

Quantum Stabilizer Codes from Maximal Curves

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Abstract—A curve attaining the Hasse-Weil bound is called a maximal curve. Usually classical error-correcting codes obtained from a maximal curve have good parameters. However, the quantum stabilizer codes obtained from such classical error-correcting codes via Euclidean or Hermitian self-orthogonality do not always possess good parameters. In this paper, the Hermitian self-orthogonality of algebraic geometry codes obtained from two maximal curves is investigated. It turns out that the stabilizer quantum codes produced from such Hermitian self-orthogonal classical codes have good parameters.

Index Terms—Algebraic geometry codes, Hermitian self-orthogonal, Quantum codes.

I. INTRODUCTION

A powerful construction of quantum codes is through classical codes with certain self-orthogonality [1], [7]. Among these self-orthogonalities, the Hermitian orthogonality produces q -ary quantum codes from q^2 -ary classical error-correcting codes, therefore Hermitian self-orthogonal classical codes may give rise to good quantum stabilizer codes. However, it is more challenging to construct Hermitian self-orthogonal classical codes than Euclidean self-orthogonal classical codes.

A good family of Hermitian self-orthogonal classical codes is from algebraic geometry codes [4], [5], [6]. For instance, in [4], a family of Hermitian self-orthogonal generalized Reed-Solomon codes is constructed and consequently a family of quantum MDS codes is produced. However, the situation is not always like this. For instance, if we consider the quantum codes produced from the Hermitian self-orthogonal classical codes based on the Hermitian curves, the parameters of these quantum codes are not satisfactory (see [11]). To show that an algebraic geometry code is Euclidean or Hermitian self-orthogonal, it is essential to construct a proper differential that satisfies certain condition (See Proposition 2.3). This is usually challenging, in particular, for the Hermitian self-orthogonality.

In this paper, we first study two maximal curves and the corresponding classical algebraic geometry codes. A useful result is that we are able to construct a suitable differential to describe their Euclidean dual codes. Then via their Euclidean self-orthogonality, we can show that these codes are Hermitian self-orthogonal for certain parameters. Finally, we apply the stabilizer method [1] to obtain quantum codes which have good parameters or even better parameters compared with those in [2], [9].

The paper is organized as follows. In Section 2, we briefly introduce some background on algebraic curves and algebraic geometry codes. Section 3 is devoted to two maximal curves

and the corresponding algebraic geometry codes with Hermitian self-orthogonality. In Section 4, we produce good quantum codes from Hermitian self-orthogonal classical codes given in Section 3. Comparisons are given as well to show that quantum codes obtained from our construction are indeed good.

II. PRELIMINARY

In this section, we briefly introduce some notations and results on algebraic curves and algebraic geometry codes. The reader may refer to [3], [12] for the details.

Let \mathcal{X} be a smooth, projective, absolutely irreducible curve of genus g defined over K , where K is a finite field. We denote by $K(\mathcal{X})$ the function field of \mathcal{X} . An element of $K(\mathcal{X})$ is called a function. The normalized discrete valuation corresponding to a point P of \mathcal{X} is written as ν_P . For every nonzero element f of $K(\mathcal{X})$, we can define a principal divisor $\text{div}(f) := \sum_P \nu_P(f)P$.

For a divisor G , the Riemann-Roch space associated to G is defined by

$$\mathcal{L}(G) = \{f \in K(\mathcal{X}) \setminus \{0\} : \text{div}(f) + G \geq 0\} \cup \{0\}.$$

Then $\mathcal{L}(G)$ is a finite-dimensional vector space over K and we denote its dimension by $\ell(G)$.

Let Ω denote the differential space of \mathcal{X} . For any nonzero differential ω , we can associate a canonical divisor $\text{div}(\omega) := \sum_P \nu_P(\omega)P$. All canonical divisors are equivalent and have degree $2g - 2$. For a divisor G , we define

$$\Omega(G) = \{\omega \in \Omega \setminus \{0\} : \text{div}(\omega) \geq G\}$$

and denote the dimension of $\Omega(G)$ by $i(G)$. Then one has

$$i(G) = \ell(H - G),$$

where H is a canonical divisor.

