

Performance Bounds on a Wiretap Network with Arbitrary Wiretap Sets

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Abstract—Consider a communication network represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} is the set of nodes and \mathcal{E} is the set of point-to-point channels in the network. On the network a secure message M is transmitted, and there may exist wiretappers who want to obtain information about the message. In secure network coding, we aim to find a network code which can protect the message against the wiretapper whose power is constrained. Cai and Yeung [5] studied the model in which the wiretapper can access any one but not more than one set of channels, called a wiretap set, out of a collection \mathcal{A} of all possible wiretap sets. In order to protect the message, the message needs to be mixed with a random key K . They proved tight fundamental performance bounds when \mathcal{A} consists of all subsets of \mathcal{E} of a fixed size r . However, beyond this special case, obtaining such bounds is much more difficult. In this paper, we investigate the problem when \mathcal{A} consists of arbitrary subsets of \mathcal{E} and obtain the following results: 1) an upper bound on $H(M)$; 2) a lower bound on $H(K)$ in terms of $H(M)$. The upper bound on $H(M)$ is explicit, while the lower bound on $H(K)$ can be computed in polynomial time when $|\mathcal{A}|$ is fixed. The tightness of the lower bound for the point-to-point communication system is also proved.

Index Terms—Information inequality, perfect secrecy, performance bounds, secure network coding.

I. INTRODUCTION

IN classical information-theoretic cryptography, when we need to send a private message to a receiver in the presence of wiretappers, in order to protect the message, we encrypt the message with a random key and send the ciphertext to the receiver. A wiretapper who has no access to the key can know nothing about the message by only observing the ciphertext, in the sense that the ciphertext and the message are statistically independent. On the other hand, the receiver obtains the key via a “secure” channel and use it to decrypt the ciphertext to recover the private message. The best known such model is the one-time pad system studied by Shannon [1], which requires the minimal amount of randomness for the key.

The one-time pad system was generalized to *secret sharing* by Blakley [2] and Shamir [3]. Ozarow and Wyner [4] also studied a similar problem which they called the *wiretap channel II*. In this model, information is sent to the receiver

through a number of point-to-point channels. It is assumed that the wiretapper can access any one but not more than one set of channels, called a wiretap set, out of a collection \mathcal{A} of all possible wiretap sets, where \mathcal{A} is specified by the problem under consideration. For example, \mathcal{A} could be the collection of all wiretap sets each containing a single channel. In this case, the wiretapper can access any one but not more than one channel. The strategy to protect the private message is the same as that in classical information-theoretic cryptography. Specifically, the private message and the random key are combined by means of a coding scheme, so that a wiretapper observes some mixtures of the message and the key, where these mixtures are statistically independent of the message. On the other hand, the receiver node can decode the message from the information received on all the channels. Note that in secret sharing and its subsequent generalizations, it is assumed that the key is available only to the transmitter and transmission in all the channels is noiseless.

Cai and Yeung [5] generalized secret sharing to secure network coding, in which a private message is sent to possibly more than one receiver through a network of point-to-point channels. The model they studied, which we refer to as the *wiretap network* (see also El Rouayheb and Soljanin [6]), is described as follows. In this model, the assumptions about the wiretapper and the strategy to protect the private message are the same as in the wiretap channel II. The significant difference is that there exist intermediate nodes in the network that can encode, and there may be more than one receiver node. The solution is that we send both the private message and the key via a network coding scheme, so that a wiretapper can only observe some mixtures of the message and the key, where the mixtures are statistically independent of the message. On the other hand, a receiver node can recover the private message by decoding the information received from its input channels. Note that when \mathcal{A} is the empty set, the wiretap network reduces to the original network coding model studied in Ahlswede *et al.* [7].

In [5], a condition for the existence of secure linear network codes was proved and a construction of such codes was proposed. The code in [5] suffers from the pitfall that the required alphabet size is larger than $|\mathcal{A}|$. Feldman *et al.* [8] generalize and simplify the method in [5]. They derived trade-off between security, the alphabet size and the multicast rate. Under their result, the alphabet size in [5] can be greatly reduced if a small amount of overall capacity is given up. In [6], El Rouayheb and Soljanin regarded the secure network coding problem as a network generalization of the model in wiretap channel II and showed that the transmitted information

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This work was partially funded by a grant from the University Grants Committee of the Hong Kong Special Administrative Region (Project No. AoE/E-02/08) and Key Laboratory of Network Coding, Shenzhen, China (ZSDY20120619151314964). This paper was presented in part at Network Coding (NetCod), 2011.

can be secured by using the coset coding scheme in [4] at the source on top of the existing network code. Moreover, their code is equivalent to the code in [5] but the required alphabet size is much smaller. The optimal code constructions in [5], [8], and [6] have a common strategy: they first construct a code on the message and the key at the source node and then transmit the source code via a network code, which depends on the code at the source node. In Cai & Yeung [9] and Zhang & Yeung [10], a general security condition for multi-source network code was presented.

The performance of a secure network coding scheme is measured by two quantities: the size of the message and the size of the key. In designing a secure network coding scheme, we want to maximize the size of the message and at the same time minimize the size of the key. The latter is necessary because in cryptography, randomness is regarded as a resource. In [5], when the collection \mathcal{A} of all wiretap sets consists of all subsets of channels whose sizes are at most some constant r , an upper bound on the size of the message and a lower bound on the size of the random key were obtained. Both of these bounds are tight for this special case. In this paper, we extend these bounds to the general case.

Cui *et al.* [11] studied secure network coding in a single-source single-sink network with unequal channel capacities. The set of wiretap sets is arbitrary and randomness can be generated at the intermediate nodes. The aim is to find the maximal source-sink communication rate, i.e., the secrecy capacity. They give a cut-set bound on the secrecy capacity and show that the cut-set bound is not achievable in general. Some achievable strategies are proposed and the computational complexity to determine the secrecy capacity is studied.

Secure network coding was also generalized from different perspectives. Bhattad and Narayanan [12] introduced weakly secure network coding, where it is required that wiretappers cannot decode any part of the source message. In this model, a weakly secure network code can be used to avoid trading off the throughput. In [13], Harada and Yamamoto studied the strongly r -secure linear network code which can protect the source message such that a wiretapper can obtain no information about any s components of the source message by accessing $n - s$ channels provided that the maximum flows to all the sink nodes are at least n , where $s \leq n - r$. A polynomial-time algorithm was proposed to construct the strongly r -secure linear network code. They also showed that strong security contains weak security as a special case.

Secure network coding with error correction was studied by Ngai and Yeung [14], where they proposed a construction of secure error-correcting (SEC) network code which can protect the message from wiretapping, random errors and errors injected by the wiretapper. They further showed the optimality of their construction.

Security network coding was also well studied from a different point of view, see [15]–[19] for other related results.

II. PROBLEM FORMULATION

In this work, we focus on the wiretap network model proposed in [5] and aim to obtain some new performance

bounds. Denote the network by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} is the set of nodes and \mathcal{E} is the set of edges, each representing a point-to-point noiseless channel in the network. In this work, we use the terms “edge” and “channel” interchangeably. On each edge e , a symbol from some transmission alphabet F can be transmitted. In this sense we say that each channel has unit capacity. We assume that \mathcal{G} is a directed acyclic multigraph, namely there can be multiple edges between each pair of nodes.

A wiretap network consists of the following components:

- 1) *Source node s* : The node set \mathcal{V} contains a node s , called the source node, where a random message M taking values in an alphabet \mathcal{M} , called the message set, is generated.
- 2) *Set of user nodes \mathcal{U}* : A user node is a node in \mathcal{V} which is fully accessed by a legal user who is required to receive the random message M with zero error. There is generally more than one user node in a network. The set of user nodes is denoted by \mathcal{U} . For each $u \in \mathcal{U}$, let $\text{maxflow}(u)$ denote the value of a maximum flow from the source node s to node u .
- 3) *Collection of sets of wiretap edges \mathcal{A}* : \mathcal{A} is a collection of arbitrary subsets of the edge set \mathcal{E} , called a wiretap set. The wiretapper can access any $A \in \mathcal{A}$ but not more than one subset in \mathcal{A} at the same time. The wiretap set A chosen by the wiretapper is fixed before communication. The sender can know \mathcal{A} before communication but cannot figure out the exact A .

We denote such a wiretap network by the tuple $(\mathcal{G}, s, \mathcal{U}, \mathcal{A})$.

A. Admissible Code

We assume that the message M is generated at the source node according to an arbitrary distribution on the message set \mathcal{M} . Let K be a random variable independent of M , called the *key*, that takes values in an alphabet \mathcal{K} according to the uniform distribution.

For each node v of the network \mathcal{G} , we denote the set of the input edges and the set of the output edges of v by $In(v)$ and $Out(v)$, respectively. A code for a wiretap network consists of a set of local encoding mappings $\{\phi_e : e \in \mathcal{E}\}$ such that for all e , ϕ_e is a function from $\mathcal{M} \times \mathcal{K}$ to F if $e \in Out(s)$, and is a function from $F^{|In(t)|}$ to F if $e \in Out(t)$ for $t \neq s$. For $e \in \mathcal{E}$, let Y_e be the random symbol in F transmitted on channel e ; i.e., the value of ϕ_e . For a subset B of \mathcal{E} , denote $(Y_e : e \in B)$ by Y_B .

To complete the description of a code, we have to specify the order in which the channels send the indices, called the *encoding order*. Since the graph \mathcal{G} is acyclic, it defines a partial order on the node set \mathcal{V} . Then the nodes in \mathcal{V} can be indexed in a way such that for two nodes t and t' , if there is a channel from node t to node t' , then $t < t'$. According to this indexing, node t sends indices in its output channels before node t' if and only if $t < t'$. The order in which the channels within the set of output channels of a node send the indices is immaterial. The important point here is that whenever a channel sends an index, all the indices necessary for encoding have already been received. A code defined as such induces a function Φ_u from $\mathcal{M} \times \mathcal{K}$ to $F^{|In(u)|}$ for all user nodes $u \in \mathcal{U}$, where the value

of Φ_u denotes the indices received by the user node u in its input channels.

In the wiretap network model, a code $\{\phi_e : e \in \mathcal{E}\}$ should satisfy the following two conditions:

- 1) *decodable condition*: For all user node $u \in \mathcal{U}$ and all $m, m' \in \mathcal{M}$ with $m \neq m'$,

$$\Phi_u(m, k) \neq \Phi_u(m', k')$$

for all $k, k' \in \mathcal{K}$. This guarantees that any two message are distinguishable at every user node.

