

Channel Capacity under Sub-Nyquist Nonuniform Sampling

Yuxin Chen, Andrea J. Goldsmith, and Yonina C. Eldar

Abstract—This paper investigates the effect of sub-Nyquist sampling upon the capacity of an analog channel. The channel is assumed to be a linear time-invariant Gaussian channel, where perfect channel knowledge is available at both the transmitter and the receiver. We consider a general class of right-invertible time-preserving sampling methods which include irregular nonuniform sampling, and characterize in closed form the channel capacity achievable by this class of sampling methods, under a sampling rate and power constraint. Our results indicate that the optimal sampling structures extract out the set of frequencies that exhibits the highest signal-to-noise ratio among all spectral sets of measure equal to the sampling rate. This can be attained through filterbank sampling with uniform sampling at each branch with possibly different rates, or through a single branch of modulation and filtering followed by uniform sampling. These results reveal that for a large class of channels, employing irregular nonuniform sampling sets, while typically complicated to realize, does not provide capacity gain over uniform sampling sets with appropriate preprocessing. Our findings demonstrate that aliasing or scrambling of spectral components does not provide capacity gain, which is in contrast to the benefits obtained from random mixing in spectrum-blind compressive sampling schemes.

Index Terms—nonuniform sampling, irregular sampling, sampled analog channels, sub-Nyquist sampling, channel capacity, Beurling density, time-preserving sampling systems

I. INTRODUCTION

The capacity of analog Gaussian channels and their capacity-achieving transmission strategies were pioneered by Shannon [2], which has provided fundamental insights for modern communication system design. Shannon’s work focused on capacity of analog channels sampled at or above twice the channel bandwidth. However, these results do not explicitly account for sub-Nyquist sampling rate constraints that may be imposed by hardware limitations. This motivates exploration of the effects of sub-Nyquist sampling upon the capacity of an analog Gaussian channel, and the fundamental capacity limits that result when considering general sampling methods that include irregular nonuniform sampling.

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A. Related Work and Motivation

Shannon introduced and derived the information theoretic metric of channel capacity for time-invariant analog waveform channels [2], which established the optimality of water-filling power allocation based on signal-to-noise ratio (SNR) over the spectral domain [3], [4]. A key idea in determining the analog channel capacity is to convert the continuous-time channel into a set of parallel discrete-time channels based on the Shannon-Nyquist sampling theorem [5]. This paradigm was employed, for example, by Medard et. al. to bound the maximum mutual information in time-varying channels [6], [7], and was used by Forney et. al. to investigate coding and modulation for Gaussian channels [8]. Most of these results focus on the analog channel capacity commensurate with uniform sampling at or above the Nyquist rate associated with the channel bandwidth. There is another line of work that characterizes the effects upon information rates of oversampling with quantization [9], [10]. In practice, however, hardware and power limitations may preclude sampling at the Nyquist rate for a wideband communication system.

More general irregular sampling methods beyond point-wise uniform sampling have been extensively studied in the sampling literature, e.g. [11]–[13]. One example is sampling on non-periodic quasi-crystal sets, which has been shown to be stable for bandlimited signals [14], [15]. These sampling approaches are of interest in some realistic situations where signals are only sampled at a nonuniformly spaced sampling set due to constraints imposed by data acquisition devices. Many sophisticated reconstruction algorithms have been developed for the class of bandlimited signals or, more generally, the class of shift-invariant signals [11], [16], [17]. For all these nonuniform sampling methods, the Nyquist sampling rate is necessary for perfect recovery of bandlimited signals [11], [18], [19].

However, for signals with certain structure, the Nyquist sampling rate may exceed that required for perfect signal reconstruction from the samples [20], [21]. For example, consider multiband signals, whose spectral contents reside within several subbands over a wide spectrum. If the spectral support is known, then the necessary sampling rate for the multiband signals is their spectral occupancy, termed the *Landau rate* [22]. Such signals admit perfect recovery when sampled at rates approaching the Landau rate, provided that the sampling sets are appropriately chosen (e.g. [23], [24]). One type of sampling mechanism that can reconstruct multiband signals sampled at the Landau rate is a filter bank followed by sampling, studied in [25]–[27]. Inspired by recent “compressive sensing” [28]–[30] ideas, spectrum-blind sub-Nyquist

sampling for multiband signals with random modulation has been developed [31] as well.

Although sub-Nyquist nonuniform sampling methods have been extensively explored in the sampling literature, they are typically investigated either under a noiseless setting, or based on statistical reconstruction measures (e.g. mean squared error (MSE)) instead of information theoretic measures. Gastpar et al [32] studied the necessary sampling density for nonuniform sampling. Recent work by Wu and Verdu [33] investigated the tradeoff between the number of samples and the reconstruction fidelity through information theoretic measures. However, these works did not explicitly consider the capacity metric for an analog channel. The most relevant capacity result to our work was by Berger et al [34], who related MSE-based optimal sampling with capacity for several special types of channels. But they did not derive the sub-Nyquist sampled channel capacity for more general channels, nor did they consider nonuniformly spaced sampling. Our recent work [35] established a new framework that characterizes sampled capacity for a broad class of sampling methods, including filter and modulation bank sampling [31], [36], [37]. For these sampling methods, we determined optimal sampling structures based on capacity as a metric, illuminated intriguing connections between MIMO channel capacity and capacity of undersampled channels, as well as a new connection between capacity and MSE. However, this prior work did not investigate analog channel capacity using more general nonuniform sampling under a sub-Nyquist sampling rate constraint.

One interesting fact discovered in [35] is the non-monotonicity of capacity with sampling rate under filter- and modulation-bank sampling, assuming an equal sampling rate per branch for a given number of branches. This indicates that more sophisticated sampling techniques, adaptive to the channel response and the sampling rate, are needed to maximize capacity under sub-Nyquist rate constraints, including both uniform and nonuniform sampling. However, none of the aforementioned work has investigated the question as to which sampling method can best exploit channel structure, thereby maximizing sampled capacity under a given sampling rate constraint. Although several classes of sampling methods were shown in [35] to have closed-form capacity solutions, the capacity limits might not even exist for general sampling methods. This raises the question as to whether there exists a capacity upper bound over a general class of sub-Nyquist sampling systems beyond the classes we discussed in [35] and, if so, when the bound is achievable. That is the question we investigate herein.

B. Contributions and Organization

Our main contribution is to derive the capacity of sub-Nyquist sampled analog channels for a general class of right-invertible time-preserving nonuniform sampling methods, under a sub-Nyquist sampling rate constraint. The channel is assumed to be a linear time-invariant (LTI) Gaussian channel, where perfect channel knowledge is available at both the transmitter and the receiver. The class of sampling systems we consider subsumes sampling structures employing irregular nonuniform sampling grids.

We first develop in Theorem 2 an upper bound on the sampled channel capacity, which corresponds to the capacity of a channel whose spectral occupancy is no larger than the sampling rate f_s . As a key step in the analysis framework for Theorem 2, we characterize in closed form the sampled channel capacity for any specific periodic sampling system, formally defined in Definition 8 (Lemma 1). We demonstrate that this fundamental capacity limit can be achieved by filterbank sampling with *varied* sampling rates at different branches, or by a single branch of modulation and filtering followed by a uniform sampling set (Theorems 3-4). In particular, the optimal sampler extracts out a spectral set of size f_s with the highest SNR, and suppresses all signal and noise components outside this spectral set.

Our results indicate that irregular nonuniform sampling sets, while typically complicated to realize in hardware, do not increase channel capacity relative to analog preprocessing with regular uniform sampling sets. We also show that when optimal filterbank or modulation sampling is employed, a mild perturbation of the optimal sampling grid does not change the capacity. Our findings demonstrate that aliasing or scrambling of spectral contents does not provide capacity gain. This is in contrast to the benefits obtained from random mixing of frequency components in many sub-Nyquist sampling schemes with unknown signal support (e.g. [31]).

The main innovation of this paper compared to our previous sub-sampled channel capacity results in [35] is as follows.

- While [35] characterizes the capacity under two types of sampling mechanisms that are widely used in practice (filter-bank sampling and modulation-bank sampling), the focus of this paper is instead to develop capacity results over a much more general class of sampling methods.
- While all results of [35] hold only under uniform sampling, our analysis herein accommodates irregular nonuniform sampling. Our results in turn corroborate the optimality of uniform sampling in achieving sampled capacity, assuming that the analog channel output is appropriately pre-processed.

The remainder of the paper is organized as follows. In Section II, we introduce our system model of sampled analog channels, and provide formal definitions of time-preserving systems, sampling rates, and sampled channel capacity. We then develop, in Section III-A, an upper bound on the sampled channel capacity ranging over all right-invertible time-preserving sampling methods, along with an approximate analysis highlighting insights into the result. The achievability of this upper bound is derived in Section III-B. The proof of Theorem 2 is provided in Appendix A. The implications of our main results are summarized in Section IV.

Before continuing, we introduce some notation that will be used throughout. We use $\mu(\cdot)$ to represent the Lebesgue measure, and denote by \mathcal{F} and \mathcal{F}^{-1} the Fourier and inverse Fourier transform, respectively. We let $[x]^+ \triangleq \max(x, 0)$, and use $\text{card}(A)$ to denote the cardinality of a set A . These and other notation in the paper are summarized in Table I.

Table I
SUMMARY OF NOTATIONS AND PARAMETERS

$\mu(\cdot)$	Lebesgue measure
Λ	sampling set $\{t_n n \in \mathbb{Z}\}$
$D^+(\Lambda)$,	upper, lower and uniform Beurling densities of Λ
$D^-(\Lambda)$, $D(\Lambda)$	
$\mathcal{L}_2(\Omega)$	set of measurable functions f supported on the set Ω such that $\int f ^2 d\mu < \infty$
\mathbb{S}_+	set of positive semidefinite matrices
$h(t), H(f)$	impulse response and frequency response of the LTI analog channel
$s_i(t), S_i(f)$	impulse response and frequency response of the i th (post-modulation) filter
$p(t), P(f)$	impulse response and frequency response of the pre-modulation filter
$\mathcal{S}_\eta(f), s_\eta(t)$	power spectral density of the noise $\eta(t)$ and $s_\eta(t) := \mathcal{F}^{-1}(\sqrt{\mathcal{S}_\eta(f)})$
f_s, T_s	aggregate sampling rate and the corresponding sampling interval ($T_s = 1/f_s$)
$q(t, \tau)$	impulse response of the sampling system, i.e. the output seen at time t due to an impulse in the input at time τ .
T_q, f_q	period of the modulating sequence $q(t)$ such that $T_q = 1/f_q$
$\mathcal{F}, \mathcal{F}^{-1}$	Fourier transform and inverse Fourier transform

II. SAMPLED CHANNEL CAPACITY

A. System Model

We consider an analog waveform channel, which is modeled as an LTI filter with impulse response $h(t)$ and frequency response $H(f) = \int_{-\infty}^{\infty} h(t) \exp(-j2\pi ft) dt$. With $x(t)$ denoting the transmitted signal, the analog channel output is given by

$$r(t) = h(t) * x(t) + \eta(t), \quad (1)$$

where the noise process $\eta(t)$ is assumed to be an additive stationary zero-mean Gaussian process with power spectral density $\mathcal{S}_\eta(f)$. We also define $s_\eta(t) := \mathcal{F}^{-1}(\sqrt{\mathcal{S}_\eta(f)})$. Unless otherwise specified, we assume throughout that *perfect channel state information* (i.e. the knowledge of both $H(f)$ and $\mathcal{S}_\eta(f)$) is available at both the transmitter and the receiver.

The analog channel output $r(t)$ is passed through M ($1 \leq M \leq \infty$) branches of linear preprocessing systems, each followed by a pointwise sampler, as illustrated in Fig. 1. The preprocessed output $y_i(t)$ at the i th branch is obtained by applying a linear bounded operator \mathcal{T}_i to the channel output $r(t)$:

$$y_i(t) = \mathcal{T}_i(r(t)) = \int q_i(t, \tau) r(\tau) d\tau, \quad (2)$$

where $q_i(t, \tau)$ denotes the impulse response of the time-varying system represented by \mathcal{T}_i , i.e. the output seen at time t due to an impulse in the input at time τ . Note that the linear operator \mathcal{T}_i can be time-varying, which subsumes filtering and modulation as special cases. For example, a modulation system $\mathcal{T}_i(x(t)) = p(t)x(t)$ for some given modulation sequence $p(t)$ has an impulse response $q(t, \tau) = p(\tau)\delta(t - \tau)$. A cascade combination of two systems \mathcal{T}_1 and \mathcal{T}_2 has an impulse response $q(t, \tau) = \int_{-\infty}^{\infty} q_2(t, \tau_1) q_1(\tau_1, \tau) d\tau_1$, with $q_1(\cdot, \cdot)$ and $q_2(\cdot, \cdot)$ denoting respectively the impulse responses of \mathcal{T}_1 and \mathcal{T}_2 [38]. When an operator \mathcal{T} is LTI, we use $q(\tau) := q(t, t - \tau)$ as shorthand to represent its impulse response.

