, \ldots, 2^{n R_{2}^{\prime}}\right]$, and $m_{2}^{\prime \prime} \in\left[1, \ldots, 2^{n R_{2}^{\prime \prime}}\right]$. Note that there is a one-to-one mapping between $\left(m_{1}^{\prime}, m_{1}^{\prime \prime}\right)$ and $m_{1}$ and between $\left(m_{2}^{\prime}, m_{2}^{\prime \prime}\right)$ and $m_{2}$.

Code construction: Divide a block of length $B n$ into $B$ blocks of length $n$. We use random coding to generate independently the code for each subblock $b$. Construct $2^{n\left(R_{1}^{\prime}+R_{2}^{\prime}\right)}$ codewords $u^{n}$ according to i.i.d. $\sim P(u)$. For every codeword $u^{n}$ construct $2^{n R_{1}^{\prime}}$ codewords $z_{1}^{n}$ according to i.i.d. $\sim P\left(z_{1} \mid u\right)$ and similarly $2^{n R_{2}^{\prime}}$ codewords $z_{2}^{n}$ according to i.i.d. $\sim P\left(z_{2} \mid u\right)$. Furthermore, generate $2^{n R_{1}^{\prime \prime}}$ codewords $x_{1}^{n}$ according to i.i.d. $\sim P\left(x_{1} \mid z_{1}, u\right)$ and similarly $2^{n R_{2}^{\prime \prime}}$ codewords $x_{2}^{n}$ according to i.i.d. $\sim P\left(x_{2} \mid z_{2}, u\right)$. The Markov structure of the code is

$$
\begin{align*}
& x_{1}^{n} \text { is determined by }\left(m_{1, b}^{\prime}, m_{1, b}^{\prime \prime}\right) \text { conditioned on }\left(m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right) \\
& x_{2}^{n} \text { is determined by }\left(m_{2, b}^{\prime}, m_{2, b}^{\prime \prime}\right) \text { conditioned on }\left(m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right) . \tag{17}
\end{align*}
$$

Encoder: At block $b \in[1, \ldots, B]$ encode the message $\left(m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right) \in\left[1, . ., 2^{n\left(R_{1}^{\prime}+R_{2}^{\prime}\right)}\right]$ using $u^{n}\left(m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right)$, encode $m_{1, b}^{\prime}$ conditioned on $\left(m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right)$ using $z_{1}^{n}\left(u^{n}, m_{1, b}^{\prime}\right)$, and encode $m_{1, b}^{\prime \prime}$ conditioned on $\left(m_{1, b}^{\prime}, m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right)$ using $x_{1}^{n}\left(z_{1}^{n}, u^{n}, m_{1, b}^{\prime \prime}\right)$. Similarly, encode $m_{2, b}^{\prime}$ conditioned on $\left(m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right)$ using $z_{2}^{n}\left(u^{n}, m_{1, b}^{\prime}\right)$, encode $m_{2, b}^{\prime \prime}$ conditioned on $\left(m_{2, b}^{\prime}, m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right)$ using $x_{2}^{n}\left(z_{2}^{n}, u^{n}, m_{2, b}^{\prime \prime}\right)$. We assume that $m_{1,0}^{\prime}=m_{2,0}^{\prime}=1$ and $m_{1, b}^{\prime}=m_{2, b}^{\prime}=1$ which allow a backward decoding as explained next.

Decoder: The receiver waits till the end of the block $B n$ and starts decoding each message in the sub-blocks going backwards $b \in[B, B-1, B-2, \ldots, 1]$. At block $b$, we assume that ( $m_{1, b}^{\prime}, m_{2, b}^{\prime}$ ) is already known to the
receiver from block $b+1$ and it needs to decode , $m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}, m_{2, b}^{\prime \prime}$ and $m_{2, b}^{\prime \prime}$. The decoder uses joint typicality decoding, hence at block $b$ it looks for $\left(\hat{m}_{1, b-1}^{\prime}, \hat{m}_{2, b-1}^{\prime}\right), \hat{m}_{2, B}^{\prime \prime}$ and $\hat{m}_{2, B}^{\prime \prime}$ for which

$$
\begin{equation*}
\left(u^{n}\left(\hat{m}_{1, b-1}^{\prime}, \hat{m}_{2, b-1}^{\prime}\right), z_{1}^{n}\left(u^{n}, m_{1, b}^{\prime}\right), z_{2}^{n}\left(u^{n}, m_{2, b}^{\prime}\right), x_{1}^{n}\left(z_{1}^{n}, u^{n}, \hat{m}_{1, b}^{\prime \prime}\right), x_{2}^{n}\left(z_{2}^{n}, u^{n}, \hat{m}_{2, b}^{\prime \prime}\right) \in T_{\epsilon}^{(n)}\left(U, Z_{1}, Z_{2}, X_{1}, X_{2}, Y\right)\right. \tag{18}
\end{equation*}
$$

If no such triplet, or more than one such triplet is found, an error is declared at block $b$ and therefore at the whole superblock $n B$ (we consider $\left(\hat{m}_{1, b-1}^{\prime}, \hat{m}_{2, b-1}^{\prime}\right)$ as one index in $\left[1, \ldots, 2^{n R_{1}^{\prime}+n R_{2}^{\prime}}\right]$. The estimated message at block $b$ sent from Encoder 1 is ( $\left.\hat{m}_{1, a}, \hat{m}_{1, b}\right)$, and the estimated message transmitted from Encoder 2 is $\left(\hat{m}_{2, a}, \hat{m}_{2, b}\right)$.

Error analysis: The following lemma will enable us to bound the probability of error of the super-block $n B$ by bounding the probability of error of each block.

Lemma 4: Let $\left\{A_{j}\right\}_{j=1}^{J}$ be a set of events and let $A_{j}^{c}$ denotes the complement of the event $A_{j}$. Then

$$
\begin{equation*}
P\left(\bigcup_{j=1}^{J} A_{j}\right) \leq \sum_{j=1}^{n} P\left(A_{j} \mid \bigcap_{i=1}^{j-1} A_{i}^{c}\right)=\sum_{j=1}^{n} P\left(A_{j} \mid A_{1}^{c}, A_{2}^{c}, \ldots, A_{j-1}^{c}\right) . \tag{19}
\end{equation*}
$$

Proof: For simplicity let us assume that $J=3$. In a straightforward manner the proof extends to any number of sets $J$. For any three sets of events $A_{1}, A_{2}, A_{3}$ we have

$$
\begin{align*}
P\left(A_{1} \cup A_{2} \cup A_{3}\right) & =P\left(A_{1} \cup\left(A_{2} \cap A_{1}^{c}\right) \cup\left(A_{3} \cap A_{1}^{c} \cap A_{2}^{c}\right)\right) \\
& =P\left(A_{1}\right)+P\left(A_{2} \cap A_{1}^{c}\right)+P\left(A_{3} \cap A_{1}^{c} \cap A_{2}^{c}\right) \\
& \leq P\left(A_{1}\right)+\frac{P\left(A_{2} \cap A_{1}^{c}\right)}{P\left(A_{1}^{c}\right)}+\frac{P\left(A_{3} \cap A_{1}^{c} \cap A_{2}^{c}\right)}{P\left(A_{1}^{c} \cap A_{2}^{c}\right)} \\
& =P\left(A_{1}\right)+P\left(A_{2} \mid A_{1}^{c}\right)+P\left(A_{3} \mid A_{1}^{c} \cap A_{2}^{c}\right) \\
& =P\left(A_{1}\right)+P\left(A_{2} \mid A_{1}^{c}\right)+P\left(A_{3} \mid A_{1}^{c}, A_{2}^{c}\right) \tag{20}
\end{align*}
$$

Using Lemma 4 we bound the probability of error in the supper block $B n$ by the sum of the probability of having an error in each block $b$ given that in previous blocks $(b+1, \ldots, B)$ the messages were decoded correctly.

