A Universal Grammar-Based Code For Lossless Compression of Binary Trees

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Abstract

We consider the problem of lossless compression of binary trees, with the aim of reducing the number of code bits needed to store or transmit such trees. A lossless grammar-based code is presented which encodes each binary tree into a binary codeword in two steps. In the first step, the tree is transformed into a context-free grammar from which the tree can be reconstructed. In the second step, the context-free grammar is encoded into a binary codeword. The decoder of the grammar-based code decodes the original tree from its codeword by reversing the two encoding steps. It is shown that the resulting grammar-based binary tree compression code is a universal code on a family of probabilistic binary tree source models satisfying certain weak restrictions.

Index Terms

grammar-based code, binary tree, lossless compression, context-free grammar, minimal DAG representation.

I. Introduction

There have been some recent initial attempts to conceptualize the notion of structure in information theory [12][3][16], with the ultimate future goal being the development of a lossless compression theory for structures. In the present paper, we put forth a general framework for this area, and then develop a lossless compression theory for binary tree structures within this framework. Our framework will permit an abstract asymptotic theory for the compression of structures to be developed, where the framework is sufficiently general to include the types of structures that have been considered in other contexts, such as in the asymptotic theory of networks [13] or the asymptotic theory of patterns [7]. The basic concepts in this framework are the notions of *structure universe*, *structure filter*, and *structure source*, which we now define; after the definitions, we give examples of the concepts relevant for the work we shall do in this paper.

Concept of Structure Universe. Broadly speaking, "structure universe" will mean the set of structures under consideration in a particular context. Each structure has a "size" assigned to it, which is a positive integer that can be a measure of how large or how complex the structure is. For example, if a structure is a finite graph g, then the size of the structure could be taken as the number of vertices of g or the number of edges of g; if a structure is a finite tree t, then the size of the structure could be taken as the number of leaves of t. We now make the notion of structure universe precise. A structure universe Ω is defined to be any countably infinite set such that for each $\omega \in \Omega$ there is defined a positive integer $|\omega|$, which we call the size of ω , such that the set $\omega \in \Omega$ is finite for each positive integer ω .

Concept of Structure Filter. A structure filter \mathcal{F} over a structure universe Ω (called Ω -filter for short) is defined to be any set of finite nonempty subsets of Ω which forms a partition of Ω . For example, given any structure universe Ω , we have the natural Ω -filter consisting of all nonempty subsets of Ω of the form $\{\omega \in \Omega : |\omega| = n\}$ $(n = 1, 2, \cdots)$. Given an Ω -filter \mathcal{F} , a real-valued function $(x_F : F \in \mathcal{F})$ defined on \mathcal{F} , and an extended real number L, the limit statement $\lim_{F \in \mathcal{F}} x_F = L$ means that for any neighborhood \mathcal{N} of L in the topology of the extended real line, the set $\{F \in \mathcal{F} : x_F \notin \mathcal{N}\}$ is finite; the limit L, if it exists, is unique, which is due to the fact that a structure filter is always countably infinite. Similarly, one can

make sense of limit statements of the form $\limsup_{F \in \mathcal{F}} x_F = L$ and $\liminf_{F \in \mathcal{F}} x_F = L$. The sets in any Ω -filter \mathcal{F} are growing in the sense that

$$\lim_{F \in \mathcal{F}} \left[\min\{ |\omega| : \omega \in F \} \right] = \infty. \tag{1.1}$$

This condition will make possible an asymptotic theory of lossless compression of structures; we will see how the condition is used in Sec. III.

Concept of Structure Source. Informally, suppose we randomly select a structure from each element of a structure filter; then these random structures constitute the output of a structure source. Formally, we define a structure source to be any triple (Ω, \mathcal{F}, P) in which Ω is a structure universe, \mathcal{F} is an Ω -filter, and P is a function from Ω into [0,1] such that

$$\sum_{\omega \in F} P(\omega) = 1, \ F \in \mathcal{F}. \tag{1.2}$$

Note that (1.2) simply tells us that P restricted to each $F \in \mathcal{F}$ yields a probability distribution on F; for any subset F' of F, we write the probability of F' under this distribution as P(F'), which is computed as the sum $\sum_{\omega \in F'} P(\omega)$.

Example 1. For each $n \geq 2$, fix an undirected graph g_n with n vertices and n(n-1)/2 edges, one edge for each pair of distinct vertices, and let G_n be the set of edge-labelings of g_n in which each edge of g_n is assigned a label from the set $\{0,1\}$. That is, G_n consists of all pairs (g_n,α) in which α is a mapping from the set of edges of g_n into the set $\{0,1\}$. Let G_n^* be a subset of G_n such that for each $(g_n,\alpha) \in G_n$, there exists a unique $(g_n,\alpha^*) \in G_n^*$ into which (g_n,α) is carried by an isomorphism (that is, there is an isomorphism of g_n onto itself which carries each edge e of g_n into an edge e' of g_n for which the edge labels $\alpha(e), \alpha^*(e')$ coincide). For example, G_3^* consists of four edge labelings of g_3 , one in which all three of the edges of g_3 are labeled 0, a second one in which all edge labels are 1, a third one in which two edge labels are 0 and the remaining one is 1, and a fourth one in which two edge labels are 1 and the remaining one is 0. Let Ω be the structure universe $\bigcup_{n\geq 2} G_n^*$, where we define the size of each labeled graph in Ω to be the number of vertices of the graph. Let \mathcal{F} be the Ω -filter $\{G_n^*: n \geq 2\}$. For each $\sigma \in (0,1)$, let $S_{\sigma} = (\Omega, \mathcal{F}, P_{\sigma})$ be the structure source such that for each $(g_n, \alpha') \in \Omega$,

$$P_{\mathbf{\sigma}}(g_n, \mathbf{\alpha}') = N(g_n, \mathbf{\alpha}') \mathbf{\sigma}^{m_1} (1 - \mathbf{\sigma})^{m_0},$$

where m_0 is the number of edges of g_n assigned α' -label 0, m_1 is the number of edges of g_n assigned α' -label 1, and $N(g_n,\alpha')$ is the number of (g_n,α) belonging to G_n for which $(g_n,\alpha^*)=(g_n,\alpha')$. For example, the P_σ probabilities assigned to the four structures in G_3^* given above are $\sigma^3,(1-\sigma)^3,3\sigma^2(1-\sigma)$, and $3\sigma(1-\sigma)^2$, respectively. In random graph theory, the structure source S_σ is called the Gilbert model [6]. Choi and Szpankowski [3] addressed the universal coding problem for the parametric family of sources $\{S_\sigma:0<\sigma<1\}$. (We discuss universal coding for general structure sources after the next two examples.)

Example 2. We consider finite rooted binary trees having at least two leaves such that each non-leaf vertex has exactly two ordered children. From now on, the terminology "binary tree" without further qualification will automatically mean such a tree. Let \mathcal{T} be a set of binary trees such that each binary tree is isomorphic as an ordered tree to a unique tree in \mathcal{T} . Then \mathcal{T} is a structure universe, where the size |t| of a tree t in the universe \mathcal{T} is taken to be

the number of leaves of t. We discuss two ways in which \mathcal{T} can be partitioned to obtain a \mathcal{T} -filter. For each $n \geq 2$, let \mathcal{T}_n be the set of trees in \mathcal{T} that have n leaves. For each $n \geq 1$, let \mathcal{T}^n be the set of trees in \mathcal{T} for which the longest root-to-leaf path consists of n edges (that is, \mathcal{T}^n consists of trees of depth n). Then $\mathcal{F}_1 = \{\mathcal{T}_n : n \geq 2\}$ and $\mathcal{F}_2 = \{\mathcal{T}^n : n \geq 1\}$ are each \mathcal{T} -filters. A structure source of the form $(\mathcal{T}, \mathcal{F}, P)$ for some \mathcal{T} -filter \mathcal{F} is called a binary tree source. In [12], binary tree sources of form $(\mathcal{T}, \mathcal{F}_1, P)$ were introduced which are called leaf-centric binary tree source models; we address the universal coding problem for such sources in Section IV of the present paper. In Section V, we address the universal coding problem for a type of binary tree source of form $(\mathcal{T}, \mathcal{F}_2, P)$ which we call a depth-centric binary tree source model.

Example 3. Let A be a finite alphabet. For each $n \ge 1$, let A^n be the set of all n-tuples of entries from A. Then $\Omega = \bigcup_{n=1}^{\infty} A^n$ is a structure universe, where we define the size of each structure in A^n to be n. Let \mathcal{F} be the Ω -filter $\{A^n : n \ge 1\}$. A structure source of the form (Ω, \mathcal{F}, P) corresponds to the classical notion of finite-alphabet information source ([8], page 14). Thus, source coding theory for structure sources will include classical finite-alphabet source coding theory as a special case.

Asymptotically Optimal Codes for Structure Sources. In the following and in the rest of the paper, \mathcal{B} denotes the set of non-empty finite-length binary strings, and L[b] denotes the length of string $b \in \mathcal{B}$. Let Ω be a structure universe. A lossless code on Ω is a pair (ψ_e, ψ_d) in which

- ψ_e (called the encoding map) is a one-to-one mapping of Ω into \mathcal{B} which obeys the prefix condition, that is, if ω_1 and ω_2 are two distinct structures in Ω , then $\psi_e(\omega_1)$ is not a prefix of $\psi_e(\omega_2)$; and
- ψ_d (called the decoding map) is the mapping from $\psi_e(\Omega)$ onto Ω which is the inverse of ψ_e .

