# Capacity of Gaussian Channels with Duty <br> Cycle and Power Constraints 

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#### Abstract

In many wireless communication systems, radios are subject to a duty cycle constraint, that is, a radio only actively transmits signals over a fraction of the time. For example, it is desirable to have a small duty cycle in some low power systems; a half-duplex radio cannot keep transmitting if it wishes to receive useful signals; and a cognitive radio needs to listen and detect primary users frequently. This work studies the capacity of scalar discrete-time Gaussian channels subject to duty cycle constraint as well as average transmit power constraint. An idealized duty cycle constraint is first studied, which can be regarded as a requirement on the minimum fraction of nontransmissions or zero symbols in each codeword. A unique discrete input distribution is shown to achieve the channel capacity. In many situations, numerically optimized on-off signaling can achieve much higher rate than Gaussian signaling over a deterministic transmission schedule. This is in part because the positions of nontransmissions in a codeword can convey information. Furthermore, a more realistic duty cycle constraint is studied, where the extra cost of transitions between transmissions and nontransmissions due to pulse shaping is accounted for. The capacity-achieving input is no longer independent over time and is hard to compute. A lower bound of the achievable rate as a function of the input distribution is shown to be maximized by a first-order Markov input process, the distribution of which is also discrete and can be computed efficiently. The results in this paper suggest that, under various duty cycle constraints, departing from the usual paradigm of intermittent packet transmissions may yield substantial gain.


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## Index Terms

Duty cycle constraint, capacity-achieving input, mutual information, entropy rate, Markov process, hidden Markov process (HMP), Monte Carlo method.

## I. Introduction

In many wireless communication systems, a radio is designed to transmit actively only for a fraction of the time, which is known as its duty cycle. For example, the ultra-wideband system in [1] transmits short bursts of signals to trade bandwidth for power savings. The physical half-duplex constraint also requires a radio to stop transmission over a frequency band from time to time if it wishes to receive useful signals over the same band. Thus wireless relays are subject to duty cycle constraint, so do cognitive radios which have to listen to the channel frequently to avoid causing interference to primary users. The de facto standard solution under duty cycle constraint is to transmit packets intermittently.

This work studies the fundamental question of what is the optimal signaling for a Gaussian channel with duty cycle constraint as well as average transmission power constraint. An important observation is that the signaling in nontransmission periods can be regarded as transmission of a special zero signal. We first make a simplistic and idealized assumption that the analog waveform corresponding to each transmitted symbol is exactly of the span of one symbol interval. We restrict our attention to discretetime scalar additive white Gaussian noise (AWGN) channels for simplicity, where the duty cycle constraint is equivalent to a requirement on the minimum fraction of zero symbols in each transmitted codeword, which is called the idealized duty cycle constraint. We then consider the case where a practical pulse shaping filter is used, e.g., for band-limited transmissions. As such, during a transition between a zero symbol and a nonzero symbol, the pulse waveform of the nonzero symbol leaks into the interval of the zero symbol. A realistic duty cycle constraint must include the extra cost incurred upon transitions between zero and nonzero symbols. The mathematical model of the preceding input-constrained channels is described in Section II,

Determining the capacity of a channel subject to various input constraints is a classical problem. It is well-known that Gaussian signaling achieves the capacity of a Gaussian channel with average input power constraint only. In addition, Zamir [2] shows that the mutual information rate achievable using a white Gaussian input never incurs a loss of more than half a bit per sample with respect to the power
constrained capacity. Furthermore, Smith [3] investigated the capacity of a scalar AWGN channel under both peak power constraint and average power constraint. The input distribution that achieves the capacity is shown to be discrete with a finite number of probability mass points. The discreteness of capacityachieving distributions for various channels, including quadrature Gaussian channels, and Rayleigh-fading channels is also established in [4]-[9]. Chan [10] studied the capacity-achieving input distribution for conditional Gaussian channels which form a general channel model for many practical communication systems. Until now, the impact of duty cycle constraint on capacity-achieving signaling is underexplored in the literature.

The main results of this paper are summarized in Section III. In the case of the idealized duty cycle constraint, because all costs associated with the constraints can be decomposed into per-letter costs, the optimal input distribution is independent and identically distributed (i.i.d.). We use a similar approach as in [3] and [10] to show that the capacity-achieving input distribution for an AWGN channel with duty cycle constraint and average power constraints is discrete. Unlike in [3] and [10], the optimal distribution has an infinite number of probability mass points, whereas only a finite number of the points are found in every bounded interval. This allows efficient numerical optimization of the input distribution.

The case of realistic duty cycle constraint is more challenging. Because the constraint concerns symbol transitions, the capacity-achieving input distribution is no longer independent over time, and becomes hard to compute. We develop a good lower bound of the input-output mutual information as a function of the input distribution. It is proved that, under the realistic duty cycle constraint, a first-order Markov process maximizes the lower bound, the distribution of which is also discrete and can be computed efficiently. The main theorems for the cases of idealized and realistic duty cycle constraints are proved in Section $I V$ and V , respectively.

We devote Section VI to the numerical methods and results. In order to compute the achievable rate when the input is a Markov Chain, a Monte Carlo method is introduced in Section VI-A to numerically compute the differential entropy rate of hidden Markov processes. Numerical results in Section VI-B demonstrate that in the case of idealize duty cycle constraint using a numerically optimized discrete signaling achieves higher rates than using Gaussian signaling over a deterministic transmission schedule. For example, if the radio is allowed to transmit no more than half the time, i.e., the duty cycle is no greater than $50 \%$, a near-optimal discrete input achieves $50 \%$ higher rate at 10 dB signal-to-noise ratio
(SNR). In the case of realistic duty cycle constraint, numerical results also show that the rate achieved by the Markov process is substantially higher than that achieved by any i.i.d. input. This suggests that, compared to intermittently transmitting packets using Gaussian or Gaussian-like signaling, it is more efficient to disperse nontransmission symbols within each packet to form codewords, which results in a form of on-off signaling.

One of the reasons for the superiority of on-off signaling is that the positions of nontransmission symbols can be used to convey information, the impact of which is particularly significant in case of low SNR or low duty cycle. This has been observed in the past. For example, as shown in [11] (see also [12], [13]), time sharing or time-division duplex (TDD) can fall considerably short of the theoretical limits in a relay network: The capacity of a cascade of two noiseless binary bit pipes through a half-duplex relay is 1.14 bits per channel use, which far exceeds the 0.5 bit achieved by TDD and even the 1 bit upper bound on the rate of binary signaling.

Besides that duty cycle constraint is frequently seen in practice, another motivation of this study is a recent work [14], in which on-off signaling is proposed for a clean-slate design of wireless ad hoc networks formed by half-duplex radios. Using this signaling scheme, which is called rapid on-off-division duplex (RODD), a node listens to the channel and receives useful signals during its own off symbols within each frame. Each node can transmit and receive messages at the same time over one frame interval, thereby achieving (virtual) full-duplex communication. Understanding the impact of duty cycle constraint is crucial to characterizing the fundamental limits of such wireless networks.

## II. System Model

Consider digital communication systems where coded data are mapped to waveforms for transmission. Usually there is a collection of pulse waveforms, where each pulse represents a symbol (or letter) from a discrete alphabet. We view nontransmission over a symbol interval as transmitting the all zero waveform. In other words, a symbol interval of nontransmission is simply regarded as transmitting a special symbol "0," which carries no energy.

As far as the capacity-achieving input is concerned it suffices to consider the baseband discrete-time
model for the AWGN channel. The received signal over a block of $n$ symbols can be described by

$$
\begin{equation*}
Y_{i}=X_{i}+N_{i} \tag{1}
\end{equation*}
$$

where $i=1, \ldots, n, X_{i}$ denotes the transmitted symbol at time $i$ and $N_{1}, \ldots, N_{n}$ are independent standard Gaussian random variables. For simplicity, we assume no inter-symbol interference is at receiver. Each symbol modulates a continuous-time pulse waveform for transmission. If the width of all pulses were exactly of one symbol interval, which is denoted by $T$, the duty cycle is equal to the fraction of nonzero symbols in a codeword. In practice, however, the pulse is usually wider than $T$, so that the support of the transmitted waveform is greater than the sum of the intervals corresponding to nonzero symbols due to leakage into intervals of adjacent zero symbols. To be specific, suppose the width of a pulse is $(1+2 c) T$, then each transition between zero and nonzero symbols incurs an additional cost of up to $c T$ in terms of actual transmission time.

Let $1-q$ denote the maximum duty cycle allowed. In this paper, we require every codeword $\left(x_{1}, x_{2}, \cdots, x_{n}\right)$ to satisfy

$$
\begin{equation*}
\frac{1}{n} \sum_{i=1}^{n} 1_{\left\{x_{i} \neq 0\right\}}+\frac{1}{n} 2 c\left(\sum_{i=1}^{n-1} 1_{\left\{x_{i}=0, x_{i+1} \neq 0\right\}}+1_{\left\{x_{n}=0, x_{1} \neq 0\right\}}\right) \leq 1-q \tag{2}
\end{equation*}
$$

where $1_{\{\cdot\}}$ is the indicator function, and the transition cost is twice that of zero-to-nonzero transitions, because the number of nonzero-to-zero transitions and the number of zero-to-nonzero transitions is equal under the cyclic transition cost configuration. From now on, we refer to (2) as duty cycle constraint $(q, c)$. Note that the idealized duty cycle constraint is the special case $(q, 0)$. If $c \in\left[0, \frac{1}{2}\right]$, then the left hand side of (2) is equal to the actual duty cycle. If $c>\frac{1}{2}$, the left hand side of 22 is an overestimate of the duty cycle. Nonetheless, we use constraint (2) for its simplicity. In addition, we consider the usual average input power constraint,

$$
\begin{equation*}
\frac{1}{n} \sum_{i=1}^{n} x_{i}^{2} \leq \gamma \tag{3}
\end{equation*}
$$

In many wireless systems, the transmitter's activity is constrained in the frequency domain as well as in the time domain. In principle, the results in this paper also apply to the more general model where the duty cycle constraint is on the time-frequency plane.

## III. Main Results

## A. The Case of Idealized Duty Cycle Constraint

Let $\mu$ denote the distribution of the channel input $X$. The set of distributions with duty cycle constraint $(q, 0)$ and power constraint $\gamma$ is denoted by

$$
\begin{equation*}
\Lambda(\gamma, q)=\left\{\mu: \mu(\{0\}) \geq q, \mathrm{E}_{\mu}\left\{X^{2}\right\} \leq \gamma\right\} \tag{4}
\end{equation*}
$$

It should be understood that $\mu$ is a probability measure defined on the Borel algebra on the real number set, denoted by $\mathcal{B}(\mathbb{R})$.

Theorem 1: The capacity of the additive white Gaussian noise channel (1) with its idealized duty cycle no greater than $1-q$ and the average power no greater than $\gamma$ is

$$
\begin{equation*}
C(\gamma, q)=\max _{\mu \in \Lambda(\gamma, q)} I(\mu) \tag{5}
\end{equation*}
$$

In particular, the following properties hold:
a) the maximum of (5) is achieved by a unique (capacity-achieving) distribution $\mu_{0} \in \Lambda(\gamma, q)$;
b) $\mu_{0}$ is symmetric about 0 and its second moment is exactly equal to $\gamma$; and
c) $\mu_{0}$ is discrete with an infinite number of probability mass points, whereas the number of probability mass points in any bounded interval is finite.