- 2) *secure condition*: the message should be information-theoretic secure, namely for all $A \in \mathcal{A}$,

$$H(M|Y_A) = H(M). \quad (1)$$

We would like to emphasize that the wiretappers can know the encoding and decoding functions of the message and the key at all the nodes.

We refer to a code satisfying 1) and 2) as an *admissible code*.

For an admissible code, we focus on the following two performance parameters, the size of the message and the size of the key:

- 1) the size of the message is measured by $H(M)$, which should be maximized;
- 2) the size of the key is measured by $H(K)$, which should be minimized.

Furthermore, we can define an achievable region for $H(M)$ and $H(K)$, and what we have done in this paper is to characterize this region.

B. Related Works

For set $A \subseteq B$, if $|A| = r$, then we refer to it as an r -subset of B . In [5], the following result was obtained.

Theorem 1. *Let q be the size of the transmission alphabet F , \mathcal{A} consist of all the r -subsets of \mathcal{E} and $n = \min_{u \in \mathcal{U}} \text{maxflow}(u)$. Then*

- 1) $H(M) \leq (n - r) \log q$;
- 2) $H(K) \geq \frac{r}{n-r} H(M)$.

Moreover, when F is a finite field such that $q > |\mathcal{A}|$, there exists a linear admissible code which can achieve equalities in these two bounds simultaneously; i.e., the size of the message is maximized and the size of the key is minimized.

If all the logarithms are in the base q , then 1) becomes

$$1') \quad H(M) \leq n - r.$$

Fig.1 illustrates the region of all $(H(M), H(K))$ that satisfy 1') and 2). If time-sharing is allowed, then this is also the region of all achievable $(H(M), H(K))$ because $(n - r, r)$ can be achieved by the code constructed in Cai & Yeung.

However, when \mathcal{A} consists of arbitrary subsets of \mathcal{E} , the problem becomes very hard and very little is known about the fundamental performance limit.

Example 1. *In Fig.2, the source node is S and there are two destination nodes U_1 and U_2 , the channel set is $\mathcal{E} = \{e_1, e_2, \dots, e_9\}$, where the channel capacity is unit. The message M and the key K are generated at S , and then are sent through the channels to U_1 and U_2 . M is required to be decodable at*

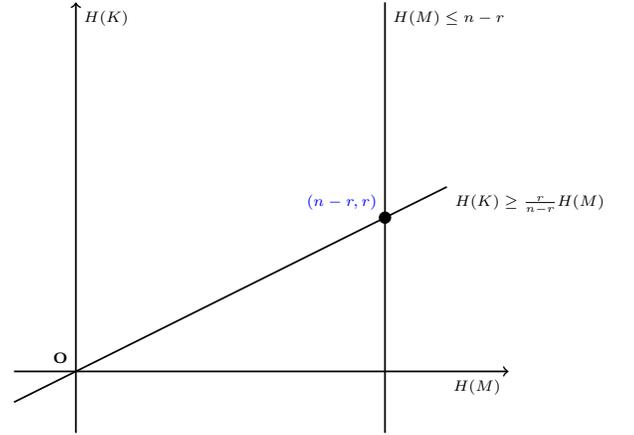


Fig. 1. The achievable region of $(H(M), H(K))$.

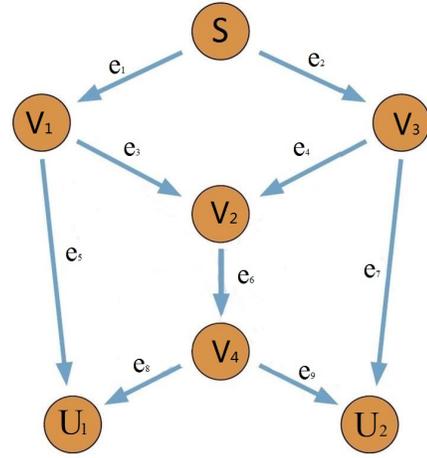


Fig. 2. Secure network coding on the butterfly network.

both U_1 and U_2 . If the set of wiretap sets \mathcal{A} is $\{W : W \subseteq \mathcal{E}, |W| = 1\}$, and for the wiretapper, it can access at most one of the sets in \mathcal{A} , then the optimal sizes of the message and the key are known in [5]. If the set of wiretap sets \mathcal{A} is arbitrary, e.g., $\mathcal{A} = \{\{e_1\}, \{e_3\}, \{e_5, e_6\}, \{e_5, e_7\}, \{e_6, e_7\}\}$, then the bounds on $H(K)$ and $H(M)$ are unknown in the literature.

C. Main Results

In this work, we investigate the performance bounds when \mathcal{A} is arbitrary. The main results are summarized as follows:

- 1) We obtain an upper bound on $H(M)$.
- 2) We propose a method to compute a lower bound on $H(K)/H(M)$, namely we obtain a lower bound on $H(K)$ in terms of $H(M)$. We first propose a brute-force algorithm for computing the lower bound. Then by refining the brute-force algorithm, we obtain an algorithm whose computational complexity is polynomial in $|\mathcal{V}|$ and $|\mathcal{E}|$ when $|\mathcal{A}|$ is fixed. The lower bound obtained by these algorithms is generally not tight. Nevertheless, we prove that is tight for the classical point-to-point communication system.

In the following sections, we first prove an upper bound on $H(M)$ in Sections III – IV and a lower bound on $H(K)$ in Sections V – IX. Then we discuss the algorithms to compute these bounds in Section X. In Section XI, we discuss the tightness of the lower bound on $H(K)$. At last, we conclude the paper in Section XII.

III. BLOCKING SETS AND WIRETAP SETS

In this section, we introduce some notations and theorems, which will be used to prove our results later.

Definition 1. For a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we denote a cut (graph cut) of \mathcal{G} by (W, W^c) , where $W \subseteq V$ contains the source node s and $W^c = V \setminus W$ contains the destination node t , and denote the set of edges from W to W^c by $E(W, W^c)$, which is also abbreviated to E_W .

We first state in the next lemma two key inequalities obtained in [5].

Lemma 1. In the network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, let (W, W^c) be a cut of \mathcal{G} . \mathcal{A} consists of all the r -subsets of \mathcal{E} . If there exists an admissible code on \mathcal{G} , then for any wiretap set $I \subseteq E_W$, we have

$$(A_1) \quad H(M) \leq H(Y_{E_W \setminus I} | Y_I);$$

$$(A_2) \quad H(K) \geq H(Y_I).$$

The inequality (A_1) was used in [5] to prove 1) and 2) of Theorem 1. The inequality (A_2) was proved but no further interpretation was provided. In this section, we extend these two inequalities to a more general situation where \mathcal{A} is arbitrary. In the following discussion, unless otherwise stated, I is assumed to be a wiretap set in \mathcal{A} .

Definition 2. In the network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a set $J \subseteq \mathcal{E}$ is called a blocking set if and only if there exists a cut (W, W^c) such that $E(W, W^c) \subseteq J$.

The blocking set is a generalization of the graph cut. Let $u \in \mathcal{U}$. Since the message M can be decoded at user node u and the symbols received at node u are functions of Y_{E_W} , where W is a cut and E_W is a subset of the blocking set J , we obtain that M is a function of Y_J , namely

$$H(M | Y_J) = 0. \quad (2)$$

Proposition 1. Let $A, B \subset \mathcal{E}$ such that $B \subset A$. If $H(M | Y_A) = H(M)$, then $H(M | Y_B) = H(M)$.

Proof: If $H(M | Y_A) = H(M)$, and $B \subseteq A$, then

$$H(M | Y_B) \geq H(M | Y_A) = H(M).$$

On the other hand,

$$H(M | Y_B) \leq H(M).$$

Hence

$$H(M | Y_B) = H(M),$$

which completes the proof. \blacksquare

The next lemma is a simple generalization of Lemma 1, which we will see is a very useful tool for obtaining performance bounds for a general secure network coding problem.

Lemma 2. In the network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, let $J \subseteq \mathcal{E}$ be a blocking set. For any admissible code on \mathcal{G} and any wiretap set $I \subseteq J$, we have

$$(B_1) \quad H(M) \leq H(Y_{J \setminus I} | Y_I);$$

$$(B_2) \quad H(K) \geq H(Y_I).$$

Proof: Since J is a blocking set, we obtain that

$$H(M | Y_J) = 0. \quad (3)$$

Since $I \subseteq J$ is a wiretap set and the code is secure, we have

$$H(M | Y_I) = H(M). \quad (4)$$

It follows that

$$\begin{aligned} H(M) &= H(M | Y_I) - H(M | Y_J) \\ &= I(M; Y_{J \setminus I} | Y_I) \\ &\leq H(Y_{J \setminus I} | Y_I), \end{aligned}$$

which completes the proof of (B_1) .

Since $H(Y_I | M, K) = 0$, $I \subseteq J$, and $H(Y_I) = H(Y_I | M)$, we obtain that

$$\begin{aligned} H(Y_I) &= H(Y_I | M) - H(Y_I | M, K) \\ &= I(Y_I; K | M) \\ &\leq H(K | M) \\ &= H(K), \end{aligned}$$

which completes the proof of (B_2) . \blacksquare

In the following, we will first prove an upper bound on $H(M)$. Then we will prove the lower bound on $H(K)$.

IV. AN UPPER BOUND ON THE MESSAGE SIZE

From Lemma 2, we can immediately obtain an upper bound on $H(M)$.

Corollary 1. Let the size of the transmission alphabet F be q . Let J be a blocking set and $I \subseteq J$ be a wiretap set. For any admissible code on \mathcal{G} ,

$$H(M) \leq \min_{J, I: I \subseteq J} |J \setminus I| \log q. \quad (5)$$

Proof: By (B_1) of Lemma 2, we have

$$\begin{aligned} H(M) &\leq H(Y_{J \setminus I} | Y_I) \\ &\leq H(Y_{J \setminus I}) \\ &\leq |J \setminus I| \log q. \end{aligned} \quad (6)$$

Then the corollary is proved by minimizing over all J, I such that $I \subseteq J$,

$$H(M) \leq \min_{J, I: I \subseteq J} |J \setminus I| \log q. \quad \blacksquare$$

From this bound, we see that if $J \setminus I = \emptyset$, then the upper bound above vanishes, which implies $H(M) = 0$. This means that if there exists a wiretap set I that contains a cut as its subset, then the network cannot send any message, because J can be taken to be I so that $|J \setminus I| = 0$.

Next we present two theorems for computing the upper bound on $H(M)$.

Lemma 3. For any fixed wiretap set I ,

$$\min_{J: I \subseteq J} |J \setminus I| = \text{mincut}(\mathcal{E} \setminus I), \quad (7)$$

where $\text{mincut}(\mathcal{E} \setminus I)$ is the minimum cut of graph $(\mathcal{V}, \mathcal{E} \setminus I)$.

