

Five Families of Three-Weight Ternary Cyclic Codes and Their Duals

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Abstract

As a subclass of linear codes, cyclic codes have applications in consumer electronics, data storage systems, and communication systems as they have efficient encoding and decoding algorithms. In this paper, five families of three-weight ternary cyclic codes whose duals have two zeros are presented. The weight distributions of the five families of cyclic codes are settled. The duals of two families of the cyclic codes are optimal.

Index Terms

Cyclic codes, weight distribution, weight enumerator, frequency hopping sequences, secret sharing.

I. INTRODUCTION

Let p be a prime. An $[n, \kappa, d]$ linear code over $\text{GF}(p)$ is a κ -dimensional subspace of $\text{GF}(p)^n$ with minimum nonzero (Hamming) weight d . A linear $[n, \kappa]$ code C over $\text{GF}(p)$ is called *cyclic* if $(c_0, c_1, \dots, c_{n-1}) \in C$ implies $(c_{n-1}, c_0, c_1, \dots, c_{n-2}) \in C$. By identifying any vector $(c_0, c_1, \dots, c_{n-1}) \in \text{GF}(p)^n$ with

$$c_0 + c_1x + c_2x^2 + \dots + c_{n-1}x^{n-1} \in \text{GF}(p)[x]/(x^n - 1),$$

any linear code C of length n over $\text{GF}(p)$ corresponds to a subset of the quotient ring $\text{GF}(p)[x]/(x^n - 1)$. A linear code C is cyclic if and only if the corresponding subset in $\text{GF}(p)[x]/(x^n - 1)$ is an ideal of the ring $\text{GF}(p)[x]/(x^n - 1)$.

It is well known that every ideal of $\text{GF}(p)[x]/(x^n - 1)$ is principal. Let $C = \langle g(x) \rangle$ be a cyclic code, where $g(x)$ is monic and has the smallest degree among all the generators of C . Then $g(x)$ is unique and called the *generator polynomial*, and $h(x) = (x^n - 1)/g(x)$ is referred to as the *parity-check polynomial* of C . If the parity check polynomial $h(x)$ of a code C of length n over $\text{GF}(p)$ is the product of t distinct irreducible polynomials over $\text{GF}(p)$, we say that the dual code C^\perp has t zeros.

Let A_i denote the number of codewords with Hamming weight i in a code C of length n . The *weight enumerator* of C is defined by $1 + A_1y + A_2y^2 + \dots + A_ny^n$. The *weight distribution* $\{A_0, A_1, \dots, A_n\}$ is an important research topic in coding theory. First, it contains crucial information as to estimate the error correcting capability and the probability of error detection and correction with respect to some algorithms [14]. Second, due to rich algebraic structures of cyclic codes, the weight distribution is often related to interesting and challenging problems in number theory.

As a subclass of linear codes, cyclic codes have been widely used in consumer electronics, data transmission technologies, broadcast systems, and computer applications as they have efficient encoding and decoding algorithms. Cyclic codes with a few weights are of special interest in authentication codes as certain parameters of the authentication codes constructed from these cyclic codes are easy to compute [9], and in secret sharing schemes as the access structures of such secret sharing schemes derived from such

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cyclic code are easy to determine [2], [8], [26]. Cyclic codes with a few weights are also of special interest in designing frequency hopping sequences [4], [7]. Three-weight cyclic codes have also applications in association schemes [1]. These are some of the motivations of studying cyclic codes with a few weights.

In this paper, five families of three-weight ternary cyclic codes whose duals have two zeros are presented. The weight distributions of the five families of cyclic codes are settled. The duals of two families of the cyclic codes proposed in this paper are optimal ternary codes. As a byproduct, three new decimation values of maximum-length sequences giving only three correlation values are also obtained in this paper.

This paper is organized as follows. Section II fixes some notations for this paper. Section III defines cyclic codes over $\text{GF}(p)$ whose duals have two zeros. Section IV-A presents two lemmas that will be needed in the sequel. Section IV defines two families of cyclic codes and determines their weight distributions. Section V describes three families cyclic codes and determines their weight distributions. Section VI makes concluding remarks.

II. SOME NOTATIONS FIXED THROUGHOUT THIS PAPER

Throughout this paper, we adopt the following notations unless otherwise stated:

- p is a prime and $q = p^m$, where m is a positive integer.
- $n = q - 1$, which is the length of a cyclic code over $\text{GF}(p)$.
- $\text{Tr}_1^j(x)$ is the trace function from $\text{GF}(p^j)$ to $\text{GF}(p)$ for any positive integer j .
- χ is the canonical additive character on $\text{GF}(q)$, i.e., $\chi(x) = e^{2\pi\sqrt{-1}\text{Tr}_1^m(x)/p}$ for any $x \in \text{GF}(q)$.
- \mathcal{C}_a denotes the p -cyclotomic coset modulo n containing a , where a is any integer with $0 \leq a \leq r - 2$, and $\ell_a := |\mathcal{C}_a|$ denote the size of the cyclotomic coset \mathcal{C}_a .
- By the Database we mean the collection of the tables of best linear codes known maintained by Markus Grassl at <http://www.codetables.de/>.