The proof of Theorem 1 is relegated to Section IV. Property (b) suggests that the capacity-achieving input always exhausts the power budget. Property (c) indicates that the capacity-achieving input can be well approximated by some discrete inputs with finite alphabet, which can be computed using numerical methods. The achievable rate of numerically optimized input distribution is studied in Section VI.

## B. The Case of Realistic Duty Cycle Constraint

In this paper, let $X_{k}^{n}$ denote the subsequence $\left(X_{k}, X_{k+1}, \cdots, X_{n}\right)$, where $X_{k}^{\infty}=\left(X_{k}, X_{k+1}, \cdots\right)$. We also use shorthand $X^{n}=X_{1}^{n}$. Let $\mu$ denote the probability distribution of the process $X_{1}, X_{2}, \cdots$. We use $\mu_{X_{i}}$ to denote the marginal distribution of $X_{i}$, and $\mu_{X_{i}, X_{j}}$ to denote the joint probability distribution of $\left(X_{i}, X_{j}\right)$. Denote the set of $n$-dimension distribution which satisfy duty cycle constraint $(q, c)$ and
power constraint $\gamma$ by

$$
\begin{gather*}
\Lambda^{n}(\gamma, q, c)=\left\{\mu: \frac{1}{n} \sum_{i=1}^{n}\left[\mu_{X_{i}}(\{0\})-2 c \mu_{X_{i}, X_{i \bmod n+1}}(\{0\} \times(\mathbb{R} \backslash\{0\}))\right] \geq q,\right. \\
\left.\mathrm{E}_{\mu}\left\{\frac{1}{n} \sum_{i=1}^{n} X_{i}^{2}\right\} \leq \gamma\right\} \tag{6}
\end{gather*}
$$

where

$$
\begin{equation*}
\mu_{X_{i}, X_{j}}(\{0\} \times(\mathbb{R} \backslash\{0\}))=P\left(X_{i}=0, X_{j} \neq 0\right) \tag{7}
\end{equation*}
$$

denotes the probability of a zero-to-nonzero transition and

$$
i \bmod n= \begin{cases}i, & \text { if } 1 \leq i<n  \tag{8}\\ 0, & \text { if } i=n\end{cases}
$$

For convenience in a subsequent proof, the duty cycle in (6) is defined in a cyclic manner using the modular operation, where a transition between $X_{n}$ and $X_{1}$ is also counted. This of course has vanishing impact as $n \rightarrow \infty$ and thus no impact on the capacity.

The capacity of the AWGN channel (1] with duty cycle constraint $(q, c)$ and power constraint $\gamma$ is

$$
\begin{equation*}
C(\gamma, q, c)=\lim _{n \rightarrow \infty} \frac{1}{n} \max _{P_{X^{n}} \in \Lambda^{n}(\gamma, q, c)} I\left(X^{n} ; Y^{n}\right) . \tag{9}
\end{equation*}
$$

The capacity is in fact achieved by a stationary input process. This is justified in Section V-A by showing that any nonstationary input process has a stationary counterpart with equal or greater inputoutput mutual information per symbol. Let us denote the set of stationary distributions which satisfy duty cycle constraint $(q, c)$ and power constraint $\gamma$ by

$$
\begin{align*}
& \Lambda(\gamma, q, c)=\left\{\mu: \mu \text { is stationary, } \mathrm{E}_{\mu}\left\{X_{1}^{2}\right\} \leq \gamma,\right.  \tag{10}\\
& \\
& \left.\mu_{X_{1}}(\{0\})-2 c \mu_{X_{1}, X_{2}}(\{0\} \times(\mathbb{R} \backslash\{0\})) \geq q\right\} .
\end{align*}
$$

Theorem 2: For any $\mu \in \Lambda(\gamma, q, c)$, let

$$
\begin{equation*}
L(\mu)=I(X ; Y)-I\left(X_{1} ; X_{2}^{\infty}\right) \tag{11}
\end{equation*}
$$

where $I(X ; Y)$ is the mutual information of the additive white Gaussian noise channel between the input
symbol $X$, which follows distribution $\mu_{X_{1}}$, and the corresponding output $Y$. The following properties hold:
a) $L(\mu)$ is a lower bound of the channel capacity;
b) The maximum of $L(\cdot)$ is achieved by a discrete first-order Markov process, denoted by $\mu^{*}$;
c) $\mu^{*}$ satisfies the following property: Define $B_{i}=1_{\left\{X_{i} \neq 0\right\}}, i=1,2, \ldots$. Then for every $i$, conditioned on $B_{i}$ and $B_{i+1}$, the variables $X_{i}$ and $X_{i+1}$ are independent, and

$$
\begin{equation*}
L\left(\mu^{*}\right)=I(X ; Y)-I\left(B_{1} ; B_{2}\right) . \tag{12}
\end{equation*}
$$

The proof of Theorem 2 is relegated to Section V . Evidently, increasing the input power by scaling the input linearly not only maintains its duty cycle, but also increases the mutual information. Therefore, the optimal input distribution must exhaust the power budget $\gamma$.

## IV. Proof of Theorem 1 (the Case of Idealized Duty Cycle Constraint)

This section is devoted to a proof of Theorem 1 for the case of the idealized duty cycle constraint $(q, 0)$. The conditional probability density function (pdf) of the output given the input of the AWGN channel (1) is

$$
\begin{equation*}
p_{Y \mid X}(y \mid x)=\phi(y-x) \tag{13}
\end{equation*}
$$

where

$$
\begin{equation*}
\phi(t)=\frac{1}{\sqrt{2 \pi}} e^{-\frac{t^{2}}{2}} \tag{14}
\end{equation*}
$$

is the standard Gaussian pdf.
With the idealized constraint, the capacity of the AWGN channel is achieved by an i.i.d. process and the duty cycle constraint reduces to a per symbol cost constraint. For given input distribution $\mu$, the pdf of the output exists and is expressed as

$$
\begin{equation*}
p_{Y}(y ; \mu)=\int p_{Y \mid X}(y \mid x) \mu(\mathrm{d} x)=\mathrm{E}_{\mu}\{\phi(y-X)\} \tag{15}
\end{equation*}
$$

Denote the relative entropy $D\left(p_{Y \mid X}(\cdot \mid x) \| p_{Y}(\cdot ; \mu)\right)$ by $d(x ; \mu)$, which is expressed as

$$
\begin{equation*}
d(x ; \mu)=\int_{-\infty}^{\infty} p_{Y \mid X}(y \mid x) \log \frac{p_{Y \mid X}(y \mid x)}{p_{Y}(y ; \mu)} \mathrm{d} y . \tag{16}
\end{equation*}
$$

The mutual information $I(\mu)=I(X ; Y)$ is then

$$
\begin{equation*}
I(\mu)=\int d(x ; \mu) \mu(\mathrm{d} x)=\mathrm{E}_{\mu}\{d(X ; \mu)\} . \tag{17}
\end{equation*}
$$

The capacity of the AWGN channel under per-letter duty cycle constraint and power constraint is evidently given by the supremum of the mutual information $I(\mu)$ where $\mu \in \Lambda(\gamma, q)$. The achievability and converse of this result can be established using standard techniques in information theory.

The proof of property (a) is presented in Section IV-A. Now suppose $\mu_{0}$ is the unique capacityachieving distribution, property (b) is established as follows. Since the mirror reflection of $\mu_{0}$ about 0 is evidently also a maximizer of (5), the uniqueness requires that $\mu_{0}$ be symmetric. Note that linear scaling of the input to increase its power maintains its duty cycle and cannot reduce the mutual information, as the receiver can add noise to maintain the same SNR. By the uniqueness of the maximizer $\mu_{0}$, the power constraint must be binding, i.e., the second moment of $\mu_{0}$ must be equal to $\gamma$. In order to prove property (c), we first establish a sufficient and necessary condition for $\mu_{0}$ in Section IV-B and then apply it to show the discreteness of $\mu_{0}$ in Section IV-C.

## A. Existence and Uniqueness of $\mu_{0}$

Let $\mathcal{P}$ denote the collection of all Borel probability measures defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, which is a topological space with the topology of weak convergence [15]. We first establish the following lemma.

Lemma 1: $\Lambda(\gamma, q)$ is compact in the topological space $\mathcal{P}$.
Proof: According to [15], the topology of weak convergence on $\mathcal{P}$ is metrizable. Therefore, by Prokhorov's theorem [16], in order to prove that $\Lambda(\gamma, q)$ is compact in $\mathcal{P}$, it suffices to show that it is both tight and closed.

For any $\epsilon>0$, there exits an $a_{\epsilon}>0$, such that for all $\mu \in \Lambda_{\gamma}$,

$$
\begin{equation*}
\mu\left(|X|>a_{\epsilon}\right) \leq \frac{\mathrm{E}_{\mu}\left\{X^{2}\right\}}{a_{\epsilon}^{2}} \leq \frac{\gamma}{a_{\epsilon}^{2}}<\epsilon \tag{18}
\end{equation*}
$$

by Chebyshev's inequality. Choose $K_{\epsilon}=\left[-a_{\epsilon}, a_{\epsilon}\right]$, then $K_{\epsilon}$ is compact in $\mathbb{R}$ and $\mu\left(K_{\epsilon}\right) \geq 1-\epsilon$ for all $\mu \in \Lambda(\gamma, q)$, thus $\Lambda(\gamma, q)$ is tight.

Let $B_{m}=\left[-\frac{1}{m}, \frac{1}{m}\right]$ for $m=1,2, \ldots$ Let $\left\{\mu_{n}\right\}_{n=1}^{\infty}$ be a convergent sequence in $\Lambda(\gamma, q)$ with limit $\mu_{0}$. Since $\mu_{n}\left(B_{m}\right) \geq q$ for every $m, n$, we have [15, Section 3.1]

$$
\begin{equation*}
q \leq \limsup _{n \rightarrow \infty} \mu_{n}\left(B_{m}\right) \leq \mu_{0}\left(B_{m}\right) \tag{19}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\mu_{0}(\{0\})=\mu_{0}\left(\bigcap_{m=1}^{\infty} B_{m}\right)=\lim _{m \rightarrow \infty} \mu_{0}\left(B_{m}\right) \geq q \tag{20}
\end{equation*}
$$

Moreover, let $f(x)=x^{2}$ which is continuous and bounded below. By weak convergence [15, Section 3.1], we have

$$
\begin{equation*}
\mathrm{E}_{\mu_{0}}\left\{X^{2}\right\}=\int f \mathrm{~d} \mu_{0} \leq \liminf _{n \rightarrow \infty} \int f \mathrm{~d} \mu_{n} \leq \gamma \tag{21}
\end{equation*}
$$

Therefore, $\mu_{0} \in \Lambda(\gamma, q)$, i.e., $\Lambda(\gamma, q)$ is closed, and the compactness of $\Lambda(\gamma, q)$ then follows.
Since the mutual information $I(\mu)$ is continuous on $\mathcal{P}$ [17, Theorem 9], it must achieve its maximum on the compact set $\Lambda(\gamma, q)$. Hence the capacity-achieving distribution $\mu_{0}$ exists.

According to [17, Corollary 2], the mutual information $I(\mu)$ is strictly concave. It is easy to see that $\Lambda(\gamma, q)$ is convex. Hence the capacity-achieving distribution $\mu_{0}$ must be unique.

