# The Group Square-Root Lasso: Theoretical Properties and Fast Algorithms

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#### Abstract

We introduce and study the Group Square-Root Lasso (GSRL) method for estimation in high dimensional sparse regression models with group structure. The new estimator minimizes the square root of the residual sum of squares plus a penalty term proportional to the sum of the Euclidean norms of groups of the regression parameter vector. The net advantage of the method over the existing Group Lasso (GL)-type procedures consists in the form of the proportionality factor used in the penalty term, which for GSRL is independent of the variance of the error terms. This is of crucial importance in models with more parameters than the sample size, when estimating the variance of the noise becomes as difficult as the original problem. We show that the GSRL estimator adapts to the unknown sparsity of the regression vector, and has the same optimal estimation and prediction accuracy as the GL estimators, under the same minimal conditions on the model. This extends the results recently established for the Square-Root Lasso, for sparse regression without group structure. Moreover, as a new type of result for Square-Root Lasso methods, with or without groups, we study correct pattern recovery, and show that it can be achieved under conditions similar to those needed by the Lasso or Group-Lasso-type methods, but with a simplified tuning strategy. We implement our method via a new algorithm, with proved convergence properties, which, unlike existing methods, scales well with the dimension of the problem. Our simulation studies support strongly our theoretical findings.

#### **Index Terms**

Group Square-Root Lasso, high dimensional regression, noise level, sparse regression, Square-Root Lasso, tuning parameter

#### I. INTRODUCTION

Variable selection in high dimensional linear regression models has become a very active area of research in the last decade. In linear models one observes independent response random variables  $Y_i \in \mathbb{R}$ ,  $1 \le i \le n$ , and assumes that each  $Y_i$  can be written as a linear function of the *i*-th observation on a *p*-dimensional predictor vector  $X_i =: (X_{i1}, \ldots, X_{ij}, \ldots, X_{ip})$ , corrupted by noise:

$$Y_i = X_i \beta^0 + \sigma \epsilon_i,\tag{1}$$

where  $\beta^0 \in \mathbb{R}^p$  is the unknown regression vector,  $\sigma \ge 0$  is the noise level, and for each  $1 \le i \le n$ , the additive term  $\epsilon_i$ , is a mean zero random noise component. Postulating that some components of  $\beta^0$  are zero is equivalent to assuming that the corresponding predictors are unrelated to the response after controlling for the predictors with non-zero components. The problem of predictor selection can be therefore solved by devising methods that estimate accurately where the zeros occur.

More recently, a large literature focusing on the selection of groups of predictors has been developed. This problem requires methods that set to zero entire groups of coefficients and is the focus of this work. Group selection arises naturally whenever it is plausible to assume, based on scientific considerations, that entire subsets of the X-variables are unrelated to the response. More generally, the need for setting groups of coefficients to zero is a building block in variable selection in general additive models and sparse kernel learning, as discussed in Meier et al. [15] and Koltchinskii and Yuan [11], among others. Another direct application is to predictor selection in the multivariate response regression model

$$Z = UA + E, (2)$$

where Z is an  $n \times m$  matrix in which each row contains measurements on an m-dimensional random response vector, U is a  $n \times p$  observed matrix whose rows are the n measurements of a p-dimensional predictor, E is the zero mean noise matrix, and A is the unknown coefficient matrix. A predictor  $U_j$  is not present in this model if the j-th row of A is equal to zero. Using the vectorization operator vec, (2) can be written as  $vec(Z') = (U \otimes I)vec(A') + vec(E')$ . Thus, if one treats rows of A as groups, predictor selection in model (2) can be regarded as group selection in linear models of type (1).

Perhaps the most popular method for group selection is the Group-Lasso, introduced by Yuan and Lin [29] and further studied theoretically in a number of works, including Lounici et al. [14], Wei and Huang [28]. The method consists in minimizing the empirical square loss plus a term proportional to the sum of the Euclidean norms of groups of coefficients. Specifically, let  $Y = (Y_1, \ldots, Y_n)'$ . We denote by  $X \in \mathbb{R}^{n \times p}$  the matrix with rows  $X_i$ ,  $1 \le i \le n$ , and refer to it in the sequel as the design matrix. We assign the individual columns of the design matrix and the corresponding entries of the regression vector

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to groups. For this, we consider a partition  $\{G_1, \ldots, G_q\}$  of  $\{1, \ldots, p\}$  into groups and denote the cardinality of a group  $G_j$ by  $T_j$  and the minimal group size by  $T_{\min} := \min_{1 \le j \le q} T_j$ . We then assign all columns of the design matrix X with indices in  $G_j$  to the group  $G_j$ . The corresponding matrix is denoted by  $X^j \in \mathbb{R}^{n \times T_j}$ . Similarly, for any vector  $\beta \in \mathbb{R}^p$ , we assign all components of  $\beta$  with indices in  $G_j$  to the group  $G_j$  and denote the corresponding vector by  $\beta^j \in \mathbb{R}^{T_j}$ . We define the active set as

$$S := \{ 1 \le j \le q : \beta^{0j} \ne 0 \}.$$
(3)

We will denote by  $||v||_2$  the Euclidean norm of a generic vector v. Let  $\lambda > 0$  be a given tuning sequence. With this notation, the Group Lasso estimator is given by

$$\bar{\beta} := \operatorname*{arg\,min}_{\beta \in \mathbb{R}^p} \left\{ \frac{\|Y - X\beta\|_2^2}{n} + \frac{\lambda}{n} \sum_{j=1}^q \sqrt{T_j} \|\beta^j\|_2 \right\}.$$

Optimal estimation of  $\beta^0$ ,  $X\beta^0$  and S via the Group-Lasso is very well understood, and we refer to Bühlmann and van de Geer [6] for an overview. However, one outstanding problem remains, and it is connected to the practical choice of  $\lambda$  that leads, respectively, to optimal estimation with respect to each of these three aspects. It is agreed upon that whereas choosing  $\lambda$  via cross-validation will yield estimates with good prediction and estimation accuracy, this choice is not optimal for correct estimation of S. A possibility is to determine first the theoretical forms of the tuning parameter that yield optimal performances, respectively, and then estimate the unknown quantities in these theoretical expressions. One important reason for which this approach has not become popular is the fact that the respective optimal values of  $\lambda$  depend on  $\sigma$ , the noise level, and the accurate estimation of  $\sigma$  when p > n may be as difficult as the original problem of selection. A step forward has been made by Belloni et al. [2], in the context of variable (not group) selection. They introduced the Square-Root Lasso (SRL) given below

$$\bar{\bar{\beta}} := \operatorname*{arg\,min}_{\beta \in \mathbb{R}^p} \left\{ \frac{\|Y - X\beta\|_2}{\sqrt{n}} + \frac{\lambda}{n} \sum_{l=1}^p |\beta_l| \right\}.$$

The consideration of the square-root form of the criterion was first proposed by Owen [18] in the statistics literature, and a similar approach is the Scaled Lasso by Sun and Zhang [22]. Belloni et al. [2] studied theoretically the estimation and prediction accuracy of the SRL estimator  $\overline{\beta}$ , and showed that it is similar to that of the Lasso, with the net advantage that optimality can be achieved for a tuning sequence independent of  $\sigma$ . This makes this version of the Lasso-type procedure much more appealing when p is large, especially when p > n, and opens the question whether the same holds true for pattern recovery, which was not studied in [2]. Moreover, given the wide applicability of group selection methods, it motivates the study of a grouped version of the Square-Root Lasso. We therefore introduce and study the Group Square-Root Lasso (GSRL)

$$\hat{\beta} := \underset{\beta \in \mathbb{R}^p}{\operatorname{arg\,min}} \left\{ \frac{\|Y - X\beta\|_2}{\sqrt{n}} + \frac{\lambda}{n} \sum_{j=1}^q \sqrt{T_j} \|\beta^j\|_2 \right\}.$$
(4)

Our contributions are:

(a) To extend the ideas behind the Square-Root Lasso for group selection and develop a new method, the Group Square-Root Lasso (GSRL).

(b) To show that the GSRL estimator has optimal estimation and prediction, achievable with a  $\sigma$ -free tuning sequence  $\lambda$ . This generalizes the results for SRL obtained by [2].

(c) To show that GSRL leads to correct pattern recovery, with a  $\sigma$ -free tuning sequence  $\lambda$ . This provides, in particular, a positive answer to the question left open in [2].

(d) To propose algorithms with guaranteed convergence properties that scale well with the size of the problem, measured by p, thereby extending the scope of the existing procedures, which are performant mainly for small and moderate values of p.

We address (a), (b) and (c) in Section 2 below, and (d) in Section 3. Section 4 contains simulation results that support strongly our findings. The proofs of all our results are collected in the Appendix.

# II. THEORETICAL PROPERTIES OF THE GROUP SQUARE-ROOT LASSO

In this section, we show that: (i) Nothing is lost by using  $||Y - X\beta||_2$  instead of  $||Y - X\beta||_2^2$  in the definition of our estimator  $\hat{\beta}$  given by (4). Specifically, the Group Square-Root Lasso has the same accuracy as the Group Lasso, under essentially the same conditions, in terms of estimation, prediction and subset recovery. (ii) The net gain is that these properties are achieved via a tuning parameter  $\lambda$  that is  $\sigma$ -free, in contrast with the Group-Lasso, which requires a tuning parameter  $\lambda$  that is a function of  $\sigma$ .

The following notation and conventions will be used throughout the paper. We assume that the design matrix is nonrandom and normalized such that the diagonal entries of the Gram matrix  $\Sigma := \frac{X'X}{n}$  are equal to 1. We denote the cardinality of the

set S defined in (3) above by s, that is |S| = s, and refer to s as the sparsity index. We set  $s^* := \sum_{j \in S} T_j$ . We denote by  $\beta_S \in \mathbb{R}^s$  (and similarly  $\beta_{S^c} \in \mathbb{R}^{p-s}$ ) the vector that consists of the entries of  $\beta \in \mathbb{R}^p$  with indices in  $\bigcup_{j \in S} G_j$  (or  $\bigcup_{j \in S^c} G_j$ ). Corresponding notation is used for matrices. For a generic vector v we denote by  $||v||_{\infty}$  its supremum norm, the maximum absolute value of its coordinates.

#### A. Estimation and Prediction

We begin with the study of the estimation and prediction accuracy of the Group Square-Root Lasso. We first state and discuss the conditions under which these results will be established.

