Errata and Corrections

Corrections to "Compressive MUSIC: Revisiting the Link Between Compressive Sensing and Array Signal Processing"

Jong Min Kim, *Member, IEEE*, and Jong Chul Ye, *Senior Member, IEEE*

***Abstract*—There are a few corrections for the above titled paper (IEEE Trans. Inf. Theory, vol. 58, no. 1, pp. 278–301, Jan. 2012). They are presented here.**

I. Main Results

The SNR analysis of the generalized MUSIC step in [1] was based on the following inequality:

$$\|P_{R(P^{\perp}_{R(A_{I_t})}Y)} - P_{R(P^{\perp}_{R(A_{I_t})}B)}\| \leq \|P_{R(Y)} - P_{R(B)}\|, \tag{I.1}$$

for any $0 \leq t \leq k-r$ and then the right-hand side of (I.1) was then upper bounded by

$$\eta_{\rm old} := \frac{2(\kappa(B)+1)}{\mathsf{SNR}_{\min}(Y)-1}$$

where

$$\mathsf{SNR}_{\min}(Y) := \frac{\sigma_{\min}(B)}{\|N\|}.$$

However, the bound (I.1) does not hold in general, so we need to modify the corresponding parts in [1]. While a corrected version of (I.1) was derived in [2], using (9) of [3], and the projection update formula (see the first equation in Section VII.B of [2]), we can derive the following bound that is also useful for our analysis:

$$\begin{aligned}&\|P_{R(P^{\perp}_{R(A_{I_t})}Y)} - P_{R(P^{\perp}_{R(A_{I_t})}B)}\|\\ &\leq \frac{1}{\mathsf{SNR}_{r+t}-1} \triangleq \eta_t,\end{aligned} \tag{I.2}$$

for $0 \leq t \leq k-r$ provided that $\mathsf{SNR}_{r+t} > 1$, where

$$\mathsf{SNR}_{r+t} := \frac{\sigma_{r+t}([A_{I_t}\ B])}{\|N\|}.$$

Here, we can see that $\eta_t \leq \eta_{k-r}$ for any $0 \leq t \leq k-r$. Then, using (I.2) instead of (I.1), we have the following result.

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J. M. Kim is with the Department of Mathematics and Computer Science, Korea Science Academy of KAIST, Busan 614-822, Korea (e-mail: franzkim@gmail.com).

J. C. Ye is with the Department of Bio and Brain Engineering, KAIST, Daejeon 305-701, Korea (e-mail: jong.ye@kaist.ac.kr).

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Theorem 1.1: For a $\mathrm{LSMMV}(m,n,k,r;\epsilon)$, if we have $I_{k-r} \subset \mathrm{supp}X$, then we can find remaining r indices of supp X with the generalized MUSIC criterion if

$$m \geq \max\left\{\frac{k(1+\delta)}{1-2\eta_{k-r}}, (1+\delta)(2k-r+1)\right\}$$

for some $\delta > 0$, and

$$\mathsf{SNR}_k = \frac{\sigma_k([A_{I_{k-r}}\ B])}{\|N\|} > 3.$$

By the similar reasoning as above, we can modify the performance analysis of the S-OMP which will be given in the followings. Note that the SNR conditions in the following theorems can be easily obtained by constraining the sampling lower bound positive.

Theorem 1.2: For $\mathrm{LSMMV}(m,n,k,r;\epsilon)$, suppose that $I_t \subset \mathrm{supp}X$ for a $0 \leq t < k-r$, and the following conditions hold:

a) r is a fixed finite number.

b)
$$\mathsf{SNR}_{r+t} = \frac{\sigma_{r+t}([A_{I_t}\ B])}{\|N\|} > 1 + 2\frac{k}{r}$$

c) we have
$$m > k(1+\delta)\frac{1}{1-2\frac{k}{r}\eta_t}\frac{2\log(n-k)}{r},$$

then we can find a correct index of $\mathrm{supp}X$ at the $(t+1)$th step of subspace S-OMP.