## B. Sufficient and Necessary Conditions

We denote the finite-power set as

$$
\begin{equation*}
\Lambda(q)=\cup_{0 \leq \gamma<\infty} \Lambda(\gamma, q) \tag{22}
\end{equation*}
$$

Let $\phi(\cdot)$ defined in 14 be extended to the complex plane. The relative entropy $d(x ; \mu)$ defined in 16 can be extended to the complex plane $\mathbb{C}$ and has the following property:

Lemma 2: For any $\mu \in \Lambda(q)$ and $z \in \mathbb{C}$,

$$
\begin{equation*}
d(z ; \mu)=\int_{-\infty}^{\infty} \phi(y-z) \log \frac{\phi(y-z)}{p_{Y}(y ; \mu)} \mathrm{d} y \tag{23}
\end{equation*}
$$

is a holomorphic function of $z$ on $\mathbb{C}$. Consequently, $d(x ; \mu)$ is a continuous function of $x$ on $\mathbb{R}$.
Proof: It can be shown that $\int_{-\infty}^{\infty} \phi(y-z) \log \phi(y-z) \mathrm{d} y$ is a constant, thus a holomorphic function of $z$ on $\mathbb{C}$. Therefore, it remains to prove that

$$
\begin{equation*}
\xi(z)=\int_{-\infty}^{\infty} \phi(y-z) \log p_{Y}(y ; \mu) \mathrm{d} y \tag{24}
\end{equation*}
$$

is a holomorphic function of $z$ on $\mathbb{C}$.
First, by Jensen's inequality, we have

$$
\begin{align*}
p_{Y}(y ; \mu) & =\mathrm{E}_{\mu}\left\{\frac{1}{\sqrt{2 \pi}} e^{-\frac{(y-X)^{2}}{2}}\right\}  \tag{25}\\
& \geq \frac{1}{\sqrt{2 \pi}} e^{-\frac{1}{2} \mathrm{E}_{\mu}\left\{(y-X)^{2}\right\}}  \tag{26}\\
& =e^{-\frac{1}{2} y^{2}-a y-b} \tag{27}
\end{align*}
$$

where $a=-\mathrm{E}_{\mu}\{X\}$ and $b=\frac{1}{2}\left(\mathrm{E}_{\mu}\left\{X^{2}\right\}+\log (2 \pi)\right)$ are real numbers due to the fact that $\mu \in \Lambda(q)$. Thus, $p_{Y}(y ; \mu) \in\left[e^{-\frac{1}{2} y^{2}-a y-b}, 1\right]$, i.e.,

$$
\begin{equation*}
\left|\log P_{Y}(y ; \mu)\right| \leq \frac{1}{2} y^{2}+a y+b \tag{28}
\end{equation*}
$$

As a result, we have

$$
\begin{align*}
\left|\phi(y-z) \log p_{Y}(y ; \mu)\right| & \leq \frac{1}{\sqrt{2 \pi}}\left|e^{-\frac{(y-z)^{2}}{2}}\right|\left(\frac{1}{2} y^{2}+a y+b\right)  \tag{29}\\
& =\frac{1}{\sqrt{2 \pi}} e^{-\frac{(y-\operatorname{Re}(z))^{2}-\operatorname{Im}^{2}(z)}{2}}\left(\frac{1}{2} y^{2}+a y+b\right) \tag{30}
\end{align*}
$$

which is integrable. (Here $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ represent the real and imaginary parts of $z$, respectively.) It follows that $\xi(z)$ given by 24 exists for any $\mu \in \Lambda(q)$ and $z \in \mathbb{C}$.

Suppose $U$ is an open and bounded subset of $\mathbb{C}$. There exists an $r>0$ such that $|\operatorname{Re}(z)| \leq r$ and $|\operatorname{Im}(z)| \leq r$ for all $z \in U$. It is easy to check that

$$
\begin{align*}
e^{-\frac{(y-\operatorname{Re}(z))^{2}}{2}} & \leq e^{-\frac{y^{2}}{2}+|y r|}  \tag{31}\\
& \leq e^{-\frac{y^{2}}{2}+y r}+e^{-\frac{y^{2}}{2}-y r}  \tag{32}\\
& =e^{\frac{r^{2}}{2}}\left[e^{-\frac{1}{2}(y-r)^{2}}+e^{-\frac{1}{2}(y+r)^{2}}\right] \tag{33}
\end{align*}
$$

Combining (29) and (33) yields that

$$
\begin{equation*}
\left|\phi(y-z) \log p_{Y}(y ; \mu)\right| \leq \frac{e^{r^{2}}}{\sqrt{2 \pi}}\left[e^{-\frac{1}{2}(y-r)^{2}}+e^{-\frac{1}{2}(y+r)^{2}}\right]\left(\frac{1}{2} y^{2}+a y+b\right) \tag{34}
\end{equation*}
$$

which is integrable. Therefore, the integral $\int_{-\infty}^{\infty} \phi(y-z) \log p_{Y}(y ; \mu) \mathrm{d} y$ is uniformly convergent for all $z \in U$. Moreover, $\phi(y-z) \log p_{Y}(y ; \mu)$ is a holomorphic function of $z$ on $U$ for each $y \in \mathbb{R}$. According to the differentiation lemma [18], $\xi(z)$ is a holomorphic function of $z$ on $U$. It then follows that it is holomorphic on the whole complex plane $\mathbb{C}$. Lemma 2 is thus established.

Let $F(\mu)$ be a real-valued function defined on the convex set $\Lambda(q)$ and $\mu_{0} \in \Lambda(q)$. Define the weak derivative of $F(\mu)$ at $\mu_{0}$ as

$$
\begin{equation*}
F_{\mu_{0}}^{\prime}(\mu)=\lim _{\theta \rightarrow 0^{+}} \frac{F\left((1-\theta) \mu_{0}+\theta \mu\right)-F\left(\mu_{0}\right)}{\theta} \tag{35}
\end{equation*}
$$

whenever the limit exists. The following result, which finds its parallel in [6], [9], [10] gives the weak derivative of the mutual information function $I(\mu)$.

Lemma 3: Let $\mu_{0}, \mu \in \Lambda(q)$, the weak derivative of the mutual information function $I(\mu)$ at $\mu_{0}$ is

$$
\begin{equation*}
I_{\mu_{0}}^{\prime}(\mu)=\int d\left(x ; \mu_{0}\right) \mu(\mathrm{d} x)-I\left(\mu_{0}\right) \tag{36}
\end{equation*}
$$

Proof: Define $\mu_{\theta}=(1-\theta) \mu_{0}+\theta \mu$ for all $\theta \in(0,1]$. It can be shown that

$$
\begin{align*}
\frac{1}{\theta}\left(I\left(\mu_{\theta}\right)-I\left(\mu_{0}\right)\right)= & \frac{1}{\theta} \int\left(d\left(x ; \mu_{\theta}\right)-d\left(x ; \mu_{0}\right)\right) \mu_{\theta}(\mathrm{d} x)+\frac{1}{\theta}\left(\int d\left(x ; \mu_{0}\right) \mu_{\theta}(\mathrm{d} x)-I\left(\mu_{0}\right)\right)  \tag{37}\\
& =-\frac{1}{\theta} \int_{-\infty}^{\infty} p_{Y}\left(y ; \mu_{\theta}\right) \log \frac{p_{Y}\left(y ; \mu_{\theta}\right)}{p_{Y}\left(y ; \mu_{0}\right)} \mathrm{d} y+\int d\left(x ; \mu_{0}\right) \mu(\mathrm{d} x)-I\left(\mu_{0}\right) \tag{38}
\end{align*}
$$

Therefore, it suffices to show that

$$
\begin{equation*}
\lim _{\theta \rightarrow 0^{+}} \int_{-\infty}^{\infty} \frac{1}{\theta} p_{Y}\left(y ; \mu_{\theta}\right) \log \frac{p_{Y}\left(y ; \mu_{\theta}\right)}{p_{Y}\left(y ; \mu_{0}\right)} \mathrm{d} y=0 \tag{39}
\end{equation*}
$$

In the remainder of this proof, we find a function independent of $\theta$ that dominates the integrand so that dominated convergence theorem can be used to establish by by exchanging the order of the limit and the integral therein.

Lemma 4: Let $\theta, a, b \in(0,1]$. Define

$$
\begin{equation*}
f(\theta)=\frac{(1-\theta) a+\theta b}{\theta} \log \frac{(1-\theta) a+\theta b}{a}, \tag{40}
\end{equation*}
$$

then

$$
\begin{equation*}
|f(\theta)| \leq b+a-b \log b-b \log a . \tag{41}
\end{equation*}
$$

Proof: It is easy to check that $f(1)=b \log \frac{b}{a}, f\left(0^{+}\right)=b-a$ and

$$
\begin{equation*}
f^{\prime}(\theta)=\frac{b-a}{\theta}-\frac{a}{\theta^{2}} \log \left(1-\theta+\frac{b}{a} \theta\right) \tag{42}
\end{equation*}
$$

Define $g(\theta)=\theta(b-a)-a \log \left(1-\theta+\frac{b}{a} \theta\right)$ for $\theta \in(0,1]$, then we have

$$
\begin{equation*}
g^{\prime}(\theta)=\frac{\theta(b-a)^{2}}{(1-\theta) a+\theta b} \geq 0 . \tag{43}
\end{equation*}
$$

Since $g\left(0^{+}\right)=0, g(\theta) \geq 0$ for all $\theta \in(0,1]$. According to (42), we have $f^{\prime}(\theta)=\frac{g(\theta)}{\theta^{2}} \geq 0$. It follows that for all $\theta \in(0,1]$,

$$
\begin{equation*}
b-a=f\left(0^{+}\right) \leq f(\theta) \leq f(1)=b \log \frac{b}{a}, \tag{44}
\end{equation*}
$$

and hence

$$
\begin{align*}
|f(\theta)| & \leq \max \left\{|b-a|,\left|b \log \frac{b}{a}\right|\right\}  \tag{45}\\
& \leq b+a-b \log b-b \log a \tag{46}
\end{align*}
$$

