# On the Capacity of the Interference Channel with a Cognitive Relay 

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#### Abstract

The InterFerence Channel with a Cognitive Relay (IFC-CR) consists of the classical interference channel with two independent source-destination pairs whose communication is aided by an additional node, referred to as the cognitive relay, that has a priori knowledge of both sources' messages. This a priori message knowledge is termed cognition and idealizes the relay learning the messages of the two sources from their transmissions over a wireless channel. This paper presents new inner and outer bounds for the capacity region of the general memoryless IFC-CR that are shown to be tight for a certain class of channels. The new outer bound follows from arguments originally devised for broadcast channels among which Sato's observation that the capacity region of channels with non-cooperative receivers only depends on the channel output conditional marginal distributions. The new inner bound is shown to include all previously proposed coding schemes and it is thus the largest known achievable rate region to date. The new inner and outer bounds coincide for a subset of channel satisfying a strong interference condition. For these channels there is no loss in optimality if both destinations decode both messages. This result parallels analogous results for the classical IFC and for the cognitive IFC and is the first known capacity result for the general IFC-CR. Numerical evaluations of the proposed inner and outer bounds are presented for the Gaussian noise case.


Index Terms-Capacity; Inner bound; Interference channel with a cognitive relay; Outer bound; Strong interference; Weak interference;

## I. Introduction

THE information theoretic study of cognitive networks networks in which a subset of the nodes has a priori knowledge of the messages of other subsets of nodes - has focused mostly on the two user Cognitive InterFerence Channel (CIFC), i.e., a variation of the classical two-user IFC where one of the transmitters has non-causal a priori knowledge of both messages to be transmitted. While idealistic, this form of genie-aided cognition has provided significant insights of the rate advantages obtainable through asymmetric or unilateral transmitter cooperation (please refer to [1] and [2], and references therein, for an extensive summary of available results for the general and Gaussian CIFC, respectively).

In this paper we study a natural extension of the CIFC where the genie-aided cognition, instead of being provided to only one of the sources of the IFC, is rather provided to a third node, referred to a the cognitive relay, that aids the communication between both source-destination pairs. One of the key challenges of this model is the issue of interference
management at the cognitive relay. Unlike in the Broadcast Channel (BC) and the CIFC, the cognitive relay in an IFC-CR has knowledge of the interference seen at each destination but has no control over the interfering signals that are sent by the sources. Gel'fand-Pinsker binning [3], or Dirty Paper Coding (DPC) for Gaussian channels [4], is a celebrated well-known technique used to mitigate interference known non-causally at a source through proper pre-coding of the message. This strategy is known to be capacity achieving for certain classes of BCs and CIFCs. In the IFC-CR, the cognitive relay can only manage the interference experienced by the destinations through its own transmissions, begging the question of how this single transmission may best be used to simultaneously aid both source-destination pairs.

The IFC-CR model encompasses many previously studied multi-terminal networks as special cases: the BC, the classical IFC and the CIFC, none of whose capacity is known in general. The generality of the IFC-CR model suggests a certain level of complexity in the analytical results, but also allows one to study whether and how results available for smaller networks may be incorporated into larger networks. For instance, the derivation of inner and outer bounds for the general memoryless IFC-CR carefully combines ideas developed for simpler networks, such as Gel'fand-Pinsker binning and genieaided outer bounds, adjusted to this more general network setting. We seek to determine whether these extensions of previously proposed techniques to our more general channel is sufficient to achieve capacity (we answer this in the positive for a subset of the strong interference regime) or whether our model is sufficiently different such that it requires new transmission techniques to achieve capacity.

## A. Past Work

The information theoretic capacity of the general memoryless IFC-CR remains an open problem for the general case. The IFC-CR was initially considered in [5] where the first achievable rate region was proposed, and was later improved upon in [6] for the Single-Input Single-Output (SISO) Gaussian channel. The authors of [6] also provided a sum-rate outer bound for the Gaussian channel based on an outer bound for the Multiple-Input Multiple-Output (MIMO)

Gaussian CIFC. In [7] a general achievable rate region was derived that contains all previously known achievable rate regions in [5], [6]. The first outer bound for a general (i.e. not necessarily Gaussian) IFC-CR was derived in [8] by using Sato's observation that the capacity region of channels with non-cooperative receivers depends only on the conditional marginal distribution of the channel outputs [9]. This general Sato-type outer bound was further tightened in [8] for a class of semi-deterministic channels in the spirit of [10]. For the special case where the sources do not interfere at the nonintended destinations, the tightened bound of [8] was shown to be capacity for the deterministic approximation of the Gaussian IFC-CR at high-SNR [11] and to be optimal to within $3 \mathrm{bits} / \mathrm{sec} / \mathrm{Hz}$ for any finite SNR [12]. Furthermore, for a subset of parameters akin to the weak interference regime for the classical IFC, the tightened bound of [8] was shown to be capacity for the general deterministic approximation of the Gaussian IFC-CR at high-SNR; the achievability in this case suggests an interesting transmission strategy where the cognitive relay is able to "pre-cancel" the interference at both destinations simultaneously.

The channel model under consideration in this work is closely related to the interference relay channel: an IFC with an additional relay node which does not have a priori knowledge of the sources' messages, but rather learns these messages over the noisy channel between the sources and the relays [13]. Although more realistic than the IFC-CR considered here, the interference relay channel is harder to study due to the causal cognition. Recently new results were derived for the interference relay channel where the relay is assumed to operate out-of-band [14], [15], i.e., a model in which the link between the relay and the destinations does not interfere with the underlaying IFC between the sources and the destinations; in this case, capacity is known to 1.15 bits/s/Hz in the symmetric Gaussian noise case [14].

The IFC-CR subsumes several well studied channel models as special case. The CIFC 2 that is, an IFC in which one transmitter has non-causal a-priori knowledge of the messages of both transmitters, may be obtained from the IFC-CR by eliminating the channel input of one of the sources. The CIFC was first considered from an information theoretic perspective in [16], where the channel was formally defined and the first achievable rate region was obtained. The largest known achievable rate region is due to Rini et al. [19], [1] and the tightest outer bound to Maric et al. [20]. Capacity has been established for channels with "very weak interference" in which (in Gaussian noise) treating interference at the primary receiver as noise is optimal [18], [21], for the "very strong interference" regime, where without loss of optimality both receivers can decode both messages and the cognitive channel reduces to a compound Multiple Access Channel (MAC) [17], for the "better cognitive decoding" regime [22], [2] where the

[^0]cognitive receiver can decode both messages without loss of optimality, for the semi-deterministic CIFC [23], [1] where a BC-type coding scheme is optimal, and for certain Gaussian CIFC without interference at the primary decoder [24], [2]. For the general Gaussian CIFC capacity is known to within $1 \mathrm{bit} / \mathrm{s} / \mathrm{Hz}$ and to within a factor 2 regardless of channel parameters [23], [2], [25].

The classical BC can be obtained from the IFC-CR by eliminating the channel inputs of both sources. The capacity of the general BC is unknown. The largest known achievable rate region is due to Marton [26] and the tightest outer bound to Nair and El Gamal [27]. In all cases where capacity is known Marton's region is optimal (see [28] and references therein for an extensive discussion of all cases where capacity is known and for the challenges in determining capacity in the open cases). Many techniques originally developed for the BC will prove useful for the derivations in this work.

Finally, the classical IFC can be obtained from the IFCCR by eliminating the channel input of the cognitive relay. The largest known achievable rate region is due to Han and Kobayashi [29], which is optimal in all cases where capacity is known (see [30] and references therein for an extensive discussion of all cases where capacity is known). In Gaussian noise, capacity is known only in strong interference [31], [32], [33] and known otherwise to within $1 \mathrm{bit} / \mathrm{s} / \mathrm{Hz}$ [34]. Some techniques originally developed for the IFC, such as rate splitting and simultaneous decoding, will be adapted to the IFC-CR model in this work.

## B. Paper Main Contributions

In this paper we determine:

1) Outer Bound:
a) Sato-type outer bound.

This outer bound uses Sato's observation [9] that the capacity of a channel with non-cooperative receivers only depends on the channel output conditional marginal distributions. This bound does not contain any auxiliary random variables and is thus computable in principle by determining the optimal distribution of the channel inpus .
b) BC-type outer bound.

This outer bound generalizes the tightest known outer bound for the general CIFC by Maric et al. [20] to the general IFC-CR. It uses a technique originally developed to prove the converse for the "more capable" BC in [35] and later generalized to obtain an outer bound for the general BC in [27]. This BC-type outer bound is the tightest known to date for the general IFC-CR. It is however expressed as a function of three auxiliary random variables for which no cardinality bound exists on the corresponding alphabets.
c) A simplification of the BC-type outer bound in the "strong interference" and "weak interference" regimes. The "strong interference" regime is defined as the regime where, loosely speaking, the non-intended destination can decode more information than the intended destination even after having removed the interfering signal. This regime parallels the "strong interference" regime for the IFC [36] and for the CIFC [37].

The "weak interference" regime is defined as the regime in which, loosely speaking, treating interference as noise is optimal. This regime parallels the "weak interference" regime for the IFC in [38], [39], [30] and for the CIFC in [18].
2) Inner Bound:
a) Largest known inner bound.

Our inner bound is shown to include all previously proposed inner bounds as special cases. This region equals the capacity region when the channel reduces to a simpler model (i.e. BC, IFC and CIFC ) for which capacity is known. The novel ingredients are a rate-split in four parts of the source messages and a very structured nesting of superposition and binning. Although the expression of the inner bound is rather involved, it provides a unifying framework to evaluate the effect of different transmission strategies on the achievable rate region.
b) The Fourier-Motzkin elimination of the proposed inner bound in several sub-cases.
The Fourier-Motzkin elimination of our general inner bound region appears difficult to reduce to a manageable number of rate bounds. We therefore proceed to analyze several simpler achievability schemes. Besides being of use in numerically evaluating regions, the simpler regions are extensions of regions known to achieve capacity when the channel reduces to an IFC or a CIFC.
3) Capacity:
a) Capacity in the "very strong interference" regime at one destination.
This is a subset of the "strong interference" regime under which our general BC-type outer bound can be simplified. In this regime both decoders can, without loss of optimality, decode both messages as in a compound MAC. The "strong interference" outer bound may be achieved using superposition coding without rate splitting or binning.
b) Capacity in the "strong interference at both receivers" regime.
A corollary of the previous capacity result where both destinations experience "very strong interference".
4) Gaussian Channels:
a) Capacity in the "very strong interference" regime at one destination and in the "strong interference at both receivers".
We determine the set of channel coefficients that satisfy the condition of "very strong interference" at one destination and of "very strong interference" at both destinations, thereby establishing capacity in these cases.
b) Outer bound for the degraded IFC-CR.

For a special class of channels that satisfies the "weak interference" condition under which our general BC-type outer bound could be simplified, we evaluate the outer bound in closed form. Unfortunately, we have not been able to find a transmission scheme that achieves this outer bound yet.
c) Numerical evaluations of the proposed simpler achievable rate regions.

These evaluations visually illustrate the relationships between the derived inner and outer bounds for the cases where capacity is open.

## C. Paper Organization

In Section [I we formally define the general memoryless IFC-CR. In Section III we proceed to derive our new outer bounds, two of which hold in general, and two of which are valid under "strong interference" and "weak interference" conditions, respectively. In Section IV] we derive a general achievable rate region for the IFC-CR and analytically show that this contains all other known inner bounds; we further simply our general inner bound in a number of simpler subcases with a limited number of auxiliary random variables and rate splits. In Section $\square$ we prove capacity for the IFCCR in the "very strong interference" regime; this is the first general capacity result for the IFC-CR and parallels results for similar regimes for the IFC and the CIFC. In Section VI we numerically illustrate the "very strong interference" capacity region and the "weak interference" outer bound for the Gaussian IFC-CR, as well as numerical results comparing several of the simplified inner bounds. We conclude the paper in Section VIII

## II. Channel Model

We consider the channel model depicted in Fig. 1 In the IFC-CR the transmission of the two independent messages $W_{i}$ uniformly distributed on $\left[1: 2^{N R_{i}}\right], i \in\{1,2\}$, block-length $N \in \mathbb{Z}^{+}$, and rates $R_{i} \in \mathbb{R}^{+}$, is aided by a single cognitive relay, whose input to the channel has subscript $c$. We define $X_{i, n}$ and $Y_{i, n}$ to be the input and output of the channel for the $i$-th source-destination pair at the $n$-th channel use, $i \in\{1,2\}$, $n \in[1: N]$, and define $X_{i, j}^{k}:=\left[X_{i, j}, X_{i, j+1}, \cdots, X_{i, k}\right]$ for $k \geq j$, and similarly for $Y_{i, j}^{k}$. The channel is assumed to be memoryless with transition probability $P_{Y_{1}, Y_{2} \mid X_{1}, X_{2}, X_{c}}$. Since the destinations do not cooperate, the capacity of the memoryless IFC-CR is only a function of the output conditional marginal distributions $P_{Y_{1} \mid X_{1}, X_{2}, X_{c}}$ and $P_{Y_{2} \mid X_{1}, X_{2}, X_{c}}$.

A non-negative rate pair $\left(R_{1}, R_{2}\right)$ is said to be achievable if there exists a sequence of encoding functions

$$
\begin{aligned}
X_{1}^{N} & =X_{1}^{N}\left(W_{1}\right), \\
X_{2}^{N} & =X_{2}^{N}\left(W_{2}\right), \\
X_{c}^{N} & =X_{c}^{N}\left(W_{1}, W_{2}\right),
\end{aligned}
$$

and a sequence of decoding functions

$$
\begin{aligned}
& \widehat{W}_{1}=\widehat{W}_{1}\left(Y_{1}^{N}\right), \\
& \widehat{W}_{2}=\widehat{W}_{2}\left(Y_{2}^{N}\right),
\end{aligned}
$$

such that

$$
\lim _{N \rightarrow \infty} \max _{i \in\{1,2\}} \operatorname{Pr}\left[\widehat{W}_{i} \neq W_{i}\right]=0
$$

The capacity region is defined as the closure of the region of all achievable $\left(R_{1}, R_{2}\right)$ pairs.

Note that the IFC-CR subsumes three well-studied channels as special cases:


Fig. 1. The general memoryless IFC-CR channel model.

- IFC: for $X_{c}=\emptyset$,
- CIFC: for $X_{1}=\emptyset$ or $X_{2}=\emptyset$, and
- BC: for $X_{1}=X_{2}=\emptyset$.

The capacity region of the general IFC-CR is unknown in general.

## III. Outer Bounds

In this section we present two new outer bounds which we term the Sato-type and the BC-type outer bound. The names of these bounds reflect the channels and/or techniques which inspired them. We then proceed to simplify the expression of these bounds in the "strong interference" and "weak interference" regime $3^{3}$ As the IFC-CR generalizes a number of multi-user channels such as the CIFC, the IFC and the BC, one expects techniques relevant in those channels to be of use in the IFC-CR, and conversely, the IFC-CR outer bounds should reduce to capacity of the simpler sub-channels when they are known. Indeed, our outer bounds generalize the underlying sub-channels, as shown in Table []

## A. Sato-type Outer Bound

We start with the outer bound for the general IFC-CR first derived by the Rini, Tuninetti and Devroye in [8, Thm.3.1]. It uses Sato's argument [9] that the capacity region of the IFCCR only depends on the channel output conditional marginal distributions since the destinations do not cooperate.

Theorem III.1. If $\left(R_{1}, R_{2}\right)$ lies in the capacity region of the IFC-CR, then the following must hold for any $\widetilde{Y}_{1}$ and $\widetilde{Y}_{2}$ having the same conditional marginal distributions as $Y_{1}$ and $Y_{2}$, respectively, but otherwise arbitrarily correlated:

$$
\begin{align*}
R_{1} & \leq I\left(Y_{1} ; X_{1}, X_{c} \mid X_{2}, Q\right), \\
R_{2} & \leq I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1}, Q\right), \\
R_{1}+R_{2} & \leq I\left(Y_{2} ; X_{1}, X_{2}, X_{c} \mid Q\right)+I\left(Y_{1} ; X_{1}, X_{c} \mid \widetilde{Y_{2}}, X_{2}, Q\right)  \tag{1c}\\
R_{1}+R_{2} & \leq I\left(Y_{1} ; X_{1}, X_{2}, X_{c} \mid Q\right)+I\left(Y_{2} ; X_{2}, X_{c} \mid \widetilde{Y_{1}}, X_{1}, Q\right), \tag{1d}
\end{align*}
$$

[^1]for some input distribution that factors as
\[

$$
\begin{equation*}
P_{Q, X_{1}, X_{2}, X_{c}}=P_{Q} P_{X_{1} \mid Q} P_{X_{2} \mid Q} P_{X_{c} \mid X_{1}, X_{2}, Q} \tag{2}
\end{equation*}
$$

\]

Proof: The proof may be found in Appendix A
The outer bound of Thm. III. 1 has the appealing feature that is does not contain any auxiliary Random Variable (RV) and is thus computable. For example (see Section VI) the "Gaussian maximizes entropy" principle suffices to show that a jointly Gaussian input exhausts the outer bound of Thm. III. 1 for the Gaussian noise channel. It also gives the capacity in several cases (please refer to Table (I). However it does not reduce to the other cases where capacity is known for simpler channels subsumed by the IFC-CR (please refer to Table】) nor to the tightest known outer bounds for the general CIFC and BC. To remedy this, we next derive an outer bound by using a bounding technique originally developed for the BC [35]. The derived bound indeed reduces to the tightest known outer bounds for the general CIFC [20] and the general BC [27] when the IFC-CR reduces to these channel models.

## B. BC-type outer bound

The outer bound in [20] for the CIFC and in [27] for the BC use in their bounding steps the Csiszár's sum identity [40]. We extend this technique here to the general IFC-CR.

Theorem III.2. If $\left(R_{1}, R_{2}\right)$ lies in the capacity region of the IFC-CR then the following must hold

$$
\begin{align*}
R_{1} & \leq I\left(Y_{1} ; X_{1}, X_{c} \mid U_{2}, X_{2}\right)  \tag{3a}\\
R_{2} & \leq I\left(Y_{2} ; X_{2}, X_{c} \mid U_{1}, X_{1}\right)  \tag{3b}\\
R_{1} & \leq I\left(Y_{1} ; V, U_{1}, X_{1}\right)  \tag{3c}\\
R_{2} & \leq I\left(Y_{2} ; V, U_{2}, X_{2}\right)  \tag{3d}\\
R_{1}+R_{2} & \leq I\left(Y_{2} ; V, U_{2}, X_{2}\right)+I\left(Y_{1} ;, X_{1}, X_{c} \mid V, U_{2}, X_{2}\right),  \tag{3e}\\
R_{1}+R_{2} & \leq I\left(Y_{1} ; V, U_{1}, X_{1}\right)+I\left(Y_{2} ;, X_{2}, X_{c} \mid V, U_{1}, X_{1}\right), \tag{3f}
\end{align*}
$$

such that

$$
\begin{equation*}
V \rightarrow\left(U_{1}, U_{2}\right) \rightarrow\left(X_{1}, X_{2}, X_{c}\right) \rightarrow\left(Y_{1}, Y_{2}\right) \tag{4}
\end{equation*}
$$

for some input distribution that factors as

$$
\begin{align*}
& P_{U_{1}, U_{2}, V, X_{1}, X_{2}, X_{c}} \\
& =P_{U_{1}} P_{U_{2}} P_{V \mid U_{1}, U_{2}} P_{X_{1} \mid U_{1}} P_{X_{2} \mid U_{2}} P_{X_{c} \mid U_{1}, U_{2}} \tag{5}
\end{align*}
$$

Proof: The proof may be found in Appendix B,
Remark III.3. Thm. III.2 is the tightest known outer bound for a general IFC-CR and

1) it reduces to the tightest known outer bound for the general BC without common rate [27] when $X_{1}=X_{2}=\emptyset$, which is tight for all cases where capacity is known.
2) it reduces to the tightest known outer bound for the general CIFC [20, Thm.4] when $X_{1}=\emptyset$, which is tight for all cases where capacity is known. The outer bound in [20, Thm.4] is tighter than the one in [18, Thm. 3.2] (see [20, Remark 6]). We can obtain the equivalent of the outer bound in [18, Thm. 3.2] by defining in Thm. III.2

TABLE I
THE OUTER BOUNDS PRESENTED IN THIS WORK AND THEIR RELATIONSHIP TO UNDERLYING SIMPLER CHANNELS WHERE CAPACITY IS KNOWN.

| Outer bound and Theorem in this work | Capacity result | Reference |
| :---: | :---: | :---: |
| Sato-type outer bound | "strong interference" IFC-CR Gaussian "strong interference" CIFC Gaussian "primary decodes cognitive" CIFC "strong interference" IFC | Thm. V.1] [37 Thm.6] [22 Thm.3.1] [36], 31$], 32]$ |
| BC-type outer bound |  | $[18$ Thm. 3.2] <br> 2 Thm. 7.1] <br> 2 Thm. 8.1] <br> [35 Sec. 3] <br> [?] 26] |

a new pair of auxiliary RVs $U_{2}^{\prime}:=\left[V, U_{2}\right], U_{1}^{\prime}:=\left[V, U_{2}\right]$ and then reasoning as in [20, Remark 6].
3) it is tighter than Thm. III.1 In fact, the region in Thm. III.2 can be enlarged by dropping (3c)-(3d). Moreover, the bound in (3a) is tighter than the one in (1a) by the "conditioning reduces entropy" principle. Similarly, to [1] Remark IV.2] the sum-bound in (3e) is tighter than the bound in (1C). However, the region in Thm. III. 2 is expressed as a function of three auxiliary RVs for which we have not obtained cardinality bounds on the respective alphabets, while the looser region in Thm. III. 1 is expressed only as a function of the inputs and is thus computable in principle.
4) Thm. III.2 neither reduces to the capacity region of a class of deterministic IFCs studied in [33] nor reduces to the outer bound for the semi-deterministic IFC in [10] when $X_{c}=\emptyset$. The difficulty in deriving outer bounds for the general IFC-CR that are tight when it reduces to an IFC is also noted in [8]. The authors of [8, Thm.3.2] were able to derive tight bounds in this scenario by imposing additional constraints on the effect of interference on the channel outputs.

## C. Simplified BC-type outer bound in the "weak interference" and "strong interference" regimes

We next proceed to simplify the proposed BC-type outer bound under specific "strong interference" and "weak interference" conditions.