III. CYCLIC CODES WHOSE DUALS HAVE TWO ZEROS

Given a positive integer m , recall that $q = p^m$ and $n = q - 1$ throughout this paper. Let α be a generator of the multiplicative group $\text{GF}(q)^*$. For any $0 \leq a \leq q - 2$, denote by $m_a(x)$ the minimal polynomial of α^{-a} over $\text{GF}(p)$.

Let $0 \leq u \leq q - 2$ and $0 \leq v \leq q - 2$ be any two integers such that $\mathcal{C}_u \cap \mathcal{C}_v = \emptyset$. Let $\mathcal{C}_{(u,v,p,m)}$ be the cyclic code over $\text{GF}(p)$ with length n whose codewords are given by

$$\mathbf{c}(a, b) = (c_0, c_1, \dots, c_{n-1}), \quad \forall (a, b) \in \text{GF}(p^{\ell_u}) \times \text{GF}(p^{\ell_v}), \quad (1)$$

where

$$c_i = \text{Tr}_1^{\ell_u}(a\alpha^{iu}) + \text{Tr}_1^{\ell_v}(b\alpha^{iv}), \quad 0 \leq i \leq n - 1.$$

By Delsarte's Theorem, the code $\mathcal{C}_{(u,v,p,m)}$ has parity-check polynomial $m_u(x)m_v(x)$ and dimension $\ell_u + \ell_v$.

In terms of exponential sums, the Hamming weight $\text{wt}(\mathbf{c}(a, b))$ of the codeword $\mathbf{c}(a, b)$ of (1) in $\mathcal{C}_{(u,v,p,m)}$ is given by

$$\text{wt}(\mathbf{c}(a, b)) = (p - 1)p^{m-1} - \frac{1}{p} \sum_{y \in \text{GF}(p)^*} T_v(ya, yb) \quad (2)$$

where

$$T_{(u,v)}(a, b) = \sum_{x \in \text{GF}(q)} \chi(ax^u + bx^v) \quad (3)$$

for each $(a, b) \in \text{GF}(q)^2$. Throughout this section, the function $T_{(u,v)}(a, b)$ is always defined as in (3) for any given u and v .

The following lemma is an extension of Lemma 6.1 in [27], and will be frequently used in the sequel when we determine the weight distributions of the five families of cyclic codes.

Lemma 3.1: Let s be any integer with $\gcd(s, q-1) = 2$. Then

$$T_{(u,v)}(a,b) = \frac{1}{2} \left(\sum_{x \in \text{GF}(q)} \chi(ax^{su} + bx^{sv}) + \sum_{x \in \text{GF}(q)} \chi(a\lambda^u x^{su} + b\lambda^v x^{sv}) \right)$$

where λ is any fixed nonsquare in $\text{GF}(q)^*$.

Proof: Let $C_0^{(2,q)}$ denote the set of all nonzero squares in $\text{GF}(q)$. Then

$$T_{(u,v)}(a,b) = 1 + \sum_{x \in C_0^{(2,q)}} \chi(ax^u + bx^v) + \sum_{x \in C_0^{(2,q)}} \chi(a\lambda^u x^u + b\lambda^v x^v). \quad (4)$$

Note that $\gcd(q-1, s) = 2$. When x runs through $\text{GF}(q)$, x^s runs twice through the nonzero squares in $\text{GF}(q)$ and takes on the value 0 once. Similarly, λx^s runs twice through all the nonsquares in $\text{GF}(q)$ and takes on the value 0 once. The conclusion then follows directly from (4) and the discussions above. ■

There are a lot of references on the codes $C_{(u,v,p,m)}$ (see for example [6], [10], [16], [12], [17], [19], [20], [21], [23], [24]). This family of cyclic codes $C_{(u,v,p,m)}$ may have many nonzero weights. It is obvious that $C_{(u,v,p,m)}$ cannot be a constant-weight code as its parity-check polynomial has two zeros. When $q = 2$, the codes $C_{(u,v,p,m)}$ cannot be a two-weight code if $C_u \cap C_v = \emptyset$ and $\ell_u > 1$ and $\ell_v > 1$. When q is odd, $C_{(u,v,p,m)}$ could be a two-weight code.