Theorem 1.3: For $\mathrm{LSMMV}(m,n,k,r;\epsilon)$, suppose that $I_t \subset \mathrm{supp}X$ for a $0 \leq t < k-r$, and the following conditions hold.

a) r is proportionally increasing with respect to k so that $\alpha := \lim_{n\to\infty} r(n)/k(n) > 0$ exist.

b)
$$\mathsf{SNR}_{r+t} = \frac{\sigma_{r+t}([A_{I_t}\ B])}{\|N\|} > 1 + \frac{2}{\alpha}$$

c) we have
$$m > k(1+\delta)^2\frac{1}{\left(1-\frac{2}{\alpha}\eta_t\right)^2}\left[2-F(\alpha)\right]^2,$$

for some $\delta > 0$, where $F(\alpha) = \frac{1}{\alpha}\int_0^{4t_1(\alpha)^2} x d\lambda_1(x)$, $d\lambda_1(x)$= $(\sqrt{(4-x)x})/(2\pi x)dx$ is the probability measure with support $[0,4]$, $0 \leq t_1(\alpha) \leq 1$ satisfies $\int_0^{2t_1(\alpha)} ds_1(x) = \alpha$, and $ds_1(x) = (1/\pi)\sqrt{4-x^2}dx$ is a probability measure with support $[0,2]$. Here, $F(\alpha)$ is an increasing function such that $F(1)=1$ and $\lim_{\alpha\to 0+} F(\alpha) = 0$. Then, we can find a correct index of $\mathrm{supp}X$ at the $(t+1)$th step of subspace S-OMP.

Remark 1.1: In another approach, $\sigma_r(P^{\perp}_{R(A_{I_t})}B)$ can be lower bounded by the restricted isometry property constant (RIP) of A_{I_t}, i.e., $\delta_k(A)$, and $\sigma_r(X^S \setminus I_t)$. Specifically, we have

$$\begin{aligned}\sigma_r(P^{\perp}_{R(A_{I_t})}B) &= \sigma_r(P^{\perp}_{R(A_{I_t})}A_{S\setminus I_t}X^{S\setminus I_t})\\ &\geq \sigma_r(P^{\perp}_{R(A_{I_t})}A_{S\setminus I_t})\sigma_r(X^{S\setminus I_t})\\ &\geq \sigma_k(A_S)\sigma_r(X^{S\setminus I_t})\\ &\geq \sqrt{1-\delta_k(A)}\sigma_r(X^{S\setminus I_t}).\end{aligned}$$

Then, we have the inequality

$$\begin{aligned}&\|P_{R(P^{\perp}_{R(A_{I_t})}Y)} - P_{R(P^{\perp}_{R(A_{I_t})}B)}\| \\ &\leq \frac{1}{\sqrt{1-\delta_k(A)}\mathsf{SNR}_{I_t}(X) - 1} \triangleq \eta_*, \end{aligned} \tag{I.3}$$

where

$$\mathsf{SNR}_{I_t}(X) = \frac{\sigma_r(X^{S\setminus I_t})}{\|N\|}.$$

Then, by using η_*, we have the similar results. Note that this expression has more direct dependence on unknown X.

II. Alternative Correction

As mentioned above, we can also modify the corresponding parts in [1] by using the corrected version of (I.1) which was given in [2]. The corrected version of (I.1) was given by the following proposition.

Proposition 2.1 ([2, Proposition 7.6]): Assume that $X \in \mathbb{R}^{n\times r}$ satisfies $\|X\|_0 = k$ with support S and the nonzero rows of X is in general position. Here, we let $\xi := \|P_{R(B)} - P_{R(Y)}\|$ and J be a proper subset of S. If sensing matrix A satisfies $\kappa(A_S) < 1/\xi$, then

$$\|P_{R(P^{\perp}_{R(A_J)}B)} - P_{R(P^{\perp}_{R(A_J)}Y)}\| \leq \frac{1}{[\xi\kappa(A_S)]^{-1} - 1}.$$

Now, we let

$$\eta := \frac{1}{[\xi\kappa(A_S)]^{-1} - 1}. \tag{II.1}$$

Then, by using above proposition, we have the following result.