The pointwise sampler following the preprocessor can be uniform or irregular [11]. Specifically, the preprocessed output $y_i(t)$ (at the i th branch) is sampled at times $t_{i,n}$ ($n \in \mathbb{Z}$), yielding a sample sequence $y_i[n] = y_i(t_{i,n})$. Here, we define the *sampling set* Λ_i at the i th branch as

$$\Lambda_i := \{t_{i,n} | n \in \mathbb{Z}\}. \quad (3)$$

In particular, if $t_{i,n} = nT_{i,s}$, then the sampling set at the i th branch is said to be uniform with period $T_{i,s}$.

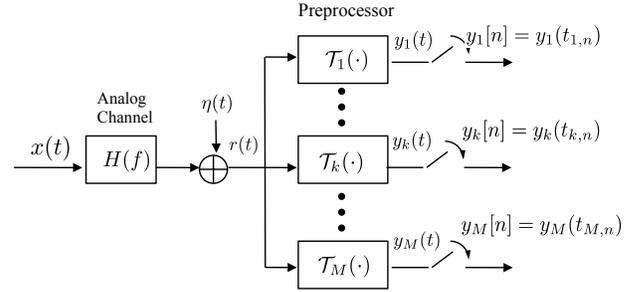


Figure 1. The input $x(t)$ is passed through the analog channel and contaminated by noise $\eta(t)$. The analog channel output $r(t)$ is then passed through M ($1 \leq M \leq \infty$) linear preprocessing system $\{\mathcal{T}_i | 1 \leq i \leq M\}$. At the i th branch, the preprocessed output $y_i(t)$ is sampled on the sampling set $\Lambda_i = \{t_{i,n} | n \in \mathbb{Z}\}$.

B. Sampling Rate Definition

Our metric of interest is the sampled channel capacity under a sampling rate constraint. We first formally define sampling rate for general nonuniform sampling mechanisms.

In general, the sampling set $\Lambda = \{t_n | n \in \mathbb{Z}\}$ may be irregular and hence aperiodic, which calls for a generalized definition of sampling rate. One notion commonly used in sampling theory is the Beurling density introduced by Beurling [18] and Landau [22], as defined below [11].

Definition 1 (Beurling Density). For a sampling set $\Lambda = \{t_k | k \in \mathbb{Z}\}$, the upper and lower Beurling density are given respectively as

$$D^+(\Lambda) = \limsup_{r \rightarrow \infty} \sup_{z \in \mathbb{R}} \frac{\text{card}(\Lambda \cap [z, z + r])}{r},$$

$$D^-(\Lambda) = \liminf_{r \rightarrow \infty} \inf_{z \in \mathbb{R}} \frac{\text{card}(\Lambda \cap [z, z + r])}{r}.$$

When $D^+(\Lambda) = D^-(\Lambda)$, the sampling set Λ is said to be of uniform Beurling density $D(\Lambda) := D^-(\Lambda)$.

When the sampling set is uniform with period T_s , the Beurling density is $D(\Lambda) = 1/T_s$, which coincides with the conventional definition of sampling rate. The notion of Beurling density allows the Shannon-Nyquist sampling theorem to be extended to nonuniform sampling. Moreover, we will use Beurling density to define sampling rate for a large class of sampling mechanisms with preprocessing.

Under a nonuniform sampling set Λ , the set of exponential functions $\{\exp(j2\pi t_n f) | n \in \mathbb{Z}\}$ forms a *non-harmonic* Fourier series [12]. Whether the class of original signals are recoverable from the nonuniform sampled sequence

is determined by the completeness of the associated non-harmonic set. In particular, when Λ is uniform, the set $\{\exp(j2\pi t_n f) \mid n \in \mathbb{Z}, t_n = n/f_s\}$ with $D(\Lambda) = f_s$ forms a Riesz basis [39] of $\mathcal{L}_2(-f_s/2, f_s/2)$ by the Shannon-Nyquist sampling theorem. For the class of sampling systems without preprocessing, a fundamental rate limit necessary for perfect reconstruction of bandlimited signals has been characterized by Landau using the definition of Beurling density, as stated in the following theorem.

Theorem 1 (Landau Rate [22]). *Consider the set \mathcal{B}_Ω of all signals whose spectral contents are supported on the frequency set Ω . Suppose that pointwise sampling without preprocessing is employed with a sampling set Λ . If all signals $f(t) \in \mathcal{B}_\Omega$ can be uniquely determined by the samples $\{f(t_n) \mid t_n \in \Lambda\}$, then one must have $D^-(\Lambda) \geq \mu(\Omega)$. The value $\mu(\Omega)$ is termed the Landau rate.*

Theorem 1 characterizes the fundamental sampling rate requirement for perfect signal reconstruction under pointwise sampling without preprocessing. In particular, when $\Omega = [-B/2, B/2]$, Theorem 1 reduces to the Shannon-Nyquist theorem.

We will thereby use Beurling density to characterize the sampling rate for a general sampling set. However, since the preprocessor might distort the time scale of the input, the resulting ‘‘sampling rate’’ might not characterize the true sampling rate applied to the original signal, as illustrated in the following example.

Example 1 (Compressor). Consider a preprocessing system defined by the relation

$$y(t) = \mathcal{T}(r(t)) = r(Lt)$$

with $L \geq 2$ being a positive integer. If we apply a uniform sampling set $\Lambda = \{t_n : t_n = n/f_s\}$ on the preprocessed output $y(t)$, the sampled sequence at a ‘‘sampling rate’’ f_s is given by

$$y[n] = y(n/f_s) = r(nL/f_s),$$

which corresponds to sampling the system input $r(t)$ at rate f_s/L . The compressor effectively time-warps the signal, thus resulting in a mismatch of the time scales between the input and output.

The compressor example illustrates that the notion of sampling rate may be misleading for systems that experience time warping. Hence, this paper will focus only on sampling that preserves time scales. One class of linear systems that preserves time scales are modulation operators ($y(t) = p(t)x(t)$), which perform pointwise scaling of the input, and hence do not change the time scale. Another class is the periodic system which includes LTI filtering, defined as follows.

Definition 2 (Periodic System). A linear preprocessing system is said to be periodic with period T_q if its impulse response $q(t, \tau)$ satisfies

$$q(t, \tau) = q(t + T_q, \tau + T_q), \quad \forall t, \tau \in \mathbb{R}. \quad (4)$$

A more general class of systems that preserve the time scale can be generated through modulation and periodic subsystems.

Specifically, we can define a general time-preserving system by connecting a set of modulation or periodic operators in parallel or in serial. This leads to the following definition.

Definition 3 (Time-preserving System). Given an index set \mathcal{I} , a preprocessing system $\mathcal{T} : x(t) \mapsto \{y_k(t), k \in \mathcal{I}\}$ is said to be time-preserving if

- (1) The system input is passed through $|\mathcal{I}|$ (possibly countably many) branches of linear preprocessors, yielding a set of analog outputs $\{y_k(t) \mid k \in \mathcal{I}\}$.
- (2) In each branch, the preprocessor comprises a set of periodic or modulation operators connected in serial.

With a preprocessing system that preserves the time scale, we can now define the aggregate sampling rate through the Beurling density.

Definition 4 (Sampling Rate for Time-preserving Systems). A sampling system is said to be time-preserving with sampling rate f_s if

- (1) Its preprocessing system \mathcal{T} is time-preserving.
- (2) The preprocessed output $y_k(t)$ is sampled by a sampling set $\Lambda_k = \{t_{l,k} \mid l \in \mathbb{Z}\}$ with a uniform Beurling density $f_{k,s}$, which satisfies¹ $\sum_{k \in \mathcal{I}} f_{k,s} = f_s$.

We note that the class of time-preserving sampling structures does not preclude random sampling schemes. For example, the preprocessing system can be a random modulator and the sampling set can be randomly spaced. Our definition also includes multibranch sampling methods. In fact, each multibranch sampling can be converted to an equivalent single branch sampling as follows.

Proposition 1. *Suppose that a multibranch sampling system has sampling rate f_s . Then there exists a single branch sampling system with sampling rate f_s that yields the same set of sampled output values as the original system. This holds simultaneously for all input signals.*

Proof: Suppose that the impulse response for the k th branch is given by $q_k(t, \tau)$ with sampling set $\Lambda_k = \{t_{k,n} \mid n \in \mathbb{Z}\}$. Without loss of generality², suppose that $\Lambda_k \cap \Lambda_{k'} = \emptyset$ for any $k \neq k'$. By ordering all sample times in $\cup_{k \in \mathcal{I}} \Lambda_k$ and renaming them to be $\{\tilde{t}_l \mid l \in \mathbb{Z}\}$ such that $\tilde{t}_l < \tilde{t}_{l+1}$ for all l , we can construct an equivalent single branch sampling system such that

$$\tilde{q}(\tilde{t}_l, \tau) = q_k(t_{k,n}, \tau)$$

if \tilde{t}_l corresponds to $\tilde{t}_{k,n}$ in the original sampling set. The sampling rate \tilde{f}_s of the new system is given by $\tilde{f}_s = \sum_{k \in \mathcal{I}} D(\Lambda_k) = f_s$. ■

¹We note that the sampling system may comprise countably many branches, each with non-zero sampling rate. For instance, if the k th branch is sampled at a rate $f_{k,s} = k^{-2}f_0$, we have an aggregate rate $f_s = \sum_{k=1}^{\infty} k^{-2}f_0 = \pi^2 f_0/6$.

²In fact, if $\Lambda_k \cap \Lambda_{k'} \neq \emptyset$, then we can introduce a new shifted pair $(q_k^\delta(t, \tau), \Lambda_k^\delta)$ such that $q_k^\delta(t + \delta, \tau) := q_k(t, \tau)$ and $\Lambda_k^\delta = \{t + \delta \mid t \in \Lambda_k\}$ for some δ such that $\Lambda_k^\delta \cap \Lambda_{k'} = \emptyset$, i.e. we can introduce certain delay to the preprocessed output and shift the sampling set correspondingly. Apparently, this new sampling structure leads to the same collection of sample outputs.

The samples obtained through this new single branch system preserve all information we can obtain from the samples of the original multibranch system. As will be seen, this proposition allows us to simplify the analysis.

C. Capacity Definition

There are two capacity definitions that are of interest in sub-Nyquist sampled channels: (1) the sampled capacity for a given sampling system; (2) the capacity for a large class of sampling systems under a sampling rate constraint. We now detail these definitions.

Suppose that the transmit signal $x(t)$ is constrained to the time interval $[-T, T]$, and the received signal $y(t)$ is sampled with sampling rate f_s and observed over the time interval $[-T, T]$. For a *given* sampling system \mathcal{P} that consists of a preprocessor \mathcal{T} and a sampling set Λ , and for a given time duration T , we define the information metric $C_T^{\mathcal{P}}(P)$ to be

$$C_T^{\mathcal{P}}(P) = \sup \frac{1}{2T} I \left(x([-T, T]), \{y[t_n]\}_{[-T, T]} \right), \quad (5)$$

where the supremum is over all input distributions subject to a power constraint $\mathbb{E}(\frac{1}{2T} \int_{-T}^T |x(t)|^2 dt) \leq P$. Here, $\{y[t_n]\}_{[-T, T]}$ denotes the set of samples obtained at times within $[-T, T] \cap \Lambda$ by the sampling system \mathcal{P} , i.e. $\{y[t_n] \mid n \in \mathbb{Z}, t_n \in [-T, T]\}$.

The capacity of the undersampled channel under a given sampling system can then be studied by taking the limit as $T \rightarrow \infty$. It was shown in [35] that $\lim_{T \rightarrow \infty} C_T^{\mathcal{P}}(P)$ exists for a broad class of sampling methods, including sampling via filter banks and sampling via periodic modulation. We caution, however, that the existence of the limit is not guaranteed for all sampling methods, e.g. the limit might not exist for irregular sampling. We therefore define the capacity for a *given* sampling system as follows.

Definition 5. $C^{\mathcal{P}}$ is said to be the information capacity of a given sampled analog channel (or *sampled channel capacity*) if $\lim_{T \rightarrow \infty} C_T^{\mathcal{P}}(P)$ exists and

$$C^{\mathcal{P}}(P) = \lim_{T \rightarrow \infty} C_T^{\mathcal{P}}(P).$$

Note that any sampled analog channel can be converted to a set of independent discrete channels via a Karhunen Loeve decomposition. The metric $C^{\mathcal{P}}(P)$ then quantifies asymptotically the maximum mutual information between the input and output of these discrete channels, or equivalently, the maximum data rate that can be conveyed reliably through these channels.

The above capacity is defined for a given sampling mechanism. Another metric of interest is the maximum data rate achievable by all sampling schemes within a general class. This motivates us to define the sub-Nyquist sampled channel capacity for a class of sampling systems as follows.

Definition 6 (Sampled Capacity under A Class of Sampling Systems). $C_{\mathcal{A}}(f_s, P)$ is said to be the capacity of an analog channel over all a class \mathcal{A} of sampling systems under a given sampling rate f_s if

$$C_{\mathcal{A}}(f_s, P) = \sup_{\mathcal{P} \in \mathcal{A}} C^{\mathcal{P}}(P).$$

The above definition of sub-sampled channel capacity characterizes the capacity limit of an analog channel over a large set of sampling mechanisms subject to a sampling rate constraint. This gives rise to the natural problem of jointly optimizing the input and the sampling architecture to maximize capacity, a goal we address in the next section.

