# The approximate maximum-likelihood certificate

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Abstract—A new property which relies on the linear programming (LP) decoder, the approximate maximum-likelihood certificate (AMLC), is introduced. When using the belief propagation decoder, this property is a measure of how close the decoded codeword is to the LP solution. Using upper bounding techniques, it is demonstrated that the conditional frame error probability given that the AMLC holds is, with some degree of confidence, below a threshold. In channels with low noise, this threshold is several orders of magnitude lower than the simulated frame error rate, and our bound holds with very high degree of confidence. In contrast, showing this error performance by simulation would require very long Monte Carlo runs. When the AMLC holds, our approach thus provides the decoder with extra error detection capability, which is especially important in applications requiring high data integrity.

### I. INTRODUCTION

Linear programming (LP) decoding has emerged in recent years as a potential candidate algorithm for approximating maximum-likelihood (ML) decoding. One reason for this is that it has been shown [1] that the LP decoding algorithm has the *ML certificate* property, i.e., that if the decoder outputs a valid codeword, it is guaranteed to be the ML codeword.

Since the discovery of LP decoding, several papers have been written on the subject of improving the performance of the decoder, e.g., by using integer programming or mixed integer linear programming [1], [2], adding constraints to the Tanner graph [3] and guessing facets of the polytope [4]. Moreover, the issue of decoding complexity has been addressed [5], [6], [7], as the complexity of linear programming techniques is in general polynomial but not linear in the block length N. Vontobel and Koetter [5] have proposed an iterative, Gauss-Seidel-type algorithm for approximate LP decoding. Based on their general approach, a linear-complexity (O(N))iterative approximate decoder [8] was suggested.

This low-complexity LP decoder was recently put to use in a framework [9] aimed at harnessing the LP decoder for tasks other than decoding. In this context, an algorithm with complexity  $O(N^2)$  was proposed which produces a lower bound on the minimum distance of a specific code. Another use is an algorithm of the same complexity for finding a tight lower bound on the fractional distance.

In this paper we propose a new application for using the LP decoder by introducing a new concept, the *approximate ML certificate* (AMLC), a tool which can improve the error detection capability of the belief propagation (BP) decoder.

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We show that if the BP decoder output satisfies the AMLC property (in particular, it is a codeword), then there is a high degree of certainty that it is the correct codeword. It is demonstrated that, when applying this technique within the error floor region, the frame error rate implied by the AMLC is several orders of magnitude lower than the average rate (the average rate in the error floor region was previously studied by Richardson [10]); ascertaining this improved reliability directly using Monte Carlo simulation would require very long simulation runs. This makes the AMLC especially useful in applications where a high level of data integrity is required.

The LP decoder is a central component in the evaluation of the AMLC, as is the aforementioned  $O(N^2)$  algorithm for obtaining a lower bound on the minimum distance. Another component used in our analysis is the generalized second version of the Duman-Salehi bound, as derived by Sason and Shamai [11]; this bound is an upper bound on the ML decoding probability. A slightly modified version of this bound is used for our purposes.

This paper is organized as follows. Section II provides some background material, related primarily to the LP decoder. In section III, we prove our main result concerning the approximate ML certificate. In Section IV numerical examples are provided, and the paper is concluded in Section V.

#### **II. PRELIMINARIES**

Consider an LDPC code  $\mathcal{C}$  described by a Tanner graph with N variable nodes and M check nodes. A codeword  $\mathbf{c} \in \mathcal{C}$  is transmitted over a discrete memoryless binary-input, output-symmetric (MBIOS) channel described by a probability transition function Q(y | c), where c is the transmitted codebit and y is the channel output (we will also use the vector notation  $Q(\mathbf{y} | \mathbf{c})$  where  $\mathbf{y}$  is the channel output vector and  $\mathbf{c}$ is the transmitted codeword, the meaning will be clear from the context). Following the notation in [1], let  $\mathcal{I}$  and  $\mathcal{J}$  be the sets of variable and constraint nodes, respectively, such that  $|\mathcal{I}| = N$  and  $|\mathcal{J}| = M$ . Define the set  $\mathcal{N}_i$  to be the set of neighbors of variable node  $i \in \mathcal{I}$ . Similarly,  $\mathcal{N}_j$  is the set of neighbors of check node  $j \in \mathcal{J}$ . Denote by  $C_j$  the constituent binary single parity check code corresponding to node  $j \in \mathcal{J}$ . Let  $\mathbf{0} \in \mathcal{C}$  denote the all-zero codeword.

The LP decoder [1] solves the following optimization problem.

$$(\boldsymbol{\lambda}, \boldsymbol{\lambda}_{\boldsymbol{\omega}}) \stackrel{\Delta}{=} \operatorname*{argmin}_{(\mathbf{c}, \boldsymbol{\omega})} \mathbf{P}(\mathbf{c})$$
 (1)

This research was supported by the Israel Science Foundation, grant no. 772/09.

(i.e.,  $\lambda$  is the optimal **c**-vector and  $\lambda_{\omega}$  is the optimal  $\omega$ -vector) subject to

$$w_{j,\mathbf{g}} \ge 0 \qquad \forall j \in \mathcal{J} , \ \mathbf{g} \in \mathcal{C}_j$$

$$\tag{2}$$

$$\sum_{\mathbf{g}\in\mathcal{C}_{j}} w_{j,\mathbf{g}} = 1 \qquad \forall j \in \mathcal{J}$$
(3)

$$c_i = \sum_{\mathbf{g} \in \mathcal{C}_j, g_i = 1} w_{j,\mathbf{g}} \qquad \forall j \in \mathcal{J}, \ i \in \mathcal{N}_j$$
(4)

where the vector  $\boldsymbol{\omega}$  is defined by

$$\boldsymbol{\omega} \stackrel{\Delta}{=} \{ w_{j,\mathbf{g}} \}_{j \in \mathcal{J}, \, \mathbf{g} \in \mathcal{C}_j}$$

and where

$$P(\mathbf{c}) \stackrel{\Delta}{=} \sum_{i \in \mathcal{I}} c_i \gamma_i \tag{5}$$

and

$$\gamma_i \stackrel{\Delta}{=} \log \frac{Q(y_i \mid 0)}{Q(y_i \mid 1)}$$

is the log-likelihood ratio (LLR). All logarithms are natural.