A number of three-weight nonbinary cyclic codes $C_{(u,v,p,m)}$ have been constructed (see for example [2], [3], [10], [15], [22], [25]). In this paper, we present five families of three-weight ternary cyclic codes $C_{(u,v,p,m)}$, determine their weight distributions and study their duals.

IV. TWO FAMILIES OF THREE-WEIGHT TERNARY CYCLIC CODES AND THEIR WEIGHT ENUMERATORS

In this section, we propose two families of three-weight cyclic codes $C_{(u,v,3,m)}$ over $\text{GF}(3)$ where u and v are some integers with $C_u \cap C_v = \emptyset$ and $(\ell_u, \ell_v) = (m, m)$. It is obvious that the code $C_{(u,v,3,m)}$ has length $3^m - 1$ and dimension $2m$ under these assumptions.

A. Some auxiliary results

In this subsection, we introduce a lemma on exponential sums over finite fields and a lemma regarding the dual code $C_{(1,v,3,m)}^\perp$ of the code $C_{(1,v,3,m)}$. Recall that χ is the canonical additive character of $\text{GF}(q)$ defined in Section II.

The following lemma will be employed in the sequel, and was proved in [27] with the help of some results from [25], [15].

Lemma 4.1: Let m be odd and h be an integer with $\gcd(m, h) = 1$. Define

$$R(a,b) = \sum_{x \in \text{GF}(q)} \chi(ax^{p^h+1} + bx^2).$$

Then, as (a,b) runs through $\text{GF}(q)^2$, the values of the sum

$$\sum_{y \in \text{GF}(p)^*} (R(ya, yb) + R(-ya, yb))$$

have the following distribution

Value	Frequency
$2(p-1)p^m$	1
$(p-1)p^{(m+1)/2}$	$(p^{m-1} + p^{(m-1)/2})(p^m - 1)$
0	$(p^m - 2p^{m-1} + 1)(p^m - 1)$
$-(p-1)p^{(m+1)/2}$	$(p^{m-1} - p^{(m-1)/2})(p^m - 1)$

The following lemma is proved in [5] and will be employed in the sequel.

TABLE I
WEIGHT DISTRIBUTION I

Weight w	No. of codewords A_w
0	1
$(p-1)p^{m-1} - \frac{p-1}{2}p^{(m-1)/2}$	$(p^m-1)(p^{m-1} + p^{(m-1)/2})$
$(p-1)p^{m-1}$	$(p^m-1)(p^m - 2p^{m-1} + 1)$
$(p-1)p^{m-1} + \frac{p-1}{2}p^{(m-1)/2}$	$(p^m-1)(p^{m-1} - p^{(m-1)/2})$

Lemma 4.2: Let $v \notin C_1$ and $\ell_v = |C_v| = m$. Then the dual $C_{(1,v,3,m)}^\perp$ of the ternary cyclic code $C_{(1,v,3,m)}$ has parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ if and only if the following conditions are satisfied:

C1: v is even;

C2: the equation $(-x-1)^v + x^v + 1 = 0$ has the only solution $x = 1$ in $\text{GF}(q)^*$; and

C3: the equation $(x+1)^v - x^v - 1 = 0$ has the only solution $x = 0$ in $\text{GF}(q)$.

B. The first family of three-weight cyclic codes

In this subsection, we study the cyclic codes $C_{(1,v,p,m)}$, where m is odd, $p = 3$, and $v = (3^{m+1} - 1)/4$. The parameters of the codes are described in the following theorem.

Theorem 4.3: Let m be odd, $p = 3$, and $v = (3^{m+1} - 1)/4$. Then $C_{(1,v,p,m)}$ is a $[p^m - 1, 2m]$ cyclic code over $\text{GF}(p)$ with the weight distribution in Table I.

Proof: Let $h = 1$ and $s = 3^h + 1$. Then $\gcd(s, 3^m - 1) = 2$ since m is odd. It is easy to check that $sv \equiv 2 \pmod{3^m - 1}$. Noticing that v is even and -1 is a nonsquare in $\text{GF}(q)$. By Lemma 3.1, we have

$$T_{(1,v)}(a, b) = \frac{1}{2} (R_{(1,v)}(a, b) + R_{(1,v)}(-a, b))$$

where

$$R_{(1,v)}(a, b) = \sum_{x \in \text{GF}(q)} \chi(ax^{3^h+1} + bx^2).$$

It then follows from (2) that

$$\text{wt}(\mathbf{c}(a, b)) = 2 \times 3^{m-1} - \frac{1}{6} \sum_{y \in \text{GF}(3)^*} (R_{(1,v)}(ya, yb) + R_{(1,v)}(-ya, yb)). \quad (5)$$