III. MAIN RESULTS

This section characterizes in closed form the sampled channel capacity for a very general class of sampling systems, under a sampling rate constraint. Specifically, we are concerned with the sampled channel capacity $C_{\mathcal{A}}(f_s, P)$, where

$$\mathcal{A} := \{\text{all right-invertible time-preserving sampling systems}\}.$$

Here, the right-invertibility represents some mild regularity constraints that ensure each sample contains innovation information, as will be formally defined later. Unless otherwise specified, all sampling systems mentioned in this section are assumed to be right-invertible time-preserving linear systems.

Before proceeding, we shall assume, throughout, that for any frequency f , the following holds

$$\begin{aligned} \mathcal{S}_{\eta}(f) &\neq 0, \\ \int_f \frac{H^2(f)}{\mathcal{S}_{\eta}(f)} df &< \infty, \\ \int_f \mathcal{S}_{\eta}(f) df &< \infty \text{ or } \mathcal{S}_{\eta}(f) \equiv 1. \end{aligned} \quad (6)$$

A. An Upper Bound on Sampled Channel Capacity

1) *The Converse:* A time-preserving sampling system preserves the time scale of the signal, and hence does not compress or expand the frequency scale. We now determine an upper limit on the sampled channel capacity for this class of general nonuniform sampling systems. Proposition 1 implies that any multibranch sampling system can be converted into a single branch sampling system without loss of information. Therefore, we restrict our analysis in this section to the class of single branch sampling systems, which provides exactly the same upper bound as the one accounting for multibranch systems. In addition, we constrain our attention to sampling methods that are right-invertible, as defined below³.

Definition 7 (Right-Invertible Sampling System). A sampling system with sampling set Λ and impulse response $q(t_i, \tau)$ ($t_i \in \Lambda$) is said to be right-invertible with respect to $\mathcal{S}_{\eta}(f)$ if

- 1) for any $k \in \mathbb{Z}$, the frequency response $\mathcal{F}(q_k(\tau))$ is bounded;
- 2) for any frequency f and any T with its associated sampling subset

$$\Lambda_{[-T, T]} = [-T, T] \cap \Lambda := \{t_1, \dots, t_{N_T}\},$$

the least singular value of the $N_T \times \infty$ -dimensional matrix $\mathbf{F}_T(f)$ is uniformly bounded away from 0.

³We impose the right-invertibility constraint primarily for the sake of mathematical convenience. We conjecture, however, that removing this constraint does not change our main result (Theorem 2).

Here, $q_k(\tau) := q(t_k, t_k - \tau)$, and \mathbf{F}_T is a $N_T \times \infty$ matrix defined by

$$[\mathbf{F}_T(f)]_{k,l} = \mathcal{F}(s_\eta \cdot q_k) \left(f + \frac{l}{T} \right), \quad 1 \leq k \leq N_T, l \in \mathbb{Z}.$$

The right invertibility of the sampling system essentially implies that for each subset of the impulse response $\{q(t_i, \tau) \mid i \in \mathcal{I}\}$, the Fourier matrix associated with any sampled response is right-invertible. This typically implies that the set of samples is a linearly independent family – each sample provides a sufficient amount of innovative information. Our main theorem is now stated as follows.

Theorem 2 (Converse). *Assume that there exists a small constant $\epsilon > 0$ such that $\mathcal{F}^{-1} \left(\frac{H(f)}{\sqrt{S_\eta(f)}} \right) (t) = O \left(\frac{1}{t^{1.5+\epsilon}} \right)$. Suppose that there exists a frequency set B_m with $\mu(B_m) = f_s$ that satisfies*

$$\int_{f \in B_m} \frac{|H(f)|^2}{S_\eta(f)} df = \sup_{B: \mu(B)=f_s} \int_{f \in B} \frac{|H(f)|^2}{S_\eta(f)} df.$$

Under any time-preserving right-invertible sampling system \mathcal{P} w.r.t. $S_\eta(f)$ with sampling rate f_s , the sampled channel capacity is upper bounded by

$$\begin{aligned} C^{\mathcal{P}}(P) &\leq C_u(f_s, P) \\ &:= \int_{f \in B_m} \frac{1}{2} \left[\log \left(\nu \frac{|H(f)|^2}{S_\eta(f)} \right) \right]^+ df, \end{aligned} \quad (7)$$

where ν is given parametrically by

$$\int_{f \in B_m} \left[\nu - \frac{S_\eta(f)}{|H(f)|^2} \right]^+ df = P. \quad (8)$$

Remark 1. *Note that $C_u(f_s, P)$ is monotonically nondecreasing in f_s and P . In fact, when the sampling rate is increased from f_s to $f_s + \delta$, $C_u(f_s, P)$ corresponds to the optimal value when considering all spectral sets of support size $f_s + \delta$. Since we are still allowed to employ (suboptimal) strategies to allocate power over smaller spectral sets with size f_s , we are optimizing over a larger set of transmission / power allocation strategies than the situation with sampling rate f_s . Therefore, $C_u(f_s, P)$ is nondecreasing in f_s .*

It can be easily shown that the upper limit $C_u(f_s, P)$ is equivalent to the maximum capacity of a channel whose spectral occupancy is no larger than f_s . The above result basically implies that even if we allow for more complex irregular sampling sets, the sampled capacity cannot exceed the one commensurate with the analog capacity when constraining all transmit signals to the interval of bandwidth f_s that experience the highest SNR. Accordingly, the optimal input distribution will lie in this maximizing frequency set. This theorem also demonstrates that the capacity is attained when aliasing is suppressed by the sampling structure, as will be seen later in our capacity-achieving scheme. When the optimal frequency interval B_m is selected, a water filling power allocation strategy is performed over the spectral domain with some water level ν determined by (8).

2) *Approximate Analysis:* To give some intuition into the results, we provide an approximate (but non-rigorous) argument relying on “noise whitening” and “orthonormal projections”.

Suppose that the Fourier transform of the analog channel output $r(t)$ is given by $H(f)X(f) + N(f)$, where $X(f)$ and $N(f)$ denote, respectively, the frequency responses of $x(t)$ and $\eta(t)$. When the sampled sequence does not collapse information, we can characterize the sampling process through a linear injective mapping \mathcal{R} from the space of linear functions $H(f)X(f) + N(f) \in \mathcal{L}_2(-\infty, \infty)$ onto the space $\mathcal{L}_2(-f_s/2, f_s/2)$ of bandlimited functions:

$$\phi(\cdot) = \mathcal{R}(HX) + \mathcal{R}(N).$$

This way the noise component $\mathcal{R}(N)$ can be treated as additive sampled noise in the frequency domain. We note, however, that this Gaussian noise $\mathcal{R}(N)$ is not necessarily independent over the spectral support $[-f_s/2, f_s/2]$. This motivates us to whiten it first without loss of information.

Denote by \mathcal{W} the whitening operator and suppose that $\mathcal{R}(N)$ is bounded away from 0. Then the prewhitening process is performed as

$$\mathcal{W}\phi(\cdot) = \mathcal{W}(\mathcal{R}(HX)) + \mathcal{W}(\mathcal{R}(N))$$

such that the noise component $\mathcal{W}(\mathcal{R}(N))$ is independent across the frequency domain. If we set $\tilde{\mathcal{R}}(\cdot) \triangleq \mathcal{W}(\mathcal{R}(\cdot))$, then we can rewrite the input-output relation as

$$\tilde{\phi}(\cdot) = \tilde{\mathcal{R}}(HX) + \tilde{N},$$

with \tilde{N} being white over $[-f_s/2, f_s/2]$ and $\tilde{\mathcal{R}}$ being an *orthonormal* operator onto $\mathcal{L}_2(-f_s/2, f_s/2)$. That said, the operator $\tilde{\mathcal{R}}$ effectively projects all spectral components $H(f)X(f) + N(f)$ onto a subspace $\mathcal{L}_2(-f_s/2, f_s/2)$. Instead of scrambling the spectral contents, the optimal projection that maximizes the SNR extracts out a spectral set B_m of support size f_s that contains the frequency components with the highest SNR. This leads to the capacity upper bound (7). As illustrated in Fig. 2, scrambling the spectral contents does not in general improve capacity. This will be formally proved in Appendix A.

3) *Proof Sketch:* We now outline the key steps underlying the proof of Theorem 2 for the white-noise scenario.

i) We start by analyzing the class of periodic sampling systems: a special type of sampling methods that allow closed-form capacity expressions. We then demonstrate that the capacity under any periodic sampling system with sampling rate f_s and transmit power P is bounded above by $C_u(f_s, P)$.

ii) The upper bound is then derived by relating general (possibly aperiodic) sampled channels with periodic sampled channels through a *finite-duration* approximation argument. In fact, instead of studying the true sampled channel response directly, we truncate the channel response so that its impulse response is nonzero only for a finite duration. The capacity bound for the resulting truncated channel is then bounded by the capacity of a new periodized channel we construct. As we show, the capacity of the truncated channel can be made arbitrarily close to the capacity of the true sampled channel.

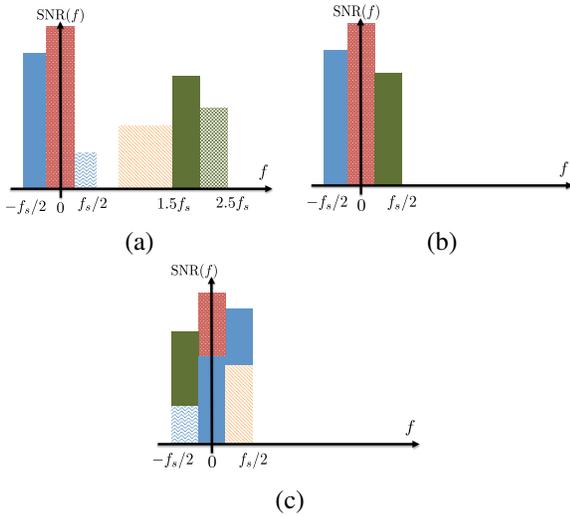


Figure 2. Projection of spectral contents from $\mathcal{L}(-\infty, -\infty)$ onto $\mathcal{L}(-f_s/2, f_s/2)$. (a) SNR of the analog channel. (b) optimal projection: it extracts out a frequency set of size f_s and zeros out all other contents. (c) a projection that scrambles the spectral contents, which does not in general improve capacity.

The most technically involved step is Step ii), which proceeds by considering two cases as follows.

- 1) **Finite-duration** $h(t)$. Consider first channels for which $h(t)$ is of finite duration, $h(t) = 0$ for any $t \notin [-L_0, L_0]$ for some $L_0 > 0$.

(a) Consider any given right-invertible time-preserving sampling system \mathcal{P} with impulse response $q(t, \tau)$, and suppose that the input $x(t)$ is time constrained to the interval $[-T, T]$. Construct a periodic channel with period $2(T + L_0)$ based on $q(t, \tau)$. Let $C_p^{\mathcal{P}}(P)$ denote the capacity of the periodized channel, whose sampling rate is bounded above by $f_s + \epsilon$ for some arbitrarily small $\epsilon > 0$.

(b) Show that $C_T^{\mathcal{P}}(P) \leq \frac{T+L_0}{T} C_p^{\mathcal{P}}\left(\frac{T}{T+L_0}P\right)$ holds uniformly for all \mathcal{P} . Since we know that $C_p^{\mathcal{P}}(P) \leq C_u(f_s + \epsilon, P)$ for any periodized channel (or, equivalently, any channel followed by a periodic sampling system), this establishes the capacity upper bound for this class of finite-duration channels, provided that T is sufficiently large.

- 2) **Infinite-duration** $h(t)$. We next extend the results to channels for which $h(t)$ is non-zero over infinite duration.

(a) Construct a truncated channel such that

$$\tilde{h}(t) = \begin{cases} h(t), & \text{if } |t| \leq L_1, \\ 0, & \text{else,} \end{cases}$$

for some sufficiently large L_1 . The capacity upper bound holds for the truncated channel, as shown in Step 1).

(b) For any given sampling system \mathcal{P} and any time interval $[-T, T]$, compare the capacity of the original channel (denoted by $C_T^{\mathcal{P}}(P)$) with the capacity of the truncated channel (denoted by $\tilde{C}_T^{\mathcal{P}}(P)$), which can be completed by investigating the spectrum of the operators associated with both sampled channels. It can be shown

that $C_T^{\mathcal{P}}(P)$ can be upper bounded by $\tilde{C}_T^{\mathcal{P}}(P + \xi) + \xi$ for some arbitrarily small constant $\xi > 0$, which holds uniformly over all sampling systems \mathcal{P} . Combining this with results shown in Step 1), we demonstrate that $\tilde{C}_T^{\mathcal{P}}(P)$ (and hence $\tilde{C}_T^{\mathcal{P}}(P)$) is bounded arbitrarily close by $C_u(f_s, P)$, which establishes the claim for the whole class of infinite-duration channels.

B. Achievability

It turns out that for most scenarios of interest, the capacity upper bound given in Theorem 2 can be attained through filterbank sampling, as stated in the following theorem.

Theorem 3 (Achievability – Sampling with a Filter Bank). Suppose that the maximizing frequency set B_m introduced in Theorem 2 exists and is piecewise continuous or, more precisely,

$$B_m = \cup_i B_i \in \mathcal{X},$$

where \mathcal{X} is an index set, and B_i 's are some non-overlapping continuous intervals. Consider the following filterbank sampling mechanism \mathcal{P}_{FB} : in the k th branch, the frequency response of the filter is given by

$$S_k(f) = \begin{cases} 1, & \text{if } f \in B_k, \\ 0, & \text{otherwise,} \end{cases} \quad (9)$$

and each filter is followed by an ideal uniform sampler with sampling rate $\mu(B_k)$. Then

$$C^{\mathcal{P}_{\text{FB}}}(P) = C_u(f_s, P),$$

where $C_u(f_s, P)$ is the upper bound given by (7).