An important observation is that the LP decoder has the ML certificate [1] in the sense that if the solution  $\lambda$  is integer (in fact an integer  $\lambda_{\omega}$  implies that  $\lambda$  is also integer by (4)) then it is the ML decision.

#### III. THE APPROXIMATE ML CERTIFICATE PROPERTY

Let the binary LDPC code C be selected uniformly from an ensemble of LDPC codes  $C^0$  (e.g., the ensemble of all (c, d)regular codes) and assume C has M codewords. Let  $\mathbf{c}_m \in C, m \in \{1, \ldots, M\}$  be the codeword selected for transmission over a MBIOS channel, and suppose that the channel output vector is  $\mathbf{y}$ . The received vector  $\mathbf{y}$  is decoded by a standard belief propagation (BP) decoder, which outputs an estimate  $\hat{\mathbf{c}}$ which may or may not be a valid codeword. Let  $P(\boldsymbol{\lambda})$  be the optimal value of the LP problem. By the fact that  $\mathbf{c}_m$  is a codeword and hence feasible in the LP we have

$$P(\boldsymbol{\lambda}) \le \mathbf{P}(\mathbf{c}_m) \tag{6}$$

Now suppose that for some  $\delta > 0$ 

$$P(\hat{\mathbf{c}}) - P(\boldsymbol{\lambda}) \le \delta \tag{7}$$

and that the decoder output  $\hat{\mathbf{c}}$  is a valid codeword. We call this event the *approximate ML certificate* and the constant  $\delta$ the *proximity gap*. Formally, the AMLC happens if and only if

$$\hat{\mathbf{c}} \in \mathrm{AMLC}(\delta)$$
 (8)

where

$$AMLC(\delta) = \{ \hat{\mathbf{c}} : \hat{\mathbf{c}} \in \mathcal{C} , P(\hat{\mathbf{c}}) - P(\boldsymbol{\lambda}) \le \delta \}$$
(9)

When  $\delta = 0$  the AMLC coincides with the standard ML certificate of [1], since in this case the codeword  $\hat{\mathbf{c}}$  is the ML solution. By (6)-(7) we conclude that

$$P(\hat{\mathbf{c}}) - P(\mathbf{c}_m) \le \delta \tag{10}$$

Now, consider the transmission of a code chosen at random from  $C^0$ . The transmitted codeword  $\mathbf{c}_m$  is also selected at random from the chosen code. If the AMLC holds, then the

word error probability given that  $\mathbf{c}_m$  was transmitted can be upper bounded as

$$\Pr\left(\hat{\mathbf{c}} \neq \mathbf{c}_{m} \middle| \begin{array}{c} \hat{\mathbf{c}} \in \mathrm{AMLC}(\delta) \\ \mathbf{c}_{m} \text{ trans.} \end{array}\right)$$

$$\leq \Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{c}_{m} \end{array} : \mathrm{P}(\mathbf{c}) \leq \mathrm{P}(\mathbf{c}_{m}) + \delta \middle| \begin{array}{c} \hat{\mathbf{c}} \in \mathrm{AMLC}(\delta) \\ \mathbf{c}_{m} \text{ trans.} \end{array}\right)$$

$$\leq \frac{\Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{c}_{m} \end{array} : \mathrm{P}(\mathbf{c}) \leq \mathrm{P}(\mathbf{c}_{m}) + \delta \middle| \mathbf{c}_{m} \text{ trans.} \right)}{\mathrm{Pr}\left(\hat{\mathbf{c}} \in \mathrm{AMLC}(\delta) \middle| \mathbf{c}_{m} \text{ trans.} \right)}$$

$$= \frac{\Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{c}_{m} \end{array} : \log\left(\frac{Q(\mathbf{y} \mid \mathbf{c}_{m})}{Q(\mathbf{y} \mid \mathbf{c})}\right) \leq \delta \middle| \mathbf{c}_{m} \text{ trans.} \right)}{\mathrm{Pr}\left(\hat{\mathbf{c}} \in \mathrm{AMLC}(\delta) \middle| \mathbf{c}_{m} \text{ trans.} \right)}$$
(11)

where in the first inequality we used (10), and in the equality we used the fact that  $P(\mathbf{c}) = \log \frac{Q(\mathbf{y} \mid \mathbf{0})}{Q(\mathbf{y} \mid \mathbf{c})}$  and  $P(\mathbf{c}_m) = \log \frac{Q(\mathbf{y} \mid \mathbf{0})}{Q(\mathbf{y} \mid \mathbf{c}_m)}$ . Note that in (11), the numerator depends on the channel probability transition function and the code chosen, while the denominator depends on the channel, the code, and also on the decoding algorithm. We can further upper bound the expression (11) by upper-bounding the numerator and lower-bounding the denominator. In the process of doing so, we eliminate the dependence on the transmitted message m.

#### A. A lower bound on the denominator

Consider the denominator in (11). The following lemma asserts its independence of the transmitted codeword.

*Lemma 1:* Consider the vector  $\lambda$  which is output by the LP decoder. Also assume that  $\mathbf{c}_m$  is the transmitted codeword and that the BP decoder is used. Then the expression

$$\Pr\left(\hat{\mathbf{c}} \in \mathsf{AMLC}(\delta) \mid \mathbf{c}_m \text{ trans.}\right) \tag{12}$$

is independent of m, and in particular it can be assumed in (12) that the all-zero codeword is transmitted.

Proof: See Appendix A.

To get a lower bound on (12), one could run Monte Carlo simulations. Consider a series of experiments conducted to estimate  $\eta \triangleq \Pr(\hat{\mathbf{c}} \in AMLC(\delta) \mid \mathbf{0} \text{ trans.})$ . In each experiment we draw a code at random from  $C^0$ , transmit the allzero codeword over the noisy channel and decode. Suppose that we run *L* experiments and find that  $\hat{\mathbf{c}} \in AMLC(\delta)$  in  $L_1$ experiments. Let

$$L_1 = \sum_{i=1}^{L} X_i$$

where if in the *i*'th experiment  $\hat{\mathbf{c}} \in \text{AMLC}(\delta)$ , then  $X_i = 1$ ; otherwise  $X_i = 0$ . If the channel is low-noise then we would expect to have  $L_1 = L(1-\epsilon)$  with small  $\epsilon$ , and in particular we would expect to have  $\epsilon < 0.5$ , which (for large *L*) would imply  $\eta > 0.5$ . Since  $\eta$  is a deterministic but unknown parameter, we cannot claim that  $\eta > 0.5$ , even if  $\epsilon$  is small; rather, this situation falls under the framework of non-bayesian hypothesis testing, so the series of experiments does allow us to say something about  $\eta$  with some degree of confidence. Consider the hypothesis

$$H_0 : \eta \le 0.5$$
 (13)