Proof: Let (W, W^c) be a graph cut and E_W be the edges across the cut. Then $J_W = E_W \cup I$ is a blocking set. If we consider only such blocking sets J_W for J in (7), we have

$$\begin{aligned} \min_{J, I: I \subseteq J} |J \setminus I| &\leq \min_{J_W} |J_W \setminus I| \\ &= \min_{E_W} |E_W \setminus I| = \text{mincut}(\mathcal{E} \setminus I). \end{aligned} \quad (8)$$

The last equation is due to the fact that $E_W \setminus I$ corresponds to the set of edges across a cut of $\mathcal{E} \setminus I$, and vice versa.

Conversely, let J_0 be a blocking set including I that minimizes $|J \setminus I|$, and $E_W \subseteq J_0$. Then

$$\begin{aligned} \min_{J, I: I \subseteq J} |J \setminus I| &= |J_0 \setminus I| \geq |E_W \setminus I| \\ &\geq \min_{E_W} |E_W \setminus I| = \text{mincut}(\mathcal{E} \setminus I). \end{aligned} \quad (9)$$

Together with (8), we can conclude the proof. ■

From Lemma 3, we obtain the following corollary.

Corollary 2.

$$\min_{J, I: I \subseteq J} |J \setminus I| = \min_I \text{mincut}(\mathcal{E} \setminus I).$$

By means of this corollary, since the mincut of a graph can be computed in $O(|\mathcal{V}| \cdot |\mathcal{E}|)$ number of steps, we can compute the upper bound on $H(M)$ in Corollary 1 in $O(|I| \cdot |\mathcal{V}| \cdot |\mathcal{E}|)$ number of steps.

V. INFORMATION INEQUALITIES FOR JOINT ENTROPY

In this section, we state and explain some information inequalities that are instrumental in proving the lower bound on $H(K)$.

Let $[n] = \{1, 2, \dots, n\}$. For a subset $\alpha \subseteq [n]$, denote $(X_i, i \in \alpha)$ by X_α . Let $\bar{\alpha} = [n] \setminus \alpha$. In information theory, the following independence bound for joint entropy (e.g., p. 29 in [20]) is well known.

$$H(X_{[n]}) = H(X_1, \dots, X_n) \leq \sum_{i=1}^n H(X_i).$$

This inequality provides an upper bound on the joint entropy $H(X_{[n]})$ in terms of the entropies of the individual random variables. It is tight when the random variables X_1, \dots, X_n are mutually independent.

A. Han's Inequalities

Han [21] generalized the independence bound to two sequences of inequalities, which are stated in the next two theorems.

Theorem 2. For $k = 1, 2, \dots, n$, let

$$H_k = \frac{1}{\binom{n}{k}} \sum_{\alpha: |\alpha|=k} \frac{H(X_\alpha)}{k}.$$

Then

$$H_n \leq H_{n-1} \leq \dots \leq H_1. \quad (10)$$

In this theorem,

$$H_n = \frac{1}{n} H(X_{[n]}) \leq H_1 = \frac{1}{n} \sum_{i=1}^n H(X_i)$$

is equivalent to the independence bound. This sequence of inequalities was used in [22] to prove a converse coding theorem in multilevel diversity coding.

Theorem 3. For $k = 1, 2, \dots, n$, let

$$H'_k = \frac{1}{\binom{n}{k}} \sum_{\alpha: |\alpha|=k} \frac{H(X_\alpha | X_{\bar{\alpha}})}{k}.$$

Then

$$H'_1 \leq H'_2 \leq \dots \leq H'_n = \frac{H(X_{[n]})}{n}. \quad (11)$$

This sequence of inequalities was used in proving 2) in Theorem 1.

B. Madiman-Tetali's Inequalities

In Han's inequalities, the term H_k (H'_k) only involves the joint entropy (conditional joint entropy) of the k -subsets of $X_{[n]}$. These inequalities have recently been generalized by Madiman and Tetali [23]. In the following, let C be an arbitrary collection of subsets of $[n]$.

Definition 3. A function $\alpha: C \rightarrow R^+$ is called a fractional covering if $\sum_{s \in C: i \in s} \alpha(s) \geq 1$ for each $i \in [n]$.

Definition 4. A function $\beta: C \rightarrow R^+$ is called a fractional packing, if $\sum_{s \in C: i \in s} \beta(s) \leq 1$ for each $i \in [n]$.

Theorem 4. For any collection C of subsets of $[n]$, any fractional covering α and any fractional packing β ,

$$\sum_{s \in C} \beta(s) H(X_s | X_{s^c}) \leq H(X_{[n]}) \leq \sum_{s \in C} \alpha(s) H(X_s). \quad (12)$$

In the rest of this work, we refer to the left hand side of the inequality as the fractional packing inequality and the right hand side of the inequality as the fractional covering inequality.

Example 2. Let $n = 3$ and $C = \{C_1, C_2, C_3\}$, where $C_1 = \{1, 2\}$, $C_2 = \{2, 3\}$ and $C_3 = \{1, 3\}$.

By Han's inequalities, we obtain that

$$\begin{aligned} \frac{1}{2} H(X_{1,2} | X_3) + \frac{1}{2} H(X_{2,3} | X_1) + \frac{1}{2} H(X_{3,1} | X_2) &\leq H(X_{1,2,3}) \\ &\leq \frac{1}{2} H(X_{1,2}) + \frac{1}{2} H(X_{2,3}) + \frac{1}{2} H(X_{3,1}). \end{aligned} \quad (13)$$

Let $\alpha_i = \alpha(C_i)$ and $\beta_i = \beta(C_i)$, $i = 1, 2, 3$. By Madiman-Tetali's inequalities, we obtain that

$$\begin{aligned} \beta_1 H(X_{1,2} | X_3) + \beta_2 H(X_{2,3} | X_1) + \beta_3 H(X_{3,1} | X_2) \\ \leq H(X_{1,2,3}) \\ \leq \alpha_1 H(X_{1,2}) + \alpha_2 H(X_{2,3}) + \alpha_3 H(X_{3,1}) \end{aligned} \quad (14)$$

holds for any fractional covering α and any fractional packing β , namely

$$\begin{aligned} \alpha_1, \alpha_2, \alpha_3 \geq 0, \alpha_1 + \alpha_3 \geq 1, \alpha_2 + \alpha_3 \geq 1, \alpha_3 + \alpha_1 \geq 1; \\ \beta_1, \beta_2, \beta_3 \geq 0, \beta_1 + \beta_3 \leq 1, \beta_2 + \beta_3 \leq 1, \beta_3 + \beta_1 \leq 1. \end{aligned}$$

In particular, when $\alpha_i = \frac{1}{2}$ and $\beta_i = \frac{1}{2}$ for all $i = 1, 2, 3$, (14) becomes (13). This shows that Madiman-Tetali's inequalities are more general than Han's inequalities.

When $C_1 = \{1,2\}$, $C_2 = \{2,3\}$, $C_3 = \{2\}$, Han's inequalities are not applicable, while by Madiman-Tetali's inequalities, we have

$$\begin{aligned} & \beta_1 H(X_{1,2}|X_3) + \beta_2 H(X_{2,3}|X_1) + \beta_3 H(X_2|X_{1,3}) \\ & \leq H(X_{1,2,3}) \\ & \leq \alpha_1 H(X_{1,2}) + \alpha_2 H(X_{2,3}) + \alpha_3 H(X_2), \end{aligned} \quad (15)$$

where

$$\begin{aligned} & \alpha_1 \geq 1, \alpha_1 + \alpha_2 + \alpha_3 \geq 1, \alpha_2 \geq 1, \text{ and } \alpha_1, \alpha_2, \alpha_3 \geq 0; \\ & \beta_1 \leq 1, \beta_1 + \beta_2 + \beta_3 \leq 1, \beta_2 \leq 1, \text{ and } \beta_1, \beta_2, \beta_3 \geq 0. \end{aligned}$$

Recently, Jiang et al. [24] have applied these inequalities to multilevel diversity coding.

VI. THE FRACTIONAL PACKING BOUND

In this section, we prove a lower bound on $H(K)$ by means of the fractional packing inequality in (12).

Theorem 5. Fix a blocking set J and let β be a fractional packing of $\{J \setminus I : I \subseteq J, I \in \mathcal{A}\}$, then

$$H(K) \geq \max_{\beta} \left(\sum_{I \subseteq J} \beta(J \setminus I) - 1 \right) H(M) \quad (16)$$

Proof: By (B_1) of Lemma 2, we have

$$H(M) \leq H(Y_{J \setminus I} | Y_I). \quad (17)$$

By inequality (12), we obtain

$$\sum_{I \subseteq J} \beta(J \setminus I) H(M) \leq \sum_{I \subseteq J} \beta(J \setminus I) H(Y_{J \setminus I} | Y_I) \leq H(Y_J).$$

Hence,

$$H(Y_J) \geq \sum_{I \subseteq J} \beta(J \setminus I) H(M). \quad (18)$$

From the definition of an admissible code, no keys are generated and used at the intermediate nodes. Hence Y_J is a function of M and K . Then,

$$\begin{aligned} H(M) + H(K) & \geq H(M, K) \\ & = H(M, K, Y_J) \\ & \geq H(Y_J) \\ & \geq \sum_{I \subseteq J} \beta(J \setminus I) H(M). \end{aligned} \quad (19)$$

This implies,

$$H(K) \geq \left(\sum_{I \subseteq J} \beta(J \setminus I) - 1 \right) H(M). \quad (20)$$

Since (20) holds for any fractional packing β , we have

$$H(K) \geq \left(\max_{\beta} \sum_{I \subseteq J} \beta(J \setminus I) - 1 \right) H(M), \quad (21)$$

which completes the proof. \blacksquare

In order to evaluate the lower bound on $H(K)$, we need to consider the following LP (linear program),

$$\begin{aligned} & \max \sum_{I \subseteq J} \beta(J \setminus I) \\ & \text{s.t. } \sum_{I \subseteq J: i \in I} \beta(J \setminus I) \leq 1, \forall i \in J. \end{aligned} \quad (22)$$

In the following discussion, we define $\tau(J) = \max_{\beta} \sum_{I \subseteq J} \beta(J \setminus I) - 1$ for a fixed blocking set J , and let $\tau = \max_J \tau(J)$. Since in (22), any $\{\beta(J \setminus I) : I \subseteq J\}$ satisfying

$$\beta(J \setminus I) \geq 0, \sum_{I \subseteq J} \beta(J \setminus I) = 1$$

is a feasible solution, we obtain that $\tau(J) \geq 0$ and $\tau \geq 0$.