Note that $\gcd(m, h) = \gcd(m, 1) = 1$, the weight distribution of the code $C_{(1,v,3,m)}$ then follows from Equation (5) and Lemma 4.1. \blacksquare

Example 4.4: Let $p = 3$ and $m = 3$. Let α be the generator of $\text{GF}(p^m)^*$ with $\alpha^3 + 2\alpha + 1 = 0$. Then $v = 20$ and $C_{(1,v,p,m)}$ is a $[26, 6, 15]$ code over $\text{GF}(3)$ with parity-check polynomial $x^6 + 2x^3 + 2x^2 + x + 2$ and weight enumerator $1 + 312y^{15} + 260y^{18} + 156y^{21}$. It has the same parameters as the best known cyclic codes in the Database, and is optimal.

Example 4.5: Let $p = 3$ and $m = 5$. Let α be the generator of $\text{GF}(p^m)^*$ with $\alpha^5 + 2\alpha + 1 = 0$. Then $v = 182$ and $C_{(1,v,p,m)}$ is a $[242, 10, 153]$ code over $\text{GF}(3)$ with parity-check polynomial $x^{10} + 2x^9 + 2x^8 + 2x^7 + 2x^5 + x^4 + 2x^3 + x^2 + x + 2$ and weight enumerator

$$1 + 21780y^{153} + 19844y^{162} + 17424y^{171}.$$

It has the same parameters as the best known cyclic codes in the Database. It is optimal or almost optimal since the upper bound on the minimal distance of any ternary linear code with length 242 and dimension 6 is 154.

The following theorem describes the parameters of the dual code $C_{(1,v,3,m)}^\perp$ of the code $C_{(1,v,3,m)}$ in Theorem 4.3.

Theorem 4.6: Let m be odd, $p = 3$, and $v = (3^{m+1} - 1)/4$. Then $C_{(1,v,p,m)}^\perp$ is a $[3^m - 1, 3^m - 1 - 2m, 4]$ cyclic code over $\text{GF}(3)$.

Proof: The dimension of $C_{(1,v,p,m)}^\perp$ follows from that of $C_{(1,v,p,m)}$. So we need to prove that the minimum distance d^\perp of $C_{(1,v,p,m)}^\perp$ is 4. To this end, we prove that the three conditions in Lemma 4.2 are met.

Obviously, v is even. So Condition C1 in Lemma 4.2 is satisfied.

We now consider Condition C2 in Lemma 4.2 and study the solutions $x \in \text{GF}(q)$ of the following equation

$$(x+1)^v + x^v = -1. \quad (6)$$

It is clear that Equation (6) is equivalent to the following

$$y - x = 1, \quad y^v + x^v = -1. \quad (7)$$

Note that -1 is a nonsquare in $\text{GF}(q)$ and $\gcd(4, q-1) = 2$ as m is odd. We now consider the solutions $(x, y) \in \text{GF}(q)^2$ of (7) by distinguishing among the following four cases.

Case 1, $x = x_1^4$ and $y = y_1^4$ for some $(x_1, y_1) \in \text{GF}(q)^2$: In this case, (x_1, y_1) is a solution of

$$y_1^4 - x_1^4 = 1, \quad y_1^2 + x_1^2 = -1. \quad (8)$$

It then follows that

$$y_1^2 + x_1^2 = -1, \quad y_1^2 - x_1^2 = -1.$$

Hence $y_1^2 = -1$, which is impossible as -1 is not a square in $\text{GF}(3^m)$.

Case 2, $x = -x_1^4$ and $y = -y_1^4$ for some $(x_1, y_1) \in \text{GF}(q)^2$: In this case, (x_1, y_1) is a solution of

$$x_1^4 - y_1^4 = 1, \quad y_1^2 + x_1^2 = -1. \quad (9)$$

It then follows that

$$y_1^2 + x_1^2 = -1, \quad y_1^2 - x_1^2 = 1.$$

Hence $y_1 = 0$ and $-1 = x_1^2$, which is impossible as -1 is not a square in $\text{GF}(q)$.

Case 3, $x = x_1^4$ and $y = -y_1^4$ for some $(x_1, y_1) \in \text{GF}(q)^2$: In this case, (x_1, y_1) is a solution of

$$y_1^4 + x_1^4 = -1, \quad y_1^2 + x_1^2 = -1. \quad (10)$$

It then follows that

$$1 = (y_1^2 + x_1^2)^2 = -1 + 2(x_1 y_1)^2$$

and $1 = (x_1 y_1)^2$. Hence x_1^2 and y_1^2 are the solutions of $z^2 - 2z + 1 = 0$. Thus, $x_1^2 = y_1^2 = 1$. Whence $x = x_1^4 = 1$.