The Riemann-Roch Theorem says that

$$\ell(G) = \deg(G) - g + 1 + \ell(H - G),$$

where H is any canonical divisor.

Before introducing algebraic geometry codes, let us fix some basic notations. Let P_1, \dots, P_n be pairwise distinct K -rational points of \mathcal{X} and $D = P_1 + \dots + P_n$. Choose a divisor G on \mathcal{X} such that $\text{supp}(G) \cap \text{supp}(D) = \emptyset$. Then $\nu_{P_i}(f) \geq 0$ for all $1 \leq i \leq n$ and any $f \in \mathcal{L}(G)$.

Consider the following two maps

$$\Psi : \mathcal{L}(G) \rightarrow K^n, \quad f \mapsto (f(P_1), \dots, f(P_n))$$

and

$$\Phi : \Omega(G - D) \rightarrow K^n, \quad \omega \mapsto (\text{res}_{P_1}(\omega), \dots, \text{res}_{P_n}(\omega)),$$

where $\text{res}_{P_i}(\omega)$ denotes the residue of ω at P_i (see [12, Chapter 2]). The images of Ψ and Φ are denoted by $C_{\mathcal{L}}(D, G)$

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and $C_\Omega(D, G)$, respectively. It is clear that both $C_{\mathcal{L}}(D, G)$ and $C_\Omega(D, G)$ are linear codes over K . They are called algebraic-geometry codes (or AG codes for short). A nice property is that the Euclidean dual $C_{\mathcal{L}}(D, G)^\perp$ (\perp denotes the Euclidean dual) of $C_{\mathcal{L}}(D, G)$ is $C_\Omega(D, G)$ (see [12, Theorem II.2.8]).

Furthermore, we have the following results.

Proposition 2.1: ([12, Theorem II.2.2 and Corollary II.2.3]) $C_{\mathcal{L}}(D, G)$ is an $[n, k, d]$ -linear code over K with parameters

$$k = \ell(G) - \ell(G - D), \quad d \geq n - \deg(G).$$

(a) If G satisfies $\deg(G) < n$, then

$$k = \ell(G) \geq \deg(G) - g + 1.$$

(b) If additionally $2g - 2 < \deg(G) < n$, then $k = \deg(G) - g + 1$.

Proposition 2.2: ([12, Theorem II.2.7]) $C_\Omega(D, G)$ is an $[n, k^\perp, d^\perp]$ -linear code over K with parameters

$$k^\perp = i(G - D) - i(G), \quad d^\perp \geq \deg(G) - (2g - 2).$$

(a) If G satisfies $\deg(G) > 2g - 2$, then

$$k^\perp = i(G - D) \geq n + g - 1 - \deg(G).$$

(b) If additionally $2g - 2 < \deg(G) < n$, then

$$k^\perp = i(G - D) = n + g - 1 - \deg(G)$$

To study Euclidean self-orthogonality, we have to investigate the relationship between $C_{\mathcal{L}}(D, G)$ and $C_\Omega(D, G)$.

Proposition 2.3: ([12, Theorem II.2.10]) Let η be a differential such that $\nu_{P_i} = -1$ and $\text{res}_{P_i}(\eta) = 1$ for all $i = 1, \dots, n$. Then

$$C_{\mathcal{L}}(D, G)^\perp = C_\Omega(D, G) = C_{\mathcal{L}}(D, D - G + \text{div}(\eta)),$$

where $C_{\mathcal{L}}(D, G)^\perp$ stands for the Euclidean dual of $C_{\mathcal{L}}(D, G)$.

To obtain good classical AG codes, one is interested in the number of K -rational points on an algebraic curve. We denote by $N_K(\mathcal{X})$ the number of K -rational points on an algebraic curve \mathcal{X} over K . A celebrated result on the number of K -rational points is the Hasse-Weil bound stating that

$$N_K(\mathcal{X}) \leq |K| + 1 + 2g\sqrt{|K|}.$$

If the number of rational points of a curve \mathcal{X} achieves the upper bound, i.e., $N_K(\mathcal{X}) = |K| + 1 + 2g\sqrt{|K|}$, then \mathcal{X} is called a maximal curve. A well-known maximal curve is the Hermitian curve over \mathbb{F}_{q^2} defined by the equation $y^q + y = x^{q+1}$, where \mathbb{F}_{q^2} denotes the finite field of q^2 elements. Lots of maximal curves can be produced by coverings of the Hermitian curve [8]. In the next section, we consider a maximal curve which is also a covering of the Hermitian curve.