Lemma 4 is thus established.
Applying Lemma 4 with $a=p_{Y}\left(y ; \mu_{0}\right)$ and $b=p_{Y}(y ; \mu)$, we have

$$
\begin{align*}
&\left|\frac{1}{\theta} p_{Y}\left(y ; \mu_{\theta}\right) \log \frac{p_{Y}\left(y ; \mu_{\theta}\right)}{p_{Y}\left(y ; \mu_{0}\right)}\right| \leq p_{Y}(y ; \mu)+p_{Y}\left(y ; \mu_{0}\right) \\
&-p_{Y}(y ; \mu) \log p_{Y}(y ; \mu)-p_{Y}(y ; \mu) \log p_{Y}\left(y ; \mu_{0}\right) \tag{47}
\end{align*}
$$

where the right hand side is an integrable function of $y$ by the result that $-\int_{-\infty}^{\infty} p_{Y}\left(y ; \mu_{2}\right) \log p_{Y}\left(y ; \mu_{1}\right) \mathrm{d} y<$ $\infty$ for any $\mu_{1}, \mu_{2} \in \Lambda(q)$. In fact, as in the proof of Lemma 2 (see (28)), there exist $a, b \in \mathbb{R}$ such that
$\left|\log p_{Y}\left(y ; \mu_{1}\right)\right| \leq \frac{1}{2} y^{2}+a y+b$. Therefore,

$$
\begin{align*}
\int_{-\infty}^{\infty}\left|p_{Y}\left(y ; \mu_{2}\right) \log p_{Y}\left(y ; \mu_{1}\right)\right| \mathrm{d} y & \leq \int_{-\infty}^{\infty} p_{Y}\left(y ; \mu_{2}\right)\left(\frac{1}{2} y^{2}+a y+b\right) \mathrm{d} y  \tag{48}\\
& =\frac{1}{2} \mathrm{E}_{\mu_{2}}\left\{X^{2}\right\}+a \mathrm{E}_{\mu_{2}}\{X\}+b+\frac{1}{2}  \tag{49}\\
& <\infty \tag{50}
\end{align*}
$$

due to the assumption that $\mu_{2} \in \Lambda(q)$.
Therefore, the dominated convergence theorem provides that

$$
\begin{align*}
\lim _{\theta \rightarrow 0^{+}} \frac{1}{\theta} \int_{-\infty}^{\infty} p_{Y}\left(y ; \mu_{\theta}\right) \log \frac{p_{Y}\left(y ; \mu_{\theta}\right)}{p_{Y}\left(y ; \mu_{0}\right)} \mathrm{d} y & =\int_{-\infty}^{\infty} \lim _{\theta \rightarrow 0^{+}} \frac{1}{\theta} p_{Y}\left(y ; \mu_{\theta}\right) \log \frac{p_{Y}\left(y ; \mu_{\theta}\right)}{p_{Y}\left(y ; \mu_{0}\right)} \mathrm{d} y  \tag{51}\\
& =\int_{-\infty}^{\infty}\left(p_{Y}(y ; \mu)-p_{Y}\left(y ; \mu_{0}\right)\right) \mathrm{d} y  \tag{52}\\
& =0 \tag{53}
\end{align*}
$$

Lemma 3 is thus proved.
We establish the following sufficient and necessary condition for the optimal input distribution.
Lemma 5: Let

$$
\begin{equation*}
f_{\lambda}(x ; \mu)=d(x ; \mu)-I(\mu)-\lambda\left(x^{2}-\gamma\right) . \tag{54}
\end{equation*}
$$

Then $\mu_{0} \in \Lambda(\gamma, q)$ achieves the capacity if and only if there exists $\lambda \geq 0$ such that $\lambda \mathrm{E}_{\mu_{0}}\left\{X^{2}-\gamma\right\}=0$ and $\mathrm{E}_{\mu}\left\{f_{\lambda}\left(X ; \mu_{0}\right)\right\} \leq 0$ for all $\mu \in \Lambda(q)$.

Proof: Define the Lagrangian

$$
\begin{equation*}
J(\mu)=I(\mu)-\lambda \mathrm{E}_{\mu}\left\{X^{2}-\gamma\right\} \tag{55}
\end{equation*}
$$

where $\lambda$ is the Lagrange multiplier. Since $\Lambda(q)$ is a convex set and $I(\mu)<\infty$ on $\Lambda(q), \mu_{0}$ is capacityachieving if and only if there exists $\lambda \geq 0$ such that the following conditions hold [19]:
(i) $\lambda \mathrm{E}_{\mu_{0}}\left\{X^{2}-\gamma\right\}=0$;
(ii) for all $\mu \in \Lambda(q), J\left(\mu_{0}\right) \geq J(\mu)$.

Due to concavity of $I(\mu), J(\mu)$ is also concave. Condition (ii) is then equivalent to that the weak derivative $J_{\mu_{0}}^{\prime}(\mu) \leq 0$ for all $\mu \in \Lambda(q)$.

By Lemma 3, the linearity of $\mathrm{E}_{\mu}\left\{X^{2}-\gamma\right\}$ with respect to (w.r.t.) $\mu$ and Condition (i), $J_{\mu_{0}}^{\prime}(\mu)$ can be easily calculated as

$$
\begin{equation*}
J_{\mu_{0}}^{\prime}(\mu)=\mathrm{E}_{\mu}\left\{f_{\lambda}\left(X ; \mu_{0}\right)\right\} \tag{56}
\end{equation*}
$$

Therefore, Condition (ii) is equivalent to $\mathrm{E}_{\mu}\left\{f_{\lambda}\left(X ; \mu_{0}\right)\right\} \leq 0$ for all $\mu \in \Lambda(q)$. Thus Lemma 5 follows.

We call $x \in \mathbb{R}$ a point of increase of a measure $\mu$ if $\mu(O)>0$ for every open subset $O$ of $\mathbb{R}$ containing $x$. Let $S_{\mu}$ be the set of points of increase of $\mu$. Based on Lemma 5, we derive another sufficient and necessary condition for the optimal input distribution, which will be used to prove Property (c) of Theorem 1 in Section IV-C.

Lemma 6: Let

$$
\begin{equation*}
g_{\lambda}(x ; \mu)=q f_{\lambda}(0 ; \mu)+(1-q) f_{\lambda}(x ; \mu) \tag{57}
\end{equation*}
$$

Then $\mu_{0} \in \Lambda(\gamma, q)$ achieves the capacity if and only if there exists $\lambda \geq 0$ such that for every $x \in \mathbb{R}$,

$$
\begin{equation*}
g_{\lambda}\left(x ; \mu_{0}\right) \leq 0 \tag{58}
\end{equation*}
$$

Furthermore, $g_{\lambda}\left(x ; \mu_{0}\right)=0$ for every $x \in S_{\mu_{0}} \backslash\{0\}$.
Proof: The necessity part is shown as follows. Suppose $\mu_{0}$ achieves the capacity, then by Lemma 5 , there exists $\lambda \geq 0$ such that $\lambda \mathrm{E}_{\mu_{0}}\left\{X^{2}-\gamma\right\}=0$ and $\mathrm{E}_{\mu}\left\{f_{\lambda}\left(X ; \mu_{0}\right)\right\} \leq 0$ for all $\mu \in \Lambda(q)$. For any $x \in \mathbb{R} \backslash\{0\}$, choose $\mu$ such that $\mu(\{0\})=q$ and $\mu(\{x\})=1-q$, so by the fact that $\mu \in \Lambda(q)$, we have

$$
\begin{equation*}
0 \geq \mathrm{E}_{\mu}\left\{f_{\lambda}\left(X ; \mu_{0}\right)\right\}=q f_{\lambda}\left(0 ; \mu_{0}\right)+(1-q) f_{\lambda}\left(X ; \mu_{0}\right) \tag{59}
\end{equation*}
$$

Due to the continuity of $d\left(x ; \mu_{0}\right)$ by Lemma 2, $f_{\lambda}\left(x ; \mu_{0}\right)$ is also continuous so that 59) holds for all $x \in \mathbb{R}$, i.e., $g_{\lambda}\left(x ; \mu_{0}\right) \leq 0$ for every $x \in \mathbb{R}$.

To finish proving the necessity, it suffices to show that $g_{\lambda}\left(x ; \mu_{0}\right)=0$ for all $x \in S_{\mu_{0}} \backslash\{0\}$. Evidently, $g_{\lambda}\left(0 ; \mu_{0}\right)=f_{\lambda}\left(0 ; \mu_{0}\right)$ and by 17) and $\lambda \mathrm{E}_{\mu_{0}}\left\{X^{2}-\gamma\right\}=0$,

$$
\begin{equation*}
\int f_{\lambda}\left(x ; \mu_{0}\right) \mu_{0}(\mathrm{~d} x)=0 \tag{60}
\end{equation*}
$$

Hence,

$$
\begin{align*}
\int_{\mathbb{R} \backslash\{0\}} g_{\lambda}\left(x ; \mu_{0}\right) \mu_{0}(\mathrm{~d} x) & =\int g_{\lambda}\left(x ; \mu_{0}\right) \mu_{0}(\mathrm{~d} x)-g_{\lambda}\left(0 ; \mu_{0}\right) \mu_{0}(\{0\})  \tag{61}\\
& \geq q f_{\lambda}\left(0 ; \mu_{0}\right)+(1-q) \int f_{\lambda}\left(x ; \mu_{0}\right) \mu_{0}(\mathrm{~d} x)-q f_{\lambda}\left(0 ; \mu_{0}\right)  \tag{62}\\
& =0 . \tag{63}
\end{align*}
$$

Since $g_{\lambda}\left(x ; \mu_{0}\right) \leq 0$ for every $x \in \mathbb{R}$, 63) implies that on $\mathbb{R} \backslash\{0\}, g_{\lambda}\left(x ; \mu_{0}\right)=0 \mu_{0}$-almost surely, so that $g_{\lambda}\left(x ; \mu_{0}\right)=0$ for all $x \in S_{\mu_{0}} \backslash\{0\}$ follows immediately.

The sufficiency part of Lemma 6 is established as follows. Suppose $g_{\lambda}\left(x ; \mu_{0}\right) \leq 0$ for every $x \in \mathbb{R}$. By integrating $g_{\lambda}\left(x ; \mu_{0}\right)$ w.r.t. $\mu_{0}$, we have

$$
\begin{align*}
q g_{\lambda}\left(0 ; \mu_{0}\right) & \geq \int g_{\lambda}\left(x ; \mu_{0}\right) \mu_{0}(\mathrm{~d} x)  \tag{64}\\
& =q g_{\lambda}\left(0 ; \mu_{0}\right)-(1-q) \lambda \mathrm{E}_{\mu_{0}}\left\{X^{2}-\gamma\right\}  \tag{65}\\
& \geq q g_{\lambda}\left(0 ; \mu_{0}\right) \tag{66}
\end{align*}
$$

where (65) is due to (17) and $g_{\lambda}\left(0 ; \mu_{0}\right)=f_{\lambda}\left(0 ; \mu_{0}\right)$, and follows from $\mathrm{E}_{\mu_{0}}\left\{X^{2}\right\} \leq \gamma$ since $\mu_{0} \in \Lambda(\gamma, q)$. Hence, $\lambda \mathrm{E}_{\mu_{0}}\left\{X^{2}-\gamma\right\}=0$ due to the fact that $q<1$. Furthermore, for any $\mu \in \Lambda(q)$, by integrating $g_{\lambda}\left(x ; \mu_{0}\right)$ w.r.t. $\mu$, we have

$$
\begin{align*}
q g_{\lambda}\left(0 ; \mu_{0}\right) & \geq \int g_{\lambda}\left(x ; \mu_{0}\right) \mu(\mathrm{d} x)  \tag{67}\\
& =q f_{\lambda}\left(0 ; \mu_{0}\right)+(1-q) \mathrm{E}_{\mu}\left\{f_{\lambda}\left(X ; \mu_{0}\right)\right\} \tag{68}
\end{align*}
$$

Because $g_{\lambda}\left(0 ; \mu_{0}\right)=f_{\lambda}\left(0 ; \mu_{0}\right)$, we have $\mathrm{E}_{\mu}\left\{f_{\lambda}\left(X ; \mu_{0}\right)\right\} \leq 0$. Together with $\lambda \mathrm{E}_{\mu_{0}}\left\{X^{2}-\gamma\right\}=0$ and Lemma 5, this implies that $\mu_{0}$ must be capacity-achieving.