Given a lossless code (ψ_e, ψ_d) on structure universe Ω and a structure source (Ω, \mathcal{F}, P) , then for each $F \in \mathcal{F}$ we define the real number

$$R(\psi_e, F, P) \stackrel{\Delta}{=} \sum_{\omega \in F, P(\omega) > 0} |\omega|^{-1} \{ L[\phi_e(\omega)] + \log_2 P(\omega) \} P(\omega),$$

which is called the F-th order average redundancy of the code (ψ_e, ψ_d) with respect to the source. We say that a lossless code (ψ_e, ψ_d) on Ω is an asymptotically optimal code for structure source (Ω, \mathcal{F}, P) if

$$\lim_{F \in \mathcal{F}} R(\psi_e, F, P) = 0. \tag{1.3}$$

Universal Codes for Structure Source Families. Let \mathcal{F} be a fixed Ω -filter for structure universe Ω . Let \mathbb{P} be a set of mappings from Ω into [0,1] such that (1.2) holds for every $P \in \mathbb{P}$. A universal code for the family of structure sources $\{(\Omega, \mathcal{F}, P) : P \in \mathbb{P}\}$ (if it exists) is a lossless code on Ω which is asymptotically optimal for every source in the family. The universal source coding problem for a family of structure sources is to determine whether the family has a universal code, and, if so, specify a particular universal code for the family.

There has been little previous work on universal coding of structure sources. One notable exception is the work of Choi and Szpankowski [3], who devised a universal code for the parametric family of Gilbert sources $\{S_{\sigma}: 0 < \sigma < 1\}$ introduced in Ex. 1. Peshkin [17]

and Busatto et al. [2] proposed grammar-based codes for compression of general graphical structures and binary tree structures, respectively; as these authors did not use a probabilistic structure source model, it is unclear whether their codes are universal in the sense of the present paper (instead, they tested performance of their codes on actual structures).

Context-Free Grammar Background. In the present paper, we further develop the idea behind the Busatto et al. code [2] to obtain a grammar-based code for binary trees which, under weak conditions, we prove to be a universal code for families of binary tree sources. In this Introduction, we describe the structure of our code in general terms; code implementation details will be given in Section II. In order to describe the grammar-based nature of our code, we need at this point to give some background information concerning deterministic context-free grammars. A deterministic context free grammar \mathbb{G} is a quadruple (S_1, S_2, s^*, P) in which

- S_1 is a finite nonempty set whose elements are called the nonterminal variables of \mathbb{G} .
- S_2 is a finite nonempty set whose elements are called the terminal variables of \mathbb{G} . ($S_1 \cup S_2$ is the complete set of variables of \mathbb{G} .)
- s^* is a designated nonterminal variable called the start variable of \mathbb{G} ;
- P is the finite set of production rules of production rules of \mathbb{G} . P has the same cardinality as S_1 . There is exactly one production rule for each nonterminal variable s, which takes the form

$$s \to (s_1, s_2, \cdots, s_n), \tag{1.4}$$

where n is a positive integer which can depend on the rule and s_1, s_2, \dots, s_n are variables of \mathbb{G} . s, (s_1, \dots, s_n) , and n are respectively called the left member, right member, and arity of the rule (1.4).

Given a deterministic context-free grammar \mathbb{G} , there is a unique up to isomorphism rooted ordered vertex-labeled tree $t(\mathbb{G})$ (which can be finite or infinite) satisfying the following properties:

- The label on the root vertex of $t(\mathbb{G})$ is the start variable of \mathbb{G} .
- The label on each non-leaf vertex of $t(\mathbb{G})$ is a nonterminal variable of \mathbb{G} .
- The label on each leaf vertex of $t(\mathbb{G})$ is a terminal variable of \mathbb{G} .
- Let s(v) be the variable of \mathbb{G} which is the label on each vertex v of $t(\mathbb{G})$. For each non-leaf vertex v of $t(\mathbb{G})$ and its ordered children v_1, v_2, \dots, v_n ,

$$s(v) \rightarrow (s(v_1), s(v_2), \cdots, s(v_n))$$

is a production rule of G.

"Unique up to isomorphism" means that for any two such rooted ordered trees there is an isomorphism between the trees as ordered trees that preserves the labeling (that is, corresponding vertices under the isomorphism have the same label). We call $t(\mathbb{G})$ the derivation tree of \mathbb{G} .

Outline of Binary Tree Compression Code. Let \mathcal{T} be the structure universe of binary trees introduced in Ex. 2. Suppose $t \in \mathcal{T}$ and suppose \mathbb{G} is a deterministic context-free grammar such that the arity of each production rule is two. Then we say that \mathbb{G} forms a representation of t if t is the unique tree in \mathcal{T} isomorphic as an ordered tree to the tree which results when all vertex labels on the derivation tree of \mathbb{G}_t are removed. In Section II, we will assign to each $t \in \mathcal{T}$ a particular deterministic context-free grammar \mathbb{G}_t which forms a representation of t.

Then we will assign to \mathbb{G}_t a binary codeword $B(\mathbb{G}_t)$ so that the prefix condition is satisfied. The grammar-based binary tree code of this paper is then the lossless code (ϕ_e, ϕ_d) on \mathcal{T} in which the encoding map ϕ_e and decoding map ϕ_d each operate in two steps as follows.

- Encoding Step 1: Given binary tree $t \in \mathcal{T}$, obtain the context-free grammar \mathbb{G}_t from t.
- Encoding Step 2: Assign to grammar \mathbb{G}_t the binary word $B(\mathbb{G}_t) \in \mathcal{B}$, and then $B(\mathbb{G}_t)$ is the codeword $\phi_e(t)$ for t.
- **Decoding Step 1:** The grammar \mathbb{G}_t is obtained from $B(\mathbb{G}_t)$, which is the inverse of the second encoding step.
- **Decoding Step 2:** \mathbb{G}_t is used to obtain the derivation tree of \mathbb{G}_t , from which t is obtained by removing all labels.

The two-step encoding/decoding maps ϕ_e and ϕ_d are depicted schematically in the following diagrams:

Encoding Map
$$\phi_e: t \in \mathcal{T} \stackrel{\text{1st step}}{\longrightarrow} \mathbb{G}_t \stackrel{\text{2nd step}}{\longrightarrow} B(\mathbb{G}_t) = \phi_e(t) \in \mathcal{B}$$

Decoding Map $\phi_d: B(\mathbb{G}_t) \stackrel{\text{1st step}}{\longrightarrow} \mathbb{G}_t \stackrel{\text{2nd step}}{\longrightarrow} t = \phi_d(B(\mathbb{G}_t))$

We point out the parallel between the grammar-based binary tree compression algorithm of this paper and the grammar-based lossless data compression methodology for data strings presented in [10]. In the grammar-based approach to compression of a data string x, one transforms x into a deterministic context-free grammar \mathbb{G}_x from which x is uniquely recoverable as the sequence of labels on the leaves of the derivation tree of \mathbb{G}_x ; one then compresses \mathbb{G}_x instead of x itself. Similarly, in the grammar-based approach to binary tree compression presented here, one transforms a binary tree t into the deterministic context-free grammar \mathbb{G}_t from which t is uniquely recoverable by stripping all labels from the derivation tree of \mathbb{G}_t ; one then compresses \mathbb{G}_t instead of t itself.

The rest of the paper is laid out as follows. In Sec. II, we present the implementation details of the grammar-based binary tree compression code (ϕ_e, ϕ_d) . In Sec. III, we present some weak conditions on a binary tree source under which (ϕ_e, ϕ_d) will be an asymptotically optimal code for the source. The remaining sections exploit these conditions to arrive at wide families of binary tree sources on which (ϕ_e, ϕ_d) is a universal code (families of leaf-centric models in Sec. IV and families of depth-centric models in Sec. V).

II. IMPLEMENTATION OF BINARY TREE COMPRESSION CODE

This section is organized as follows. In Section II-A, we give some background regarding binary trees that shall be used in the rest of the paper. Then, in Sec II-B, we explain how to transform each binary tree $t \in \mathcal{T}$ into the deterministic context-free grammar \mathbb{G}_t ; this is Step 1 of encoding map ϕ_e . In Section II-C, there follows an explanation on how the codeword $B(\mathbb{G}_t)$ is obtained from \mathbb{G}_t ; this is Step 2 of encoding map ϕ_e . Examples illustrating the workings of the encoding map ϕ_e and the decoding map ϕ_d are presented in Section II-D. Theorem 1 is then presented in Section II-E, which gives a performance bound for the code (ϕ_e, ϕ_d) . Finally, in Section II-F, we discuss a sense in which the grammar \mathbb{G}_t is minimal and unique among all grammars which form a representation of $t \in \mathcal{T}$.

A. Binary Tree Background

We take the direction along each edge of a binary tree to be away from the root. The root vertex of a binary tree is the unique vertex which is not the child of any other vertex, the leaf vertices are the vertices that have no child, and each of the non-leaf vertices has exactly two ordered children. We regard a tree consisting of just one vertex to be a binary tree, which we call a trivial binary tree; all other binary trees have at least two leaves and are called non-trivial. Given a binary tree t, V(t) shall denote the set of its vertices, and $V^{1}(t)$ shall denote the set of its non-leaf vertices. Each edge of t is an ordered pair (a,b) of vertices in V(t), where a is the vertex at which the edge begins and b is the vertex at which the edge ends (a is the parent of b and b is a child of a). A path in a binary tree is defined to be any sequence (v_1, v_2, \dots, v_k) of vertices of length $k \ge 2$ in which each vertex from v_2 onward is a child of the preceding vertex. For each vertex v of a binary tree which is not the root, there is a unique path which starts at the root and ends at v. We define the depth level of each non-root vertex v of a binary tree to be one less than the number of vertices in the unique path from root to v (this is the number of edges along the path); we define the depth level of the root to be zero. Vertex v_2 is said to be a descendant of vertex v_1 if there exists a (necessarily unique) path leading from v_1 to v_2 . If a binary tree has n leaf vertices, then it has n-1 non-leaf vertices and therefore 2(n-1) edges.