III. AG CODES FROM MAXIMAL CURVES

Throughout the rest of this paper, we consider the finite field $K = \mathbb{F}_{q^2}$, where q is a power of 2.

A. AG codes from the first maximal curve

Let $F = \mathbb{F}_{q^2}(\mathcal{X})$ be the function field of \mathcal{X} over \mathbb{F}_{q^2} , where \mathcal{X} is defined by the following equation

$$y^2 + y = x^{q+1}.$$

The genus g of \mathcal{X} is $g = q/2$ and the number of rational points is $2q^2 + 1$. The set of these $2q^2 + 1$ rational points consists of a point at infinity P_∞ and the other $2q^2$ ‘‘finite’’ rational points.

Let $n = 2q^2$ and let $\{P_1, \dots, P_n\}$ be all n ‘‘finite’’ rational points. Put $D = P_1 + \dots + P_n$.

Lemma 3.1: For a positive integer m , the Euclidean dual $C_{\mathcal{L}}(D, mP_\infty)^\perp$ of $C_{\mathcal{L}}(D, mP_\infty)$ is $C_{\mathcal{L}}(D, (n + 2g - 2 - m)P_\infty)$.

Proof: Consider the differential $\eta = \frac{dx}{x-x^q}$. Then one can verify that $\text{div}(\eta) = -D + (n + 2g - 2)P_\infty$ and $\text{res}_{P_i}(\eta) = 1$ for all $i = 1, \dots, n$. Thus, by Proposition 2.3, we have

$$\begin{aligned} C_{\mathcal{L}}(D, mP_\infty)^\perp &= C_\Omega(D, mP_\infty) \\ &= C_{\mathcal{L}}(D, D - mP_\infty + \text{div}(\eta)) \\ &= C_{\mathcal{L}}(D, (n + 2g - 2 - m)P_\infty). \end{aligned}$$

This completes the proof. \blacksquare

Remark 3.2: From Lemma 3.1, the dual of the AG code $C_{\mathcal{L}}(D, mP_\infty)$ can be represented as another AG code by choosing suitable differential. Therefore, self-orthogonality of the AG code can be described in the term of the degree of divisor G , i.e., m in our case. However, this is not always the case for other curves. Actually it is a challenging task to find the proper differential needed.

For simplicity, let us denote by C_m the AG code $C_{\mathcal{L}}(D, mP_\infty)$. Then, the above result says that $C_m^\perp = C_{n+2g-2-m}$. Hence, Lemma 3.1 gives the following result.

Corollary 3.3: C_m is Euclidean self-orthogonal if $m \leq n/2 + g - 1$.

Recall that the Hermitian inner product for two vectors $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$ in $\mathbb{F}_{q^2}^n$ is defined by $\langle \mathbf{a}, \mathbf{b} \rangle_H := \sum_{i=1}^n a_i b_i^q$. For a linear code C over \mathbb{F}_{q^2} , the *Hermitian dual* of C is defined by

$$C^{\perp_H} := \{\mathbf{v} \in \mathbb{F}_{q^2}^n : \langle \mathbf{v}, \mathbf{c} \rangle_H = 0 \ \forall \mathbf{c} \in C\}.$$

Then C is Hermitian self-orthogonal if $C \subseteq C^{\perp_H}$. by the definition of Hermitian self-orthogonality, one can easily obtain a useful fact, namely $C \subseteq C^{\perp_H}$ if and only if $C^q \subseteq C^\perp$.

Theorem 3.4: C_m is Hermitian self-orthogonal if $m \leq 2q - 2$.

Proof: If $m \leq 2q - 2$, then we have $mq \leq n + 2g - 2 - m$. Thus, one has $C_{mq} \subseteq C_{n+2g-2-m}$. Hence, the desired result follows from the fact that

$$C_m^\perp = C_{n+2g-2-m} \quad \text{and} \quad C_m^q \subseteq C_{mq}.$$

\blacksquare

B. AG codes from the second maximal curve

By abuse of notations, we still use the same notations as in the previous section for our second maximal curve and corresponding AG codes.