Corollary 3. $\tau(J) > 0$ if and only if for each edge $e \in J$, e is covered by some wiretap sets.

Proof: If $e \in J$ is not covered by any wiretap set, then for all wiretap set I , $e \in J \setminus I$. By the LP in (22), we obtain that the constraint from edge e is $\sum_{i=1}^d \beta_i \leq 1$, where d is the number of wiretap sets. This constraint dominates any other constraint, and the maximum is attained when this bound is tight. Hence, $\tau(J) = \sum_{i=1}^d \beta_i - 1 = 0$.

Conversely, assume that for all $e \in J$, it is covered by at least one wiretap set. Fix e , and we can assume that, without loss of generality, $e \in I_1$. Then we have $e \notin J \setminus I_1$, implying that the number of sets $J \setminus I_j$ ($j \neq 1$) which cover e is at most $d-1$. Let $\beta_i = \frac{1}{d-1}$ for $1 \leq i \leq d$. Then β_i is a feasible solution, and hence $\tau(J) \geq \sum_{i=1}^d \beta_i - 1 = 1/(d-1) > 0$. \blacksquare

Corollary 3 has the following implication. For a fixed J , if there exists an edge $e \in J$ such that e is not covered by any wiretap set, then $\tau(J) = 0$, and so

$$\tau = \max_{J'} \tau(J') = \max_{J' \neq J} \tau(J').$$

On the other hand, if every edge $e \in J$ is covered by at least one wiretap set, then $\tau(J) > 0$, and so

$$\tau = \max_{J'} \tau(J') \geq \tau(J) > 0.$$

Therefore, for the purpose of computing τ , we assume without loss of generality that every edge $e \in J$ is covered by at least one wiretap set.

VII. AN ALTERNATIVE BOUND

In the last section, we proved a lower bound on $H(K)$ in terms of fractional packings of $\{J \setminus I : I \subseteq J, I \in \mathcal{A}\}$ for all blocking sets J . In this section, we prove an alternative lower bound on $H(K)$ in terms of fractional coverings of $\{I : I \subseteq J, I \in \mathcal{A}\}$. In the next section, we prove a duality result between fractional packing and fractional covering that implies the equivalence of these two bounds.

Fix a blocking set J . By (B_2) of Lemma 2, for any wiretap set $I \subseteq J$, we have

$$H(K) \geq H(Y_I). \quad (23)$$

Let α be a fractional covering of $\{I : I \subseteq J\}$. By the fractional covering inequality in (12), we obtain that

$$H(Y_J) \leq \sum_{I \subseteq J} \alpha(I) H(Y_I) \leq \sum_{I \subseteq J} \alpha(I) H(K). \quad (24)$$

Together with (18), we further obtain

$$\sum_{I \subseteq J} \beta(J \setminus I) H(M) \leq H(Y_J) \leq \sum_{I \subseteq J} \alpha(I) H(K). \quad (25)$$

Then

$$H(K) \geq \frac{\sum_{I \subseteq J} \beta(J \setminus I)}{\sum_{I \subseteq J} \alpha(I)} H(M). \quad (26)$$

Maximizing over all β and minimizing over all α , we obtain another lower bound on $H(K)$ for a fixed J :

$$H(K) \geq \frac{\max_{\beta} \sum_{I \subseteq J} \beta(J \setminus I)}{\min_{\alpha} \sum_{I \subseteq J} \alpha(I)} H(M). \quad (27)$$

The maximization in the above has been considered in Section VI. Thus in order to evaluate the above lower bound on $H(K)$, we also need to consider the following LP:

$$\begin{aligned} \min \quad & \sum \alpha(I) \\ \text{s.t.} \quad & \sum_{I \subseteq J: i \in I} \alpha(I) \geq 1, \forall i \in J. \end{aligned} \quad (28)$$

VIII. A DUALITY RESULT

In this section, we prove that (27) is equivalent to (16).

Theorem 6. For a given blocking set J ,

$$\max_{\beta} \left(\sum_{I \subseteq J} \beta(J \setminus I) - 1 \right) = \frac{\max_{\beta} \sum_{I \subseteq J} \beta(J \setminus I)}{\min_{\alpha} \sum_{I \subseteq J} \alpha(I)},$$

where α is a fractional covering of $\{I : I \subseteq J, I \in \mathcal{A}\}$ and β is a fractional packing of $\{J \setminus I : I \subseteq J, I \in \mathcal{A}\}$.

In the following discussions, let

$$l_C(J) = \min_{\alpha} \sum_{I \subseteq J} \alpha(I) \quad (29)$$

and

$$l_P(J) = \max_{\beta} \sum_{I \subseteq J} \beta(J \setminus I), \quad (30)$$

where α is a fractional covering of $\{I : I \subseteq J\}$ and β is a fractional packing of $\{J \setminus I : I \subseteq J\}$.

Proof (Theorem 6): In this proof, since J is fixed, we can use l_C and l_P instead of $l_C(J)$ and $l_P(J)$ without ambiguity. We need to prove

$$l_P - 1 = \frac{l_P}{l_C}, \quad (31)$$

namely

$$l_C = \frac{l_P}{l_P - 1} \quad \text{or} \quad l_P = \frac{l_C}{l_C - 1}. \quad (32)$$

Let I_1, I_2, \dots, I_d be the wiretap sets in J .

- (1) Let $\alpha^* = \operatorname{argmin} \left\{ \sum_{I \subseteq J} \alpha(I) \right\}$ and for each wiretap set I_i , $\alpha_i^* = \alpha^*(I_i)$. For $1 \leq i \leq d$, define

$$\text{sum} = \sum_{i=1}^d \alpha_i^* \quad \text{and} \quad \beta_i^* = \frac{\alpha_i^*}{\text{sum} - 1}. \quad (33)$$

Next, we prove that $\{\beta_i^* : 1 \leq i \leq d\}$ is a feasible solution to the LP in (22); i.e., there exists a fractional packing β^* on $\{J \setminus I_i : 1 \leq i \leq d\}$ such that $\beta^*(J \setminus I_i) = \beta_i^*$.

For each $e \in J$, we can assume without loss of generality that I_1, \dots, I_j are the sets containing e and I_{j+1}, \dots, I_d be the sets not containing e . Since $\{\alpha_i^* : 1 \leq i \leq d\}$ is a fractional covering, $\sum_{i=1}^j \alpha^*(I_i) \geq 1$. For every $e \in J$, since $e \notin J \setminus I_s$, for $1 \leq s \leq j$ and $e \in J \setminus I_s$, for $j+1 \leq s \leq d$, we have

$$\begin{aligned} \sum_{i=j+1}^d \beta^*(J \setminus I_i) &= \sum_{i=j+1}^d \frac{\alpha^*(I_i)}{\text{sum} - 1} = \frac{\sum_{i=j+1}^d \alpha^*(I_i)}{\text{sum} - 1} \\ &= \frac{\text{sum} - \sum_{i=1}^j \alpha^*(I_i)}{\text{sum} - 1} \leq \frac{\text{sum} - 1}{\text{sum} - 1} = 1. \end{aligned}$$

Since l_P is the maximum of the summation in (30) over all fractional packing β , together with (33), we have

$$l_P \geq \sum_{i=1}^d \beta_i^* = \frac{\text{sum}}{\text{sum} - 1} = \frac{l_C}{l_C - 1}. \quad (34)$$

- (2) Let $\beta^* = \operatorname{argmax} \left\{ \sum_{I \subseteq J} \beta(J \setminus I) \right\}$ and for each wiretap set I_i , $\beta_i^* = \beta^*(J \setminus I_i)$. For $1 \leq i \leq d$, define

$$\text{sum} = \sum_{i=1}^d \beta_i^* \quad \text{and} \quad \alpha_i^* = \frac{\beta_i^*}{\text{sum} - 1}. \quad (35)$$

Next, we prove that $\{\alpha_i^* : 1 \leq i \leq d\}$ is a feasible solution to the LP in (28); i.e., there exists a fractional covering α^* on $\{I_i : 1 \leq i \leq d\}$ such that $\alpha^*(I_i) = \alpha_i^*$. For each $e \in J$, we can assume without loss of generality that I_1, \dots, I_j are the sets containing e and I_{j+1}, \dots, I_d be the sets not containing e . Since $\{\beta_i^* : 1 \leq i \leq d\}$ is a fractional packing, $\sum_{i=j+1}^d \beta^*(J \setminus I_i) \leq 1$. For every $e \in J$, since $e \notin J \setminus I_s$, for $1 \leq s \leq j$ and $e \in J \setminus I_s$, for $j+1 \leq s \leq d$, we have

$$\begin{aligned} \sum_{i=1}^j \alpha^*(I_i) &= \sum_{i=1}^j \frac{\beta^*(J \setminus I_i)}{\text{sum} - 1} = \frac{\sum_{i=1}^j \beta^*(J \setminus I_i)}{\text{sum} - 1} \\ &= \frac{\text{sum} - \sum_{i=j+1}^d \beta^*(J \setminus I_i)}{\text{sum} - 1} \geq \frac{\text{sum} - 1}{\text{sum} - 1} = 1. \end{aligned}$$

Since l_C is the minimum of the summation in (29) over all fractional covering α , together with (35), we have

$$l_C \leq \sum_{i=1}^d \alpha_i^* = \frac{\text{sum}}{\text{sum} - 1} = \frac{l_P}{l_P - 1}. \quad (36)$$

By (34) and (36), we obtain $l_C l_P \geq l_C + l_P \geq l_C l_P$, namely $l_C l_P = l_C + l_P$, which completes the proof. ■

By Theorem 5 and 6, the following bound is equivalent to the bound in Theorem 5.

Theorem 7. Fix a blocking set J and let α be a fractional covering of $\{I : I \subseteq J, I \in \mathcal{A}\}$. Then

$$H(K) \geq \max_{\alpha} \frac{1}{\sum_{I \subseteq J} \alpha(I) - 1} H(M). \quad (37)$$

By (32), we can write the lower bound in Theorem 5 or 7 as $\frac{H(K)}{H(M)} \geq \frac{1}{l_C - 1}$ and consider only the LP in (28). Since $\tau = \max_J 1/(l_C(J) - 1) = \max_J (l_P(J) - 1)$, we need to find $\min_J l_C(J)$ or $\max_J l_P(J)$. In the following sections, we refer to these two equivalent bounds as the *fractional covering bound* and the *fractional packing bound*, respectively.