As shown in Theorem II.1 below, our results hold under the general Compatibility Condition on the design matrix, introduced for the Lasso in [23], and extended to this setting in [6, Page 255]. This condition is a slight relaxation of the widely used Cone or Restricted Eigenvalues Condition (see [3]). We refer to [24] and [6, Chapter 6.13] for a detailed comparison between these two and other related conditions.

a) Compatibility Condition (CC): We say that the Compatibility Condition is met for  $\kappa > 0$  and  $\gamma > 1$  if

$$\sum_{j \in S} \sqrt{T_j} \|\delta^j\|_2 \le \frac{\sqrt{s^*} \|X\delta\|_2}{\sqrt{n\kappa}}$$
(5)

for all  $\delta \in \Delta_{\gamma}$ , where

$$\Delta_{\gamma} := \{\delta \in \mathbb{R}^p : \sum_{j \in S^c} \sqrt{T_j} \|\delta^j\|_2 \le \gamma \sum_{j \in S} \sqrt{T_j} \|\delta^j\|_2 \}.$$
(6)

We refer to  $\kappa$  and  $\gamma$  as the compatibility constants and write

$$(\kappa, \gamma) \in C(X, S).$$

The compatibility constant  $\kappa$  measures the correlations in the design matrix: the smaller the value of  $\kappa$ , the larger the correlations.

For clarity of exposition, we will assume for the rest of the paper that the additive noise terms  $\epsilon_i$  have a standard Gaussian distribution.

The second ingredient in our analysis is the definition of the appropriate noise component that needs to be compensated for by the tuning parameter  $\lambda$ . The proofs of our results reveal that it is

$$V := \max_{1 \le j \le q} \left\{ \frac{\sqrt{n} \| (X'\epsilon)^j \|_2}{\sqrt{T_j} \| \epsilon \|_2} \right\}.$$
 (7)

For  $\gamma > 1$  given by condition **CC** above , let  $\overline{\gamma} := \frac{\gamma+1}{\gamma-1}$ . For given  $\lambda > 0$  define the set

$$\mathcal{A} := \{ V \le \lambda/\overline{\gamma} \} \,. \tag{8}$$

We first establish our result over the set A. We then show, in Lemma II.1 below, that the set A has probability  $1 - \alpha$ , for any  $\alpha$  close to zero, for an appropriate choice of the tuning parameter  $\lambda$ . Since  $\lambda$  will be chosen relative to the ratio of the random variables that define V, the factor  $\sigma$  cancels out. This is the key for obtaining a tuning parameter  $\lambda$  independent of the variance of the noise.

With  $\gamma > 1$  given by **CC** above and  $\kappa > 0$  given by **CC**, we assume in what follows that the sparsity index  $s^*$  is not larger than the sample size n. Specifically, we assume that

$$s^* < \frac{n^2 \kappa^2}{\lambda^2}.\tag{9}$$

We will show in Lemma II.1 below that the value of  $\lambda$  for which the event  $\mathcal{A}$  has high probability is, in terms of orders of magnitude, no larger than  $\lambda = O(\sqrt{n \log q})$ . Therefore, and using the notation  $\leq$  for inequalities that hold up to multiplicative constants, the condition on the sparsity index becomes

$$s^* \lesssim \frac{n}{\log q},$$

which re-emphasizes the introduction of  $s^*$  in this analysis to start with: whereas we allow p > n, we cannot expect good performance of any method from a limited sample size n, unless the true model has essentially fewer parameters than n.

The following result summarizes the prediction and estimation properties of the Group Square-Root Lasso estimator. It generalizes [2, Theorem 1], where the Square-Root Lasso is treated, corresponding in our set-up to the special case q = p.

**Theorem II.1.** Assume that  $(\kappa, \gamma) \in C(X, S)$  and that (9) holds. Then, on the event  $\mathcal{A}$ , the following hold:

$$\|X(\widehat{\beta} - \beta^{0})\|_{2} \lesssim \frac{\sigma\lambda\sqrt{s^{*}}}{\kappa\sqrt{n}}$$

$$\sum_{j=1}^{q} \sqrt{T_{j}} \|(\widehat{\beta} - \beta^{0})^{j}\|_{2} \lesssim \frac{\sigma\lambda s^{*}}{\kappa^{2}n}.$$
(10)

and

The precise constants in the statements above are given in the proof of this theorem, presented in the appendix. Theorem II.1 is the crucial step in showing that the GSRL estimator, which has a tuning parameter free of  $\sigma$ , has the same optimal rates of convergence as the Group Lasso estimator, see for instance Lounici et al. [14] or Bühlmann and van de Geer [6]. We will determine the size of  $\lambda$  in Lemma II.1 below and state the resulting rates in Corollary II.1.

#### **Remark II.1.** For prediction, the condition

$$\sum_{j \in S} \sqrt{T_j} \|\delta^j\|_2 - \sum_{j \in S^c} \sqrt{T_j} \|\delta^j\|_2 \le \frac{\sqrt{s^*} \|X\delta\|_2}{\sqrt{n\kappa}}$$

for  $\delta \in \Delta_1$  could replace the CC condition (5), cf. [1]. We additionally note that prediction (in contrast to correct subset recovery and estimation) is even possible for highly correlated design matrices, see [10, 27]. However, a detailed discussion of prediction for correlated design matrices is not in the scope of this paper.

**Remark II.2.** Inequality (10) directly implies correct subset recovery for the Group Square-Root Lasso in the special case  $\sigma = 0$ , cf. [1]. In contrast,  $\sigma = 0$  and the conditions of Theorem II.1 are not sufficient to ensure correct subset recovery for the Lasso and the Group Lasso.

#### B. Correct subset recovery

We study below the subset recovery properties of the Group Square-Root Lasso. Similarly to the analysis of all other Lasso-type procedures, subset recovery can only be guaranteed under additional assumptions on the model.

The first condition is the Group Irrepresentable Condition, which is an additional condition on the the design matrix X. To introduce it, we decompose the Gram matrix  $\Sigma$  with  $\Sigma_{1,1} := \frac{X'_S X_S}{n}$ ,  $\Sigma_{1,2} := \frac{X'_S X_{S^c}}{n}$ ,  $\Sigma_{2,1} := \frac{X'_{S^c} X_S}{n}$ , and  $\Sigma_{2,2} := \frac{X'_{S^c} X_{S^c}}{n}$ . We define  $\widetilde{\Sigma}_{2,1} := (0 \ \Sigma_{1,2})'$  and  $\widetilde{\Sigma}_{1,1}^{-1} := (0 \ \Sigma_{1,1}^{-1})'$ .

b) Group Irrepresentable Condition (GIR): We say that the Group Irrepresentable Condition is met for  $0 < \eta < 1$  if  $\Sigma_{1,1}$  is invertible and

$$\max_{v:\|v^k\|_2 \le \sqrt{T_k}} \max_{1 \le j \le q} \frac{\|(\Sigma_{2,1} \Sigma_{1,1}^{-1} v)^j\|_2}{\sqrt{T_j}} < \eta.$$
(11)

We refer to  $\eta$  as the group irrepresentable constant and write

 $\eta \in I(X,S).$ 

The Group Irrepresentable Condition implies the Compatibility Condition discussed above, see for instance [6], and it is therefore more restrictive. However, it is essentially a necessary and sufficient condition for consistent support recovery via Lasso-type procedures, see [30]. We refer to [6, 16, 30, 31] for different versions of the Irrepresentable Condition and further discussion of these versions.

The second condition needed for precise support recovery regards the strength of the signal  $\beta^0$ . Because the noise can conceal small components of the regression vector  $\beta^0$ , some of its nonzero components need to be sufficiently large to be detectable. We formulate this in the Beta Min Condition, similarly to [7] and [19, 25]:

c) Beta Min Condition (BM): We say that the Beta Min Condition is met for  $m \in \mathbb{R}^s$  if

$$\|\beta^{0j}\|_{\infty} \ge m_j,\tag{12}$$

for all  $j \in S$ . We then write  $m \in B(\beta^0)$ .

Note that only one component of  $\beta^0$  in each non-zero group has to be sufficiently large, because we aim to select whole groups, and not individual components.

A slightly different tuning parameter, still independent of  $\sigma$  is needed for consistent subset recovery. Let  $\tilde{\eta} := \frac{1+\eta}{1-\eta}$ , for  $\eta$  given by GIR above, and recall that  $\overline{\gamma} = \frac{\gamma+1}{\gamma-1}$ , with  $\gamma$  defined in condition CC above. Define the event

$$\mathcal{A}_1 = \{ V \le \lambda / (\overline{\gamma} \lor 2\widetilde{\eta}) \}. \tag{13}$$

Finally, we introduce the following notation

$$\xi_{\|\cdot\|_{\infty}} := \max_{v:\|v^k\|_2 \le \sqrt{T_k}} \max_{1 \le j \le q} \frac{\|(\Sigma_{1,1}^{-1}v)^j\|_{\infty}}{\sqrt{T_j}}.$$

Note that for orthonormal design matrices,  $\xi_{\|\cdot\|_{\infty}} = 1$ . Let  $\alpha \in (0,1)$  be given.

**Theorem II.2.** Assume that the conditions CC, GIR and BM are met, and that (9) holds. Assume that  $(\kappa, \gamma) \in C(X, S)$  and  $\eta \in I(X, S)$ . Let D > 0 be a dominating constant. Then, on the set  $A_1$ , we have, with probability greater than  $1 - \alpha$ :

- (1)  $\widehat{\beta}_{S^c} = 0;$
- (2) For all  $1 \le j \le q$ ,

$$\|(\widehat{\beta} - \beta^0)^j\|_{\infty} \le D \frac{\sqrt{T_j}\sigma\lambda}{n}.$$

(3) If there exists an  $m \in B(\beta^0)$  such that  $m_j \ge D \frac{\sqrt{T_j \sigma \lambda}}{n}$ , for each  $j \in S$ , then  $S = \widehat{S}$ .

**Remark II.3.** The constant D depends on  $\gamma, \eta, \kappa$  and  $\xi_{\|\cdot\|_{\infty}}$ , but not on n, p, q. Its exact form is given in the proof of Theorem II.2. The results above show that the Group Square Root Lasso will recover the sparsity pattern consistently, as long as  $A_1$  has high probability, which we show in Lemma II.1 below. Theorem II.2 holds under slightly more general conditions on the design than the variant on the mutual coherence condition employed in Lounici et al. [14], for pattern recovery with the Group Lasso. Moreover, the recovery is guaranteed for signals of minimal strength, just above noise level, which we quantify precisely in Corollary II.1 below.

**Remark II.4.** Theorem **II.2** can be proved only under GIR and BM, as GIR implies CC. However, using only GIR would require the derivation of the corresponding constants under which CC holds, as we will appeal to the conclusion of Theorem **II.1** in the course of the proof of Theorem **II.2**. Given that the arguments are already technical, we opted for stating both assumptions separately, for transparency.

**Remark II.5.** The Group Square-Root Lasso can be shown to lead to correct subset recovery under sharper Beta Min Conditions, for a constant D independent of  $\xi_{\|\cdot\|_{\infty}}$ , if we impose stricter conditions on the design. For example, one can invoke the Group Mutual Coherence Condition (GMC) and apply ideas developed in [7] to find the condition  $m_j \gtrsim \sqrt{T_j}\lambda/n$ , which is of the same order as above, but holds up to universal constants, independent of the conditions on the design. We do not detail this approach here, since the GMC implies GIR, and the proof would follow very closely the ideas in [7].