Case 4, $x = -x_1^4$ and $y = y_1^4$ for some $(x_1, y_1) \in \text{GF}(q)^2$: In this case, (x_1, y_1) is a solution of

$$y_1^4 + x_1^4 = 1, \quad y_1^2 + x_1^2 = -1. \quad (11)$$

It then follows that

$$1 = (y_1^2 + x_1^2)^2 = 1 + 2(x_1 y_1)^2.$$

Hence $0 = (x_1 y_1)^2$. Therefore $x_1 = 0$ or $y_1 = 0$. Hence $-1 = y_1^2$ or $-1 = x_1^2$, which are impossible.

Summarizing the four cases above, we proved that Condition C3 in Lemma 4.2 is satisfied.

One can similarly prove that Condition C3 of Lemma 4.2 is also met.

The desired conclusion of the parameters of the code then follows from Lemma 4.2. ■

Note that the codes $C_{(1,v,p,m)}^\perp$ of Theorem 4.6 are optimal in the sense that the minimum distance of any ternary linear code of length $3^m - 1$ and dimension $3^m - 1 - 2m$ is at most 4 due to the sphere-packing bound.

Example 4.7: Let $p = 3$ and $m = 3$. Let α be the generator of $\text{GF}(p^m)^*$ with $\alpha^3 + 2\alpha + 1 = 0$. Then $v = 20$ and $C_{(1,v,p,m)}^\perp$ is a $[26, 20, 4]$ code over $\text{GF}(3)$ with generator polynomial $x^6 + 2x^3 + 2x^2 + x + 2$.

C. The second family of three-weight cyclic codes

In this subsection, we analyze the cyclic codes $C_{(1,v,p,m)}$, where $m \equiv 7 \pmod{8}$, $p = 3$, and $v = (3^{(m+1)/8} - 1)(3^{(m+1)/4} + 1)(3^{(m+1)/2} + 1)$. The parameters of the codes are described in the following theorem.

Theorem 4.8: Let $m \equiv 7 \pmod{8}$, $p = 3$ and $v = (3^{(m+1)/8} - 1)(3^{(m+1)/4} + 1)(3^{(m+1)/2} + 1)$. Then $C_{(1,v,3,m)}$ is a $[3^m - 1, 2m]$ cyclic code over $\text{GF}(3)$ with the weight distribution in Table I.

Proof: Let $h = (m+1)/8$ and $s = 3^h + 1$. Since $m \equiv 7 \pmod{8}$, $\gcd(s, 3^m - 1) = 2$ and v is even. It is easy to check that $sv \equiv 2 \pmod{3^m - 1}$. Select $\lambda = -1$ as a nonsquare in $\text{GF}(p^m)$. Applying Lemma 3.1, we have

$$T_{(1,v)}(a, b) = \frac{1}{2} (R_{(1,v)}(a, b) + R_{(1,v)}(-a, b)) \quad (12)$$

where

$$R_{(1,v)}(a, b) = \sum_{x \in \text{GF}(q)} \chi(ax^{3^h+1} + bx^2).$$

It then follows from (2) and (12) that

$$\text{wt}(\mathbf{c}(a, b)) = 2 \times 3^{m-1} - \frac{1}{6} \sum_{y \in \text{GF}(3)^*} (R_{(1,v)}(ya, yb) + R_{(1,v)}(-ya, yb)). \quad (13)$$

The weight distribution of the code $C_{(1,v,3,m)}$ then follows from (13) and Lemma 4.1. \blacksquare

Example 4.9: Let $p = 3$ and $m = 7$. Let α be a generator of $\text{GF}(q)^*$ with $\alpha^7 + 2\alpha^2 + 1 = 0$. Then $v = 1640$ and $C_{(1,v,p,m)}$ is a $[2186, 14, 1431]$ code over $\text{GF}(3)$ with parity-check polynomial

$$x^{14} + 2x^{13} + x^{12} + x^{11} + x^9 + 2x^8 + 2x^7 + x^6 + 2x^3 + x^2 + x + 2$$

and weight enumerator

$$1 + 1652616y^{1431} + 1595780y^{1458} + 1534572y^{1485}.$$

Now we determine the parameters of the dual code $C_{(1,v,3,m)}^\perp$ of $C_{(1,v,3,m)}$ in Theorem 4.8. To this end, we need the following lemmas whose proofs are omitted.