Proof: The spectral components in B_i can be perfectly reconstructed from the sequence that is obtained by first extracting out a subinterval B_i and then uniformly sampling the filtered output with sampling rate $f_{s,i}$. The capacity under \mathcal{P}_{FB} is commensurate to the analog capacity when constraining the transmit signal to $\cup_i B_i$, which is equivalent to $C_u(f_s, P)$. ■

Note that the bandwidth of B_i may be irrational and the system may require an infinite number of filters. Theorem 3 indicates that filterbank sampling with *varied sampling rates* in different branches maximizes capacity.

The optimality of filterbank sampling immediately leads to another optimal sampling structure under mild conditions. As we have shown in [35], filterbank sampling with equal rates on different branches is equivalent to a single branch of modulation, as illustrated in Fig. 3. This approach attains the sampled capacity achievable by filterbank sampling if the SNRs of the analog channel are piecewise constant in frequency. Although the filterbank sampling we derive in (9) does not employ equal rates on different branches, for most channels of physical interest we can simply divide each branch further into a number of sub-branches to allow the rates at different branches to be reasonably close to each other. Therefore, for most channels of physical interest (say, the channels whose SNRs in frequency are Riemann-integrable), the capacity achievable through filterbank sampling can be approached arbitrarily closely by a single branch of sampling

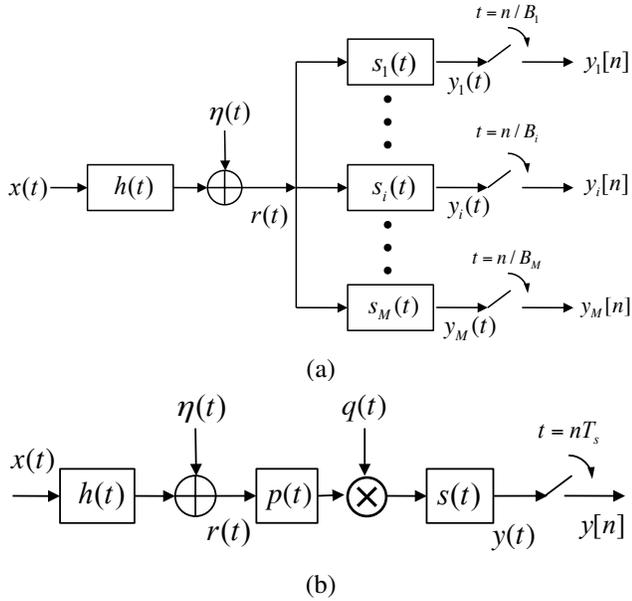


Figure 3. (a) Filterbank sampling: each branch filters out a frequency interval of bandwidth B_k , and samples it with rate $f_{k,s} = B_k$; (b) A single branch of modulation and filtering: the channel output is prefiltered by a filter with impulse response $p(t)$, modulated by a sequence $q(t)$, post-filtered by another filter of impulse response $s(t)$, and finally sampled by a uniform sampler at a rate f_s . If the SNR $|H(f)|^2/S_{\eta}(f)$ is piecewise flat, then $p(t)$, $q(t)$ and $s(t)$ can be chosen such that the two systems are equivalent in terms of sampled capacity.

with modulation. This achievability result is formally stated in the following theorem.

Theorem 4 (Achievability – A Single Branch of Sampling with Modulation and Filtering). *Under the assumptions of Theorem 3, suppose further that $|H(f)|^2/S_{\eta}(f)$ is constant within each set B_i . Then for any $\epsilon > 0$, there exists a time-preserving sampling system \mathcal{P}_{MF} with sampling rate f_s using a single branch of sampling with modulation and filtering such that $C^{\mathcal{P}_{MF}}(P) \geq C_u(f_s, P) - \epsilon$, where $C_u(f_s, P)$ is defined in (7).*

Proof: It is straightforward to see that there exists a set of non-overlapping intervals $\{\tilde{B}_k\}$ each with *equal measure* that can approximate the original sets $\{B_i \mid 1 \leq i \leq n\}$ arbitrarily well. Employing the sampling method described in [35, Section V.D] achieves a sampled capacity arbitrarily close to $C_u(f_s, P)$. ■

A channel of physical interest can often be approximated as piecewise constant over frequency in this way. Given the maximizing frequency set B_m , the sampling structure suggested in [35, Section V.D] first suppresses the frequency components outside B_m using an optimal LTI prefilter. A modulation module is then applied to scramble all frequency components within B_m . The aliasing effect can be significantly mitigated by appropriate choices of modulation weights for different spectral subbands. We then employ another band-pass filter to suppress out-of-band signals, and sample the output using a pointwise uniform sampler. Compared with filterbank sampling, a single branch of modulation and filtering only requires the design of a lowpass filter, a band-pass filter, and

a multiplication module, which might be of lower complexity to implement than a filter bank.

IV. DISCUSSION

Some properties of the capacity and capacity-achieving strategies are now discussed.

- **Monotonicity.** It can be seen from Theorem 2 that increasing the sampling rate from f_s to \tilde{f}_s results in another frequency set \tilde{B}_m of support size \tilde{f}_s that has the highest SNRs. By definition, the original frequency set B_m must be a subset of \tilde{B}_m . Therefore, the sampled capacity with rate \tilde{f}_s is no lower than the sampled capacity with rate f_s .
- **Irregular sampling set.** Sampling with irregular nonuniform sampling sets, while requiring complicated reconstruction and interpolation techniques [11], does not outperform filterbank or modulation bank sampling with regular uniform sampling sets in maximizing capacity for the channels considered herein.
- **Alias suppression.** We have seen that aliasing does not allow a higher capacity to be achieved when perfect channel state information is known at both the transmitter and the receiver. The optimal sampling method corresponds to the optimal alias-suppression strategy. This is in contrast to the benefits obtained through random mixing of spectral components in many sub-Nyquist sampling schemes with unknown signal supports. When we are allowed to jointly optimize over both input and sampling schemes with perfect channel state information, scrambling of spectral contents does not in general maximize capacity.
- **Perturbation of the sampling set.** If optimal filterbank or modulation sampling is employed, then mild perturbation of post-filtering uniform sampling sets does not degrade the sampled capacity. One surprisingly general example was proved by Kadec [40]. Suppose that a sampling rate \hat{f}_s is used in any branch and the sampling set satisfies $|\hat{t}_n - n/\hat{f}_s| \leq \hat{f}_s/4$. Then $\{\exp(j2\pi\hat{t}_n f) \mid n \in \mathbb{Z}\}$ also forms a Riesz basis of $\mathcal{L}_2(-\hat{f}_s/2, \hat{f}_s/2)$, thereby preserving information integrity. These nonuniform sampling and reconstruction schemes, while generally complicated to implement in practice, significantly broaden the class of sampling mechanisms that allow perfect reconstruction of bandlimited signals, and indicate stability and robustness of the sampling sets. Kadec's result immediately implies that the sampled capacity is invariant under mild perturbation of the sampling sets.
- **Hardware implementation.** When the sampling rate is increased from f_{s1} to f_{s2} , we need only to insert an additional filter bank of overall sampling rate $f_{s2} - f_{s1}$ to extract out another set of spectral components with bandwidth $f_{s2} - f_{s1}$. Thus, the adjustment of the sampling hardware system for filterbank sampling is incremental with no need to rebuild the whole system from scratch.
- **Spectrum Blind Sampling.** This paper focuses on the scenario with perfect channel state information known at the transmitter, the receiver, and the sampler. This is

different from the setting of compressed sensing, where the signal spectrum is unknown to the sampler and the decoder. In fact, the alias-suppressing sampler requires knowledge of the channel. If this knowledge is not available, then alias-suppressing samplers might result in low capacity. When the sampler is spectrum blind and the channel realization is uncertain, random sampling that scrambles the spectral contents [31], [41] outperforms alias-suppressing sampling in minimizing the rate loss due to channel-independent sampling design. We investigate the capacity of sub-Nyquist sampled channels with unknown CSI in our companion paper [42].

V. CONCLUDING REMARKS

We developed the maximum achievable information rate for a general class of right-invertible time-preserving nonuniform sampling methods under a sampling rate constraint. It is shown that nonuniformly spaced sampling sets, while requiring fairly complicated reconstruction / approximation algorithms, do not provide any capacity gain. Encouragingly, filterbank sampling with varied sampling rates on different branches, or a single branch of sampling with modulation and filtering, are sufficient to achieve the sampled channel capacity. In addition, both strategies suppress aliasing effects. In terms of maximizing capacity, there is no need to employ irregular sampling sets that are more complicated to implement in practical hardware systems. The resulting sampled capacity is shown to be monotonically increasing in sampling rate.

Our results in this paper are based on the assumption that perfect channel state information is known at both the transmitter and the receiver. It remains to be seen what sampling strategies can optimize information rates when only partial channel state information is known. It is unclear whether anti-aliasing methods are still optimal in maximizing capacity. Moreover, when it comes to the multi-user information theory setting, anti-aliasing methods might not outperform other spectral-mixing approaches in the entire capacity region. It would be interesting to see how to optimize the sampling schemes in multi-user channels, for example, joint sampling schemes in sampled multiple access analog channels.

ACKNOWLEDGMENTS

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APPENDIX A PROOF OF THEOREM 2

For simplicity of presentation, we assume throughout that the noise is white, i.e. $S_\eta(f) \equiv 1$. In fact, under the assumption (6), we can always split the channel filter $H(f)$ into two parts with respective frequency response $H(f)/\sqrt{S_\eta(f)}$ and $\sqrt{S_\eta(f)}$. Since the colored noise is equivalent to a white Gaussian noise passed through a filter with transfer function $\sqrt{S_\eta(f)}$, the original system can be redrawn as in Fig. 4. The filter with frequency response $\sqrt{S_\eta(f)}$ can then

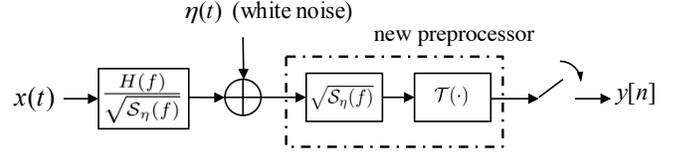


Figure 4. Equivalent representation of sampling systems in the presence of colored noise.

be incorporated into the preprocessing system to generate a new time-preserving preprocessor.

Recall that Proposition 1 indicates that any multibranch sampling system can be converted into a single branch sampling system without loss of information. As a result, we restrict our proof to the class of single branch sampling systems.

A. Capacity under Periodic Sampling Systems

Recall that for a time varying system, the impulse response $q(t, \tau)$ is defined as the output seen at time t due to an impulse in the input at time τ . The sampling system may not be time-invariant, but a broad class of sampling mechanisms applied in practice exhibit block-wise time invariance properties. Specifically, we introduce the notion of periodic sampling systems as follows.

Definition 8 (Periodic Sampling). Consider a sampling system with a preprocessing system of impulse response $q(t, \tau)$ followed by a sampling set $\Lambda = \{t_k \mid k \in \mathbb{Z}\}$. A linear sampling system is said to be periodic with period T_q and sampling rate f_s ($f_s T_q \in \mathbb{Z}$) if the preprocessing system is periodic with period T_q and the sampling set satisfies

$$t_{k+f_s T_q} = t_k + T_q, \quad \forall k \in \mathbb{Z}. \quad (10)$$

In short, a periodic sampling system consists of a periodic preprocessor followed by a pointwise sampler with a periodic sampling set, as illustrated in Fig. 5. Since the impulse response can be arbitrary within a period, this allows us to model multibranch sampling methods with each branch using the same sampling rate. Periodic sampling schemes subsume as special cases a broad class of sampling techniques, e.g. sampling via filter banks, sampling via periodic modulation, and recurrent nonuniform sampling [13], [43].

The periodicity of the sampling system renders the linear operator associated with the whole system to be block Toeplitz. The asymptotic spectral properties of block Toeplitz operators (e.g. [44]) guarantee the existence of $\lim_{T \rightarrow \infty} C_T^P(f_s, P)$ for a given periodic sampling system \mathcal{P} , and allows a capacity expression to be obtained in terms of the Fourier representation. Denote by $Q_k(f)$ the Fourier transform of the impulse response $q(t_k, t_k - t)$ of the sampling system, i.e. $Q_k(f) := \int_{-\infty}^{\infty} q(t_k, t_k - t) \exp(-j2\pi ft) dt$. We further introduce an $f_s T_q \times \infty$ dimensional Fourier series matrix $\mathbf{F}_q(f)$ associated with the sampling system, and another infinite diagonal square matrix $\mathbf{F}_h(f)$ associated with the channel response. For all $m, l \in \mathbb{Z}$ and $1 \leq k \leq f_s T_q$, we

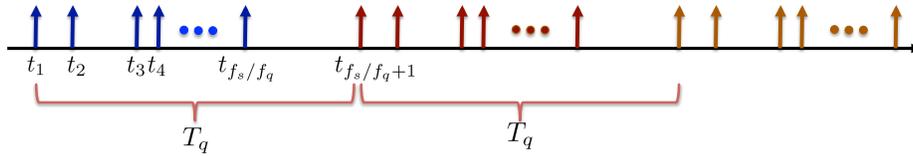


Figure 5. The sampling set of a periodic sampling system with period $1/f_q$ and sampling rate f_s .

set

$$\begin{cases} (\mathbf{F}_q)_{k,l}(f) & := Q_k(f + lf_q), \\ (\mathbf{F}_h)_{l,l}(f) & := \frac{H(f+lf_q)}{\sqrt{S(f+lf_q)}}. \end{cases} \quad (11)$$

We can then express the sampled analog capacity for a given periodic system \mathcal{P} in closed form as follows⁴.