We have a locally defined order on each binary tree t in which each sibling pair of child vertices of t is ordered. From this locally defined order, one can infer various total orders on V(t) which are each consistent with the local orders on the sets of children. The most useful of the possible total orders for us will be the *breadth-first* order. If we list the vertices of a binary tree in breadth-first order, we first list the root vertex at depth level 0, then its two ordered children at depth level 1, then the vertices at depth level 2, depth level 3, etc. Two vertices v_1, v_2 at depth level j > 0 are consecutive in breadth-first order if and only if either (a) v_1, v_2 have the same parent and v_1 precedes v_2 in the local ordering of children, or (b) the parent of v_1 and the parent of v_2 are consecutive in the breadth-first ordering of the non-leaf vertices at depth level j - 1. It is sometimes convenient to represent a tree t pictorially via a "top down" picture, where the root vertex of t appears at the top of the picture (depth level 0) and edges extend downward in the picture to reach vertices of increasing depth level; the vertices at each depth level will appear horizontally in the picture with their left-right order corresponding to the breadth-first order. Fig. 1 depicts two binary trees with their vertices labeled in breadth-first order.

The structure universe \mathcal{T} consists only of nontrivial binary trees. Sometimes we need to consider a trivial binary tree consisting of just one vertex. Fix such a trivial tree t^* . Then $\mathcal{T}^* = \mathcal{T} \cup \{t^*\}$ can be taken as our structure universe of binary trees both trivial and nontrivial. For each $n \geq 1$, letting \mathcal{T}_n be the set of trees in \mathcal{T}^* having n leaves, and letting K_n be the cardinality of \mathcal{T}_n , it is well known [18] that $\{K_n : n \geq 1\}$ is the Catalan sequence, expressible by the formula

$$K_n = \frac{1}{n} {2(n-1) \choose n-1}, \ n \ge 1.$$

For example, using this formula, we have

$$K_1 = K_2 = 1$$
, $K_3 = 2$, $K_4 = 5$, $K_5 = 14$.

Fig. 1 depicts one of the $(1/8)\binom{14}{7} = 429$ binary trees in \mathcal{T}_8 , and one of the $(1/16)\binom{30}{15} = 9,694,845$ binary trees in \mathcal{T}_{16} .

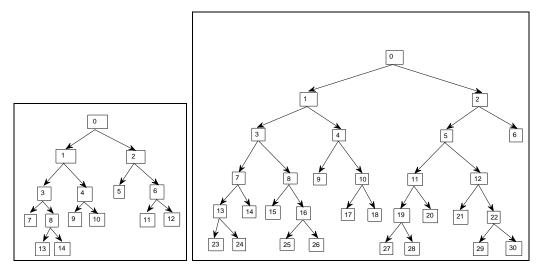


Fig. 1: Binary trees in \mathcal{T}_8 (left) and \mathcal{T}_{16} (right) with breadth-first ordered vertices

A subtree of a binary tree t is a tree whose edges and vertices are edges and vertices of t; by convention, we require also that a subtree of a binary tree should be a (nontrivial or trivial) binary tree. There are two special types of subtrees of a binary tree that shall be of interest to us, namely final subtrees and initial subtrees. Given a binary tree t, a final subtree of t is a subtree of t whose root is some fixed vertex of t and whose remaining vertices are all the descendants of this fixed vertex in t; an initial subtree of t is any subtree of t whose root coincides with the root of t. If t is any nontrivial binary tree and $v \in V(t)$, we define t(v) to be the unique binary tree in T^* which is isomorphic to the final subtree of t rooted at v. Note that $t(v) = t^*$ if v is a leaf of t, and that t(v) = t if $t \in T$ and v is the root of t. There are also two other trees of the t(v) type which appear often enough that we give them a special name; letting v_1, v_2 be the ordered children of the root of nontrivial binary tree t, we define $t_L = t(v_1)$ and $t_R = t(v_2)$ to reflect the respective left and right positions of these trees in the top down pictorial representation of tree t.

B. Encoding Step 1

Given $t \in T$, we explain how to transform t into the grammar \mathbb{G}_t , which is Step 1 of the encoding map ϕ_e . Define N = N(t) to be the cardinality of the set $\{t(v) : v \in V(t)\}$. Note that $N \geq 2$ since t^* and t are distinct and both belong to this set. The set of nonterminal variables of \mathbb{G}_t is the nonempty set of integers $\{0, 1, \dots, N-2\}$. The set of terminal variables of \mathbb{G}_t is the singleton set $\{T\}$, where we have denoted the unique terminal variable as the special symbol T. The start variable of \mathbb{G}_t is 0. All that remains to complete the definition of \mathbb{G}_t is to specify the production rules of \mathbb{G}_t . We do this indirectly by first labeling the vertices of t in a certain way and then extracting the production rules from the labeled tree. This labeling takes place as follows. The root of t is labeled 0 and each leaf of t is labeled t. The vertices of t

are traversed in breadth-first order. Whenever a vertex v is thus encountered which as yet has no label, one checks to see whether t(v) coincides with t(v') for some previously traversed vertex v'. If this is the case, v is assigned the same label as v'; otherwise, v is assigned label equal to the smallest member of the set $\{0,1,\cdots,N-2\}$ which has so far not been used as a label. For each nonterminal variable $i \in \{0,1,\cdots,N-2\}$, we can then extract from the labeled tree the unique production rule of \mathbb{G}_t of form $i \to (i_1,i_2)$ by finding any vertex of the labeled tree whose label is i; the entries i_1,i_2 are then the respective labels on the ordered children of this vertex. Incidentally, the labeled tree we employed in this construction turns out to be the derivation tree of \mathbb{G}_t .

Figures 2-3 illustrate the results of Encoding Step 1 for the binary trees in Fig. 1.

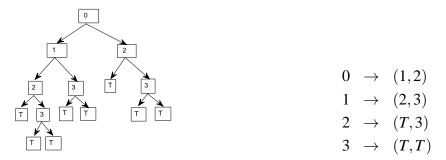


Fig. 2: Encoding Step 1 For Left Figure 1 tree

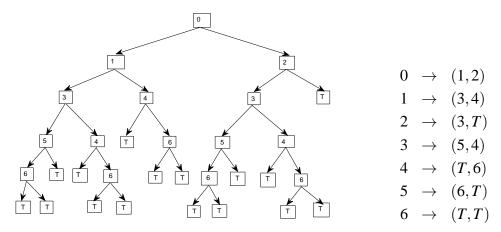


Fig. 3: Encoding Step 1 For Right Figure 1 tree

C. Encoding Step 2

Fix $t \in \mathcal{T}$. We now explain Step 2 of the encoding of t which is to obtain from the grammar \mathbb{G}_t a string $B(\mathbb{G}_t) \in \mathcal{B}$ which is taken as the codeword $\phi_e(t)$ of t. We will be employing two sequences S(t) and $S_1(t)$ defined as follows:

• Let N = N(t). For each $i = 0, \dots, N-2$, let ordered pair (a_{2i+1}, a_{2i+2}) be the right member of the production rule of \mathbb{G}_t whose left member is i. Then S(t) is the sequence of length 2N-2 defined by

$$S(t) \stackrel{\Delta}{=} (a_1, a_2, \cdots, a_{2N-3}, a_{2N-2}).$$

The alphabet of S(t) is $A(t) = \{1, 2, \dots, N-2\} \cup \{T\}$. Note that \mathbb{G}_t is fully recoverable from S(t).

• $S_1(t)$ is the sequence of length N remaining after one deletes from S(t) the first left-to-right appearance in S(t) of each member of the set $\{1, 2, \dots, N-2\}$.

Note that N = N(t) = 2 if and only if t is the unique tree in \mathcal{T}_2 ; in this case, \mathbb{G}_t has only one production rule $0 \to (T,T)$, and $S(t) = S_1(t) = (T,T)$. If N = 2, define $B(\mathbb{G}_t) = 1$. Now assume N > 2. The codeword $B(\mathbb{G}_t)$ will be obtained via processing of the sequence S(t). Note that S(t) partitions into the two subsequences $S_1(t)$ (defined previously) and $S_2(t) = (1,2,\cdots,N-2)$. For each $a \in A(t)$, define f_a to be the positive integer

$$f_a \stackrel{\Delta}{=} \operatorname{card} \{ 1 \le i \le 2N - 2 : a_i = a \},$$

that is, $(f_a : a \in A(t))$ is the un-normalized first-order empirical distribution of S(t). Let $S_1(t)$ be the set of all possible permutations of $S_1(t)$; the cardinality of $S_1(t)$ is then computable as

$$\operatorname{card}(S_1(t)) = \frac{N!}{f_T! \prod_{i=1}^{N-2} (f_i - 1)!}.$$

 $B(\mathbb{G}_t)$ is defined to be the left-to-right concatenation of the binary strings B_1, B_2, B_3, B_4 obtained as follows:

- B_1 is the binary string of length N-1 consisting of N-2 zeroes followed by 1.
- B_2 is the binary string of length 2N-2 in which there are exactly N-2 entries equal to 1, where these entries correspond to the first left-to-right appearances in S(t) of the members of the set $\{1, 2, \dots, N-2\}$. Given B_2 , one can reconstruct S(t) from its two subsequences $S_1(t)$ and $S_2(t)$.
- B_3 is the binary string consisting of N-1 alternate runs of ones and zeroes, where the lengths of the runs (left-to-right) are taken to be $f_1, f_2, \dots, f_{N-2}, 1$, respectively. Since $f_T > 1$, B_3 is of length less than 2N-2.
- Let M(t) = [log₂ card(S₁(t))]. If M(t) = 0, B₄ is the empty string. Otherwise, list all members of S₁(t) in the lexicographical ordering resulting from the ordering 1,···, N − 2, T of the alphabet A(t). Assign each member of the list an index, starting with index 0. Let I be the index of S₁(t) in this list. B₄ is the length M(t) binary expansion of integer I.