First let us bound the probability that for some $b$, Transmitter 1 decodes the message $m_{2, b}^{\prime}$ incorrectly or Transmitter 2 decodes the message $m_{1, b}^{\prime}$ incorrectly at the end of block $b$. Using Lemma 4 it suffices to show that the probability of error-decoding in each block $b$ goes to zero, assuming that all previous messages in block $(1,2, \ldots, b-1)$ were decoded correctly.

Let $E_{1, b}$ be the event that Transmitter 1 has an error in decoding $m_{2, b}^{\prime}$ and let $E_{2, b}$ be the event that Transmitter 2 has an error in decoding $m_{1, b}^{\prime}$. The term $P\left(E_{1, b} \cup E_{2, b} \mid E_{0, b-1}^{c}\right)$ is the probability that Transmitter 1 or 2 incorrectly decoded $m_{2, b}^{\prime}$ and $m_{1, b}^{\prime}$, respectively, given that $m_{1, b-1}^{\prime}$ and $m_{2, b-1}^{\prime}$ were decoded correctly. Without loss of generality let's assume that $m_{1, b}^{\prime}=m_{2, b}^{\prime}=1$. An error occurs if and only if there is another message $m_{1, b}^{\prime}>1$ that maps to the same codeword as $z_{1}^{n}\left(1, u^{n}\right)$ or there is another message $m_{2, b}^{\prime}>1$ that maps to the same codeword as $z_{2}^{n}\left(1, u^{n}\right)$. The probability that $z_{1}^{n}\left(i, u^{n}\right)=z_{1}^{n}\left(1, u^{n}\right)$ where $i>1$ and where $\left(z_{1}^{n}\left(1, u^{n}\right), u^{n}\right) \in T_{\epsilon}^{(n)}\left(Z_{1}, U\right)$ is bounded by $2^{-n\left(H\left(Z_{1} \mid U\right)-\delta(\epsilon)\right)}$, where $\delta(\epsilon)$ goes to zero as $\epsilon$ goes to zero. Hence

$$
P\left(E_{1, b} \cup E_{2, b} \mid E_{1, b-1}^{c}, E_{2, b-1}^{c}\right) \stackrel{(a)}{\leq} P\left(E_{1, b} \mid E_{1, b-1}^{c}, E_{2, b-1}^{c}\right)+P\left(E_{2, b} \mid E_{1, b-1}^{c}, E_{2, b-1}^{c}\right)
$$

$$
\begin{align*}
& \leq \sum_{i=2}^{2^{n R_{1}^{\prime}}} 2^{-n\left(H\left(Z_{1} \mid U\right)-\delta(\epsilon)\right)}+\sum_{i=2}^{2^{n R_{2}^{\prime}}} 2^{-n\left(H\left(Z_{2} \mid U\right)-\delta(\epsilon)\right)} \\
& \leq \quad 2^{n\left(R_{1}^{\prime}-n\left(H\left(Z_{1} \mid U\right)\right)+\delta(\epsilon)\right)}+2^{n\left(R_{2}^{\prime}-n\left(H\left(Z_{2} \mid U\right)+\delta(\epsilon)\right)\right)} \tag{21}
\end{align*}
$$

where inequality (a) follows from the union bounds. Now we bound the probability that the receiver decodes the messages $\left(m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right)$, or $m_{2, b}^{\prime \prime}$ or $m_{2, b}^{\prime \prime}$ incorrectly at block $b$ given that at block $b+1$ the messages $\left(m_{1, b}^{\prime}, m_{2, b}^{\prime}\right)$ were decoded correctly and given that Transmitter 1 and 2 encodes the right messages ( $m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}$ ) in block $b$. Without loss of generality assume $\left(m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right)=1$ (for simplicity we index both messages by one index), $m_{2, b}^{\prime \prime}=1$ and $m_{2, b}^{\prime \prime}=1$. Let us define the event

$$
\begin{equation*}
E_{i, j, k, b} \triangleq\left\{\left(u^{n}(i), z_{1}^{n}\left(u^{n}, m_{1, b}^{\prime}\right), z_{2}^{n}\left(u^{n}, m_{2, b}^{\prime}\right), x_{1}^{n}\left(u^{n}, z_{1}^{n}, j\right), x_{2}^{n}\left(u^{n}, z_{2}^{n}, k\right), y^{n}\right) \in T_{\epsilon}^{(n)}\left(U, Z_{1}, Z_{2}, X_{1}, X_{2}, Y\right)\right\} \tag{22}
\end{equation*}
$$

An error occurs if either the correct codewords are not jointly typical with the received sequences, i.e., $E_{1,1,1, b}^{c}$, or there exists a different $(i, j, k) \neq(1,1,1)$ such that $E_{i, j, k, b}$ occurs. Let $P_{e, b}^{(n)}$ be the error-decoding at block $b$ given that in blocks $(b+1, \ldots, B)$ there was no error-decoding. From the union of bounds we obtain that
$P_{e, b}^{(n)} \leq \operatorname{Pr}\left(E_{1,1,1, b}^{c}\right)+\sum_{i=1, j=1, k>1} \operatorname{Pr}\left(E_{i, j, k, b}\right)+\sum_{i=1, j>1, k=1} \operatorname{Pr}\left(E_{i, j, k, b}\right)+\sum_{i=1, j>1, k>1} \operatorname{Pr}\left(E_{i, j, k, b}\right)+\sum_{i>1, j \geq 1, k \geq 1} \operatorname{Pr}\left(E_{i, j, k, b}\right)$.

Now let us show that each term in 23) goes to zero as the blocklength of the code $n$ goes to infinity.

- Upper-bounding $\operatorname{Pr}\left(E_{1,1,1}^{c}\right)$ : Since we assume that the Transmitter 1 and 2 encode the right $\left(m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right)$ and the receiver decoded the right $\left(m_{1, b}^{\prime}, m_{2, b}^{\prime}\right)$ in block $b+1$, by the LLN $\operatorname{Pr}\left(E_{1,1,1, b}^{c}\right) \rightarrow 0$.
- Upper-bounding $\sum_{i=1, j=1, k>1} \operatorname{Pr}\left(E_{i, j, k}\right)$ : The probability that $Y^{n}$, which is generated according to $P\left(y \mid x_{1}, x_{2}\right)=P\left(y \mid x_{1}, x_{2}, u, z\right)$, is jointly typical with $x_{2}^{n}$, which was generated according to $P\left(x_{2} \mid z_{2}, u\right)=$ $P\left(x_{2} \mid u, z_{1}, z_{2}, x_{1}\right)$, where $\left(x_{1}^{n}, z_{1}^{n}, z_{2}^{n}, u^{n}\right) \in T_{\epsilon}^{(n)}\left(X_{1}, Z_{1}, Z_{2}, U\right)$ is bounded by (Lemma 3)

$$
\begin{equation*}
\operatorname{Pr}\left\{\left(x_{1}^{n}, z_{1}^{n}, X_{2}^{n}, z_{2}^{n}, u^{n}, Y^{n}\right) \in T_{\epsilon}^{(n)} \mid\left(x_{1}^{n}, z_{1}^{n}, z_{2}^{n}, u^{n}\right) \in T_{\epsilon}^{(n)}\right\} \leq 2^{-n\left(I\left(X_{2} ; Y \mid X_{1}, Z_{2}, U\right)-\delta(\epsilon)\right)} \tag{24}
\end{equation*}
$$