Lemma 1. *Suppose the sampling system \mathcal{P} is periodic with period T_q and sampling rate f_s , where $f_s T_q \in \mathbb{Z}$. Let $f_q := 1/T_q$. Assume that $\mathcal{S}_\eta(f) \neq 0$ for every f , $|H(f)Q_k(f)|^2/\mathcal{S}_\eta(f)$ is bounded and integrable for all $1 \leq k \leq f_s T_q$. We also assume that the smallest singular value of $\mathbf{F}_q \mathbf{F}_q^*(f)$ is uniformly bounded away from 0 over all $f \in [-f_q/2, f_q/2]$.*

(1) *The sampled channel capacity under optimal power allocation is given by*

$$C^{\mathcal{P}}(P) = \frac{1}{2} \int_{-f_q/2}^{f_q/2} \sum_{i=1}^{f_s T_q} \log[\nu \cdot \lambda_i]^+ df, \quad (12)$$

where ν satisfies

$$\int_{-f_q/2}^{f_q/2} \sum_{i=1}^{f_s T_q} \left[\nu - \frac{1}{\lambda_i} \right]^+ df = P.$$

Here, λ_i denotes the i th largest eigenvalue of the matrix $(\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}} \mathbf{F}_q \mathbf{F}_h \mathbf{F}_h^* \mathbf{F}_q^* (\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}}$.

(2) *Suppose further that $H(f) = 0$ for any $f \notin [0, W]$, and that the transmitter employs equal power allocation over $[0, W]$. Then the sampled channel capacity is given by*

$$C_{\text{eq}}^{\mathcal{P}}(P) = \int_{-f_q/2}^{f_q/2} \frac{1}{2} \log \left(\mathbf{I} + \frac{P}{W} (\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}} \mathbf{F}_q \cdot \mathbf{F}_h \mathbf{F}_h^* \mathbf{F}_q^* (\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}} \right) df. \quad (13)$$

Proof: See Appendix B. ■

In Lemma 1, ν is the water-level with respect to the optimal water-filling power allocation strategy over the eigenvalues of the matrix $(\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}} \mathbf{F}_q \mathbf{F}_h \mathbf{F}_h^* \mathbf{F}_q^* (\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}}$. The capacity expression (12) admits a simple upper bound, as stated below.

⁴We note that a periodic sampling system can be equivalently converted to a filterbank sampling system, and hence [35, Theorem 4] immediately leads to (12). While the analysis framework in [35] relies on a discretization argument, we provide here a more general approach based on operator analysis that no longer requires discretization. More importantly, this approach allows to accommodate any sampled channel with integrable $|H(f)Q_k(f)|^2/\mathcal{S}_\eta(f)$, which subsumes channels under band-limited filterbank sampling. In comparison, [35] requires $\mathcal{F}^{-1}(|H(f)Q_k(f)|/\sqrt{\mathcal{S}_\eta(f)})$ to lie in \mathcal{L}_1 , which is far more restrictive (e.g. it does not subsume band-limited filterbank sampling as the sinc function is not absolutely integrable).

Corollary 1. (a) *Consider the setup and assumptions in Lemma 1. Under all periodic sampling systems with period T_q and sampling rate f_s , the sampled channel capacity can be bounded above by*

$$C_{f_q}(f_s, P) = \frac{1}{2} \int_{-f_q/2}^{f_q/2} \sum_{i=1}^{f_s T_q} [\log(\nu_p \lambda_i \{\mathbf{F}_h \mathbf{F}_h^*\})]^+ df, \quad (14)$$

where ν_p satisfies

$$\int_{-f_q/2}^{f_q/2} \sum_{i=1}^{f_s T_q} \left[\nu_p - \frac{1}{\lambda_i \{\mathbf{F}_h \mathbf{F}_h^*\}} \right]^+ df = P.$$

(b) *Suppose that there exists a frequency set B_m that satisfies $\mu(B_m) = f_s$ and*

$$\int_{f \in B_m} \frac{|H(f)|^2}{\mathcal{S}_\eta(f)} df = \sup_{B: \mu(B) = f_s} \int_{f \in B} \frac{|H(f)|^2}{\mathcal{S}_\eta(f)} df.$$

Then

$$C_{f_q}(f_s, P) \leq C_u(f_s, P), \quad (15)$$

where $C_u(f_s, P)$ is given by (7).

Proof: (a) Following the same steps as in [35, Proposition 1], we can see that the i th largest eigenvalue satisfies

$$\lambda_i \left\{ (\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}} \mathbf{F}_q \mathbf{F}_h \mathbf{F}_h^* \mathbf{F}_q^* (\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}} \right\} \leq \lambda_i(\mathbf{F}_h \mathbf{F}_h^*),$$

which immediately leads to (14).

(b) For any given f_q , the upper bound (14) is obtained by extracting out a certain frequency set B that has measure $\mu(B) = f_s$ and suppressing all spectral components outside B . By our definition of B_m , any choice of B with spectral size f_s will not outperform B_m . Hence, choosing $B = B_m$ leads to a universal upper bound. ■

Corollary 1 reveals that the capacity under any periodic sampling system, no matter what its period is, cannot exceed the upper bound $C_u(f_s, P)$ in Theorem 2. Our remaining proof is then established by observing that any aperiodic sampling system can be related to a periodic sampling system by truncation and periodization, as elaborated in the next two subsections.

B. General Upper Bound: Finite-duration $h(t)$

In this subsection, we focus on the channel whose impulse response is of finite duration $2L_0$, i.e.

$$h(t) = 0, \quad \forall t (|t| > L_0).$$

Our goal is to prove that the capacity upper bound (7) holds for this type of channel.

For any transmission block of duration $2T$, we call the transmit signal $x(t)$ over this block a codeword (or symbol) of code length $2T$. The information conveyed through such finite-duration codewords can be bounded via certain analog channel capacity, as long as we can preclude inter-symbol interference. The key idea here is to separate consecutive codewords with a guard zone with sufficient length and then use capacity-achieving strategies separately for each codeword. When the code length $2T$ is sufficiently large, the transmission time wasted on the guard zones becomes negligible, which in turn allows us to approach the true capacity arbitrarily well. The detailed analysis proceeds as follows.

Step 1. Consider an input $x(t)$ that is constrained to the interval $[-T, T]$. Since $h(t)$ is of finite duration $2L_0$, the channel output $r(t) = h(t) * x(t) + \eta(t)$ will be affected by the input only when $t \in [-T - L_0, T + L_0]$. Define a *window operator* and its complement operator such that

$$w_T(f(t)) = \begin{cases} f(t), & \text{if } |t| \leq T + L_0, \\ 0, & \text{else;} \end{cases} \quad (16)$$

and

$$w_T^\perp(f(t)) = \begin{cases} 0, & \text{if } |t| \leq T + L_0, \\ f(t), & \text{else.} \end{cases} \quad (17)$$

Then for any linear sampling operator \mathcal{P} with impulse response $q(t, \tau)$, the sampled output is $\mathcal{P}(r(t)) = \mathcal{P}(w_T(r(t))) + \mathcal{P}(w_T^\perp(r(t)))$. One can easily observe that the component $\mathcal{P}(w_T^\perp(r(t)))$ contains no information about $x(t)$, and is statistically *independent* of $\mathcal{P}(w_T(r(t)))$ due to the whiteness assumption of the noise. In other words, the sampling input outside the interval $[-T - L_0, T + L_0]$ does not improve capacity at all. Consequently, it suffices to restrict attention to the class of sampling systems whose system input is constrained to the interval $[-T - L_0, T + L_0]$.

Step 2. Construct a periodization of the above sampled channel model with finite input duration. Set the impulse response $q_{T+L_0}^p(t, \tau)$ of the preprocessor of the periodized sampling system to be a periodic extension of $q(t, \tau)$ in the block $[-T - L_0, T + L_0] \times [-T, T]$. Specifically, if $\tau = k \cdot 2(T + L_0) + \tau_r$ for some $k \in \mathbb{Z}$ and $\tau_r \in [-T - L_0, T + L_0]$, then

$$q_{T+L_0}^p(t, \tau) = \begin{cases} q(t - 2k(T + L_0), \tau_r), & \text{if } |t - 2k(T + L_0)| \\ & \leq T + L_0, \\ 0, & \text{else.} \end{cases} \quad (18)$$

Apparently, $q_{T+L_0}^p(t, \tau)$ corresponds to a periodic preprocessing system with period $2(T + L_0)$.

Suppose without loss of generality that the indices of the sample times that fall in $[-T - L_0, T + L_0]$ are $0, 1, \dots, K - 1$, i.e. $\{k \mid t_k \in [-T - L_0, T + L_0]\} = \{0, 1, \dots, K - 1\}$. We can then set the sampling set $\Lambda_{T+L_0}^p$ of the periodized system such that for any sampling time $t_k \in \Lambda_{T+L_0}^p$, we have

$$t_k = t_{k \bmod K} + 2(T + L_0) \cdot \left\lfloor \frac{k}{K} \right\rfloor, \quad (19)$$

where $\lfloor x \rfloor \triangleq \max\{n \mid n \in \mathbb{Z}, n \leq x\}$. Clearly, this forms a periodic sampling set with period $2(T + L_0)$. The definition

of Beurling density ensures that for any $\epsilon > 0$, there exists a T_D such that for every $T > T_D$,

$$f_s - \epsilon \leq D(\Lambda_{T+L_0}^p) \leq f_s + \epsilon.$$

Due to the finite-duration assumption of $h(t)$, our construction (18) guarantees that the input $x(t)$ within time interval $[2k(T + L_0) - T, 2k(T + L_0) + T]$ will only affect the sampled output at the k th time block $[(2k - 1)(T + L_0), (2k + 1)(T + L_0)]$, as illustrated in Fig. 6. Since the noise $\eta(t)$ is assumed to be white, the noise components across different time blocks are independent. In fact, the intervals $[2k(T + L_0) + T, (2k + 1)(T + L_0) - T]$ ($k \in \mathbb{Z}$) act effectively as *guard zones* in order to avoid leakage of signals across different time blocks.

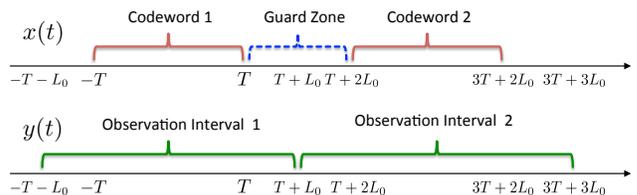


Figure 6. The code words of duration $2T$ are separated by guard zones of duration $2L_0$. There is no inter-symbol interference among different observation intervals.

Based on the above argument, we can separate codewords of duration $2T$ in $[2k(T + L_0) + T, (2k + 1)(T + L_0) - T]$ ($k \in \mathbb{Z}$) on the analog channel by a guard zone $2L_0$ (as illustrated in Fig. 6). The ratio of guard space to the length of the time block vanishes as $T \rightarrow \infty$, and there is no intersymbol interference under the new system we construct. By our capacity definition, for any $\delta > 0$, there exists a T_0 such that $\forall T > T_0$, we have

$$\frac{T + L_0}{T} < 1 + \delta, \quad \text{and} \quad \frac{T}{T + L_0} > 1 - \delta.$$

Consequently,

$$\begin{aligned} C_T^p(P) &\stackrel{(i)}{\leq} \frac{T + L_0}{T} C_p^p\left(\frac{T}{T + L_0} P\right) \\ &\leq (1 + \delta) C_p^p((1 - \delta) P), \end{aligned} \quad (20)$$

where C_p^p denotes the capacity under our periodized sampling system. The inequality (i) follows from the following three arguments:

- C_T^p is the information rate when we observe the samples within the interval $[-T, T]$, which is smaller than the information rate, termed \hat{C}_T^p , when we observe all samples within $[-T - L_0, T + L_0]$;
- \hat{C}_T^p is equivalent to the maximum information rate achievable by the periodized system, under the constraint that there is no input signal transmitted over the guard zones. Clearly, this rate will be smaller than the capacity without this transmission constraint, which is $\frac{T+L_0}{T} C_p^p$. Here, the multiplication factor $\frac{T+L_0}{T}$ arises from the fact that we only use a portion $\frac{T}{T+L_0}$ of time for transmission;
- Since the total energy over each transmission block is PT and each guard zone has zero power, the average power allocated to the transmitted signal is $\frac{T}{T+L_0} P$.