## C. Discreteness of $\mu_{0}$

With Lemma 6 established, we now prove Property (c) in Theorem 1.
Let $\lambda \geq 0$ satisfy condition (58) and $d(z ; \mu)$ be defined in (23). We extend functions $f_{\lambda}(x ; \mu)$ in Lemma 5 and $g_{\lambda}(x ; \mu)$ in Lemma 6 to be defined on the whole complex plane $\mathbb{C}$ as 54) and (57), respectively, with $x$ replaced by $z \in \mathbb{C}$. By Lemma $2, d(z ; \mu)$ is a holomorphic function of $z$ on $\mathbb{C}$,
hence so is $g_{\lambda}(z ; \mu)$. According to Lemma 6 , each element in the set $S_{\mu_{0}} \backslash\{0\}$ is a zero of the function $g_{\lambda}\left(z ; \mu_{0}\right)$.

Next we show that for any bounded interval $L$ of $\mathbb{R}, S_{\mu_{0}} \bigcap L$ is a finite set. Suppose, to the contrary, $S_{\mu_{0}} \cap L$ is infinite, then it has a limit point in $\mathbb{R}$ by the Bolzano-Weierstrass Theorem [18] and hence, $g_{\lambda}\left(z ; \mu_{0}\right)=0$ on the whole complex plane $\mathbb{C}$ by the Identity Theorem [20]. Then, by (16], (54) and (57), for every $x \in \mathbb{R}$,

$$
\begin{equation*}
\int_{-\infty}^{\infty} \phi(y-x) r(y) \mathrm{d} y=0 \tag{69}
\end{equation*}
$$

where

$$
\begin{equation*}
r(y)=\log p_{Y}\left(y ; \mu_{0}\right)+\lambda y^{2}+c \tag{70}
\end{equation*}
$$

and $c=\frac{1}{2} \log (2 \pi e)+I\left(\mu_{0}\right)-\frac{q}{1-q} d(0)-\lambda(\gamma+1)$ is a constant.
As in the proof of Lemma 2, there exist $a, b \in \mathbb{R}$ such that $\left|\log p_{Y}\left(y ; \mu_{0}\right)\right| \leq \frac{1}{2} y^{2}+a y+b$. As a result, there exist some $\alpha, \beta>0$ such that $|r(y)| \leq \alpha y^{2}+\beta$. Since the convolution of $r(y)$ and the Gaussian density is equal to the zero function by (69), $r(y)$ must be the zero function according to [10, Corollary 9]. This requires the capacity-achieving output distribution $p_{Y}\left(y ; \mu_{0}\right)$ be Gaussian, which cannot be true unless $X$ is Gaussian, which contradicts the assumption that $X$ has a probability mass at 0 . Therefore, $S_{\mu_{0}} \bigcap L$ must be a finite set for any bounded interval $L$, which further implies that $S_{\mu_{0}}$ is at most countable.

Finally, we show that $S_{\mu_{0}}$ is countably infinite. Suppose, to the contrary, $S_{\mu_{0}}=\left\{x_{i}\right\}_{i=1}^{N}$ is a finite set with $\mu_{0}\left(\left\{x_{i}\right\}\right)=p_{i}$ and $\left|x_{i}\right| \leq B_{1}$ for all $i=1,2, \ldots, N$. For any $y>B_{1}$,

$$
\begin{equation*}
p_{Y}\left(y ; \mu_{0}\right)=\sum_{i=1}^{N} p_{i} \phi\left(y-x_{i}\right) \leq e^{-\frac{\left(y-B_{1}\right)^{2}}{2}} . \tag{71}
\end{equation*}
$$

For any $\epsilon>0$, choose $B_{2}>0$ such that $\int_{-B_{2}}^{B_{2}} \phi(x) \mathrm{d} x>1-\epsilon$. By (16), (54), (57) and (58), for any
$x>B_{1}+B_{2}$, we have

$$
\begin{align*}
0 & \geq-\int_{-\infty}^{\infty} \phi(y-x) \log p_{Y}\left(y ; \mu_{0}\right) \mathrm{d} y-\lambda x^{2}-(c+\lambda)  \tag{72}\\
& \geq \int_{x-B_{2}}^{x+B_{2}} \phi(y-x) \frac{1}{2}\left(y-B_{1}\right)^{2} \mathrm{~d} y-\lambda x^{2}-(c+\lambda)  \tag{73}\\
& =\int_{B_{2}}^{B_{2}} \phi(t) \frac{1}{2}\left(x-B_{1}+t\right)^{2} \mathrm{~d} t-\lambda x^{2}-(c+\lambda)  \tag{74}\\
& \geq \frac{1}{2}\left(x-B_{1}\right)^{2}(1-\epsilon)-\lambda x^{2}-(c+\lambda) . \tag{75}
\end{align*}
$$

For (72) to hold for large $x, \lambda$ must satisfy $\lambda \geq \frac{1}{2}$.
To finish the proof, it suffices to show that $\lambda<\frac{1}{2}$ for any $\gamma>0$, so that contradiction arises, which implies that $S_{\mu_{0}}$ must be countably infinite. For fixed $q \in(0,1)$, denote the Lagrange multiplier in (58) as $\lambda(\gamma)$. Denote $C_{G}(\gamma)=\frac{1}{2} \log (1+\gamma)$, which is the channel capacity of a Gaussian channel with the average power constraint only. By the envelope theorem [19], $\lambda(\gamma)$ is the derivative of $C(\gamma, q)$ w.r.t. $\gamma$. Since $C(0, q)=C_{G}(0)=0$ and the derivative of $C_{G}(\gamma)$ at $\gamma=0$ is $\frac{1}{2}$, we have $\lambda(0) \leq \frac{1}{2}$, otherwise we could find a small enough $\gamma$ such that $C(\gamma, q)$ would exceed $C_{G}(\gamma)$ which is obviously impossible. Next we show that $C(\gamma, q)$ is strictly concave for $\gamma \geq 0$. Suppose $\mu_{1}$ and $\mu_{2}$ are the capacity-achieving input distributions of (5) for different power constraints $\gamma_{1}$ and $\gamma_{2}$, respectively. Due to Property (b) in Theorem 1, $\mu_{1}$ and $\mu_{2}$ must be different. Define $\mu_{\theta}=\theta \mu_{1}+(1-\theta) \mu_{2}$ for $\theta \in(0,1)$. It is easy to see that $\mu_{\theta}$ satisfies that the duty cycle is no greater than $1-q$ and the average input power is no greater than $\theta \gamma_{1}+(1-\theta) \gamma_{2}$. Now we have

$$
\begin{align*}
C\left(\theta \gamma_{1}+(1-\theta) \gamma_{2}, q\right) & \geq I\left(\mu_{\theta}\right)  \tag{76}\\
& >\theta I\left(\mu_{1}\right)+(1-\theta) I\left(\mu_{2}\right)  \tag{77}\\
& =\theta C\left(\gamma_{1}, q\right)+(1-\theta) C\left(\gamma_{2}, q\right) \tag{78}
\end{align*}
$$

where (77) is due to the strict concavity of $I(\mu)$. Therefore, the strict concavity of $C(\gamma, q)$ for $\gamma \geq 0$ follows, which implies that $\lambda(\gamma)<\lambda(0)=\frac{1}{2}$ for all $\gamma>0$.

## V. Proof of Theorem 2 (the Case of Realistic Duty Cycle Constraint)

## A. Stationarity of the Capacity-achieving Input Distribution

We first establish the fact that a stationary distribution achieves the capacity of the AWGN channel with the realistic duty cycle constraint and power constraint.

Proposition 1: A stationary distribution ${ }^{11}$ achieves

$$
\begin{equation*}
\max _{\mu \in \Lambda^{n}(\gamma, q, c)} I\left(X^{n} ; Y^{n}\right) . \tag{79}
\end{equation*}
$$

Proof: Let $T_{k}(\cdot)$ as a $k$-cyclic-shift operator on $\mu \in \Lambda^{n}(\gamma, q, c)$, defined as

$$
\begin{equation*}
T_{k}(\mu)=\mu_{X_{k+1}, \cdots, X_{n}, X_{1}, \cdots X_{k}} \tag{80}
\end{equation*}
$$

where $k=1, \cdots, n-1$, and specifically $T_{0}(\mu)=\mu$. For any distribution $\mu$ in $\Lambda^{n}(\gamma, q, c)$, a distribution $\nu$ on $X^{n}$ can be defined as

$$
\begin{equation*}
\nu=\frac{1}{n} \sum_{k=0}^{n-1} T_{k}(\mu) . \tag{81}
\end{equation*}
$$

According the concavity of the mutual information $I(\cdot)$,

$$
\begin{align*}
I(\nu) & =I\left(\frac{1}{n} \sum_{k=0}^{n-1} T_{k}(\mu)\right)  \tag{82}\\
& \geq \frac{1}{n} \sum_{k=0}^{n-1} I\left(T_{k}(\mu)\right)  \tag{83}\\
& =I(\mu) \tag{84}
\end{align*}
$$

where $I\left(T_{k}(\mu)\right)=I(\mu)$ since the AWGN channel (1) is a memoryless and time-invariant. Obviously $\nu$ is a stationary distribution and satisfied the duty cycle constraint and power constraint, i.e., $\nu \in \Lambda^{n}(\gamma, q, c)$, hence Proposition 1 established.
${ }^{1}$ The stationarity of distribution $\nu$ on $X^{n}$ satisfies

$$
\nu_{X_{s}, \cdots, X_{t}}=\nu_{X_{s+k}}, \cdots, X_{t+k}
$$

for any index $s, t, k$ satisfied

$$
1 \leq s \leq t \leq n \quad 1 \leq s+k \leq t+k \leq n
$$

According to Proposition 1, for any $n, I\left(X^{n} ; Y^{n}\right)$ is maximized by a stationary distribution. Therefore with $n$ converges to infinity, the capacity in 9 is achieved by a stationary input distribution.

## B. The Input-output Mutual Information

Proposition 2: Let the input follows a stationary distribution $\mu \in \Lambda(\gamma, q, c)$. The limit of the inputoutput mutual information per symbol as a function of $\mu$ can be expressed as

$$
\begin{equation*}
I(\mu)=I(X ; Y)-h(Y)+h(\mathscr{Y}) \tag{85}
\end{equation*}
$$

where $I(X ; Y)$ is the mutual information of the AWGN channel between the input $X$, which follows distribution $\mu_{X_{1}}$ and the corresponding output $Y, h(Y)$ is the differential entropy of $Y$ and $h(\mathscr{Y})$ is the differential entropy rate of output process $\left\{Y_{i}\right\}$.

Proof: The mutual information between $X^{n}$ and $Y^{n}$ can be expressed using relative entropies

$$
\begin{align*}
I\left(X^{n} ; Y^{n}\right) & =D\left(P_{Y^{n} \mid X^{n}} \| P_{Y^{n}} \mid P_{X^{n}}\right)  \tag{86}\\
& =D\left(P_{Y^{n} \mid X^{n}} \| P_{Y_{1}} \times \cdots \times P_{Y_{n}} \mid P_{X^{n}}\right)-D\left(P_{Y^{n}} \| P_{Y_{1} \times \cdots \times P_{Y_{n}}}\right)  \tag{87}\\
& =\sum_{k=1}^{n} D\left(P_{Y_{k} \mid X_{k}}\left|P_{Y_{k}}\right| P_{X^{n}}\right)-\mathrm{E}\left\{\log P_{Y^{n}}\left(Y^{n}\right)-\sum_{i=1}^{n} \log P_{Y_{i}}\left(Y_{i}\right)\right\}  \tag{88}\\
& =n I(X ; Y)-n h(Y)+h\left(Y^{n}\right) \tag{89}
\end{align*}
$$

Then

$$
\begin{align*}
I(\mu) & =\lim _{n \rightarrow \infty} \frac{1}{n} I\left(X^{n} ; Y^{n}\right)  \tag{90}\\
& =I(X ; Y)-h(Y)+\lim _{n \rightarrow \infty} \frac{1}{n} h\left(Y^{n}\right)  \tag{91}\\
& =I(X ; Y)-h(Y)+h(\mathscr{Y}) \tag{92}
\end{align*}
$$

Proposition 2 is established.
When the input is an i.i.d. random process, the output process is also i.i.d., $h(Y)=h(\mathscr{Y})$. This implies the following corollary.