Hence, we obtain

$$
\begin{equation*}
\sum_{i=1, j=1, k>1} \operatorname{Pr}\left(E_{i, j, k, b}\right) \leq 2^{n R_{2}^{\prime \prime}} 2^{-n\left(I\left(X_{2} ; Y \mid X_{1}, Z_{1}, Z_{2}, U\right)-\delta(\epsilon)\right)} \tag{25}
\end{equation*}
$$

- Upper-bounding $\sum_{i=1, j>1, k=1} \operatorname{Pr}\left(E_{i, j, k, b}\right)$ : Similarly, to (25) we obtain

$$
\begin{equation*}
\sum_{i=1, j>1, k=1} \operatorname{Pr}\left(E_{i, j, k, b}\right) \leq 2^{n R_{1}^{\prime \prime}} 2^{-n\left(I\left(X_{1} ; Y \mid X_{2}, Z_{1}, Z_{2}, U\right)-\delta(\epsilon)\right)} \tag{26}
\end{equation*}
$$

- Upper-bounding $\sum_{i=1, j>1, k>1} \operatorname{Pr}\left(E_{i, j, k, b}\right)$ by

$$
\begin{equation*}
\sum_{i=1, j>1, k>1} \operatorname{Pr}\left(E_{i, j, k, b}\right) \leq 2^{n\left(R_{2}^{\prime \prime}+R_{1}^{\prime \prime}\right)} 2^{-n\left(I\left(X_{2}, X_{1} ; Y \mid Z_{1}, Z_{2}, U\right)-\delta(\epsilon)\right)} \tag{27}
\end{equation*}
$$

- Upper-bounding $\sum_{i>1, j \geq 1, k \geq 1} \operatorname{Pr}\left(E_{i, j, k, b}\right)$ by

$$
\sum_{i>1, j \geq 1, k \geq 1} \operatorname{Pr}\left(E_{i, j, k, b}\right) \leq 2^{n\left(R_{1}^{\prime \prime}+R_{1}^{\prime}+R_{2}\right)} 2^{-n\left(I\left(X_{2}, X_{1}, U, Z_{1}, Z_{2} ; Y\right)-\delta(\epsilon)\right)}
$$

$$
\begin{equation*}
=2^{n\left(R_{1}+R_{2}-I\left(X_{2}, X_{1} ; Y\right)-\delta(\epsilon)\right)} \tag{28}
\end{equation*}
$$

To summarize we obtained that if $R_{1}^{\prime}=R_{1}-R_{1}^{\prime \prime}, R_{1}^{\prime \prime}, R_{2}^{\prime}=R_{2}-R_{2}^{\prime \prime}, R_{2}^{\prime \prime}$ and $R_{2}$ satisfy

$$
\begin{align*}
R_{1}-R_{1}^{\prime \prime} & \leq H\left(Z_{1} \mid U\right) \\
R_{2}-R_{2}^{\prime \prime} & \leq H\left(Z_{2} \mid U\right) \\
R_{1}^{\prime \prime} & \leq I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right) \\
R_{2}^{\prime \prime} & \leq I\left(X_{2} ; Y \mid X_{1}, Z_{2}, U\right) \\
R_{1}^{\prime \prime}+R_{2}^{\prime \prime} & \leq I\left(X_{1}, X_{2} ; Y \mid Z_{1}, Z_{2}, U\right) \\
R_{1}+R_{2} & \leq I\left(X_{2}, X_{1} ; Y\right) \tag{29}
\end{align*}
$$

then there exists a sequence of code with a probability of error that goes to zero as the block length goes to infinity. Using Fourier-Motzkin elimination [17] first for $R_{1}^{\prime \prime}$ we obtain

$$
\begin{align*}
R_{1}-H\left(Z_{1} \mid U\right) & \leq I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right) \\
R_{2}-R_{2}^{\prime \prime} & \leq H\left(Z_{2} \mid U\right) \\
R_{2}^{\prime \prime} & \leq I\left(X_{2} ; Y \mid X_{1}, Z_{2}, U\right), \\
R_{1}-H\left(Z_{1} \mid U\right)+R_{2}^{\prime \prime} & \leq I\left(X_{1}, X_{2} ; Y \mid Z_{1}, Z_{2}, U\right), \\
R_{1}+R_{2} & \leq I\left(X_{2}, X_{1} ; Y\right), \tag{30}
\end{align*}
$$

and applying Fourier-Motzkin elimination also for $R_{2}^{\prime \prime}$ we obtain

$$
\begin{align*}
R_{1}-H\left(Z_{1} \mid U\right) & \leq I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right) \\
R_{2}-H\left(Z_{2} \mid U\right) & \leq I\left(X_{2} ; Y \mid X_{1}, Z_{2}, U\right) \\
R_{1}-H\left(Z_{1} \mid U\right)+R_{2}-H\left(Z_{2} \mid U\right) & \leq I\left(X_{1}, X_{2} ; Y \mid Z_{1}, Z_{2}, U\right) \\
R_{1}+R_{2} & \leq I\left(X_{2}, X_{1} ; Y\right) \tag{31}
\end{align*}
$$

which is equivalent to the region of Case $A$ in (7).
Achievability for Case B: The achievability of case B is very similar to case A, only that the codewords of $X_{2}$ needs to be generated according to shannon strategy (or a code-trees ) rather than codewords. This is due to the fact that $Z_{1, i}$ is known causally and $X_{2}$ is generated according to a distribution $P\left(x_{2} \mid u, z_{1}\right)$.

## V. COMMON MESSAGE

Let us now consider the case where a common message, $m_{0} \in\left\{1,2, \ldots, 2^{n R_{0}}\right\}$, is known to encoders 1 and 2 and needs to be transmitted to the decoder in addition to the private messages $m_{1}, m_{2}$. Hence Encoder 1 is given by the function

$$
\begin{equation*}
\text { Case A, B, } \quad f_{1, i}:\left\{1, \ldots, 2^{n R_{0}}\right\} \times\left\{1, \ldots, 2^{n R_{1}}\right\} \times \mathcal{Z}_{2}^{i-1} \mapsto X_{1, i} \tag{32}
\end{equation*}
$$

and Encoder 2 is given by the functions

$$
\text { Case A, } \quad f_{2, i}:\left\{1, \ldots, 2^{n R_{0}}\right\} \times\left\{1, \ldots, 2^{n R_{2}}\right\} \times \mathcal{Z}_{1}^{i-1} \mapsto X_{1, i},
$$

$$
\begin{equation*}
\text { Case B, } \quad f_{2, i}:\left\{1, \ldots, 2^{n R_{0}}\right\} \times\left\{1, \ldots, 2^{n R_{2}}\right\} \times \mathcal{Z}_{1}^{i} \mapsto X_{1, i} \tag{33}
\end{equation*}
$$