## Corollary III.4. "Strong interference at Rx 1" outer bound.

 If$$
\begin{equation*}
I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1}\right) \leq I\left(Y_{1} ; X_{2}, X_{c} \mid X_{1}\right) \tag{6}
\end{equation*}
$$

for all distributions that factor as

$$
\begin{equation*}
P_{X_{1}, X_{2}, X_{c}}=P_{X_{1}} P_{X_{2}} P_{X_{c} \mid X_{1}, X_{2}}, \tag{7}
\end{equation*}
$$

then, if $\left(R_{1}, R_{2}\right)$ lies in the capacity region of the IFC-CR, the following must hold

$$
\begin{align*}
R_{1} & \leq I\left(Y_{1} ; X_{1}, X_{c} \mid X_{2}, Q\right)  \tag{8a}\\
R_{2} & \leq I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1}, Q\right)  \tag{8b}\\
R_{1}+R_{2} & \leq I\left(Y_{1} ; X_{1}, X_{2}, X_{c} \mid Q\right), \tag{8c}
\end{align*}
$$

for some distribution that factors as in (2).
Proof: The proof follows from showing that under the condition in (6) the sum-rate bounds in Thm. III.2 simplify
to 8 Cc . The details of the proof may be found in Appendix C

Note that, given the symmetry of the channel model, Cor. III. 4 also holds by reversing the role of the sources. Although not valid for a general IFC-CR, Cor. III. 4 is expressed only as a function of the channel inputs and does not contain auxiliary RVs as Thm. III.1, which simplifies both the calculation of the outer bound and the derivation of a capacity achieving encoding strategy.

## Corollary III.5. "Weak interference at Rx 2 " outer bound.

 If$$
\begin{equation*}
I\left(Y_{2} ; U \mid X_{2}\right) \leq I\left(Y_{1} ; U \mid X_{2}\right) \tag{9}
\end{equation*}
$$

holds for all distributions

$$
\begin{equation*}
P_{U, X_{1}, X_{2}, X_{c}}=P_{X_{1}} P_{X_{2}} P_{X_{c} \mid X_{1}, X_{2}} P_{U \mid X_{1}, X_{2}, X_{c}}, \tag{10}
\end{equation*}
$$

such that $U \rightarrow\left(X_{1}, X_{2}, X_{c}\right) \rightarrow\left(Y_{1}, Y_{2}\right)$, then, if $\left(R_{1}, R_{2}\right)$ lies in the capacity region of the IFC-CR, the following must hold:

$$
\begin{align*}
& R_{1} \leq I\left(Y_{1} ; X_{1}, X_{c} \mid X_{2}, U\right)  \tag{11a}\\
& R_{2} \leq I\left(Y_{2} ; X_{2}, U\right)  \tag{11b}\\
& R_{2} \leq I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1}\right) \tag{11c}
\end{align*}
$$

for some distribution that factors as in (10).
Proof: The proof may be found in Appendix D
Again, given the symmetry of the channel model, Cor. III.5 also holds when the sources are reversed.

## IV. Inner Bounds

In this section we derive an inner bound for a general IFCCR , then analytically show that this region contains all other previously derived regions, and finally derive simple and easy-to-understand expressions for a number of sub-schemes of our general inner bound. Given the generality of the IFCCR channel model, the coding scheme we propose contains a large number of rate bounds and several auxiliary RVs. Unfortunately, this is unavoidable if one wishes the achievable scheme to be capacity in all the cases when the channel reduces to one where capacity is known. Our aim in deriving this achievable rate region is therefore mainly to provide a unified framework to efficiently investigate the rate advantages provided by different transmission strategies.

## A. General achievable rate region

The achievable scheme is obtained as a combination of the following well established random coding techniques:

- Rate-splitting: This refers to splitting the message of a source into different independent sub-messages, one for each possible subset of destinations. Rate splitting was first introduced by Han and Kobayashi for the classical IFC [29] (referred to as the Han and Kobayashi region or rate-splitting from now on) and is a fundamental tool in achieving capacity in a number of cases when combined with superposition coding and binning. In our achievable scheme we rate split each message into private and public parts at the intended transmitter and at the cognitive relay.
- Superposition coding: Superposition coding was first introduced in [41] for the degraded BC and intuitively consists of generating codewords conditional on other ones, or "stacking" codewords on top of each other. Destinations in the system decode (some of the) codewords starting from the bottom of the stack, while treating the remaining codewords as noise. Thus, a given message may be decoded at one destination but treated as noise at another. Here we superpose public messages to broadcast messages and the messages known at the cognitive relay over the messages at the two sources.
- Gel'fand-Pinsker binning: Often simply referred to as binning [42], it allows a transmitter to "pre-code" (portions of) the message against the interference that message is known to experience at a destination. Binning is also used in Marton's largest known achievable rate region for the general memoryless BC [26]. It is also a crucial element in other channels, usually with some form of "broadcast" element, including the CIFC [1]. In this achievable scheme the cognitive relay performs binning against the private messages of the sources.
- Simultaneous decoding: As at the destination of a MAC, a destination jointly decodes its intended message and some of the sub-messages of non-intended sources with the objective to reduce the level of interference. Simultaneous or joint decoding is optimal in many cases of "strong" interference.

We next derive a transmission scheme that contains a general combination these encoding techniques. By removing certain features from this general scheme, one can quickly obtain simpler and analytically more tractable sub-schemes that can be compared to each other and to outer bounds, as we shall do in the next subsection. We shall also show that this general inner bound includes all known to-date achievable rate regions. The novelty of our proposed region, which will allow us to show inclusion in all known regions, is a rate split into four parts for each source message (as opposed to the classical rate split in two parts for the classical IFC [29] and to the rate split in three parts for the CIFC [1]).

Theorem IV.1. Region $\mathcal{R}^{(\mathrm{RTDG})}$. The region $\mathcal{R}^{(\mathrm{RTDG})}$ is defined as the set of non-negative rate pairs $\left(R_{1}, R_{2}\right)$ for which
there exists a non-negative rate vector
$\left(R_{1 \mathrm{c}}, R_{2 \mathrm{c}}, R_{1 \mathrm{p}}, R_{2 \mathrm{p}}, R_{1 \mathrm{cb}}, R_{2 \mathrm{cb}}, R_{1 \mathrm{pb}}, R_{2 \mathrm{pb}}, R_{0 \mathrm{cb}}^{\prime}, R_{1 \mathrm{pb}}^{\prime}, R_{2 \mathrm{pb}}^{\prime}\right)$
$\in \bigcup_{P}\left\{\mathcal{R}_{0} \cap \mathcal{R}_{1} \cap \mathcal{R}_{2}\right\}$
such that

$$
\begin{equation*}
R_{i}=R_{i \mathrm{c}}+R_{i \mathrm{p}}+R_{i \mathrm{cb}}+R_{i \mathrm{pb}}, i \in\{1,2\} \tag{13}
\end{equation*}
$$

where the union in (12) is over all input distributions $P$ given by

$$
\begin{align*}
P= & P_{Q} P_{U_{1 \mathrm{c}}, X_{1} \mid Q} P_{U_{2 \mathrm{c}}, X_{2} \mid Q} \\
& P_{U_{1 \mathrm{pb}}, U_{2 \mathrm{pb}}, U_{0 \mathrm{cb}}, X_{c} \mid U_{1 \mathrm{c}}, X_{1}, U_{2 \mathrm{c}}, X_{2}, Q}, \tag{14}
\end{align*}
$$

where the "binning rate region" $\mathcal{R}_{0}$ in (12) is given in (15) and the "decoding rate region at destination 1 " $\mathcal{R}_{1}$ in (12) is given in (16) for

$$
\begin{aligned}
& L_{i \mathrm{pb}}=R_{i \mathrm{pb}}+R_{i \mathrm{pb}}^{\prime}, \quad i \in\{1,2\} \\
& L_{0 \mathrm{cb}}=R_{1 \mathrm{cb}}+R_{2 \mathrm{cb}}+R_{0 \mathrm{cb}}^{\prime}
\end{aligned}
$$

and where the "decoding rate region at destination 2 " $\mathcal{R}_{2}$ in (12) is obtained permuting the indices 1 and 2 in the "decoding rate region at destination 1 " $\mathcal{R}_{1}$ in (16).

Moreover, in the "decoding rate region at destination 1 " $\mathcal{R}_{1}$ in (16) (and similarly for $\mathcal{R}_{2}$ but with the role of the sources swapped) the following rate bounds can be dropped

- (16a) and (16b): when $R_{1}=R_{1 \mathrm{c}}=R_{1 \mathrm{p}}=R_{1 \mathrm{cb}}=$ $R_{1 \mathrm{pb}}=0$,
- (16c) and (16d): when $R_{1 \mathrm{p}}=R_{1 \mathrm{cb}}=R_{1 \mathrm{pb}}=0$,
- (16e) and (16f): when $R_{1 \mathrm{cb}}=R_{1 \mathrm{pb}}=0$,
- 16 g ): when $R_{1 \mathrm{p}}=R_{1 \mathrm{pb}}=0$,
- (16h): when $R_{1 \mathrm{pb}}=0$,
because these bounds correspond to an error event in which a non-intended common message or a bin index is incorrectly decoded and no other intended message is incorrectly decoded.

Proof: The achievable rate region in (12) may be obtained using the result in [43] by specifying how rate splitting, binning and superposition coding are performed. The details of the proof are reported in Appendix E for completeness. In what follows we sketch the main elements of the encoding and decoding procedures and we give an intuitive explanation about the proposed choices. We do not consider the time sharing RV $Q$ to simplify the description.

Rate Splitting: The message $W_{i}, i \in\{1,2\}$, is split into four sub-messages:

- Private message $W_{i \mathrm{p}}$ of rate $R_{i \mathrm{p}}$,
- Common message $W_{i c}$ of rate $R_{i c}$,
- Common Broadcasted message $W_{i \mathrm{cb}}$ of rate $R_{i \mathrm{cb}}$, and
- Private Broadcasted message $W_{i \mathrm{pb}}$ of rate $R_{i \mathrm{pb}}$, so that (13) holds.

Codebook Generation: The sources and the cognitive relay generate the following codebooks:

- Common message: $w_{i c} \in\left[1: 2^{N R_{i c}}\right]$ is encoded into $U_{i c}^{N}\left(w_{i c}\right)$ with iid distribution $P_{U_{i c}}, i \in\{1,2\}$.
- Private message: for a given $w_{i c}, w_{i \mathrm{p}} \in\left[1: 2^{N R_{i \mathrm{p}}}\right]$ is encoded into $X_{i}^{N}\left(w_{i \mathrm{p}} \mid w_{i c}\right)$ with iid distribution $P_{X_{i} \mid U_{i c}}$ (i.e., $X_{i}^{N}$ is superimposed to $U_{i c}^{N}$ ), $i \in\{1,2\}$.

$$
\begin{align*}
& R_{0 \mathrm{cb}}^{\prime} \geq I\left(X_{1}, X_{2} ; U_{0 \mathrm{cb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, Q\right)  \tag{15a}\\
& R_{1 \mathrm{pb}}^{\prime} \geq I\left(X_{2} ; U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, X_{1}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}, Q\right)  \tag{15b}\\
& R_{2 \mathrm{pb}}^{\prime} \geq I\left(X_{1} ; U_{2 \mathrm{pb}} \mid U_{1 \mathrm{c}}, X_{2}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}, Q\right)  \tag{15c}\\
& R_{1 \mathrm{pb}}^{\prime}+R_{2 \mathrm{pb}}^{\prime} \geq I\left(X_{2} ; U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}, Q\right)+I\left(X_{1} ; U_{2 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{2}, U_{0 \mathrm{cb}}, Q\right) \\
&+I\left(U_{1 \mathrm{pb}} ; U_{2 \mathrm{pb}} \mid U_{1 \mathrm{c}}, X_{1}, U_{2 \mathrm{c}}, X_{2}, U_{0 \mathrm{cb}}, Q\right), \tag{15d}
\end{align*}
$$

$$
\begin{align*}
R_{1 \mathrm{c}}+R_{1 \mathrm{p}}+R_{2 \mathrm{c}}+L_{0 \mathrm{cb}}+L_{1 \mathrm{pb}} & \leq I\left(U_{0 \mathrm{cb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, Q\right)+I\left(Y_{1} ; U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}, U_{1 \mathrm{pb}}, Q\right)  \tag{16a}\\
R_{1 \mathrm{c}}+R_{1 \mathrm{p}}+L_{0 \mathrm{cb}}+L_{1 \mathrm{pb}} & \leq I\left(U_{0 \mathrm{cb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, Q\right)+I\left(Y_{1} ; U_{1 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}, U_{1 \mathrm{pb}} \mid U_{2 \mathrm{c}}, Q\right)  \tag{16b}\\
R_{1 \mathrm{p}}+R_{2 \mathrm{c}}+L_{0 \mathrm{cb}}+L_{1 \mathrm{pb}} & \leq I\left(U_{0 \mathrm{cb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, Q\right)+I\left(Y_{1} ; U_{2 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}, U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, Q\right)  \tag{16c}\\
R_{1 \mathrm{p}}+L_{0 \mathrm{cb}}+L_{1 \mathrm{pb}} & \leq I\left(U_{0 \mathrm{cb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, Q\right)+I\left(Y_{1} ; X_{1}, U_{0 \mathrm{cb}}, U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, Q\right)  \tag{16d}\\
R_{2 \mathrm{c}}+L_{0 \mathrm{cb}}+L_{1 \mathrm{pb}} & \leq I\left(U_{0 \mathrm{cb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, Q\right)+I\left(Y_{1} ; U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}, U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, X_{1}, Q\right) \\
L_{0 \mathrm{cb}}+L_{1 \mathrm{pb}} & \leq I\left(U_{0 \mathrm{cb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, Q\right)+I\left(Y_{1} ; U_{0 \mathrm{cb}}, U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}, Q\right)  \tag{16f}\\
R_{1 \mathrm{p}}+L_{1 \mathrm{pb}} & \left.\leq I\left(U_{0 \mathrm{cb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, Q\right)+I\left(Y_{1} ; X_{1}, U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}, Q\right)\right)  \tag{16~g}\\
L_{1 \mathrm{pb}} & \leq I\left(Y_{1} ; U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}, Q\right) \tag{16h}
\end{align*}
$$

- Common broadcasted messages: for a given pair $\left(w_{1 \mathrm{c}}, w_{2 \mathrm{c}}\right)$, the pair $\left(w_{1 \mathrm{cb}}, w_{2 \mathrm{cb}}\right) \in\left[1: 2^{N R_{1 \mathrm{cb}}}\right] \times[1:$ $\left.2^{N R_{2 \mathrm{cb}}}\right]$ is encoded into $U_{0 \mathrm{cb}}^{N}\left(w_{1 \mathrm{cb}}, w_{2 \mathrm{cb}}, b_{0 \mathrm{cb}} \mid w_{1 \mathrm{c}}, w_{2 \mathrm{c}}\right)$,

- Private broadcasted message: for a given $\quad\left(w_{1 \mathrm{c}}, w_{2 \mathrm{c}}, w_{1 \mathrm{cb}}, w_{2 \mathrm{cb}}, b_{0 \mathrm{cb}},, w_{i \mathrm{p}}\right)$, $w_{i \mathrm{pb}} \in\left[1 \quad: \quad 2^{N R_{i \mathrm{pb}}}\right] \quad$ is encoded into $U_{i \mathrm{pb}}^{N}\left(w_{i \mathrm{pb}}, b_{i \mathrm{pb}} \mid w_{1 \mathrm{c}}, w_{2 \mathrm{c}}, w_{1 \mathrm{cb}}, w_{2 \mathrm{cb}}, b_{0 \mathrm{cb}}, w_{i \mathrm{p}}\right)$,
$b_{i \mathrm{pb}} \in\left[1 \quad: \quad 2^{N R_{i \mathrm{pb}}^{\prime}}\right]$, with distribution $P_{U_{i \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{\mathrm{zc}}, U_{0 \mathrm{cb}}, X_{i}}^{N}, i \in\{1,2\}$.
Encoding: The cognitive relay has knowledge of both messages $W_{1}, W_{2}$ and is thus able to perform binning with the goal to create the most general distribution among conditionally independent RVs /codebooks. It does the following:
- $U_{0 \mathrm{cb}}^{N}$ was generated only based on $\left(U_{1 \mathrm{c}}^{N}, U_{2 \mathrm{c}}^{N}\right)$. The cognitive relay bins $U_{0 \mathrm{cb}}^{N}$ against $\left(X_{1}^{N}, X_{2}^{N}\right)$, as for channel with states known non-causally at the encoder [42], to make it look like it were generated iid with distribution $P_{U_{\text {0cb }} \mid X_{1}, X_{2}, U_{1 \mathrm{c}}, U_{2 \mathrm{c}}}$. For this to be possible, the "binning rate" $R_{0 \mathrm{cb}}^{\prime}$ must satisfy (15a).
- $U_{1 \mathrm{pb}}^{N}$, resp. $U_{2 \mathrm{pb}}^{N}$, was generated independently of $\left(X_{2}^{N}, U_{2 \mathrm{pb}}^{N}\right)$, resp. $\left(X_{1}^{N}, U_{1 \mathrm{pb}}^{N}\right)$, conditioned on the "common" RVs $\left(U_{1 \mathrm{c}}^{N}, U_{2 \mathrm{c}}^{N}, U_{0 \mathrm{cb}}^{N}\right)$. The cognitive relay bins $U_{1 \mathrm{pb}}^{N}$ and $U_{2 \mathrm{pb}}^{N}$ against each other, as in Marton's region for the general BC [26], and against $\left(X_{1}^{N}, X_{2}^{N}\right)$ to make them look like they were generated iid with distribution $P_{U_{1 \mathrm{pb}}, U_{2 \mathrm{pb}} \mid X_{1}, X_{2}, U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}}$. For this to be possible, the "binning rate" pair ( $R_{1 \mathrm{pb}}^{\prime}, R_{2 \mathrm{pb}}^{\prime}$ ) must satisfy (15b)(15d).
- to send $w_{i}=\left(w_{i c}, w_{i \mathrm{p}}, w_{i \mathrm{cb}}, w_{i \mathrm{pb}}\right)$ source $i$ sends $X_{i}^{N}\left(w_{i \mathrm{p}} \mid w_{i c}\right), i \in\{1,2\}$.


Fig. 2. A graphical representation of the coding scheme for the inner bound region in Section IV] The RVs for message 1 are in blue diamond boxes while the RVs for message 2 are in red square boxes. A solid line among RVs indicates that the RVs are superposed while a dashed line that the RVs are binned against each other.

- to send $\left(w_{1}, w_{2}\right)=\left(\left(w_{1 \mathrm{c}}, w_{1 \mathrm{p}}, w_{1 \mathrm{cb}}, w_{1 \mathrm{pb}}\right)\right.$, $\left.\left(w_{2 \mathrm{c}}, w_{2 \mathrm{p}}, w_{2 \mathrm{cb}}, w_{2 \mathrm{pb}}\right)\right)$ the cognitive relay sends $X_{c}^{N}$ obtained as a deterministic function of the tuplet $\left(U_{1 \mathrm{c}}^{N}, U_{2 \mathrm{c}}^{N}, X_{1}^{N}, X_{2}^{N}, U_{0 \mathrm{cb}}^{N}, U_{1 \mathrm{pb}}^{N}, U_{2 \mathrm{pb}}^{N}\right)$ found after the different binning operations.

Fig. 2 is a graphical representation of the proposed achievable scheme. Each box represents an auxiliary RV/codebook carrying the sub-message with the same subscript (note that the RVs $X_{1}$ and $X_{2}$ carry the sub-messages $W_{1 \mathrm{p}}$ and $W_{2 \mathrm{p}}$, respectively, and $U_{0 \mathrm{cb}}$ carries the pair of sub-messages ( $\left.W_{1 \mathrm{cb}}, W_{2 \mathrm{cb}}\right)$ ).

Decoding: Destination $i, i \in\{1,2\}$, simultaneously decodes all RVs/codebooks except $\left(X_{i}^{N}, U_{\bar{i} \mathrm{pb}}^{N}\right)$ with $\bar{i} \neq i$. This is successful with high probability if the rates belong to the
"decoding rate region at destination $i$ " $\mathcal{R}_{i}$ defined in (16), $i \in\{1,2\}$.

Remark IV. 2 (Intuitive interpretation of the proposed coding scheme). Loosely speaking the achievable rate region is obtained by considering a Han and Kobayashi transmission scheme for the IFC among the two source-destination pairs and extending this coding scheme with the scheme for the CIFC [19] for each source-destination pair. The RVs $U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}, X_{2}$ correspond to the Han and Kobayashi scheme [29] for the IFC. The common broadcasted message $U_{0 \text { cb }}$ is superposed to both the common messages $U_{1 \mathrm{c}}, U_{2 \mathrm{c}}$ and carries the common broadcasted messages for both users, $W_{1 \mathrm{cb}}$ and $W_{2 \mathrm{cb}}$. Since these messages are to be decoded at both decoders, there is no rate advantage in assigning a different RV to each rate split. Note that $U_{0 c b}$ cannot be stacked over to the private messages $\left(X_{1}, X_{2}\right)$ since these messages are not decoded at the non-intended destinations. To achieve the most general input distribution, the cognitive relay performs binning of $U_{0 c b}$ against the known interfering signals $\left(X_{1}, X_{2}\right)$. The private broadcasted message $U_{1 \mathrm{pb}}$ is stacked onto $\left(U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}, X_{1}\right)$ - this can be done since this RV is to be decoded only at destination 1 which also decodes $X_{1}$. The same procedure is applied to $U_{2 \mathrm{pb}}$. At the last encoding step at the cognitive relay, $U_{1 \mathrm{pb}}$ and $U_{2 \mathrm{pb}}$ are binned against each other and against the non-intended private messages to achieve the most general distribution.

Finally, note that the proposed scheme with only the "broadcast" RVs ( $\left.U_{0 \mathrm{cb}}, U_{1 \mathrm{pb}}, U_{2 \mathrm{pb}}\right)$ corresponds to Marton's achievable rate region for the general BC [26], without the "broadcast" RVs it corresponds to Han and Kobayashi's achievable rate region for the general IFC [29], and with the "broadcast" RVs only for one source it corresponds to Rini et al's achievable rate region for the general CIFC [1]. Therefore, our proposed achievable rate region reduces to the largest known achievable rate regions for the simpler channels subsumed by the IFC-CR, which are capacity-achieving for all cases where capacity is known.
B. Inclusion of the Jiang et al. region [7] for the IFC-CR: scheme with $\left(U_{1 \mathrm{c}}, X_{1}, U_{2 \mathrm{c}}, X_{2}, U_{1 \mathrm{pb}}, U_{2 \mathrm{pb}}\right)$

We now show that the achievable rate region in Thm. IV. 1 includes all previously proposed achievable rate regions for the IFC-CR by showing that the region in Thm. IV. 1 includes the region in [7] as a special case, which is currently the largest known region for this channel and contains the regions of [5] and [6].