Corollary 1: Among all i.i.d. distributions, the one that maximizes the mutual information under duty
cycle constraint $(q, c)$ and average power constraint $\gamma$ can be solved from the following optimization:

$$
\begin{array}{cl}
\underset{P_{X}}{\operatorname{maximize}} & I(X ; Y) \\
\text { subject to } & P_{X}(0)-2 c P_{X}(0)\left(1-P_{X}(0)\right) \leq q  \tag{93}\\
& \mathrm{E}\left\{X^{2}\right\} \leq \gamma
\end{array}
$$

In the special case of no transition cost, i.e., $c=0$, the result of (93) is equal to that of (5).

## C. Proof of Theorem 2

The mutual information expressed by (85) is hard to optimize, even if the input is restricted to Markov processes. To simply the matter, we introduce a lower bound of $I(\mu)$, which is given by $L(\mu)$ in 11.

Property (a): Using the fact that processing reduce relative entropy and $\mu$ is specified as a stationary probability distribution, we have

$$
\begin{align*}
\frac{1}{n} D\left(P_{Y^{n}} \| P_{Y_{1}} \times P_{Y_{2}} \times \cdots \times P_{Y_{n}}\right) & \leq \frac{1}{n} D\left(P_{X^{n}} \| P_{X_{1}} \times P_{X_{2}} \times \cdots \times P_{X_{n}}\right)  \tag{94}\\
& =\frac{1}{n} \sum_{k=1}^{n-1} D\left(P_{X_{k} \mid X_{k+1}^{n}} \| P_{X_{k}} \mid P_{X_{k+1}^{n}}\right)  \tag{95}\\
& =\frac{1}{n} \sum_{k=2}^{n} I\left(X_{1} ; X_{2}^{k}\right) . \tag{96}
\end{align*}
$$

Therefore

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n} D\left(P_{X^{n}} \| P_{X_{1}} \times P_{X_{2}} \times \cdots \times P_{X_{n}}\right)=I\left(X_{1} ; X_{2}^{\infty}\right) \tag{97}
\end{equation*}
$$

using the fact that the Cesáro mean of sequence $I\left(X_{1}, X_{2}^{k}\right)$ is $I\left(X_{1} ; X_{2}^{\infty}\right)$. Applying 85, 87) and 97),

$$
\begin{equation*}
L(\mu)=I(X ; Y)-I\left(X_{1} ; X_{2}^{\infty}\right) \leq I(\mu) \leq C(\gamma, q, c) \tag{98}
\end{equation*}
$$

Thus Property (a) is established.
Property (b): For any $\mu \in \Lambda(\gamma, q, c)$, which is not Markov in general, its first-order Markov approximation $\nu$ is defined by

$$
\begin{equation*}
\nu_{X_{1}, \cdots, X_{n}}=\mu_{X_{1}} \mu_{X_{2} \mid X_{1}} \mu_{X_{3} \mid X_{2}} \cdots \mu_{X_{n} \mid X_{n-1}} . \tag{99}
\end{equation*}
$$

Evidently, $\nu$ and $\mu$ have identical marginal distributions: $\nu_{X_{i}}=\mu_{X_{i}}$, and also identical joint distributions of any consecutive pairs: $\nu_{X_{i}, X_{i+1}}=\mu_{X_{i}, X_{i+1}}$. Therefore

$$
\begin{equation*}
\nu_{X_{i}}(\{0\})=\mu_{X_{i}}(\{0\}) \tag{100}
\end{equation*}
$$

and

$$
\begin{equation*}
\nu_{X_{i}, X_{i+1}}\left(\left\{x_{i}=0, x_{i+1} \neq 0\right\}\right)=\mu_{X_{i}, X_{i+1}}\left(\left\{x_{i}=0, x_{i+1} \neq 0\right\} .\right. \tag{101}
\end{equation*}
$$

Since $\mu \in \Lambda(\gamma, q, c)$, we have $\nu \in \Lambda(\gamma, q, c)$. Let $\left\{X_{i}\right\}$ follow distribution $\mu$ and $\left\{Z_{i}\right\}$ follow distribution $\nu$. Then

$$
\begin{align*}
I\left(Z_{1} ; Z_{2}^{\infty}\right) & =I\left(Z_{1} ; Z_{2}\right)+I\left(Z_{1} ; Z_{3}^{\infty} \mid Z_{2}\right)  \tag{102}\\
& =I\left(Z_{1} ; Z_{2}\right)  \tag{103}\\
& =I\left(X_{1} ; X_{2}\right)  \tag{104}\\
& \leq I\left(X_{1} ; X_{2}^{\infty}\right) \tag{105}
\end{align*}
$$

where equality holds if and only if $\left\{X_{i}\right\}$ is a first-order Markov process. By 11] and 105 , $L(\nu) \geq L(\mu)$. So for any $\mu$ which maximizes $L(\mu), \nu$ can be generated from $\mu$ by with $L(\nu) \geq L(\mu) . L(\mu)$ must be maximized by a first-order Markov process.

Property (c): Suppose $\nu$ is a stationary fist-order Markov process, sufficiently denote as $\nu=$ $\left\{\mathcal{X}, P_{X_{2} \mid X_{1}}\right\}$, where $\mathcal{X}$ is the state space of $\nu$ and $P_{X_{2} \mid X_{1}}$ is the transition probability distribution. Define a new first-order Markov process $\bar{\nu}$ from $\nu$ as follows.

Definition 1: Let $\bar{\nu}$, defined on the same state space $\mathcal{X}$ as $\nu$, be a first-order Markov process denoted by $\left(\mathcal{X}, P_{Z_{2} \mid Z_{1}}\right)$, where

$$
P_{Z_{2} \mid Z_{1}}\left(z_{2} \mid z_{1}\right)= \begin{cases}\alpha & z_{1}=0 z_{2}=0  \tag{106}\\ 1-\beta & z_{1} \neq 0 z_{2}=0 \\ \frac{1-\alpha}{\eta} P_{X}\left(z_{2}\right) & z_{1}=0 z_{2} \neq 0 \\ \frac{\beta}{\eta} P_{X}\left(z_{2}\right) & z_{1} \neq 0 z_{2} \neq 0\end{cases}
$$

where

$$
\begin{equation*}
S_{1}=\mathcal{X} \backslash\{0\} \tag{107}
\end{equation*}
$$

and

$$
\begin{align*}
& \alpha=P_{X_{2} \mid X_{1}}(0 \mid 0)  \tag{108}\\
& \beta=P\left(X_{2} \in S_{1} \mid X_{1} \in S_{1}\right)  \tag{109}\\
& \eta=P\left(X \in S_{1}\right) . \tag{110}
\end{align*}
$$

The process $\bar{\nu}$ is described by $\left(\mathcal{X}, \alpha, \beta, P_{X}\right)$. It is easy to prove that the stationary distribution $P_{Z}$ of $\bar{\nu}$ is equal to $P_{X}$ of $\nu, \bar{\nu} \in \Lambda(\gamma, q, c)$. Moreover, $\bar{\nu}$ satisfies the same power and duty cycle constraint $\nu$ satisfies, i.e., $\bar{\nu} \in \Lambda(\gamma, q, c)$. Furthermore let $B_{i}=1_{\left\{X_{i} \neq 0\right\}}$, then

$$
\begin{align*}
& P_{B_{2} \mid B_{1}}(0 \mid 0)=\alpha  \tag{111}\\
& P_{B_{2} \mid B_{1}}(1 \mid 1)=\beta . \tag{112}
\end{align*}
$$

Let $b_{i}=1_{\left\{z_{i} \neq 0\right\}}$. Since

$$
\begin{equation*}
P_{Z_{2} \mid Z_{1}}\left(z_{2} \mid z_{1}\right)=P_{B_{2} \mid B_{1}}\left(b_{1} \mid b_{2}\right) \frac{P_{X}\left(z_{2}\right)}{P_{B_{2}}\left(b_{2}\right)}, \tag{113}
\end{equation*}
$$

$Z_{i}$ and $Z_{i+1}$ are independent given $B_{i}=1_{\left\{Z_{i} \neq 0\right\}}$ and $B_{i+1}=1_{\left\{Z_{i+1} \neq 0\right\}}$.
Based on 106 to (113), it is easy to see that

$$
\begin{align*}
I\left(Z_{1} ; Z_{2}\right) & =\mathrm{E}\left\{\log \frac{P_{Z_{2} \mid Z_{1}}\left(Z_{2} \mid Z_{1}\right)}{P_{Z_{2}}\left(Z_{2}\right)}\right\}  \tag{114}\\
& =\mathrm{E}\left\{\log \frac{P_{B_{2} \mid B_{1}}\left(B_{2} \mid B_{1}\right)}{P_{B_{2}}\left(B_{2}\right)}\right\}  \tag{115}\\
& =I\left(B_{1} ; B_{2}\right)  \tag{116}\\
& \leq I\left(X_{1} ; X_{2}\right) . \tag{117}
\end{align*}
$$

The inequality in 117) follows since $X_{1} \rightarrow X_{2} \rightarrow B_{2}$ forms a Markov chain then $I\left(X_{1} ; B_{2}\right) \leq I\left(X_{1} ; X_{2}\right)$ [21] and $B_{2} \rightarrow X_{1} \rightarrow B_{1}$ also forms a Markov chain then $I\left(B_{2} ; B_{1}\right) \leq I\left(B_{2} ; X_{1}\right)$.