Lemma 4.10: Let $m \equiv 7 \pmod{8}$ and $q = 3^m$. Define $h = (m+1)/8$ and $s = 3^h + 1$. Then all solutions $(x_1, y_1) \in \text{GF}(q) \times \text{GF}(q)$ of the equation $y_1^2 - x_1^2 = 1$ can be expressed as

$$x_1 = \theta - \theta^{-1}, \quad y_1 = -\theta - \theta^{-1} \quad (14)$$

where $\theta \in \text{GF}(q)^*$. In addition, we have

$$\begin{cases} x_1^s &= \theta^s + \theta^{-s} - (\theta^{3^{(m+1)/8}-1} + \theta^{1-3^{(m+1)/8}}) \\ y_1^s &= \theta^s + \theta^{-s} + (\theta^{3^{(m+1)/8}-1} + \theta^{1-3^{(m+1)/8}}). \end{cases} \quad (15)$$

Lemma 4.11: Let $m \equiv 7 \pmod{8}$. Define $h = (m+1)/8$ and $s = 3^h + 1$. If $m \equiv 7 \pmod{16}$, then

$$\begin{cases} s \equiv 4 \pmod{8} \\ \gcd(s, q^2 - 1) = 4 \\ \gcd(3^h - 1, q^2 - 1) = 2. \end{cases}$$

If $m \equiv -1 \pmod{16}$, then

$$\begin{cases} s \equiv 2 \pmod{8} \\ \gcd(s, q^2 - 1) = 2 \\ \gcd(3^h - 1, q^2 - 1) = 8. \end{cases}$$

Lemma 4.12: Let m be odd and $q = 3^m$. Let γ be a generator of $\text{GF}(q^2)$ and let $\varepsilon = \gamma^{(q^2-1)/4}$. Then All solutions of $(x_1, y_1) \in \text{GF}(q^2) \times \text{GF}(q^2)$ of the equation $x_1^2 + y_1^2 = -1$ can be expressed as

$$x_1 = \varepsilon(\theta + \theta^{-1}), \quad y_1 = \theta - \theta^{-1} \quad (16)$$

for some $\theta \in \text{GF}(q^2)^*$, where $\varepsilon^2 = -1$.

Furthermore,

$$\begin{aligned} x_1^s &= (\theta^s + \theta^{-s} + \theta^{3^h-1} + \theta^{1-3^h})\varepsilon^s \\ y_1^s &= \theta^s + \theta^{-s} - (\theta^{3^h-1} + \theta^{1-3^h}). \end{aligned}$$

The following theorem describes the parameters of the dual code $C_{(1,v,3,m)}^\perp$ of $C_{(1,v,3,m)}$ in Theorem 4.8.

Theorem 4.13: Let $m \equiv 7 \pmod{8}$, $p = 3$, and $v = (3^{(m+1)/8} - 1)(3^{(m+1)/4} + 1)(3^{(m+1)/2} + 1)$. Then $C_{(1,v,p,m)}^\perp$ is a $[3^m - 1, 3^m - 1 - 2m, 4]$ cyclic code over $\text{GF}(3)$.

Proof: The dimension of $C_{(1,v,p,m)}^\perp$ follows from that of $C_{(1,v,p,m)}$. So we need to prove that the minimum distance d^\perp of $C_{(1,v,p,m)}^\perp$ is 4. To this end, we prove that the three conditions in Lemma 4.2 are met.

Define $h = (m+1)/8$ and $s = 3^h + 1$. It is easily seen that $\gcd(s, q-1) = 2$ and $sv \equiv 2 \pmod{q-1}$. Note that -1 is a nonsquare in $\text{GF}(q)$ as m is odd.

Obviously, v is even. So Condition C1 in Lemma 4.2 is satisfied. We now consider Condition C3, and study the solutions $x \in \text{GF}(q)$ of the following equation

$$(x+1)^v - x^v = 1. \quad (17)$$

It is clear that Equation (17) is equivalent to the following system of equations

$$y - x = 1, \quad y^v - x^v = 1. \quad (18)$$

We now consider the solutions $(x, y) \in \text{GF}(q)^2$ of (18) by distinguishing among the following four cases.

Case 1, $x = x_1^s$ and $y = y_1^s$ for some $(x_1, y_1) \in \text{GF}(q)^2$: In this case, (x_1, y_1) is a solution of

$$y_1^s - x_1^s = 1, \quad y_1^2 - x_1^2 = 1. \quad (19)$$

Let $\theta = y_1 - x_1$. Clearly, $\theta \neq 0$. Then plugging the expressions of (x_1, y_1) in Lemma 4.10 into the first equation of (19) yields

$$\theta^{3^{(m+1)/8}-1} + \theta^{1-3^{(m+1)/8}} = -1. \quad (20)$$

Obviously, the equation $z + z^{-1} = -1$ has the unique solution $z = 1$. In this case, it follows from (20) that $\theta^{3^{(m+1)/8}-1} = 1$. Note that $\gcd(3^{(m+1)/8} - 1, 3^m - 1) = 2$. Hence $\theta = \pm 1$. It then follows from the first equation in (14) that $x_1 = \theta - \theta^{-1} = 0$. Hence $x = x_1^s = 0$.