For  $\epsilon < 0.5$ , the following inequality holds

$$\Pr(L_1 \ge (1-\epsilon)L \mid H_0 \text{ valid}) < 2^{-L} \binom{L}{\epsilon L} \cdot (\epsilon L + 1) \quad (14)$$

since the RHS is an upper bound on the tail of a binomial distribution. Now suppose that in a Monte Carlo simulation we get  $L_1 = L(1 - \epsilon)$  with  $\epsilon < 0.5$ . By (14) we observe that if  $\epsilon$  is very small, then the RHS of (14) is very low, and thus we can reject  $H_0$  with a high degree of confidence. Conversely, if in our simulation  $\epsilon$  is close to 0.5, we cannot reject  $H_0$  with high confidence.

Define

$$\xi(L,\epsilon) \stackrel{\Delta}{=} 1 - 2^{-L} \binom{L}{\epsilon L} \cdot (\epsilon L + 1) \tag{15}$$

Given the Monte Carlo result discussed above, one may conclude that

$$\Pr\left(\hat{\mathbf{c}} \neq \mathbf{c}_{m} \mid \hat{\mathbf{c}} \in \text{AMLC}(\delta) , \mathbf{c}_{m} \text{ trans.}\right)$$

$$\leq 2\Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{c}_{m} \end{array} : \log\left(\frac{Q(\mathbf{y} \mid \mathbf{c}_{m})}{Q(\mathbf{y} \mid \mathbf{c})}\right) \leq \delta \mid \mathbf{c}_{m} \text{ trans.}\right)$$
(16)

which reflects the assertion  $\eta > 0.5$ . This assertion holds with confidence level  $\xi(L, \epsilon)$ . Note that for fixed  $\epsilon$ , the likelihood that the bound (16) does not hold decays exponentially with L.

The standard approach to estimating the frame error rate performance is to use a Monte Carlo simulation. The result is a confidence interval on the actual error rate. In our method we also use a Monte Carlo simulation. However, in the following we derive an analytic bound on the RHS of (16) which, combined with the simulation, enables us to obtain extremely large confidence levels for very small frame error rates whenever the AMLC holds.

#### B. An upper bound on the numerator

Consider now the RHS of (16) (disregarding the constant 2). Recalling that C is chosen at random from  $C^0$ , one may write

$$\Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{c}_{m} \end{array} : \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{c}_{m})} e^{\delta} \ge 1 \mid \mathbf{c}_{m} \text{ trans.} \right)$$
$$= \sum_{\mathcal{C}_{i} \in \mathcal{C}^{0}} \Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{c}_{m} \end{array} : \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{c}_{m})} e^{\delta} \ge 1 \mid \mathbf{c}_{m} \text{ trans.} \right)$$
$$\cdot \Pr\left(\mathcal{C} = \mathcal{C}_{i} \mid \mathbf{c}_{m} \text{ trans.}\right)$$
(17)

Clearly,  $\Pr(\mathcal{C} = \mathcal{C}_i \mid \mathbf{c}_m \text{ trans.}) = \Pr(\mathcal{C} = \mathcal{C}_i)$  as the selection of the message is independent of the selection of the code. In addition, we have the following result regarding the independence of the inner expression in the sum (17) on m.

Lemma 2: The expression

$$\Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{c}_m \end{array} : \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{c}_m)} e^{\delta} \ge 1 \left| \begin{array}{c} \mathcal{C} = \mathcal{C}_i \\ \mathbf{c}_m \text{ trans.} \end{array} \right)$$
(18)

appearing within the sum (17) is independent of m.

*Proof:* See Appendix B. Due to Lemma 2, we can assume without loss of generality that the all-zero codeword  $\mathbf{0}$  is transmitted and rewrite (17) as

$$\Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{0} \end{array} : \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \ge 1 \mid \mathbf{0} \text{ trans.} \right)$$
$$= \sum_{\mathcal{C}_i \in \mathcal{C}^0} \Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{0} \end{array} : \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \ge 1 \mid \mathbf{0} \text{ trans.} \end{array} \right)$$
$$\cdot \Pr\left(\mathcal{C} = \mathcal{C}_i\right)$$
(19)

Sason and Shamai [11] have proposed a tight upper bound on the ML decoding error probability using the generalized second version of the Duman-Salehi bound, referred to as the DS2 bound. Using a slightly modified version of this bound, we can find an upper bound on

$$\Pr\left(\begin{array}{cc} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{0} \end{array} : \left. \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \ge 1 \right| \begin{array}{c} \mathcal{C} = \mathcal{C}_i \\ \mathbf{0} \text{ trans.} \end{array} \right)$$

To this end, one may write

$$\Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{0} \end{array} : \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \ge 1 \left| \begin{array}{c} \mathcal{C} = \mathcal{C}_{i} \\ \mathbf{0} \text{ trans.} \end{array} \right) \\ \le \sum_{\mathbf{y}} Q(\mathbf{y} \mid \mathbf{0}) \left( \sum_{\substack{\mathbf{c} \neq \mathbf{0} \\ \mathbf{c} \in \mathcal{C}_{i}}} \left( \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \right)^{\lambda} \right)^{\rho} \\ = \sum_{\mathbf{y}} \Psi_{N}^{0}(\mathbf{y}) \Psi_{N}^{0}(\mathbf{y})^{-1} Q(\mathbf{y} \mid \mathbf{0}) \left( \sum_{\substack{\mathbf{c} \neq \mathbf{0} \\ \mathbf{c} \in \mathcal{C}_{i}}} \left( \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \right)^{\lambda} \right)^{\rho} \\ = \sum_{\mathbf{y}} \Psi_{N}^{0}(\mathbf{y}) \left( \Psi_{N}^{0}(\mathbf{y})^{-\frac{1}{\rho}} Q(\mathbf{y} \mid \mathbf{0})^{\frac{1}{\rho}} \sum_{\substack{\mathbf{c} \neq \mathbf{0} \\ \mathbf{c} \in \mathcal{C}_{i}}} \left( \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \right)^{\lambda} \right)^{\rho}$$
(20)