IX. SOME PROPERTIES OF THE LOWER BOUND

Consider the matrix form of the LP in (28) for the fractional covering. Let I_1, I_2, \dots, I_d be the wiretap sets. For each blocking set J , construct a $|J| \times d$ matrix A_J to represent the edges in J as follows. Let $e_1^J, e_2^J, \dots, e_{|J|}^J$ be the edges in J . If $e_i^J \in I_j$, then $A_J(i, j) = 1$, else $A_J(i, j) = 0$. Each column of A_J corresponds to a wiretap set, and each row of A_J corresponds to an edge in J .

We can now write the LP in (28) and its dual as

$$\begin{array}{ll} \text{LP : } \min & 1^T x \\ \text{s.t.} & A_J x \geq \mathbf{1} \\ & x \geq 0 \end{array} \quad \text{Dual : } \begin{array}{ll} \max & 1^T y \\ \text{s.t.} & A_J^T y \leq \mathbf{1} \\ & y \geq 0 \end{array}$$

The strong duality theorem in linear programming (Theorem 14 in the appendix) states that the LP and its dual problem have the same optimal value.

When we try to solve the above LP, we need to consider some special relations among the wiretap sets and the blocking sets, namely a wiretap set is a subset of another wiretap set, or a blocking set is a subset of another blocking set. We discuss these issues in the following.

Corollary 4. For a given blocking set J , if wiretap sets I_i and I_j satisfy $I_i \subseteq I_j \subseteq J$, then I_i can be ignored in the model.

Proof: For wiretap sets $I_i, I_j \subseteq J$, if $I_i \subseteq I_j$, then the i th and j th column of A_J satisfy $A_J^i \leq A_J^j$ componentwise, which implies in the dual problem the constraint $(A_J^i)^T y \leq 1$ is dominated by the constraint $(A_J^j)^T y \leq 1$. Thus we can ignore the column A_J^i in A_J , or equivalently, the wiretap set I_i . ■

In the following discussion, we assume that I_i ($1 \leq i \leq d$) is not a subset of any other wiretap sets.

Corollary 5. If the blocking sets J', J satisfy $J' \subseteq J$, then $\tau(J) \leq \tau(J')$.

Proof: By definition, if $J' \subseteq J$, then $A_{J'}$ is a submatrix of A_J . By comparing the linear programs for J' and J , we notice that the two objective functions are the same, but the feasible region of J is a subset of that of J' , because $A_{J'}$ is a submatrix of A_J . Since we need to obtain the minimum value of the objective function, we have $l_C(J') \leq l_C(J)$, where $l_C(J')$ and $l_C(J)$ are the optimal values for J' and J , respectively. Then $\tau(J') = 1/(l_C(J') - 1) \geq 1/(l_C(J) - 1) = \tau(J)$, which concludes the proof. ■

This corollary implies that toward computing $\tau = \max_J \tau(J)$, if $J' \subseteq J''$, then J'' can be ignored. In particular, since each blocking set contains a graph cut (also a blocking set), toward computing τ , it is attained over all graph cuts between the source and destination nodes.

In the following sections, we will discuss the algorithms on computing the bound on $H(K)/H(m)$ and the tightness of our bound.

X. ALGORITHMS FOR COMPUTING THE LOWER BOUND

A. A Brute Force Algorithm

Based on the above discussion, we propose a brute force algorithm, namely that we enumerate all the graph cuts and solve the corresponding LPs (e.g., by the simplex algorithm). Then the time complexity is $2^{|J|} O(LP)$, where $O(LP)$ is the time complexity of the LP; e.g., the interior point algorithm can terminate in $O(m^2 n + m^3)$ arithmetic operations, where m is number of constraints and n is the number of variables.

Theorem 8. Sperner's Theorem [25]: If A_1, A_2, \dots, A_m are subsets of $N := \{1, 2, \dots, n\}$ such that A_i is not a subset of A_j if $i \neq j$, then $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

When solving the LP, the primary factors of the complexity are the number of variables and constraints, namely the number of wiretap sets d and $|J|$. By Theorem 8, since for every two wiretap sets I_i and I_j , I_i is not a subset of I_j if $i \neq j$, we obtain $d \leq \binom{|J|}{\lfloor \frac{|J|}{2} \rfloor}$.

In this algorithm, the total complexity is exponential, which is not practical when the problem size is large. Next we propose an algorithm which is polynomial when d is constant.

B. A Polynomial-Time Algorithm

In this part we show that when the number of wiretap sets, d , is a constant, there exists a polynomial algorithm for computing the lower bound. In the following discussion, we use some definitions and theorems in linear optimization which are given in the appendix.

In the above brute force algorithm, we consider every blocking set J and solve the following linear program for J :

$$\text{LP}(J) : \begin{array}{ll} \min & 1^T x \\ \text{s.t.} & A_J x \geq \mathbf{1} \\ & x \geq 0, x \in R^d. \end{array}$$

If we let $A'_J = \begin{pmatrix} A_J \\ I_{d \times d} \end{pmatrix}$ and $b_J = \begin{pmatrix} 1_{|J|} \\ 0_d \end{pmatrix}$, where $I_{d \times d}$ is the $d \times d$ identity matrix, then the above constraints can be written as $A'_J x \geq b_J$.

Let $P = \{x \in \mathbb{R}^d \mid A_J x \geq 1, x \geq 0\}$. Since $x = 1_d \in P$, P is a nonempty polyhedron. Since A'_J contains $I_{d \times d}$ as a submatrix, we see that there exist d rows of A'_J which are linearly independent. So by Theorem 12 (in Appendix A), the polyhedron P has at least one extreme point. Since $x \geq 0$, the optimal value is nonnegative, and hence not equal to $-\infty$. By Theorem 13, there exists an extreme point which is optimal. Let $x^*(J)$ denote an extreme point (not necessary unique) that gives the optimal solution. Then by Theorem 11, $x^*(J)$ is a basic feasible solution. A straightforward method to find $x^*(J)$ is to enumerate all the basic solutions of $LP(J)$, and check whether the basic solutions are feasible or not. In order to enumerate all the basic feasible solutions, we consider all $d \times d$ submatrices of A'_J . For such a submatrix S , there is a corresponding basic solution if and only if $\text{rank}(S) = d$, and if so, denote this basic solution by x_S . Then a basic solution x_S is feasible if $A'_J x_S \geq b_J$. Among all these basic feasible solutions, $x^*(J)$ is the one that attains the minimum value.

To sum up, we draw the following conclusion.

Conclusion 1. *For blocking set J , the optimal solution can be obtained by solving one of the equations: $Sx = b_S$, where S is a $d \times d$ submatrix of A'_J and b_S is the corresponding subvector of b_J .*

Furthermore, to obtain the best lower bound on $\frac{H(K)}{H(M)}$, we need to solve the linear program to obtain the optimal value for each blocking set. Then take the minimum of these optimal values over all blocking sets to obtain the best lower bound. This can be achieved by repeating the procedure in Conclusion 1.

The method described above is inefficient because if S is a submatrix of both A'_{J_1} and A'_{J_2} for two different blocking sets J_1 and J_2 , the exact same processing of S would be performed twice. In the remaining of this section, we aim to improve the method by removing such redundant operations.

In the method described above, if we obtain the best lower bound on $\frac{H(K)}{H(M)}$ from blocking set J , we refer to the optimal value and the optimal solution of $LP(J)$ as the best optimal value and the best optimal solution. Recall that for each blocking set J , since $J \subseteq \mathcal{E}$, A'_J is a submatrix of $A'_\mathcal{E}$ (\mathcal{E} is a blocking set so $A'_\mathcal{E}$ is defined accordingly). Then we can draw another conclusion.

Conclusion 2. *Consider the best lower bound on $\frac{H(K)}{H(M)}$ in network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The best optimal solution can be obtained by solving one of the equations $Sx = b_S$, where S is a $d \times d$ submatrix of $A'_\mathcal{E}$ and b_S is the corresponding subvector of $b_\mathcal{E}$.*

Definition 5. *For each blocking set J , let Q_J be the set of all basic feasible solutions of $LP(J)$, and let $Q = \bigcup_J Q_J$.*

Let $\gamma = \binom{|\mathcal{E}|+d}{d}$. By Conclusion 2, the best optimal value is $\min_{x \in Q} 1^T x$. If we compute the set Q by means of the prescription in Definition 5, we need to enumerate all the blocking sets, and hence the computational complexity is exponential in $|\mathcal{E}|$. But

we notice that matrix $A'_\mathcal{E}$ has γ submatrices with dimension $d \times d$ and each of them corresponds to at most one basic feasible solution, and so $|Q| \leq \gamma$. When d is a constant, γ is polynomial in $|\mathcal{E}|$, which suggests that if we compute Q by enumerating these γ $d \times d$ submatrices, we may obtain an algorithm which is polynomial in $|\mathcal{E}|$. By the definition of Q , for each $d \times d$ submatrix S , if $\text{rank}(S) < d$, we cannot obtain a basic solution from $Sx = b_S$. Therefore, we only need to consider S such that

1) $\text{rank}(S) = d$.

When S satisfies 1), $Sx = b_S$ has a unique solution, which we denote by x_S . In the sequel, whenever we discuss x_S , we implicitly assume that S satisfies 1), otherwise x_S is undefined. If x_S is feasible for some blocking set J , namely $A'_J x_S \geq b_J$, then x_S satisfies

2) $x_S \geq 0$.

Let Q' be the set of all x_S satisfying 2). Then $Q \subseteq Q'$ and Q' can be computed in polynomial time. Now we need to solve the following problem: if $x \in Q'$, what is the necessary and sufficient condition for $x \in Q$?

For each edge $e \in \mathcal{E}$, let $(a^e)^T$ denote the row of $A_\mathcal{E}$ corresponding to e . For each $x_S \in Q'$, let $F(S) = \{e \in \mathcal{E} \mid (a^e)^T x_S \geq 1\}$.

Theorem 9. *Let $x_S \in Q'$. Then $x_S \in Q$ if and only if $F(S)$ is a blocking set.*

Proof: “ \Rightarrow ” For $x_S \in Q'$, if $x_S \in Q$, then x_S is a basic feasible solution of $LP(J)$ for some blocking set J . By $A'_J x_S \geq b_J$, we obtain that for each $e \in J$, $(a^e)^T x_S \geq 1$, which means $e \in F(S)$, implying $J \subseteq F(S)$. Hence $F(S)$ is a blocking set.