Remarkably, no additional auxiliary random variable is needed to characterizes the capacity region, since the partial cribbing is used for generating a common message. Let the rate regions $\mathcal{R}_{A}^{0}$ and $\mathcal{R}_{B}^{0}$ be defined exactly as $\mathcal{R}_{A}$ and $\mathcal{R}_{B}$ only that the last inequality in (7), i.e., $R_{1}+R_{2} \leq I\left(X_{1}, X_{2} ; Y\right)$, is replaced by

$$
\begin{equation*}
R_{0}+R_{1}+R_{2} \leq I\left(X_{1}, X_{2} ; Y\right) \tag{34}
\end{equation*}
$$

Theorem 5 (Capacity region in the case of a common message): The capacity regions of the MAC with strictlycausal (Case A), and mixed causal and strictly-causal (Case B) partial cribbing with a common message are $\mathcal{R}_{A}^{0}$ and $\mathcal{R}_{B}^{0}$, respectively.

Note that if there is no cribbing, i.e., $Z_{1}$ and $Z_{2}$ are constant, we obtain the capacity region of the MAC with a common message as derived by Slepian and Wolf [18]. We sketch here only the differences between the proof of Theorem 5 and Theorem 1

Proof of Theorem 5
Converse: Similar to the sequence of inequalities in we have

$$
\begin{align*}
n\left(R_{0}+R_{1}+R_{2}\right) & =H\left(M_{0}, M_{1}, M_{2}\right) \\
& \leq \sum_{i=1}^{n} I\left(X_{1, i}, X_{2, i} ; Y_{i}\right)+n \epsilon_{n} \tag{35}
\end{align*}
$$

Adding conditioning on $M_{0}$ in the sequence of inequalities we obtain

$$
\begin{align*}
n\left(R_{1}+R_{2}\right) & =H\left(M_{1}, M_{2}\right) \\
& =H\left(M_{1}, M_{2} \mid M_{0}\right) \\
& \leq \sum_{i=1}^{n} H\left(Z_{1, i}, Z_{2, i} \mid Z_{1}^{i-1}, Z_{2}^{i-1}, M_{0}\right)+I\left(X_{1, i}, X_{2, n} ; Y_{i} \mid Z_{1}^{i}, Z_{2}^{i}, M_{0}\right)+n \epsilon_{n} \\
& =\sum_{i=1}^{n} H\left(Z_{1, i}, Z_{2, i} \mid U_{i}\right)+I\left(X_{1, i}, X_{2, n} ; Y_{i} \mid Z_{1, i}, Z_{2, i}, U_{i}\right)+n \epsilon_{n} \tag{36}
\end{align*}
$$

where the last step is due to the new definition of $U_{i}$ as

$$
\begin{equation*}
U_{i} \triangleq\left(M_{0}, Z_{1}^{i-1}, Z_{2}^{i-1}\right) \tag{37}
\end{equation*}
$$

Similarly, adding conditioning on $M_{0}$ in the sequence of inequalities (13) we obtain

$$
\begin{aligned}
n R_{1} & =H\left(M_{1}\right) \\
& =H\left(M_{1} \mid M_{2}, M_{0}\right) \\
& \leq \sum_{i=1}^{n} H\left(Z_{1, i} \mid Z_{1}^{i-1}, Z_{2}^{i-1}, M_{0}\right)+I\left(Y_{i} ; X_{1, i} \mid X_{2, i}, Z_{1}^{i}, Z_{2}^{i}, M_{0}\right)+n \epsilon_{n}
\end{aligned}
$$

$$
\begin{equation*}
=\sum_{i=1}^{n} H\left(Z_{1, i} \mid U_{i}\right)+I\left(Y_{i} ; X_{1, i} \mid X_{2, i}, U_{i}, Z_{1, i}\right)+n \epsilon_{n} \tag{38}
\end{equation*}
$$

In a similar way, we obtain the inequality for $R_{2}$ as in (14).
Achievability: The achievability proof is similar to that in Theorem 1 except that we generate $2^{n\left(R_{1}^{\prime}+R_{2}^{\prime}+R_{0}\right)}$ codewords $u^{n}$ according to i.i.d. $\sim P(u)$, rather than $2^{n\left(R_{1}^{\prime}+R_{2}^{\prime}\right)}$, and wherever we have in the achievability proof of Theorem $1\left(m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right)$ we should now have $\left(m_{0}, m_{1, b-1}^{\prime}, m_{2, b-1}^{\prime}\right)$. Hence we obtain the same sequence of inequalities as in 29) except that in the last inequality which corresponds to an error in all messages we have

$$
\begin{equation*}
R_{0}+R_{1}+R_{2} \leq I\left(X_{2}, X_{1} ; Y\right) \tag{39}
\end{equation*}
$$

## VI. Special case of partial cribbing: Semi-Deterministic relay channel

As a special case of the partial cribbing encoders, let us consider the case where Encoder 2 has no message to send, i.e., $R_{2}=0$, and only Encoder 2 cribs from Encoder 1, i.e., $Z_{2}$ is a constant. We show here that indeed the region obtained via partial cribbing when $R_{2}=0$ and the region obtained via semi-deterministic relay channel coincide.

Case $A$, semi-deterministic relay with a delay: This case become a special case of the semi-deterministic relay channel which was introduced and solved by El-Gamal [5], where Encoder 2 plays the role of the relay. In such a case the region $\mathcal{R}_{A}$ becomes

$$
\mathcal{R}_{A}=\left\{\begin{array}{l}
R_{1} \leq H\left(Z_{1} \mid U\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right)  \tag{40}\\
R_{1} \leq I\left(X_{1}, X_{2} ; Y \mid U, Z_{1}\right)+H\left(Z_{1} \mid U\right) \\
R_{1} \leq I\left(X_{1}, X_{2} ; Y\right), \text { for } \\
P(u) P\left(z_{1} \mid u\right) P\left(x_{1} \mid z_{1}, u\right) P\left(x_{2} \mid u\right) P\left(y \mid x_{1}, x_{2}\right)
\end{array}\right\}
$$