where the expression on the second line, which holds for all  $\lambda, \rho \geq 0$ , is an adaptation of the 1965 Gallager bound [12] to our purposes, and  $\Psi_N^0(\mathbf{y})$  is a probability measure on  $\mathbf{y}$  called a *tilting measure* [11], which is allowed in general to depend on the transmitted codeword. By invoking Jensen's inequality in (20), we get

$$\Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{0} \end{array} : \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \ge 1 \left| \begin{array}{c} \mathcal{C} = \mathcal{C}_{i} \\ \mathbf{0} \text{ trans.} \end{array} \right) \\ \le \left( \sum_{\substack{\mathbf{c} \neq \mathbf{0} \\ \mathbf{c} \in \mathcal{C}_{i}}} \sum_{\mathbf{y}} Q(\mathbf{y} \mid \mathbf{0})^{\frac{1}{\rho}} \Psi_{N}^{0}(\mathbf{y})^{1-\frac{1}{\rho}} \left( \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \right)^{\lambda} \right)^{\rho}$$
(21)

which holds for  $\lambda \ge 0$ ,  $0 \le \rho \le 1$ . Now let us restrict our discussion to tilting measures which do not depend on the transmitted codeword and which also decompose as N-fold products of the same single-letter measure, i.e.,

$$\Psi_N^0(\mathbf{y}) = \prod_{i=1}^N \psi(y_i)$$

Also recall that the channel is memoryless and thus also decomposes as an N-fold product. Using this in (21) yields

$$\Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{0} \end{array} : \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \ge 1 \left| \begin{array}{c} \mathcal{C} = \mathcal{C}_{i} \\ \mathbf{0} \text{ trans.} \end{array} \right)$$
$$\le e^{\delta \rho \lambda} \left[ \sum_{h=1}^{N} A_{h} \left( \sum_{y} \psi(y)^{1-\frac{1}{\rho}} Q(y|0)^{\frac{1}{\rho}} \right)^{N-h} \left( \sum_{y} \psi(y)^{1-\frac{1}{\rho}} Q(y|0)^{\frac{1-\lambda\rho}{\rho}} Q(y|1)^{\lambda} \right)^{h} \right]^{\rho}$$
$$\left( \sum_{y} \psi(y)^{1-\frac{1}{\rho}} Q(y|0)^{\frac{1-\lambda\rho}{\rho}} Q(y|1)^{\lambda} \right)^{h} \right]^{\rho}$$

where  $A_h$  is the distance spectrum of the code  $C_i$ . Now we partition the code  $C_i$  into constant Hamming weight subcodes where  $C_{i,h}$  contains all words in  $C_i$  of weight h (note that in general these subcodes are nonlinear). By applying a union bound over the subcodes on the LHS of (22) we get

$$\Pr\left(\begin{array}{c} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{0} \end{array} : \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \ge 1 | \mathcal{C} = \mathcal{C}_{i} , \mathbf{0} \text{ trans.} \right)$$

$$\le \sum_{h=1}^{N} \Pr\left(\exists \mathbf{c} \in \mathcal{C}_{i,h} : \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \ge 1 \middle| \begin{array}{c} \mathcal{C} = \mathcal{C}_{i} \\ \mathbf{0} \text{ trans.} \end{array} \right)$$

$$\triangleq \sum_{h=1}^{N} P_{1}(h)$$
(23)

where by (22)

$$P_{1}(h) \leq e^{\delta\rho\lambda} (A_{h})^{\rho} \left[ \left( \sum_{y} \psi(y)^{1-\frac{1}{\rho}} Q(y|0)^{\frac{1}{\rho}} \right)^{N-h} \right. \\ \left. \left. \left( \sum_{y} \psi(y)^{1-\frac{1}{\rho}} Q(y|0)^{\frac{1-\lambda\rho}{\rho}} Q(y|1)^{\lambda} \right)^{h} \right]^{\rho} \right]^{\rho}$$
(24)

Let  $\overline{A_h}$  and  $\overline{P_1(h)}$  denote the averages of the distance distribution  $A_h$  and  $P_1(h)$ , respectively, taken over the ensemble  $\mathcal{C}^0$ . By Jensen's inequality (applied as  $\overline{[(A_h)^{\rho}]} \leq \overline{[A_h]}^{\rho}$ ,  $0 \leq \rho \leq 1$ ) we have

$$\overline{P_{1}(h)} \leq e^{\delta\rho\lambda} (\overline{A_{h}})^{\rho} \left[ \left( \sum_{y} \psi(y)^{1-\frac{1}{\rho}} Q(y|0)^{\frac{1}{\rho}} \right)^{N-h} \right. \\ \left. \cdot \left( \sum_{y} \psi(y)^{1-\frac{1}{\rho}} Q(y|0)^{\frac{1-\lambda\rho}{\rho}} Q(y|1)^{\lambda} \right)^{h} \right]^{\rho}$$
(25)

The overall bound is given by

$$\Pr\left(\hat{\mathbf{c}} \neq \mathbf{c}_{m} \mid \hat{\mathbf{c}} \in \text{AMLC}(\delta) , \ \mathbf{c}_{m} \text{ trans.}\right) \leq 2\sum_{h=1}^{N} \overline{P_{1}(h)}$$
(26)

where  $\overline{P_1(h)}$  is given by (25). This upper bound only depends on the average distance spectrum, which is known for many code ensembles and in particular for LDPC codes. Now, we can optimize the bound (25) over  $\lambda \ge 0$ ,  $0 \le \rho \le 1$  and the tilting measure  $\psi(\cdot)$ . This optimization is performed for every value of *h* separately. Some additional technical details regarding this optimization are provided in Appendix C.

## C. Application of the AMLC to expurgated LDPC ensembles

In this subsection we consider the application of the upper bound on the error probability given the AMLC to expurgated ensembles of LDPC codes. The expurgated ensemble  $C^{\gamma}$  is obtained from the original ensemble  $C^0$  by removing all codes with minimum distance  $\gamma$  or less. The reason for dealing with this ensemble rather than  $C^0$  is that the decoding error probability over  $C^0$  is dominated [13], [14] by a small set of "bad" codes with small minimum distance; we will show that if we can avoid these "bad" codes, then the occurrence of the AMLC implies very low error rates.