Verification of Prefix Condition. Suppose $t \in \mathcal{T}$ has been processed by the encoding map ϕ_e to yield codeword $\phi_e(t) = B(\mathbb{G}_t)$. Step 1 of the decoding map ϕ_d is to determine the grammar \mathbb{G}_t from $B(\mathbb{G}_t)$. More generally, we discuss here how S(t) and hence \mathbb{G}_t is recoverable from any binary word w of which codeword $B(\mathbb{G}_t) = B_1B_2B_3B_4$ is a prefix; this will establish that the encoding map $\phi_e : \mathcal{T} \to \mathcal{B}$ satisfies the prefix condition. Scanning w left-to-right to find the first 1, one determines B_1 and N = N(t). B_2 is then determined from the fact that its length is 2N - 2, and then B_3 is determined from the fact that it consists of N - 1 runs. Knowledge

of B_3 allows one to determine the set $S_1(t)$ and to compute M(t), the length of B_4 , whence B_4 can be extracted from w. From B_4 , one is able to locate $S_1(t)$ in the list of the members of $S_1(t)$. Using B_2 , one is able to put together S(t) from $S_1(t)$ and $S_2(t)$.

D. Encoding/Decoding Examples

We present two examples. Example 4 illustrates how the encoding map ϕ_e works, and Example 5 illustrates how the decoding map ϕ_d works.

Example 4. Let t be the tree on the right in Fig. 1. Fig. 3 illustrates the results of Step 1 of encoding map ϕ_e . We then obtain

$$N = N(t) = 8,$$

$$S(t) = (1,2,3,4,3,T,5,4,T,6,6,T,T,T),$$

$$S_1(t) = (3,T,4,T,6,T,T,T),$$

$$S_2(t) = (1,2,3,4,5,6),$$

$$f_1 = f_2 = f_5 = 1, f_3 = f_4 = f_6 = 2, f_T = 5,$$

$$B_1 = 0000001,$$

$$B_2 = 11110010010000,$$

$$B_3 = 1011001001.$$

We now list the 8!/5! = 336 members of $S_1(t)$ in lexicographical order until $S_1(t)$ is obtained:

index	sequence	index	sequence
0	(3,4,6,T,T,T,T,T)	7	(3,6,T,4,T,T,T,T)
1	(3,4,T,6,T,T,T,T)	8	(3,6,T,T,4,T,T,T)
2	(3,4,T,T,6,T,T,T)	9	(3,6,T,T,T,4,T,T)
3	(3,4,T,T,T,6,T,T)	10	(3,6,T,T,T,T,4,T)
4	(3,4,T,T,T,T,6.T)	11	(3,6,T,T,T,T,T,4)
5	(3,4,T,T,T,T,T,6)	12	(3, T, 4, 6, T, T, T, T)
6	(3,6,4,T,T,T,T,T)	13	(3, T, 4, T, 6, T, T, T)

The index of $S_1(t)$ is thus I = 13. (Alternatively, one can use the method of Cover [4] to compute I directly without forming the above list.) To obtain B_4 , we expand the index I = 13 into its $\lceil \log_2 336 \rceil = 9$ bit binary expansion, which yields

$$B_4 = 000001101.$$

The codeword $\phi_e(t) = B_1B_2B_3B_4$ is of length 7 + 14 + 10 + 9 = 40. Example 5. Let binary tree $t \in \mathcal{T}$ be such that

$$\phi_e(t) = B(\mathbb{G}_t) = 00011101000010011000001.$$

We employ the decoding map ϕ_d to find t from $B(\mathbb{G}_t)$. In Decoding Step 1, the grammar \mathbb{G}_t must be determined, which, as remarked earlier, is equivalent to finding the sequence

S(t). $B(\mathbb{G}_t) = B_1B_2B_3B_4$ must be parsed its constituent parts B_1, B_2, B_3, B_4 . B_1 is the unique prefix of $B(\mathbb{G}_t)$ belonging to the set $\{1,01,001,001,0001,\cdots\}$, whence $B_1 = 0001$, and hence N = N(t) = 4 + 1 = 5. Thus, S(t) and B_2 are both of length 2N - 2 = 8, whence

$$B_2 = 11010000$$

and S(t) is of the form

$$S(t) = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8).$$

The positions of symbol 1 in B_2 tell us that

$$S_2(t) = (a_1, a_2, a_4) = (1, 2, 3),$$

and therefore $S_1(t)$ is made up of the remaining entries in S(t), giving us

$$S_1(t) = (a_3, a_5, a_6, a_7, a_8).$$

Since B_3 consists of N-1=4 runs of ones and zeroes, with the last run of length 1, we must have

$$B_3 = 100110.$$

The alphabet of S(t) is $\{1, 2, \dots, N-2, T\} = \{1, 2, 3, T\}$, and so from B_3 the frequencies of 1,2,3 in S(t) are the lengths of the first three runs in B_3 , respectively, whence

$$f_1 = 1$$
, $f_2 = 2$, $f_3 = 2$.

The remaining entries of S(t) are all equal to T, giving us $f_T = 8 - (1+2+2) = 3$. It follows that $S_1(t)$ consists of $f_1 - 1 = 0$ entries equal to 1, $f_2 - 1 = 1$ entry equal to 2, $f_3 - 1 = 1$ entry equal to 3, and $f_T = 3$ entries equal to T. Consequently, $S_1(t)$ is the set of all permutations of (2,3,T,T,T). The cardinality of this set is 5!/3! = 20, and so B_4 is of length $\lceil \log_2 20 \rceil = 5$. This checks with what is left of $B(\mathbb{G}_t) = B_1B_2B_3B_4$ after B_1, B_2, B_3 are removed, namely

$$B_4 = 00001$$
.

The index of $S_1(t)$ in the list of the members of $S_1(t)$ is thus I = 1. This list starts with (2,3,T,T,T), which has index 0, and the sequence following this must therefore by $S_1(t)$. We conclude that

$$S_1(t) = (2, T, 3, T, T).$$

 $S_1(t)$ and $S_2(t)$ now both being known, we put them together to obtain

$$S(t) = (1, 2, 2, 3, T, 3, T, T).$$

Partitioning S(t) into blocks of length two, we obtain the four production rules of \mathbb{G}_t in Fig. 3, whereupon \mathbb{G}_t is determined, completing Decoding Step 1. In Decoding Step 2, one grows the derivation tree of \mathbb{G}_t from the production rules of \mathbb{G}_t as explained in the Introduction, giving us the derivation tree in Fig. 3; stripping the labels from this tree, we obtain the binary tree t on the left in Fig. 1, completing Decoding Step 2.

E. Performance Bound

We present Theorem 1, which gives us an upper bound on the lengths of the binary codewords assigned by the encoding map ϕ_e which shall be useful in later sections. Theorem 1 uses the notion of the first order empirical probability distribution of a sequence (s_1, s_2, \dots, s_n) whose entries are selected from a finite alphabet A, which is the probability distribution $p = (p_a : a \in A)$ defined by

$$p_a \stackrel{\Delta}{=} n^{-1} \operatorname{card} \{ 1 \le i \le n : s_i = a \}, \ a \in A.$$

The Shannon entropy H(p) of this first order empirical distribution p is defined as

$$H(p) \stackrel{\Delta}{=} \sum_{a \in A} -p_a \log_2 p_a,$$

which is also expressible as

$$H(p) = n^{-1} \sum_{i=1}^{n} -\log_2 p_{s_i}.$$

Theorem 1. Let t be any binary tree in \mathcal{T} . Let p_t be the first order empirical probability distribution of the sequence $S_1(t)$. Then

$$L[\phi_e(t)] \le 5(N(t) - 1) + N(t)H(p_t). \tag{2.5}$$

Proof. Let N = N(t). We have $N \ge 2$. If N = 2, then t is the unique tree in \mathcal{T}_2 and $L[\phi_e(t)] = 1$, whence (2.5) holds because the right side is 5. Assume N > 2. Recall that $\mathcal{S}_1(t)$ is the set of all permutations of $S_1(t)$. From the relationships

$$L[\phi_e(t)] = \sum_{i=1}^4 L[B_i] = 3(N-1) + L[B_3] + \lceil \log_2(\operatorname{card}(\mathcal{S}_1(t))) \rceil,$$

$$L[B_3] \le 2N - 3,$$

$$\lceil \log_2(\operatorname{card}(\mathcal{S}_1(t))) \rceil \le \log_2(\operatorname{card}(\mathcal{S}_1(t))) + 1,$$

we obtain

$$L[\phi_e(t)] \le 5(N-1) + \log_2(\operatorname{card}(\mathcal{S}_1(t))).$$

Since $S_1(t)$ is a type class of sequences of length N in the sense of Chapter 2 of [5], Lemma 2.3 of [5] tells us that

$$\log_2(\operatorname{card}(S_1(t))) \leq NH(p_t).$$

Inequality (2.5) is now evident.