### C. Choice of the Tuning Parameter

As discussed above, the novel property of the Group Square-Root Lasso method is that its tuning parameter  $\lambda$  can be chosen independently of the noise level  $\sigma$ . This is particularly interesting in the high-dimensional setting  $p \gg n$ , where good estimates of  $\sigma$  are not usually available. In determining  $\lambda$  for this method, we recall that it has to be sufficiently large to overrule the noise component, which is independent of  $\sigma$ ,

$$V = \max_{1 \le j \le q} \left\{ \frac{\sqrt{n} \| (X'\epsilon)^j \|_2}{\sqrt{T_j} \| \epsilon \|_2} \right\}$$

in that the events  $\mathcal{A}$  and  $\mathcal{A}_1$ , given above by (8) and (13), respectively, hold with high probability. At the same time, the bounds in Theorem II.2 and II.1 become sharper for smaller values of  $\lambda$ . To incorporate these two constraints, we choose the tuning parameter as the smallest value that overrules the noise part with high probability. For this, we fix  $\alpha \in (0, 1)$  and choose the smallest value  $\lambda$  such that with probability at least  $1 - \alpha$  it still holds that  $\lambda/\overline{\gamma} \geq V$  or  $\lambda/(\overline{\gamma} \vee 2\tilde{\eta}) \geq V$ , depending on the type of results we are interested in. Standard values for  $\alpha$  are 0.05 and 0.01.

For each j, let  $\zeta_j = \|X^j\|^2/n$  and  $\zeta = \max_j \zeta_j$ , where  $\|A\|$  is the operator norm of a generic matrix A.

**Lemma II.1.** Assume that the noise terms  $\epsilon_i$ ,  $1 \le i \le n$ , are i.i.d. standard Gaussian random variables, and assume that  $T_j < n$ , for all  $1 \le j \le q$ . Let  $\alpha \in (0,1)$  be given such that  $16 \log(2q/\alpha) \le n - T_{max}$ . Then, if

$$\lambda_0 \ge \frac{\sqrt{2\zeta}n}{\sqrt{n-T_{max}}} \left(1 + \sqrt{\frac{2\log(2q/\alpha)}{T_{min}}}\right),$$

it holds that

$$\mathbb{P}(V \ge \lambda_0) \le \alpha.$$

As an immediate consequence, the following corollary summarizes the expressions of  $\lambda$  for which the events  $\mathcal{A}$  and  $\mathcal{A}_1$  hold with probability  $1 - \alpha$ , for each given  $\alpha$ . Notice that  $\lambda$  is independent of  $\sigma$ , as claimed. Corollary II.1 also shows that the sharp rates of convergence and subset recovery properties of the Group Lasso are also enjoyed by the Group Square-Root Lasso, with the important added benefit that the new method's tuning parameter is  $\sigma$ -free.

**Corollary II.1.** Assume that the noise terms  $\epsilon_i$ ,  $1 \le i \le n$ , are i.i.d. standard Gaussian random variables and assume that  $T_j < n$ , for all  $1 \le j \le q$ . Let  $\alpha \in (0, 1)$  be given such that  $16 \log(2q/\alpha) \le n - T_{max}$ .

(i) If  

$$\lambda \ge \frac{\sqrt{2\zeta}n\overline{\gamma}}{\sqrt{n - T_{max}}} \left(1 + \sqrt{\frac{2\log(2q/\alpha)}{T_{min}}}\right)$$

then  $\mathbb{P}(\mathcal{A}) \geq 1 - \alpha$ . (ii) If

$$\lambda \geq \frac{\sqrt{2\zeta}n(\overline{\gamma} \vee 2\widetilde{\eta})}{\sqrt{n - T_{max}}} \left(1 + \sqrt{\frac{2\log(2q/\alpha)}{T_{min}}}\right)$$

then  $\mathbb{P}(\mathcal{A}_1) \geq 1 - \alpha$ .

(iii) Under the assumptions of Theorem II.1, its conclusion holds with probability at least  $1 - 2\alpha$  and  $\lambda = O(\sqrt{\frac{n}{T_{min}}} \log q)$ . (iv) Under the assumptions of Theorem II.2, its conclusion holds with probability at least  $1 - 2\alpha$  and  $\lambda = O(\sqrt{\frac{n}{T_{min}}} \log q)$ .

The first two claims follow immediately from Lemma II.1 and the definitions of A and  $A_1$ , respectively. The third and forth claims follow directly from the first two, by invoking Theorems II.1 and II.2, respectively. We only considered Gaussian noise above for clarity of exposition. However, more general results can be established applying different deviation inequalities, for instance [4, 13, 26]. For example, if the  $\epsilon_i$ 's belong to a general sub-exponential family, the order of magnitude of  $\lambda$  remains the same. We also refer to [1], where the analysis involving non-Gaussian noise makes use of moderate deviation theory for self-normalized sums, leading in some cases to results similar to those obtained for Gaussian noise. Additionally, an analysis that takes into account correlations between the groups is expected to lead to results similar to those established for the Lasso, see [10, 27].

#### **III. COMPUTATIONAL ALGORITHM**

In this section we show that the Group Square-Root Lasso can be implemented very efficiently. We consider estimators of a form slightly more general than (4):

$$\hat{\beta} := \underset{\beta \in \mathbb{R}^p}{\operatorname{arg\,min}} \left\{ \|Y - X\beta\|_2 + \sum_{j=1}^q \lambda_j \|\beta^j\|_2 \right\},\tag{14}$$

where  $\lambda_1, \ldots, \lambda_q > 0$  are arbitrary given constants. For convenience, we will implement, without loss of generality, the following variant

$$\hat{\beta} := \underset{\beta \in \mathbb{R}^p}{\operatorname{arg\,min}} \left\{ \|Y - X\beta\|_2 / K + \sum_{j=1}^q \lambda_j \|\beta^j\|_2 \right\},\tag{15}$$

where K is a fixed, sufficiently large constant. A global minimum of (15), for given constants  $\lambda_1, \ldots, \lambda_q$ , is also a global minimum of (14) with constants  $K\lambda_1, \ldots, K\lambda_q$ .

When q = p and  $\lambda_1 = \cdots = \lambda_q = \lambda$ , (15) reduces to the Square-Root Lasso, which was formulated in the form [2]:

$$\min_{t,v,\beta^+,\beta^-} \frac{t}{K} + \lambda \sum_{j=1}^{p} (\beta^{j+} + \beta^{j-})$$
s.t.  $v_i = Y_i - x_i^T \beta^+ + x_i^T \beta^-, \ 1 \le i \le n, \ t \ge \|v\|_2, \ \beta^+ \ge 0, \ \beta^- \ge 0.$ 
(16)

The last three constraints are second-order cone constraints. Based on this conic formulation, Belloni et al. [2], have derived three computational algorithms for solving the Square-Root Lasso:

- 1) First order methods by calling the TFOCS Matlab package, or TFOCS for short;
- 2) Interior point method by calling the SDPT3 Matlab package, or IPM for short;
- 3) Coordinatewise optimization, or COORD for short.

According to our experience, TFOCS is very slow and inaccurate. COORD is reasonably fast, but not as accurate as IPM, especially in applications with a large number of parameters p. In computing a solution path, COORD is still much slower than the, perhaps most popular, coordinate descent algorithm for solving the Lasso [9]. Therefore, even for the Square-Root Lasso, without groups, a fast and accurate algorithm is still needed.

We propose a scaled thresholding-based iterative selection procedure (S-TISP) for solving the general Group Square-Root Lasso problem (15). Assume the scaling step

$$Y \leftarrow Y/K, \quad X \leftarrow X/K$$
 (17)

has been performed. Starting from an arbitrary  $\beta(0) \in \mathbb{R}^p$ , S-TISP performs the following iterations to update  $\beta(t), t = 0, 1, \ldots$ :

$$\beta^{j}(t+1) = \vec{\Theta}(\beta^{j}(t) + (X^{j})'(Y - X\beta(t)); \lambda_{j} || X\beta(t) - Y ||_{2}), \quad 1 \le j \le q.$$
(18)

Here,  $\vec{\Theta}$  is the multivariate soft-thresholding operator [21] defined through  $\vec{\Theta}(0; \lambda) := 0$  and  $\vec{\Theta}(a; \lambda) := a\Theta(||a||_2; \lambda)/||a||_2$ for  $a \neq 0$ , where  $\Theta(t; \lambda) := \operatorname{sign}(t)(|t| - \lambda)_+$  is the soft-thresholding rule. S-TISP is extremely simple to implement and does not resort to any optimization packages.

The following theorem guarantees the global convergence of  $\beta(t)$ . The result is considerably stronger than those 'every accumulation point'-type conclusions that are often seen in numerical analysis.

**Theorem III.1.** Suppose  $\lambda_j > 0$  and the following regularity condition holds:  $\inf_{\xi \in A} ||X\xi - Y||_2 > 0$ , where  $A = \{\vartheta \beta(t) + \xi \}$  $(1-\vartheta)\beta(t+1): \vartheta \in [0,1], t=0,1,\ldots$  From for K large enough, the sequence of iterates  $\beta(t)$  generated by (18) starting with any  $\beta^{(0)}$  converges to a global minimum of (15).

According to our experience, smaller values of K lead to faster convergence if the algorithm converges. The choice K = $||X||/\sqrt{2}$ , motivated by display (39) in the proof of Theorem III.1, works well in the simulation studies; we recall that ||X||is the operator norm of the matrix X. The associated objective function is  $||Y - X\beta||_2 + \sum_j K \cdot \lambda_j ||\beta^j||_2$  which reduces to the specific form (4) if we set

$$\lambda_j = \frac{\lambda}{K} \cdot \sqrt{\frac{T_j}{n}}.$$

Other choices of  $\lambda_j$  are allowable in our computational algorithm. We suggest using warm starts so that the convexity of the problem can be well exploited. Concretely, after specifying a decreasing grid for  $\lambda$ , denoted by  $\Lambda = \{\lambda_1, \dots, \lambda_l\}$ , we use the converged solution  $\beta_{\lambda_l}$  as the initial point  $\beta(0)$  in (18) for the new optimization problem associated with  $\lambda_{l+1}$ .

## **IV. SIMULATIONS**

## A. Computational Time Comparison for the Square-Root Lasso

As explained in Section III above, the Square-Root Lasso can be computed using one of the algorithms TFOCS, COORD, or IPM [2]. As the Square-Root Lasso is a special case of the Group Square-Root Lasso (GSRL), corresponding to q = p, it can also be implemented via our proposed S-TISP algorithm. In this section we compare the three existing methods with ours in terms of computational time. We are particularly interested in high-dimensional, sparse problems, when p is large and  $\beta^0$  is sparse. Since no competing GSRL algorithms exist, we only consider the non-grouped version of S-TISP in the experiments below, for transparent comparison with published literature on algorithms for the Square-Root Lasso, which is only devoted to variable selection, and not to group selection.