The discreteness of the optimized input distribution is proved in the following. According to Properties (b) and (c), lower bound $L(\cdot)$ is maximized by a first-order Markov process, the transition probability distribution of which $P_{X_{2} \mid X_{1}}$ can be expressed as

$$
\begin{equation*}
P_{X_{2} \mid X_{1}}\left(x_{2} \mid x_{1}\right)=P_{B_{2} \mid B_{1}}\left(b_{2} \mid b_{1}\right) \frac{P_{X}\left(x_{2}\right)}{P_{B_{2}}\left(b_{2}\right)} \tag{118}
\end{equation*}
$$

where $b_{i}=1_{\left\{x_{i} \neq 0\right\}} P_{X}=\mu_{X}$ and $P_{X_{2} \mid X_{1}}=\mu_{X_{2} \mid X_{1}}$. Then the maximum of $L(\mu)$ can be achieved by the follow optimization

$$
\begin{array}{cc}
\underset{q_{0}}{\operatorname{maximize}} & I_{X}\left(q_{0}\right)-I_{B}\left(q_{0}\right) \\
\text { subject to } & I_{X}\left(q_{0}\right)=\underset{P_{X}}{\operatorname{maximize}} I(X ; Y) \\
& I_{B}\left(q_{0}\right)=\underset{P\left(B_{2} \mid B_{1}\right)}{\operatorname{minimize}} I\left(B_{1} ; B_{2}\right) \\
& P_{X}(0)=P_{B_{1}}(0)=P_{B_{2}}(0)=q_{0} \\
& q_{0}-2 c q_{0} P_{B_{2} \mid B_{1}}(1 \mid 0) \geq q \tag{123}
\end{array}
$$

Since given any $q_{0} \geq q>0, I_{X}\left(q_{0}\right)-I_{B}\left(q_{0}\right)$ can be maximized by the maximum of $I_{X}\left(q_{0}\right)$ and the minimum of $I_{B}\left(q_{0}\right)$ respectively, the maximization of 119 must be achieved by $P_{X}$, which maximizes $I(X ; Y)$ for given $q_{0}$. Therefore given $q_{0}$, the maximization in 120 is similar to the problem in Theorem 1. The difference to Theorem 1 is that in 120 the distribution $P_{X}$ satisfies $P_{X}(0)=q_{0} \geq q$, however in Theorem 1 the distribution $P_{X}$ satisfies $P_{X}(0) \geq q$. Define

$$
\begin{equation*}
\Lambda_{0}\left(\gamma, q_{0}\right)=\left\{\mu: \mu(\{0\})=q_{0}, \mathrm{E}_{\mu}\left\{X^{2}\right\} \leq \gamma\right\} \tag{124}
\end{equation*}
$$

where $\mu$ is the marginal input distribution of the first-order Markov process. We can establish the following lemma.

Lemma 7: $\Lambda_{0}\left(\gamma, q_{0}\right)$ is compact in the topological space $\mathcal{P}$.

Proof: As mentioned in Lemma 1, the topology of weak convergence on $\mathcal{P}$ is metrizable with the Lévy-Prohorov metric [15] and defined as

$$
\begin{align*}
& L(\mu, \nu)=\inf \left\{\delta: \mu(A) \leq \nu\left(A^{(\delta)}\right)+\delta\right. \text { and } \\
& \left.\qquad \nu(A) \leq \mu\left(A^{(\delta)}\right)+\delta \text { for all } A \subseteq \mathcal{B}\right\} \tag{125}
\end{align*}
$$

for any $\mu, \nu \in \mathcal{P}$, where $A^{(\delta)}$ denotes the set of all $x \in \mathbb{R}$ which lie a $d$-distance less than $\delta$ from $A$.
Similarly as in the proof of Lemma 1, it suffices to show that $\Lambda_{0}\left(\gamma, q_{0}\right)$ is both tight and closed in $\mathcal{P}$. The tightness can be shown by the same arguments as in Lemma 1. In the following, we prove that $\Lambda_{0}\left(\gamma, q_{0}\right)$ is closed in $\mathcal{P}$.

Let $B_{m}=\left[-\frac{1}{m}, \frac{1}{m}\right]$ for $m=1,2, \ldots$ Let $\left\{\mu_{n}\right\}_{n=1}^{\infty}$ be a convergent sequence in $\Lambda_{0}\left(\gamma, q_{0}\right)$ with limit $\mu_{0}$. For any $m \in \mathbb{N}$, there exists an $n_{m}$ such that $L\left(\mu_{n}, \mu_{0}\right)<\frac{1}{m}$ for all $n>n_{m}$. By the definition of $L$ in (125), we have for any $m \in \mathbb{N}$ and $n>n_{m}$,

$$
\begin{equation*}
\mu_{0}(\{0\}) \leq \mu_{n}\left(B_{m}\right)+\frac{1}{m} \tag{126}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu_{n}(\{0\}) \leq \mu_{0}\left(B_{m}\right)+\frac{1}{m} \tag{127}
\end{equation*}
$$

For any $n \in \mathbb{N} \bigcup\{0\}$, we have

$$
\begin{equation*}
\mu_{n}(\{0\})=\mu_{n}\left(\bigcap_{m=1}^{\infty} B_{m}\right)=\lim _{m \rightarrow \infty} \mu_{n}\left(B_{m}\right) \tag{128}
\end{equation*}
$$

so for any $m \in \mathbb{N}$, there exists an $n_{m}^{\prime}$ such that $\mu_{n}\left(B_{m}\right) \leq \mu_{n}(\{0\})+\frac{1}{m}$. Therefore, according to (126) and (127), for all $m \in \mathbb{N}$ and $n>\max \left\{n_{m}, n_{m}^{\prime}\right\}$,

$$
\begin{equation*}
q_{0}-\frac{2}{m} \leq \mu_{0}(\{0\}) \leq q_{0}+\frac{2}{m} \tag{129}
\end{equation*}
$$

Thus we have $\mu_{0}(\{0\})=q_{0}$ by letting $m \rightarrow \infty$.
Moreover, let $f(x)=x^{2}$ which is continuous and bounded below. By weak convergence [15, Section 3.1], we have

$$
\begin{equation*}
\mathrm{E}_{\mu_{0}}\left\{X^{2}\right\}=\int f \mathrm{~d} \mu_{0} \leq \liminf _{n \rightarrow \infty} \int f \mathrm{~d} \mu_{n} \leq \gamma \tag{130}
\end{equation*}
$$

Together with $\mu_{0}(\{0\})=q_{0}$, we have $\mu_{0} \in \Lambda_{0}\left(\gamma, q_{0}\right)$, i.e., $\Lambda_{0}\left(\gamma, q_{0}\right)$ is closed, and the compactness of $\Lambda_{0}\left(\gamma, q_{0}\right)$ then follows.

Now $P_{X}$ can be proved to be discrete by following the same development as in the proof of Theorem 1 with Lemma 1 substituted by Lemma 7. Because $P_{X}$ is the stationary distribution of the Markov process, the maximum of the lower bound $L(\cdot)$ is achieved by a discrete first-order Markov process.

Based on Theorem 2, in order to find the lower bound of the capacity, we can maximize $L(\mu)$ and obtain an optimized discrete first-order Markov input $\mu^{*}=\left\{\mathcal{X}, \alpha, \beta, P_{X}\right\}$ in $\Lambda(\gamma, q, c)$. Let $\mu_{0}$ denote the capacity-achieving distribution, then

$$
\begin{equation*}
I\left(\mu_{0}\right) \geq I\left(\mu^{*}\right) \geq L\left(\mu^{*}\right) \tag{131}
\end{equation*}
$$

In Section VI-A, we develop a computationally efficient scheme to determine $\mu^{*}$, which is a good approximation of the capacity-achieving input $\mu_{0}$.

## VI. NUMERICAL METHODS AND RESULTS

## A. Computation of the entropy of Hidden Markov Processes

In order to numerically calculate the mutual information (85), it is important to compute the differential entropy rate of a HMP generated by Markov input through the AWGN channel. Computing the (differential) entropy rate of HMPs is a hard problem. Most works in this area focus on the entropy rate of the binary Markov input through various channels. Reference [22] solves a linear system for the stationary distribution of the quantized Markov process to obtain a good approximation of the entropy rate for the HMP output generated by binary Markov input through a binary symmetric channel. In [23], the entropy rate of HMP generated by binary-symmetric Markov input through arbitrary memoryless channels is studied and a numerical method is presented based on quantizing a fixed-point functional equation. Based on these existing studies, a Monte Carlo algorithm is provided in this paper to compute the differential entropy rate of HMPs generated from a $m$-state Markov chain ( $m \geq 3$ ) through the AWGN channel. We sketch the main ideas in our algorithm for computing the differential entropy rate in this subsection.

Based on Blackwell's work [24], the entropy of HMPs can be expressed as an expectation on the distribution of the conditional distribution of $X_{0}$ given the past observations $Y_{-\infty}^{0}$. In order to estimate
$P_{X_{0} \mid Y_{-\infty}^{0}}$, first define the log-likelihood ratio:

$$
\begin{equation*}
L_{n}^{(i)}=\log \frac{P_{X_{n} \mid Y^{n}}\left(X^{(i)} \mid Y^{n}\right)}{P_{X_{n} \mid Y^{n}}\left(X^{(0)} \mid Y^{n}\right)}, \quad i=0,1, \cdots, m-1 \tag{132}
\end{equation*}
$$

where $m$ is the number of the states of Markov Chain, $X^{(i)} \in \mathcal{X}$ is the $i$ th state and $\mathcal{X}$ is the state space of Markov Chain. It is obviously that $L_{n}^{(0)} \equiv 0$. Then given $L_{n}=\left\{L_{n}^{(0)}, L_{n}^{(1)}, \ldots, L_{n}^{(m-1)}\right\}$, $P_{X_{n} \mid Y^{n}}\left(X_{n} \mid Y^{n}\right)$ can be calculated as

$$
\begin{equation*}
P_{X_{n} \mid Y^{n}}\left(X^{(i)} \mid Y^{n}\right)=\frac{e^{L_{n}^{(i)}}}{\sum_{i=0}^{m-1} e^{L_{n}^{(i)}}} \tag{133}
\end{equation*}
$$

and when $n \rightarrow \infty, 133$ converges to $P_{X_{0} \mid Y_{-\infty}^{0}}\left(X^{(i)} \mid Y_{-\infty}^{0}\right)$.
In addition, $L_{n+1}^{(i)}$ can be calculated from $\boldsymbol{L}_{n}$ iteratively as

$$
\begin{equation*}
L_{n+1}^{(i)}=R^{(i)}\left(Y_{n+1}\right)+F^{(i)}\left(\boldsymbol{L}_{n}\right) \tag{134}
\end{equation*}
$$

where

$$
\begin{align*}
R^{(i)}\left(Y_{n+1}\right) & =\left(X^{(i)}-X^{(0)}\right) Y_{n+1}-\frac{1}{2}\left(\left(X^{(i)}\right)^{2}-\left(X^{(0)}\right)^{2}\right)  \tag{135}\\
F^{(i)}\left(\boldsymbol{L}_{\boldsymbol{n}}\right) & =\log \frac{\sum_{k=0}^{m-1} P_{X_{2} \mid X_{1}}\left(X^{(i)} \mid X^{(k)}\right) e^{L_{n}^{(k)}}}{\sum_{k=0}^{m-1} P_{X_{2} \mid X_{1}}\left(X^{(0)} \mid X^{(k)}\right) e^{L_{n}^{(k)}}} \tag{136}
\end{align*}
$$