Let  $A_h^{\gamma}$  denote the average distance spectrum over  $C^{\gamma}$ . It was shown [14] that if  $\gamma > 0$  is selected small enough, then with probability 1 - o(1), a randomly-selected code from  $C^0$  is also in  $C^{\gamma}$ ; this implies that for large enough N, so that less than half the codes are expurgated, the following bound holds:

$$\overline{A_h^{\gamma}} \le \begin{cases} 2\overline{A_h}, & h > \gamma \\ 0, & h \le \gamma \end{cases}$$
(27)

When using the DS2 bound we can plug  $\overline{A_h^{\gamma}}$  instead of  $\overline{A_h}$  in (25). In practice, when applying the Monte Carlo procedure outlined in Section III-A, we draw codes at random from  $\mathcal{C}^0$  and thus we need to test whether these codes are also in  $\mathcal{C}^{\gamma}$ . To do this, we use the procedure described in [9, Section 5] which obtains a lower bound  $LB(\mathcal{C}_1)$  on the minimum distance of the randomly-drawn code  $\mathcal{C}_1$ . If  $LB(\mathcal{C}_1) > \gamma$ , then  $\mathcal{C}_1 \in \mathcal{C}^{\gamma}$ . Note, however, that the converse is not necessarily true, i.e., we could have  $\mathcal{C}_1 \in \mathcal{C}^{\gamma}$  but with  $LB(\mathcal{C}_1) \leq \gamma$ . Define the ensemble

$$\tilde{\mathcal{C}}^{\gamma} = \{ \mathcal{C} \in \mathcal{C}^0 : LB(\mathcal{C}) > \gamma \}$$
(28)

then clearly  $\tilde{C}^{\gamma} \subseteq C^{\gamma}$ . Let  $\overline{A_h^{\gamma}}$  be the average distance spectrum over  $\tilde{C}^{\gamma}$ . We will obtain an upper bound on  $\overline{A_h^{\gamma}}$  which is similar to (27) using a Monte Carlo approach, similar to the argument made in Section III-A. Suppose we run *L* experiments. In each experiment we randomly pick a code  $C \in C^0$  and calculate LB(C). Now suppose that in  $L_2 = L(1 - \epsilon_2)$  experiments we obtain that  $LB(C) > \gamma$ , and  $\epsilon_2 < 0.5$  is small. From this set of experiments, we conclude as we did in Section III-A that

$$\overline{\tilde{A}_{h}^{\gamma}} \leq \begin{cases} 2\overline{A_{h}}, & h > \gamma \\ 0, & h \le \gamma \end{cases}$$
(29)

with high confidence level.

Consider the following procedure for obtaining a bound on the confidence level of (25)-(26) when  $\tilde{A}_h^{\gamma}$  (upper-bounded in (29)) is used as the distance spectrum. The confidence level output by this algorithm is a combination of the confidence level associated with  $\eta \geq 0.5$  (see Section III-A) and the statement (29). That is, the null hypothesis in this case is

$$H_0 : \{\eta \le 0.5 \text{ or } \Pr(LB(\mathcal{C}) > \gamma) \le 0.5\}$$
 (30)

Algorithm 1: Given an ensemble of codes  $C^0$ , a channel probability distribution  $Q(\cdot|\cdot)$  and number of trials L, do:

- 1) Initialize: Set E = 0.
- 2) Loop L times:
  - Pick a code C uniformly from  $C^0$ .

- Calculate  $LB(\mathcal{C})$ .
- If LB(C) ≤ γ, E ← E + 1 and skip to next loop iteration.
- Transmit the all-zero codeword through the channel.
- Decode using the BP decoder and the LP decoder.
- If the BP decoder output ĉ is not a codeword, or if P(ĉ) – P(λ) > δ, set E ← E + 1
- Output confidence level of bound: Define ε = E/L. If ε < 0.5, output ξ(L, ε) defined in (15). Otherwise, output "error".</li>

Algorithm 1 is introduced for the purpose of jointly assessing the possibility of rejecting the hypothesis (13), and the validity of (29) as an upper bound on the distance spectrum using the same confidence level-based Monte Carlo based method from Section III-A. The algorithm counts the number of failed attempts E out of L experiments, where a failure consists of either having a code C not pass the test  $LB(C) > \gamma$ , or, having passed this test, getting a BP decoder output which does not satisfy the AMLC. The algorithm is correct because if the null hypothesis (30) holds, then in any single experiment we would have a probability of failure at least 0.5. If the total number of failures E is small (i.e., less than half the total number of experiments) then the confidence level, following the derivation in Section III-A, is output. On the other hand, if  $E \ge 0.5L$ , the result is deemed unreliable.

#### D. Statement of Main Result

The analysis in this section leads to the following result.

Theorem 1: Consider the transmission of a codeword from an LDPC code drawn at random from the ensemble  $C^0$  over an MBIOS channel. Fix the proximity gap  $\delta > 0$  and the expurgation depth  $\gamma > 0$ . Then the probability of frame error with BP decoding given that the AMLC (8)-(9) holds is upper-bounded by (25)-(26). This bound holds with confidence level  $\xi(L, \epsilon)$  which can be obtained using the L Monte Carlo experiments, as detailed in Algorithm 1.

#### IV. NUMERICAL RESULTS AND DISCUSSION

Figure 1 shows a comparison between the frame error rate (FER) obtained by a simulation of the BP decoder over the binary symmetric channel (BSC) and the DS2 bound (26), calculated for various values of  $\delta$ . In this example, we consider the ensemble  $C^0$  of (3,4)-regular LDPC codes with block length N = 1000. For the calculation of the DS2 bound and the distance spectrum we use  $\gamma = 20$  as the expurgation depth.

We conducted two experiments to determine the confidence level of the bound, using Algorithm 1. In the first experiment, 150 randomly-generated codes were tested over a BSC with crossover probability p = 0.14. In the second experiment, 600 randomly-generated codes were tested over a BSC with p = 0.1. In both experiments, all the codes belonged to the ensemble  $C^{\gamma}$ . The results of the first experiment are summarized in Table I. These results indicate that in this case, the null hypothesis (30) can be rejected with very high confidence level even for  $\delta = 0$ . Consequently, the conditional frame error probability given that the the AMLC holds for  $\delta = 0$ , is (with very high confidence) lower than  $3 \cdot 10^{-5}$ ,

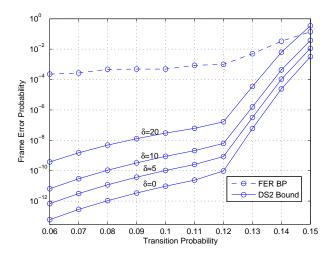


Fig. 1. A comparison between simulated frame error rate of the BP decoder and the DS2 bound (26), assuming the AMLC, for the (3,4)-regular LDPC ensemble and various values of  $\delta$ .