Case 2, $x = -x_1^s$ and $y = -y_1^s$ for some $(x_1, y_1) \in \text{GF}(q)^2$: In this case, (x_1, y_1) is a solution of

$$y_1^s - x_1^s = -1, \quad y_1^2 - x_1^2 = 1. \quad (21)$$

Let $\theta = y_1 - x_1$. Clearly, $\theta \neq 0$. Then plugging the expressions of (x_1, y_1) in Lemma 4.10 into the first equation of (21) yields

$$\theta^{3^{(m+1)/8}-1} + \theta^{1-3^{(m+1)/8}} = 1. \quad (22)$$

The equation $z + z^{-1} = 1$ has the unique solution $z = -1$. It then follows from (22) that $\theta^{3^{(m+1)/8}-1} = -1$. This is impossible as -1 is not a square in $\text{GF}(q)$.

Case 3, $x = -x_1^s$ and $y = y_1^s$ for some $(x_1, y_1) \in \text{GF}(q)^2$: In this case, (x_1, y_1) is a solution of

$$y_1^s + x_1^s = 1, \quad y_1^2 - x_1^2 = 1. \quad (23)$$

Let $\theta = y_1 - x_1$. Clearly, $\theta \neq 0$. Then plugging the expressions of (x_1, y_1) in Lemma 4.10 into the first equation of (23) gives $2(\theta^s + \theta^{-s}) = 1$. Note that $\gcd(s, 3^m - 1) = 2$. We obtain that $\theta = \pm 1$. It then follows that $x_1 = 0$ and $x = x_1^s = 0$.

Case 4, $x = x_1^s$ and $y = -y_1^s$ for some $(x_1, y_1) \in \text{GF}(q)^2$: In this case, (x_1, y_1) is a solution of

$$y_1^s + x_1^s = -1, \quad y_1^2 - x_1^2 = 1. \quad (24)$$

Let $\theta = y_1 - x_1$. Clearly, $\theta \neq 0$. Then plugging the expressions of (x_1, y_1) in Lemma 4.10 into the first equation of (24) gives

$$\theta^s + \theta^{-s} = 1,$$

which is impossible, as $z + z^{-1} = 1$ has no solution $x \in \text{GF}(q)$.

Summarizing the conclusions in the four cases above, we proved that Condition C3 in Lemma 4.2 is satisfied.

One can similarly prove that Condition C2 in Lemma 4.2 is satisfied.

Finally, the desired conclusions on the parameters of this code follow from Lemma 4.2. ■

Note that the codes $C_{(1,v,p,m)}^\perp$ of Theorem 4.13 are optimal in the sense that the minimum distance of any ternary linear code of length $3^m - 1$ and dimension $3^m - 1 - 2m$ is at most 4 due to the sphere-packing bound.

V. THREE FAMILIES OF THREE-WEIGHT TERNARY CYCLIC CODES AND THEIR WEIGHT ENUMERATORS

In this section, we present three families of three-weight codes whose weight distributions are given in Table II and are different from the one in Table I. To this end, we need the following lemma proved by Feng and Luo [11].

Lemma 5.1: Let $m \geq 3$ be odd, p be an odd prime and let h be a positive integer with $\gcd(m, h) = 1$. Then the code $C_{(p^h+1, 2, p, m)}$ has dimension $2m$ and the weight distribution in Table II.

TABLE II
WEIGHT DISTRIBUTION II

Weight w	No. of codewords A_w
0	1
$(p-1)(p^{m-1} - p^{(m-1)/2})$	$\frac{1}{2}(p^m - 1)(p^{m-1} + p^{(m-1)/2})$
$(p-1)p^{m-1}$	$(p^m - 1)(p^m - p^{m-1} + 1)$
$(p-1)(p^{m-1} + p^{(m-1)/2})$	$\frac{1}{2}(p^m - 1)(p^{m-1} - p^{(m-1)/2})$

Theorem 5.2: Let $m \geq 3$ be odd and $p = 3$. Then $C_{(1,v,p,m)}$ is a $[p^m - 1, 2m]$ ternary cyclic code with the weight distribution in Table II if

- $v = (3^{m+1} - 1)/(3^h + 1) + (3^m - 1)/2$, where $(m+1)/h$ is even; or
- $v = (3^{(m+1)/8} - 1)(3^{(m+1)/4} + 1)(3^{(m+1)/2} + 1) + (3^m - 1)/2$, where $m \equiv 7 \pmod{8}$; or
- $v = (3^{(m+1)/4} - 1)(3^{(m+1)/2} + 1) + (3^m - 1)/2$, where $m \equiv 3 \pmod{4}$.