Theorem IV.3. The achievable rate region in Thm. IV. 1 contains the achievable rate region in [7(21)-(31)].

Proof: Set $U_{0 \mathrm{cb}}=\emptyset$ in Thm. IV.1. The resulting achievable rate region includes the region in [7], (20)-(31)] (which includes the region in [7, (1)-(19)]) as shown in Appendix $F$
C. Sub-schemes from the general achievable rate region in Thm. IV. 1

The inner bound of Thm. IV.1 provides a unified framework from which we may derive simpler inner bounds that may be more easily manipulated and understood. In particular one would like an achievable rate region to be expressed in terms of the rate bounds directly on $R_{1}$ and $R_{2}$ rather than on the rates corresponding to the rate-split messages. Such a region may be obtained by eliminating the sub-rates from the rate region expression using the Fourier-Motzkin elimination procedure. Fourier-Motzkin elimination yields an analytically manageable number of rate bounds only for a relatively small number of rate splits. In this section we introduce a series of sub-schemes containing a limited number of auxiliary RVs and derive the corresponding Fourier-Motzkin eliminated rate regions (resulting in $\left(R_{1}, R_{2}\right)$ rate regions) which are then compared to the outer bounds derived in Section III) In addition to these sub-schemes being more analytically tractable due to the small number of auxiliary random variables and rate-splits, these particular sub-schemes were chosen as they are natural extensions of schemes that achieve capacity when the IFC-CR reduces to specific classes of CIFC, IFC and BC channels. Table $\Pi$ illustrates the different sub-schemes and for which classes of channels this reduces to capacity.

1) All private messages: scheme with only $\left(X_{1}, X_{2}, U_{1 \mathrm{pb}}, U_{2 \mathrm{pb}}\right)$ : This sub-scheme is obtained by setting the rate of the common messages to zero. It illustrates the effect of binning performed at the cognitive relay to pre-code against the interference due to the non-intended message at each destination.

Corollary IV.4. By considering $U_{1 \mathrm{c}}=U_{2 \mathrm{c}}=U_{0 \mathrm{cb}}=\emptyset \mathrm{in}$ Thm. IV. 1 the following rate region is achievable

$$
\begin{align*}
R_{1} & \leq I\left(Y_{1} ; X_{1}, U_{1 \mathrm{pb}} \mid Q\right)-I\left(X_{2} ; U_{1 \mathrm{pb}} \mid X_{1}, Q\right)  \tag{17a}\\
R_{2} & \leq I\left(Y_{2} ; X_{2}, U_{2 \mathrm{pb}} \mid Q\right)-I\left(X_{1} ; U_{2 \mathrm{pb}} \mid X_{2}, Q\right)  \tag{17b}\\
R_{1}+R_{2} & \leq I\left(Y_{1} ; X_{1}, U_{1 \mathrm{pb}} \mid Q\right)+I\left(Y_{2} ; X_{2}, U_{2 \mathrm{pb}} \mid Q\right) \\
& -I\left(X_{2} ; U_{1 \mathrm{pb}} \mid X_{1}, Q\right)-I\left(X_{1} ; U_{2 \mathrm{pb}} \mid X_{2}, Q\right) \\
& -I\left(U_{1 \mathrm{pb}} ; U_{2 \mathrm{pb}} \mid X_{1}, X_{2}, Q\right) \tag{17c}
\end{align*}
$$

for all the distributions that factors as

$$
P_{Q} P_{X_{1} \mid Q} P_{X_{2} \mid Q} P_{X_{c}, U_{1 \mathrm{pb}}, U_{2 \mathrm{pb}} \mid X_{1}, X_{2}, Q}
$$

Proof: The proof may be found in Appendix G
The graphical representation of the achievable scheme in Cor. IV. 4 is provided in Fig 3(a)

The scheme in Cor. IV. 4 achieves capacity (see Table II) when the channel reduces to a semi-deterministic BC [45], [42] and to a semi-deterministic CIFC [1]; in these two cases the private broadcasted RV for the destination with noiseless output must equal the noiseless channel output; if both destination outputs are noiseless, the optimal assignment is $U_{1 \mathrm{pb}}=Y_{1}$ and $U_{2 \mathrm{pb}}=Y_{2}$.
2) All common messages: scheme with only $\left(U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}\right)$ : We now consider an achievability scheme where both decoders decode both messages and where, therefore, no binning or rate splitting is necessary.


Fig. 3. Specific choice of RVs for the general coding scheme in Fig. 2 The missing nodes in each figure indicates that the associated auxiliary RV has rate zero. The remaining nodes are encoded as prescribed by Th. IV. 1

TABLE II
THE CAPACITY RESULTS AVAILABLE FOR BC, IFC and CIFC AND THE ASSIGNMENT OF RVS in the region in 12 THAT ACHIEVE THE CORRESPONDING REGION.

| Sub-scheme \# | RVs used | Capacity result | Reference |
| :--- | :--- | :--- | :--- |
| 1 (all private) | $X_{1}, X_{2}, U_{1 \mathrm{pb}}, U_{2 \mathrm{pb}}$ | semi-deterministic BC, semi-det. CIFC | $[?],[1]$ |
| 2 (all common) | $U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}$ | very strong interference CIFC, IFC, IFC-CR | $[37],[31]$ |
| 3 (one common, one private) | $U_{1 \mathrm{p}}, U_{2 \mathrm{c}}, U_{1 \mathrm{pb}}$ | very weak CIFC | $[18]$ |
| 4 (common from sources, private from relay) | $U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{1 \mathrm{pb}}$ | very weak CIFC | $[18]$ |
| Han and Kobayashi region | $X_{1}, X_{2}, U_{1 \mathrm{c}}, U_{2 \mathrm{c}}$ | a class of deterministic IFC | $[33]$ |
| Marton region | $U_{0 \mathrm{cb}}, U_{1 \mathrm{pb}}, U_{1 \mathrm{pb}}$ | a More capable BC, BC with degraded message set | $[35],[44]$ |

Corollary IV.5. By considering $X_{1}=U_{1 \mathrm{c}}, X_{2}=U_{2 \mathrm{c}}, X_{c}=$ $U_{0 \mathrm{cb}}$ and $U_{1 \mathrm{pb}}=U_{2 \mathrm{pb}}=\emptyset$ in Thm. IV.I] the following rate region is achievable

$$
\begin{align*}
R_{1} & \leq I\left(Y_{1} ; X_{1}, X_{c} \mid X_{2}, Q\right)  \tag{18a}\\
R_{2} & \leq I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1}, Q\right)  \tag{18b}\\
R_{1}+R_{2} & \leq I\left(Y_{1} ; X_{1}, X_{2}, X_{c} \mid Q\right)  \tag{18c}\\
R_{1}+R_{2} & \leq I\left(Y_{2} ; X_{1}, X_{2}, X_{c} \mid Q\right) \tag{18d}
\end{align*}
$$

for all distribution that factors as

$$
P_{Q} P_{X_{1} \mid Q} P_{X_{2} \mid Q} P_{X_{c} \mid X_{1}, X_{2}, Q}
$$

Proof: The proof may be found in Appendix $H$
A graphical representation of the achievable rate region in Cor. IV. 5 is depicted in Fig. 3(b).

This scheme achieves capacity (see Table III) when the channel reduces to a CIFC in the "very strong interference" regime of [37] and to a IFC in the "strong interference" regime of [31].
3) One common and one private message: scheme with only $\left(X_{1}, U_{2 \mathrm{c}}, U_{1 \mathrm{pb}}\right):$ For a CIFC in the "very weak interference" regime, capacity is achieved by a fully common primary message and full private cognitive message [18]. We extend this transmission strategy to the IFC-CR by considering the case where one of the two source messages is private while the other is common.
Corollary IV.6. By considering $U_{1 \mathrm{c}}=\emptyset, X_{2}=U_{2 \mathrm{c}}=$ $U_{0 \mathrm{cb}}, U_{2 \mathrm{pb}}=\emptyset, U_{1 \mathrm{pb}}=X_{c}$ in Thm. IV. 1 the following rate region is achievable

$$
\begin{align*}
R_{1} & \leq I\left(Y_{1} ; X_{1}, X_{c} \mid X_{2}, Q\right)  \tag{19a}\\
R_{2} & \leq I\left(Y_{2} ; X_{2} \mid Q\right)  \tag{19b}\\
R_{2} & \left.\left.\leq I\left(Y_{1} ; X_{c}, X_{2} \mid X_{1}, Q\right)\right\}\right\}  \tag{19c}\\
R_{1}+R_{2} & \leq I\left(Y_{1} ; X_{2}, X_{1}, X_{c} \mid Q\right) \tag{19d}
\end{align*}
$$

for all distribution that factors as

$$
P_{Q} P_{X_{1} \mid Q} P_{X_{2} \mid Q} P_{X_{c}, U_{1 \mathrm{pb}} \mid X_{1}, X_{2}, Q}
$$

Proof: The proof may be found in Appendix I
A graphical representation of the achievable rate region of Cor. IV. 6 is depicted in Fig. 3(c),

This scheme achieves capacity (see Table 【I) when the channel reduces to a CIFC in the very weak interference regime [18].
4) Common messages for the sources and private messages from the cognitive relay: scheme with only $\left(U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{1 \mathrm{pb}}\right)$ :
Here we aim to expand the scheme that achieves capacity it the "very weak interference" regime for the CIFC [18] (see Table II) by having the two sources transmit common messages while the cognitive relay sends part of a private message for source 1 .

Corollary IV.7. By considering $X_{1}=U_{1 \mathrm{c}}, X_{2}=U_{2 \mathrm{c}}, X_{c}=$ $U_{1 \mathrm{pb}}, U_{0 \mathrm{cb}}=U_{1 \mathrm{pb}}, U_{2 p b}=\emptyset$ in Thm. NV.1 the following rate region is achievable

$$
\begin{align*}
R_{1} & \leq I\left(Y_{1} ; X_{1}, X_{c} \mid X_{2}, Q\right)  \tag{20a}\\
R_{1} & \leq I\left(Y_{1} ; X_{c} \mid X_{1}, X_{2}\right)+I\left(Y_{2} ; X_{1} \mid X_{2}, Q\right)  \tag{20b}\\
R_{2} & \leq I\left(Y_{1} ; X_{2}, X_{c} \mid X_{1}, Q\right)  \tag{20c}\\
R_{2} & \leq I\left(Y_{2} ; X_{2} \mid X_{1}, Q\right)  \tag{20d}\\
R_{1}+R_{2} & \leq I\left(Y_{1} ; X_{1}, X_{2}, X_{c}, Q\right)  \tag{20e}\\
R_{1}+R_{2} & \leq I\left(Y_{1} ; X_{2}, X_{c} \mid X_{1}, Q\right)+I\left(Y_{2} ; X_{1} \mid X_{2}, Q\right)  \tag{20f}\\
R_{1}+R_{2} & \leq I\left(Y_{1} ; X_{c} \mid X_{1}, X_{2}, Q\right)+I\left(Y_{2} ; X_{2}, X_{2}, Q\right)  \tag{20~g}\\
R_{1}+2 R_{2} & \leq I\left(Y_{1} ; X_{2}, X_{c} \mid X_{1}\right)+I\left(Y_{2} ; X_{1}, X_{2}, Q\right) \tag{20h}
\end{align*}
$$

for some distributions that factor as

$$
P_{Q} P_{X_{1} \mid Q} P_{X_{1} \mid Q} P_{X_{c} \mid X_{1}, X_{2}, Q}
$$

Proof: The proof may be found in Appendix J
A graphical representation of the achievable rate region of Cor. IV.7 is depicted in Fig. 3(d),

## V. CAPACITY IN "VERY STRONG INTERFERENCE AT RX 1" AND IN "STRONG INTERFERENCE AT BOTH RXS"

In this section we show the achievability of the outer bound in Cor. III.4 in the "very strong interference at Rx 1" and the "strong interference at both Rxs" regime (to be defined later), which are two subsets of the "strong interference" regime defined by (6). These results parallel the "very strong interference" capacity result for the IFC [36] and the CIFC [37], where, the channel reduces to a compound two-user MAC. For this class of channels the interfering signal at each receiver can be decoded without loss of optimality. Since the interference can always be distinguished from the intended signal, there is no need to perform interference pre-coding at the cognitive relay. This greatly simplifies the achievable scheme required to match the outer bound in Cor. III. 4 and the simple superposition coding scheme in Cor. IV.5 will be shown to be optimal.
Theorem V.1. Capacity in "very strong interference at Rx 1". If

$$
\begin{align*}
& I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1}\right) \leq I\left(Y_{1} ; X_{2}, X_{c} \mid X_{1}\right)  \tag{21a}\\
& I\left(Y_{1} ; X_{1}, X_{2}, X_{c}\right) \leq I\left(Y_{2} ; X_{1}, X_{2}, X_{c}\right) \tag{21b}
\end{align*}
$$

holds for all distributions that factor as $P_{X_{1}, X_{2}, X_{c}}=$ $P_{X_{1}} P_{X_{2}} P_{X_{c} \mid X_{1}, X_{2}}$ (same factorization as in (7)), then the region in Cor. III.4 is capacity.

Proof: Under the condition in 21a) (which is the same as the "strong interference at Rx 1 " condition in (6) the region in (8) is an outer bound for the considered IFC-CR. Consider now the achievable rate region in Cor. IV.5 given by (18). Under the condition in (21b) the sum-rate bound in (18d) is redundant and the resulting region coincides with the outer bound in (8).

Theorem V.2. Capacity in "strong interference at both Rxs". If

$$
\begin{align*}
& I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1}\right) \leq I\left(Y_{1} ; X_{2}, X_{c} \mid X_{1}\right)  \tag{22a}\\
& I\left(Y_{1} ; X_{1}, X_{c} \mid X_{2}\right) \leq I\left(Y_{2} ; X_{1}, X_{c} \mid X_{2}\right) \tag{22b}
\end{align*}
$$

holds for all distributions that factor as $P_{X_{1}, X_{2}, X_{c}}=$ $P_{X_{1}} P_{X_{2}} P_{X_{c} \mid X_{1}, X_{2}}$ (same factorization as in (7)), then the region in (18) is capacity.

Proof: The proof follows similarly to that of Thm. V.1,

## VI. The Gaussian Case

In the following, to obtain more of a feel for the channel model and the conditions under which capacity holds, we evaluate the "strong interference" outer bound conditions and the region in Cor. III.4, as well as the "very strong interference" capacity conditions and the region in Thm. V. 1 for the Gaussian IFC-CR (G-IFC-CR).

## A. Channel Model

The G-IFC-CR is shown in Fig. 4 Without loss of generality (see Appendix K) we can restrict our attention to the G-IFCCR in standard form given by:

$$
\begin{align*}
& Y_{1}=\left|h_{11}\right| X_{1}+\left|h_{2 c}\right| X_{c}+h_{12} X_{2}+Z_{1}  \tag{23a}\\
& Y_{2}=\left|h_{22}\right| X_{2}+\left|h_{2 c}\right| X_{c}+h_{21} X_{1}+Z_{2} \tag{23b}
\end{align*}
$$

where $h_{i} \in \mathbb{C}, i \in\{11,1 c, 12,22,2 c, 21\}$, are constant and known to all terminals, $Z_{i} \sim \mathcal{N}_{\mathbb{C}}(0,1), i \in\{1,2\}$, and $\mathbb{E}\left[\left|X_{i}\right|^{2}\right] \leq 1, i \in\{1,2, c\}$. The channel links $h_{i}, i \in$ $\{11,22,1 c, 2 c\}$ can be taken to be real-valued without loss of generality because receivers and transmitters can compensate for the phase of the signals. The correlation among the noises is irrelevant because the capacity of the channel without receiver cooperation only depends on the noise marginal distributions.

## B. Gaussian Channel under "strong interference at Rx 1"

We now evaluate Cor. III. 4 and Thm. V. 1 for the G-IFC-CR.

## Theorem VI.1. The "strong interference at Rx 1" outer bound for the G-IFC-CR. If

$$
\begin{equation*}
\left|\left|h_{22}\right|+\widetilde{\beta}_{2}\right| h_{2 c}| |^{2} \leq\left|h_{12}+\widetilde{\beta}_{2}\right| h_{1 c}| |^{2} \tag{24}
\end{equation*}
$$



Fig. 4. The Gaussian IFC-CR in standard form.
for

$$
\begin{align*}
\measuredangle \widetilde{\beta}_{2} & =\measuredangle\left(\left|h_{2 c}\right|\left|h_{22}\right|-\left|h_{1 c}\right| h_{12}\right),  \tag{25a}\\
\left|\widetilde{\beta}_{2}\right|^{2} & = \begin{cases}1 & \text { if }\left|h_{2 c}\right| \geq\left|h_{1 c}\right| \\
\min \left\{1, \frac{\left|\left|h_{2 c}\right|\right| h_{22}\left|-\left|h_{1 c}\right| h_{12}\right|}{\left|\left|h_{2 c}\right|^{2}-\left|h_{1 c}\right|^{2}\right|}\right\} & \text { if }\left|h_{2 c}\right|<\left|h_{1 c}\right|\end{cases} \tag{25b}
\end{align*}
$$

the capacity of a G-IFC-CR is contained in the set:

$$
\begin{align*}
& R_{1} \leq \mathcal{C}\left(\left\|h_{11}\left|+\beta_{1}^{*}\right| h_{1 c}\right\|^{2}+\left|h_{1 c}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}\right)\right)  \tag{26a}\\
& R_{2} \leq \mathcal{C}\left(\left\|h_{22}\left|+\beta_{2}^{*}\right| h_{2 c}\right\|^{2}+\left|h_{2 c}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}\right)\right) \tag{26b}
\end{align*}
$$

$$
\begin{align*}
R_{1}+R_{2} \leq & \mathcal{C}\left(\left\|h_{11}\left|+\beta_{1}^{*}\right| h_{1 c}\right\|^{2}+\left|h_{12}+\beta_{2}^{*}\right| h_{1 c} \|^{2}\right. \\
& \left.+\left|h_{1 c}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}\right)\right), \tag{26c}
\end{align*}
$$

taken over the union of all $\left(\beta_{1}, \beta_{2}\right) \in \mathbb{C}^{2}:\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2} \leq 1$, where $\mathcal{C}(x):=\log (1+x)$.

Proof: The proof may be found in Appendix L

## Theorem VI.2. Capacity in "very strong interference at Rx 1" for the Gaussian IFC-CR.

If in addition to the condition in (24) the following also holds

$$
\begin{align*}
& \left\{\left(\left|h_{11}\right|^{2}+\left|h_{1 c}\right|^{2}+\left|h_{12}\right|^{2}\right)-\left(\left|h_{21}\right|^{2}+\left|h_{2 c}\right|^{2}+\left|h_{22}\right|^{2}\right)\right. \\
& \left.\quad+2 \sqrt{\left.| | h_{11}| | h_{1 c}\left|-h_{21}\right| h_{2 c}\right|^{2}+\left.\left|h_{12}\right| h_{1 c}\left|-\left|h_{22}\right|\right| h_{2 c}\right|^{2}}\right\} \leq 0 \tag{27}
\end{align*}
$$

then the region in (26) is capacity.
Proof: The proof may be found in Appendix M
Remark VI.3. Thm. VI.2 reduce to known capacity results in the "very strong interference" regime when the IFC-CR reduces to a simpler channel:

- When the IFC-CR reduces to an IFC, i.e., $\left|h_{1 c}\right|=\left|h_{2 c}\right|=$ 0 , the condition in (24) reduces to the well-known "strong interference at Rx 1" $\left|h_{22}\right|^{2} \leq\left|h_{12}\right|^{2}$, and the condition in (27) to $\left|h_{11}\right|^{2}+\left|h_{12}\right|^{2} \leq\left|h_{21}\right|^{2}+\left|h_{22}\right|^{2}$ (larger total received power at Rx 2 than at Rx 1 ).
- When the IFC-CR reduces to a C-IFC with user 1 as primary user, i.e., $\left|h_{22}\right|=h_{12}=0$, the condition in (24)
reduces to $\left|h_{2 c}\right|^{2} \leq\left|h_{1 c}\right|^{2}$ (strong interference at the primary receiver) and the condition in (27) to

$$
\begin{aligned}
& \left|h_{11}\right|^{2}+\left|h_{1 c}\right|^{2}-\left|h_{21}\right|^{2}-\left|h_{2 c}\right|^{2} \\
& \quad+2| | h_{11}| | h_{1 c}\left|-h_{21}\right| h_{2 c}| | \leq 0
\end{aligned}
$$

which is the same as the condition in [2, Thm.II.3].

- When the IFC-CR reduces to a C-IFC with user 2 as primary user, i.e., $\left|h_{11}\right|=h_{21}=0$, the conditions in (24) and (27) are equivalent to $I\left(Y_{1} ; X_{2}, X_{c}\right)=$ $I\left(Y_{2} ; X_{2}, X_{c}\right)$ for all input distributions, that is,

$$
\begin{aligned}
& \left\{h_{12}=\left|h_{22}\right|,\left|h_{1 c}\right|=\left|h_{2 c}\right|\right\} \\
& \quad \text { or } \quad\left\{h_{12}=\left|h_{2 c}\right|,\left|h_{22}\right|=\left|h_{1 c}\right|\right\}
\end{aligned}
$$

- When the IFC-CR reduces to a BC. i.e., $\left|h_{11}\right|=h_{21}=$ $\left|h_{22}\right|=h_{12}=0$ the conditions in (24) and (27) are equivalent to $I\left(Y_{1} ; X_{c}\right)=I\left(Y_{2} ; X_{c}\right)$ for all input distributions, that is, a BC with statistically equivalent receivers, i.e., $\left|h_{2 c}\right|=\left|h_{1 c}\right|$.
Theorem VI.4. Capacity in "strong interference at both Rxs" for the G-IFC-CR. When the condition in (24) along with the symmetric condition for source-destination pair 2 hold, the region

$$
\begin{align*}
R_{1} \leq & \mathcal{C}\left(\left|\left|h_{11}\right|+\beta_{1}^{*}\right| h_{1 c} \|^{2}+\left|h_{1 c}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}\right)\right) \\
R_{2} \leq & \mathcal{C}\left(\left|\left|h_{22}\right|+\beta_{2}^{*}\right| h_{2 c} \|^{2}+\left|h_{2 c}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}\right)\right)  \tag{28b}\\
R_{1}+R_{2} \leq & \mathcal{C}\left(\left|\left|h_{11}\right|+\beta_{1}^{*}\right| h_{1 c}\left\|^{2}+\left|h_{12}+\beta_{2}^{*}\right| h_{1 c}\right\|^{2}\right. \\
& \left.+\left|h_{1 c}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}\right)\right)  \tag{28c}\\
R_{1}+R_{2} \leq & \mathcal{C}\left(\left|h_{21}+\beta_{1}^{*}\right| h_{2 c}\left\|^{2}+\left|\left|h_{22}\right|+\beta_{2}^{*}\right| h_{2 c}\right\|^{2}\right. \\
& \left.+\left|h_{2 c}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}\right)\right) \tag{28d}
\end{align*}
$$

taken over the union of all $\left(\beta_{1}, \beta_{2}\right) \in \mathbb{C}^{2}:\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2} \leq 1$ is capacity.