δ	$\epsilon$	$1 - \xi(L, \epsilon)$
0	0.1667	$7.13 \cdot 10^{-31}$
5	0.0733	$4.95 \cdot 10^{-37}$
10	0.02	$1.6 \cdot 10^{-42}$
20	0	$7 \cdot 10^{-46}$

TABLE I Confidence level bounds for different values of  $\delta$ 

which is about 1000 times lower than the simulated frame error rate at p = 0.14. In the second experiment, both the BP and LP decoders succeeded in decoding all transmissions, and thus in Algorithm 1 we get  $\epsilon = 0$ . This puts the confidence level of all the DS2 bound curves in Figure 1 at an extremely high level of

$$\xi(600,0) = 1 - 2^{-600} \tag{31}$$

In this case, the conditional frame error probability given that the AMLC holds is more than 7 orders of magnitude smaller than the simulated frame error rate (the difference between the BP curve and the  $\delta = 0$  curve for p = 0.1). The confidence level in this case is also much higher than in the first experiment. Due to the high confidence levels observed in the first experiment with p = 0.14, the confidence level result of the second experiment with p = 0.1 is not surprising, and in general we expect the confidence level to increase as the channel noise level decreases.

The strength of this result is that it demonstrates that the LP decoder can provide the BP decoder with extra error detection capability. This capability is especially useful in applications where a codeword should be rejected unless it is decoded correctly, and rejection should occur with high probability (as in data applications requiring high reliability). Achieving codeword reliability results of this order via simple Monte Carlo simulation would require very long simulation runs. In fact, our technique for upper bounding the frame error rate given that the AMLC property holds, has a common feature with the importance sampling method, since both attempt

to alleviate the computational burden associated with simple Monte Carlo simulation. We also note that in both experiments described above, the AMLC was satisfied in a large percentage of the trials, implying that it is not only capable of increasing the reliability of the decoder output, but it also does so very frequently. Using the AMLC provides an alternative to external error-detection codes, such as cyclic redundancy checks, which cause some coding rate loss. This comes at the expense of extra processing, in the shape of an LP decoder at the receiver. We note that this decoder can be implemented in linear time [8]. There is also a one-time task of computing the confidence level, which can be performed off-line. The computational complexity of calculating the lower bound [9] on the minimum distance  $LB(\mathcal{C})$  is quadratic in the block length. Thus the task of obtaining a confidence level using Algorithm 1 is performed with complexity  $O(N^2L)$ , where L is the number of simulated blocks. We also note the following points.

- It is possible to tune the AMLC result to obtain different error rates and confidence levels by varying the value of the proximity gap δ and the expurgation depth γ. Higher values of δ will produce higher values of the DS2 bound (this is evident from Figure 1), but on the other hand will increase the confidence level (as can be seen in Table I), as the requirement (7) becomes more lax. Higher values of γ will yield lower values of the DS2 bound. This, however, comes at the expense of a lower confidence level because while running Algorithm 1 more codes will be rejected as having low minimum distance.
- One may observe that in the example above, the AMLC result is applied to a random selection of a code from an ensemble. In many applications, it would be desirable to apply the AMLC result to a specific code rather than an ensemble. The difficulty is that while ensemble averages of distance spectra are typically known or can be easily upper-bounded, for specific codes this is not the case in general. Naturally, if one obtains for a specific code the exact distance spectrum (or an upper bound), it is straightforward to plug it in the DS2 bound (25)-(26). Another alternative is to use known concentration results [15] for the distance spectrum which enable one to give upper bounds on the distance spectrum of a specific code, which themselves hold with some confidence level. This confidence level can be integrated with our confidence bound  $\xi(L,\epsilon)$ . The result would be a looser bound (as compared to (25)-(26)) which applies with confidence level worse than  $\xi(L, \epsilon)$ , but it would apply to specific codes.
- Application of the AMLC result is not restricted to the BP decoder. The result extends trivially to any decoder which satisfies the symmetry condition (32). In particular, this condition is fulfilled by standard message-passing algorithms, e.g., min-sum, Gallager-A, Gallager-B.
- In a more general context, the AMLC result can be applied to any LP formulation. In particular, the LP program proposed by Feldman [16] for general Turbo codes can be used to achieve better error detection under

standard iterative decoding schemes <sup>1</sup>. The same goes for nonbinary LDPC codes when represented using the LP formulation proposed by Flanagan *et al.* [17].

Finally, it may be observed that our bound can be improved by any method which tightens the LP relaxation, e.g., the check node merging technique [9], lifting methods [1], and others. By using any of these methods, we can obtain a vector  $\lambda$  such that  $P(\lambda)$  is *larger* than that obtained by the standard LP decoder, essentially because the optimization (1) is performed over a smaller domain. The result is that for any BP-decoded codeword  $\hat{c}$ , we can use a smaller value of  $\delta$  in the AMLC (9), which gives an exponential improvement in the DS2 bound, as can be clearly seen in (25)-(26).

#### V. CONCLUSION

A new property, the approximate maximum-likelihood certificate, is introduced. This property of a BP-decoded codeword enables to increase its reliability, i.e., to increase the error detection capability. This is achieved for LDPC codes using tools related to linear programming decoding, including a recently-proposed algorithm for finding a lower bound on the minimum distance of a specific code which serves to improve the result. By applying the AMLC in the error floor region, it was demonstrated that the property can imply a frame error rate several orders of magnitude lower than a simulated error rate. While the increased frame error detection capability only holds with a certain confidence level, it was shown that this level is extremely high in the error floor region.