Let q be an odd power of 2. Thus, 3 divides $q + 1$. Let $F = \mathbb{F}_{q^2}(\mathcal{X})$ be the function field of \mathcal{X} over \mathbb{F}_{q^2} , where \mathcal{X} is defined by the following equation

$$y^q + y = x^3.$$

The genus g of \mathcal{X} is $g = q - 1$ and the number of rational points is $3q^2 - 2q + 1$. The set of these $3q^2 - 2q + 1$ rational points consists of a point at infinity P_∞ and the other $3q^2 - 2q$ “finite” rational points.

Let $n = 3q^2 - 2q$ and let $\{P_1, \dots, P_n\}$ be all n “finite” rational points. Put $D = P_1 + \dots + P_n$.

Lemma 3.5: For a positive integer m , the Euclidean dual $C_{\mathcal{L}}(D, mP_\infty)^\perp$ of $C_{\mathcal{L}}(D, mP_\infty)$ is $C_{\mathcal{L}}(D, (n + 2g - 2 - m)P_\infty)$.

Proof: Let α be a $(q^2 - 1)$ th primitive root of unity in \mathbb{F}_{q^2} and define the polynomial

$$h(x) := x \prod_{j=0}^{3(q-1)-1} (\alpha^{j(q+1)/3} - x) = x(1 - x^{3(q-1)}) = x - x^{3q-2}.$$

It is easy to see that $x - \alpha^i$ splits completely in F if and only if i is divisible by $(q+1)/3$. Furthermore, x splits completely in F . This implies that the principal divisor $\text{div}(h(x))$ is $D - (3q^2 - 2q)P_\infty$.

Consider the differential $\eta = \frac{dx}{h(x)}$. Then one can verify that $\text{div}(\eta) = -D + (n + 2g - 2)P_\infty$ and $\text{res}_{P_i}(\eta) = 1$ for all $i = 1, \dots, n$. Thus, by Proposition 2.3, we have

$$\begin{aligned} C_{\mathcal{L}}(D, mP_\infty)^\perp &= C_{\Omega}(D, mP_\infty) \\ &= C_{\mathcal{L}}(D, D - mP_\infty + \text{div}(\eta)) \\ &= C_{\mathcal{L}}(D, (n + 2g - 2 - m)P_\infty). \end{aligned}$$

This completes the proof. \blacksquare

For simplicity, let us denote by C_m the AG code $C_{\mathcal{L}}(D, mP_\infty)$. Then, the above result says that $C_m^\perp = C_{n+2g-2-m}$. Hence, Lemma 3.5 gives the following results.

Corollary 3.6: C_m is Euclidean self-orthogonal if $m \leq n/2 + g - 1$.

Theorem 3.7: C_m is Hermitian self-orthogonal if $m \leq 3q - 4$.

Proof: If $m \leq 2q - 2$, then we have $mq \leq n + 2g - 2 - m$. Thus, one has $C_{mq} \subseteq C_{n+2g-2-m}$. Hence, the desired result follows from the fact that

$$C_m^\perp = C_{n+2g-2-m} \quad \text{and} \quad C_m^q \subseteq C_{mq}.$$

This completes the proof. \blacksquare

IV. QUANTUM STABILIZER CODES

In this section, we apply the Hermitian self-orthogonality of the classical AG codes C_m constructed in the previous section to produce quantum stabilizer codes and then analyze their parameters.

Let us first recall a result on quantum codes obtained from Hermitian self-orthogonal classical codes.

Lemma 4.1: (see [1]) There is a q -ary $[[n, n - 2k, d^\perp]]$ -quantum stabilizer code whenever there exists a q -ary classical Hermitian self-orthogonal $[n, k]$ -linear code with dual distance d^\perp .

Using the connection of quantum codes with classical Hermitian self-orthogonal codes in Lemma 4.1, we can derive our main result stated as below. Then we use some numerical results to show that the quantum codes produced from our results are indeed good.

Example 4.2: **Theorem 4.3:** If q is a power of 2, then there exists a q -ary $[[2q^2, k_Q := 2q^2 - 2m + q - 2, d_Q \geq m + 2 - q]]_q$ quantum code for any positive integer m satisfying $q - 1 \leq m \leq 2q - 2$.