Proof: The proof follows similarly to the one of Thm. VI. 1
Remark VI.5. Thm. VI. 4 reduce to known capacity results when the IFC-CR reduces to a simpler channel:

- When the IFC-CR reduces to an IFC, i.e., $\left|h_{1 c}\right|=\left|h_{2 c}\right|=$ 0 , the condition in (24) reduces to the well-known "strong interference" regime, $\left\{\left|h_{22}\right|^{2} \leq\left|h_{12}\right|^{2},\left|h_{11}\right|^{2} \leq\left|h_{21}\right|^{2}\right\}$.
- When the IFC-CR reduces to a C-IFC with user 1 as primary user, i.e., $\left|h_{22}\right|=h_{12}=0$ or $X_{2}=\emptyset$, interestingly, the "very strong interference at Rx 1 " condition is equivalent to the "strong interference at both Rx's" condition. This can be seen by noticing that for $X_{2}=\emptyset$ the conditions in (21) coincide with the conditions in (22).
- When the IFC-CR reduces to a C-IFC with user 2 as primary user, i.e., $\left|h_{11}\right|=h_{21}=0$, we have the equivalent of case $\left|h_{22}\right|=h_{12}=0$ in the above bullet point but with the role of the users swapped.
- When the IFC-CR reduces to a BC. i.e., $\left|h_{1 c}\right|=\left|h_{2 c}\right|=$ $\left|h_{22}\right|=h_{12}=0$ the "strong interference at both Rx's" condition and the "very strong interference at

Rx 1" conditions are the same and are equivalent to $I\left(Y_{1} ; X_{c}\right)=I\left(Y_{2} ; X_{c}\right)$ for all input distributions, that is, $\left|h_{1 c}\right|=\left|h_{2 c}\right|$.

## C. Gaussian Channel under "weak interference"

The condition in (9) for the "weak interference at Rx 2" outer bound in Cor. III.5 is, in general, very hard to verify as it must hold for a large set of distribution involving an auxiliary RV. In this section we restrict attention to a special class of G-IFC-CR in which the condition in (9) is easily verified, namely a class of "degraded" G-IFC-CR defined by

$$
\begin{equation*}
\frac{h_{21}}{\left|h_{11}\right|}=\frac{\left|h_{2 c}\right|}{\left|h_{1 c}\right|}:=|\rho| \in[0,1] \tag{29}
\end{equation*}
$$

so that the channel input/output relationship becomes

$$
\begin{align*}
& Y_{1}=\left|h_{11}\right| X_{1}+\left|h_{1 c}\right| X_{c}+h_{12} X_{2}+Z_{1}  \tag{30a}\\
& Y_{2}=|\rho|\left(\left|h_{11}\right| X_{1}+\left|h_{1 c}\right| X_{c}\right)+\left|h_{22}\right| X_{2}+Z_{2} \tag{30b}
\end{align*}
$$

Since the noise correlation among the noises is irrelevant for capacity, conditioned on $X_{2}$ we have the following Markov chain

$$
\begin{align*}
X_{\mathrm{eq}} & \rightarrow Y_{1} \rightarrow Y_{2}  \tag{31}\\
& X_{\mathrm{eq}}:=\left|h_{11}\right| X_{1}+\left|h_{1 c}\right| X_{c} \\
& Y_{2} \sim|\rho| Y_{1}+\sqrt{1-|\rho|^{2}} Z_{0} \\
& Z_{0} \sim \mathcal{N}_{\mathbb{C}}(0,1) \text { independent of everything else },
\end{align*}
$$

in other words, conditioned on $X_{2}$, the channel in (29) is equivalent to a SISO degraded BC with input $X_{\text {eq. }}$. From (31) and for any $P_{U, X_{1}, X_{2}, X_{c}}$ such that $U \rightarrow\left(X_{1}, X_{2}, X_{c}\right) \rightarrow$ $\left(Y_{1}, Y_{2}\right)$ we have that

$$
I\left(U ; Y_{2} \mid X_{2}\right) \leq I\left(U ; Y_{1} \mid X_{2}\right)
$$

which is exactly the "weak interference at Rx 2 " condition in (9).

Theorem VI.6. The "weak interference at Rx 2" outer bound for the degraded G-IFC-CR. For the degraded $G$ -IFC-CR in the capacity region is contained into the region

$$
\begin{align*}
R_{1} \leq & \mathcal{C}\left(\left|\left|h_{11}\right|+\left|h_{1 c}\right| \beta_{1}^{*}\right|^{2} \alpha\right)  \tag{32a}\\
R_{2} \leq & \mathcal{C}\left(|\rho|^{2}| | h_{11}\left|+\left|h_{1 c}\right| \beta_{1}^{*}\right|^{2}+\left(\left|h_{22}\right|+|\rho|\left|h_{1 c}\right| \beta_{2}^{*}\right)^{2}\right) \\
& -\mathcal{C}\left(|\rho|^{2}| | h_{11}\left|+\left|h_{1 c}\right| \beta_{1}^{*}\right|^{2} \alpha\right)  \tag{32b}\\
R_{2} \leq & \mathcal{C}\left(\left(\left|h_{22}\right|+|\rho|\left|h_{1 c}\right| \beta_{2}^{*}\right)^{2}\right) \tag{32c}
\end{align*}
$$

taken over the union of all $\alpha \in[0,1]$ and $\left(\beta_{1}, \beta_{2}\right)$ such that $\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}=1$.

Proof: The proof can be found in Appendix N
Remark VI.7. Special cases for the outer bound in Thm.VI.6.

- When $\left|h_{1 c}\right|=0$, the channel in (30) reduces to an IFC with "weak interference" at receiver 2 whose capacity is not known. The outer bound in Thm. VI.6 in this case is looser than the outer bounds in [46], [34]. However, the

Sato-type outer bound in Thm. [II.1] reduces to [46] and the tightened outer bound in [8] reduces to [34].

- When the IFC-CR reduces to a C-IFC with user 1 as primary user, i.e., $\left|h_{22}\right|=h_{12}=0$, the channel in (30) reduces to a Gaussian degraded CIFC [1] whose capacity is not known. The outer bound in Thm. VI. 6 in this case is looser that the outer bound in [1, Cor. 3.5]. In this case, the best known outer bound in [1, Cor. 3.5] is still of BC-type, from a MIMO BC with degraded message set however.
- When the IFC-CR reduces to a C-IFC with user 2 as primary user, i.e., $\left|h_{11}\right|=0$, the channel in (30) reduces to a Gaussian CIFC in weak interference [18] whose capacity is known [18], [21]. The outer bound in Thm. VI. 6 in this case reduces to capacity.
- When the IFC-CR reduces to a BC. i.e., $\left|h_{11}\right|=h_{21}=$ $\left|h_{22}\right|=0$, the channel in (30) reduces to a degraded SISO BC whose capacity is known [47]. The outer bound in Thm. VI. 6 in this case reduces to capacity.


## VII. NumERICAL Evaluations

In this section we present a series of numerical evaluations of the results presented in the paper for the G-IFC-CR with real-valued inputs and real-valued channel coefficients. Using numerical examples, we investigate the relationship between inner and outer bounds as well as the position and extension of the "strong", "weak" and "very strong" interference regimes.

In Fig. 5 we depict

- Fig. 5(a) the "strong interference at Rx 1 " regime of (24) and the "very strong interference at Rx 1 " regime of (27),
- Fig. 5(b), the "strong interference at Rx 2" regime of (24) and the "very strong interference at Rx 2 " regime of (27),
- Fig. 5(c) the "strong interference" regime of (24) at Rx 1 and at Rx 2 and the "strong interference at both Rxs" regime of Thm. VI.4,
- Fig. 5(d) the degraded G-IFC-CR of (29) and the "weak interference" regime of Thm. VI.6,
for fixed $h_{11}=h_{22}=h_{1 c}=h_{2 c}=1$ on the plane $\left[h_{12}, h_{21}\right] \in$ $[-10,10] \times[-10,10]$.
Since $\left|h_{c}\right|=\left|h_{1 c}\right|=\left|h_{2 c}\right|$, from (25) we have that the "strong interference" condition becomes linear in $h_{21}$ and $h_{12}$, i.e. condition (24) becomes:

$$
\begin{array}{r}
\left|\left|h_{11}\right|+\left|h_{2 c}\right|\right|^{2} \leq\left|h_{21}+\left|h_{2 c}\right|\right|^{2} \\
\left(\left|h_{11}\right|-h_{21}\right)\left(\left|h_{11}\right|+h_{21}+2\left|h_{c}\right|\right) \leq 0 \tag{34}
\end{array}
$$

Similarly, since $\left|h_{c}\right|=\left|h_{1 c}\right|=\left|h_{2 c}\right|$, the degraded condition at destination 1 in (29) coincides with $\left|h_{11}\right|=h_{21}$ : from this consideration and given (34), we have that the degraded channel at destination 1 is also in "strong interference" at destination 2. Given the symmetry of the channel, we also have that the degraded channel at destination 2 is also in "strong interference" at destination 1.

In Fig. 6 we plot the conditions

$$
\begin{align*}
I\left(Y_{1} ; X_{1}, X_{2}, X_{c}\right) & =I\left(Y_{2} ; X_{1}, X_{2}, X_{c}\right)  \tag{35a}\\
I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1}\right) & =I\left(Y_{1} ; X_{2}, X_{c} \mid X_{1}\right)  \tag{35b}\\
I\left(Y_{1} ; X_{1}, X_{c} \mid X_{2}\right) & =I\left(Y_{2} ; X_{1}, X_{c} \mid X_{2}\right) \tag{35c}
\end{align*}
$$


(a) The "strong" (blue, hatched) and the "very strong interference at Rx 1 " (blue, cross-hatched) regimes

(c) The "strong interference at Rx 1 " (green hatched) and the "strong interference at Rx 2" (blue-hatched) regimes.

(b) The "strong" (green, hatched) and the "very strong interference at Rx 2 " (green, cross-hatched) regimes.

(d) The degraded the G-IFC-CR for Rx 1 (blue, dotted) and Rx 2 (green, dotted ) and the "weak interference" regime for Rx 1 (blue solid) and Rx 2 (green solid)

Fig. 5. Different parameter regimes for G-IFC-CR with $h_{11}=h_{22}=1, h_{1 c}=h_{2 c}=2$ and $\left[h_{12}, h_{21}\right] \in[-10,10] \times[-10,10]$.
for increasing values of $\left|h_{c}\right|=\left|h_{1 c}\right|=\left|h_{2 c}\right| \in[1,5]$ for fixed $\left|h_{11}\right|=\left|h_{22}\right|=1$ on the plane $\left[h_{12}, h_{21}\right] \in[-10,10] \times$ $[-10,10]$. The line corresponding to each condition marks the boundary of the "strong interference" and the "very strong interference" conditions at destination 1 and 2. The darker hues are associated with smaller values of $\left|h_{c}\right|$ while lighter hues with larger values. While the boundaries of the "strong interference" regime are always linear in $h_{12}, h_{21}$ for any $\left|h_{c}\right|$, the "very strong interference" condition is approximated by an hyperbole for large $h_{21}$ and $h_{12}$.

In Figs. 7, 8 and 9 we compare inner and outer bounds for three points in the plane $\left[h_{12}, h_{21}\right] \in[-10: 10] \times[-10,10]$ for fixed $\left|h_{11}\right|=\left|h_{22}\right|=\left|h_{1 c}\right|=\left|h_{2 c}\right|=1$ :

- Fig. $7\left(h_{12}, h_{21}\right)=(-2,-2)$, where the Sato type outer bound of Thm. III.1 holds, but not the outer bounds of Thm. VI. 1 or Thm. VI.6,
- Fig. $8\left(h_{12}, h_{21}\right)=(-2,+1)$, where the Sato type outer bound of Thm. III. 1 and the "strong interference at Rx 2" outer bound of Thm. VI.1 hold;
- Fig. $9\left(h_{12}, h_{21}\right)=(0.5,+1)$, where the Sato type outer bound of Thm. III.1, the "strong interference at Rx 2" outer bound of Thm. VI. 1 and the "weak interference at Rx 1" outer bound of Thm. VI. 6 hold.
In Fig. 7 we notice that a combination of common and private message, the scheme in Sec. IV-C3, outperforms the
schemes that utilize only common or only private messages, the schemes in Sec. IV-C1 and Sec. IV-C2] respectively. Despite of the good performance of the scheme in Sec. IV-C3, a substantial distance between inner and outer bound can be observed. The outer bound of Thm. III. 2 is known to be capacity for the CIFC in "weak" interference, "very strong" interference and for the "primary decodes cognitive" regime [2]. This result shows that the outer bound in Thm. III.2 is not tight far all the parameter region.

Fig. 8 shows that the "strong interference" outer bound of Cor. III. 4 is tighter than the Sato-type outer bound in Thm. III. 2 for some rate pairs. The scheme with one common and one private message in Sec. IV-C3 outperforms the schemes in Sec. IV-C2, Sec. IV-C1 and Sec. IV-C4 although the performance is comparable for some parameter values.
In Fig. 9 we observe that the "weak interference" outer bound in VI. 6 is tighter than the Sato-type outer bound in III. 1 for some rate pairs, although the "strong interference" outer bound of VI. 1 remains the tightest in this case. For this specific choice of parameters the channel is both in "weak interference" at destination 1 as well as in "strong interference" at destination 2 . In this specific regime the scheme in IV-C3 approaches the strong interference outer bound for some parameter values. Since $Y_{2}$ is a degraded version of $Y_{1}$ conditioned on $X_{2}$, loosely speaking, there is no loss of generality in having receiver 1 decode the message in $X_{2}$; for

(a) The condition $I\left(Y_{1} ; X_{1}, X_{2}, X_{c}\right)=I\left(Y_{2} ; X_{1}, X_{2}, X_{c}\right)$ for increasing $h_{1 c}=h_{2 c}$

(b) The condition $I\left(Y_{1} ; X_{2}, X_{c} \mid X_{1}\right)=I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1}\right)$ for increasing $h_{1 c}=h_{2 c}$

(c) The condition $I\left(Y_{1} ; X_{1}, X_{c} \mid X_{2}\right)=I\left(Y_{2} ; X_{1}, X_{c} \mid X_{2}\right)$ for increasing $h_{1 c}=h_{2 c}$

Fig. 6. The conditions in 35) for $\left|h_{11}\right|=\left|h_{22}\right|=1$ and $h_{1 c}=h_{2 c} \in\{1 \ldots 5\}$.


Fig. 7. A plot for $\left|h_{11}\right|=\left|h_{22}\right|=\left|h_{1 c}\right|=\left|h_{2 c}\right|=1$ and $h_{12}=h_{21}=-2$.


Fig. 8. A plot for $\left|h_{11}\right|=\left|h_{22}\right|=\left|h_{1 c}\right|=\left|h_{2 c}\right|=1$ and $h_{12}=-2, h_{21}=+1$.


Fig. 9. A plot for $\left|h_{11}\right|=\left|h_{22}\right|=\left|h_{1 c}\right|=\left|h_{2 c}\right|=1$ and $h_{12}=.5, h_{21}=+1$.
this reasons one expects the scheme in Sec. IV-C3 to perform well in this case.

## VIII. Conclusion and Future Work

We introduce new, general outer bounds for the IFC-CR that are inspired by capacity results available for the broadcast channel and the cognitive interference channel. We show the achievability of one outer bound in the "very strong interference" regime by having both decoders decode both messages as in a compound multiple access channel. This result is very similar in nature to the "very strong interference" capacity results for the interference channel and the cognitive interference channel. We also derive the provably largest achievable rate region for this channel model by using classical random coding arguments such as rate splitting, superposition coding and binning. This region contains all the key transmission features using in achieving capacity in channels and classes of channels for which capacity is known. As such, this general achievable rate region is algebraically complex, but fairly general, and is shown to reduce to capacity for all sub-channels for which capacity is known. The contributions of this paper are a first step to a better understanding of the capacity region of the cognitive interference channel with a cognitive relay which remains largely undiscovered.

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## Appendix A <br> Proof of Theorem III. 1

From Fano's inequality, if $P_{e} \rightarrow 0$ as $N \rightarrow \infty$ then

$$
H\left(W_{i} \mid Y_{i}^{N}\right) \leq N \epsilon_{N} \quad \text { with } \quad \epsilon_{N} \rightarrow 0 \quad \text { as } \quad N \rightarrow \infty
$$

with $i \in\{1,2\}$ and thus
$N\left(R_{i}-\epsilon_{N}\right) \leq I\left(W_{i} ; Y_{i}^{N}\right) \leq I\left(W_{i} ; Y_{i}^{N} \mid W_{\bar{i}}\right), i \in\{1,2\}, \bar{i} \neq i$,
where the last inequality in the above expression follows from the independence of the source messages.

The rate $R_{1}$ can be bounded as in (1a) (and similarly for $R_{2}$ in (1b)) since

$$
\begin{aligned}
& N\left(R_{1}-\epsilon_{N}\right) \\
& \leq I\left(W_{1} ; Y_{1}^{N} \mid W_{2}\right)
\end{aligned}
$$

Fano's inequality
$=H\left(Y_{1}^{N} \mid W_{2}\right)-H\left(Y_{1}^{N} \mid W_{1}, W_{2}\right)$
Definition of mutual information
$\begin{aligned}= & H\left(Y_{1}^{N} \mid W_{2}, X_{2}^{N}\left(W_{2}\right)\right) \\ & -H\left(Y_{1}^{N} \mid W_{1}, W_{2}, X_{1}^{N}\left(W_{1}\right), X_{2}^{N}\left(W_{2}\right), X_{c}^{N}\left(W_{1}, W_{2}\right)\right)\end{aligned}$
Deterministic encoding
$=H\left(Y_{1}^{N} \mid W_{2}, X_{2}^{N}\right)-\sum_{t} H\left(Y_{1, t} \mid X_{1, t}, X_{2, t}, X_{c, t}\right)$
Memoryless channel
$=\sum_{t} H\left(Y_{1, t} \mid W_{2}, X_{2}^{N}, Y_{1}^{t-1}\right)-\sum_{t} H\left(Y_{1, t} \mid X_{1, t}, X_{2, t}, X_{c, t}\right)$
Chain rule for entropy
$\leq \sum_{t} H\left(Y_{1, t} \mid X_{2, t}\right)-\sum_{t} H\left(Y_{1, t} \mid X_{1, t}, X_{2, t}, X_{c, t}\right)$
Conditioning reduces entropy
$=\sum_{t} I\left(Y_{1, t} ; X_{1, t}, X_{c, t} \mid X_{2, t}\right)$
Definition of mutual information
$=N I\left(Y_{1} ; X_{1}, X_{c} \mid X_{2}, Q\right)$,
Introduction of time-sharing RV
where, in the last equality, $Q$ is a time sharing RV that is independent of all other RVs and uniformly distributed on $[1$ : $N]$.

Next, let $\widetilde{Y}_{i}^{N}$ have the same conditional marginal distribution as $Y_{i}^{N}, i \in\{1,2\}$. The Sato-type bound [9] sum-rate bounds in (1c) and (1d) follow since

$$
\begin{aligned}
& N\left(R_{1}+R_{2}-2 \epsilon_{N}\right) \\
& \leq I\left(Y_{1}^{N} ; W_{1} \mid W_{2}\right)+I\left(Y_{2}^{N} ; W_{2}\right)
\end{aligned}
$$

Fano's inequality

$$
\leq I\left(Y_{1}^{N}, \widetilde{Y}_{2}^{N} ; W_{1} \mid W_{2}\right)+I\left(Y_{2}^{N} ; W_{2}\right)
$$

Non-negativity of mutual information
$=I\left(Y_{1}^{N} ; W_{1} \mid W_{2}, \widetilde{Y}_{2}^{N}\right)+I\left(Y_{2}^{N} ; W_{1}, W_{2}\right)$
$\widetilde{Y}_{2}^{N}$ and $Y_{2}^{N}$ have the same marginal cdf
$\leq I\left(Y_{1}^{N} ; X_{1}^{N}, X_{c}^{N} \mid \widetilde{Y}_{2}^{N}, X_{2}^{N}\right)+I\left(Y_{2}^{N} ; X_{1}^{N}, X_{2}^{N}, X_{c}^{N}\right)$
$\leq N\left(I\left(Y_{1} ; X_{1}, X_{c} \mid \widetilde{Y}_{2}, X_{2}, Q\right)+I\left(Y_{2} ; X_{1}, X_{2}, X_{c} \mid Q\right)\right)$,
and where the last two inequalities follows from steps similar to the derivation of the bound on $R_{1}$ above.