Theorem 2.2: For $\mathrm{LSMMV}(m,n,k,r;\epsilon)$, suppose that $I_t \subset \mathrm{supp}X$ for a $0 \leq t < k-r$, the following conditions hold:

$$m \geq \max\left\{\frac{k(1+\delta)}{1-2\eta}, (1+\delta)(2k-r+1)\right\}$$

for some $\delta > 0$, and for $\eta < 1/2$ or

$$\xi := \|P_{R(P^{\perp}_{R(A_J)}B)} - P_{R(P^{\perp}_{R(A_J)}Y)}\| < \frac{1}{3\kappa(A_S)}.$$

For the sufficient condition for partial support recovery using subspace S-OMP, we have the following theorems.

Theorem 2.3: For $\mathrm{LSMMV}(m,n,k,r;\epsilon)$, suppose that $I_t \subset \mathrm{supp}X$ for a $0 \leq t < k-r$ and the following conditions hold:

a) r is a fixed finite number.
b) Assume that we have

$$\xi < \frac{1}{\left(1+\frac{2k}{r}\right)\kappa(A_S)}.$$

If we have

$$m > k(1+\delta)\frac{1}{1-\frac{2k}{r}\eta}\frac{2\log(n-k)}{r},$$

then we can find $k-r$ correct indices of $\mathrm{supp}X$ by applying subspace S-OMP.

Theorem 2.4: For $\mathrm{LSMMV}(m,n,k,r;\epsilon)$, suppose that $I_t \subset \mathrm{supp}X$ for a $0 \leq t < k-r$, and the following conditions hold:

a) r is proportionally increasing with respect to k so that $\alpha := \lim_{n\to\infty} r(n)/k(n) > 0$ exist.
b) Assume that we have

$$\xi < \frac{1}{\left(1+\frac{2}{\alpha}\right)\kappa(A_S)}.$$

Then, if we have

$$m > k(1+\delta)^2\frac{1}{\left(1-\frac{2}{\alpha}\eta\right)^2}\left[2-F(\alpha)\right]^2,$$

for some $\delta > 0$, where $F(\alpha) = \frac{1}{\alpha}\int_0^{4t_1(\alpha)^2} x\,d\lambda_1(x)$, $d\lambda_1(x) = (\sqrt{(4-x)x})/(2\pi x)dx$ is the probability measure with support $[0,4]$, $0 \leq t_1(\alpha) \leq 1$ satisfies $\int_0^{2t_1(\alpha)} ds_1(x) = \alpha$, and $ds_1(x) = (1/\pi)\sqrt{4-x^2}dx$ is a probability measure with support $[0,2]$. Here, $F(\alpha)$ is an increasing function such that $F(1) = 1$ and $\lim_{\alpha\to 0^+} F(\alpha) = 0$. Then, we can find $k-r$ correct indices of $\mathrm{supp}X$ by applying subspace S-OMP.

III. Comparison of the Two Bounds

In this section, we are interested in comparing the implication of two bounds: (I.2) and that of Proposition 7.6 in [2]. As can be seen in Section II, the results using the bounds in Proposition 7.6 in [2] are simpler than those of using (I.2). More specifically, the η in (II.1) is only dependent on the condition number of partial matrix A_S, whereas the η_t in (I.2) depends on both A_{I_t} and B, or A_S and X due to (I.3). Hence, the new bounds (I.2) and (I.3) may not be practical to compute because we do not know B and X *a priori*.

However, the new bound (I.2) provides a novel error correction scheme as described in [3], as the SNR condition (10) in [3] becomes more relaxed thanks to the inclusion of a correct index.

However, we admit that this argument is not rigorous since the SNR requirement in (10) of [3] is just a sufficient condition for successful recovery, and neither the tightness of the sufficient condition nor a corresponding necessary condition has been shown. Moreover, the strict increase of $\sigma_k[A_{k-r+l}\quad B]$ with l is not guaranteed when the newly found atom is orthogonal to the singular vector corresponding $\sigma_k([A_{k-r+(l-1)}\quad B])$. However, such *sequential subspace estimation* was empirically shown useful to improve the noise robustness in a subspace-based sequential joint sparse recovery algorithm [3], so we believe that the new bound (I.2) described in this correction is useful in its own right.

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References

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