Detail deduction of 134 is shown in 137

$$
\begin{align*}
L_{n+1}^{(i)} & =\log \frac{P_{X_{n+1} \mid Y_{1}^{n+1}}\left(X^{(i)} \mid Y_{1}^{n+1}\right)}{P_{X_{n+1} \mid Y_{1}^{n+1}}\left(X^{(0)} \mid Y_{1}^{n+1}\right)}  \tag{137}\\
& =\log \frac{\sum_{k=0}^{m-1} P_{Y_{n+1} \mid X_{n+1}}\left(Y_{n+1} \mid X^{(i)}\right) P_{X_{2} \mid X_{1}}\left(X^{(i)} \mid X^{(k)}\right) P_{X_{n} \mid Y_{1}^{n}}\left(X^{(k)} \mid Y_{1}^{n}\right)}{\sum_{k=0}^{m-1} P_{Y_{n+1} \mid X_{n+1}}\left(Y_{n+1} \mid X^{(0)}\right) P_{X_{2} \mid X_{1}}\left(X^{(0)} \mid X^{(k)}\right) P_{X_{n} \mid Y_{1}^{n}}\left(X^{(k)} \mid Y_{1}^{n}\right)}  \tag{138}\\
& =\log \frac{P_{Y_{n+1} \mid X_{n+1}}\left(Y_{n+1} \mid X^{(i)}\right)}{P_{Y_{n+1} \mid X_{n+1}}\left(Y_{n+1} \mid X^{(0)}\right)}+\log \frac{\sum_{k=0}^{m-1} P_{X_{2} \mid X_{1}}\left(X^{(i)} \mid X^{(k)}\right) e^{L_{n}^{(k)}}}{\sum_{k=0}^{m-1} P_{X_{2} \mid X_{1}}\left(X^{(0)} \mid X^{(k)}\right) e^{L_{n}^{(k)}}}  \tag{139}\\
& =R^{(i)}\left(Y_{n+1}\right)+F^{(i)}\left(\boldsymbol{L}_{n}\right) \tag{140}
\end{align*}
$$

For the hidden Markov processes observed through the AWGN channel 11 , the entropy of HMPs can be computed as 24]

$$
\begin{equation*}
h(\mathscr{Y})=\lim _{n \rightarrow \infty}-\iint r\left(y, \boldsymbol{l}_{n}\right) \log r\left(y, \boldsymbol{l}_{n}\right) d y d P_{\boldsymbol{L}_{n}}\left(\boldsymbol{l}_{n}\right) \tag{141}
\end{equation*}
$$

where

$$
\begin{equation*}
r\left(y, \boldsymbol{L}_{n}\right)=\sum_{i=0}^{m-1} \phi\left(y-x^{(i)}\right) \sum_{k=0}^{m-1} \frac{e^{L_{n}^{(i)}}}{\sum_{i=0}^{m-1} e^{L_{n}^{(i)}}} P_{X_{2} \mid X_{1}}\left(x^{(i)} \mid x^{(k)}\right) \tag{142}
\end{equation*}
$$

In order to compute the entropy rate of HMPs based on 141 , the key is to estimate the probability distribution of $\boldsymbol{L}_{n}, P_{\boldsymbol{L}_{n}}$. In [22] for binary Markov input and the binary symmetric channel, $\boldsymbol{L}_{n}$ is considered as a 1 -dim $M$-state Markov chain by quantizing the dynamic system expressed in (134). Then the distribution of $\boldsymbol{L}_{\infty}$ is the stationary distribution of the quantized Markov process and can be computed easily through eigenvector solving method. In this paper because the number of states of the Markov input, $m$ is larger than 2 and the HMPs is observed through the AWGN channel, directly quantizing the dynamic system will generate a quantized Markov chain with $M^{m-1}$ states, which is very difficult to deal with when large $M$ is selected for good estimation precision.

According to 134 , since $\boldsymbol{L}_{n+1}$ is only dependent on $\boldsymbol{L}_{n}$ and $Y_{n+1},\left\{\boldsymbol{L}_{n}\right\}$ can be considered as a Markov process. In order to compute the stationary probability distribution $P_{\boldsymbol{L}_{\infty}}$, we can evolve the distribution of $\boldsymbol{L}_{n}$ based on 134 from any initial distribution $P_{\boldsymbol{L}_{0}}$. When $n$ is large enough, the distribution $P_{\boldsymbol{L}_{n}}$ converges to $P_{\boldsymbol{L}_{\infty}}$. A Monte Carlo algorithm for approximating $h(\mathscr{Y})$ is introduced as follows:

1) Initialize $M$ particles $\left\{\boldsymbol{L}_{0,1}, \cdots, \boldsymbol{L}_{0, M}\right\}, \boldsymbol{L}_{0, k}$ can be simply sampled from the ( $m-1$ )-dim Uniform distribution with each dimension on $\left[-\max \left(X^{(i)}\right), \max \left(X^{(i)}\right)\right]$.
2) for $n=0,1,2, \cdots, N$, iteratively evolve the particles $\left\{\boldsymbol{L}_{0,1}, \cdots, \boldsymbol{L}_{0, M}\right\}$ based on 134$\}$, where each $y_{n+1, k}$ is sampled according to $r\left(y, \boldsymbol{L}_{n, k}\right)$.
3) when $N$ is large enough, $\left\{\boldsymbol{L}_{N, k}\right\}$ can be used to estimate $h(\mathscr{Y})$ as

$$
\begin{equation*}
h(\mathscr{Y}) \approx-\frac{1}{M} \sum_{k=1}^{M} \int r\left(y, \boldsymbol{L}_{N, k}\right) \log r\left(y, \boldsymbol{L}_{N, k}\right) d y \tag{143}
\end{equation*}
$$

When $M$ is very large, histogram method can be used to describe $\left\{L_{N, k}\right\}$ and reduce the computational load.

## B. Numerical Results

1) Idealized duty cycle constraint $(q, 0)$ : One implication of Theorem 1 is that directly computing the capacity-achieving input distribution requires solving an optimization problem with infinite variables
which is prohibitive. Assuming any upper bound on the number of probability mass points, however, a numerical optimization over the mutual information can yield a suboptimal input distribution and a lower bound on the channel capacity. As we increase the number of mass points, the lower bound can be further refined. We take this approach to numerically compute a good approximation of the channel capacity by optimizing over a sufficient number of probability mass points.

Given the duty cycle and power constraints, we first numerically optimize the mutual information by a 3-point input distribution (including a mass at 0 ), then increase the number of probability mass points by 2 at a time to improve the mutual information, until the improvement is less than $10^{-3}$.

First consider the case that the duty cycle is no greater than $70 \%$, i.e., $P(X=0) \geq q=0.3$. For different SNRs, the mass points of the near-optimal input distribution with finite support along with the corresponding probability masses are shown in Fig. 1. Due to symmetry, only the positive half of the input distribution is plotted. We can see that as the SNR increases, more masses are put on higher-amplitude points, whereas the probability mass at zero achieves its lower bound 0.3 eventually.

In Fig 2, we compare the rate achieved by the near-optimal input distribution and the rate achieved by a conventional scheme using Gaussian signaling over a deterministic schedule, which is $(1-q)$ times the Gaussian channel capacity without duty cycle constraint. It is shown in the figure that there is substantial gain for both 0 dB and 10 dB SNRs by using discrete input over Gaussian signaling with a deterministic schedule. For example, when the SNR is 10 dB , given the duty cycle is no more than $50 \%$, the discrete input distribution achieves $50 \%$ higher rate. Hence departing from the usual paradigm of intermittent packet transmissions may yield significant gains.

We also plot in Fig 2 the achievable rate by a superposition coding, where the input distribution is a mixture of Gaussian and a point mass at 0 . We first decode the support of the input to find out the positions of nonzero symbols, and then the Gaussian codeword conditioned on the support. It is shown in the figure that the near-optimal discrete input achieves higher rate compared with the mixture input.
2) Realistic duty cycle constraint $(q, c)$ : In this subsection the numerical results of lower bound of capacity and suboptimal distribution are provided based on the results in Section V and VI-A.

We first seek a discrete Markov chain with finite alphabet that maximizes the objective $L(\mu)$ defined in (11). Once the optimal Markov distribution $\mu^{*}$ is determined, we compute the achievable rate $I\left(\mu^{*}\right)$ according to 85).


Fig. 1. Suboptimal input distribution for $P(X=0) \geq q=0.3$.

|  |  | $X_{2}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 0.0000 | 3.9281 | -3.9281 | 7.1398 | -7.1398 |
| $X_{1}$ | 0.0000 | 0.8342 | 0.0605 | 0.0605 | 0.0224 | 0.0224 |
|  | 3.9281 | 0.4923 | 0.1852 | 0.1852 | 0.0687 | 0.0687 |
|  | -3.9281 | 0.4923 | 0.1852 | 0.1852 | 0.0687 | 0.0687 |
|  | 7.1398 | 0.4923 | 0.1852 | 0.1852 | 0.0687 | 0.0687 |
|  | -7.1398 | 0.4923 | 0.1852 | 0.1852 | 0.0687 | 0.0687 |
|  | $P_{X}$ | 0.7481 | 0.0919 | 0.0919 | 0.0341 | 0.0341 |
| TABLE I |  |  |  |  |  |  |

In this paper $\mu^{*}=\left(\mathcal{X}, \alpha, \beta, P_{X}\right)$ is used to approximate the optimum distribution $\mu_{0}$ through the maximizing $L(\cdot)$. It is obvious that the optimized $\mu^{*}$ is symmetric about 0 . Table $\mathbb{I}$ is the transition


Fig. 2. Achievable rates under duty cycle constraint for 0 dB and 10 dB SNRs.
probability matrix $P_{X_{2} \mid X_{1}}$ and stationary probability $P_{X}$ for $q=0.5, c=1.0$ and $\operatorname{SNR}=8 \mathrm{~dB}$. The symmetry of the transition probability matrix is evident, as conditioned on that two consecutive symbols are nonzero, they are independent.

Fig. 3 ] shows the stationary (marginal) distribution for suboptimal Markov input. In order to compensate the transition cost, additional fraction of zero symbol should be transmitted, $P_{X}(0)>q$. As the SNR increases, more and more weights are put on distant constellation points, where less and less weights are put on the zero letter.

In Fig. 4, the rates achieved by various optimized input distributions are plotted against the SNR. The rate achieved by the optimized Markov input is larger than that of suboptimal i.i.d. input calculated by


Fig. 3. The marginal distribution of the stationary Markov input. Duty cycle $\leq 0.5$, transition cost $c=1.0$.
formula (93) with duty cycle constraint $(q, c)$. The lower bound $L(\mu)$ is quite tight and can be regarded as a good approximation of mutual information of first-order Markov inputs.

Figs. 5 and 6 demonstrate the sensitivity of the achievable rates to the duty cycle parameter $q$ and the transition cost $c$, respectively. The performance of Markov inputs is superior to i.i.d. inputs as well as Gaussian signaling with deterministic schedule. Fig 5 shows that the performance of i.i.d. input is similar to the deterministic schedule, which implies that different from the case under the idealized duty cycle constraint, i.i.d. input is not a good choice under the realistic duty cycle constraint.


Fig. 4. The achievable rate vs. the SNR. Duty cycle $\leq 0.5$, transition cost $c=1.0$.

## VII. Concluding Remarks

In this paper we have studied the impact of duty cycle constraint on the capacity of AWGN channels. Under the idealize duty cycle constraint, the optimal distribution has an infinite number of probability mass points in a bounded interval. This allows efficient numerical optimization of the input distribution. Under the realistic duty cycle constraint, the capacity-achieving input is hard to compute. We develop techniques for computing a near-optimal input distribution. This input takes the form of a discrete firstorder Markov process, which matches the "Markov" nature of the duty cycle constraint. The numerical results show that under the duty cycle constraint, departing from the usual paradigm of intermittent packet transmissions may yield substantial gain.


Fig. 5. The achievable rate vs. the duty cycle, $\mathrm{SNR}=10 \mathrm{~dB}$ and transition $\operatorname{cost} c=1.0$.

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Fig. 6. The achievable rate vs. transition cost, with $\operatorname{SNR}=10 \mathrm{~dB}$ and $q=0.5$.
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