For uniformity of comparison, we used a Toeplitz design as in Belloni et al. [2] with correlation matrix  $[0.5^{|i-j|}]_{p \times p}$ . The noise variance is fixed at 1 and the true signal is the p-dimensional, sparse vector  $\beta^0 = (2.5 \ 0 \ 2.5 \ 2.5 \ 0 \ \cdots \ 0)'$ . The first four components of  $\beta^0$  are fixed. The rest are all equal to zero, and their number varies as we vary the dimension of  $\beta^0$  by setting p = 25, 50, 100, 200, 500, 1000, in order to investigate the computational scalability of each of the algorithms under consideration. We set n = 50 for all values of p. We perform the following computations: (i) *PATH*. Solution paths are computed for  $\lambda/(\sqrt{nK}) = 2^{-6}, 2^{-5.8}, \dots, 2^{-0.2}, 2^0$ . This grid is empirically constructed to cover

all potentially interesting solutions as p varies.

(ii) TH. We use the theoretical choice  $\lambda/\sqrt{n} = 1.1\Phi^{-1}(1 - 0.05/(2p))$  recommended in [2] to compute a specific coefficient estimate. In both cases, the error tolerance is 1e-6. Each experiment is repeated 50 times, and we report the average CPU time.

We used the Matlab codes downloaded from Belloni's website and installed some further required Matlab packages, with necessary changes to rescale  $\lambda$ . We made consistent termination criteria, and suppressed the outputs. In particular, we implemented the warm start initiation in COORD which boosts its convergence substantially. The original initialization in the COORD relies on a ridge regression estimate and is slow in computing a solution path. Table I shows the average computational time for 50 runs of each of the algorithms under comparison.

Table I: Computational time comparison (CPU time in seconds) of the first order method by calling the TFOCS package, the interior
point method (IPM) based on SDPT3, the coordinatewise optimization (COORD), and the S-TISP.

PATH	p = 25	p = 50	p = 100	p = 200	p = 500	p = 1000
S-TISP	0.09	0.34	0.64	0.77	1.28	3.42
COORD	0.24	0.67	0.68	0.69	2.37	32.13
IPM	3.84	4.42	4.99	6.09	9.07	15.36
TFOCS	119.08	245.74	452.82	685.25	749.55	696.45
TH	p = 25	p = 50	p = 100	p = 200	p = 500	p = 1000
S-TISP	0.02	0.04	0.13	0.30	0.66	1.80
COORD	0.03	0.07	0.14	0.31	1.03	2.75
IPM	0.13	0.15	0.20	0.28	0.67	2.16
TFOCS	1.00	1.45	1.42	4.52	3.23	5.74

As we can see from Table I, TFOCS and IPM do not scale well for growing p, especially when p > n. After comparing the COORD estimates to those obtained by interior point methods (SDPT3 and SeDuMi), we found that, unfortunately, COORD is a very crude and inaccurate approach. Its inaccuracy is exacerbated by warm starts. We also found that the solutions obtained by calling the TFOCS package are not trustworthy for moderate or large values of p, and that TFOCS is very slow. Our S-TISP achieves comparable accuracy to IPM in the above experiments, and its computational costs scale well with the problem size. In fact, it provides an impressive computational gain over the aforementioned algorithms for high-dimensional data, that is, large p.

## B. Tuning Comparison

In this part of the experiments, we provide empirical evidence of the advantages of the Group Square-Root Lasso in parameter tuning.

We use the same Toeplitz design as before and set  $\sigma = 1$ . The true coefficient vector is generated as  $\beta^0 = (\{2.5\}^3, \{0\}^3, \{2.5\}^3, \{2.5\}^3, \{0\}^3, \dots, \{0\}^3)'$  consisting of three 2.5's, three 0's, three 2.5's, three 2.5's, and finally a sequence of three 0's. Hence, S = 3 and the group sizes are equal to 3. We fix n = 100 and vary p at 60, 300, 600. Each setup is simulated 50 times, and at each run, the Group Square-Root Lasso algorithm, implemented through our proposed S-TISP, is called with three parameter tuning strategies.

(a) Theoretical choice, denoted by TH. This is based on a simplified version of the sequence  $\lambda_0$  given by Lemma II.1. To motivate our choice, we first recall the notation  $\zeta_j = ||X^j||^2/n$  and  $\zeta = \max \zeta_j$ , where ||A|| is the spectral norm of a generic matrix A. Define  $T_{\min} = \min T_j$ ,  $T_{\max} = \max T_j$ . With this notation, we showed in the course of the proof of Lemma II.1 that the sequence  $\lambda_0$  needs to be chosen such that, for given  $\alpha$ ,

$$\mathbb{P}\left(V \ge \lambda_0\right) \le \sum_{j=1}^q \mathbb{P}\left(\chi_{T_j}^2 \ge \frac{\frac{\lambda_0^2}{n^2}(n - T_{\max})}{(\zeta - \frac{\lambda_0^2 T_{\min}}{n^2})_+} \cdot \chi_{n-T_j}^2\right) \le \alpha,$$

where  $\chi^2_{T_j}$  and  $\chi^2_{n-T_j}$  are independent  $\chi^2$  variables. Since the ratio of these two variables has a *F*-distribution, and with the notation  $\tau := \frac{\frac{\lambda_0^2}{n^2}(n-T_{\max})}{(\zeta - \frac{\lambda_0^2 T_{\min}}{n^2})_+}$ , we further have

$$\mathbb{P}(V \ge \lambda_0) \le \sum_{j=1}^{q} (1 - F_{T_j, n-T_j}(\tau))$$
$$\le q (1 - F_{T_{\min}, n-T_{\min}}(\tau))$$

where  $F_{n_1,n_2}$  denotes the cumulative distribution function of a F-distribution with  $n_1$  and  $n_2$  degrees of freedom. Hence,  $\mathbb{P}(V \ge \lambda_0) \le \alpha$  if  $\tau \ge F_{T_{\min},n-T_{\min}}^{-1}(1-\alpha/q) =: \tau_0$  or, equivalently, if

$$\lambda_0 \ge n \sqrt{\frac{\zeta \tau_0}{T_{\min} \tau_0 + n - T_{\max}}}.$$
(19)

The proof of Lemma II.1, in which control of the event  $(V \ge \lambda_0)$  and the determination of  $\lambda_0$  is done via deviation inequalities for  $\chi^2$  random variables, can be used to show that  $\lambda_0$  given by (19) above has the correct order of magnitude. Since the

calculation involving the F-distribution leading to (19) is more precise, we advocate this choice for practical use, for models with Gaussian errors. We further use Corollary II.1 to choose  $\lambda = \lambda_0$ , for our particular design.

Therefore, we use the form (15) in our implementation, with

$$\lambda_j = \sqrt{\zeta \tau_0 / (T_{\min} \tau_0 + n - T_{\max})} \sqrt{nT_j} / K,$$

and  $\tau_0 = F_{T_{\min},n-T_{\min}}^{-1}(1-\alpha/q)$ ,  $K = ||X||_2/\sqrt{2}$  and  $\alpha = 0.01$ . After the optimal estimate is located, bias correction is conducted by fitting a local OLS restricted to the selected dimensions, to boost the prediction accuracy.

(b) Cross-Validation (CV). We use 5-fold CV to determine the optimal value of  $\lambda$  and the associated estimate. Similarly, bias-correction is performed at the end.

(c) SCV-BIC [21]. We cross-validate the sparsity patterns instead of the values of  $\lambda$ . Unlike *K*-fold CV, only one penalized solution path needs to be generated by running the Group Square-Root Lasso on the *entire* dataset. This determines the candidate sparsity patterns. Then, we fit restricted OLS in each CV training to evaluate the validation error of the associated sparsity pattern and append a BIC correction term to the total validation error. SCV-BIC is much less expensive than CV, noting that the OLS fitting is cheap, and has been shown to bring significant performance improvement, see [21] for details and [8] for a similar approach.

To measure the prediction accuracy, we generated additional test data with  $N_{test} = 1e+4$  observations in each simulation. The effective prediction error is given by  $\mathbf{MSE} = 100 \cdot (\sum_{i=1}^{N_{test}} (y_i - x_i^T \hat{\beta})^2 / (N_{test} \sigma^2) - 1)$ . We found the histogram of MSE is highly asymmetric and far from Gaussian. Therefore, the 40% trimmed-mean (instead of the mean or the somewhat crude median) of MSEs was reported as the goodness of fit of the obtained model. We characterize the selection consistency by computing the **Miss** (M) rate – the mean of  $|\{j : \beta^{j0} \neq 0, \hat{\beta}^{j0} = 0\}|/|\{j : \beta^{j0} \neq 0\}|$  over all simulations, where  $|\cdot|$  is the cardinality of a set, and **False Alarms** (FA) rate – the mean of  $|\{j : \beta^{j0} = 0, \hat{\beta}^{j0} = 0, \hat{\beta}^{j0} \neq 0\}|/|\{j : \beta^{j0} = 0\}|$  over all simulations. Correct selection occurs when M = FA = 0.

Table II: Performance of Group Square-Root Lasso Tunings—CV, SCV-BIC, and the theoretical choice (TH), in terms of miss rate (M), false alarm rate (FA), and prediction error (MSE).

	p = 60			p = 300			p = 600		
	М							FA	
		12.75%							
SCV-BIC	0%	0%							
TH	0%	0%	9.82	0%	0%	9.99	0.67%	0%	9.20

The missing rates are very low, which indicates that all truly relevant predictors are detected most of the time. We point out that this will typically happen when the signal strength is moderate to high (2.5 in our simulations), and it supports our theoretical findings. We expect a lesser performance when the signal strength is weaker. We conclude from Table II that the selection by CV is acceptable, especially in high-dimensional, sparse problems, but it has the worst behavior relative to the other tuning strategies. SCV-BIC gives excellent prediction accuracy and recovers the true sparsity pattern successfully. It is much more efficient than CV but still requires the computation of one Group Square-Root Lasso solution path. The theoretical choice (TH) directly specifies the value for the regularization parameter and there is no need for a time-consuming grid search. For Gaussian errors, this particular TH gives almost comparable performance to SCV-BIC in terms of both prediction and variable selection accuracy.

#### APPENDIX

## Proofs for Section 2

Throughout this section we will make use of the following basic fact.