“ \Leftarrow ” Recall that $A'_\mathcal{E} = \begin{pmatrix} A_J \\ I_{d \times d} \end{pmatrix}$. For a $d \times d$ submatrix S of $A'_\mathcal{E}$, let E_S be the set consisting of all $e \in \mathcal{E}$ such that e corresponds to a row of S . By the definition of x_S , we have that for each $e \in E_S$, $(a^e)^T x_S = 1$, which means that $e \in F(S)$, implying that $E_S \subseteq F(S)$. Let $J = F(S)$. Then J is a blocking set. For $e \in J$, $(a^e)^T x_S \geq 1$, namely $A_J x_S \geq 1$. Together with $x_S \geq 0$, we have $A'_J x_S \geq b_J$. Since $Sx_S = b_S$ and S is a $d \times d$ submatrix of A'_J , x_S is a basic feasible solution of $LP(J)$, and hence $x_S \in Q$. ■

By Theorem 9, for $x_S \in Q'$, in order to determine whether $x_S \in Q$, we only need to check whether $F(S)$ is a blocking set. This can be done in polynomial time as follows. In the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, upon deleting all the edges in $F(S)$, we need to check whether the source node and the destination node are connected in the residual graph, which can be achieved via a Depth-First Search (DFS) algorithm (e.g., in [26]) with time complexity $O(|\mathcal{V}| + |\mathcal{E}|)$. Based on the these results, we propose Algorithm 1 on the next page for computing the lower bound on $H(K)/H(M)$.

The time complexity analysis of Algorithm 1 is as follows:

1. In step a), the time for calculating all x_S is $O(\gamma * d^3)$, where d^3 is the time for matrix inversion by Gaussian elimination.
2. In step b), in the worst case, we need to enumerate all the γ submatrices. For each submatrix S , there are at most $|\mathcal{E}|$

Algorithm 1 Algorithm for computing a lower bound on $\frac{H(K)}{H(M)}$

- a) For each $d \times d$ submatrix S of $A'_\mathcal{E}$, keep the matrix provided that it satisfies $\text{rank}(S) = d$ and $x_S \geq 0$.
 - b) For each S that survives in a), calculate $F(S)$, and determine whether $F(S)$ is a blocking set. If so, calculate $\text{val}(S) = 1_d^T x_S$, else ignore S .
 - c) Output S and x_S that attain the minimum $\text{val}(S)$.
-

edges in $F(S)$, and so we have to delete at most $|\mathcal{E}|$ edges in graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The complexity for determining whether $F(S)$ is a blocking set is $O(|\mathcal{V}| + |\mathcal{E}|)$. In sum, the time complexity of this step is $O(\gamma * (|\mathcal{V}| + |\mathcal{E}|))$.

3. With steps a) and b) together, the total complexity is $O(\gamma * d^3 + \gamma * (|\mathcal{V}| + |\mathcal{E}|)) = O(|\mathcal{E}|^d (|\mathcal{V}| + |\mathcal{E}|))$, which is polynomial when d is a constant.

XI. TIGHTNESS OF THE LOWER BOUND

In this section, we discuss tightness of the lower bound on $H(K)/H(M)$ obtained by Algorithm 1. In Cai and Yeung [9], a security condition for multi-source linear network coding was proved. This condition, stated in the next theorem, is instrumental in the discussion in this section. For the sake of completeness, we include in Appendix B a proof of this theorem which is somewhat simpler than the proof in [9].

In the sequel, let F_q be a finite field of size q and $F_q^r = \underbrace{F_q \times F_q \dots \times F_q}_r$. For a matrix A , we also write the number of rows and columns of A as $\text{row}(A)$ and $\text{col}(A)$, respectively.

Theorem 10. *Let A and B be given matrices defined on F . Let M be a random vector with positive probability distribution on F_q^m and K be a uniformly distributed random vector on F_q^k . Let $Y = \begin{pmatrix} A & B \\ & M \\ & K \end{pmatrix}$ and $C = \begin{pmatrix} A & B \\ & M \\ & K \end{pmatrix}$ and assume that $\text{rank}(C)$ is equal to the number of rows of C . Then the following are equivalent:*

- a) M and Y are independent, namely $I(Y; M) = 0$;
- b) $\text{rank}(B) = \text{row}(B)$, or equivalently, $\text{rank}(B) = \text{rank}(C)$.

In practice, when $q \rightarrow \infty$, the matrix C can be generated randomly. With high probability approaching 1, $\text{rank}(C)$ is equal to the rows of C .

A. When the Best Lower bound is Zero

In this case, the lower bound on $H(K)/H(M)$ is tight as we now show. By $\tau = \max_J \tau(J) = 0$, we obtain that for each blocking set J , $\tau(J) = 0$. In Corollary 3, by letting J be an arbitrary graph cut (W, W^c) of network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we see that there exists an edge $e \in E(W, W^c)$ such that e is not contained in any wiretap set. Hence in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, if we delete all the edges which are contained in some wiretap sets, then the number of remaining edges in each graph cut is at least 1. By the max-flow min-cut theorem, there exists a path P from the source node to the destination node and all the

edges in P are not contained in any wiretap sets. So we can send a message M along P without mixing it with a random key. For such a scheme, $H(M) > 0$ and $H(K) = 0$, implying that the bound $H(K)/H(M) \geq 0$ is tight.

B. Point-to-Point Communication System

In this section, we prove that in a point-to-point communication system, the lower bound on $H(K)/H(M)$ is tight. Consider such a system. Let s and u be the source node and the destination node, respectively. Let h be the number of edges from node s to node u and I_1, I_2, \dots, I_d be the wiretap sets.

We now write the LP in (28) and its dual as follows

$$\begin{aligned} \text{Primal : } \min \quad & 1^T x & \text{Dual : } \max \quad & 1^T y \\ \text{s.t.} \quad & A_J x \geq \mathbf{1} & \text{s.t.} \quad & A_J^T y \leq \mathbf{1} \\ & x \geq 0, x \in R^d & & y \geq 0, y \in R^h \end{aligned} \quad (38)$$

Since the primal has an optimal solution x^* , by the strong duality theorem in linear optimization (Theorem 14 in the appendix), the dual also has an optimal solution y^* and $1^T x^* = 1^T y^*$. Next we prove that the lower bound on $\frac{H(K)}{H(M)}$ can be achieved, namely there exists a code such that $H(M) = (1^T y^* - 1)H(K)$.

Proposition 2. *There exists an optimal solution y^* such that all its entries are rational numbers.*

Proof: By Conclusion 1, there exists an extreme point y^* which is optimal. This extreme point can be obtained by solving a particular set of linear equations, whose coefficients are rational numbers. Hence we conclude that y^* is also rational. ■

Let $y^* = (a_1/b_1, a_2/b_2, \dots, a_h/b_h)^T$, where $a_i, b_i \in \mathbb{N}$ and $\text{gcd}(a_i, b_i) = 1$, $1 \leq i \leq h$. Let $g = \text{lcm}(b_1, b_2, \dots, b_h)$, and $w_i = g \cdot a_i/b_i$, $w_i \in \mathbb{N}$. Let $w_{\max} = \max_{1 \leq i \leq h} w_i$ and

$w = \left(\sum_{i=1}^h w_i - g \right)$. Then $1^T y^* - 1 = \frac{w}{g}$. Let M and K be uniformly distributed on F_q^g and F_q^w , respectively. Next, we prove that there exists a linear code with transmission alphabet $F = F_q^{w_{\max}}$ such that $H(K) = g$ and $H(M) = w$ (where the logarithm is in the base q), and on each edge e_i ($1 \leq i \leq h$), the codeword is a vector defined on $F_q^{w_i}$. By appending to the codeword a zero vector of length $w_{\max} - w_i$, the codeword becomes a vector in F . When $w_i = 0$, we transmit nothing on edge e_i , so we can ignore edge e_i . In the following, without loss of generality, we assume that $w_i > 0$.

Proposition 3. *There exists a wiretap set I such that*

$$\sum_{e_i \in I} w_i = g.$$

Proof: Since y^* is a basic feasible solution of the dual problem in (38), we can find matrix C such that

$$C y^* = \begin{pmatrix} 1_{n_1} \\ 0_{n_2} \end{pmatrix}, \quad (39)$$

where C is an invertible $h \times h$ submatrix of $\begin{pmatrix} A_J^T \\ I_{h \times h} \end{pmatrix}$ and $n_1 + n_2 = h$. In the dual problem, we can see that $y_0 = (1, 0, \dots, 0) \in R^h$ is a feasible solution and $1^T y_0 = 1$. Therefore, $1^T y^* \geq 1^T y_0 = 1$. If $n_1 = 0$, then $y^* = 0$, so that $1^T y^* = 0$, a contradiction. Hence, $n_1 > 0$. Then we obtain from (39) that

$$C \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_h \end{pmatrix} = \begin{pmatrix} g \\ g \\ \vdots \\ 0 \end{pmatrix}. \quad (40)$$

Letting I be the wiretap set that corresponds to the first row of C , we have $\sum_{e_i \in I} w_i = g$. ■

Without loss of generality, we can let the wiretap set I prescribed in Proposition 3 be $I_d = \{e_{t+1}, e_{t+2}, \dots, e_h\}$, so that the edges apart from those in I_d are e_1, e_2, \dots, e_t . Then for each I_i where $1 \leq i \leq d-1$, by $A_J^T y^* \leq 1$ and $y^* = (w_1/g, w_2/g, \dots, w_h/g)$, we have

$$\sum_{j: e_j \in I_i} w_j \leq g \quad (41)$$

for $1 \leq i \leq d-1$.

We assume

$$M = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_t \end{pmatrix} \in F_q^w, \quad (42)$$

where $m_i \in F_q^{w_i}$ ($1 \leq i \leq t$). Let B_i ($1 \leq i \leq h$) be a $w_i \times g$ matrix defined on F_q to be specified later. Let the symbol transmitted on edge e_i be

$$Y_i = m_i + B_i K, \quad (43)$$

where $Y_i \in F_q^{w_i}$, $1 \leq i \leq t$, and let

$$Y_{I_d} = B_{I_d} K, \quad (44)$$

where

$$B_{I_d} = \begin{pmatrix} B_{t+1} \\ B_{t+2} \\ \vdots \\ B_h \end{pmatrix} \quad (45)$$

is the $g \times g$ identity matrix on F_q . Namely, for $t+1 \leq i \leq h$, the symbol transmitted on edge e_i is

$$Y_i = B_i K. \quad (46)$$

Let Y be the symbols transmitted on all the edges. Then we

can write

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_t \\ Y_{I_d} \end{pmatrix} = \begin{pmatrix} D_1 & 0 & 0 & \dots & 0 & B_1 \\ 0 & D_2 & 0 & \dots & 0 & B_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & D_t & B_t \\ 0 & 0 & \dots & 0 & 0 & B_{I_d} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_t \\ K \end{pmatrix} \quad (47)$$

where D_i , $1 \leq i \leq t$, is the $w_i \times w_i$ identity matrix.