Proof: We now prove the conclusion of this theorem for the first v . Let $p = 3$ and $q = p^m - 1$. Define $\lambda = \alpha^{(q-1)/(p-1)}$, where α is a generator of $\text{GF}(q)^*$. Clearly λ is a generator of $\text{GF}(p)^*$. Since m is odd, λ is a nonsquare in $\text{GF}(q)$. It is easy to verify that $\lambda^v = \lambda$.

Let $s = p^h + 1$. Then $\gcd(s, p^m - 1) = 2$ since m is odd. It is easy to verify that v is odd and $sv \equiv 2 \pmod{q-1}$. Applying Lemma 3.1, we have

$$\begin{aligned} T_{(1,v)}(a,b) &= \frac{1}{2} \left(\sum_{x \in \text{GF}(q)} \chi(ax^s + bx^{sv}) + \sum_{x \in \text{GF}(q)} \chi(a\lambda x^s + b\lambda^v x^{sv}) \right) \\ &= \frac{1}{2} \left(\sum_{x \in \text{GF}(q)} \chi(ax^s + bx^{sv}) + \sum_{x \in \text{GF}(q)} \chi(a\lambda x^s + b\lambda x^{sv}) \right) \\ &= \frac{1}{2} \left(\sum_{x \in \text{GF}(q)} \chi(ax^{p^h+1} + bx^2) + \sum_{x \in \text{GF}(q)} \chi(a\lambda x^{p^h+1} + b\lambda x^2) \right). \end{aligned}$$

Hence,

$$\sum_{y \in \text{GF}(p)^*} T_{(1,v)}(ay, by) = \sum_{y \in \text{GF}(p)^*} \sum_{x \in \text{GF}(q)} \chi(ayx^{p^h+1} + byx^2).$$

It then follows from (2) that the Hamming weight of the codeword $\mathbf{c}(a,b)$ in $\mathcal{C}_{(1,v,p,m)}$ is equal to that of the codeword $\mathbf{c}(a,b)$ in $\mathcal{C}_{(p^h+1,2,p,m)}$. Hence the two codes have the same weight distribution. The desired conclusion on the weight distribution of the code $\mathcal{C}_{(1,v,p,m)}$ follows from Lemma 5.1 as $\gcd(h,m) = 1$.

The proof of the conclusion for $v = (3^{(m+1)/8} - 1)(3^{(m+1)/4} + 1)(3^{(m+1)/2} + 1) + (3^m - 1)/2$ is similar to that for the first v except that we need to set $h = (m+1)/8$ and is omitted.

The proof of the conclusion for $(3^{(m+1)/4} - 1)(3^{(m+1)/2} + 1) + (3^m - 1)/2$ is similar to that for the first v except that we need to set $h = (m+1)/4$ and is omitted. \blacksquare

The dual codes $\mathcal{C}_{(1,v,p,m)}^\perp$ of the codes $\mathcal{C}_{(1,v,p,m)}$ in Theorem 5.2 has parameters $[p^m - 1, p^m - 1 - 2m, 2]$. The minimum distance of $\mathcal{C}_{(1,v,p,m)}^\perp$ is 2 as v is odd.

VI. SUMMARY AND CONCLUDING REMARKS

In this paper, we presented five families of three-weight ternary cyclic codes and settled their weight distributions. The duals of the first two families of ternary codes are optimal. The first two families of cyclic codes have the weight distribution in Table I, while the last three families have the weight distribution of table II. It would be interesting to investigate the applications of these cyclic codes in authentication codes, secret sharing and frequency hopping sequences using the frameworks developed in [2], [4], [8], [9], [26].

The key technique for settling the weight distributions of these cyclic codes in this paper is the application of the noninvertible transformations $x \mapsto x^s$ from $\text{GF}(q)$ to $\text{GF}(q)$ with $\gcd(s, q-1) = 2$ and Lemma 3.1. With these innovations we were able to determine the weight distributions of the cyclic codes with the help of known results on certain exponential sums.

Note that $\gcd(v, q-1) = 1$ and $v-1 \equiv 0 \pmod{p-1}$ for all the v 's listed in Theorem 5.2. It follows from the discussions in Section A2 in [13] and the weight distribution of the code $\mathcal{C}_{(1,v,p,m)}$ of Theorem 5.2 that the crosscorrelation function of any maximum-length sequence of period $q-1$ over $\text{GF}(p)$ and its v -decimated version takes on only the following three correlation values:

$$-1 - p^{(m+1)/2}, \quad -1, \quad -1 + p^{(m+1)/2}.$$

These three-level decimation values v should be new and form another contribution of this paper to the theory of sequences.

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