Clearly, $H\left(Z_{1} \mid U\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right) \leq I\left(X_{1}, X_{2} ; Y \mid U, Z_{1}\right)+H\left(Z_{1} \mid U\right)$ hence the region we obtained is $R_{1} \leq$ $\min \left(H\left(Z_{1} \mid U\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right), I\left(X_{1}, X_{2} ; Y\right)\right)$ for some $P(u) P\left(z_{1} \mid u\right) P\left(x_{1} \mid z_{1}, u\right) P\left(x_{2} \mid u\right)$. Now consider

$$
\begin{align*}
H\left(Z_{1} \mid U\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right) & \stackrel{(a)}{=} H\left(Z_{1} \mid U, X_{2}\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right) \\
& \stackrel{(b)}{\leq} H\left(Z_{1} \mid X_{2}\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}\right) \tag{41}
\end{align*}
$$

where step (a) follows from the Markov chain $X_{2}-U-Z_{1}$ and step (b) from the fact that conditioning reduces entropy and from the Markov chain $Y-\left(X_{1}, Z_{1}, X_{2}\right)-U$. By choosing $U=X_{2}$ we obtain the upper bound of (41) and the expression $I\left(X_{1}, X_{2} ; Y\right)$ does not decrease. Hence the capacity region is

$$
\begin{equation*}
R_{1} \leq \min \left(H\left(Z_{1} \mid X_{2}\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}\right), I\left(X_{1}, X_{2} ; Y\right)\right) \tag{42}
\end{equation*}
$$

for some $P\left(x_{1}, x_{2}\right)$. Eq. 42) coincides with the result in [5].
Case B, semi-deterministic relay without delay: In this case $\mathcal{R}_{B}$ become the set of rates $R_{1}$ that satisfies

$$
\begin{equation*}
R_{1} \leq \min \left(H\left(Z_{1} \mid U\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right), I\left(X_{1}, X_{2} ; Y\right)\right) \tag{43}
\end{equation*}
$$

for some $P\left(x_{1}, z_{1}, u\right) P\left(x_{2} \mid u, z_{1}\right)$. The case of relays without delay was investigated by El-Gamal et. al. in [19] where it was shown that the capacity region for the semi-deterministic relay without delay which is denoted by $C_{0, \text { semi-det }}$ is

$$
\begin{equation*}
C_{0, \text { semi-det }}=\max _{P\left(u, x_{1}\right), x_{2}=f\left(u, z_{1}\right)} \min \left(I\left(X_{1} ; Y, Z_{1} \mid U\right), I\left(U, X_{1} ; Y\right)\right) \tag{44}
\end{equation*}
$$

At first glance, the expressions in (43) seem to be different from the expression in (44), but with some simple manipulations one can show that the expression are equivalent. In particular, the first term in (44) may be written as

$$
\begin{align*}
I\left(X_{1} ; Y, Z_{1} \mid U\right) & =I\left(X_{1} ; Z_{1} \mid U\right)+I\left(X_{1} ; Y \mid U, Z_{1}\right) \\
& \stackrel{(a)}{=} H\left(Z_{1} \mid U\right)+I\left(X_{1} ; Y \mid U, Z_{1}\right) \\
& \stackrel{(b)}{=} H\left(Z_{1} \mid U\right)+I\left(X_{1} ; Y \mid U, Z_{1}, X_{2}\right) \tag{45}
\end{align*}
$$

where step (a) follows from the fact that $Z_{1}$ is a function of $X_{1}$ and step (b) from the fact that $X_{2}$ is a function of $\left(U, Z_{1}\right)$. The second term in (44) may be written as

$$
\begin{align*}
I\left(U, X_{1} ; Y\right) & \stackrel{(a)}{=} I\left(U, X_{1}, X_{2} ; Y\right) \\
& \stackrel{(b)}{=} I\left(X_{1}, X_{2} ; Y\right) \tag{46}
\end{align*}
$$

where step (a) follows from the fact that $X_{2}$ is a function of $\left(U, X_{1}\right)$ and step (b) follows from the Markov chain $Y-\left(X_{1}, X_{2}\right)-U$. Now, to conclude that (43) and (44) are equivalent we need to show that it suffices to consider only distributions where $X_{2}$ is a function of $\left(U, Z_{1}\right)$ in (43). It follows from [20, Lemma 1] that there exists a random variable $W$ independent of $\left(U, Z_{1}\right)$ and satisfies $W-\left(X_{2}, U, Z_{1}\right)-\left(Y, X_{1}\right)$ such that $X_{2}$ is a deterministic function of $\left(U, Z_{1}, W\right)$. Therefore

$$
\begin{align*}
H\left(Z_{1} \mid U\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U\right) & =H\left(Z_{1} \mid U, W\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U, W\right) \\
& =H\left(Z_{1} \mid \tilde{U}\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, \tilde{U}\right) \tag{47}
\end{align*}
$$

where $\tilde{U}=(U, W)$. Hence it suffices to consider $X_{2}$ that is a function of $\left(\tilde{U}, Z_{1}\right)$ and it emerges that (43) is equivalent to 44).

## VII. Gaussian MAC with quantized cribbing

In this section we consider the additive Gaussian noise MAC, i.e., $Y=X_{1}+X_{2}+W$, where $W$ is a memoryless Gaussian noise with variance $N$, i.e., $W \sim \operatorname{Norm}(0, N)$. We assume a power constraints $P_{1}$ and $P_{2}$ on the inputs from Encoder 1 and Encoder 2, respectively. If the encoders do not cooperate than the capacity is given by

$$
\begin{align*}
R_{1} & \leq \frac{1}{2} \log \left(1+\frac{P_{1}}{N}\right) \\
R_{2} & \leq \frac{1}{2} \log \left(1+\frac{P_{2}}{N}\right) \\
R_{1}+R_{2} & \leq \frac{1}{2} \log \left(1+\frac{P_{1}+P_{2}}{N}\right) . \tag{48}
\end{align*}
$$

If there is perfect cribbing from Encoder 1 to Encoder 2, either with delay or without the capacity is the same as


Fig. 2. Gaussian MAC with quantized cribbing. The cribbing that Encoder 2 observes is the quantized signal from Encoder 1. There exist power constraints $\sum_{i=1}^{n} E\left[X_{1, i}^{2}\right] \leq P_{1}$ and $\sum_{i=1}^{n} E\left[X_{1, i}^{2}\right] \leq P_{2}$.
if Encoder 2 knows the message of Encoder 1 since Encoder 1 can send the message in one epoch time. Hence, the capacity is the union over $0 \leq \rho \leq 1$ of the regions

$$
\begin{align*}
R_{2} & \left.\leq \frac{1}{2} \log \left(1+\frac{P_{2}}{N}\left(1-\rho^{2}\right)\right)\right) \\
R_{1}+R_{2} & \leq \frac{1}{2} \log \left(1+\frac{P_{1}+2 \rho \sqrt{P_{1} P_{2}}+P_{2}}{N}\right) . \tag{49}
\end{align*}
$$

Now, let us consider the case where Encoder 2 observes a quantized version of the signal from Encoder 1 without delay. The setting is depicted in Fig. 2] We assume that the quantizer is a scalar quantizer designed such that under a Gaussain input with variance $P_{1}=1$ the discrete values have the same probability (see Fig. 3 for an example of 2-bit quantizer).


Fig. 3. The 2-bit quantizer's boundaries are designed such that if the input signal has a normal distribution with variance $P_{1}=1$ the output values from the quantizer have equal probability. The input to the 2-bit quantizer is $X_{1}$ and the output is $Q \in\{1,2,3,4\}$.

Next, we consider a simple achievable scheme for the Gaussian MAC with a quantizer cribbing without delay,
where the power constraints are $P_{1}=P_{2}=1$ and the noise variance is $N=\frac{1}{2}$. We evaluate the region $\mathcal{R}_{B}$ given by (77) and (8) for the case where $U_{1}, U_{2}, Z_{2}$ are constants, $X_{1} \sim N(0,1), Z_{1}$ is a quantized version of $X_{1}$ such that each value has equal probability. The input distribution is $P_{X_{2} \mid Z_{1}}\left(x_{2} \mid z_{1}\right)=\rho P_{V}\left(x_{2}\right)+(1-\rho) P_{X_{1} \mid Z_{1}}\left(x_{2} \mid z_{1}\right)$, where


Fig. 4. Achievable regions of Gaussian MAC with a quantizer cribbing.
$V \sim N(0,1)$ and is independent of $X_{1}$ and $Z$. Note that under these assumptions $X_{2} \sim N(0,1)$ and therefore satisfies the power constraint. Fig. 4 depicts the simple achievable scheme for different quantizers. The blue line in Fig. 4 is the capacity region where there is no cribbing, evaluated according to 48). The red line is the capacity region where there is perfect cribbing, evaluated according to 49). The lines in between are achievable regions according to the simple scheme we have described above. One can see that the main gain is already due to 1-bit quantizer and that the difference between the achievable scheme with a 4-bit quantizer and the capacity region where there is perfect cribbing is negligible.