For a matrix A , we denote the vector space spanned by the rows of A by $\text{rowspan}(A)$. For each e_i ($1 \leq i \leq h$), let $V_i = \text{rowspan}(0, \dots, D_i, \dots, 0, B_i)$ (the row space of the i th row in (47)). From the above construction, we have $\dim(V_i) = w_i$

for $1 \leq i \leq h$ and $\dim(V_1 \oplus V_2 \oplus \dots \oplus V_h) = \sum_{i=1}^h w_i$.

In the code we have constructed, we see from (43) that the g symbols of the key K are sent on the edges in I_d . Therefore, $I(Y_{I_d}; M) = 0$. The following lemma, which is a refinement of Lemma 3 in [5], is instrumental for constructing B_i , $1 \leq i \leq t$.

Lemma 4. *Let V_1, V_2, \dots, V_m be vector subspaces on F_q^n , and $\dim(V_i) = d_i$ ($1 \leq i \leq m$). If $d \geq 0$ and $d + d_i \leq n$ ($1 \leq i \leq m$), then for $q > m$, there exists a vector subspace V of F_q^n , such that $\dim(V) = d$ and $\dim(V \oplus V_i) = \dim(V) + \dim(V_i)$ ($1 \leq i \leq m$).*

Proof: Let $\{b_1, b_2, \dots, b_d\}$ be a basis of V . For all $1 \leq i \leq m$, let $\{v_{i1}, v_{i2}, \dots, v_{id_i}\}$ be a maximally independent set of vectors in V_i . We construct $\{b_1, b_2, \dots, b_d\}$ by induction. It suffices to show that for $1 \leq j \leq d$, if b_1, b_2, \dots, b_{j-1} have been chosen such that for all V_i , $1 \leq i \leq m$,

$$b_1, b_2, \dots, b_{j-1}, v_{i1}, v_{i2}, \dots, v_{id_i} \quad (48)$$

are linearly independent, then it is possible to choose b_j such that for all $1 \leq i \leq m$,

$$b_1, b_2, \dots, b_{j-1}, b_j, v_{i1}, v_{i2}, \dots, v_{id_i} \quad (49)$$

are linearly independent. Specifically, b_j is chosen such that it is independent of the set of vectors in (48) for all $1 \leq i \leq m$; i.e.,

$$b_j \in F_q^n \setminus \cup_{1 \leq i \leq m} \langle b_1, b_2, \dots, b_{j-1}, v_{i1}, v_{i2}, \dots, v_{id_i} \rangle. \quad (50)$$

Since the cardinality of a subspace in F_q^n is finite, we need to

show that the set above is nonempty.

$$\begin{aligned}
& \left| \bigcup_{1 \leq i \leq m} \langle b_1, b_2, \dots, b_{j-1}, v_{i1}, v_{i2}, \dots, v_{id_i} \rangle \right| \\
& \leq \sum_{1 \leq i \leq m} \left| \langle b_1, b_2, \dots, b_{j-1}, v_{i1}, v_{i2}, \dots, v_{id_i} \rangle \right| \\
& = \sum_{1 \leq i \leq m} q^{d_i+j-1} \\
& \leq \sum_{1 \leq i \leq m} q^{n-1} \text{ (for } d_i + j \leq d_i + d \leq n) \\
& = mq^{n-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \left| F_q^n \setminus \bigcup_{1 \leq i \leq m} \langle b_1, b_2, \dots, b_{j-1}, v_{i1}, v_{i2}, \dots, v_{id_i} \rangle \right| \\
& \geq q^n - mq^{n-1} \\
& = q^{n-1}(q - m) \\
& > 0,
\end{aligned}$$

since $q > m$. Hence b_j can be chosen for all $1 \leq j \leq m$. ■

In the following, we construct $B_i, 1 \leq i \leq t$ to satisfy the secure condition: for each wiretap set $I, I(Y_I; M) = 0$. Since the symbols transmitted on the edges in wiretap set $I_i = \{e_{i1}, e_{i2}, \dots, e_{i|I_i|}\}$ ($1 \leq i \leq d-1$) are

$$\begin{pmatrix} Y_{i_1} \\ Y_{i_2} \\ \vdots \\ Y_{i_{|I_i|}} \end{pmatrix} = \quad (51)$$

$$\begin{pmatrix} 0 & \dots & D_{i_1} & \dots & \dots & \dots & B_{i_1} \\ 0 & \dots & \dots & D_{i_2} & \dots & \dots & B_{i_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & D_{i_{|I_i|}} & \dots & B_{i_{|I_i|}} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_t \\ K \end{pmatrix}, \quad (52)$$

by Theorem 10, for each wiretap set I_i ($1 \leq i \leq d-1$), if

$$T_i = \begin{pmatrix} B_{i_1} \\ B_{i_2} \\ \vdots \\ B_{i_{|I_i|}} \end{pmatrix} \quad (53)$$

satisfies b) of Theorem 10, namely

$$\dim(T_i) = \text{row}(T_i) = \sum_{j=1}^{|I_i|} \text{row}(B_{i_j}) = \sum_{j=1}^{|I_i|} w_{i_j}, \quad (54)$$

then for I_i , the secure condition holds.

For $1 \leq i \leq d-1$, we define matrix T_i^0 as follows: if $I_i \cap I_d = \{e_{j_1}, e_{j_2}, \dots, e_{j_r}\}$, then

$$T_i^0 = \begin{pmatrix} B_{j_1} \\ B_{j_2} \\ \vdots \\ B_{j_r} \end{pmatrix}, \quad (55)$$

else T_i^0 is the empty matrix. For each i, T_i^l for $1 \leq l \leq t$ are defined inductively as follows: if $e_l \in I_i$, then

$$T_i^l = \begin{pmatrix} T_i^{l-1} \\ B_l \end{pmatrix},$$

else

$$T_i^l = T_i^{l-1}.$$

We can verify that for $1 \leq i \leq d-1$, the rows of T_i^t are a permutation of the rows of T_i . Hence, (54) holds if and only if

$$\dim(T_i^t) = \text{row}(T_i^t). \quad (56)$$

Now, we construct $B_i, 1 \leq i \leq t$ one by one starting from B_1 . For each $l, 1 \leq l \leq t$, we need to construct B_l such that T_i^l satisfies b) of Theorem 10; i.e.,

$$\dim(T_i^l) = \text{row}(T_i^l), \quad (57)$$

for $1 \leq i \leq d-1$.

Before we construct B_1 , for wiretap set I_i ($1 \leq i \leq d-1$), since B_{I_d} is an identity matrix, if $I_i \cap I_d \neq \emptyset$, then

$$\dim(T_i^0) = \sum_{j: e_j \in I_i \cap I_d} w_j = \text{row}(T_i^0),$$

else $\dim(T_i^0) = 0$. For either case, (57) holds.

For B_1 , $\text{row}(B_1) = w_1$, and it is required that if $e_1 \in I_i$,

$$\begin{aligned}
\dim(T_i^1) &= \text{row}(T_i^1) \\
&= \text{row}(T_i^0) + \text{row}(B_1) \\
&= \text{row}(T_i^0) + w_1,
\end{aligned} \quad (58)$$

for $1 \leq i \leq d-1$. By (41), if $e_1 \in I_i$ ($1 \leq i \leq d-1$), we have

$$\begin{aligned}
\text{row}(T_i^0) + w_1 &= \sum_{j: e_j \in I_i \cap I_d} w_j + w_1 \\
&\leq \sum_{j: e_j \in I_i} w_j \\
&\leq g.
\end{aligned} \quad (59)$$

By (59) and Lemma 4, we can construct a $w_1 \times g$ matrix B_1 to satisfy (58), and hence (57) is satisfied for $l = 1$.

We assume that for a fixed l' , where $1 \leq l' \leq t-1$, $B_1, B_2, \dots, B_{l'}$ have been constructed so that (57) is satisfied for $1 \leq l \leq l'$. Then

$$\dim(T_i^{l'}) = \text{row}(T_i^{l'}) = \sum_{j: e_j \in I_i \cap I_d} w_j + \sum_{j: e_j \in I_i, j \leq l'} w_j. \quad (60)$$

For $B_{l'+1}$, $\text{row}(B_{l'+1}) = w_{l'+1}$, and it is required that if $e_{l'+1} \in I_i$,

$$\dim(T_i^{l'+1}) = \text{row}(T_i^{l'+1}) = \text{row}(T_i^{l'}) + w_{l'+1}. \quad (61)$$

By (41) and (60), if $e_{l'+1} \in I_i$,

$$\begin{aligned}
\text{row}(T_i^{l'}) + w_{l'+1} &= \sum_{j: e_j \in I_i \cap I_d} w_j + \sum_{j: e_j \in I_i, j \leq l'+1} w_j \\
&\leq \sum_{j: e_j \in I_i} w_j \\
&\leq g.
\end{aligned} \quad (62)$$

By Lemma 4 and (62), we can construct a $w_{l'+1} \times g$ matrix $B_{l'+1}$ such that (61) holds, and hence (57) is satisfied for $l = l' + 1$. By mathematical induction, we can construct $B_i, 1 \leq i \leq t$.

The decoding can be done as follows. We first obtain K from wiretap set I_d . Then y_i can be solved for all $1 \leq i \leq h$ and by (43) we obtain that $m_i = Y_i - B_i K$ for $1 \leq i \leq t$.

For the code we have constructed, $H(M) = w$ and $H(K) = g$, so that $H(M)/H(K) = w/g = 1^T y^* - 1$ as desired. Hence the lower bound on $H(K)/H(M)$ by Algorithm 1 is tight.

Now, we give an example to demonstrate our lower bound on $H(K)/H(M)$.

Example 3. Let $\{e_1, e_2, e_3\}$ be a cut-set. The set of wiretap sets $\mathcal{A} = \{A_1, A_2\}$, where $A_1 = \{e_1, e_2\}$, $A_2 = \{e_2, e_3\}$ are two wiretap sets. By the fractional covering bound, we have

$$\max \quad x_1 + x_2 + x_3 \quad (63)$$

$$\text{s.t.} \quad x_1 + x_2 \leq 1; \quad (64)$$

$$x_2 + x_3 \leq 1; \quad (65)$$

$$0 \leq x_1, x_2, x_3 \leq 1; \quad (66)$$

It is easy to see $x_1 = x_3 = 1, x_2 = 0$ is an optimal solution. Hence

$$\frac{H(K)}{H(M)} \geq \frac{1}{x_1 + x_2 + x_3 - 1} = 1. \quad (67)$$

Let $H(K) = 1$. From our construction of the code that achieves the lower bound, we see that x_i ($i = 1, 2, 3$) can be interpreted as the information rate on channel e_i , with the information transmitted on channel e_1, e_2 , and e_3 being mutually independent. The constraints (64) and (65) mean the size of the symbols in each wiretap set cannot exceed the size of the key, which is similar to Shannon's perfect secrecy.