We know from Corollary 1(b) that

$$\begin{aligned} C_p^{\mathcal{P}}((1-\delta)P) &\leq C_u(D(\Lambda_{T+L_0}^{\mathcal{P}}), (1-\delta)P) \\ &\leq C_u(f_s + \epsilon, (1-\delta)P), \end{aligned} \quad (21)$$

where the last inequality arises from observing that $C_u(f_s, P)$ is monotonically non-decreasing in f_s and P . Putting (20) and (21) together yields

$$C_T^{\mathcal{P}}(P) \leq (1+\delta)C_u(f_s + \epsilon, P) \quad (22)$$

as soon as $T > \max\{T_0, T_D\}$. Since ϵ and δ can be chosen arbitrarily small, we have that

$$\limsup_{T \rightarrow \infty} C_T^{\mathcal{P}}(P) \leq C_u(f_s, P)$$

when $h(t)$ is of finite duration and $\eta(t)$ is white.

C. General Upper Bound: Infinite-Duration $h(t)$

We now investigate the capacity bound when $h(t)$ is not time-limited. We would like to prove that for any given sampling system \mathcal{P} and any $\epsilon > 0$, there exists T_1 such that for any $T > T_1$, one has

$$C_T^{\mathcal{P}} \leq C_u(f_s, P) + \epsilon.$$

Our proof proceeds by comparing the original channel with a truncated channel whose channel response $\tilde{h}(f)$ satisfies

$$\tilde{h}(t) \triangleq \begin{cases} h(t), & \text{if } |t| \leq L_1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\xi > 0$ be an arbitrary small constant, and L_1 chosen such that

$$\int_{-\infty}^{-L_1} |h(t)|^2 dt + \int_{L_1}^{\infty} |h(t)|^2 dt \leq \xi. \quad (23)$$

We further constrain the input and the observed sampled output to the time interval $[-T, T]$. For both the original and truncated channel, the sampled noise is not white, which motivates us to first perform prewhitening.

Suppose without loss of generality that the sampled times within $[-T, T]$ are $\{t_i \mid 1 \leq i \leq K_T\}$. For convenience of notation, we introduce a linear operator $\hat{\mathcal{P}}_T$ associated with the sampling system such that

$$\hat{\mathcal{P}}_T(\hat{r}(t)) = [y[1], y[2], \dots, y[K_T]],$$

where $\hat{r}(t) = g(x) * x(t) + \hat{\eta}(t)$ is the sampling system input, $\hat{\eta}(t)$ is white, and $\{y[n]\}$ are the corresponding sampled output. Thus, for the original channel, one can write

$$\begin{bmatrix} y[1] \\ y[2] \\ \vdots \\ y[K_T] \end{bmatrix} = \hat{\mathcal{P}}_T(g(t) * x(t)) + \hat{\mathcal{P}}_T(\hat{\eta}(t)).$$

Denote by $\hat{q}(t_i, \tau)$ the impulse response associated with this sampling system. Then, the noise component $\hat{\mathcal{P}}_T(\hat{\eta}(t))$ can be whitened by left-multiplying it with a K_T -dimensional square matrix $\mathbf{W}_{\hat{\mathcal{P}}}^{-1/2}$ defined by

$$\mathbf{W}_{\hat{\mathcal{P}}}(i, j) = \int_{-\infty}^{\infty} \hat{q}(t_i, \tau) \hat{q}^*(t_j, \tau) d\tau.$$

The invertibility is guaranteed by our assumptions. To see this, if we denote by $\tilde{\eta} \triangleq \mathbf{W}_{\hat{\mathcal{P}}}^{-1/2} \hat{\mathcal{P}}_T(\hat{\eta}(t))$ the K_T -dimensional ‘‘prewhitened’’ noise, then one can verify that for every i and j ,

$$\begin{aligned} &\left[\mathbb{E} \left(\hat{\mathcal{P}}_T(\hat{\eta}(t)) \left(\hat{\mathcal{P}}_T(\hat{\eta}(t)) \right)^\top \right) \right]_{ij} \\ &= \mathbb{E} \left[\left(\int_{-\infty}^{\infty} \hat{q}(t_i, \tau) \hat{\eta}(\tau) d\tau \right) \left(\int_{-\infty}^{\infty} \hat{q}^*(t_j, \tau) \hat{\eta}(\tau) d\tau \right) \right] \\ &= \int_{-\infty}^{\infty} \hat{q}(t_i, \tau) \hat{q}^*(t_j, \tau) d\tau \\ &= \mathbf{W}_{\hat{\mathcal{P}}}(i, j) \end{aligned}$$

or, equivalently,

$$\mathbb{E} \left[\hat{\mathcal{P}}_T(\hat{\eta}(t)) \left(\hat{\mathcal{P}}_T(\hat{\eta}(t)) \right)^\top \right] = \mathbf{W}_{\hat{\mathcal{P}}}.$$

As a result, the covariance of $\tilde{\eta}$ obeys

$$\begin{aligned} \mathbb{E} [\tilde{\eta} \tilde{\eta}^\top] &= \mathbf{W}_{\hat{\mathcal{P}}}^{-1/2} \mathbb{E} \left(\hat{\mathcal{P}}_T(\hat{\eta}(t)) \left(\hat{\mathcal{P}}_T(\hat{\eta}(t)) \right)^\top \right) \mathbf{W}_{\hat{\mathcal{P}}}^{-1/2} \\ &= \mathbf{I}. \end{aligned}$$

If we denote by $\hat{\mathcal{P}}_w \triangleq \mathbf{W}_{\hat{\mathcal{P}}}^{-1/2} \cdot \hat{\mathcal{P}}_T$ and let $\hat{q}_w(t_i, \tau)$ represent its associated impulse response, then the above calculation reveals that

$$\int_{-\infty}^{\infty} \hat{q}_w(t_i, \tau) \hat{q}_w^*(t_j, \tau) d\tau = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{else,} \end{cases} \quad (24)$$

indicating that $\{\hat{q}_w(t_i, \cdot), 1 \leq i \leq K_T\}$ forms a set of orthonormal sequences in the corresponding Hilbert space.

For an operator \mathcal{A} with an impulse response $a(t, \tau)$ ($-T \leq \tau \leq T$, $t \in \{t_i \mid 1 \leq i \leq K_T\}$) and input domain $\mathcal{D}(\mathcal{A})$, we denote by $\|\mathcal{A}\|_{\mathbb{F}}$ the generalized Frobenius norm of the operator \mathcal{A} with respect to its associated domain, namely,

$$\|\mathcal{A}\|_{\mathbb{F}} := \begin{cases} \sqrt{\sum_{i=1}^{K_T} \int_{-T}^T |a(t_i, \tau)|^2 d\tau}, & \text{if } \mathcal{D}(\mathcal{A}) = \{t_i \mid 1 \leq i \leq K_T\} \times [-T, T], \\ \sqrt{\sum_{i=1}^{K_T} \int_{-\infty}^{\infty} |a(t_i, \tau)|^2 d\tau}, & \text{if } \mathcal{D}(\mathcal{A}) = \{t_i \mid 1 \leq i \leq K_T\} \times [-\infty, \infty], \\ \sqrt{\int_{-\infty}^{\infty} \int_{-T}^T |a(t, \tau)|^2 d\tau dt}, & \text{if } \mathcal{D}(\mathcal{A}) = [-\infty, \infty] \times [-T, T]. \end{cases}$$

Recall that $\hat{q}_w(t_i, \cdot)$ ($1 \leq i \leq K_T$) forms orthonormal sequences. By Bessel’s inequality [45], an operator \mathcal{A} with $\mathcal{D}(\mathcal{A}) = [-\infty, \infty] \times [-T, T]$ satisfies

$$\sum_{i=1}^{K_T} |\langle \hat{q}_w(t_i, \cdot), a(\cdot, \tau) \rangle|^2 \leq \int_{-\infty}^{\infty} |a(\tau_1, \tau)|^2 d\tau_1$$

for every $\tau \in [-T, T]$, which immediately gives

$$\begin{aligned} \|\hat{\mathcal{P}}_w \mathcal{A}\|_{\mathbb{F}}^2 &= \sum_{i=1}^{K_T} \int_{-T}^T \left| \int_{-\infty}^{\infty} \hat{q}_w(t_i, \tau_1) a(\tau_1, \tau) d\tau_1 \right|^2 d\tau \\ &= \int_{-T}^T \sum_{i=1}^{K_T} |\langle \hat{q}_w(t_i, \cdot), a(\cdot, \tau) \rangle|^2 d\tau \\ &\leq \int_{-T}^T \int_{-\infty}^{\infty} |a(\tau_1, \tau)|^2 d\tau_1 d\tau \leq \|\mathcal{A}\|_{\mathbb{F}}^2. \end{aligned}$$

Denote by $\{\lambda_i\}$ and $\{\tilde{\lambda}_i\}$ the set of *squared* singular values associated with the original sampled channel operator $\hat{P}_w\mathcal{G}$ and the operator $\hat{P}_w\tilde{\mathcal{G}}$ of the truncated sampled channel, respectively. Here, \mathcal{G} and $\tilde{\mathcal{G}}$ represent respectively the operator associated with the original channel response and the truncated channel response. We can obtain some properties of $\{\lambda_i\}$ and $\{\tilde{\lambda}_i\}$ as stated in the following lemma.

Lemma 2. *Suppose that $\int_{-\infty}^{\infty} |g(t)|^2 dt < C_g < \infty$ for some constant C_g . For any $\xi > 0$, there exists T_0 such that for every $T > T_0$, one has*

$$(1) \left| \frac{1}{2T} \sum_i \lambda_i - \frac{1}{2T} \sum_i \tilde{\lambda}_i \right| \leq \xi + 2\sqrt{\xi C_g}.$$

$$(2) \frac{1}{2T} \sum_i \lambda_i \leq \int_{-\infty}^{\infty} |g(t)|^2 dt < \infty.$$

(3) *Suppose that $h(t) = O\left(\frac{1}{t^{1.5+\varepsilon}}\right)$ for some small $\varepsilon > 0$. Then there exists $T_{0,\varepsilon}$ such that for every $T > T_{0,\varepsilon}$, one has $|\lambda_i - \tilde{\lambda}_i| \leq \xi$.*

Proof: See Appendix C. ■

For notational simplicity, define two functions as follows

$$C_T^P(\nu, \{\lambda_i\}) := \frac{1}{2T} \sum_{i=1}^{K_T} \frac{1}{2} [\log(\nu \lambda_i)]^+ \quad (25)$$

and

$$F_T(\nu, \{\lambda_i\}) := \frac{1}{2T} \sum_{i=1}^{K_T} \left[\nu - \frac{1}{\lambda_i} \right]^+ \quad (26)$$

for some water level ν . Note that if ν is chosen such that $F_T(\nu, \{\lambda_i\}) = P$, then

$$C_T^P(\nu, \{\lambda_i\}) = C_T^P(P).$$

Apparently, both $C_T^P(P)$ and $C_T^P(\nu, \{\lambda_i\})$ are non-decreasing functions of $\{\lambda_i\}$, which implies that

$$C_T^P(\nu, \{\lambda_i\}) \leq C_T^P\left(\nu, \left\{ \max\left\{ \lambda_i, \xi^{\frac{1}{3}} \right\} \right\}\right) \quad (27)$$

and

$$C_T^P(P) \leq C_T^P\left(\nu, \left\{ \max\left\{ \lambda_i, \xi^{\frac{1}{3}} \right\} \right\}\right),$$

where ν is determined by

$$F_T\left(\nu, \left\{ \max\left\{ \lambda_i, \xi^{\frac{1}{3}} \right\} \right\}\right) = P. \quad (28)$$

Here, $\xi > 0$ is some arbitrarily small constant. In fact, one can easily verify that $C_T^P\left(\nu, \left\{ \max\left\{ \lambda_i, \xi^{\frac{1}{3}} \right\} \right\}\right)$ with ν determined by (28) is no larger than the sum capacity of two separate channels with respective eigenvalues $\{\lambda_i\}$ and $\{\tilde{\lambda}_i := \xi^{\frac{1}{3}}\}$ each with power allocation P . In other words,

$$\begin{aligned} C_T^P\left(\nu, \left\{ \max\left\{ \lambda_i, \xi^{\frac{1}{3}} \right\} \right\}\right) &\leq C_T^P(P) + C_T^P\left(\nu, \left\{ \xi^{\frac{1}{3}} \right\}_{1 \leq i \leq K_T}\right) \\ &\leq C_T^P(P) + \frac{K_T}{2T} \log\left(1 + \frac{PT}{K_T} \xi^{\frac{1}{3}}\right) \\ &\leq C_T^P(P) + \frac{K_T}{2T} \cdot \frac{PT}{K_T} \xi^{\frac{1}{3}} \\ &= C_T^P(P) + \frac{P}{2} \xi^{\frac{1}{3}}. \end{aligned} \quad (29)$$

$$\begin{aligned} &\leq C_T^P(P) + \frac{K_T}{2T} \log\left(1 + \frac{PT}{K_T} \xi^{\frac{1}{3}}\right) \\ &\leq C_T^P(P) + \frac{K_T}{2T} \cdot \frac{PT}{K_T} \xi^{\frac{1}{3}} \\ &= C_T^P(P) + \frac{P}{2} \xi^{\frac{1}{3}}. \end{aligned} \quad (30)$$