Lemma IV.1. For a given 
$$\alpha \in (0,1)$$
, let  $t = \sqrt{\frac{4\ln(1/\alpha)}{n}} + \frac{4\ln(1/\alpha)}{n}$  and define  
$$\mathcal{B} := \{ \|\epsilon\|_2 / \sqrt{n} \le \sqrt{1+t} \}.$$
 (20)

Then,

$$\mathbb{P}\left(\mathcal{B}\right) \geq 1 - \alpha.$$

The proof of this result is a direct application of Lemma 8.1 in [6]. Notice that, on  $\mathcal{B}$ , we have  $\|\epsilon\|_2/\sqrt{n} \leq C$ , for a dominating constant C. We will make implicit use of this fact throughout.

*Proof of Theorem II.1.* In the first step of the proof, we show that  $\hat{\delta} := \hat{\beta} - \beta^0 \in \Delta_{\gamma}$ . The desired bounds are then derived in a second step.

For the first step, we note that the definition of the estimator (4) implies

$$\frac{\|Y - X\widehat{\beta}\|_2}{\sqrt{n}} - \frac{\|Y - X\beta^0\|_2}{\sqrt{n}} \le \frac{\lambda}{n} \sum_{j=1}^q \sqrt{T_j} \left( \|\beta^{0j}\|_2 - \|\widehat{\beta}^j\|_2 \right),$$

and simple algebra yields

$$\frac{\lambda}{n} \sum_{j=1}^{q} \sqrt{T_j} \left( \|\beta^{0j}\|_2 - \|\widehat{\beta}^j\|_2 \right) = \frac{\lambda}{n} \sum_{j \in S} \sqrt{T_j} \left( \|\beta^{0j}\|_2 - \|\widehat{\beta}^j\|_2 \right) - \frac{\lambda}{n} \sum_{j \in S^c} \sqrt{T_j} \|\widehat{\beta}^j\|_2$$
$$\leq \frac{\lambda}{n} \sum_{j \in S} \sqrt{T_j} \|\beta^{0j}\|_2 - \|\widehat{\beta}^j\|_2 |-\frac{\lambda}{n} \sum_{j \in S^c} \sqrt{T_j} \|\widehat{\beta}^j\|_2$$
$$\leq \frac{\lambda}{n} \sum_{j \in S} \sqrt{T_j} \|\widehat{\delta}^j\|_2 - \frac{\lambda}{n} \sum_{j \in S^c} \sqrt{T_j} \|\widehat{\delta}^j\|_2.$$

These two inequalities give

$$\frac{\|Y - X\widehat{\beta}\|_2}{\sqrt{n}} \le \frac{\|Y - X\beta^0\|_2}{\sqrt{n}} + \frac{\lambda}{n} \sum_{j \in S} \sqrt{T_j} \|\widehat{\delta}^j\|_2 - \frac{\lambda}{n} \sum_{j \in S^c} \sqrt{T_j} \|\widehat{\delta}^j\|_2.$$
(21)

Next, we bound the error term. We obtain, via an application of the Cauchy-Schwarz's inequality, and recalling the definition of the error term V:

$$\begin{aligned} |\epsilon' X \widehat{\delta}| &= |\sum_{j=1}^{q} \epsilon' X^{j} \widehat{\delta}^{j}| \\ &\leq \sum_{j=1}^{q} \|(\epsilon' X^{j})'\|_{2} \|\widehat{\delta}^{j}\|_{2} \\ &\leq \max_{1 \leq j \leq q} \left\{ \frac{\sqrt{n} \|(\epsilon' X^{j})'\|_{2}}{\sqrt{T_{j}} \|\epsilon\|_{2}} \right\} \frac{\|\epsilon\|_{2}}{\sqrt{n}} \sum_{j=1}^{q} \sqrt{T_{j}} \|\widehat{\delta}^{j}\|_{2} \\ &= V \frac{\|\epsilon\|_{2}}{\sqrt{n}} \sum_{j=1}^{q} \sqrt{T_{j}} \|\widehat{\delta}^{j}\|_{2}. \end{aligned}$$

$$(22)$$

•

We then observe that

$$\frac{\nabla \|Y - X\beta\|_2|_{\beta = \beta^0}}{\sqrt{n}} = \frac{-X'\epsilon}{\sqrt{n}\|\epsilon\|_2}$$

and use Inequality (22) and the fact that any norm is convex to obtain

$$\frac{\|Y - X\hat{\beta}\|_2}{\sqrt{n}} - \frac{\|Y - X\beta^0\|_2}{\sqrt{n}} \ge \frac{|\epsilon' X\hat{\delta}|}{\sqrt{n}} \\ \ge \frac{V}{n} \sum_{j=1}^q \sqrt{T_j} \|\hat{\delta}^j\|_2$$

Since on the set  $\mathcal{A}$  we have  $\lambda/\overline{\gamma} \geq V$ , we further obtain

$$\frac{\|Y - X\widehat{\beta}\|_2}{\sqrt{n}} - \frac{\|Y - X\beta^0\|_2}{\sqrt{n}} \ge -\frac{\lambda}{n\overline{\gamma}} \sum_{j=1}^q \sqrt{T_j} \|\widehat{\delta}^j\|_2.$$
(23)

Combining (21) and (23), we find

$$-\frac{\lambda}{n\overline{\gamma}}\sum_{j=1}^{q}\sqrt{T_{j}}\|\widehat{\delta}^{j}\|_{2} \leq \frac{\lambda}{n}\sum_{j\in S}\sqrt{T_{j}}\|\widehat{\delta}^{j}\|_{2} - \frac{\lambda}{n}\sum_{j\in S^{c}}\sqrt{T_{j}}\|\widehat{\delta}^{j}\|_{2},$$

and thus

$$\left(1 - \frac{1}{\overline{\gamma}}\right) \frac{\lambda}{n} \sum_{j \in S^c} \sqrt{T_j} \|\widehat{\delta}^j\|_2 \le \left(1 + \frac{1}{\overline{\gamma}}\right) \frac{\lambda}{n} \sum_{j \in S} \sqrt{T_j} \|\widehat{\delta}^j\|_2$$

This implies  $\frac{\lambda}{n} \sum_{j \in S^c} \sqrt{T_j} \|\widehat{\delta}^j\|_2 \le \left(\frac{\overline{\gamma}+1}{\overline{\gamma}-1}\right) \frac{\lambda}{n} \sum_{j \in S} \sqrt{T_j} \|\widehat{\delta}^j\|_2$  and since  $\lambda > 0$ , we obtain

$$\sum_{j \in S^c} \sqrt{T_j} \|\widehat{\delta}^j\|_2 \le \gamma \sum_{j \in S} \sqrt{T_j} \|\widehat{\delta}^j\|_2, \tag{24}$$

or equivalently,  $\widehat{\delta} \in \Delta_{\gamma}$ , as desired.

To derive the bounds stated in the theorem we begin by observing that

$$\frac{\|Y - X\widehat{\beta}\|_2}{\sqrt{n}} - \frac{\|Y - X\beta^0\|_2}{\sqrt{n}} \le \frac{\lambda}{n} \frac{\sqrt{s^*} \|X\widehat{\delta}\|_2}{\sqrt{n\kappa}}$$
(25)

by (21) and the Compatibility Condition (5). Next, we write

$$\frac{\|Y - X\widehat{\beta}\|_2^2}{n} - \frac{\|Y - X\beta^0\|_2^2}{n} = \frac{\|X\widehat{\delta} - \sigma\epsilon\|_2^2}{n} - \frac{\|\sigma\epsilon\|_2^2}{n} = \frac{\|X\widehat{\delta}\|_2^2}{n} - \frac{2\sigma\epsilon' X\widehat{\delta}}{n},$$

and we use (5), (22), and (25) to obtain

$$\begin{split} &\frac{\|X\widehat{\delta}\|_{2}^{2}}{n} = \frac{\|Y - X\widehat{\beta}\|_{2}^{2}}{n} - \frac{\|Y - X\beta^{0}\|_{2}^{2}}{n} + \frac{2\sigma\epsilon' X\widehat{\delta}}{n} \\ &= \left(\frac{\|Y - X\widehat{\beta}\|_{2}}{\sqrt{n}} - \frac{\|Y - X\beta^{0}\|_{2}}{\sqrt{n}}\right) \left(\frac{\|Y - X\widehat{\beta}\|_{2}}{\sqrt{n}} + \frac{\|Y - X\beta^{0}\|_{2}}{\sqrt{n}}\right) + \frac{2\sigma\epsilon' X\widehat{\delta}}{n} \\ &\leq \frac{\lambda}{n} \frac{\sqrt{s^{*}} \|X\widehat{\delta}\|_{2}}{\sqrt{n\kappa}} \left(\frac{2\|Y - X\beta^{0}\|_{2}}{\sqrt{n}} + \frac{\lambda}{n} \frac{\sqrt{s^{*}} \|X\widehat{\delta}\|_{2}}{\sqrt{n\kappa}}\right) + \frac{2V\|\sigma\epsilon\|_{2}}{n^{3/2}} \sum_{j=1}^{q} \sqrt{T_{j}} \|\widehat{\delta}^{j}\|_{2} \\ &\leq \frac{s^{*}\lambda^{2}}{\kappa^{2}n^{2}} \frac{\|X\widehat{\delta}\|_{2}^{2}}{n} + \frac{2\lambda}{n} \frac{\sqrt{s^{*}} \|X\widehat{\delta}\|_{2}}{\sqrt{n\kappa}} \frac{\|\sigma\epsilon\|_{2}}{\sqrt{n}} + \frac{2(1+\gamma)\lambda\|\sigma\epsilon\|_{2}}{\overline{\gamma}n^{3/2}} \frac{\sqrt{s^{*}} \|X\widehat{\delta}\|_{2}}{\sqrt{n\kappa}} \\ &= \frac{s^{*}\lambda^{2}}{\kappa^{2}n^{2}} \frac{\|X\widehat{\delta}\|_{2}^{2}}{n} + \gamma \frac{2\lambda}{n} \frac{\sqrt{s^{*}} \|X\widehat{\delta}\|_{2}}{\sqrt{n\kappa}} \frac{\|\sigma\epsilon\|_{2}}{\sqrt{n}} \end{split}$$

since on  $\mathcal{A}$  we have  $\lambda/\overline{\gamma} \geq V$ . Consequently,

$$\frac{\|X\widehat{\delta}\|_2^2}{n} \le \frac{u\sqrt{s^*}\lambda\|\sigma\epsilon\|_2\|X\widehat{\delta}\|_2}{n^2\kappa},$$

where  $u := \frac{2\gamma}{1 - \frac{\lambda^2 s^*}{n^2 \kappa^2}} \in (0, \infty)$  by assumption (9). Since on  $\mathcal{B}$  we have

$$\frac{\|\sigma\epsilon\|_2}{\sqrt{n}} \le \sigma\sqrt{1+t}$$

the first statement of the theorem follows:

$$\|X(\widehat{\beta} - \beta^0)\|_2 \le \sigma\sqrt{1+t}\frac{\lambda\sqrt{s^*u}}{\kappa\sqrt{n}} \lesssim \sqrt{1+t}\frac{\lambda\sqrt{s^*}}{\kappa\sqrt{n}}.$$

For the second claim, we use the fact that  $\delta \in \Delta_{\gamma}$  and the Compatibility Condition (5) to deduce that

$$\sum_{j=1}^q \sqrt{T_j} \|\delta^j\|_2 \le (\gamma+1) \sum_{j \in S} \sqrt{T_j} \|\delta^j\|_2 \le \frac{(\gamma+1)\sqrt{s^*} \|X(\widehat{\beta}-\beta^0)\|_2}{\sqrt{n\kappa}} \le \frac{\lambda(\gamma+1)us^*}{n\kappa^2} \frac{\|\sigma\epsilon\|_2}{\sqrt{n}}.$$

Therefore, again on the set  $\mathcal{B}$ , we have

$$\sum_{j=1}^{q} \sqrt{T_j} \| (\hat{\beta} - \beta)^j \|_2 \le \sigma \sqrt{1 + t} \frac{\lambda(\gamma + 1)us^*}{n\kappa^2} \lesssim \sqrt{1 + t} \frac{\lambda s^*}{n\kappa^2},$$

which concludes the proof of this theorem.