F. Minimality/Uniqueness of \mathbb{G}_t

Given $t \in \mathcal{T}$, we discuss what distinguishes \mathbb{G}_t among the possibly many deterministic context-free grammars which form a representation of t. First, we explain what it means for a directed acyclic graph (DAG) to be a representation of t. Let D be a finite rooted DAG with at least two vertices such that each non-leaf vertex has exactly two ordered edges. Define $\mathbb{G}(D)$ to be the deterministic context-free grammar whose set of nonterminal variables is the set of non-leaf vertices of D, whose set of terminal variables is the set of leaf vertices of D, whose start variable is the root vertex of D, and whose production rules are all the rules of the form $v \to (v_1, v_2)$ in which v is a non-leaf vertex of D, and v_1, v_2 are the respective vertices of D at the terminus of the edges 1,2 emanating from v. Then we say that D is a representation of $t \in \mathcal{T}$ if the grammar $\mathbb{G}(D)$ forms a representation of t. It is known that each binary tree in \mathcal{T} has a unique DAG representation up to isomorphism with the minimal number of vertices [14]; we call this DAG the minimal DAG representation of the binary tree. One particular choice of minimal DAG representation of $t \in T$ is the DAG $D^*(t)$ defined as follows. The set of vertices of $D^*(t)$ is $\{t(v): v \in V(t)\}$. The root vertex of $D^*(t)$ is t, and t^* is the unique leaf vertex of $D^*(t)$. If u is a non-leaf vertex of $D^*(t)$, then there are exactly two ordered edges emanating from u, edge 1 terminating at u_L and edge 2 terminating at u_R . Note that the number of vertices of the minimal DAG representation $D^*(t)$ of t is N(t), which coincides with the number of variables of \mathbb{G}_t . (Recall that the complete set of variables of \mathbb{G}_t is $\{0,1,\cdots,N(t)-2\}\cup\{T\}$, of cardinality N(t).) The paper [2] gives a linear-time algorithm for computing $D^*(t)$. Fig. 4 illustrates a binary tree together with its minimal DAG representation.

Lemma 1. Let $t \in \mathcal{T}$. Then \mathbb{G}_t has the smallest number of variables among all deterministic context-free grammars which form a representation of t.

Proof. Let \mathbb{G} be a deterministic context-free grammar which forms a representation of t. The proof consists in showing that the number of variables of \mathbb{G} is at least N(t), the number of variables of \mathbb{G}_t . In the following, we explain how to extract from the derivation tree $t(\mathbb{G})$ of \mathbb{G} a rooted ordered DAG D(t) which is a representation of t. The set of vertices of D(t)is the set of labels on the vertices of $t(\mathbb{G})$. The root vertex of D(t) is the label on the root vertex of $t(\mathbb{G})$, the set of non-leaf vertices of D(t) is the set of labels on the non-leaf vertices of $t(\mathbb{G})$, and the set of leaf vertices of D(t) is the set of labels on the leaf vertices of $t(\mathbb{G})$. Let s be any non-leaf vertex of D(t). Find a vertex v of $t(\mathbb{G})$ whose label is s, and let s_1, s_2 be the respective labels on the ordered children of v in $t(\mathbb{G})$; the pair (s_1, s_2) thus derived will be the same no matter which vertex v of $t(\mathbb{G})$ with label s is chosen. There are exactly two ordered edges of D(t) emanating from s, namely, edge 1 which terminates at s_1 and edge 2 which terminates at s_2 . This completes the specification of the DAG D(t). By construction of D(t), the number of variables of \mathbb{G} is at least as much as the number of vertices of D(t). Since D(t) is a DAG representation of t, the number of vertices of D(t) is at least as much as the number of vertices N(t) of the minimal DAG representation of t. Thus, the number of variables of \mathbb{G} is at least N(t), completing the proof.

Remark. With some more work, one can show that any deterministic context-free grammar which forms a representation of $t \in \mathcal{T}$ and has the same number of variables as \mathbb{G}_t must be isomorphic to \mathbb{G}_t , using the known fact mentioned earlier that the minimal DAG representation of t is unique up to isomorphism. This gives us a sense in which \mathbb{G}_t is unique.

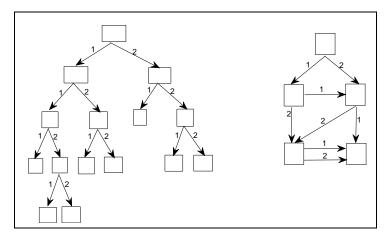


Fig. 4: A binary tree (left) and its minimal DAG representation (right)

III. Sources For Which (ϕ_e, ϕ_d) is Asymptotically Optimal

This section examines the asymptotic performance of the code (ϕ_e, ϕ_d) on a binary tree source. We put forth weak sufficient conditions on a binary tree source so that our two-step grammar-based code (ϕ_e, ϕ_d) will be an asymptotically optimal code for the source. Before doing that, we need to first establish a lemma giving an asymptotic average redundancy lower bound for general structure sources.

Suppose (Ω, \mathcal{F}, P) be an arbitrary structure source. Let (ψ_e, ψ_d) be a lossless code on Ω , and let $F \in \mathcal{F}$ be such that every structure $\omega \in F$ is of the same size. The well-known entropy lower bound for prefix codes tells us that

$$\sum_{\omega \in F} L[\psi_e(\omega)] P(\omega) \geq \sum_{\omega \in F, \, P(\omega) > 0} -P(\omega) \log_2 P(\omega),$$

from which it follows that

$$R(\psi_{e}, F, P) > 0$$

that is, the F-th order average redundancy of the code with respect to the source is non-negative. Although this redundancy non-negativity property fails for a general structure source, the following result gives us an asymptotic sense in which average redundancy is non-negative.

Lemma 2. Let (Ω, \mathcal{F}, P) be a general structure source. Then

$$\liminf_{F \in \mathcal{F}} R(\psi_e, F, P) \ge 0$$
(3.6)

for any lossless code (ψ_e, ψ_d) on Ω .

Proof. Fix a general structure source (Ω, \mathcal{F}, P) . Let \mathbb{Q} be the set of all $Q: \Omega \to (0,1)$ such that the restriction of Q to each $F \in \mathcal{F}$ is a probability distribution on F. In the first part of the proof, we show that

$$\liminf_{F \in \mathcal{F}} \sum_{\omega \in F} |\omega|^{-1} P(\omega) \log_2 \left(\frac{P(\omega)}{Q(\omega)} \right) \ge 0, \quad Q \in \mathbb{Q}, \tag{3.7}$$

where in (3.7) and henceforth, any expected value of the form $\sum_{\omega \in F} g(\omega) P(\omega)$ is computed by summing only over those $\omega \in F$ for which $P(\omega) > 0$. The proof of (3.7) exploits the concept of divergence. If $p = (p_j : j \in A)$ and $q = (q_j : j \in A)$ are any two probability distributions on a finite set A, with all q_j probabilities > 0, we let D(p|q) denote the divergence of p with respect to q, defined by

$$D(p|q) \stackrel{\Delta}{=} \sum_{j \in A} p_j \log_2 \left(\frac{p_j}{q_j}\right).$$

It is well-known that $D(p|q) \ge 0$ [5]. Fix an arbitrary $Q \in \mathbb{Q}$. Given $F \in \mathcal{F}$, let $I_F = \{|\omega| : \omega \in F\}$, and for each $i \in I_F$, let $F_i = \{\omega \in F : |\omega| = i\}$. Furthermore, let P_F, Q_F be the probability distributions on I_F such that

$$P_F(i) = P(F_i), i \in I_F,$$

$$Q_F(i) = Q(F_i), i \in I_F,$$

and for each $i \in I_F$, let P_F^i, Q_F^i be probability distributions on F_i such that

$$P(\omega) = P_F(i)P_F^i(\omega), \ \omega \in F_i,$$

$$Q(\omega) = Q_F(i)Q_F^i(\omega), \ \omega \in F_i.$$

It is easy to show that

$$\sum_{\omega \in F} |\omega|^{-1} P(\omega) \log_2 \left(\frac{P(\omega)}{Q(\omega)} \right) = \sum_{i \in I_F} i^{-1} P_F(i) D(P_F^i | Q_F^i) + \sum_{i \in I_F} i^{-1} P_F(i) \log_2 \left(\frac{P_F(i)}{Q_F(i)} \right),$$

and therefore

$$\sum_{\omega \in F} |\omega|^{-1} P(\omega) \log_2 \left(\frac{P(\omega)}{Q(\omega)} \right) \ge \sum_{i \in I_F} i^{-1} P_F(i) \log_2 \left(\frac{P_F(i)}{Q_F(i)} \right).$$