On the other hand, we cannot directly apply the bounds in Cai & Yeung [5] since \mathcal{A} does not contain the set $\{e_1, e_3\}$. If we consider a weaker set of wiretap sets $\mathcal{A}' = \{A'_1, A'_2, A'_3\}$, where $A'_1 = \{e_1\}$, $A'_2 = \{e_2\}$, and $A'_3 = \{e_3\}$. By the bounds in Cai & Yeung [5], we have

$$\frac{H(K)}{H(M)} \geq \frac{1}{2}, \quad (68)$$

which is strictly less than our bound.

XII. CONCLUSION

In this paper, we have obtained an upper bound on the size of the message and a lower bound on the size of the key for a secure network code on a wiretap network, when the set of wiretap sets \mathcal{A} is arbitrary. The lower bound on the size of the key is obtained via a set of entropy inequalities by Madiman and Tetali [23]. The bound on $H(K)$ consists of a fractional covering bound and a fractional packing bound, which can be proved to be equivalent. Computation of this bound can be achieved in polynomial time when $|\mathcal{A}|$ is fixed, and it is tight for the special case of the point-to-point communication system. That is, from the perspective of cut-set bound, our lower bound on $H(K)$ is optimal. Compared to the existing bounds, our bound is more general to outperform all of them. Consider the region of points $(H(M), H(K))$, our result has

established an outer bound on the achievable region. Moreover, our bounds have characterized the performance of routing, which is a special network code and can be simplified as a point-to-point communication system.

APPENDIX A LINEAR OPTIMIZATION

In this appendix, we present some standard definitions and theorems in linear optimization taken from [27].

Definition 6. A polyhedron is a set that can be described in the form $\{x \in R^n | Ax \geq b\}$, where A is an $m \times n$ matrix and b is a vector in R^m .

Definition 7. Let P be a polyhedron. A vector $x \in P$ is an extreme point of P if we cannot find two vectors $y, z \in P$, both different from x , and a scalar $\lambda \in [0, 1]$, such that $x = \lambda y + (1 - \lambda)z$.

Definition 8. Let P be a polyhedron. A vector $x \in P$ is a vertex of P if there exists some c' such that $cx' < c'y$ for all y satisfying $y \in P$ and $y \neq x$.

Definition 9. Consider a polyhedron P defined by linear equality and inequality constraints, and let x^* be an element of R^n .

- (a) The vector x^* is a basic solution if:
 - 1) All equality constraints are active.
 - 2) Out of the constraints that are active at x^* , there are n of them that are linearly independent.
- (b) If x^* is a basic solution that satisfies all of the constraints, we say that it is a basic feasible solution.

Theorem 11. Let P be a nonempty polyhedron and let $x^* \in P$. Then, the following are equivalent:

- (a) x^* is a vertex;
- (b) x^* is an extreme point;
- (c) x^* is a basic feasible solution.

Definition 10. A polyhedron $P \subset R^n$ contains a line if there exists a vector $x \in P$ and a nonzero vector $d \in R^n$ such that $x + \lambda d \in P$ for all scalars λ .

Theorem 12. Suppose that the polyhedron $P = \{x \in R^n | a'_i x \geq b_i, i = 1, \dots, m\}$ is nonempty. Then, the following are equivalent:

- (a) The polyhedron P has at least one extreme point.
- (b) The polyhedron P does not contain a line.
- (c) There exists n vectors out of the family a_1, \dots, a_m , which are linearly independent.

Theorem 13. Consider the linear programming problem of minimizing $c'x$ over a polyhedron P . Suppose that P has at least one extreme point. Then, either the optimal cost is equal to $-\infty$, or there exists an extreme point which is optimal.

Theorem 14 (Strong duality). If a linear programming problem has an optimal solution, so does its dual, and the respective optimal costs are equal.

APPENDIX B
PROOF TO THEOREM 10

Proof: “ $a \Rightarrow b$ ” Since $\text{rank}(C) = \text{row}(C)$, we have

$$\text{row}(C) \leq \text{col}(C).$$

Then for each $Y = y$, the equation $y = AM + BK$ has at least one solution for (M, K) , which means

$$\Pr(Y = y) > 0.$$

Together with $\Pr(M = m) > 0$ and $I(Y; M) = 0$, we obtain that

$$\Pr(Y = y, M = m) = \Pr(Y = y)\Pr(M = m) > 0,$$

namely for each y and m , the equation $y = Am + BK$ has at least one solution for K . Since $BK = y - Am$ has at least one solution for arbitrary (y, m) , we obtain $\text{rank}(B) = \text{row}(B)$.

“ $b \Rightarrow a$ ” Let $W = AM$, $V = BK$ and $r = \text{rank}(B)$. Since K is uniformly distributed, V is uniformly distributed on F_q^r . Since $\text{row}(Y) = \text{row}(V)$,

$$H(Y) \leq \log |F_q^r| = H(V) = H(BK).$$

On the other hand,

$$\begin{aligned} H(Y) &= H(Y|M) + I(Y; M) \\ &\geq H(Y|M) \\ &= H(AM + BK|M) \\ &= H(BK|M) \\ &= H(BK), \end{aligned}$$

which means that $H(Y) \geq H(BK)$ and the equality holds if and only if $I(Y; M) = 0$. ■

REFERENCES

- [1] C. E. Shannon, “Communication theory of secrecy systems,” *Bell Sys. Tech. Journal* 28, pp. 656-715, 1949.
- [2] G. R. Blakley, “Safeguarding cryptographic keys,” in *Proceedings of the National Computer Conference*, 48: 313-317, 1979.
- [3] A. Shamir, “How to share a secret,” *Comm. ACM*, 22: 612-613, 1979.
- [4] L. H. Ozarow and A. D. Wyner, “Wire-tap Channel II,” *AT&T Bell Labs. Tech. J.*, 63: 2135-2157, 1984.
- [5] N. Cai and R. W. Yeung, “Secure Network Coding on a Wiretap Network,” *IEEE Trans. Inform. Theory*, 57(1): 424-435, Jan. 2011.
- [6] S. Y. E. Rouayheb and E. Soljanin, “On wiretap networks II,” *IEEE International Symposium on Information Theory*, Nice, France, pp. 551-555, Jun. 24-29, 2007.
- [7] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, “Network information flow,” *IEEE Trans. Inform. Theory*, IT-46: 1204-1216, 2000.
- [8] J. Feldman, T. Malkin, C. Stein, and R. A. Servedio, “On the capacity of secure network coding,” 42nd Annual Allerton Conference on Communication, Control, and Computing, Monticello, IL, Sept 29-Oct 1, 2004.
- [9] N. Cai and R. W. Yeung, “A Security Condition for Multi-Source Linear Network Coding,” *IEEE International Symposium on Information Theory*, Nice, France, June 24-29, 2007.
- [10] Z. Zhang and R. W. Yeung, “A General Security Condition for Multi-Source Linear Network Coding,” *IEEE International Symposium on Information Theory*, Seoul, Korea, June 28-July 3, 2009.
- [11] T. Cui, T. Ho, and J. Kliewer, “On Secure Network Coding with Nonuniform or Restricted Wiretap Sets,” *IEEE Trans. Inform. Theory*, 59(1): 166-176, Jan. 2013.
- [12] K. Bhattad and K. R. Narayanan, “Weakly secure network coding,” in Proc. First Workshop on Network Coding, Theory, Appl. (NetCod’05), Apr. 2005.
- [13] K. Harada and H. Yamamoto, “Strongly secure linear network coding,” *EICE Trans. Fundament.*, vol. E91-A, no. 10, pp. 2720-2728, Oct. 2008.

- [14] C.-K. Ngai and R. W. Yeung, “Secure error-correcting (SEC) network codes,” presented at the 2009 Workshop on Network Coding, Theory and Appl., Lausanne, Switzerland, 2009.
- [15] L. Lima, M. Médard, and J. Barros, “Random Linear Network Coding: A free cipher?,” *IEEE International Symposium on Information Theory*, Nice, France, Jun 24-Jun 29, 2007.
- [16] S. Jaggi, M. Langberg, S. Katti, T. Ho, D. Katabi, and M. Médard, “Resilient network coding in the presence of byzantine adversaries,” in Proc. IEEE INFOCOM 2007, Anchorage, AK, pp. 616-624, May 2007.
- [17] T. Ho, B. Leong, R. Koetter, M. Médard, M. Effros, and D. R. Karger, “Byzantine modification detection in multicast networks using randomized network coding,” *IEEE International Symposium on Information Theory*, Chicago, USA, Jun. 2004.
- [18] L. Lima, S. Gheorghiu, J. Barros, M. Médard, A. L. Toledo, “Secure Network Coding for Multi-Resolution Wireless Video Streaming,” *Journal of Selected Areas in Communications*, Vol. 28, No. 3, Apr. 2010.
- [19] J. Tan and M. Médard, “Secure Network Coding with a Cost Criterion,” Proc. 4th International Symposium on Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt’06), Boston MA, April, 2006.
- [20] R. W. Yeung, *Information Theory and Network Coding*, Springer, 2008.
- [21] T. S. Han, “Nonnegative entropy measures of multivariate symmetric correlations,” *Info. Contr.*, 36: 133-156, 1978.
- [22] R. W. Yeung and Z. Zhang, “On symmetrical multilevel diversity coding,” *IEEE Trans. Inform. Theory*, vol. 45, pp. 609-621, Mar. 1999.
- [23] M. Madiman and P. Tetali, “Information inequalities for joint distributions, with interpretations and applications,” *IEEE Trans. Inform. Theory*, 56(6): 2699-2713, 2010.
- [24] J. Jiang, N. Marukala and T. Liu, “Symmetrical Multilevel Diversity Coding with an All-Access Encoder,” submitted to *IEEE Trans. Inform. Theory*.
- [25] J. H. van Lint and R. M. Wilson, *A course in combinatorics*, Cambridge University Press, Cambridge, second edition, 2001.
- [26] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, Cambridge, MA: MIT Press, 2001.
- [27] D. Bertsimas and J. N. Tsitsiklis, *Introduction to Linear Optimization*, Belmont, MA: Athena Scientific, 1997.

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