Appendix B
Proof of Theorem III. 2

The bound in (3d), and similarly for (3c) but with the role of the users swapped, is obtained as follows

$$
\begin{aligned}
& N\left(R_{2}-\epsilon_{N}\right) \\
& \leq I\left(Y_{2}^{N} ; W_{2}\right) \\
& \quad \text { Fano's inequality } \\
& =\sum_{i=1}^{N} H\left(Y_{2, i} \mid Y_{2, i+1}^{N}\right)-H\left(Y_{2, i} \mid Y_{2, i+1}^{N}, W_{2}\right) \\
& \quad \text { Chain rule for entropy } \\
& \leq \sum_{i=1}^{N} H\left(Y_{2, i}\right)-H\left(Y_{2, i} \mid Y_{1}^{i-1}, Y_{2, i+1}^{N}, W_{2}, X_{2, i}\right)
\end{aligned}
$$

Conditioning reduces entropy
$=\sum_{i=1}^{N} I\left(Y_{2, i} ; V_{i}, U_{2, i}, X_{2, i}\right)$,
where we defined

$$
\begin{aligned}
& U_{u, i}:=\left[W_{u}\right], u \in\{1,2\}, \\
& V_{i}:=\left[Y_{2, i+1}^{N}, Y_{1}^{i-1}\right] .
\end{aligned}
$$

The bound of (3b), and similarly for (3a) but with the role
of the users swapped, is obtained as follows

$$
\begin{aligned}
N & \left(R_{2}-\epsilon_{N}\right) \\
\leq & I\left(Y_{2}^{N} ; W_{2} \mid W_{1}\right) \\
= & \sum_{i=1}^{N} H\left(Y_{2, i} \mid Y_{2, i+1}^{N}, W_{1}, X_{1, i}\right) \\
& -H\left(Y_{2, i} \mid Y_{2, i+1}^{N}, W_{2}, X_{2, i}, W_{1}, X_{1, i}, X_{c, i}\right) \\
\leq & \sum_{i=1}^{N} H\left(Y_{2, i} \mid W_{1}, X_{1, i}\right) \\
& -H\left(Y_{2, i} \mid Y_{1}^{i-1}, Y_{2, i+1}^{N}, W_{2}, X_{2, i}, W_{1}, X_{1, i}, X_{c, i}\right) \\
= & \sum_{i=1}^{N} I\left(Y_{2, i} ; V_{i}, U_{2, i}, X_{2, i}, X_{c, i} \mid U_{1, i}, X_{1, i}\right) \\
= & \sum_{i=1}^{N} I\left(Y_{2, i} ; X_{2, i}, X_{c, i} \mid U_{1, i}, X_{1, i}\right)
\end{aligned}
$$

The sum-rate bound in (3e), and similarly for (3f) but with the role of the users swapped, is obtained as

$$
\begin{aligned}
N & \left(R_{1}+R_{2}-2 \epsilon_{N}\right) \\
\leq & I\left(Y_{1}^{N} ; W_{1} \mid W_{2}\right)+I\left(Y_{2}^{N} ; W_{2}\right) \\
\leq & \sum_{i=1}^{N} I\left(Y_{1, i} ; W_{1}, Y_{2, i+1}^{N} \mid Y_{1}^{i-1}, W_{2}, X_{2, i}\right) \\
& +I\left(Y_{2, i} ; W_{2}, X_{2, i}, Y_{2, i+1}^{N}\right) \\
= & \sum_{i=1}^{N} I\left(Y_{1, i} ; Y_{2, i+1}^{N} \mid Y_{1}^{i-1}, W_{2}, X_{2, i}\right) \\
& -I\left(Y_{2, i} ; Y_{1}^{i-1} \mid W_{2}, X_{2, i}, Y_{2, i+1}^{N}\right) \\
& +I\left(Y_{1, i} ; W_{1} \mid Y_{1}^{i-1}, Y_{2, i+1}^{N}, W_{2}, X_{2, i}\right) \\
& +I\left(Y_{2, i} ; W_{2}, X_{2, i}, Y_{2, i+1}^{N}, Y_{1}^{i-1}\right) \\
\stackrel{(a)}{=} & \sum_{i=1}^{N} I\left(Y_{1, i} ; W_{1} \mid Y_{1}^{i-1}, Y_{2, i+1}^{N}, W_{2}, X_{2, i}\right) \\
& +I\left(Y_{2, i} ; Y_{2, i+1}^{N}, Y_{1}^{i-1}, W_{2}, X_{2, i}\right) \\
= & \sum_{i=1}^{N} I\left(Y_{1, i} ; U_{1, i}, X_{1, i}, X_{c, i} \mid V_{i}, U_{2, i}, X_{2, i}\right) \\
& +I\left(Y_{2, i} ; V_{i}, U_{2, i}, X_{2, i}\right) \\
= & \sum_{i=1}^{N} I\left(Y_{1, i} ; X_{1, i}, X_{c, i} \mid V_{i}, U_{2, i}, X_{2, i}\right)+I\left(Y_{2, i} ; V_{i}, U_{2, i}, X_{2, i}\right)
\end{aligned}
$$

where the equality in (a) follows from the "Csiszár's sum identity" [40]. Note that the Markov chain in (4) holds since for all $i \in[1: N]$ we have

$$
V_{i} \rightarrow\left(U_{1, i}, U_{2, i}\right) \rightarrow\left(X_{1, i}, X_{2, i}, X_{c, i}\right) \rightarrow\left(Y_{1, i}, Y_{2, i}\right)
$$

owing to the cognition structure and the memoryless channel that imply

$$
\begin{aligned}
& P_{W_{1}, W_{2}, X_{1}^{N}, X_{2}^{N}, X_{c}^{N}, Y_{1}^{N}, Y_{2}^{N}} \\
& =P_{W_{1}} P_{W_{2}} \prod_{i=1}^{N} \delta\left(W_{1}-U_{1, i}\right) \delta\left(W_{2}-U_{2, i}\right) P_{X_{1, i} \mid U_{1, i}} P_{X_{2, i} \mid U_{2, i}} \\
& P_{X_{c, i} \mid U_{1, i}, U_{2, i}} P_{Y_{1, i}, Y_{2, i} \mid X_{1, i}, X_{2, i}, X_{c, i}}
\end{aligned}
$$

from which the factorization in (5) also follows.
Note that we do not need a time sharing RV here since $Q$ can be incorporated in the RV $V$ without loss of generality.

## Appendix C <br> Proof of Corollary III. 4

Similar to [48, Lem. 4] and [36, Lem. 1], if the condition in (6) holds for all distributions in (7), then

$$
\begin{equation*}
I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1}, U\right) \leq I\left(Y_{1} ; X_{2}, X_{c} \mid X_{1}, U\right) \tag{36}
\end{equation*}
$$

for all $P_{X_{1}, X_{2}, X_{c}, U}=P_{X_{1}} P_{X_{2}} P_{X_{c} \mid X_{1}, X_{2}} P_{U \mid X_{1}, X_{2}, X_{c}}$. From this, it follows that when condition (6) holds, the bound in (3f) may be upper bounded as:

$$
\begin{aligned}
& I\left(Y_{1} ; V, U_{1}, X_{1}\right)+I\left(Y_{2} ; U_{2}, X_{2}, X_{c} \mid V, U_{1}, X_{1}\right) \\
& =I\left(Y_{1} ; V, U_{1}, X_{1}\right)+I\left(Y_{2} ; X_{2}, X_{c} \mid V, U_{1}, X_{1}\right) \\
& \leq I\left(Y_{1} ; V, U_{1}, X_{1}\right)+I\left(Y_{1} ; X_{2}, X_{c} \mid V, U_{1}, X_{1}\right) \\
& =I\left(Y_{1} ; X_{2}, X_{c}, V, U_{1}, X_{1}\right) \\
& =I\left(Y_{1} ; X_{1}, X_{2}, X_{c}\right)
\end{aligned}
$$

where the last equality follows from the Markov chain in (4)

## Appendix D <br> Proof of Corollary III. 5

Consider dropping from the outer bound in Thm. III. 2 all rate constraints but (3a), 3d) and (3e), i.e., consider the outer bound

$$
\begin{align*}
R_{1} & \leq I\left(Y_{1} ; X_{1}, X_{c} \mid U_{2}, X_{2}\right)  \tag{37a}\\
R_{2} & \leq I\left(Y_{2} ; V, U_{2}, X_{2}\right)  \tag{37b}\\
R_{1}+R_{2} & \leq I\left(Y_{2} ; V, U_{2}, X_{2}\right)+I\left(Y_{1} ; X_{1}, X_{c} \mid V, U_{2}, X_{2}\right) \tag{37c}
\end{align*}
$$

We intend to show that when the condition in (9) holds for all distributions in (10), the region in (37) can be rewritten as

$$
\begin{align*}
& R_{1} \leq I\left(Y_{1} ; X_{1}, X_{c} \mid V, U_{2}, X_{2}\right)  \tag{38a}\\
& R_{2} \leq I\left(Y_{2} ; V, U_{2}, X_{2}\right) \tag{38b}
\end{align*}
$$

which is equivalent to the region in 11a)-11b by defining $U=\left[V, U_{2}\right]$. Successively we show how the rate bound in (11c) can be added to the region in (38) to obtain a tighter outer bound.

For any fixed $P_{V, U_{2}, X_{1}, X_{2}, X_{c}}$, the region in (37) has three Pareto optimal points:

- $P_{1}=(0,37 b)$,
- $P_{2}=(37 \mathrm{c})-37 \mathrm{~b},(37 \mathrm{~b})$,
- $P_{3}=(\sqrt{37 a}$, (37c $-37 \mathrm{a})$.
- $P_{4}=(\sqrt{37 a}, 0)$.

We now show that the outer bound in (38) contains each of these points. By considering the union over all the possible distributions $P_{V, U_{2}, X_{1}, X_{2}, X_{c}}$ we can conclude that the outer bound in (38) is looser than (37). The corner points $P_{1}, P_{2}$ and $P_{4}$ are also corner points of the region in (38) for the
same $P_{V, U_{2}, X_{1}, X_{2}, X_{c}}$. Consider the region of (38) for $V=\emptyset$, then the corner point $P_{3}$ is included in such region when

$$
\begin{align*}
& I\left(Y_{2} ; U_{2}, X_{2}\right) \geq I\left(Y_{2} ; V, U_{2}, X_{2}\right)+I\left(Y_{1} ; X_{1}, X_{c} \mid V, U_{2}, X_{2}\right) \\
& \quad-I\left(Y_{1} ; X_{1}, X_{c} \mid U_{2}, X_{2}\right) \\
& I\left(Y_{2} ; V \mid U_{2}, X_{2}\right) \geq I\left(Y_{1} ; V \mid U_{2}, X_{2}\right)-I\left(Y_{1} ; V \mid U_{2}, X_{2}, X_{1}, X_{c}\right) \\
& I\left(Y_{2} ; V \mid U_{2}, X_{2}\right) \geq I\left(Y_{1} ; V \mid U_{2}, X_{2}\right) \tag{39}
\end{align*}
$$

where (39) follows from the Markov chain in (4). As for the App. C] the result of [48, Lem. 4] and [36, Lem. 1] assures that condition in (9) for $U=V$ implies that

$$
I\left(Y_{2} ; V \mid U_{2}, X_{2}\right) \geq I\left(Y_{1} ; V \mid U_{2}, X_{2}\right)
$$

for any $P_{X_{2}, U_{2}, V}$, from which it follows that (37) is contained into (38) when (9) holds. Finally the rate bound in (11c) is obtained from 11b) by noticing that

$$
\begin{equation*}
R_{2} \leq I\left(Y_{2} ; X_{2}, X_{c} \mid U_{1}, X_{1}\right) \leq I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1}\right) \tag{40}
\end{equation*}
$$

The bound in (40) is not required to prove capacity for the CIFC in "weak interference" [18], [21] but it can be tighter than 11b for Gaussian IFC-CR in "weak interference" of Sec. VI-C

## Appendix E Proof of Theorem IV. 1

For easy of notation we omit the time sharing $\operatorname{RV} Q$ in the following. The coding scheme is as follows.

## - Class of input distributions

Consider a distribution from (14).

- Rate-splitting

Each independent message $W_{i}, i \in\{1,2\}$, uniformly distributed on $\left[1: 2^{N R_{i}}\right]$, is split into four sub-messages:

- $W_{i c}$ : a common message transmitted by source $i$ for both destinations,
- $W_{i \mathrm{p}}$ : a private message transmitted by source $i$ for destination $i$,
- $W_{i c b}$ : a common message transmitted by the cognitive relay to both destinations,
- $W_{i \mathrm{pb}}$ : a private message transmitted by the cognitive relay to destination $i$.
The sub-messages $\left\{W_{k}\right\}_{k \in\{1 \mathrm{c}, 2 \mathrm{c}, 1 \mathrm{p}, 2 \mathrm{p}, 1 \mathrm{cb}, 2 \mathrm{cb}, 1 \mathrm{pb}, 2 \mathrm{pb}\}}$, are independent with $W_{k}$ uniformly distributed on [1: $\left.2^{N R_{k}}\right]$ so that

$$
\begin{align*}
W_{1}= & \left(W_{1 \mathrm{c}}, W_{1 \mathrm{p}}, W_{1 \mathrm{cb}}, W_{1 \mathrm{pb}}\right)  \tag{41a}\\
& R_{1}=R_{1 \mathrm{c}}+R_{1 \mathrm{p}}+R_{1 \mathrm{cb}}+R_{1 \mathrm{pb}}  \tag{41b}\\
W_{2}= & \left(W_{1 \mathrm{c}}, W_{2 \mathrm{p}}, W_{2 \mathrm{cb}}, W_{2 \mathrm{pb}}\right)  \tag{41c}\\
& R_{2}=R_{2 \mathrm{c}}+R_{2 \mathrm{p}}+R_{2 \mathrm{cb}}+R_{2 \mathrm{pb}} \tag{41d}
\end{align*}
$$

## - Code-book generation

Given any distribution in (14), the sources and the cognitive relay generate the following codebooks:

- Common message: $w_{i c} \in\left[1: 2^{N R_{i c}}\right]$ is encoded into $U_{i \mathrm{c}}^{N}\left(w_{i c}\right)$ with iid distribution $P_{U_{i c}}, i \in\{1,2\}$.
- Private message: for a given $w_{i c}, w_{i \mathrm{p}} \in\left[1: 2^{N R_{i \mathrm{p}}}\right]$ is encoded into $X_{i}^{N}\left(w_{i \mathrm{p}} \mid w_{i c}\right)$ with iid distribution $P_{X_{i} \mid U_{i c}}, i \in\{1,2\}$.
- Common broadcasted messages: for a given pair $\left(w_{1 \mathrm{c}}, w_{2 \mathrm{c}}\right)$, the pair $w_{1 \mathrm{cb}} \in$ $\left[1: 2^{N R_{1 \mathrm{cb}}}\right], w_{2 \mathrm{cb}} \in\left[1: 2^{N R_{2 \mathrm{cb}}}\right]$ is encoded into $\quad U_{0 \mathrm{cb}}^{N}\left(w_{1 \mathrm{cb}}, w_{2 \mathrm{cb}}, b_{0 \mathrm{cb}} \mid w_{1 \mathrm{c}}, w_{2 \mathrm{c}}\right)$, $b_{0 \mathrm{cb}} \in\left[1: 2^{N R_{0 \mathrm{cb}}^{\prime}}\right]$, with iid distribution $P_{U_{0 \mathrm{cb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}}$.
- Private broadcasted message: for a given $\quad\left(w_{1 \mathrm{c}}, w_{2 \mathrm{c}}, w_{1 \mathrm{cb}}, w_{2 \mathrm{cb}}, b_{0 \mathrm{cb}}, w_{i \mathrm{p}}\right)$, $w_{i \mathrm{pb}} \in\left[1: \quad 2^{N R_{i \mathrm{pb}}}\right] \quad$ is encoded into $U_{i \mathrm{pb}}^{N}\left(w_{i \mathrm{pb}}, b_{i \mathrm{pb}} \mid w_{1 \mathrm{c}}, w_{2 \mathrm{c}}, w_{1 \mathrm{cb}}, w_{2 \mathrm{cb}}, b_{0 \mathrm{cb}}, w_{i \mathrm{p}}\right)$, $b_{i \mathrm{ipb}} \in\left[1: \quad 2^{N R_{i \mathrm{pb}}^{\prime}}\right]$, with distribution $P_{U_{i \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{\mathrm{ocb}}, X_{i}}^{N}, i \in\{1,2\}$.


## - Encoding

Given $w_{1}=\left(w_{1 \mathrm{p}}, w_{1 \mathrm{c}}, w_{1 \mathrm{cb}}, w_{1 \mathrm{pb}}\right)$ and $w_{2}=$ $\left(w_{2 \mathrm{p}}, w_{2 \mathrm{c}}, w_{2 \mathrm{cb}}, w_{2 \mathrm{pb}}\right)$ :

- source 1 sends $X_{1}^{N}\left(w_{1 \mathrm{p}} \mid w_{1 \mathrm{c}}\right)$.
- source 2 sends $X_{2}^{N}\left(w_{2 \mathrm{p}} \mid w_{2 c}\right)$.
- First binning step: the cognitive relay looks for an index $b_{0 \text { cb }}$ such that

$$
\begin{align*}
& \left(U_{1 \mathrm{c}}^{N}\left(w_{1 \mathrm{c}}\right), X_{1}^{N}\left(w_{1 \mathrm{p}} \mid w_{1 \mathrm{c}}\right), U_{2 \mathrm{c}}^{N}\left(w_{2 \mathrm{c}}\right), X_{2}^{N}\left(w_{2 \mathrm{p}} \mid w_{2 \mathrm{c}}\right),\right. \\
& \left.U_{0 \mathrm{cb}}^{N}\left(w_{1 \mathrm{cb}}, w_{1 \mathrm{cb}}, b_{0 \mathrm{cb}} \mid w_{1 \mathrm{c}}, w_{2 \mathrm{c}}\right)\right) \\
& \quad \in T_{\epsilon}^{N}\left(P_{U_{0 \mathrm{cb}}, X_{1}, X_{2}, U_{1 \mathrm{c}}, U_{2 \mathrm{c}}}\right) \tag{42}
\end{align*}
$$

If more than one such index satisfies the relationship in (42), it selects one uniformly at random; if no such index exists, it sets $b_{0 c b}=1$ and in this case we say that a encoding error at the first binning step has occurred.

- Second binning step: Let $b_{0 c b}^{*}$ be the index determined at the first binning step. The cognitive relay looks for a pair of indexes $\left(b_{1 \mathrm{pb}}, b_{2 \mathrm{pb}}\right)$ such that

$$
\begin{align*}
& \left(U_{1 \mathrm{c}}^{N}\left(w_{1 \mathrm{c}}\right), X_{1}^{N}\left(w_{1 \mathrm{p}} \mid w_{1 \mathrm{c}}\right), U_{2 \mathrm{c}}^{N}\left(w_{2 \mathrm{c}}\right), X_{2}^{N}\left(w_{2 \mathrm{p}} \mid w_{2 \mathrm{c}}\right),\right. \\
& U_{0 \mathrm{cb}}^{N}\left(w_{1 \mathrm{cb}}, w_{1 \mathrm{cb}}, b_{0 \mathrm{cb}}^{*} \mid w_{1 \mathrm{c}}, w_{2 c}\right), \\
& U_{1 \mathrm{pb}}^{N}\left(w_{1 \mathrm{pb}}, b_{1 \mathrm{pb}} \mid w_{1 \mathrm{c}}, w_{2 \mathrm{c}}, w_{1 \mathrm{cb}}, w_{2 \mathrm{cb}}, b_{0 \mathrm{cb}}^{*}, w_{1 \mathrm{p}}\right), \\
& \left.U_{2 \mathrm{pb}}^{N}\left(w_{2 \mathrm{pb}}, b_{2 \mathrm{pb}} \mid w_{1 \mathrm{c}}, w_{2 \mathrm{c}}, w_{1 \mathrm{cb}}, w_{2 \mathrm{cb}}, b_{0 \mathrm{cb}}^{*}, w_{2 \mathrm{p}}\right)\right) \\
& \quad \in T_{\epsilon}^{N}\left(P_{\left.U_{1 \mathrm{pb}}, U_{2 \mathrm{pb}}, U_{0 \mathrm{cb}}, X_{1}, X_{2}, U_{1 \mathrm{c}}, U_{2 \mathrm{c}}\right) .}\right. \tag{43}
\end{align*}
$$

If more than one such pair of indices satisfies the relationship in (43), it selects one uniformly at random; if no such pair exists, it sets $\left(b_{1 \mathrm{pb}}, b_{2 \mathrm{pb}}\right)=(1,1)$ and in this case we say that a encoding error at the second binning step has occurred.

- For the found triplet $\left(b_{0 \mathrm{cb}}^{*}, b_{1 \mathrm{pb}}^{*}, b_{2 \mathrm{pb}}^{*}\right)$ the cognitive relay sends a codeword

$$
\begin{aligned}
& X_{c}^{N}\left(w_{1 \mathrm{pb}}, b_{1 \mathrm{pb}}^{*}, w_{2 \mathrm{pb}}, b_{2 \mathrm{pb}}^{*}\right. \\
& \left.\quad w_{1 \mathrm{cb}}, w_{2 \mathrm{cb}}, b_{0 \mathrm{cb}}^{*}, w_{1 \mathrm{c}}, w_{2 \mathrm{c}}, w_{1 \mathrm{p}}, w_{2 \mathrm{p}}\right)
\end{aligned}
$$

jointly typical with all the selected codewords.