Let E_{P}^{F}, E_{O}^{F} be the expected values defined by

$$E_P^F \stackrel{\Delta}{=} \sum_{i \in I_F} i^{-1} P_F(i),$$

$$E_Q^F \stackrel{\Delta}{=} \sum_{i \in I_F} i^{-1} Q_F(i).$$

Note that E_P^F and E_Q^F both belong to the interval (0,1]. Let P_F^*, Q_F^* be the probability distributions on I_F defined by

$$P_F^*(i) \stackrel{\Delta}{=} i^{-1} P_F(i) / E_P^F, i \in I_F,$$

$$Q_F^*(i) \stackrel{\Delta}{=} i^{-1} Q_F(i) / E_O^F, \ i \in I_F.$$

Then we have

$$\sum_{i \in I_F} i^{-1} P_F(i) \log_2 \left(\frac{P_F(i)}{Q_F(i)} \right) = E_P^F D(P_F^* | Q_F^*) + E_P^F \log_2 (1/E_Q^F) + E_P^F \log_2 E_P^F.$$

Since $1/E_Q^F \ge 1$, the first two terms on the right side of the preceding equality are non-negative, whence

$$\liminf_{F \in \mathcal{F}} \sum_{\omega \in F} |\omega|^{-1} P(\omega) \log_2 \left(\frac{P(\omega)}{Q(\omega)} \right) \ge \liminf_{F \in \mathcal{F}} E_P^F \log_2 E_P^F. \tag{3.8}$$

Note that

$$0 < E_P^F \le \frac{1}{\min\{|\omega| : \omega \in F\}},$$

and so by (1.2)

$$\lim_{F \in \mathcal{F}} E_P^F = 0,\tag{3.9}$$

the right side of (3.8) is zero, and (3.7) holds. To finish the proof, let (ψ_e, ψ_d) be any lossless code on Ω . By Kraft's inequality for prefix codes, there exists $Q \in \mathbb{Q}$ such that

$$L[\psi_e(\omega)] \ge -\log_2 Q(\omega), \ \ \omega \in \Omega,$$

and hence

$$R(\psi_e, F, P) = \sum_{\omega \in F} |\omega|^{-1} \{ L[\phi_e(\omega)] + \log_2 P(\omega) \} P(\omega) \ge \sum_{\omega \in F} |\omega|^{-1} P(\omega) \log_2 \left(\frac{P(\omega)}{Q(\omega)} \right).$$

(3.6) then follows by appealing to (3.7).

Remark. In view of Lemma 2, given a general structure source (Ω, \mathcal{F}, P) , a lossless code (ψ_e, ψ_d) on Ω is an asymptotically optimal code for the source if and only if

$$\limsup_{F \in \mathcal{F}} R(\psi_e, F, P) \le 0. \tag{3.10}$$

We now turn our attention to properties of a binary tree source under which the grammar-based code (ϕ_e, ϕ_d) on \mathcal{T} will be asymptotically optimal for the source. There are two of these properties, the Domination Property and the Representation Ratio Negligibility Property, which are discussed in the following.

Domination Property. We define Λ to be the set of all mappings $\lambda: \mathcal{T}^* \to (0,1]$ such that

- (a): $\lambda(t) \leq \lambda(t_L)\lambda(t_R), t \in \mathcal{T}.$
- (b): There exists a positive integer $K(\lambda)$ such that

$$1 \le \sum_{t \in \mathcal{T}_n} \lambda(t) \le n^{K(\lambda)}, \ n \ge 1.$$
 (3.11)

An element λ of Λ dominates a binary tree source $(\mathcal{T}, \mathcal{F}, P)$ if $P(t) \leq \lambda(t)$ for all $t \in \mathcal{T}$. A binary tree source satisfies the Domination Property if there exists an element of Λ which dominates the source.

Representation Ratio Negligibility Property. Let $t \in \mathcal{T}$. We define the representation ratio of t, denoted r(t), to be the ratio between the number of variables of the grammar \mathbb{G}_t and the number of leaves of t. That is, r(t) = N(t)/|t|. Since

$$N(t) = \operatorname{card}\{t(v) : v \in V(t)\} = 1 + \operatorname{card}\{t(v) : v \in V^{1}(t)\} \le 1 + (|t| - 1) = |t|,$$

the representation ratio is at most 1. In the main result of this section, Theorem 2, we will see that our ability to compress $t \in \mathcal{T}$ via the code (ϕ_e, ϕ_d) becomes greater as r(t)

becomes smaller. We say that a binary tree source $(\mathcal{T}, \mathcal{F}, P)$ obeys the Representation Ratio Negligibility Property (RRN Property) if

$$\lim_{F \in \mathcal{F}} \sum_{t \in F} r(t)P(t) = 0. \tag{3.12}$$

Definition. Henceforth, $\gamma:[0,1]\to[0,\infty)$ is the function defined by

$$\gamma(x) \stackrel{\Delta}{=} \left\{ \begin{array}{cc} -(x/2)\log_2(x/2), & x > 0\\ 0, & x = 0 \end{array} \right.$$

Theorem 2. The following statements hold:

(a): For each $\lambda \in \Lambda$,

$$|t|^{-1} \{ L[\phi_e(t)] + \log_2 \lambda(t) \} \le (2K(\lambda) + 10)\gamma(r(t)), \ t \in \mathcal{T}.$$
 (3.13)

(b): Let $(\mathcal{T}, \mathcal{F}, P)$ be a binary tree source satisfying the Domination Property, where \mathcal{F} can be any \mathcal{T} -filter. There exists a positive real number C, depending only on the source, such that

$$R(\phi_e, F, P) \le C\gamma\left(\sum_{t \in F} r(t)P(t)\right), \ F \in \mathcal{F}.$$
 (3.14)

(c): (ϕ_e, ϕ_d) is an asymptotically optimal code for any binary tree source which satisfies both the Domination Property and the RRN Property.

Proof. It suffices to prove part (a). (Part (b) follows from part (a) and the fact that γ is a concave function; part(c) follows from part(b) and (3.10).) Let $\lambda \in \Lambda$ be arbitrary. Fix $t \in \mathcal{T}$ and let N = N(t). There is an initial binary subtree t^{\dagger} of t such that

- There are N leaf vertices of t^{\dagger} .
- The subtrees t(v) are distinct as v ranges through the N-1 non-leaf vertices of t^{\dagger} .

(One can obtain t^{\dagger} either by pruning the derivation tree of \mathbb{G}_t or by growing it using the production rules of \mathbb{G}_t so that in the growth process each production rule is used to extend a leaf exactly once; see Fig. 5.) Let v_1, v_2, \dots, v_N be an enumeration of the leaves of t^{\dagger} . There is a one-to-one correspondence between the set $\{t(v): v \in V(t)\}$ and the set of variables of \mathbb{G}_t , and under this correspondence, the sequence $s^* = (t(v_1), t(v_2), \dots, t(v_N))$ is carried into a sequence which is a permutation of the sequence $S_1(t)$, and the first order empirical distribution p^* of s^* is carried into the first order empirical distribution p_t of $S_1(t)$. Thus, the Shannon entropies $H(p^*)$, $H(p_t)$ coincide, and appealing to Theorem 1, we have

$$L[\phi_e(t)] \le 5(N-1) + \sum_{i=1}^{N} -\log_2 p^*(t(v_i)).$$

Define

$$M_j \stackrel{\Delta}{=} \sum_{u \in \mathcal{T}_j} \lambda(u), \ j \ge 1.$$

There is a unique real number D > 1/2 such that

$$q(u) \stackrel{\Delta}{=} DM_j^{-1} |u|^{-2} \lambda(u), \quad u \in \mathcal{T}_j, \quad j \ge 1$$
(3.15)

defines a probability distribution on \mathcal{T}^* . Shannon's Inequality ([1], page 37) then gives us

$$\sum_{i=1}^{N} -\log_2 p^*(t(v_i)) \leq \sum_{i=1}^{N} -\log_2 q(t(v_i)).$$

Using formula (3.15) and the fact that $-\log_2 D \le 1$, we obtain

$$\sum_{i=1}^{N} -\log_2 q(t(v_i)) = N(-\log_2 D) + Q_1 + 2Q_2 + Q_3$$

$$\leq N + Q_1 + 2Q_2 + Q_3,$$

where

$$Q_1 = \sum_{i=1}^{N} \log_2 M_{|t(v_i)|},$$
 $Q_2 = \sum_{i=1}^{N} \log_2 |t(v_i)|,$
 $Q_3 = -\sum_{i=1}^{N} \log_2 \lambda(t(v_i)).$

We bound each of these quantities in turn. By (3.11), we obtain

$$Q_1 \leq K(\lambda)Q_2$$
.

By concavity of the logarithm function, and recalling that r(t) = N/|t|, we have

$$Q_2 \leq N \log_2 \left(\frac{\sum_{i=1}^N |t(v_i)|}{N} \right) = N \log_2(|t|/N) = 2|t|\gamma(r(t)) - N.$$

By property (a) for membership of λ in Λ , we have

$$Q_3 \leq -\log_2 \lambda(t)$$
.

Combining previous bounds, and writing $K = K(\lambda)$, we see that

$$L[\phi_e(t)] + \log_2 \lambda(t) \le 6N - (K+2)N + 2(K+2)|t|\gamma(r(t))$$

 $< 3|t|r(t) + 2(K+2)|t|\gamma(r(t))$

holds, whence (3.13) holds because $r(t) \leq 2\gamma(r(t))$, completing the proof of part (a) of Theorem 2.