We next prove Theorem II.2. We begin with two preparatory results.

**Lemma IV.2.** Assume that  $Y \neq X\hat{\beta}$  over some set C. Then, on C, the quantity  $\hat{\beta}$  is a solution of the criterion (4) if and only if for every  $1 \leq j \leq q$ 

$$\widehat{\beta}^{j} \neq 0 \Rightarrow \frac{(X'(Y - X\widehat{\beta}))^{j}}{\|Y - X\widehat{\beta}\|_{2}} = \frac{\lambda \sqrt{T_{j}}}{\sqrt{n} \|\widehat{\beta}^{j}\|_{2}} \widehat{\beta}^{j}$$
(26)

$$\widehat{\beta}^{j} = 0 \Rightarrow \frac{\|(X'(Y - X\widehat{\beta}))^{j}\|_{2}}{\|Y - X\widehat{\beta}\|_{2}} \le \frac{\lambda\sqrt{T_{j}}}{\sqrt{n}}.$$
(27)

*Proof.* Since all terms of the criterion (4) are convex, and thus, the criterion is convex, we can apply standard subgradient calculus. The subgradient  $\partial f|_x$  of a convex function  $f : \mathbb{R}^p \to \mathbb{R}$  at a point  $x \in \mathbb{R}^p$  is defined as the set of vectors  $v \in \mathbb{R}^p$  such that for all  $y \in \mathbb{R}^p$ 

$$f(y) \ge f(x) + v'(y - x).$$

From this, one derives easily that subgradients are linear and additive and that the subgradient  $\partial f|_x$  is equal to the gradient  $\nabla f|_x$  if the function f is differentiable at x. Moreover,  $x \in \mathbb{R}^p$  is a minimum of the function f if and only if  $0 \in \partial f|_x$ . Since  $Y \neq X\hat{\beta}$ , the first term of the criterion (4) is differentiable and we have

$$\nabla \|Y - X\beta\|_2|_\beta = \frac{\nabla \|Y - X\beta\|_2^2|_\beta}{2\|Y - X\beta\|_2} = \frac{-X'(Y - X\beta)}{\|Y - X\beta\|_2}.$$
(28)

For the remaining terms, we observe that for any vector  $w \in \mathbb{R}^T \setminus \{0\}, T \in \mathbb{N}$ ,

$$\nabla \|w\|_2|_w = \frac{w}{\|w\|_2} \tag{29}$$

and for w = 0

$$v \in \partial \|w\|_2|_{w=0} \Leftrightarrow \|z\|_2 \ge \|0\|_2 + (z-0)'v = z'v \quad \text{for all } z \in \mathbb{R}^T$$

$$(30)$$

and, consequently,  $\partial ||w||_2|_{w=0} = \{v \in \mathbb{R}^T : ||v||_2 \le 1\}$ . The claim follows then from Equations (28), (29), and (30).

**Lemma IV.3.** Under the conditions of Theorem 11.1, it holds that, on the set  $\mathcal{A} \cap \mathcal{B}$ , we have

$$\left(1 - \frac{\lambda\sqrt{s^*u}}{n\kappa}\right)\|\sigma\epsilon\|_2 \le \|Y - X\widehat{\beta}\|_2 \le \left(1 + \frac{\lambda\sqrt{s^*u}}{n\kappa}\right)\|\sigma\epsilon\|_2$$

for  $u := \frac{2\gamma}{1 - \frac{\lambda^2 s^*}{n^2 \kappa^2}}$ .

*Proof.* By the triangle inequality

$$\|\sigma \epsilon\|_{2} - \|X(\widehat{\beta} - \beta_{0})\|_{2} \le \|Y - X\widehat{\beta}\|_{2} \le \|\sigma \epsilon\|_{2} + \|X(\widehat{\beta} - \beta_{0})\|_{2}.$$

The claim follows immediately by Theorem II.1 above.

Proof of Theorem II.2. The crucial step in this proof is to use the KKT Conditions in Lemma IV.2 in order to show that, on  $\mathcal{A}_1 \cap \mathcal{B}$ , we have  $\hat{\beta}_{S^c} = 0$ .

First, we observe that Lemma IV.3 implies that  $Y - X\hat{\beta} \neq 0$  on  $\mathcal{A} \cap \mathcal{B}$ , and we can consequently apply the KKT Conditions derived in Lemma IV.2 for  $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ . Moreover, since by definition,  $\mathcal{A}_1 \cap \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B}$ , the results also hold on the smaller set. Thus, there exists a vector  $\tau \in \mathbb{R}^p$  such that  $\|\tau^j\|_2 \leq \sqrt{T_j}$  for all  $1 \leq j \leq q$  and, additionally,  $\tau^j = \frac{\sqrt{T_j}\hat{\beta}^j}{\|\hat{\beta}^j\|_2}$ , for all  $1 \leq j \leq q$  such that  $\hat{\beta}^j \neq 0$ , and  $\tau$  satisfies the equality

$$\frac{X'(Y - X\widehat{\beta})}{\|Y - X\widehat{\beta}\|_2} = \frac{\lambda}{\sqrt{n}}\tau.$$

We rewrite this with

$$\widehat{\psi}:=\|Y-X\widehat{\beta}\|_2$$
 and  $\widehat{\delta}:=\widehat{\beta}-\beta^0$ 

as

$$\sigma X' \epsilon - X' X \widehat{\delta} = \frac{\widehat{\psi} \lambda}{\sqrt{n}} \tau.$$

So, on the one hand, we have

$$-n^{2}\Sigma_{1,1}\widehat{\delta}_{S} - n^{2}\Sigma_{1,2}\widehat{\delta}_{S^{c}} = \sqrt{n}\widehat{\psi}\lambda\tau_{S} - n\sigma(X'\epsilon)_{S},$$
(31)

or, equivalently,

and finally

$$-n^{2}\widehat{\delta}_{S} - n^{2}\Sigma_{1,1}^{-1}\Sigma_{1,2}\widehat{\delta}_{S^{c}} = \sqrt{n}\widehat{\psi}\lambda\Sigma_{1,1}^{-1}\tau_{S} - n\sigma\Sigma_{1,1}^{-1}(X'\epsilon)_{S},$$

$$n^{2}\hat{\delta}_{S^{c}}\Sigma_{2,1}\hat{\delta}_{S} - n^{2}\hat{\delta}_{S^{c}}^{\prime}\Sigma_{2,1}\Sigma_{1,1}^{-1}\Sigma_{1,2}\hat{\delta}_{S^{c}} = \sqrt{n}\hat{\psi}\lambda\hat{\delta}_{S^{c}}^{\prime}\Sigma_{2,1}\Sigma_{1,1}^{-1}\tau_{S} - n\sigma\hat{\delta}_{S^{c}}^{\prime}\Sigma_{2,1}\Sigma_{1,1}^{-1}(X^{\prime}\epsilon)_{S}.$$
(32)

On the other hand, we have

$$-n^2 \Sigma_{2,1} \widehat{\delta}_S - n^2 \Sigma_{2,2} \widehat{\delta}_{S^c} = \sqrt{n} \widehat{\psi} \lambda \tau_{S^c} - n \sigma(X' \epsilon)_{S^c}.$$

Since for  $j \in S^c$ 

$$\begin{split} \widehat{\beta}^{j} &\neq 0 \Rightarrow \widehat{\delta}^{j} \cdot \tau^{j} = \frac{\sqrt{T_{j}} \ \widehat{\delta}^{j} \cdot \widehat{\delta}^{j}}{\|\widehat{\delta}\|_{2}} = \sqrt{T_{j}} \|\widehat{\delta}^{j}\|_{2} \\ \widehat{\beta}^{j} &= 0 \Rightarrow \widehat{\delta}^{j} \cdot \tau^{j} = 0 = \sqrt{T_{j}} \|\widehat{\delta}^{j}\|_{2}, \end{split}$$

this implies that

$$-n^{2}\widehat{\delta}_{S^{c}}\Sigma_{2,1}\widehat{\delta}_{S} - n^{2}\widehat{\delta}_{S^{c}}^{\prime}\Sigma_{2,2}\widehat{\delta}_{S^{c}} = \sqrt{n}\widehat{\psi}\lambda\widehat{\delta}_{S^{c}}^{\prime}\tau_{S^{c}} - n\sigma\widehat{\delta}_{S^{c}}^{\prime}(X^{\prime}\epsilon)_{S^{c}}$$
$$= \sqrt{n}\widehat{\psi}\lambda\sum_{j\in S^{c}}\sqrt{T_{j}}\left(\|\widehat{\delta}^{j}\|_{2} - \frac{\sigma\sqrt{n}\ \widehat{\delta}^{j}\cdot(X^{\prime}\epsilon)^{j}}{\sqrt{T_{j}}\lambda\widehat{\psi}}\right)$$

The right-hand side can be bounded from below, using Cauchy-Schwarz's Inequality, by

$$\sqrt{n}\widehat{\psi}\lambda\sum_{j\in S^c}\sqrt{T_j}\left(\|\widehat{\delta}^j\|_2-\|\widehat{\delta}^j\|_2\frac{\sigma\sqrt{n}\|(X'\epsilon)^j\|_2}{\sqrt{T_j}\lambda\widehat{\psi}}\right).$$

Lemma IV.3 implies that  $\lambda/\widetilde{\eta} \geq \widehat{V}$  for

$$\widehat{V} := \max_{1 \le j \le q} \left\{ \frac{\sigma \sqrt{n} \| (X' \epsilon)^j \|_2}{\sqrt{T_j} \widehat{\psi}} \right\},\,$$

and thus, the above term can be bounded from below by

$$\left(1 - \frac{1}{\widetilde{\eta}}\right) \sqrt{n}\widehat{\psi}\lambda \sum_{j \in S^c} \sqrt{T_j} \|\widehat{\delta}^j\|_2$$