Theorem 4.4: If q is an odd power of 2, then there exists a q -ary $[[3q^2 - 2q^2, k_Q := 3q^2 - 2m - 4, d_Q \geq m + 4 - 2q]]_q$ quantum code for any positive integer m satisfying $2q - 3 \leq m \leq 3q - 4$.

The proof of Theorems 4.3 and 4.8 directly follows from Theorems 3.4, 3.7 and Lemma 4.1.

For $q = 2$ and $1 \leq m \leq 2$, by Theorem 4.3 we can obtain binary quantum codes with parameters $[[8, 4, 2]]_2$ and $[[8, 2, 3]]_2$ which are optimal from the online table [9].

Example 4.5: For $q = 4$ and $3 \leq m \leq 6$, Theorem 4.3 produces 4-ary $[[32, 34 - 2m, m - 2]]_4$ quantum codes. Namely, $[[32, 28, 1]]_4$, $[[32, 26, 2]]_4$, $[[32, 24, 3]]_4$, $[[32, 22, 4]]_4$ quantum codes can be derived. These codes have good parameters. For instance, in the online table [2], a $[[36, 22, 4]]_4$ quantum code is given. This implies that our quantum code has a smaller length for the same dimension and distance.

Example 4.6: Let $q = 8$ and $7 \leq m \leq 14$. Then by Theorem 4.3, we can derive 8-ary $[[126, 134 - 2m, m - 6]]_8$ quantum codes. For instance, new quantum codes with parameters $[[128, 108, 6]]_8$, $[[128, 106, 7]]_8$, $[[128, 104, 8]]_8$ can be produced. They have reasonably better parameters compared with the quantum codes with parameters $[[134, 108, 6]]_8$, $[[134, 106, 7]]_8$, $[[134, 96, 8]]_8$ given in [2].

Example 4.7: Let $q = 8$ and $13 \leq m \leq 20$. Then we can derive 8-ary $[[176, 188 - 2m, m - 12]]_8$ quantum codes. For instance, new quantum codes with parameters $[[176, 154, 5]]_8$, $[[176, 152, 6]]_8$, $[[176, 150, 7]]_8$, $[[176, 148, 8]]_8$ can be produced. They have reasonably better parameters compared with the quantum codes with parameters $[[185, 149, 5]]_8$, $[[185, 125, 7]]_8$, $[[185, 113, 8]]_8$ given in [2].

The above examples show that we can derive quantum codes from Theorem 4.3 which are optimal or even have better parameters compared with [2], [9]. However, for large q , it is difficult to find explicit known codes to compare with ours since there are no suitable tables for reference. Nevertheless, we can still illustrate our result by comparing it with some bounds for large q . We only discuss the quantum codes given in Theorem 4.3.

Remark 4.8: Let us analyze the parameters of the quantum codes given in Theorem 4.3.

- (i) From the quantum Singleton bound and Theorem 4.3, the quantum codes given in Theorem 4.3 satisfy

$$n + 2 - q \leq k_Q + 2d_Q \leq n + 2,$$

where n is the length $2q^2$. So the difference of our quantum codes from the Singleton bound is q .

(ii) Let us consider the quantum Hamming bound [7]

$$q^{n-k_Q} \geq \sum_{j=0}^{\lfloor (d_Q-1)/2 \rfloor} \binom{n}{j} (q^2 - 1)^j.$$

For instance, we just consider the case where $m = 2q - 3$. Then, $d_Q = q - 1$. Thus, if take logarithm of the right-hand side of the above Hamming bound, we get the following limit

$$\frac{1}{q} \log_q \left(\sum_{j=0}^{(q-2)/2} \binom{n}{j} (q^2 - 1)^j \right) \rightarrow \frac{3}{2}$$

as q tends to ∞ , i.e., the right-hand side of the above Hamming bound is $q^{3q/2+o(q)}$. The left-hand side of the above Hamming bound is q^{2q-2} . If we take logarithm of both the sides with base q , then one can see the difference is about $q/2 + o(q)$. This difference is smaller than the one compared with the Singleton bound.

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