

# A Constrained Coding Approach to Error-Free Half-Duplex Relay Networks

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**Abstract**—We show that the broadcast capacity of an infinite-depth tree-structured network of error-free half-duplex-constrained relays can be achieved using constrained coding at the source and symbol forwarding at the relays.

**Index Terms**—Relay channels, constrained coding, half-duplex constraint.

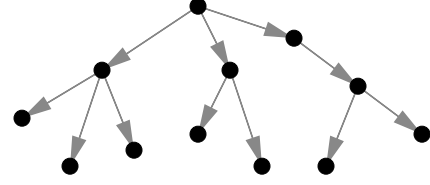


Fig. 1. A broadcast tree of depth  $D = 3$ .

## I. INTRODUCTION

INFORMATION transmission through a relay channel or network with error-free and/or half-duplex-constrained relays is a problem that has been considered by several authors [1]–[7]. In this paper the focus is on directed trees of error-free half-duplex-constrained relays, as shown in Fig. 1. Such networks include a chain of relays as a special case. The transmission objective is to broadcast information from a source (situated at the root of the tree) to all network nodes, each of which is half-duplex constrained. In each time slot, a node either receives (without error) the transmission of its parent, or broadcasts information to its children, but it may not do both.

More precisely, we assume that transmission between nodes in the network occurs in discrete time-slots. Let  $\mathcal{Q} := \{0, \dots, q-1\}$  be a  $q$ -ary transmission alphabet, let  $\mathbf{N}$  be an additional symbol which indicates a channel use without transmission, and let  $\mathcal{X} := \mathcal{Q} \cup \{\mathbf{N}\}$ . In any given time-slot, each node of the network broadcasts a symbol  $x \in \mathcal{X}$  to its children; the node is said to be ON if  $x \in \mathcal{Q}$ ; otherwise  $x = \mathbf{N}$  and the node is said to be OFF.

The half-duplex constraint is captured as follows. When a relay is OFF, it is connected to its parent through a noiseless  $(q+1)$ -ary channel with alphabet  $\mathcal{X}$ , and so receives the transmission from its parent without error. When a relay is ON, it cannot receive, so the symbol sent by its parent is erased.

The simplest approach to information broadcasting is to require each network node to be OFF half of the time, organized in deterministic fashion so that a node is OFF whenever its parent might be ON. Nodes simply forward what they receive, resulting in a transmission rate of  $0.5 \log_2(q+1)$  bits per

symbol (b/sym). The broadcast capacity, on the other hand, approaches [7]

$$C(q) := \log_2 \left( \frac{1 + \sqrt{4q+1}}{2} \right) \text{ b/sym} \quad (1)$$

as the tree-depth becomes large. In the binary case, deterministic store-and-forward achieves 0.5 b/sym whereas  $C(1) = \log_2 \phi = 0.6924$ , where  $\phi$  is the golden ratio. For trees of finite depth, even greater rates are possible. For example, for trees of depth  $D = 2$ , a rate of 0.7729 b/sym is achievable in the binary case. It is clear that deterministic store-and-forward falls short of the maximum possible transmission rate. However, to achieve the broadcast capacity of trees of finite depth requires a sophisticated coding approach based on coding additional information in the ON-OFF patterns of the nodes [7].

We note that ON-OFF patterns have also been exploited for neighbor discovery in half-duplex-constrained networks using a compressed sensing approach [8]–[10]. Another problem, namely a line of three nodes where the first two nodes are half-duplex sources and where all nodes are connected by packet erasure channels, was addressed in [11] within a queuing-theoretic framework. In [12] a Gaussian point-to-point channel with a sender subject to a duty cycle constraint (e.g., a half-duplex constraint) and an average power constraint is considered. Interestingly, the optimal input distribution is shown to be discrete, i.e., a modulated ON-OFF signaling scheme is capacity-achieving.

In this paper we will present a broadcasting scheme, based on constrained coding, that preserves the simplicity of the store-and-forward approach, but achieves a higher transmission rate than deterministic store-and-forward. In particular, we show that we can achieve a broadcast rate of  $C(q)$  in any error-free half-duplex-constrained tree network using constrained coding at the source and symbol forwarding at the relays.

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## II. CONSTRAINED CODING BACKGROUND

The approach we take to broadcasting in a tree of half-duplex-constrained nodes uses tools from constrained coding (or symbolic dynamics); see, e.g., [13], [14]. In a nutshell, the field of constrained coding studies mappings from unconstrained input sequences to output sequences obeying certain constraints. The constraints are often expressed by specifying forbidden sub-blocks, i.e., subsequences that are not permitted to occur in any output sequence. A classical example is the *golden mean shift*, which is the set of binary sequences in which the sub-block 11 never occurs. Constrained coding has found many applications in magnetic and optical recording systems.