## - Encoding error analysis

Given the symmetry of the codebook generation, we can assume without loss of generality that the messages

$$
\begin{aligned}
& W_{1}=\left(W_{1 \mathrm{c}}, W_{1 \mathrm{p}}, W_{1 \mathrm{cb}}, W_{1 \mathrm{pb}}\right)=(1,1,1,1), \\
& W_{2}=\left(W_{2 \mathrm{c}}, W_{2 \mathrm{p}}, W_{2 \mathrm{cb}}, W_{2 \mathrm{pb}}\right)=(1,1,1,1),
\end{aligned}
$$

were transmitted. We now derive the conditions under which encoding is successful with high probability. Let also $\left(B_{0 \mathrm{cb}}^{*}, B_{1 \mathrm{pb}}^{*}, B_{2 \mathrm{pb}}^{*}\right)$ be the triplet found by the cognitive relay during the two binning steps of the encoding process.
Let $E_{\mathrm{cb}}$, resp. $E_{\mathrm{pb}}$, denote the event that the first binning step in (42), resp. the second binning step in (43), is not successful. The probability of encoding error is bounded by:

$$
\operatorname{Pr}[\text { encoding error }] \leq \operatorname{Pr}\left[E_{\mathrm{cb}}\right]+\operatorname{Pr}\left[E_{\mathrm{pb}} \mid E_{\mathrm{cb}}^{c}\right]
$$

where $E_{\mathrm{cb}}^{c}$ denotes the complement of the event $E_{\mathrm{cb}}$. We start by noting that the encoded sequences are generated iid according to

$$
\begin{align*}
P^{(\text {gen })} \triangleq & P_{U_{1 \mathrm{c}}, X_{1}} P_{U_{2 \mathrm{c}}, X_{2}} P_{U_{0 \mathrm{cb}} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}} \\
& P_{U_{1 \mathrm{pb}} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}, U_{0 \mathrm{cb}}, X_{1}} P_{U_{2 \mathrm{pb}} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}, U_{0 \mathrm{cb}}, X_{2}} \tag{44}
\end{align*}
$$

but after binning they look as if generated iid according to

$$
\begin{gather*}
P^{(\mathrm{enc})} \triangleq P_{U_{1 \mathrm{c}}, X_{1}} P_{U_{2 \mathrm{c}}, X_{2}} P_{U_{0 c \mathrm{cb}} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}, X_{1}, X_{2}} \\
P_{U_{1 \mathrm{pb}}, U_{2 \mathrm{pb}} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}, U_{0 c \mathrm{cb}}, X_{1}, X_{2}} \tag{45}
\end{gather*}
$$

we thus expect the encoding error probability to be of the form

$$
\begin{align*}
& \mathbb{E}\left[\log \frac{P^{(\text {gen })}}{P^{(\mathrm{enc})}}\right] \\
& =I\left(U_{0 \mathrm{cb}} ; X_{1}, X_{2} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}\right) \\
& +I\left(U_{1 \mathrm{pb}} ; X_{2} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}\right)+I\left(U_{2 \mathrm{pb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}\right) \\
& +I\left(U_{1 \mathrm{pb}} ; U_{2 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}, X_{1}, X_{2}\right) \tag{46}
\end{align*}
$$

The rigorous error analysis is as follows.

- First binning step. $E_{\mathrm{cb}}$ is the event that for all $b_{0 \mathrm{cb}} \in$ $\left[1: 2^{N R_{0 c}^{\prime}}\right]$
$\left(U_{1 \mathrm{c}}^{N}(1), X_{1}^{N}(1 \mid 1), U_{2 \mathrm{c}}^{N}(1), X_{2}^{N}(1 \mid 1), U_{0 \mathrm{cb}}^{N}\left(1,1, b_{1 \mathrm{c}} \mid 1,1\right)\right.$ $\notin T_{\epsilon}^{N}\left(P_{U_{1 \mathrm{c}}, X_{1}, U_{2 \mathrm{c}}, X_{2}, U_{0 \mathrm{cb}}}\right)$,

By standard arguments, $\operatorname{Pr}\left[E_{\mathrm{cb}}\right] \rightarrow 0$ as $N \rightarrow \infty$ if

$$
R_{0 \mathrm{cb}} \geq I\left(X_{1}, X_{2} ; U_{0 \mathrm{cb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}\right)
$$

as in (15a).

- Second binning step. Let $b_{0 \text { cb }}^{*}$ be the index that was found to satisfy (42) at the first decoding step. We bound the probability of error in the second encoding step as

$$
\operatorname{Pr}\left[E_{\mathrm{pb}} \mid E_{\mathrm{cb}}^{c}\right]=\operatorname{Pr}\left[\bigcap _ { b _ { 1 } = 1 } ^ { 2 ^ { N R _ { 1 \mathrm { pb } } ^ { \prime } } } \bigcap _ { b _ { 2 } = 1 } ^ { 2 ^ { N R _ { 2 \mathrm { pb } } ^ { \prime } } } \left(U_{1 \mathrm{c}}^{N}(1), X_{1}^{N}(1 \mid 1),\right.\right.
$$

$$
U_{2 \mathrm{c}}^{N}(1), X_{2}^{N}(1 \mid 1), U_{0 \mathrm{cb}}^{N}\left(1,1, b_{0 \mathrm{cb}}^{*} \mid 1,1\right)
$$

$$
U_{1 \mathrm{pb}}^{N}\left(1, b_{1} \mid 1,1,1,1, b_{0 \mathrm{cb}}^{*}\right),
$$

$$
\left.\left.U_{2 \mathrm{pb}}^{N}\left(1, b_{2} \mid 1,1,1,1, b_{0 \mathrm{cb}}^{*}\right)\right) \notin T_{\epsilon}^{N}\left(P^{(\mathrm{enc})}\right)\right]
$$

$$
=\operatorname{Pr}[K=0] \leq \frac{\operatorname{Var}[K]}{\mathbb{E}[K]^{2}},
$$

where $P^{(\mathrm{enc})}$ is given in 45), where

$$
K=\sum_{b_{1}=1}^{2^{N R_{1 \mathrm{pb}}^{\prime}}} \sum_{b_{2}=1}^{2^{N R_{2 \mathrm{pb}}^{\prime}}} K_{b_{1}, b_{2}}
$$

with $K_{b_{1}, b_{2}}$ the indicator function of the event
$\left(U_{1 \mathrm{c}}^{N}(1), X_{1}^{N}(1 \mid 1), U_{2 \mathrm{c}}^{N}(1), X_{2}^{N}(1 \mid 1), U_{0 \mathrm{cb}}^{N}\left(1,1, b_{0 \mathrm{cb}}^{*} \mid 1,1\right)\right.$, $\left.U_{1 \mathrm{pb}}^{N}\left(1, b_{1} \mid 1,1,1,1, b_{0 \mathrm{cb}}^{*}\right), U_{2 \mathrm{pb}}^{N}\left(1, b_{2} \mid 1,1,1,1, b_{0 \mathrm{cb}}^{*}\right)\right)$ $\in T_{\epsilon}^{N}\left(P^{(\mathrm{enc})}\right)$

The mean value of $K$ (neglecting all terms that depend on $\epsilon$ and that eventually go to zero as $N \rightarrow \infty)$ is:

$$
\begin{aligned}
\mathbb{E}[K] & =\sum_{b_{1}=1}^{2^{N R_{1 \mathrm{pb}}^{\prime}}} \sum_{b_{2}=1}^{N R_{2 \mathrm{pb}}^{\prime}} \operatorname{Pr}\left[K_{b_{1}, b_{2}}=1\right] \\
& =2^{N\left(R_{1 \mathrm{pb}}^{\prime}+R_{2 \mathrm{pb}}^{\prime}-A\right)}
\end{aligned}
$$

with

$$
\begin{aligned}
& 2^{-N A}=\operatorname{Pr}\left[K_{b_{1}, b_{2}}=1\right]=\mathbb{E}\left[K_{b_{1}, b_{2}}\right] \\
& =\sum_{\left(u_{1 \mathrm{pb}}^{N}, u_{2 \mathrm{pb}}^{N}\right) \in T_{\epsilon}^{N}\left(P^{(\mathrm{enc})} \mid u_{1 \mathrm{c}}^{N}, x_{1}^{N}, u_{2 c}^{N}, x_{2}^{N}\right)} \\
& P_{U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}}^{N} P_{U_{2 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{2}, U_{0 \mathrm{cb}}}^{N} \\
& =2^{-N\left[I\left(U_{1 \mathrm{pb}} ; X_{2} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}\right)+I\left(U_{2 \mathrm{pb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{2}, U_{0 \mathrm{cb}}\right)\right]} \\
& \quad 2^{\left.-N I\left(U_{1 \mathrm{pb}} ; U_{2 \mathrm{pb}} \mid U_{1 \mathrm{c}}, X_{1}, U_{2 \mathrm{c}}, X_{2}, U_{0 \mathrm{cb}}\right)\right]} .
\end{aligned}
$$

The variance of $K$ (neglecting all terms that depend on $\epsilon$ and that eventually go to zero as $N \rightarrow \infty$ ) is:

$$
\begin{aligned}
& \operatorname{Var}[K]=\sum_{b_{1}=1}^{2^{N R_{1 \mathrm{pb}}^{\prime}}} \sum_{b_{2}=1}^{2^{N R_{2 \mathrm{pb}}^{\prime}}} \sum_{b_{1}^{\prime}=1}^{2^{N R_{1 \mathrm{pb}}^{\prime}}} \sum_{b_{2}^{\prime}=1}^{2^{N R_{2 \mathrm{pb}}^{\prime}}} \\
& \left(\operatorname{Pr}\left[K_{b_{1}, b_{2}}=1, K_{b_{1}^{\prime}, b_{2}^{\prime}}=1\right]-\operatorname{Pr}\left[K_{b_{1}, b_{2}}=1\right] \operatorname{Pr}\left[K_{b_{1}^{\prime}, b_{2}^{\prime}}=1\right]\right) \\
& \leq \underbrace{\sum_{b_{1}=b_{1}^{\prime}, b_{2}=b_{2}^{\prime}} \operatorname{Pr}\left[K_{b_{1}, b_{2}}=1\right]}_{=\mathbb{E}[K]} \\
& +\underbrace{\sum_{b_{1}=b_{1}^{\prime}, b_{2} \neq b_{2}^{\prime}} \operatorname{Pr}\left[K_{b_{1}, b_{2}}=1\right] \operatorname{Pr}\left[K_{b_{1}, b_{2}^{\prime}}=1 \mid K_{b_{1}, b_{2}}=1\right]}_{=\mathbb{E}[K] 2^{N\left(R_{2 \mathrm{pb}}^{\prime}-B\right)}} \\
& +\underbrace{}_{b_{b_{1} \neq b_{1}^{\prime}, b_{2}=b_{2}^{\prime}} \operatorname{Pr}\left[K_{b_{1}, b_{2}}=1\right] \operatorname{Pr}\left[K_{b_{1}^{\prime}, b_{2}}=1 \mid K_{b_{1}, b_{2}}=1\right]} \\
& =\underbrace{-\mathbb{E}[K] 2^{N\left(R_{1 \mathrm{pb}}^{\prime}-C\right)}} \\
& +\underbrace{\sum_{b_{1} \neq b_{1}^{\prime}, b_{2} \neq b_{2}^{\prime}} \operatorname{Pr}\left[K_{b_{1}, b_{2}}=1\right] \operatorname{Pr}\left[K_{b_{1}^{\prime}, b_{2}^{\prime}}=1 \mid K_{b_{1}, b_{2}}=1\right]} \\
& =\mathbb{E}[K] 2^{N\left(R_{1 \mathrm{pb}}^{\prime}+N R_{2 \mathrm{pb}}^{\prime}-D\right)}
\end{aligned}
$$

with

$$
\begin{aligned}
2^{-N B} & =\operatorname{Pr}\left[K_{b_{1}, b_{2}^{\prime}}=1 \mid K_{b_{1}, b_{2}}=1\right] \\
& =\sum_{u_{2 \mathrm{pb}}^{N} \in T_{\epsilon}^{N}\left(P^{(\mathrm{enc})} \mid u_{1 \mathrm{c}}^{N}, x_{1}^{N}, u_{2 c}^{N}, x_{2}^{N}, u_{0 \mathrm{cb}}^{N}, u_{1 \mathrm{pb}}^{N}\right)} \\
& P_{U_{2 \mathrm{pb}} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}, U_{0 \mathrm{cb}}, X_{2}}^{N} \\
& =2^{-N I\left(U_{2 \mathrm{pb}} ; X_{1}, U_{1 \mathrm{pb}} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}, X_{2}, U_{0 \mathrm{cb}}\right)},
\end{aligned}
$$

and similarly, i.e., swap the role of the users in the expression above,

$$
2^{-N C}=2^{-N I\left(U_{1 \mathrm{pb}} ; X_{2}, U_{2 \mathrm{pb}} \mid U_{1 \mathrm{c}}, X_{1}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}\right)}
$$

and finally

$$
\begin{aligned}
& 2^{-N D}=\operatorname{Pr}\left[K_{b_{1}^{\prime}, b_{2}^{\prime}}=1 \mid K_{b_{1}, b_{2}}=1\right] \\
& =\sum_{\left(u_{1 \mathrm{pb}}^{N}, u_{2 \mathrm{pb}}^{N}\right) \in T_{\epsilon}^{N}\left(P^{(\mathrm{enc}) \mid} \mid u_{1 \mathrm{c}}^{N}, x_{1}^{N}, u_{2 c}^{N}, x_{2}^{N}, u_{0 \mathrm{ob}}^{N}\right)} \\
& P_{U_{2 \mathrm{pb}}^{N} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{2}, U_{0 \mathrm{cb}}}^{N} P_{U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}} \\
& =2^{-N A} .
\end{aligned}
$$

Hence, we can bound $\operatorname{Pr}[K=0]$ as:

$$
\begin{aligned}
& \operatorname{Pr}[K=0] \\
& \leq \frac{1+2^{N\left(R_{1 \mathrm{pb}}^{\prime}-C\right)}+2^{N\left(R_{2 \mathrm{pb}}^{\prime}-B\right)}+2^{N\left(R_{1 \mathrm{pb}}^{\prime}+R_{2 \mathrm{pb}}^{\prime}-A\right)}}{2^{N\left(R_{1 \mathrm{pb}}^{\prime}+R_{2 \mathrm{pb}}^{\prime}-A\right)}} \\
& \text { and } \operatorname{Pr}[K=0] \rightarrow 0 \text { if } \\
& \qquad R_{1 \mathrm{pb}}^{\prime}+R_{2 \mathrm{pb}}^{\prime}-A>0 \\
& \quad R_{1 \mathrm{pb}}^{\prime}+R_{2 \mathrm{pb}}^{\prime}-A-\left(R_{2 \mathrm{pb}}^{\prime}-B\right)>0 \\
& \quad R_{1 \mathrm{pb}}^{\prime}+R_{2 \mathrm{pb}}^{\prime}-A-\left(R_{1 \mathrm{pb}}^{\prime}-C\right)>0
\end{aligned}
$$

that is, if

$$
\begin{aligned}
R_{1 \mathrm{pb}}^{\prime}+R_{2 \mathrm{pb}}^{\prime}>A & =\mathrm{eq} \cdot(15 \mathrm{~d}) \\
R_{1 \mathrm{pb}}^{\prime}>A-B & =\mathrm{eq} \cdot(15 \mathrm{~b} \\
R_{2 \mathrm{pb}}^{\prime}>A-C & =\mathrm{eq} \cdot(15 \mathrm{c}
\end{aligned}
$$

since

$$
\begin{aligned}
A & =I\left(U_{1 \mathrm{pb}} ; X_{2} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}\right) \\
& +I\left(U_{2 \mathrm{pb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{2}, U_{0 \mathrm{cb}}\right) \\
& +I\left(U_{1 \mathrm{pb}} ; U_{2 \mathrm{pb}} \mid U_{1 \mathrm{c}}, X_{1}, U_{2 \mathrm{c}}, X_{2}, U_{0 \mathrm{cb}}\right) \\
& =I\left(U_{1 \mathrm{pb}} ; X_{2} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}\right)+B \\
& =I\left(U_{2 \mathrm{pb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{2}, U_{0 \mathrm{cb}}\right)+C .
\end{aligned}
$$

- Decoding. We only describe the decoding at destination 1 as the same applies to destination 2 with the role of the users swapped. Destination 1 looks for a unique quadruplet $\left(w_{1 \mathrm{p}}, w_{1 \mathrm{c}}, w_{1 \mathrm{cb}}, w_{1 \mathrm{pb}}\right)$ and for some quadruplet $\left(w_{2 \mathrm{c}}, w_{2 \mathrm{cb}}, b_{0 \mathrm{cb}}, b_{1 \mathrm{pb}}\right)$ such that

$$
\begin{align*}
& \left(U_{1 \mathrm{c}}^{N}\left(w_{1 \mathrm{c}}\right), X_{1}^{N}\left(w_{1 \mathrm{p}} \mid w_{1 \mathrm{c}}\right), U_{2 \mathrm{c}}^{N}\left(w_{2 \mathrm{c}}\right)\right. \\
& U_{0 \mathrm{cb}}^{N}\left(w_{1 \mathrm{cb}}, w_{1 \mathrm{cb}}, b_{0 \mathrm{cb}} \mid w_{1 \mathrm{c}}, w_{2 \mathrm{c}}\right) \\
& U_{1 \mathrm{pb}}^{N}\left(w_{1 \mathrm{pb}}, b_{1 \mathrm{pb}} \mid w_{1 \mathrm{c}}, w_{2 \mathrm{c}}, w_{1 \mathrm{cb}}, w_{2 \mathrm{cb}}, b_{0 \mathrm{cb}}, w_{1 \mathrm{p}}\right), \\
& ) \in T_{\epsilon}^{N}\left(P^{(\text {dest.1) }}\right) \tag{47}
\end{align*}
$$

where

$$
\begin{align*}
P^{(\text {dest.1) }} & =\sum_{X_{2}, U_{2 \mathrm{pb}}, X_{c}} P_{U_{1 \mathrm{c}}, X_{1}} P_{U_{2 \mathrm{c}}, X_{2}} \\
& =P_{U_{1 \mathrm{pb}}, U_{2 \mathrm{pb}}, U_{0 \mathrm{cb}}, X_{c} \mid U_{1 \mathrm{c}}, X_{1}, U_{2 \mathrm{c}}, X_{2}} \\
& =P_{U_{1 \mathrm{c}}, X_{1}} P_{U_{2 \mathrm{c}}} P_{U_{1 \mathrm{pb}}, U_{0 \mathrm{cb}} \mid U_{1 \mathrm{c}}, X_{1}, U_{2 \mathrm{c}}} \tag{48}
\end{align*}
$$

If none or more than one quadruplet $\left(w_{1 \mathrm{p}}, w_{1 \mathrm{c}}, w_{1 \mathrm{cb}}, w_{1 \mathrm{pb}}\right)$ is found an error has occurred.

## - Decoding Error Analysis.

Let $E_{\text {dest.1 }}$ denote the event that the relationship in (47) is not satisfied by any $\left(w_{1 \mathrm{p}}, w_{1 \mathrm{c}}, w_{1 \mathrm{cb}}, w_{1 \mathrm{pb}}\right)$ or that is it satisfied by more than one such a quadruplet. We have

$$
\begin{aligned}
\operatorname{Pr}[\text { decoding error }] & \leq \operatorname{Pr}[\text { encoding error }] \\
& +\operatorname{Pr}\left[E_{\text {dest. } 1} \mid \text { encoding successful }\right]
\end{aligned}
$$

where $\operatorname{Pr}[$ encoding error $] \rightarrow 0$ if the rates are chosen form the "binning rate region" $\mathcal{R}_{0}$ defined by (12). Hence we only need to analyze the probability of decoding error assuming the encoding was successful.
Table III summarizes the possible error events at destination 1, where a " 0 " means that the corresponding message index is in error, a " $\checkmark$ " that the corresponding message index, and bin index if any, is correct, and the ". .." that is does not matter whether the corresponding message index is correct or not as in either case the joint density needed to evaluate the error event probability factorizes as if the message were in error (because of superposition to at least one codeword with a message index in error). For the cases where $U_{0 \mathrm{cb}}$ does not have the correct dependency on $\left(U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}\right)$, i.e., for all cases listed in Table III but for event $E_{8}$ which is marked as "special", an intuitive analysis of the probability of error is as follows. Depending on which messages are wrongly decoded at destination 1, and assuming the encoding steps were successful, the decoded codewords and the received $Y_{1}^{N}$ are iid jointly distributed according to
$P_{1 \mid \star} \triangleq P_{U_{1 \mathrm{c}}, X_{1}} P_{U_{2 \mathrm{c}}} P_{U_{0 \mathrm{cb}} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}} P_{U_{1 \mathrm{pb}} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}} P_{Y_{1} \mid \star}$,
where " $\star$ " in (49) indicates the set of correctly decoded messages. However, the actual transmitted codewords and the received $Y_{1}^{N}$ considered at destination 1 look as if they were generated iid according to

$$
\begin{equation*}
P_{1} \triangleq P_{U_{1 \mathrm{c}}, X_{1}} P_{U_{2 \mathrm{c}}} P_{U_{0 \mathrm{cb}}, U_{1 \mathrm{pb}} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}, X_{1}} P_{Y_{1} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}, U_{1 \mathrm{pb}}} \tag{50}
\end{equation*}
$$

Hence we expect the probability of error at destination 1 to depend on terms of the type

$$
\begin{align*}
& I_{1 \mid \star}=\mathbb{E}\left[\log \frac{P_{1}}{P_{1 \mid \star}}\right] \\
& =\mathbb{E}\left[\log \frac{P_{U_{0 \mathrm{cb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}} P_{Y_{1} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}, U_{1 \mathrm{pb}}}}{P_{U_{0 \mathrm{cb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}} P_{Y_{1} \mid \star}}\right] \\
& =I\left(U_{0 \mathrm{cb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}\right)+I\left(Y_{1} ; U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}, U_{0 \mathrm{cb}}, U_{1 \mathrm{pb}} \mid \star\right) . \tag{51}
\end{align*}
$$

When $U_{0 \mathrm{cb}}$ has the correct dependency on $\left(U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}\right)$, i.e., only for the "special" event $E_{8}$ in Table III) the
density $P_{1 \mid \star}$ in (49) must be modified as follows. We must use $P_{U_{0 c b} \mid U_{2 c}, U_{1 c}, X_{1}}$ (i.e., correct dependency on $\left(U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, X_{1}\right)$ ) rather than $P_{U_{\text {ocb }} \mid U_{2 \mathrm{c}}, U_{1 \mathrm{c}}}$. This results in the absence of the term $I\left(U_{0 \mathrm{cb}} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}\right)$ in (51).
The rigorous analysis of the error probability is as follows.

- $\operatorname{Pr}\left[E_{1}\right]$ and $\operatorname{Pr}\left[E_{2}\right]: U_{1 c}$ is in error.