# APPENDIX A Proof of Lemma 1

Consider first the BP decoder. From the symmetry of the BP algorithm over MBIOS channels [18], we know that

$$\hat{\mathbf{c}}_i((-1)^{\mathbf{c}_m} \cdot \mathbf{y}) = \begin{cases} \hat{\mathbf{c}}_i(\mathbf{y}), & \mathbf{c}_{m,i} = 0\\ 1 - \hat{\mathbf{c}}_i(\mathbf{y}), & \mathbf{c}_{m,i} = 1 \end{cases}$$
(32)

where  $(-1)^{\mathbf{c}_m}$  is a vector of  $\pm 1$  corresponding to the codeword  $\mathbf{c}_m$ , the multiplication  $(-1)^{\mathbf{c}_m} \cdot \mathbf{y}$  is componentwise, and  $\hat{\mathbf{c}}_i$  (resp.  $\mathbf{c}_{m,i}$ ) is the *i*'th bit of  $\hat{\mathbf{c}}$  (resp.  $\mathbf{c}_m$ ). Now consider the LP decoder, which is used to produce the vector  $\boldsymbol{\lambda}$ . Fix  $\epsilon > 0$ . The vector  $\boldsymbol{\lambda} = \{\lambda_i\}_{i \in \mathcal{I}}$  satisfies the symmetry condition (see [1],[8, Lemma 6])

$$\lambda_i((-1)^{\mathbf{c}_m} \cdot \mathbf{y}) = \begin{cases} \lambda_i(\mathbf{y}), & \mathbf{c}_{m,i} = 0\\ 1 - \lambda_i(\mathbf{y}), & \mathbf{c}_{m,i} = 1 \end{cases}$$
(33)

<sup>1</sup>For standard parallel concatenated Turbo codes, no expurgation is needed because all codes in the ensemble do not have codewords of very low weight.

Now,

$$\begin{aligned} &\Pr\left(\hat{\mathbf{c}} \in \mathrm{AMLC}(\delta) \mid \mathbf{c}_{m} \text{ trans.}\right) \\ &= \Pr\left(\mathrm{P}\left(\hat{\mathbf{c}}\right) - \mathrm{P}\left(\boldsymbol{\lambda}\right) \leq \delta, \ \hat{\mathbf{c}} \in \mathcal{C} \mid \mathbf{c}_{m} \text{ trans.}\right) \\ &= \Pr\left(\sum_{i \in \mathcal{I}} \log\left(\frac{Q(y_{i}|0)}{Q(y_{i}|1)}\right) \left(\hat{\mathbf{c}}_{i}(\mathbf{y})\right. \\ &\left. -\lambda_{i}(\mathbf{y})\right) \leq \delta, \ \hat{\mathbf{c}} \in \mathcal{C} \mid \mathbf{c}_{m} \text{ trans.}\right) \end{aligned} \\ &= \Pr\left(\sum_{i \in \mathcal{I}} \log\left(\frac{Q(y_{i}|0)}{Q(y_{i}|1)}\right) \left(-1\right)^{\mathbf{c}_{m,i}} \left(\hat{\mathbf{c}}_{i}((-1)^{\mathbf{c}_{m}} \cdot \mathbf{y})\right. \\ &\left. -\lambda_{i}((-1)^{\mathbf{c}_{m}} \cdot \mathbf{y})\right) \leq \delta, \ \hat{\mathbf{c}} \in \mathcal{C} \mid \mathbf{0} \text{ trans.} \right) \end{aligned} \\ &= \Pr\left(\sum_{i \in \mathcal{I}} \log\left(\frac{Q(y_{i}|0)}{Q(y_{i}|1)}\right) \left(\hat{\mathbf{c}}_{i}(\mathbf{y}) - \lambda_{i}(\mathbf{y})\right) \leq \delta, \\ &\left. \hat{\mathbf{c}} \in \mathcal{C} \mid \mathbf{0} \text{ trans.} \right) \end{aligned} \\ &= \Pr\left(\mathrm{Pr}\left(\mathrm{P}(\hat{\mathbf{c}}) - \mathrm{P}(\boldsymbol{\lambda}) \leq \delta, \ \hat{\mathbf{c}} \in \mathcal{C} \mid \mathbf{0} \text{ trans.}\right) \\ &= \Pr\left(\hat{\mathbf{c}} \in \mathrm{AMLC}(\delta) \mid \mathbf{0} \text{ trans.}\right) \end{aligned}$$

where

- the first inequality is by the definition (9).
- in the second equality we use the definition (5) and stress the dependence on y.
- in the third equality we use the symmetry of the channel as well as the symmetry of the BP decoder (32).
- in the fourth equality we use the symmetry of the LP and BP decoders ((32),(33)).
- in the fifth equality we again use the definition (5).
- the final equality is again by the definition (9).

The above series of equalities hold for all m. This proves the claim.

## APPENDIX B Proof of Lemma 2

For any two codewords  $c_1$  and  $c_2$ , define the sets

$$A_{1}(\mathbf{c}_{1}, \mathbf{c}_{2}) \stackrel{\Delta}{=} \{i : (\mathbf{c}_{1})_{i} = 0, (\mathbf{c}_{2})_{i} = 1\}$$
  
$$A_{2}(\mathbf{c}_{1}, \mathbf{c}_{2}) \stackrel{\Delta}{=} \{i : (\mathbf{c}_{1})_{i} = 1, (\mathbf{c}_{2})_{i} = 0\}$$
(34)

where  $(\mathbf{c})_i$  is the *i*'th bit of codeword  $\mathbf{c}$ . We now have