The *capacity* of a constrained system, which is the maximum rate at which unconstrained binary data may be mapped to constrained output data, is defined as

$$C = \limsup_{n \rightarrow \infty} \frac{1}{n} \log_2 N(n) \text{ b/sym},$$

where  $N(n)$  denotes the number of sequences in the output alphabet having length  $n$  and satisfying the given constraint. For example, the golden mean shift satisfies the Fibonacci recurrence: for  $n \geq 2$ ,

$$N(n) = N(n-1) + N(n-2), \text{ with } N(0) = 1, N(1) = 2.$$

From this it can be shown that the golden mean shift has  $C = \log_2 \phi$ , where  $\phi = (1 + \sqrt{5})/2$  is the golden ratio (a result that explains the name “golden mean shift”). Interestingly, the golden ratio also arises in the analysis of the trapdoor channel [15] with feedback, where it is shown that the capacity equals  $\log_2 \phi$ .

It is well known that the capacity of certain constrained systems can be obtained via an irreducible, lossless graph presentation of the constraint [13]. If  $G$  is such a presentation, and  $A_G$  is the adjacency matrix of  $G$ , then

$$C = \log_2 \lambda(A_G),$$

where  $\lambda$  is the largest of the absolute values of the eigenvalues of  $A_G$ . This formulation of capacity will be used in the sequel.

## III. CODE CONSTRUCTION

We now describe the constrained coding approach taken in this paper. The transmission protocol is trivial, amounting to simple symbol-forwarding: during any given time-slot, every non-source node simply forwards (to all of its children) the symbol it has received from its parent during the previous time-slot. Correct forwarding is achieved provided that nodes obey the half-duplex constraint, i.e., that they are never ON when their parent node might be ON. Under the symbol-forwarding protocol, this is accomplished if and only if the source is never itself ON in two adjacent time-slots.

Thus we arrive naturally at a constrained coding problem: the source may emit any sequence of symbols drawn from  $\mathcal{X}$  satisfying the constraint that no two adjacent symbols are drawn from  $\mathcal{Q}$ . In the language of symbolic dynamics, every transmitted sequence is drawn from the shift of finite type denoted as  $\mathcal{X}_{\mathcal{Q}^2}$  having forbidden sub-block set  $\mathcal{Q}^2 := \mathcal{Q} \times$

$\mathcal{Q}$ . An irreducible, lossless graph presentation of this shift is shown in Fig. 2. When  $q = 1$ , this shift is equivalent to the golden mean shift.

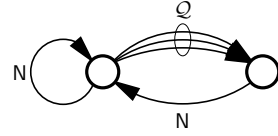


Fig. 2. Graph presentation of half-duplex constraint under symbol-forwarding.

The adjacency matrix of this presentation is given, as a function of  $q$ , as

$$A(q) = \begin{bmatrix} 1 & q \\ 1 & 0 \end{bmatrix},$$

which has characteristic polynomial

$$p_q(\lambda) = \lambda^2 - \lambda - q.$$

The eigenvalues of  $A(q)$  (the roots of  $p_q(\lambda)$ ) are given as

$$\lambda = \frac{1}{2} \left( 1 \pm \sqrt{1 + 4q} \right),$$

and the constrained capacity (the logarithm of the largest eigenvalue) is given as

$$C(q) = \log_2 \left( \frac{1 + \sqrt{1 + 4q}}{2} \right).$$

Remarkably—and this is the central result of this paper—the constrained coding approach achieves the broadcast capacity of infinite-depth trees, but without the necessity of designing sophisticated timing codes as in [7].

The capacity  $C(q)$  can be approached using methods (e.g., the state-splitting algorithm) from constrained coding. Fig. 3 provides two examples. The first, in Fig. 3(a), is a standard example in constrained coding [14] and gives a rate-(2/3) encoder for  $q = 1$ , which achieves more than 96% of the capacity  $C(1) = \log_2(\phi)$ . The second, in Fig. 3(b), is a rate-(3/2) encoder for  $q = 6$ , which achieves more than 94% of the capacity  $C(6) = \log_2(3)$ . The encoder can be constructed in three steps. First, the second power graph is generated from the golden mean shift shown in Fig. 2. Subsequently, the state with most outgoing edges is split (according to the splitting criteria of the state-splitting algorithm). In the final step, sufficiently many edges are deleted so that eight outgoing edges per state remain. Similar examples can readily be constructed for other values of  $q$ . For any given  $q$ , if the number of encoder states is allowed to grow,  $C(q)$  can be approached arbitrarily closely.

## IV. DISCUSSION

Table I compares, for  $q = 1$ , rates achievable in networks of finite depth  $D$  using three approaches: the timing codes presented in [7], the constrained coding approach of this paper, and the deterministic store-and-forward approach. The row labelled  $C$ , which gives the maximum achievable rate (using timing codes), serves as a benchmark for the other schemes. We observe that  $C(q)/C$ , the relative efficiency of constrained