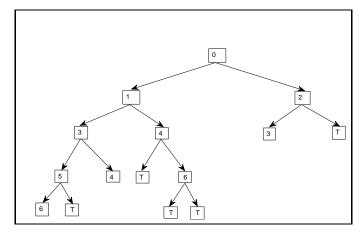


Fig. 5: Initial subtree of Fig. 3 derivation tree used in Theorem 2 proof

IV. Universal Coding of Leaf-Centric Binary Tree Sources

We fix throughout this section the \mathcal{T} -filter $\mathcal{F}_1 = \{\mathcal{T}_n : n \geq 2\}$. We now formally define the set of leaf-centric binary tree sources, which are certain binary tree sources of the form $(\mathcal{T}, \mathcal{F}_1, P)$. Let \mathbb{N} be the set of positive integers, and let Σ_1 be the set of all functions σ from $\mathbb{N} \times \mathbb{N}$ into [0,1] such that

$$\sum_{\{(i,j): i,j \ge 1, \ i+j=n\}} \sigma(i,j) = 1, \ \ n \ge 2.$$

For each $\sigma \in \Sigma_1$, let P_{σ} be the mapping from T into [0,1] such that

$$P_{\sigma}(t) = \prod_{v \in V^{1}(t)} \sigma(|t(v)_{L}|, |t(v)_{R}|), \ t \in \mathcal{T}.$$

Since

$$\sum_{t\in\mathcal{T}_n}P_{\sigma}(t)=1,\ n\geq 2,$$

 $S(\sigma) = (\mathcal{T}, \mathcal{F}_1, P_{\sigma})$ is a binary tree source. The sources in the family $\{S(\sigma) : \sigma \in \Sigma_1\}$ are called leaf-centric binary tree sources, the reason being that the probability of each tree is computed based purely upon the number of leaves in each of its final subtrees. Leaf-centric binary tree sources were first considered in the paper [12].

Example 6. Let Σ_1^{\dagger} be the subset of Σ_1 consisting of all $\sigma \in \Sigma_1$ for which

$$\{(i,j): i,j \geq 1, \ i+j=n, \ \sigma(i,j)>0\} \subset \{(1,n-1),(n-1,1)\}, \ n \geq 2.$$

If $\sigma \in \Sigma_1^{\dagger}$, then a tree $t \in \mathcal{T}$ with positive P_{σ} probability must satisfy the property that there exist only two vertices of t at each depth level of t beyond level 0; we call such a binary tree a one-dimensional tree. Consider the structure universe of binary strings \mathcal{B} , in which the size of a string $b \in \mathcal{B}$ is taken to be its length L[b]. For each $n \geq 1$, let \mathcal{B}_n be the set of strings in \mathcal{B} of length n, and let $\mathcal{F}(\mathcal{B})$ be the \mathcal{B} -filter $\{\mathcal{B}_n : n \geq 1\}$. Let $[0,1]^{\infty}$ be the set of all sequences $\alpha = (\alpha_i : i \geq 1)$ in which each α_i belongs to the interval [0,1], and for each $\alpha \in [0,1]^{\infty}$, let

 $(\mathcal{B}, \mathcal{F}(\mathcal{B}), Q_{\alpha})$ be the one-dimensional source in which for each string $b_1b_2\cdots b_n$ belonging to \mathcal{B} we have

$$Q_{\alpha}(b_1b_2\cdots b_n)=\prod_{i=1}^n q(\alpha_i,b_i),$$

where $q(\alpha_i, b_i)$ is taken to α_i if $b_i = 0$ and taken to be $1 - \alpha_i$, otherwise. It is easy to see that the family of sources $\{(\mathcal{T}, \mathcal{F}_1, P_\sigma) : \sigma \in \Sigma_1^\dagger\}$ has a universal code if and only if the family of one-dimensional sources $\{(\mathcal{B}, \mathcal{F}(\mathcal{B}), Q_\alpha) : \alpha \in [0, 1]^\infty\}$ has a universal code. The third author has shown that this latter family of one-dimensional sources has no universal code. Therefore, the family $\{S(\sigma) : \sigma \in \Sigma_1^\dagger\}$ has no universal code, and so the bigger family of all leaf-centric binary tree sources also has no universal code.

The following result shows that (ϕ_e, ϕ_d) is a universal code for a suitably restricted subfamily of the family of leaf-centric binary tree sources.

Theorem 3. Let Σ_1^* be the uncountable set consisting of all $\sigma \in \Sigma_1$ such that

$$\sup\left\{\frac{i+j}{\min(i,j)}: i,j \ge 1, \ \sigma(i,j) > 0\right\} < \infty. \tag{4.16}$$

Then (ϕ_e, ϕ_d) is a universal code for the family of sources $\{S(\sigma) : \sigma \in \Sigma_1^*\}$.

Before proceeding with the proof of Theorem 3, we provide an example of a source in $\{S(\sigma) : \sigma \in \Sigma_1^*\}$.

Example 7. Given a general structure source (Ω, \mathcal{F}, P) , then for each $F \in \mathcal{F}$, the F-th order entropy of the source is defined by

$$H_F(P) \stackrel{\Delta}{=} \sum_{\omega \in F} -|\omega|^{-1} P(\omega) \log_2 P(\omega).$$

 $\lim_{F \in \mathcal{F}} H_F(P)$ is defined to be the entropy rate of the source, if the limit exists; otherwise, the source has no entropy rate. In universal source coding theory for families of classical one-dimensional sources (see Ex. 3), the sources are typically assumed to be stationary sources or finite-state sources, which are types of sources which have an entropy rate. In the universal coding of binary tree sources, however, one very often deals with sources which have no entropy rate. We illustrate a particular source of this type in the family $\{S(\sigma) : \sigma \in \Sigma_1^*\}$. Let $\sigma \in \Sigma_1^*$ be the function such that for each even $n \geq 2$,

$$\sigma(n/2, n/2) = 1,$$

and for each odd $n \geq 3$,

$$\sigma(\lfloor n/2 \rfloor, \lceil n/2 \rceil) = \sigma(\lceil n/2 \rceil, \lfloor n/2 \rfloor) = 1/2.$$

The resulting leaf-centric binary tree source $S(\sigma)$, introduced in [12], is called the *bisection tree source model*. In [9], it is shown that there is a unique nonconstant continuous periodic function $f: \mathbb{R} \to [0,1]$, with period 1, such that

$$-\log_2 P_{\sigma}(t) = |t| f(\log_2 |t|), \ t \in \mathcal{T}, \tag{4.17}$$

and the restriction of f to [0,1] is characterized as the attractor of a specific iterated function system on [0,1]; because of this property, the source $S(\sigma)$ has no entropy rate.

Proof of Theorem 3. If $\sigma \in \Sigma_1$, let $\lambda : \mathcal{T}^* \to [0,1]$ be the function such that $\lambda(t^*) = 1$ and

$$\lambda(t) = \max(K_n^{-1}, P_{\sigma}(t)), \ t \in \mathcal{T}_n, \ n \ge 2.$$

Then $\lambda \in \Lambda$ and λ dominates P_{σ} . Thus, every source in the family $\{S(\sigma) : \sigma \in \Sigma_1^*\}$ satisfies the Domination Property. By Theorem 2, our proof will be complete once it is shown that every source in this family satisfies the RRN Property. More generally, we show that the RRN Property holds for any binary tree source $(\mathcal{T}, \mathcal{F}, P)$ for which

$$\sup_{t \in \mathcal{T}, P(t) > 0} \left\{ \max_{v \in V^{1}(t)} \left[\frac{|t(v)|}{\min(|t(v)_{L}|, |t(v)_{R}|)} \right] \right\} < \infty.$$
 (4.18)

(The \mathcal{T} -filter \mathcal{F} in the given source $(\mathcal{T},\mathcal{F},P)$ need not be equal to \mathcal{F}_1 .) Let C be a positive integer greater than or equal to the supremum on the left side of (4.18). Fix $t \in \mathcal{T}$ for which P(t) > 0. As in the proof of Theorem 2, let t^{\dagger} be an initial binary subtree of t with N = N(t) leaves such that $\{t(v): v \in V^1(t^{\dagger})\} = \{t(v): v \in V^1(t)\}$. Let v_1, v_2, \cdots, v_N be an enumeration of the leaves of t^{\dagger} and for each $i = 1, 2, \cdots, N$, let $u_i \in V^1(t^{\dagger})$ be the parent vertex of v_i . We have

$$\frac{|t(u_i)|}{|t(v_i)|} \le C, \ i = 1, 2, \dots, N,$$

and therefore

$$\frac{|t(u_1)|+|t(u_2)|+\cdots+|t(u_N)|}{|t(v_1)|+|t(v_2)|+\cdots+|t(v_N)|} \leq C.$$

The sum in the denominator is |t|, and so

$$\frac{|t(u_1)| + |t(u_2)| + \dots + |t(u_N)|}{|t|} \le C. \tag{4.19}$$

Each $u \in \{u_1, \dots, u_N\}$ can be the parent of at most two elements of the set $\{v_1, \dots, v_N\}$, and so

$$\operatorname{card}(\{u_1, \dots, u_N\}) \ge (1/2)\operatorname{card}(\{v_1, \dots, v_N\}) = N/2.$$

The mapping $u \to t(u)$ from the set $V^1(t^{\dagger})$ into the set $\{t(v) : v \in V^1(t)\}$ is a one-to-one onto mapping (both sets have cardinality N-1). Therefore,

$$\operatorname{card}(\{t(u_1), t(u_2), \dots, t(u_N)\}) \ge N/2. \tag{4.20}$$

Let $k = \lceil N/2 \rceil$. We conclude from (4.19)-(4.20) that there are k distinct trees t_1, t_2, \dots, t_k in T whose total number of leaves is $\leq |t|C$, where we suppose that these k trees have been enumerated so that

$$|t_1| \leq |t_2| \leq \cdots \leq |t_k|$$
.