For any positive water level ν and some small constant $\xi > 0$, the Lipschitz constants of the functions

$$f_1(x) := \frac{1}{2} \left[\log\left(\nu \max\left\{x, \xi^{\frac{1}{3}}\right\}\right) \right]^+$$

$$f_2(x) := \left[\nu - \max\left\{x, \xi^{\frac{1}{3}}\right\} \right]^+$$

are bounded above in magnitude by $\frac{1}{2}\xi^{-1/3}$ and $\xi^{-2/3}$, respectively. Using the same water level ν , the corresponding power for both channels can be computed as

$$\begin{aligned} P &= \frac{1}{2T} \sum_{i=1}^{K_T} \left[\nu - \frac{1}{\max\left\{ \lambda_i, \xi^{\frac{1}{3}} \right\}} \right]^+, \\ \tilde{P} &= \frac{1}{2T} \sum_{i=1}^{K_T} \left[\nu - \frac{1}{\max\left\{ \tilde{\lambda}_i, \xi^{\frac{1}{3}} \right\}} \right]^+. \end{aligned}$$

Combining Lemma 2 and the Lipschitz constants of $f_2(x)$ immediately suggests that: there exists $T_{0,\varepsilon}$ such that for any $T > T_{0,\varepsilon}$, one has

$$\begin{aligned} |\tilde{P} - P| &= \frac{1}{2T} \sum_{i=1}^{K_T} \frac{1}{\xi^{\frac{2}{3}}} |\lambda_i - \tilde{\lambda}_i| \leq \frac{K_T}{2T \xi^{\frac{2}{3}}} \xi \\ &\leq (f_s + \varepsilon) \xi^{\frac{1}{3}}. \end{aligned} \quad (31)$$

Similarly, we can bound

$$\begin{aligned} &\left| C_T^P\left(\nu, \left\{ \max\left\{ \lambda_i, \xi^{\frac{1}{3}} \right\} \right\}\right) - C_T^P\left(\nu, \left\{ \max\left\{ \tilde{\lambda}_i, \xi^{\frac{1}{3}} \right\} \right\}\right) \right| \\ &\leq \frac{1}{2T} \sum_{i=1}^{K_T} \frac{1}{2\xi^{\frac{1}{3}}} |\lambda_i - \tilde{\lambda}_i| \\ &\leq \frac{1}{4} (f_s + \varepsilon) \xi^{\frac{2}{3}}. \end{aligned} \quad (32)$$

Combining (31), (32) and (22) suggests that

$$\begin{aligned} C_T^P(P) &\leq C_T^P\left(\nu, \left\{ \max\left\{ \lambda_i, \xi^{\frac{1}{3}} \right\} \right\}\right) \\ &\leq C_T^P\left(\nu, \left\{ \max\left\{ \tilde{\lambda}_i, \xi^{\frac{1}{3}} \right\} \right\}\right) + \frac{1}{4} (f_s + \varepsilon) \xi^{\frac{2}{3}} \\ &\leq C_T^{\tilde{P}}(\tilde{P}) + \frac{\tilde{P}}{2} \xi^{\frac{1}{3}} + \frac{1}{4} (f_s + \varepsilon) \xi^{\frac{2}{3}} \\ &\leq C_T^{\tilde{P}}\left(P + (f_s + \varepsilon) \xi^{\frac{1}{3}}\right) + \frac{\tilde{P}}{2} \xi^{\frac{1}{3}} + \frac{1}{4} (f_s + \varepsilon) \xi^{\frac{2}{3}} \\ &\leq \frac{P + (f_s + \varepsilon) \xi^{\frac{1}{3}}}{2} \xi^{\frac{1}{3}} + \frac{1}{4} (f_s + \varepsilon) \xi^{\frac{2}{3}} + (1 + \delta) \\ &\quad \cdot C_u\left(f_s + \varepsilon, P + (f_s + \varepsilon) \xi^{\frac{1}{3}}\right), \end{aligned} \quad (33)$$

where (33) is a consequence of (30). Since δ, ε , and ξ can all be made arbitrarily small, it follows that

$$\limsup_{T \rightarrow \infty} C_T^P(P) \leq C_u(f_s, P),$$

completing the proof.

APPENDIX B
PROOF OF LEMMA 1

The proof is restricted to the channel with white noise, i.e. $\mathcal{S}_\eta(f) \equiv 1$. It is straightforward to extend the analysis to colored noise through the argument presented in the first paragraph of Appendix A.

Our proof proceeds in the following three steps.

- 1) We first introduce several correlation functions and compute the Fourier series associated with them. These quantities are crucial in deriving the capacity expression. In particular, when the sampling system is periodic, the infinite correlation matrices are block Toeplitz.
- 2) When constrained to a finite time interval $[-nT_q, nT_q]$, the sampled output is a finite vector. The sampled noise is in general not white, which motivates us to whiten it first. In fact, the covariance matrix of the sampled noise can be easily derived in terms of the proposed correlation functions.
- 3) For any time interval $[-nT_q, nT_q]$, the capacity is obtained through the Karhunen Loeve expansion. Specifically, the capacity depends on the eigenvalues of the associated system operator, which is related to the correlation functions. The asymptotic properties of block Toeplitz matrices guarantee the convergence when $n \rightarrow \infty$, which allow us to derive in closed form the sampled channel capacity.

A. Correlation functions and Fourier series

For a concatenated linear system consisting of the channel filter followed by the sampling system, we denote by

$$s(t_0, t_i) := \int_{-\infty}^{\infty} h(\tau - t_i) q(t_0, \tau) d\tau \quad (34)$$

its system output seen at time t_0 due to an impulse input at time t_i . For notational convenience, we define $q_k(\tau) := q(t_k, \tau)$ as the sampling output response at time t_k due to an impulse input to the sampling system at time τ . Two *output autocorrelation functions* are defined as follows

$$\mathcal{R}_{hq}(t_k, t_l) \triangleq \int_{-\infty}^{\infty} s(t_k, \tau) s^*(t_l, \tau) d\tau \quad (35)$$

and

$$\mathcal{R}_q(t_k, t_l) \triangleq \int_{-\infty}^{\infty} q(t_k, \tau) q^*(t_l, \tau) d\tau. \quad (36)$$

For notational simplicity, we use $\mathcal{R}_{hq}(k, l)$ (resp. $\mathcal{R}_q(k, l)$) and $\mathcal{R}_{hq}(t_k, t_l)$ (resp. $\mathcal{R}_q(t_k, t_l)$) interchangeably. When the sampling system is periodic with period T_q , one can easily see that both $[\mathcal{R}_{hq}(k, l)]_{k, l=-\infty}^{\infty}$ and $[\mathcal{R}_q(k, l)]_{k, l=-\infty}^{\infty}$ are infinite block Toeplitz matrices.

The spectral properties associated with the system operators are captured by *Fourier series matrices* \mathbf{F}_{hq} , \mathbf{F}_{qq} , \mathbf{F}_h and \mathbf{F}_q . Specifically, \mathbf{F}_{hq} is an $f_s T_q$ -dimensional square matrix such that: for any frequency f and all $1 \leq k, i \leq f_s T_q$,

$$(\mathbf{F}_{hq})_{k,i}(f) := \sum_{l=-\infty}^{\infty} \mathcal{R}_{hq}(t_k, t_i + l f_s T_q) \exp(j2\pi l f) \quad (37)$$

and

$$(\mathbf{F}_{qq})_{k,i}(f) := \sum_{l=-\infty}^{\infty} \mathcal{R}_q(t_k, t_i + l f_s T_q) \exp(j2\pi l f). \quad (38)$$

Besides, for every frequency f , we define an $f_q T_s \times \infty$ dimensional matrix $\mathbf{F}_q(f)$ and an infinite square diagonal matrix $\mathbf{F}_h(f)$ such that for all $l \in \mathbb{Z}$ and $1 \leq k \leq f_q T_s$:

$$(\mathbf{F}_q)_{k,l}(f) := Q_k(f + l f_q), \quad (39)$$

$$(\mathbf{F}_h)_{l,l}(f) := H(f + l f_q), \quad (40)$$

where $Q_k(f) \triangleq \mathcal{F}(q_k(\cdot)) = \mathcal{F}(q(t_k, \cdot))$.

The key properties of the above autocorrelation functions and Fourier series are summarized in the following lemma.

Lemma 3. *The Fourier series matrices satisfy:*

$$\mathbf{F}_{hq} = \mathbf{F}_q \mathbf{F}_h \mathbf{F}_q^* \quad (41)$$

and

$$\mathbf{F}_{qq} = \mathbf{F}_q \mathbf{F}_q^*. \quad (42)$$

Proof: See Appendix D. ■

B. Noise whitening

Denote by $\mathcal{Q}_k(\cdot)$ the sampling operator associated with the sample time t_k such that $\mathcal{Q}_k(x) \triangleq \int_{-\infty}^{\infty} q(t_k, \tau) x(\tau) d\tau$. The correlation of noise components $\mathcal{Q}_k(\eta)$ at different times can be calculated as

$$\begin{aligned} \mathbb{E}[\mathcal{Q}_k(\eta) \mathcal{Q}_l^*(\eta)] &= \mathbb{E} \left[\int_{-\infty}^{\infty} q(t_k, \tau_k) \eta(\tau_k) d\tau_k \left(\int_{-\infty}^{\infty} q(t_l, \tau_l) \eta(\tau_l) d\tau_l \right)^* \right] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q(t_k, \tau_k) q^*(t_l, \tau_l) \mathbb{E}(\eta(\tau_k) \eta^*(\tau_l)) d\tau_k d\tau_l \\ &= \int_{-\infty}^{\infty} q(t_k, \tau) q^*(t_l, \tau) d\tau, \end{aligned}$$

which immediately implies that $\mathcal{Q}(\eta) = [\dots, \mathcal{Q}_1(\eta), \mathcal{Q}_2(\eta), \dots]^T$ is a zero-mean Gaussian vector with covariance matrix \mathcal{R}_q .

We now constrain both the transmit interval and the observation interval to $[-nT_q, nT_q]$. Let

$$\mathbf{y}_n = [y[-nf_s T_q + 1], \dots, y[nf_s T_q - 1] y[nf_s T_q]]^T,$$

where the sampled output sequence satisfies

$$y[k] = \mathcal{Q}_k(h(t) * x(t)) + \mathcal{Q}_k(\eta(t)). \quad (43)$$

Introduce two $2nf_s T_q$ -dimensional *truncated* autocorrelation matrices \mathcal{R}_{hq}^n and \mathcal{R}_q^n such that for all $-nf_s T_q < k, l \leq nf_s T_q$,

$$(\mathcal{R}_{hq}^n)_{k,l} = \mathcal{R}_{hq}(t_k, t_l),$$

$$(\mathcal{R}_q^n)_{k,l} = \mathcal{R}_q(t_k, t_l).$$

Clearly, the noise components of \mathbf{y}_n exhibit a covariance matrix \mathcal{R}_q^n , which motivates to whiten it first.

By left multiplying \mathbf{y}_n with $(\mathcal{R}_q^n)^{-\frac{1}{2}}$, we obtain a new input-output relation as

$$\tilde{\mathbf{y}}_n[k] = \tilde{\mathcal{Q}}_k(h(t) * x(t)) + \tilde{\eta}[k], \quad \forall k (|k| \leq n f_s T_q),$$

where $\{\tilde{\eta}[k]\}$ are i.i.d. Gaussian random variables each of unit variance. Denote by $\tilde{q}(t_k, \tau)$ the equivalent impulse response of this new system. The truncated output autocorrelation function $\mathcal{R}_{\tilde{q}}^n$ is given as $(\mathcal{R}_{\tilde{q}}^n)_{k,l} = \mathcal{R}_{\tilde{q}}(t_k, t_l) = \int_{-\infty}^{\infty} \tilde{q}(t_k, \tau) \tilde{q}^*(t_l, \tau) d\tau$, satisfying

$$\mathcal{R}_{\tilde{q}}^n = (\mathcal{R}_q^n)^{-\frac{1}{2}} \mathcal{R}_{h_q} (\mathcal{R}_q^n)^{-\frac{1}{2}} \quad (44)$$

by construction.

C. Capacity via asymptotic properties of block Toeplitz matrices

While both \mathcal{R}_q^n and $\mathcal{R}_{h_q}^n$ are block Toeplitz matrices, $\mathcal{R}_{\tilde{q}}^n$ is in general not a block Toeplitz matrix. By exploiting the asymptotic equivalence in Toeplitz matrix theory [46], one can see that $\mathcal{R}_{\tilde{q}}^n$ is asymptotically equivalent to a block-Toeplitz matrix generated by the Fourier series

$$\begin{aligned} & \mathcal{F} \left(\mathcal{R}_q^{-\frac{1}{2}} \right) \mathcal{F} \left(\mathcal{R}_{h_q} \right) \mathcal{F} \left(\mathcal{R}_q^{-\frac{1}{2}} \right) \\ &= (\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}} \mathbf{F}_q \mathbf{F}_h \mathbf{F}_h^* \mathbf{F}_q^* (\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}}. \end{aligned}$$

Therefore, the asymptotic spectral properties of a block-Toeplitz matrix (e.g. [44]) state that for any nondecreasing continuous function $g(t)$ with a bounded slope, one has

$$\lim_{n \rightarrow \infty} \frac{1}{2nT_q} \sum_{i=1}^{2nf_s T_q} g(\lambda_i(\mathcal{R}_{\tilde{q}}^n)) = \frac{1}{2\pi T_q} \int_{-\pi}^{\pi} \sum_{i=1}^{f_s T_q} g(\hat{\lambda}_i) d\omega, \quad (45)$$

where $\hat{\lambda}_i$ represents the i th eigenvalue of $(\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}} \mathbf{F}_q \mathbf{F}_h \mathbf{F}_h^* \mathbf{F}_q^* (\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}}$.