If the decoding of $U_{1 c}$ fails, the codewords ( $X_{1}, U_{1 \mathrm{cb}}, U_{2 \mathrm{cb}}, U_{1 \mathrm{pb}}$ ) cannot be successfully decoded since they are superposed to a wrong $U_{1 c}$. $U_{2 \mathrm{c}}$, which is generated independently of $U_{1 \mathrm{c}}$, can be in error or not and we shall distinguish the two cases in the following.
Event $E_{1}$ in Table III corresponds to the case where both $U_{1 c}$ and $U_{2 c}$ are in error (and thus all the messages superposed to them are in error too); its probability can be bounded as

$$
\begin{aligned}
& \operatorname{Pr}\left[E_{1}\right]=\operatorname{Pr}\left[\bigcup_{\widetilde{w}_{1 \mathrm{c}} \neq 1, \widetilde{w}_{2 \mathrm{c}} \neq 1, \widetilde{w}_{1 \mathrm{p}}, \widetilde{w}_{1 \mathrm{cb}}, \widetilde{w}_{2 \mathrm{cb}}, \widetilde{w}_{1 \mathrm{pb}}, \widetilde{b}_{\mathrm{bcb}}, \widetilde{b}_{1 \mathrm{pb}}}\right. \\
& \left(Y_{1}^{N}, U_{1 \mathrm{c}}^{N}\left(\widetilde{w}_{1 \mathrm{c}}\right), X_{1}^{N}\left(\widetilde{w}_{1 \mathrm{p}} \mid \widetilde{w}_{1 \mathrm{c}}\right), U_{2 \mathrm{c}}^{N}\left(\widetilde{w}_{2 \mathrm{c}}\right),\right. \\
& U_{0 \mathrm{cb}}^{N}\left(\widetilde{w}_{1 \mathrm{cb}}, \widetilde{w}_{2 \mathrm{cb}}, \widetilde{b}_{0 \mathrm{cb}} \mid \widetilde{w}_{1 \mathrm{c}}, \widetilde{w}_{2 \mathrm{c}}\right) \text {, } \\
& U_{1 \mathrm{pb}}^{N}\left(\widetilde{w}_{1 \mathrm{pb}}, \widetilde{b}_{1 \mathrm{pb}} \mid \widetilde{w}_{1 \mathrm{c}}, \widetilde{w}_{1 \mathrm{p}}, \widetilde{w}_{1 \mathrm{cb}}, \widetilde{w}_{2 \mathrm{cb}}, \widetilde{b}_{0 \mathrm{cb}}\right) \\
& \left.\in T_{\epsilon}^{N}\left(P^{\text {(dest.1) }}\right)\right] \\
& \leq 2^{N\left(R_{1 \mathrm{c}}+R_{1 \mathrm{p}}+R_{2 \mathrm{c}}+L_{0 \mathrm{cb}}+L_{1 \mathrm{pb}}\right)} \\
& \left.\sum_{\left(y_{1}^{N}, u_{1 \mathrm{c}}^{N}, u_{2 c}^{N}, x_{1}^{N}, u_{0 \mathrm{cb}}^{N}, u_{1 \mathrm{pb}}^{N}\right) \in T_{\epsilon}^{N}\left(P^{(\text {dest.1) })}\right.} P_{1 \mid \star}\right|_{\star=\emptyset} \\
& \leq 2^{N\left(R_{1 \mathrm{c}}+R_{1 \mathrm{p}}+R_{2 \mathrm{c}}+L_{0 \mathrm{cb}}+L_{1 \mathrm{pb}}-\left.I_{1 \mid \star}\right|_{*=\emptyset}\right)} \text {, }
\end{aligned}
$$

for $P_{1 \mid \star}$ given in (50) and $I_{1 \mid \star}$ given in (51) evaluated for $\star=\emptyset$. Hence $\operatorname{Pr}\left[E_{1}\right] \rightarrow 0$ as $N \rightarrow \infty$ if (16a) holds.
Event $E_{2}$ in Table III corresponds to the case where $U_{1 \mathrm{c}}$ is in error (and thus all the messages superposed to it are in error too) and $U_{2 \mathrm{c}}$ is correctly decoded. Similarly to what done for event $E_{1}$, the probability of event $E_{2}$ goes to zero if 16 b holds.

- $\operatorname{Pr}\left[E_{3}\right], \operatorname{Pr}\left[E_{4}\right]$ and $\operatorname{Pr}\left[E_{5}\right]: X_{1}$ is in error.

Similarly to what done for event $E_{1}$, the probability of event $E_{3}$ goes to zero if 16 c holds, the probability of event $E_{4}$ goes to zero if 16 d holds, and the probability of event $E_{5}$ goes to zero if (16f) holds.

- $\operatorname{Pr}\left[E_{6}\right]$ and $\operatorname{Pr}\left[E_{7}\right]: U_{0 \mathrm{cb}}$ is in error.

Similarly to what done for event $E_{1}$, the probability of event $E_{6}$ goes to zero if 16 g ) holds, and the probability of event $E_{7}$ goes to zero if (16e) holds.
$-\operatorname{Pr}\left[E_{8}\right]: U_{1 \mathrm{pb}}$ is in error.
Similarly to what done for event $E_{1}$, the probability of event $E_{8}$ goes to zero if (16h) holds.

## Appendix F <br> Proof of Thm. IV. 3

Without loss of generality we may introduce in Thm. IV. 1 a new RV $U_{i \text { p }}$ and let $X_{i}$ be a deterministic function of $\left(U_{i \mathrm{c}}, U_{i \mathrm{p}}\right)$, i.e. $X_{i}=X_{i}\left(U_{i \mathrm{c}}, U_{i \mathrm{p}}\right), i \in\{1,2\}$.

TABLE III
Possible decoding errors at destination 1. Legend: a " 0 " means that the corresponding message is in error, a " $\checkmark$ " THAT THE CORRESPONDING MESSAGE IS CORRECT, AND THE ". . ." THAT IS DOES NOT MATTER WHETHER THE CORRESPONDING MESSAGE IS CORRECT OR NOT AS IN EITHER CASE THE JOINT DENSITY NEEDED TO EVALUATE THE ERROR EVENT PROBABILITY FACTORIZES AS IF THE MESSAGE WERE IN ERROR (BECAUSE OF SUPERPOSITION TO AT LEAST ONE MESSAGE IN ERROR). THE EVENT $E_{8}$ IS "SPECIAL" IN THAT THE TERM $I\left(U_{0 c b} ; X_{1} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}\right)$ IN $(51)$ MUST BE OMITTED.

|  | $U_{1 \mathrm{c}}$ <br> $w_{1 \mathrm{c}}$ | $U_{2 \mathrm{c}}$ <br> $w_{2 \mathrm{c}}$ | $X_{1}$ <br> $w_{1 \mathrm{p}}$ | $U_{0 \mathrm{cb}}$ <br> $\left(w_{1 \mathrm{cb}}, w_{2 \mathrm{cb}}, b_{0 \mathrm{cb}}\right)$ | $U_{1 \mathrm{pb}}$ <br> $\left(w_{1 \mathrm{pb}}, b_{1 \mathrm{pb}}\right)$ | Set $\star$ to be used in (51) |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $E_{1}$ | 0 | 0 | $\ldots$ | $\ldots$ | $\ldots$ | $\emptyset$ |
| $E_{2}$ | 0 | $\checkmark$ | $\ldots$ | $\ldots$ | $\ldots$ | $U_{2 \mathrm{c}}$ |
| $E_{3}$ | $\checkmark$ | 0 | 0 | $\ldots$ | $\ldots$ | $U_{1 \mathrm{c}}$ |
| $E_{4}$ | $\checkmark$ | $\checkmark$ | 0 | 0 | $\ldots$ | $U_{1 \mathrm{c}}, U_{2 \mathrm{c}}$ |
| $E_{5}$ | $\checkmark$ | $\checkmark$ | 0 | $\checkmark$ | $\ldots$ | $U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}$ |
| $E_{6}$ | $\checkmark$ | 0 | $\checkmark$ | $\ldots$ | $\ldots$ | $U_{1 \mathrm{c}}, X_{1}$ |
| $E_{7}$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | 0 | $\ldots$ | $U_{1 \mathrm{c}}, X_{1}, U_{2 \mathrm{c}}$ |
| $E_{8}$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | 0 | $U_{1 \mathrm{c}}, X_{1}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}$ (special) |

TABLE IV
THE CORRESPONDENCE OF RVS IN THE COMPARISON BETWEEN THE REGION IN [7] AND THE REGION IN 52].

| Region in [7] | Region in [52] |
| :--- | :--- |
| $U_{1}$ | $U_{1 \mathrm{c}}$ |
| $U_{2}$ | $U_{2 \mathrm{c}}$ |
| $V_{1}$ | $U_{1 \mathrm{p}}$ |
| $V_{2}$ | $U_{2 \mathrm{p}}$ |
| $W_{1}$ | $U_{1 \mathrm{pb}}$ |
| $W_{2}$ | $U_{2 \mathrm{pb}}$ |

With

$$
R_{0 \mathrm{cb}}^{\prime}=R_{1 \mathrm{cb}}=R_{2 \mathrm{cb}}=R_{1 \mathrm{pb}}=R_{2 \mathrm{pb}}=0, \quad U_{0 \mathrm{cb}}=\emptyset,
$$

the achievable rate region in Thm. IV.1 given by (12) becomes

$$
\begin{align*}
& R_{1 \mathrm{pb}}^{\prime} \geq I\left(U_{1 \mathrm{pb}} ; U_{2 \mathrm{p}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{1 \mathrm{p}}\right)  \tag{52a}\\
& R_{2 \mathrm{pb}}^{\prime} \geq I\left(U_{2 \mathrm{pb}} ; U_{1 \mathrm{p}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{2 \mathrm{p}}\right)  \tag{52b}\\
& R_{1 \mathrm{pb}}^{\prime}+R_{2 \mathrm{pb}}^{\prime} \geq I\left(U_{1 \mathrm{pb}}^{\prime} ; U_{2 \mathrm{p}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{1 \mathrm{p}}\right) \\
&+I\left(U_{2 \mathrm{pb}} ; U_{1 \mathrm{p}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{2 \mathrm{p}}\right) \\
&+I\left(U_{1 \mathrm{pb}} ; U_{2 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{1 \mathrm{p}}, U_{2 \mathrm{c}}, U_{2 \mathrm{p}}\right) \tag{52c}
\end{align*}
$$

$$
\begin{align*}
& R_{1 \mathrm{c}}+R_{2 \mathrm{c}}+L_{1 \mathrm{p}} \leq I\left(Y_{1} ; U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{1 \mathrm{p}}, U_{1 \mathrm{pb}} \mid Q\right)  \tag{52d}\\
& R_{2 \mathrm{c}}+L_{1 \mathrm{p}} \leq I\left(Y_{1} ; U_{2 \mathrm{c}}, U_{1 \mathrm{p}}, U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, Q\right)  \tag{52e}\\
& R_{1 \mathrm{c}}+L_{1 \mathrm{p}} \leq I\left(Y_{1} ; U_{1 \mathrm{c}}, U_{1 \mathrm{p}}, U_{1 \mathrm{pb}} \mid U_{2 \mathrm{c}}, Q\right)  \tag{52f}\\
& L_{1 \mathrm{p}} \leq I\left(Y_{1} ; U_{1 \mathrm{p}}, U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, Q\right)  \tag{52~g}\\
& L_{1 \mathrm{p}}=R_{1 \mathrm{p}}+R_{1 \mathrm{pb}}^{\prime} \\
& R_{1 \mathrm{c}}+R_{2 \mathrm{c}}+L_{2 \mathrm{p}} \leq I\left(Y_{2} ; U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{2 \mathrm{p}}, U_{2 \mathrm{pb}} \mid Q\right)  \tag{52h}\\
& R_{2 \mathrm{c}}+L_{2 \mathrm{p}} \leq I\left(Y_{2} ; U_{2 \mathrm{c}}, U_{2 \mathrm{p}}, U_{2 \mathrm{pb}} \mid U_{1 \mathrm{c}}, Q\right)  \tag{52i}\\
& R_{1 \mathrm{c}}+L_{2 \mathrm{p}} \leq I\left(Y_{2} ; U_{1 \mathrm{c}}, U_{2 \mathrm{p}}, U_{2 \mathrm{pb}} \mid U_{2 \mathrm{c}}, Q\right)  \tag{52j}\\
& L_{2 \mathrm{p}} \leq I\left(Y_{2} ; U_{2 \mathrm{p}}, U_{2 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, Q\right)  \tag{52k}\\
& L_{2 \mathrm{p}}=R_{2 \mathrm{p}}+R_{2 \mathrm{pb}}^{\prime}
\end{align*}
$$

for all distributions that factors as

$$
\begin{align*}
& P_{Q} P_{U_{1 \mathrm{c}}, U_{1 \mathrm{p}}, X_{1} \mid Q} P_{U_{2 \mathrm{c}}, U_{2 \mathrm{p}}, X_{2} \mid Q} \\
& P_{U_{1 \mathrm{pb}}, U_{2 \mathrm{p}}, X_{c} \mid U_{1 \mathrm{c}}, U_{1 \mathrm{p}}, U_{2 \mathrm{c}}, U_{2 \mathrm{p}}, X_{1}, X_{2}, Q} . \tag{53}
\end{align*}
$$

In order to compare the special case of our achievable rate region given by (52) with the region in [7], consider the correspondence of RVs in Table IV With this correspondence we see that the regions in [7] (20)-(31)] and (52) have the same
rate bounds and holds for the same set of input distributions. Since the region in (52) is a special case of our general achievable rate region, we conclude that the region in (52) contains the region in [7].

## Appendix G <br> Proof of Corollary IV. 4

Let $R_{1}=R_{1 \mathrm{p}}$ and $R_{2}=R_{2 \mathrm{p}}$, i.e.,

$$
R_{1 \mathrm{c}}=R_{2 \mathrm{c}}=R_{0 \mathrm{cb}}^{\prime}=R_{1 \mathrm{cb}}=R_{2 \mathrm{cb}}=R_{1 \mathrm{pb}}=R_{2 \mathrm{pb}}=0 .
$$

The region in (16) becomes

$$
\begin{align*}
R_{1 \mathrm{pb}}^{\prime} & \geq I\left(X_{2} ; U_{1 \mathrm{pb}} \mid X_{1}\right)  \tag{54a}\\
R_{2 \mathrm{pb}}^{\prime} & \geq I\left(X_{1} ; U_{2 \mathrm{pb}} \mid X_{2}\right)  \tag{54b}\\
R_{1 \mathrm{pb}}^{\prime}+R_{2 \mathrm{pb}}^{\mathrm{p}} & \geq I\left(X_{2} ; U_{1 \mathrm{pb}} \mid X_{1}\right)+I\left(X_{1} ; U_{2 \mathrm{pb}} \mid X_{2}\right) \\
& +I\left(U_{1 \mathrm{pb}} ; U_{2 \mathrm{pb}} \mid X_{1}, X_{2}\right)  \tag{54c}\\
R_{1 \mathrm{p}}+R_{1 \mathrm{pb}}^{\prime} & \leq I\left(Y_{1} ; X_{1}, U_{1 \mathrm{pb}}\right)  \tag{54d}\\
R_{2 \mathrm{p}}+R_{2 \mathrm{pb}}^{\prime} & \leq I\left(Y_{2} ; X_{2}, U_{2 \mathrm{pb}}\right) \tag{54e}
\end{align*}
$$

With

$$
\begin{aligned}
R_{1 \mathrm{pb}}^{\prime} & =I\left(X_{2} ; U_{1 \mathrm{pb}} \mid X_{1}\right)+a_{1}, a_{1} \geq 0, \\
R_{2 \mathrm{pb}}^{\prime} & =I\left(X_{1} ; U_{2 \mathrm{pb}} \mid X_{2}\right)+a_{2}, a_{2} \geq 0, \\
a_{1}+a_{2} & =I\left(U_{1 \mathrm{pb}} ; U_{2 \mathrm{pb}} \mid X_{1}, X_{2}\right),
\end{aligned}
$$

the achievable rate region in (54) becomes

$$
\bigcup\left\{\begin{array}{l}
R_{1 \mathrm{p}} \leq I\left(Y_{1} ; X_{1}, U_{1 \mathrm{pb}}\right)-I\left(X_{2} ; U_{1 \mathrm{pb}} \mid X_{1}\right)-a_{1}, \\
R_{2 \mathrm{p}} \leq I\left(Y_{2} ; X_{2}, U_{2 \mathrm{pb}}\right)-I\left(X_{1} ; U_{2 \mathrm{pb}} \mid X_{2}\right)-a_{2},
\end{array}\right.
$$

where the union is over all $\left(a_{1}, a_{2}\right) \in \mathbb{R}_{+}^{2}$ such that $a_{1}+a_{2}=I\left(U_{1 \mathrm{pb}} ; U_{2 \mathrm{pb}} \mid X_{1}, X_{2}\right)$, which coincides with (17).

Interestingly, we point out that the Fourier-Motzkin elimination of the region with only ( $X_{1}, X_{2}, U_{1 \mathrm{pb}}, U_{2 \mathrm{pb}}$ ) and with $R_{1 \mathrm{pb}} \geq 0$ and $R_{2 \mathrm{pb}} \geq 0$ is the same as with $R_{1 \mathrm{pb}}=$ $0, R_{2 \mathrm{pb}}=0$.

## Appendix H Proof of Corollary IV. 5

Let $R_{1}=R_{1 \mathrm{c}}$ and $R_{2}=R_{2 \mathrm{c}}$, that is

$$
R_{1 \mathrm{p}}=R_{2 \mathrm{p}}=L_{1 \mathrm{pb}}=L_{2 \mathrm{pb}}=L_{0 \mathrm{cb}}=0
$$

The region in (16) with $U_{1 \mathrm{pb}}=U_{2 \mathrm{pb}}=\emptyset$ and $I\left(X_{1}, X_{2} ; U_{0 \mathrm{cb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}\right)=0$ becomes

$$
\begin{align*}
R_{1 \mathrm{c}}+R_{2 \mathrm{c}} & \leq I\left(Y_{1} ; U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}, X_{1}\right)  \tag{55a}\\
R_{1 \mathrm{c}} & \leq I\left(Y_{1} ; U_{1 \mathrm{c}}, U_{0 \mathrm{cb}}, X_{1} \mid U_{2 \mathrm{c}}\right)  \tag{55b}\\
R_{2 \mathrm{c}}+R_{1 \mathrm{c}} & \leq I\left(Y_{2} ; U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}, X_{2}\right)  \tag{55c}\\
R_{2 \mathrm{c}} & \leq I\left(Y_{2} ; U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}, X_{1} \mid U_{1 \mathrm{c}}\right) \tag{55d}
\end{align*}
$$

which coincides with the region in 18 by choosing $X_{1}=$ $U_{1 \mathrm{c}}, X_{2}=U_{2 \mathrm{c}}, X_{c}=U_{0 \mathrm{cb}}$.

## Appendix I

Proof of Corollary IV. 6
Let $R_{1}=R_{1 \mathrm{p}}+R_{1 \mathrm{pb}}$ and $R_{2}=R_{2 \mathrm{c}}$, that is

$$
R_{1 \mathrm{c}}=R_{2 \mathrm{p}}=L_{2 \mathrm{pb}}=L_{0 \mathrm{cb}}=0
$$

The region in (16) with $U_{1 \mathrm{c}}=\emptyset, X_{2}=U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}=U_{2 \mathrm{c}}$ and $U_{2 \mathrm{pb}}=U_{0 \mathrm{c}}$ becomes

$$
\begin{align*}
& R_{2 \mathrm{c}}+R_{1 \mathrm{p}}+R_{1 \mathrm{pb}} \leq I\left(Y_{1} ; X_{1}, U_{2 \mathrm{c}}, U_{1 \mathrm{pb}}\right)  \tag{56a}\\
& R_{2 \mathrm{c}}+  \tag{56b}\\
& R_{1 \mathrm{pb}} \leq I\left(Y_{1} ; U_{2 \mathrm{c}}, U_{1 \mathrm{pb}} \mid U_{1 \mathrm{p}}\right)  \tag{56c}\\
& R_{1 \mathrm{p}}+R_{1 \mathrm{pb}} \leq I\left(Y_{1} ; U_{1 \mathrm{p}}, U_{1 \mathrm{pb}} \mid U_{2 \mathrm{c}}\right)  \tag{56d}\\
& R_{1 \mathrm{pb}} \leq I\left(Y_{1} ; U_{1 \mathrm{pb}} \mid X_{1}, U_{2 \mathrm{c}}\right)  \tag{56e}\\
& R_{2 \mathrm{c}} \leq I\left(Y_{2} ; U_{2 \mathrm{c}}\right)
\end{align*}
$$

which coincides with the region in (19) by choosing $X_{2}=$ $U_{2 \mathrm{c}}, X_{c}=U_{1 \mathrm{pb}}$.

## Appendix J

Proof of Corollary IV. 7
Let $R_{1}=R_{1 \mathrm{c}}+R_{1 \mathrm{pb}}$ and $R_{2}=R_{2 \mathrm{c}}$, that is

$$
R_{1 \mathrm{p}}=R_{2 \mathrm{p}}=L_{2 \mathrm{pb}}=L_{2 \mathrm{pb}}=L_{0 \mathrm{cb}}=0
$$

The region in (16) with $X_{1}=U_{1 \mathrm{c}}, X_{2}=U_{2 \mathrm{c}}, U_{0 \mathrm{cb}}=U_{2 \mathrm{c}}$ and $U_{2 \mathrm{pb}}=U_{0 \mathrm{c}}$ becomes

$$
\begin{align*}
R_{1 \mathrm{c}}+R_{2 \mathrm{c}}+R_{1 \mathrm{pb}} & \leq I\left(Y_{1} ; U_{1 \mathrm{c}}, U_{2 \mathrm{c}}, U_{1 \mathrm{pb}}\right)  \tag{57a}\\
R_{2 \mathrm{c}}+R_{1 \mathrm{pb}} & \leq I\left(Y_{1} ; U_{2 \mathrm{c}}, U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}\right)  \tag{57b}\\
R_{1 \mathrm{c}} \quad R_{1 \mathrm{pb}} & \leq I\left(Y_{1} ; U_{1 \mathrm{c}}, U_{1 \mathrm{pb}} \mid U_{2 \mathrm{c}}\right)  \tag{57c}\\
R_{1 \mathrm{pb}} & \leq I\left(Y_{1} ; U_{1 \mathrm{pb}} \mid U_{1 \mathrm{c}}, U_{2 \mathrm{c}}\right)  \tag{57d}\\
R_{1 \mathrm{c}}+R_{2 \mathrm{c}} & \leq I\left(Y_{2} ; U_{1 \mathrm{c}}, U_{2 \mathrm{c}}\right)  \tag{57e}\\
R_{2 \mathrm{c}} & \leq I\left(Y_{2} ; U_{2 \mathrm{c}} \mid U_{1 \mathrm{c}}\right)  \tag{57f}\\
R_{1 \mathrm{c}} & \leq I\left(Y_{2} ; U_{1 \mathrm{c}} \mid U_{2 \mathrm{c}}\right) . \tag{57~g}
\end{align*}
$$

which coincides with the region in (20) by choosing $X_{1}=$ $U_{1 \mathrm{c}}, X_{2}=U_{2 \mathrm{c}}, X_{c}=U_{1 \mathrm{pb}}$.