Let $t(1), t(2), t(3), \cdots$ be an enumeration of all trees in \mathcal{T} such that t(1) is the unique tree in \mathcal{T}_2 , t(2), t(3) are the two trees in \mathcal{T}_3 , t(4), t(5), t(6), t(7), t(8) are the five trees in \mathcal{T}_4 , and so forth. We clearly have $|t(i)| \leq |t_i|$ for $i = 1, \dots, k$. Therefore,

$$|t(1)| + |t(2)| + \dots + |t(k)| \le |t|C.$$
 (4.21)

The sequence $m_i = |t(i)|$ can be characterized as the sequence in which $m_1 = 2$ and for each $j \ge 3$, $m_i = j$ for all integers i satisfying

$$K_2 + K_3 + \cdots + K_{j-1} < i \le K_2 + K_3 + \cdots + K_j$$
.

Define

$$k(M) \stackrel{\Delta}{=} \max\{k \ge 1 : m_1 + m_2 + \dots + m_k \le M\}, M \ge 2.$$

Since the sequence $\{K_j: j \ge 2\}$ grows exponentially fast, it follows that $k(M)/M = O(1/\log_2 M)$ by an argument similar to an argument on page 753 of [10], and hence

$$\lim_{M \to \infty} k(M)/M = 0. \tag{4.22}$$

From (4.21), we have shown that

$$\lceil N(t)/2 \rceil < k(|t|C), \ t \in \mathcal{T}, \ P(t) > 0.$$

Dividing both sides by |t| and summing, we then have

$$\sum_{t \in F} r(t)P(t) \le 2\sum_{t \in F} |t|^{-1}k(|t|C))P(t), \ F \in \mathcal{F}.$$
(4.23)

Let $n_F = \min\{|t| : t \in F\}$, and define

$$\delta(J) \stackrel{\Delta}{=} \sup\{k(j)/j : j \ge J\}, J \ge 2.$$

From (4.23), we then have

$$\sum_{t \in \mathcal{T}} r(t)P(t) \le 2C\delta(n_F C), \ F \in \mathcal{F}. \tag{4.24}$$

By (1.1), $\lim_{F \in \mathcal{F}} n_F = \infty$, and we also have $\lim_{J \to \infty} \delta(J) = 0$. Taking the limit along filter \mathcal{F} on both sides of (4.24), we then obtain (3.12), which is the RRN Property for the source $(\mathcal{T}, \mathcal{F}, P)$.

V. Universal Coding of Depth-Centric Binary Tree Sources

For each $t \in \mathcal{T}^*$, define d(t) to be the depth of t, which is the number of edges in the longest root-to-leaf path in t. We have $d(t^*) = 0$ and as defined in Ex. 2, for each $n \geq 1$ we let \mathcal{T}^n be the set of trees $\{t \in \mathcal{T}: d(t) = n\}$. We fix throughout this section the \mathcal{T} -filter $\mathcal{F}_2 = \{\mathcal{T}^n: n \geq 1\}$. We now formally define the set of depth-centric binary tree sources, which are certain binary tree sources of the form $(\mathcal{T}, \mathcal{F}_2, P)$. Let \mathbb{Z}^+ be the set of nonnegative integers, and let Σ_2 be the set of all functions σ from $\mathbb{Z}^+ \times \mathbb{Z}^+$ into [0,1] such that

$$\sum_{\{(i,j): i,j \geq 0, \max(i,j) = n-1\}} \sigma(i,j) = 1, \ n \geq 1.$$

For each $\sigma \in \Sigma_2$, let P_{σ} be the mapping from \mathcal{T} into [0,1] such that

$$P_{\sigma}(t) = \prod_{v \in V^{1}(t)} \sigma(d(t(v)L), d(t(v)R)), \ t \in \mathcal{T}.$$

Since

$$\sum_{t\in\mathcal{T}^n} P_{\sigma}(t) = 1, \ n\geq 1,$$

 $S(\sigma) = (\mathcal{T}, \mathcal{F}_2, P_{\sigma})$ is a binary tree source. The sources in the family $\{S(\sigma) : \sigma \in \Sigma_2\}$ are called depth-centric binary tree sources, the reason being that the probability of each tree is based purely upon the depths of its final subtrees.

Example 8. Let Σ_2^{\dagger} be the subset of Σ_2 consisting of all $\sigma \in \Sigma_2$ for which

$$\{(i,j): i,j \geq 0, \ \max(i,j) = n-1, \ \sigma(i,j) > 0\} \subset \{(0,n-1),(n-1,0)\}, \ n \geq 1.$$

If $\sigma \in \Sigma_2^{\dagger}$, then a tree $t \in \mathcal{T}$ has positive P_{σ} probability if and only if t is a one-dimensional tree. The family of sources $\{S(\sigma) : \sigma \in \Sigma_2^{\dagger}\}$ has no universal code by the same argument given in Ex. 6. Thus, the bigger family of all depth-centric binary tree sources also has no universal code.

Our final result shows that (ϕ_e, ϕ_d) is a universal code for a suitably restricted subfamily of the family of depth-centric binary tree sources.

Theorem 4. Let Σ_2^* be the uncountable set consisting of all $\sigma \in \Sigma_2$ such that

$$\sup\{|i-j|: i, j \ge 0, \ \sigma(i,j) > 0\} < \infty$$
 (5.25)

and

$$\operatorname{card}\{|i-j|: i, j \ge 0, \, \max(i,j) = n-1, \, \sigma(i,j) > 0\} = 1, \, n \ge 1.$$
 (5.26)

Then (ϕ_e, ϕ_d) is a universal code for the family of sources $\{S(\sigma) : \sigma \in \Sigma_2^*\}$.

Proof. Each source in the family $\{S(\sigma) : \sigma \in \Sigma_2^*\}$ satisfies the Domination Property, by the same argument given in the proof of Theorem 3. Appealing to Theorem 2, our proof will be complete once we verify that each source in this family also satisfies the RRN Property. Fix the source $S(\sigma)$, where $\sigma \in \Sigma_2^*$. By the last part of the proof of Theorem 3, $S(\sigma)$ will satisfy the RRN Property if

$$\sup_{t \in \mathcal{T}, P_{\sigma}(t) > 0} \left\{ \max_{v \in V^{1}(t)} \left[\frac{|t(v)|}{\min(|t(v)_{L}|, |t(v)_{R}|)} \right] \right\} < \infty.$$
 (5.27)

By (5.26), for each $n \ge 1$, there exists $k_n \in \{0, 1, \dots, n-1\}$ such that

$$\{(i,j): i,j \ge 0, \max(i,j) = n-1, \ \sigma(i,j) > 0\} \subset \{(k_n,n-1), (n-1,k_n)\}.$$

Let $(x(n): n \ge 0)$ be the sequence of real numbers such that x(0) = 1 and

$$x(n) = x(n-1) + x(k_n), \ n \ge 1.$$

We prove the statement

$$|t| = x(d(t)), \ t \in \{t^*\} \cup \{t' \in \mathcal{T} : P_{\sigma}(t') > 0\}$$
 (5.28)

by induction on |t|, starting with |t| = 1. If |t| = 1, then $t = t^*$ and |t| = x(d(t)) is the true statement 1 = x(0). Now fix $u \in \mathcal{T}$ for which $P_{\sigma}(u) > 0$ and we assume as our induction hypothesis that |t| = x(d(t)) holds for every $t \in \{t^*\} \cup \{t' \in \mathcal{T} : P_{\sigma}(t') > 0\}$ for which |t| < |u|.

Note that $(d(u_L), d(u_R))$ belongs to the set $\{(d(u) - 1, k_{d(u)}), (k_{d(u)}, d(u) - 1)\}$. The induction hypothesis holds for both u_L and u_R , and so

$$|u| = |u_L| + |u_R| = x(d(u_L)) + x(d(u_R)) = x(d(u) - 1) + x(k_{d(u)}) = x(d(u)),$$

completing the proof of statement (5.28). We conclude from (5.28) that for every $t \in \mathcal{T}$ for which $P_{\sigma}(t) > 0$,

$$\frac{|t(v)|}{\min(|t(v)_L|,|t(v)_R|)} \in \{x(n)/x(k_n) : n \ge 1\}, \ v \in V^1(t).$$

By (5.25), let $m \in \mathbb{Z}^+$ be the supremum on the left side of (5.25); then $n-1-k_n \le m$ for $n \ge 1$. Since the sequence (x(n)) is nondecreasing, $x(n)/x(n-1) \le 2$ for $n \ge 1$, and so

$$\frac{x(n)}{x(k_n)} = \prod_{i=k_n+1}^n \frac{x(i)}{x(i-1)} \le 2^{n-k_n} \le 2^{m+1}, \ n \ge 1.$$

Thus, the left side of (5.27) is at most 2^{m+1} and (5.27) holds, completing our proof.

VI. CONCLUSIONS

We have shown that the grammar-based code (ϕ_e, ϕ_d) on the set \mathcal{T} of binary tree structures defined in this paper is asymptotically optimal for any binary tree source satisfying the Domination Property and the Representation Ratio Negligibility Property. In typical cases, we have found that the Domination Property is easy to verify for a binary tree source, whereas the RRN Property is more troublesome to verify. In a subsequent paper [11], we investigate more scenarios in which the RRN Property will hold. (The one-dimensional binary trees discussed in Example 6 need to be avoided in a binary tree source model, as well as some trees derived from these.) In [11], we also show that (ϕ_e, ϕ_d) is universal for some families of binary tree sources induced by branching processes (including families of sources which were considered in [15] from an entropy point of view but not from a compression point of view).

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