So, in summary, we have

$$-n^{2}\widehat{\delta}_{S^{c}}'\Sigma_{2,1}\widehat{\delta}_{S} - n^{2}\widehat{\delta}_{S^{c}}'\Sigma_{2,2}\widehat{\delta}_{S^{c}} \ge \left(1 - \frac{1}{\widetilde{\eta}}\right)\sqrt{n}\widehat{\psi}\lambda\sum_{j\in S^{c}}\sqrt{T_{j}}\|\widehat{\delta}^{j}\|_{2}.$$
(33)

Subtracting Equation (33) from Equation (32) then yields

$$n^{2}\widehat{\delta}_{S^{c}}^{\prime}(\Sigma_{2,2}-\Sigma_{2,1}\Sigma_{1,1}^{-1}\Sigma_{1,2})\widehat{\delta}_{S^{c}}$$

$$\leq \sqrt{n}\widehat{\psi}\lambda\widehat{\delta}_{S^{c}}^{\prime}\Sigma_{2,1}\Sigma_{1,1}^{-1}(\tau_{S}-\frac{\sqrt{n}\sigma}{\lambda\widehat{\psi}}(X^{\prime}\epsilon)_{S}) - \left(1-\frac{1}{\widetilde{\eta}}\right)\sqrt{n}\widehat{\psi}\lambda\sum_{j\in S^{c}}\sqrt{T_{j}}\|\widehat{\delta}^{j}\|_{2}.$$
(34)

The first term of the right-hand side above can be bounded via the Cauchy-Schwarz's inequality by

$$\begin{split} \sqrt{n}\widehat{\psi}\lambda\widehat{\delta}_{S^c}\Sigma_{2,1}\Sigma_{1,1}^{-1}(\tau_S - \frac{\sqrt{n}\sigma}{\lambda\widehat{\psi}}(X'\epsilon)_S) &= \sqrt{n}\widehat{\psi}\lambda\sum_{j\in S^c}\widehat{\delta}^j \cdot (\widetilde{\Sigma}_{2,1}\Sigma_{1,1}^{-1}(\tau_S - \frac{\sqrt{n}\sigma}{\lambda\widehat{\psi}}(X'\epsilon)_S))^j \\ &\leq \sqrt{n}\widehat{\psi}\lambda\sum_{j\in S^c}\sqrt{T_j}\|\widehat{\delta}^j\|_2 \frac{\|(\widetilde{\Sigma}_{2,1}\Sigma_{1,1}^{-1}(\tau_S - \frac{\sqrt{n}\sigma}{\lambda\widehat{\psi}}(X'\epsilon)_S))^j\|_2}{\sqrt{T_j}}. \end{split}$$

Now, we observe that if  $\lambda/\tilde{\eta} \geq \hat{V}$ , then  $\frac{\sigma\sqrt{n}}{\tilde{\psi}\lambda} ||(X'\epsilon)^j||_2 \leq \frac{\sqrt{T_j}}{\tilde{\eta}}$  for all  $0 \leq j \leq q$ , and thus, the above expression can be bounded by

$$\sqrt{n}\widehat{\psi}\lambda \max_{\substack{v:\|v^k\|_2 \leq \left(1+\frac{1}{\eta}\right)\sqrt{T_k}}} \sum_{j\in S^c} \sqrt{T_j} \|\widehat{\delta}^j\|_2 \frac{\|(\Sigma_{2,1}\Sigma_{1,1}^{-1}v)^j\|_2}{\sqrt{T_j}} \\
= \left(1+\frac{1}{\widetilde{\eta}}\right)\sqrt{n}\widehat{\psi}\lambda \max_{\substack{v:\|v^k\|_2 \leq \sqrt{T_k}}} \sum_{j\in S^c} \sqrt{T_j} \|\widehat{\delta}^j\|_2 \frac{\|(\widetilde{\Sigma}_{2,1}\Sigma_{1,1}^{-1}v)^j\|_2}{\sqrt{T_j}}$$

If  $\hat{\beta}_{S^c} \neq 0$ , this is strictly smaller than

$$\left(1+\frac{1}{\widetilde{\eta}}\right)u\sqrt{n}\widehat{\psi}\lambda\sum_{j\in S^c}\sqrt{T_j}\|\widehat{\delta}^j\|_2 = \left(1-\frac{1}{\widetilde{\eta}}\right)\sqrt{n}\widehat{\psi}\lambda\sum_{j\in S^c}\sqrt{T_j}\|\widehat{\delta}^j\|_2$$

by our Group Irrepresentable Condition. Then, by Inequality (34), this yields

$$n^{2}\widehat{\delta}_{S^{c}}'(\Sigma_{2,2}-\Sigma_{2,1}\Sigma_{1,1}^{-1}\Sigma_{1,2})\widehat{\delta}_{S^{c}}<0.$$

But since  $\Sigma_{2,2} - \Sigma_{2,1} \Sigma_{1,1}^{-1} \Sigma_{1,2} \ge 0$ , this leads to a contradiction. Hence,  $\hat{\delta}_{S^c} = 0$  and the first claim is proved. For the second claim, we invoke  $\hat{\delta}_{S^c} = 0$  to obtain, using Equation (31),

$$-n^2 \Sigma_{1,1} \widehat{\delta}_S = \sqrt{n} \widehat{\psi} \lambda \tau_S - n \sigma (X' \epsilon)_S$$

This implies

$$-n^2 \widehat{\delta}_S = \sqrt{n} \widehat{\psi} \lambda \Sigma_{1,1}^{-1} \left( \tau_S - \frac{\sqrt{n} \sigma(X'\epsilon)_S}{\widehat{\psi} \lambda} \right)$$

and, using  $\lambda/\widetilde{\eta} \leq \widehat{V}$  and bounding the norms as above,

$$\begin{split} \|\widehat{\delta}^{j}\|_{\infty} &\leq \max_{v:\|v^{k}\|_{2} \leq \sqrt{T_{j}}} \frac{\left(1 + \frac{1}{\overline{\eta}}\right)\lambda}{n} \frac{\widehat{\psi}}{\sqrt{n}} \|(\widetilde{\Sigma}_{1,1}^{-1}v)^{j}\|_{\infty} \\ &\leq \frac{\left(1 + \frac{1}{\overline{\eta}}\right)\sqrt{T_{j}\lambda}}{n} \frac{\widehat{\psi}}{\sqrt{n}} \xi_{\|\cdot\|_{\infty}} \\ &\leq \frac{\left(1 + \frac{1}{\overline{\eta}}\right)\sqrt{T_{j}\lambda}}{n} \frac{\|\sigma\epsilon\|_{2}}{\sqrt{n}} \left(1 + \frac{\lambda\sqrt{s^{*}u}}{n\kappa}\right) \xi_{\|\cdot\|_{\infty}} \\ &\leq \frac{2\sigma\sqrt{1+t}}{1 + \overline{\eta}} (1+u)\xi_{\|\cdot\|_{\infty}} \frac{\lambda\sqrt{T_{j}}}{n} \\ &\leq D\frac{\lambda\sqrt{T_{j}}}{n}, \text{ for all } 1 \leq j \leq q, \end{split}$$

which is the second claim of this theorem. In the above derivation we used Lemma IV.3 above for the third inequality, and assumption (9) for the forth and the fact that  $\sqrt{1+t}$  is bounded by a constant, by the definition of the set  $\mathcal{B}$  in Lemma IV.1 above. We also recall that under (9), the quantity u is a positive constant.

The third claim follows immediately from the first two and the Beta Min Condition. This concludes the prof of this theorem.

Proof of Lemma II.1. We first observe that

$$\mathbb{P}(V \ge \lambda_0) = \mathbb{P}\left(\max_{1 \le j \le q} \left\{\frac{\sqrt{n} \|(\epsilon' X^j)'\|_2}{\sqrt{T_j} \|\epsilon\|_2}\right\} \ge \lambda_0\right)$$
$$= \mathbb{P}\left(\max_{1 \le j \le q} \epsilon' \left(\frac{X^j (X^j)'}{n} - \frac{\lambda_0^2 T_j}{n^2} I\right) \epsilon \ge 0\right)$$
$$\le \sum_{j=1}^q \mathbb{P}\left(\epsilon' \left(\frac{X^j (X^j)'}{n} - \frac{\lambda_0^2 T_j}{n^2} I\right) \epsilon \ge 0\right).$$

Let U'(j)D(j)U(j) be a spectral decomposition of  $X^j(X^j)'/n$  such that U(j) is orthogonal and D(j) is diagonal with diagonal entries  $\xi_1(j) \ge \cdots \ge \xi_{T_j}(j) \ge \xi_{T_j+1}(j) = \cdots = \xi_n(j) = 0$ . With the notation  $\zeta_j = ||X^j||^2/n$ , where ||A|| is the spectral norm of a generic matrix A, we have  $\zeta_j = \xi_1(j)$ . It follows that

$$\begin{aligned} \epsilon' \left( \frac{X^j (X^j)'}{n} - \frac{\lambda_0^2 T_j}{n^2} I \right) \epsilon &= \epsilon' \left( U'(j) D(j) U(j) - \frac{\lambda_0^2 T_j}{n^2} I \right) \epsilon \\ &= (U(j)\epsilon)' \left( D(j) - \frac{\lambda_0^2 T_j}{n^2} I \right) U(j)\epsilon \\ &\leq \|\epsilon_1\|_2^2 \left( \zeta_j - \frac{\lambda_0^2 T_j}{n^2} \right)_+ - \|\epsilon_2\|_2^2 \frac{\lambda_0^2 T_j}{n^2}, \end{aligned}$$

where  $\epsilon_1$  and  $\epsilon_2$  are independent with  $\epsilon_1 \sim \mathcal{N}(0, I_{T_j})$  and  $\epsilon_2 \sim \mathcal{N}(0, I_{(n-T_j)})$ . Thus, for any fixed  $r \in (0, 1)$  we have

$$\mathbb{P}(V \ge \lambda_{0}) \le \sum_{j=1}^{q} \mathbb{P}\left(\frac{\|\epsilon_{1}\|_{2}^{2}}{T_{j}} \cdot (\zeta_{j} - \frac{\lambda_{0}^{2}T_{j}}{n^{2}})_{+} \ge \frac{\lambda_{0}^{2}}{n^{2}} \cdot \|\epsilon_{2}\|_{2}^{2}\right) \\
\le \sum_{j=1}^{q} \mathbb{P}\left(\frac{\|\epsilon_{1}\|_{2}^{2}}{T_{j}} \cdot (\frac{n^{2}\zeta_{j}}{\lambda_{0}^{2}} - T_{j})_{+} \cdot \frac{1}{n - T_{j}} \ge 1 - r\right) \\
+ \sum_{j=1}^{q} \mathbb{P}\left(\frac{1}{n - T_{j}} \cdot \|\epsilon_{2}\|_{2}^{2} \le 1 - r\right).$$
(35)