## Appendix K

## THE IFC-CR IN STANDARD FORM

A general IFC-CR is expressed as

$$
\begin{align*}
& \widetilde{Y}_{1}=\widetilde{h}_{11} \widetilde{X}_{1}+\widetilde{h}_{1 c} \widetilde{X}_{c}+\widetilde{h}_{12} \widetilde{X}_{2}+\widetilde{Z}_{1},  \tag{58a}\\
& \widetilde{Y}_{2}=\widetilde{h}_{22} \widetilde{X}_{1}+\widetilde{h}_{2 c} \widetilde{X}_{c}+\widetilde{h}_{21} \widetilde{X}_{1}+\widetilde{Z}_{2}, \tag{58b}
\end{align*}
$$

for $\widetilde{h}_{i}, \quad i \in\{11,22,1 c, 2 c, 12,21\}, \mathbb{E}\left[\left|\widetilde{X}_{j}\right|^{2}\right] \leq \widetilde{P}_{j}, \quad j \in$ $\{1,2, c\}$ and $\mathbb{E}\left[\left|\widetilde{Z}_{k}\right|^{2}\right]=\sigma_{k}^{2}, k \in\{1,2\}$. Assuming without
loss of generality that all the entries of $\left(\widetilde{P}_{1}, \widetilde{P}_{2}, \widetilde{P}_{c}, \sigma_{1}^{2}, \sigma_{2}^{2}\right)$ are strictly positive 4 consider now the transformation

$$
\begin{array}{ll}
Y_{1}=\frac{\widetilde{Y}_{1}}{\sigma_{1}} \mathrm{e}^{-j \angle \widetilde{h}_{1 c}} & Y_{2}=\frac{\widetilde{Y}_{2}}{\sigma_{2}} \mathrm{e}^{-j \angle \widetilde{h}_{2 c}} \\
X_{1}=\frac{\widetilde{X}_{1}}{\sqrt{\widetilde{P}_{1}}} \mathrm{e}^{-j\left(\angle \widetilde{h}_{11}+\angle \widetilde{h}_{1 c}\right)} & X_{2}=\frac{\widetilde{X}_{2}}{\sqrt{\widetilde{P}_{2}}} \mathrm{e}^{-j\left(\angle \widetilde{h}_{22}+\angle \widetilde{h}_{2 c}\right)} \\
X_{c}=\frac{\widetilde{X}_{c}}{\sqrt{\widetilde{P}_{c}}} & \left|h_{22}\right|=\frac{\sqrt{\widetilde{P}_{2}}\left|\widetilde{h}_{22}\right|}{\sigma_{2}} \\
\left|h_{11}\right|=\frac{\sqrt{\widetilde{P}_{1}}\left|\widetilde{h}_{11}\right|}{\sigma_{1}} & \left|h_{2 c}\right|=\frac{\sqrt{\widetilde{P}_{c}}\left|\widetilde{h}_{2 c}\right|}{\sigma_{2}} \\
\left|h_{1 c}\right|=\frac{\sqrt{\widetilde{P}_{c}}\left|\widetilde{h}_{1 c}\right|}{\sigma_{1}} & \left|h_{21}\right|=\frac{\sqrt{\widetilde{P}_{1}} \widetilde{h}_{21}}{\sigma_{2}} \mathrm{e}^{-j \angle \widetilde{h}_{22}}
\end{array}
$$

Since the above transformation is invertible, the channel in (58) is equivalent to the channel in (23).

## Appendix L <br> Proof of Theorem VI. 1

Given the "Gaussian maximizes entropy" property [50] we have that the union over all the distributions in (2) of the region in (8) is equal to the union over all distributions with $Q=\emptyset$ and $\left[X_{1}, X_{2}, X_{c}\right]$ zero-mean proper-complex Gaussian with covariance matrix

$$
\operatorname{Cov}\left(X_{1}, X_{2}, X_{c}\right)=\left(\begin{array}{ccc}
1 & 0 & \beta_{1}  \tag{59}\\
0 & 1 & \beta_{2} \\
\beta_{1}^{*} & \beta_{2}^{*} & 1
\end{array}\right):=\mathbf{S}
$$

for $\left(\beta_{1}, \beta_{2}\right) \in \mathbb{C}^{2}$ such that $\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2} \leq 1$. With (59) we write:

$$
X_{c}=\beta_{1}^{*} X_{1}+\beta_{2}^{*} X_{2}+\sqrt{1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}} X_{c, i n .}
$$

for $X_{1}, X_{2}, X_{c, \text { in. }}$ iid $\mathcal{N}(0,1)$, from which

$$
\begin{aligned}
Y_{j} & =\left[h_{j 1}+\beta_{1}^{*}\left|h_{j c}\right|\right] X_{1}+\left[h_{j 2}+\beta_{2}^{*}\left|h_{j c}\right|\right] X_{2} \\
& +\left|h_{j c}\right| \sqrt{1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}} X_{c, i n .}+Z_{j}, \quad j \in\{1,2\} .
\end{aligned}
$$

and thus, conditioned on $X_{1}$, we have that $Y_{j}$ is distributed as

$$
\left[h_{j 2}+\beta_{2}^{*}\left|h_{j c}\right|\right] X_{2}+\left|h_{j c}\right| \sqrt{1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}} X_{c, i n .}+Z_{j}, j \in\{1,2\}
$$

${ }^{4}$ If $\widetilde{P}_{1}=0, \widetilde{P}_{2}=0, \widetilde{P}_{c}=0$ the channel capacity is trivially $R_{1}=R_{2}=$ 0. If $\widetilde{P}_{1}=0, \widetilde{P}_{2}=0, \widetilde{P}_{c}>0$ the channel is equivalent to a Gaussian BC with input $X_{c}$ whose capacity is known [47]. If $\widetilde{P}_{1}=0, \widetilde{P}_{2}>0, \widetilde{P}_{c}=0$, and similarly if $\widetilde{P}_{1}>0, \widetilde{P}_{2}=0, \widetilde{P}_{c}=0$, the channel is a Gaussian point-topoint channel whose capacity is known [49]. If $\widetilde{P}_{1}=0, \widetilde{P}_{2}>0, \widetilde{P}_{c}>0$, and similarly if $\widetilde{P}_{1}>0, \widetilde{P}_{2}=0, \widetilde{P}_{c}>0$, the channel is equivalent to a Gaussian CIFC whose capacity is known to within 1 bit [2]. If $P_{1}>0, \widetilde{P}_{2}>0, \widetilde{P}_{c}=$ 0 , the channel is a Gaussian IFC whose capacity is known to within 1 bit [34]. If either of the noise variances is zero, the corresponding channel has infinite capacity, which does not have any physical meaning.

Since the condition in (6) must hold for all $\left(\beta_{1}, \beta_{2}\right) \in \mathbb{C}^{2}$ such that $\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2} \leq 1$, we obtain
for all Gaussian inputs $I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1}\right) \leq I\left(Y_{1} ; X_{2}, X_{c} \mid X_{1}\right)$

$$
\begin{aligned}
& \Longleftrightarrow \mathcal{C}\left(\left|\left|h_{22}\right|+\beta_{2}\right) \in \mathbb{C}_{2}^{*}:\left|h_{2 c}\right|^{2}+\left|h_{1}\right|^{2}+\left.\left|\beta_{2 c}\right|^{2}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}\right)\right) \\
& \leq \mathcal{C}\left(\left|h_{12}+\beta_{2}^{*}\right| h_{1 c}| |^{2}+\left|h_{1 c}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}\right)\right) \\
& \Longleftrightarrow \Longleftrightarrow\left(\beta_{1}, \beta_{2}\right) \in \mathbb{C}^{2}:\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2} \leq 1 \\
&\left|h_{2 c}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}\right)+\left|h_{22}\right|^{2}+2\left|h_{2 c}\right|\left|h_{22}\right| \Re\left\{\beta_{2}\right\} \\
& \leq\left|h_{1 c}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}\right)+\left|h_{12}\right|^{2}+2\left|h_{1 c}\right| \Re\left\{h_{12} \beta_{2}\right\} \\
& \Longleftrightarrow \max _{\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2} \leq 1}\left\{\left(\left|h_{2 c}\right|^{2}-\left|h_{1 c}\right|^{2}\right)\left(1-\left|\beta_{1}\right|^{2}\right)\right. \\
&\left.+2 \Re\left\{\left(\left|h_{2 c}\right|\left|h_{22}\right|-\left|h_{1 c}\right| h_{12}\right) \beta_{2}\right\}\right\} \leq\left|h_{12}\right|^{2}-\left|h_{22}\right|^{2} \\
& \Longleftrightarrow \max ^{2}\left\{\left(\left|h_{2 c}\right|^{2}-\left|h_{1 c}\right|^{2}\right)\left(1-\left|\beta_{1}\right|^{2}\right)\right. \\
&\left.+2| | h_{2 c}| | h_{22}\left|-\left|h_{1 c}\right| h_{12}\right| \sqrt{1-\left|\beta_{1}\right|^{2}}\right\} \leq\left|h_{12}\right|^{2}-\left|h_{22}\right|^{2},
\end{aligned}
$$

where in the last step the optimal $\beta_{2}$ is

$$
\beta_{2}=\mathrm{e}^{-\mathrm{j} \measuredangle\left(\left|h_{2 c}\right|\left|h_{22}\right|-\left|h_{1 c}\right| h_{12}\right)} \sqrt{1-\left|\beta_{1}\right|^{2}} .
$$

Let now

$$
\begin{aligned}
& \sqrt{1-\left|\beta_{1}\right|^{2}}=x \\
& \left|h_{2 c}\right|^{2}-\left|h_{1 c}\right|^{2}=a \\
& \left|\left|h_{2 c}\right|\right| h_{22}\left|-\left|h_{1 c}\right| h_{12}\right|=|b|
\end{aligned}
$$

The quadratic function $f(x)=a x^{2}+2|b| x$ is non-decreasing in $x \in[0,1]$ if $a x+|b| \geq 0$. If $a \geq 0:+|a| x+|b| \geq 0$ for all $x \in[0,1]$ hence $x=1$ is optimal. Else (i.e., if $a<0$ ): $-|a| x+|b| \geq 0$ for $x \leq|b| /|a|$. Thus, if $a<0,|b| /|a| \leq 1$ : $x=|b| /|a| \in[0,1]$ is optimal, and if $a<0,|b| /|a|>1: x=1$ is optimal. This shows the optimal $\beta_{2}$ is the one given in (25).

## Appendix M <br> Proof of Theorem VI. 2

With the parameterization in (59) the condition in (21b) can be rewritten as
for all Gaussian inputs : $I\left(Y_{1} ; X_{1}, X_{2}, X_{c}\right) \leq I\left(Y_{2} ; X_{1}, X_{2}, X_{c}\right)$

$$
\begin{aligned}
& \Longleftrightarrow \forall\left(\beta_{1}, \beta_{2}\right) \in \mathbb{C}^{2}:\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2} \leq 1 \\
& \mathcal{C}\left(\left.\left|\left|h_{11}\right|+\beta_{1}^{*}\right| h_{1 c}\right|^{2}+\left.\left|h_{12}+\beta_{2}^{*}\right| h_{1 c}\right|^{2}\right. \\
&\left.+\left|h_{1 c}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}\right)\right) \\
& \leq \mathcal{C}\left(\left.\left|h_{21}+\beta_{1}^{*}\right| h_{2 c}\right|^{2}+\left.\left|\left|h_{22}\right|+\beta_{2}^{*}\right| h_{2 c}\right|^{2}\right. \\
&\left.+\left|h_{2 c}\right|^{2}\left(1-\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2}\right)\right) \\
& \Longleftrightarrow\left(\left|h_{11}\right|^{2}+\left|h_{1 c}\right|^{2}+\left|h_{12}\right|^{2}\right)-\left(\left|h_{21}\right|^{2}+\left|h_{2 c}\right|^{2}+\left|h_{22}\right|^{2}\right) \\
&+\max _{\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2} \leq 1} 2 \Re\left(\beta_{1}\left(\left|h_{1 c}\right|\left|h_{11}\right|-\left|h_{2 c}\right| h_{21}\right)\right. \\
&\left.+\beta_{2}\left(\left|h_{1 c}\right| h_{12}-\left|h_{2 c}\right|\left|h_{22}\right|\right)\right) \leq 0 \\
& \Longleftrightarrow\left(\left|h_{11}\right|^{2}+\left|h_{1 c}\right|^{2}+\left|h_{12}\right|^{2}\right)-\left(\left|h_{21}\right|^{2}+\left|h_{2 c}\right|^{2}+\left|h_{22}\right|^{2}\right) \\
&+2 \underset{\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2} \leq 1}{ }\left\{\left|\beta_{1}\right|| | h_{1 c}| | h_{11}\left|-\left|h_{2 c}\right| h_{21}\right|\right. \\
&\left.+\left|\beta_{2}\right|| | h_{1 c}\left|h_{12}-\left|h_{2 c}\right|\right| h_{22}| |\right\} \leq 0 \\
& \Longleftrightarrow\left(\left|h_{11}\right|^{2}+\left|h_{1 c}\right|^{2}+\left|h_{12}\right|^{2}\right)-\left(\left|h_{21}\right|^{2}+\left|h_{2 c}\right|^{2}+\left|h_{22}\right|^{2}\right) \\
&+2 \sqrt{| | h_{1 c}| | h_{11}\left|-\left|h_{2 c}\right| h_{21}\right|^{2}+\left|\left|h_{1 c}\right| h_{12}-\left|h_{2 c}\right|\right| h_{22}| |^{2}} \leq 0 .
\end{aligned}
$$

We next show that, given $A \geq 0$ and $B \geq 0$ :

$$
\sqrt{A^{2}+B^{2}}=\max \{x A+y B\} \quad \text { s.t. } \quad x^{2}+y^{2} \leq 1
$$

Indeed, for $t \geq 0$ let the Lagrangian be:

$$
L=x A+y B-2 / t\left(x^{2}+y^{2}-1\right)
$$

then at the optimal point

$$
\begin{aligned}
& d L / d x=A-x / t=0 \Longleftrightarrow x=A t \\
& d L / d y=B-y / t=0 \Longleftrightarrow y=B t
\end{aligned}
$$

hence the optimal Lagrangian multiplier is

$$
x^{2}+y^{2}=\left(A^{2}+B^{2}\right) t^{2}=1 \Longleftrightarrow t=\frac{1}{\sqrt{A^{2}+B^{2}}}
$$

## Appendix N

"WEAK INTERFERENCE" OUTER BOUND FOR THE IFC-CR
We now evaluate the "weak interference at Rx 1" outer bound in Cor. III.5 for the channel model in (30). We proceed as in [18]. We must evaluate the region

$$
\begin{aligned}
R_{1} & \leq I\left(Y_{1} ; X_{1}, X_{c} \mid X_{2}, U\right) \\
& =h\left(Y_{1}-h_{12} X_{2} \mid X_{2}, U\right)-\log (\pi \mathrm{e}) \\
R_{2} & \leq I\left(Y_{2} ; X_{2}, U\right)=h\left(Y_{2}\right)-h\left(Y_{2} \mid X_{2}, U\right) \\
& \leq \log \left(\operatorname{Var}\left[Y_{2}\right]\right)-\left[h\left(Y_{2}-\left|h_{22}\right| X_{2} \mid X_{2}, U\right)-\log (\pi \mathrm{e})\right]
\end{aligned}
$$

for all distribution that factors as in (10). As for the El Gamal's converse for the degraded BC we have

$$
\begin{aligned}
& h\left(Z_{1}\right)=h\left(Y_{1} \mid X_{1}, X_{2}, X_{c}\right) \leq h\left(Y_{1} \mid X_{2} U\right) \leq h\left(Y_{1}-h_{12} X_{2} \mid X_{2}\right) \\
& \Longleftrightarrow \log (1) \leq h\left(Y_{1} \mid X_{2} U\right)-\log (\pi \mathrm{e}) \\
& \quad \leq \log \left(1+\operatorname{Var}\left[X_{\mathrm{eq}} \mid X_{2}\right]\right)
\end{aligned}
$$

where $X_{\mathrm{eq}}:=\left|h_{11}\right| X_{1}+\left|h_{1 c}\right| X_{c}$ as defined in (31).
Hence there must exist an $\alpha \in[0,1]$ such that

$$
\begin{aligned}
& h\left(Y_{1} \mid X_{2} U\right)-\log (\pi \mathrm{e}) \\
& =\log \left(1+\alpha \operatorname{Var}\left[X_{\mathrm{eq}} \mid X_{2}\right]\right) .
\end{aligned}
$$

Moreover, since conditioned on $X_{2}$ the channel in (30) is degraded, the (scalar) Entropy Power Inequality (EPI) [51] for complex-valued RVs grants

$$
\begin{aligned}
2^{h\left(Y_{2} \mid X_{2}, U\right)} & =2^{h\left(|\rho| X_{\mathrm{eq}}+\sqrt{1-|\rho|^{2}} Z_{0} \mid X_{2}, U\right)} \\
& \geq|\rho|^{2} 2^{h\left(Y_{1} \mid X_{2}, U\right)}+\left(1-|\rho|^{2}\right) 2^{h\left(Z_{0}\right)}
\end{aligned}
$$

which implies

$$
h\left(Y_{2} \mid X_{2}, U\right)-\log (\pi \mathrm{e}) \geq \log \left(1+\alpha \operatorname{Var}\left[X_{\mathrm{eq}} \mid X_{2}\right]\right)
$$

With this we obtain

$$
\begin{aligned}
& R_{1} \leq \mathcal{C}\left(\alpha \operatorname{Var}\left[X_{\mathrm{eq}} \mid X_{2}\right]\right) \\
& R_{2} \leq \mathcal{C}\left(\operatorname{Var}\left[|\rho| X_{\mathrm{eq}}+\left|h_{22}\right| X_{2}\right]\right)-\mathcal{C}\left(\alpha|\rho|^{2} \operatorname{Var}\left[X_{\mathrm{eq}} \mid X_{2}\right]\right)
\end{aligned}
$$

Moreover, from (1b) we also have

$$
\begin{aligned}
R_{2} & \leq I\left(Y_{2} ; X_{2}, X_{c} \mid X_{1} Q\right) \\
& \leq \log \left(1+\operatorname{Var}\left[Y_{2}-|\rho|\left|h_{11}\right| X_{1} \mid X_{1}\right]\right)
\end{aligned}
$$

By considering the input covariance $\mathbf{S}$ defined in (59), for a fixed $\left(\beta_{1}, \beta_{2}\right):\left|\beta_{1}\right|^{2}+\left|\beta_{2}\right|^{2} \leq 1$ we obtain

$$
\begin{aligned}
& \operatorname{Var}\left[X_{1}+|\rho| X_{c} \mid X_{2}\right] \\
& \quad=1+|\rho|^{2}\left(1-\left|\beta_{2}\right|^{2}\right)+2|\rho| \Re\left\{\beta_{1}\right\} \\
& \quad \leq 1+|\rho|^{2}\left|\beta_{1}\right|^{2}+2|\rho| \Re\left\{\beta_{1}\right\} \\
& \quad=\left|1+|\rho| \beta_{1}\right|^{2} \\
& \quad \operatorname{Var}\left[\left|h_{21}\right|\left(X_{1}+|\rho| X_{c}\right)+\left|h_{22}\right| X_{2}\right] \\
& \quad=\left|h_{21}\right|^{2}\left(1+|\rho|^{2}\right)+\left|h_{22}\right|^{2}+2\left|h_{21}\right|\left|h_{22}\right||\rho| \Re\left\{\beta_{2}\right\} \\
& \quad \leq\left|h_{21}\right|^{2}\left(1+|\rho|^{2}\right)+\left|h_{22}\right|^{2}+2\left|h_{21}\right|\left|h_{22}\right||\rho|\left|\beta_{2}\right| \\
& \quad \leq\left|h_{21}\right|^{2}\left(1+|\rho|^{2}\right)+\left|h_{22}\right|^{2}+2\left|h_{21}\right|\left|h_{22}\right||\rho| \sqrt{1-\left|\beta_{1}\right|^{2}} \\
& \operatorname{Var}\left[\left|h_{21}\right||\rho| X_{c}+\left|h_{22}\right|^{2} \mid X_{1}\right] \\
& \quad=\left|h_{21}\right|^{2}|\rho|^{2}\left(1-\left|\beta_{1}\right|^{2}\right)+\left|h_{22}\right|^{2}+2\left|h_{21}\right||\rho|\left|h_{22}\right| \Re\left\{\beta_{2}\right\} \\
& \quad \leq\left|h_{21}\right|^{2}|\rho|^{2}\left(1-\left|\beta_{1}\right|^{2}\right)+\left|h_{22}\right|^{2}+2\left|h_{21}\right||\rho|\left|h_{22}\right|\left|\beta_{2}\right| \\
& \quad \leq\left|h_{21}\right|^{2}|\rho|^{2}\left(1-\left|\beta_{1}\right|^{2}\right)+\left|h_{22}\right|^{2}+2\left|h_{21}\right||\rho|\left|h_{22}\right| \sqrt{1-\left|\beta_{1}\right|^{2}} \\
& \quad=\left(\left|h_{21}\right||\rho| \sqrt{1-\left|\beta_{1}\right|^{2}}+\left|h_{22}\right|\right)^{2}
\end{aligned}
$$

Note that the above shows that we can only consider $\left|\beta_{1}\right|^{2}+$ $\left|\beta_{2}\right|^{2}=1$ without loss of generality. With this, we obtain the region in (32).


[^0]:    ${ }^{1}$ The authors of [7] refer to the IFC-CR as "broadcast channel with cognitive relays", arguing that the model can also be obtained by adding two partially cognitive relays to a broadcast channel.
    ${ }^{2}$ The CIFC has also been referred to as the cognitive channel [16], an interference channel with "unidirectional cooperation" [17] and an interference channel with "degraded message sets" [18].

[^1]:    ${ }^{3}$ We note that our naming convention is not entirely consistent with past uses of the term "strong/weak interference". Here, as in our previous work on the CIFC [1], [2], we use "strong/weak interference" to denote regimes inspired by similar results for the IFC under which we may obtain either a tighter or simpler outer bound for the channel of interest, and use the terms "very strong/very weak" to denote regimes in which additional conditions (therefore forming subsets of the "strong/weak" regimes) are imposed on top of the "strong/weak" conditions that allow these outer bounds to be achieved.