$$\Pr\left(\begin{array}{l} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{c}_{m} \end{array} : \frac{Q(\mathbf{y} \mid \mathbf{c})}{Q(\mathbf{y} \mid \mathbf{c}_{m})} e^{\delta} \ge 1 \left| \begin{array}{l} \mathcal{C} = \mathcal{C}_{i} \\ \mathbf{c}_{m} \operatorname{trans.} \end{array} \right) \\ = \Pr\left(\begin{array}{l} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{c}_{m} \end{array} : \prod_{i \in A_{1}(\mathbf{c}, \mathbf{c}_{m})} \frac{Q(y_{i} \mid 0)}{Q(y_{i} \mid 1)} \\ \cdot \prod_{i \in A_{2}(\mathbf{c}, \mathbf{c}_{m})} \frac{Q(y_{i} \mid 1)}{Q(y_{i} \mid 0)} e^{\delta} \ge 1 \left| \begin{array}{c} \mathcal{C} = \mathcal{C}_{i} \\ \mathbf{c}_{m} \operatorname{trans.} \end{array} \right) \\ = \Pr\left(\begin{array}{l} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{c}_{m} \end{array} : \prod_{i \in A_{1}(\mathbf{c}, \mathbf{c}_{m})} \frac{Q(-y_{i} \mid 0)}{Q(-y_{i} \mid 1)} \\ \cdot \prod_{i \in A_{2}(\mathbf{c}, \mathbf{c}_{m})} \frac{Q(y_{i} \mid 1)}{Q(y_{i} \mid 0)} e^{\delta} \ge 1 \left| \begin{array}{c} \mathcal{C} = \mathcal{C}_{i} \\ \mathbf{0} \operatorname{trans.} \end{array} \right) \\ = \Pr\left(\begin{array}{l} \exists \mathbf{c} \in \mathcal{C} \\ \mathbf{c} \neq \mathbf{c}_{m} \end{array} : \prod_{i \in A_{1}(\mathbf{c}, \mathbf{c}_{m}) \cup A_{2}(\mathbf{c}, \mathbf{c}_{m})} \frac{Q(y_{i} \mid 1)}{Q(y_{i} \mid 0)} \\ \cdot e^{\delta} \ge 1 \left| \begin{array}{c} \mathcal{C} = \mathcal{C}_{i} \\ \mathbf{0} \operatorname{trans.} \end{array} \right) \\ = \Pr\left(\begin{array}{c} \exists \mathbf{c}' \in \mathcal{C} \\ \mathbf{c}' \neq \mathbf{0} \end{array} : \frac{Q(\mathbf{y} \mid \mathbf{c}')}{Q(\mathbf{y} \mid 0)} e^{\delta} \ge 1 \left| \begin{array}{c} \mathcal{C} = \mathcal{C}_{i} \\ \mathbf{0} \operatorname{trans.} \end{array} \right) \\ = \Pr\left(\begin{array}{c} \exists \mathbf{c}' \in \mathcal{C} \\ \mathbf{c}' \neq \mathbf{0} \end{array} : \frac{Q(\mathbf{y} \mid \mathbf{c}')}{Q(\mathbf{y} \mid \mathbf{0})} e^{\delta} \ge 1 \left| \begin{array}{c} \mathcal{C} = \mathcal{C}_{i} \\ \mathbf{0} \operatorname{trans.} \end{array} \right) \\ \end{array}\right)$$

where

- the first equality is due to the definition (34) of the sets  $A_1$  and  $A_2$ .
- the second equality is due to the same definitions of  $A_1$ and  $A_2$  as well as the symmetry of the channel (Q(y|x) = Q(-y|1-x)).
- the third equality is due to the symmetry of the channel.
- the final equality, which is the desired result, is due to the linearity of the code.

# Appendix C

# Optimization of the DS2 bound

Consider the DS2 bound (25) for fixed h. Let  $\beta \stackrel{\Delta}{=} \frac{h}{N}$ . First, rewrite the bound in exponential form as

$$\overline{P_{1}(h)} \leq e^{-NE_{DS2}(\delta,\beta,\rho,\lambda,\psi(\cdot))}$$

$$E_{DS2}(\delta,\beta,\rho,\lambda,\psi(\cdot)) \triangleq -\frac{\delta}{N}\rho\lambda - \frac{\rho}{N}\ln\left(\overline{A_{h}}\right)$$

$$-\rho(1-\beta)\ln\left(\sum_{y}\psi(y)^{1-\frac{1}{\rho}}Q(y|0)^{\frac{1}{\rho}}\right)$$

$$-\rho\beta\ln\left(\sum_{y}\psi(y)^{1-\frac{1}{\rho}}Q(y|0)^{\frac{1-\lambda\rho}{\rho}}Q(y|1)^{\lambda}\right)$$

Assuming fixed values of  $\beta$  and  $\delta$ , the exponent  $E_{\text{DS2}}(\delta, \beta, \rho, \lambda, \psi(\cdot))$  should be maximized over

$$\lambda \ge 0, \quad 0 \le \rho \le 1, \quad \left\{ \psi(y) : \sum_{y} \psi(y) = 1 \right\}$$
(36)

For fixed values of  $\lambda$  and  $\rho$ , we use calculus of variations to find the optimum tilting measure  $\psi$ ; this analysis yields the

optimality condition

$$\psi(y)^{-\frac{1}{\rho}} \left( \frac{(1-\beta)(1-\frac{1}{\rho})g_1(y)}{\sum_y \psi(y)^{1-\frac{1}{\rho}}g_1(y)} + \frac{\beta(1-\frac{1}{\rho})g_2(y)}{\sum_y \psi(y)^{1-\frac{1}{\rho}}g_2(y)} \right) + \mu = 0 \quad (37)$$

where  $\mu$  is a Lagrange multiplier and

$$g_1(y) \stackrel{\scriptscriptstyle \Delta}{=} Q(y|0)^{\frac{1}{\rho}} \quad g_2(y) \stackrel{\scriptscriptstyle \Delta}{=} Q(y|0)^{\frac{1}{\rho}} \left(\frac{Q(y|1)}{Q(y|0)}\right)^{\lambda}$$

The solution to (37) is given in the following implicit form

$$\psi(y) = \zeta \left(g_1(y) + \kappa g_2(y)\right)^{\rho} = \zeta Q(y|0) \left[1 + \kappa \left(\frac{Q(y|1)}{Q(y|0)}\right)^{\lambda}\right]^{\rho}$$

where

$$\kappa = \frac{\beta}{1-\beta} \frac{\sum_{y} Q(y|0) \left(1 + \kappa \left(\frac{Q(y|1)}{Q(y|0)}\right)^{\lambda}\right)^{\rho-1}}{\sum_{y} Q(y|0) \left(\frac{Q(y|1)}{Q(y|0)}\right)^{\lambda} \left(1 + \kappa \left(\frac{Q(y|1)}{Q(y|0)}\right)^{\lambda}\right)^{\rho-1}}$$
(38)

The appropriate normalizing constant  $\zeta$  is given by

$$\zeta = \left[\sum_{y} Q(y|0) \left(1 + \kappa \left(\frac{Q(y|1)}{Q(y|0)}\right)^{\lambda}\right)^{\rho}\right]^{-1}$$
(39)

To find the optimized tilting measure, we solve (38) numerically, and determine  $\zeta$  by (39). The optimal values of  $\lambda$  and  $\rho$  are then found numerically.

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