If  $\zeta_j \leq \frac{\lambda_0^2 T_j}{n^2}$ , then the first sum in the inequality above is trivially equal to zero, therefore the argument below is needed only when the reverse inequality holds. From Laurent and Massart [12, Lemma 1],  $\mathbb{P}(X - d \geq dt) \leq \exp\left(-\frac{d}{4}(\sqrt{(1+2t}-1)^2)\right)$  and  $P(X \leq d - dt) \leq \exp\left(-\frac{d}{4}t^2\right)$ , for  $X \sim \chi^2(d)$ . Therefore, for the first term in (35) we obtain, for each j:

$$\mathbb{P}\left(\frac{\|\epsilon_{1}\|_{2}^{2}}{T_{j}} \cdot \left(\frac{n^{2}\zeta_{j}}{\lambda_{0}^{2}} - T_{j}\right) \cdot \frac{1}{n - T_{j}} \ge 1 - r\right)$$

$$\leq \exp\left(-\frac{T_{j}}{4}\left(\sqrt{\frac{2(1 - r)(n - T_{j})}{\frac{n^{2}\zeta_{j}}{\lambda_{0}^{2}} - T_{j}} - 1} - 1\right)^{2}\right)$$

$$\leq \exp\left(-\frac{T_{\min}}{4}\left(\sqrt{\frac{2(1 - r)(n - T_{\max})}{\frac{n^{2}\zeta}{\lambda_{0}^{2}} - T_{\min}} - 1} - 1\right)^{2}\right)$$

To bound the last term in (35) we first obtain, for each *j*:

$$\mathbb{P}\left(\frac{\|\epsilon\|_2^2}{n-T_j} < 1-r\right) \le \exp\left(-\frac{(n-T_j)r^2}{4}\right) \le \exp\left(-\frac{(n-T_{\max})r^2}{4}\right).$$

Hence,

$$\mathbb{P}\left(V \ge \lambda_0\right)$$

$$\leq q \cdot \exp\left(-\frac{T_{\min}}{4}\left(\sqrt{\frac{2(1-r)(n-T_{\max})}{\frac{n^2\zeta}{\lambda_0^2}-T_{\min}}}-1-1\right)^2\right) + q \cdot \exp\left(-\frac{(n-T_{\max})r^2}{4}\right).$$

For  $r = 2\sqrt{\frac{\log(2q/\alpha)}{n-T_{\text{max}}}}$  the last term is bounded by  $\alpha/2$ . For this value of r and with

$$\lambda_0 = \frac{\sqrt{2\zeta}n}{\sqrt{n - T_{\max}}} \left( 1 + \sqrt{\frac{2\log(2q/\alpha)}{T_{\min}}} \right).$$

the first term is also bounded by  $\alpha/2$ . This concludes the proof.

## Proofs for Section III

**Lemma IV.4.** Given any  $\lambda$ ,  $\vec{\Theta}(\cdot; \lambda)$  is nonexpansive:  $\|\vec{\Theta}(x; \lambda) - \vec{\Theta}(\tilde{x}; \lambda)\|_2 \le \|x - \tilde{x}\|_2$ ,  $\forall x, \tilde{x} \in \mathbb{R}^p$ .

*Proof.* Define  $\Delta = \|x - \tilde{x}\|_2^2 - \|\vec{\Theta}(x;\lambda) - \vec{\Theta}(\tilde{x};\lambda)\|_2^2$ ,  $a = \|x\|_2$ ,  $b = \|\tilde{x}\|_2$ , and  $c = x'\tilde{x}/(ab)$ . Obviously,  $|c| \leq 1$  and  $c = \vec{\Theta}(x;\lambda)'\vec{\Theta}(\tilde{x};\lambda)/(ab)$ . By the Cosine Rule,

$$\|x - \tilde{x}\|_{2}^{2} = a^{2} + b^{2} - 2abc$$
  
$$\|\vec{\Theta}(x;\lambda) - \vec{\Theta}(\tilde{x};\lambda)\|_{2}^{2} = ((a-\lambda)_{+})^{2} + ((b-\lambda)_{+})^{2} - 2(a-\lambda)_{+}(b-\lambda)_{+}c.$$

(i) Suppose  $a > \lambda$  and  $b > \lambda$ . Then  $\Delta = -2\lambda^2 + 2(a+b)\lambda + 2\lambda^2c - 2\lambda(a+b)c = 2(1-c)\lambda(a+b-\lambda) \ge 0$ . (ii) Suppose  $a < \lambda$  and  $b > \lambda$ . Then  $\Delta = a^2 + b^2 - 2abc - (b-\lambda)^2 = a^2 - 2abc - \lambda^2 + 2b\lambda \ge a^2 - 2ab - \lambda^2 + 2b\lambda = (2b-a-\lambda)(\lambda-a) \ge 0$ . Therefore,  $\|\vec{\Theta}(x;\lambda) - \vec{\Theta}(\tilde{x};\lambda)\|_2 \le \|x - \tilde{x}\|_2$ .

*Proof of Theorem III.1.* By Lemma IV.4, the mapping (18) is nonexpansive. We use Opial's conditions [17, 20] for studying nonexpansive operators to prove the strict convergence of  $\beta(t)$ . The key of the proof is to show the mapping is *asymptotically* regular:  $\|\beta(t+1) - \beta(t)\| \to 0$  as  $t \to \infty$ , for any starting point  $\beta(0)$ .

Assume the scaling operations (17) have performed beforehand. Let  $F(\beta) = ||Y - X\beta||_2 + \sum_{j=1}^q \lambda_j ||\beta^j||_2$  be the objective function. Introduce a surrogate function

$$G(\beta,\gamma) = \|Y - X\beta\|_{2} + \frac{1}{\|X\beta - Y\|_{2}}(\gamma - \beta)'X'(X\beta - Y) + \frac{1}{2\|X\beta - Y\|_{2}}\|\beta - \gamma\|_{2}^{2} + \sum_{j}\lambda_{j}\|\gamma^{j}\|_{2}.$$
(36)

Given  $\beta$ , algebraic manipulations show that minimizing G over  $\gamma$  is equivalent to

$$\min_{\gamma} \frac{1}{\|X\beta - Y\|_{2}} (\gamma - \beta)' X' (X\beta - Y) + \frac{1}{2\|X\beta - Y\|_{2}} \|\beta - \gamma\|_{2}^{2} + \sum_{j} \lambda_{j} \|\gamma^{j}\|_{2} \iff \\
\min_{\gamma} \frac{1}{\|X\beta - Y\|_{2}} \left( \frac{1}{2} \|\gamma - [\beta + X'Y - X'X\beta]\|_{2}^{2} + \|X\beta - Y\|_{2} \sum_{j} \lambda_{j} \|\gamma^{j}\|_{2} \right).$$
(37)

Applying Lemma 1 and Lemma 2 in [21], we have the optimal  $\gamma_o$  given by

$$\gamma_{o}^{j} = \vec{\Theta}(\beta^{j} + (X^{j})'Y - (X^{j})'X\beta; \|X\beta - Y\|_{2}\lambda_{j}), \quad 1 \le j \le q,$$
(38)

and further obtain

$$G(\beta, \gamma_o + \delta) - G(\beta, \gamma_o) \ge \frac{\|\delta\|_2^2}{2\|X\beta - Y\|_2}.$$

On the other hand, a Taylor series expansion gives

$$\|Y - X\beta\|_{2} + \frac{1}{\|X\beta - Y\|_{2}} (\gamma - \beta)' X' (X\beta - Y) - \|Y - X\gamma\|_{2}$$
  
=  $-\frac{1}{2} (\beta - \gamma)' \left[ \frac{1}{\|X\xi - Y\|_{2}} X' X - \frac{1}{\|X\beta - Y\|_{2}^{3}} X' (X\xi - Y) (X\xi - Y)' X \right] (\beta - \gamma),$ 

for some  $\xi = \vartheta \beta + (1 - \vartheta) \beta$  with  $\vartheta \in (0, 1)$ .

Now, for the iterates defined by (18), we obtain

$$\begin{split} F(\beta(t+1)) &+ \frac{1}{2} (\beta(t+1) - \beta(t))' \left( \frac{1}{\|X\beta(t) - Y\|_2} I - \frac{1}{\|X\xi(t) - Y\|_2} X' X \right) (\beta(t+1) - \beta(t)) \\ &= G(\beta(t), \beta(t+1)) \\ &\leq G(\beta(t), \beta(t)) - \frac{1}{2} \frac{1}{\|X\beta(t) - Y\|_2} (\beta(t+1) - \beta(t))' (\beta(t+1) - \beta(t)) \\ &= F(\beta(t)) - \frac{1}{2} \frac{1}{\|X\beta(t) - Y\|_2} (\beta(t+1) - \beta(t))' (\beta(t+1) - \beta(t)), \end{split}$$

for some  $\xi(t) = \vartheta(t)\beta(t) + (1 - \vartheta(t))\beta(t+1)$  with  $\vartheta(t) \in (0, 1)$ . Therefore, with ||X|| standing for the operator norm of X,

$$F(\beta(t)) - F(\beta(t+1)) \ge \frac{1}{2} \left( \frac{2}{\|X\beta(t) - Y\|_2} - \frac{\|X\|^2}{\|X\xi(t) - Y\|_2} \right) \|\beta(t+1) - \beta(t)\|_2^2.$$
(39)

Under the regularity condition and for K large enough,  $F(\beta(t))$  is monotonically decreasing. In fact, with  $||X\xi(t) - Y||_2 > \epsilon$ and  $M \triangleq F(\beta(0))$ ,  $||X||_2 < 2\epsilon/M$  suffices. It follows that

$$F(\beta(t+1)) \leq F(\beta(t)) \leq M, \forall t, \text{ and } \left(\frac{2}{\|X\beta(t) - Y\|_2} - \frac{\|X\|^2}{\|X\xi(t) - Y\|_2}\right) \|\beta(t+1) - \beta(t)\|_2^2 \to 0 \text{ as } t \to \infty.$$

This, together with the conditions in the theorem, implies that  $\beta(t)$  is uniformly bounded and asymptotically regular. Finally, the fixed point set of the mapping is non-empty because it is a nonexpansive mapping into a bounded closed convex subset [5].

With all of Optial's conditions satisfied,  $\beta(t)$  has a unique limit point  $\beta^*$ . It is easy to verify that  $\beta^*$  as a fixed point of (18) satisfies the KKT equations (26) and (27). This means  $\beta^*$  is a global minimum.

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