## VIII. CONTROLLED CRIBBING

Here we consider the case where the cribbing is controlled by an action which depends on previously cribbed signals. In this study, only Encoder 2 cribs causally or strictly causally. More precisely, at time $i$ there is a controller which takes action $a_{1, i}$ and the cribbed signals from Encoder 1 to Encoder 2 at time $i$ is $z_{1, i}=f\left(x_{1, i}, a_{1, i}\right)$ as shown in Fig. [5] The action at time $i$ depends on past cribbed observation, i.e., $a_{1, i}\left(z_{1}^{i-1}\right)$ and the action is a limited resource, namely, there is a restriction that $\frac{1}{n} \sum_{i=1}^{n} E\left[\Lambda\left(A_{1, i}\right)\right] \leq \Gamma$, where $\Lambda\left(a_{1}\right)$ is a cost of taking action $a_{1}$.

Let us now formally define a controlled code.
Definition 2: A $\left(2^{n R_{1}}, 2^{n R_{2}}, n\right)$ code with controlled partial cribbing, as shown in Fig. 5] consists at time $i$ of an encoding function at Encoder 1

$$
\begin{equation*}
\text { Case A, B, } \quad f_{1, i}:\left\{1, \ldots, 2^{n R_{1}}\right\} \mapsto X_{1, i} \tag{50}
\end{equation*}
$$

and an encoding function at Encoder 2 that changes according to the following case settings
Case A $\quad f_{2, i}:\left\{1, \ldots, 2^{n R_{2}}\right\} \times \mathcal{Z}_{1}^{i-1} \mapsto X_{1, i}$,


Fig. 5. Partial cribbing with actions. The action at time $i$ is $a_{1, i}$ and is determined by previous cribbed observations i.e., $z_{1}^{i-1}$. The cribbed signal $z_{1, i}$ from Encoder 1 to Encoder 2 is given by the deterministic function $z_{1, i}=g_{1}\left(a_{1, i}, x_{1, i}\right)$. There exists a constraint on the actions of the form $\frac{1}{n} \sum_{i=1}^{n} E\left[\Lambda\left(A_{1, i}\right)\right] \leq \Gamma$.

$$
\begin{equation*}
\text { Case B } \quad f_{2, i}:\left\{1, \ldots, 2^{n R_{2}}\right\} \times \mathcal{Z}_{1}^{i} \mapsto X_{1, i}, \tag{51}
\end{equation*}
$$

and a controlled action

$$
\begin{equation*}
g_{1, i}: \mathcal{Z}_{1}^{i-1} \mapsto A_{1, i} \tag{52}
\end{equation*}
$$

and a decoding function,

$$
\begin{equation*}
h: \mathcal{Y}^{n} \mapsto\left\{1, \ldots, 2^{n R_{1}}\right\} \times\left\{1, \ldots, 2^{n R_{2}}\right\} \tag{53}
\end{equation*}
$$

The code needs to satisfy the constraint $\frac{1}{n} \sum_{i=1}^{n} E\left[\Lambda_{1}\left(A_{1, i}\right)\right] \leq \Gamma_{1}$. The probability of error, achievable pair-rates and capacity region are defined in the usual way for MAC as presented in Def. 1 .

Let us now define the following regions $\mathcal{R}_{A}^{a}, \mathcal{R}_{B}^{a}$, which are contained in $\mathbb{R}_{+}^{2}$, namely, contained in the set of non negative two dimensional real numbers.

$$
\mathcal{R}_{A}^{a}=\left\{\begin{array}{l}
R_{1} \leq H\left(Z_{1} \mid U, A_{1}\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, U, A_{1}\right)  \tag{54}\\
R_{2} \leq I\left(X_{2} ; Y \mid X_{1}, U, A_{1}\right) \\
R_{1}+R_{2} \leq I\left(X_{1}, X_{2} ; Y \mid U, A_{1}, Z_{1}\right)+H\left(Z_{1} \mid U, A_{1}\right) \\
R_{1}+R_{2} \leq I\left(X_{1}, X_{2} ; Y\right), \text { for } \\
P\left(u, a_{1}\right) P\left(x_{1}, z_{1} \mid u, a_{1}\right) P\left(x_{2} \mid u, a_{1}\right) P\left(y \mid x_{1}, x_{2}\right) \text { s.t. } E\left[\Lambda_{1}\left(A_{1}\right)\right] \leq \Gamma_{1} .
\end{array}\right\}
$$

The region $\mathcal{R}_{B}^{a}$ is defined with the same set of inequalities as in (54), but the joint distribution is of the form

$$
\begin{equation*}
P\left(u, a_{1}\right) P\left(x_{1}, z_{1} \mid u, a_{1}\right) P\left(x_{2} \mid z_{1}, u, a_{1}\right) P\left(y \mid x_{1}, x_{2}\right) \text { s.t. } E\left[\Lambda\left(A_{1}\right)\right] \leq \Gamma . \tag{55}
\end{equation*}
$$

Theorem 6 (Capacity region): The capacity regions of the MAC with actions and with strictly-causal (Case A), and mixed causal and strictly-causal (Case B), as described in Def. 2] are $\mathcal{R}_{A}^{a}$, and $\mathcal{R}_{B}^{a}$, respectively.

The proof is based on minor modification of the proof of the capacity region of the MAC with partial cribbing presented in Theorem 1

## Proof:

Achievability: Consider the achievability proof of Theorem 1 and just replace $U_{i}$ by the pair $\left(U_{i}, A_{1, i}\right)$. Note that the proof holds since at the end of block $b$ the controller is able to decode $m_{1, b}^{\prime}$.

Converse: Consider the converse proof of Theorem 1 and just replace $U_{i}$. Note that $U_{i} \triangleq Z_{1}^{i-1}$. since $A_{i}$ is a function of $Z_{1}^{i-1}$ its also a function of $U_{i}$ and by replacing $U_{i}$ by $U_{i}, A_{1, i}$ we obtain the converse proof

Example 1 (Deterministic Relay with actions): Consider the case where only Encoder 1 has a message to transmit and Encoder 2 has no message of its own to transmit, but helps to increase the rate of Encoder 1. Encoder 2, which plays the role of a relay, takes an action $A_{i}$ that is a function of the observed signal up to time $i-1$, i.e., $Z^{i-1}$. If $A_{i}=1$, then $Z_{i}=X_{i}$, and otherwise $Z_{i}$ is a constant. The cribbing signal $Z_{i}$ is observed at Encoder 2 with a delay. There exists a constraint that $\frac{1}{n} \sum_{i=1}^{n} E\left[A_{i}\right] \leq \Gamma$. In addition, Encoder 2 transmits a signal $X_{2, i}$ through the channel at time $i$, where $X_{2, i}$ is a function of $Z^{i-1}$. The output channel $Y$ is randomly chosen with equal probability to be either $X_{1}$ or $X_{2}$. This example is illustrated in Fig. 6 and is a special case of the setting presented in Fig. 5


Fig. 6. An example of deterministic cribbing with actions. The relay (Encoder 2) take an action $A_{i}$ at time $i$ that depends on previous cribbing, i.s., $Z^{i-1}$. The cribbing signal $Z_{i}$ equals to $X_{1, i}$ if $A_{i}=1$ and is constant otherwise. The cribbing is a limited resource hence there exists a constraint that on the portion of time that Encoder 2 can crib the signal from Encoder 1, namely, $\frac{1}{n} \sum_{i=1}^{n} E\left[A_{i}\right] \leq \Gamma$. The output channel $Y$ is randomly chosen with equal probability to be either $X_{1}$ or $X_{2}$

The next lemma establishes the capacity region of a deterministic relay with actions which is a special case of the cribbing with actions.

Lemma 7: The capacity region of partial deterministic cribbing with actions where only Encoder 1 sends a message, i.e., $R_{2}=0$ and there exists a delay in the cribbing (Case A ) is

$$
\begin{equation*}
R_{1}=\max _{P_{X_{1}, X_{2}, A: E[c(A)] \leq \Gamma}} \min \left\{H\left(Z \mid X_{2}, A\right)+I\left(X_{1} ; Y \mid X_{2}, Z_{1}, A\right), I\left(Y ; X_{1}, X_{2}\right)\right\} \tag{56}
\end{equation*}
$$

If there is no delay in the cribbing (Case B), i.e., $X_{2, i}\left(Z^{i}\right)$, then

$$
\begin{equation*}
R_{1}=\max _{P_{U, X_{1}, A} P_{X_{2} \mid U, Z, A}: E[c(A)] \leq \Gamma} \min \left\{H(Z \mid U, A)+I\left(X_{1} ; Y \mid X_{2}, Z, U, A\right), I\left(Y ; X_{1}, X_{2}\right)\right\} \tag{57}
\end{equation*}
$$

Proof: Since $R_{2}=0$ follows from (54) that

$$
\begin{equation*}
R_{1} \leq \max _{\mathcal{P}} \min \left\{H(Z \mid U, A)+I\left(X_{1} ; Y \mid X_{2}, Z, U, A\right), I\left(X_{1}, X_{2} ; Y\right)\right\} \tag{58}
\end{equation*}
$$

For the case where there is a delay in the cribbing (case A) the set of joint distributions $\mathcal{P}$ is of the form $P\left(u, a, x_{1}\right) P\left(x_{2} \mid u, a\right) P\left(y \mid x_{1}, x_{2}\right)$ and $Z$ is a function of $A$ and $X$. Using mathematical manipulation on the first term in the minimum in we obtain

$$
\begin{align*}
R_{1} & \stackrel{(a)}{\leq} H\left(Z \mid U, A, X_{2}\right)+I\left(X_{1} ; Y \mid X_{2}, Z, U, A\right) \\
& \stackrel{(b)}{\leq} H\left(Z \mid A, X_{2}\right)+I\left(X_{1} ; Y \mid X_{2}, Z, A\right) \tag{59}
\end{align*}
$$

where step (a) follows from the Markov chain $X_{2}-(U, A)-X_{1}-Z$ and step (b) from the fact that conditioning reduces entropy and the Markov chain $Y-\left(X_{1}, X_{2}, Z_{A}\right)-U$. By choosing $U=X_{2}$ the first term of 58) become the upper bound in (59); hence (56) is the capacity region.

In the case that there is no delay in the cribbing the capacity region is simply (58) where the set of joint distribution $\mathcal{P}$ is of the form $P\left(u, a, x_{1}\right) P\left(x_{2} \mid u, a, z\right) P\left(y \mid x_{1}, x_{2}\right)$ and $z$ is a deterministic function of $a$ and $x$.

For the case of delay in the cribbing, the action $A_{i}$ can be seen as part of the output signal from Encoder 2 to the channel, and indeed by replacing $X_{2}$ in (42) with $\left(X_{2}, A\right)$, we obtain (56). However, in the case of no delay in the cribbing i.e., $X_{2}\left(Z^{i}\right)$, the replacement of $X_{2}$ is not possible since the action must have a delay i.e., $A_{i}\left(Z^{i-1}\right)$.

For obtaining a numerical solution when there is a delay in the cribbing, namely, evaluating (56) for the example in Fig. 6 we can assume without loss of optimality that

$$
\begin{align*}
\operatorname{Pr}(A=1) & =\Gamma \\
\operatorname{Pr}\left(X_{1}=X_{2} \mid A=0\right) & =\alpha_{0} \\
\operatorname{Pr}\left(X_{1}=X_{2} \mid A=1\right) & =\alpha_{1} . \tag{60}
\end{align*}
$$

The reason one can assume that $\operatorname{Pr}(A=1)=\Gamma$ is because if this is not the case, and one has a code where the portion of $\operatorname{Pr}(A=1)$ is smaller than $\Gamma$, then one can add actions $A=1$ for some portion of time without decreasing the performance of the code. Furthermore, since the channel is symmetric with respect to 0 and 1 (by exchanging 0 and 1 for the inputs to the channels the performance of the code remains the same) only the probability $\operatorname{Pr}\left(X_{1}=X_{2}\right)$ is important. Furthermore, from the same reasons one can also assume that $P\left(x_{1}\right)$ and $P\left(x_{2}\right)$ are Bernoulli $\left(\frac{1}{2}\right)$ without loss of optimality. Now we shall compute the terms in (56)

$$
\begin{align*}
I\left(Y ; X_{1}, X_{2}\right) & =H(Y)-H\left(Y \mid X_{1}, X_{2}\right) \\
& =1-\Gamma+\alpha_{1} \Gamma-(1-\Gamma)\left(1-\alpha_{0}\right) \\
& =\alpha_{1} \Gamma+\alpha_{0}(1-\Gamma)  \tag{61}\\
H\left(Z \mid X_{2}, A\right) & =\Gamma H_{b}\left(\alpha_{1}\right) \tag{62}
\end{align*}
$$

$$
\begin{align*}
I\left(X_{1} ; Y \mid X_{2}, Z, A\right) & =H\left(Y \mid X_{2}, Z, A\right)-H\left(Y \mid X_{1}, X_{2}, A\right) \\
& \stackrel{(a)}{=} \Gamma\left(1-\alpha_{1}\right)+(1-\Gamma) H_{b}\left(\frac{1+\alpha_{0}}{2}\right)-\Gamma\left(1-\alpha_{1}\right)-(1-\Gamma)\left(1-\alpha_{0}\right) \\
& =(1-\Gamma)\left(H_{b}\left(\frac{1+\alpha_{0}}{2}\right)+\alpha_{0}-1\right), \tag{63}
\end{align*}
$$

where step (a) in (63) is due to the fact that $\operatorname{Pr}\left(Y=X_{2} \mid X_{2}, a=0\right)=\alpha_{0}+\frac{1-\alpha_{0}}{2}$ and therefore $H\left(Y \mid X_{2}, Z, A\right)=$ $\Gamma\left(1-\alpha_{1}\right)+(1-\Gamma) H_{b}\left(\frac{1+\alpha_{0}}{2}\right)$ where $H_{b}(p)$ is the binary entropy, i.e., $-p \log p-(1-p) \log (1-p)$ for $0 \leq p \leq 1$. Hence the capacity of the setting in Fig. 6 as a function on the constrain of the action $\Gamma$ is

$$
\begin{equation*}
C(\Gamma)=\max _{0 \leq \alpha_{0}, \alpha_{1} \leq 1} \min \left(\Gamma H_{b}\left(\alpha_{1}\right)+(1-\Gamma)\left(H_{b}\left(\frac{1+\alpha_{0}}{2}\right)+\alpha_{0}-1\right), \alpha_{1} \Gamma+\alpha_{0}(1-\Gamma)\right) \tag{64}
\end{equation*}
$$

The capacity $C(\Gamma)$ is depicted in Fig. 7 and can be found simply using a grid-search on $0 \leq \alpha_{0}, \alpha_{1} \leq 1$ or by


Fig. 7. Capacity of setting in Fig. 6as a function of the action constraint $\Gamma$. For the case where $\Gamma=0$ the capacity can be solved analytically since it is the capacity of the $Z$ channel. The capacity where $\Gamma=1$ is the simple expression $\max _{\alpha_{1}} \min \left(\alpha_{1}, H_{b}\left(\alpha_{1}\right)\right)$ which can be solved numerically by solving $\alpha=H_{b}(\alpha)$.
convex optimization tools. In the case that $\Gamma=0, X_{2, i}$ is independent of the message $m_{1}$ and therefore we obtain that at any time $i$ the channel from Encoder 1 to the output behaves as a $Z$-channel if $X_{2, i}=0$ and as an $S$ channel if $X_{2, i}=1$ and the capacity of those two channels are $H_{b}\left(\frac{1}{5}\right)-\frac{2}{5}$, and therefore $C(0)=H_{b}\left(\frac{1}{5}\right)-\frac{2}{5}$. For the case that $\Gamma=1$ we obtain from (64) that $C(1)=\max _{\alpha_{1}} \min \left(\alpha_{1}, H_{b}\left(\alpha_{1}\right)\right)$. The $\alpha$ that maximizes the expression of $C(1)$ is the one that solves the equation $\alpha_{1}=H_{b}\left(\alpha_{1}\right)$.

## IX. CONCLUSIONS AND FURTHER RESEARCH DIRECTIONS

We have considered the problem of MACs with partial cribbing encoders, namely, in a two encoder MAC the observed cribbed signal is a deterministic function of the other encoder output. We have characterized the capacity region for the two cases where the partial cribbing is causal or strictly causal. Rate splitting is the main additional
technique used in the achievability proof over the techniques used for perfect cribbing. The extension of perfect cribbing to partial cribbing resemble to the extension of the decode-and-forward technique for the relay to the partial-decode-and-forward technique [15]. The method we used for partial cribbing may be also used for noisy cribbing, although in general the capacity region of noisy cribbing is an open question. Another question that has not been solved yet is the non causal partial cribbing. For the perfect cribbing case Willems [16] solved the noncausal case simply by showing that causal and non-causal perfect cribbing results in the same capacity region.

Solving the partial cribbing setting allowed us to solve an action dependent cribbing problem. In this paper we considered the case where the action is only a function of the previously observed cribbing. However, the case in Fig. 5] where the action is a function of the previously observed cribbing and the message of the cribbing encoder, i.e., $a_{1, i}\left(z_{1}^{i-1}, m_{1}\right)$ is yet to be solved. Issues of this nature may be raised in the sphere of cognitive communication systems where sensing other users' signals is a resource with a cost.

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## REFERENCES

[1] F. M. J. Willems. Information-Theoretical Results for the Discrete Memoryless Multiple Access Channel. Ph.D. dissertation, Katholieke Universiteit Leuven, Haverlee, Belgium, 1982.
[2] S. I. Bross, Y. Steinberg, and S. Tinguely. The causal cognitive interference channel. In International Zurich Seminar on Communications (IZS),, March 3-5, 2010.
[3] S. I. Bross and A. Lapidoth. The state-dependent multiple-access channel with states available at a cribbing encoder. In 2010 IEEE $26-t h$ Convention of Electrical and Electronics Engineers in Israel (IEEEI 2010).
[4] F. M.J. Willems. The multiple-access channel with cribbing encoders revisited. Tutorial Lecture at MSRI, Berkeley, Workshop Mathematics of Relaying and Cooperation in Communication Networks, April 10-12, 2006.
[5] A. A. El Gamal and M. R. Aref. The capacity of the semideterministic relay channel. IEEE Trans. Inf. Theory, 28(3):536, 1982.
[6] T. M. Cover and A. El Gamal. Capacity theorems for the relay channel. IEEE Trans. Inf. Theory, 25(5):572-584, 1979.
[7] S. I. Gelfand and M. S. Pinsker. Capacity of a broadcast channel with one deterministic component. IEEE Trans. Inf. Theory, 16:24-34, 1980.
[8] T. Weissman. Capacity of channels with action-dependent states. IEEE Trans. Inf. Theory, 56(11):5396-5411, 2010.
[9] T. Weissman and H. H. Permuter. Source coding with a side information "vending machine". submitted to IEEE Trans. Inf. Theory. Available at arxiv.org/abs/0904.2311, 2009.
[10] H. Asnani, H. H. Permuter, and T. Weissman. Probing capacity. 2010. submitted to IEEE Trans. Inf. Theory. Available at arxiv.org/abs/1010.1309.
[11] H. Asnani, H. H. Permuter, and T. Weissman. To feed or not to feed back. 2010. submitted to IEEE Trans. Inf. Theory. Available at arxiv.org/abs/1011.1607.
[12] I. Csiszár and J. Körner. Information Theory: Coding Theorems for Discrete Memoryless Systems. Academic, New York, 1981.
[13] T. M. Cover and J. A. Thomas. Elements of Information Theory. Wiley, New-York, 2nd edition, 2006.
[14] G. Kramer. Topics in multi-user information theory. Foundations and Trends in Communications and Information Theory, 4(4/5):265-444, 2007.
[15] A. El Gamal and Y.-H. Kim. Lecture notes on network information theory. 2010. availble at http://arxiv.org/abs/1001.3404
[16] F. M. J. Willems and E. C. van der Meulen. The discrete memoryless multiple-access channel with cribbing encoders. IEEE Trans. Inf. Theory, 31(3):313-327, 1985.
[17] N. Lauritzen. Lectures on convex sets. availble at http://home.imf.au.dk/niels/lecconset.pdf 2010.
[18] D. Slepian and J. K. Wolf. A coding theorem for multiple-access channel with correlated sources. Bell Syst. Tech. J., 51:10371076, 1973.
[19] A. El Gamal, N. Hassanpour, and J. P. Mammen. Relay networks with delays. IEEE Trans. Inf. Theory, 53(10):3413-3431, 2007.
[20] J. Wang, J. Chen, L. Zhao, P. Cuff, and H. H. Permuter. A random variable substitution lemma with applications to multiple description coding. submitted to IEEE Trans. Inf. Theory. Available at arxiv.org/abs/0909.3135, 2009.