(1) The capacity of the sampled channel with an optimal water level ν_p can now be calculated as

$$C^{\mathcal{P}}(P) = \lim_{n \rightarrow \infty} \frac{1}{2nT_q} \sum_{i=1}^{2nf_s T_q} \frac{1}{2} [\log(\nu_p \lambda_i(\mathcal{R}_{\tilde{q}}^n))]^+ \quad (46)$$

$$= \frac{1}{2\pi T_q} \int_{-\pi}^{\pi} \sum_{i=1}^{f_s T_q} \frac{1}{2} [\log(\nu_p \hat{\lambda}_i)]^+ d\omega \quad (47)$$

$$= \int_{-f_q/2}^{f_q/2} \sum_{i=1}^{f_s T_q} \frac{1}{2} [\log(\nu_p \hat{\lambda}_i)]^+ df,$$

where (47) is a consequence of (45).

The water level ν is computed through the following parametric equation

$$\lim_{n \rightarrow \infty} \frac{1}{2nT_q} \sum_{i=1}^{2nf_s T_q} \left[\nu_p - \frac{1}{\lambda_i(\mathcal{R}_{\tilde{q}}^n)} \right]^+ df = P,$$

which by (45) is asymptotically equivalent to

$$\frac{1}{2\pi T_q} \int_{-\pi}^{\pi} \sum_{i=1}^{f_s T_q} \left[\nu_p - \frac{1}{\hat{\lambda}_i} \right]^+ d\omega = P,$$

or

$$\int_{-f_q/2}^{f_q/2} \sum_{i=1}^{f_s T_q} \left[\nu_p - \frac{1}{\hat{\lambda}_i} \right]^+ df = P.$$

by change of variables. This establishes the claim.

(2) We consider now the scenario where equal power allocation is employed. Classical MIMO channel capacity results [47] indicate that the optimal power allocation for the transmitter is to allocate equal amount of power in all transmit branches. It remains to see how much power is allocated to the branch associated with λ_i ($\mathcal{R}_{\tilde{q}}^n$).

In fact, if the transmitter knows the channel bandwidth, almost all power (except for negligible leakage due to finite-time approximation) will be allocated inside the channel bandwidth $[0, W]$. Therefore, by the Shannon-Nyquist sampling theorem, all transmit signals can be equivalently transformed to a delta train $\sum_{i=-\infty}^{\infty} x_i \delta(t-i/W)$, where x_i 's are randomly generated transmit signals. Consider the input time block $[-nT_q, nT_q]$, then there are equivalently $2nT_q W$ transmit branches inside this time block. Since the total power is $P_{\text{tot}} = 2nT_q P$, the power allocated to each transmit branch is given by

$$P_0 = \lim_{n \rightarrow \infty} \frac{P_{\text{tot}}}{2nT_q / (\frac{1}{W})} = \lim_{n \rightarrow \infty} \frac{2nT_q P}{2nWT_q} = \frac{P}{W}.$$

As a result, the sampled capacity under equal power allocation is computed as

$$\begin{aligned} C_{\text{eq}}^{\mathcal{P}}(P) &= \lim_{n \rightarrow \infty} \frac{1}{2nT_q} \sum_{i=1}^{2nf_s T_q} \frac{1}{2} \log \left(1 + \frac{P}{W} \lambda_i(\mathcal{R}_{\tilde{q}}^n) \right) \\ &= \frac{1}{2\pi T_q} \int_{-\pi}^{\pi} \sum_{i=1}^{f_s T_q} \frac{1}{2} \log \left(1 + \frac{P}{W} \hat{\lambda}_i \right) d\omega \\ &= \int_{-f_q/2}^{f_q/2} \frac{1}{2} \log \left(\mathbf{I} + \frac{P}{W} (\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}} \mathbf{F}_q \mathbf{F}_h \right. \\ &\quad \left. \cdot \mathbf{F}_h^* \mathbf{F}_q^* (\mathbf{F}_q \mathbf{F}_q^*)^{-\frac{1}{2}} \right) df. \end{aligned}$$

APPENDIX C PROOF OF LEMMA 2

(1) Let \mathcal{G} and $\tilde{\mathcal{G}}$ denote respectively the operators associated with $g(t)$ and $\tilde{g}(t)$. Then, the triangle inequality yields

$$\begin{aligned} \|\hat{\mathcal{P}}_w \mathcal{G}\|_{\text{F}} &\leq \|\hat{\mathcal{P}}_w \tilde{\mathcal{G}}\|_{\text{F}} + \|\hat{\mathcal{P}}_w (\mathcal{G} - \tilde{\mathcal{G}})\|_{\text{F}} \\ &\leq \|\hat{\mathcal{P}}_w \tilde{\mathcal{G}}\|_{\text{F}} + \|\mathcal{G} - \tilde{\mathcal{G}}\|_{\text{F}}, \end{aligned}$$

and hence

$$\|\hat{\mathcal{P}}_w \mathcal{G}\|_{\text{F}}^2 \leq \|\hat{\mathcal{P}}_w \tilde{\mathcal{G}}\|_{\text{F}}^2 + \|\mathcal{G} - \tilde{\mathcal{G}}\|_{\text{F}}^2 + 2 \|\tilde{\mathcal{G}}\|_{\text{F}} \|\mathcal{G} - \tilde{\mathcal{G}}\|_{\text{F}}. \quad (48)$$

From (23) one can easily show that for any $\xi > 0$, there exists a T_0 such that for every $T > T_0$, one has

$$\|\mathcal{G} - \tilde{\mathcal{G}}\|_{\text{F}} \leq \sqrt{2T \left(\int_{-\infty}^{-T} + \int_T^{\infty} \right) |h(t)|^2 dt} \leq \sqrt{2T\xi}.$$

Additionally, suppose that $\int_{-\infty}^{\infty} |h(t)|^2 dt \leq C_g < \infty$. Then, where

$$\|\tilde{\mathcal{G}}\|_{\text{F}} \leq \sqrt{2T \int_{-\infty}^{\infty} |h(t)|^2 dt} \leq \sqrt{2TC_g}.$$

This together with (48) immediately gives us

$$\|\hat{\mathcal{P}}_w \mathcal{G}\|_{\text{F}}^2 \leq \|\hat{\mathcal{P}}_w \tilde{\mathcal{G}}\|_{\text{F}}^2 + 2T\xi + 4T\sqrt{\xi C_g}.$$

Similar to [3, Theorem 8.4.1], we can obtain that

$$\sum_i \lambda_i = \|\hat{\mathcal{P}}_w \mathcal{G}\|_{\text{F}}^2 \quad \text{and} \quad \sum_i \tilde{\lambda}_i = \|\hat{\mathcal{P}}_w \tilde{\mathcal{G}}\|_{\text{F}}^2.$$

Therefore,

$$\begin{aligned} \frac{1}{2T} \sum_i \lambda_i - \frac{1}{2T} \sum_i \tilde{\lambda}_i &= \frac{1}{2T} \|\hat{\mathcal{P}}_w \mathcal{G}\|_{\text{F}}^2 - \frac{1}{2T} \|\hat{\mathcal{P}}_w \tilde{\mathcal{G}}\|_{\text{F}}^2 \\ &\leq \xi + 2\sqrt{\xi C_g}. \end{aligned}$$

Similarly,

$$\frac{1}{2T} \sum_i \lambda_i - \frac{1}{2T} \sum_i \tilde{\lambda}_i \geq -\xi - 2\sqrt{\xi C_g}.$$

(2) We can also bound the sum of eigenvalues as follows

$$\begin{aligned} \frac{1}{2T} \sum_i \lambda_i &= \frac{1}{2T} \|\hat{\mathcal{P}}_w \mathcal{G}\|_{\text{F}}^2 \leq \frac{1}{2T} \|\mathcal{G}\|_{\text{F}}^2 \\ &= \int_{-\infty}^{\infty} |h(t)|^2 dt < \infty. \end{aligned}$$

(3) If $g(t) = O\left(\frac{1}{t^{1+\epsilon}}\right)$, then one can further control

$$\begin{aligned} \|\mathcal{G} - \tilde{\mathcal{G}}\|_{\text{F}}^2 &\leq 2T \left(\int_{-\infty}^{-T} + \int_T^{\infty} \right) |h(t)|^2 dt \\ &\leq 2TO \left(\frac{1}{T^{2+2\epsilon}} \right) = O \left(\frac{1}{T^{1+2\epsilon}} \right). \end{aligned}$$

Therefore, applying Weyl's Theorem [48] yields that

$$\begin{aligned} |\lambda_i - \tilde{\lambda}_i| &\leq \left\| \hat{\mathcal{P}}_w \mathcal{G} \left(\hat{\mathcal{P}}_w \mathcal{G} \right)^* - \hat{\mathcal{P}}_w \tilde{\mathcal{G}} \left(\hat{\mathcal{P}}_w \tilde{\mathcal{G}} \right)^* \right\|_{\text{F}} \\ &\leq \left\| \hat{\mathcal{P}}_w (\mathcal{G} - \tilde{\mathcal{G}}) \right\|_{\text{F}} \left(\left\| \hat{\mathcal{P}}_w \mathcal{G} \right\|_{\text{F}} + \left\| \hat{\mathcal{P}}_w \tilde{\mathcal{G}} \right\|_{\text{F}} \right) \\ &\leq \left\| \mathcal{G} - \tilde{\mathcal{G}} \right\|_{\text{F}} \left(\|\mathcal{G}\|_{\text{F}} + \|\tilde{\mathcal{G}}\|_{\text{F}} \right) \\ &\leq O \left(\frac{1}{T^\epsilon} \right). \end{aligned}$$

Therefore, for any small $\xi > 0$, there exists a constant $T_{0,\epsilon}$ such that for every $T > T_{0,\epsilon}$, one has $|\lambda_i - \tilde{\lambda}_i| < \xi$.

APPENDIX D PROOF OF LEMMA 3

Simple manipulation yields

$$\begin{aligned} \mathcal{R}_{hq}(t_k, t_l) &= \int_{-\infty}^{\infty} s(t_k, \tau) s^*(t_l, \tau) d\tau \\ &= \iiint q(t_k, \tau_k) h(\tau_k - \tau) h^*(\tau_l - \tau) q^*(t_l, \tau_l) d\tau_k d\tau_l d\tau \\ &= \iint q_k(\tau_k) \mathcal{R}_h(\tau_l - \tau_k) q_l^*(\tau_l) d\tau_k d\tau_l, \end{aligned}$$

$$\begin{aligned} \mathcal{R}_h(\tau_l - \tau_k) &:= \int_{\tau} h(\tau_k - \tau) h^*(\tau_l - \tau) d\tau \\ &= \int_{\tau} h(\tau_k - \tau_l + \tau) h^*(\tau) d\tau \\ &= (h * h^{-*})(\tau_k - \tau_l). \end{aligned}$$

Here, for any function $f(t)$, we use $f^-(t)$ to denote $f(-t)$.

By the periodicity assumption of the sampling system, one can derive

$$\begin{aligned} \mathcal{R}_{hq}(t_k + af_s T_q, t_l + bf_s T_q) &= \iint q(t_k + aT_q, \tau_k) \mathcal{R}_h(\tau_l - \tau_k) q^*(t_l + bT_q, \tau_l) d\tau_k d\tau_l \\ &= \iint q(t_k, \tau_k - aT_q) \mathcal{R}_h(\tau_l - \tau_k) \\ &\quad \cdot q^*(t_l + (b-a)T_q, \tau_l - aT_q) d\tau_k d\tau_l \\ &= \mathcal{R}_{hq}(t_k, t_l + (b-a)f_s T_q). \end{aligned}$$

Observing that

$$\begin{aligned} \mathcal{R}_{hq}(t_k, t_i + lf_s T_q) &= \iint q(t_k, \tau_k) \mathcal{R}_h(\tau_i - \tau_k) q^*(t_i + lT_q, \tau_i) d\tau_k d\tau_i \\ &= \iint q_k(\tau_k) \mathcal{R}_h(\tau_i + lT_q - \tau_k) q_i^*(\tau_i) d\tau_k d\tau_i \\ &= (\mathcal{R}_h * q_k * q_i^{-*})(lT_q), \end{aligned}$$

we can see that $(\mathbf{F}_{hq})_{k,i}$ is simply the Fourier transform of the sampled sequence of $\mathcal{R}_h * q_k * q_i^{-*}$. The properties of the Fourier transform suggest that

$$\begin{aligned} \mathcal{F}(\mathcal{R}_h * q_k * q_i^{-*})(f) &= \mathcal{F}(\mathcal{R}_h)(f) \cdot Q_k(f) \cdot Q_i^*(f) \\ &= |H(f)|^2 Q_k(f) \cdot Q_i^*(f), \end{aligned}$$

where $Q_k(f) := \mathcal{F}(q_k)$. By definition in (37), one can write

$$\begin{aligned} (\mathbf{F}_{hq})_{k,i}(f) &:= \sum_{l=-\infty}^{\infty} \mathcal{R}_{hq}(t_k, t_i + lf_s T_q) \exp(j2\pi lf) \\ &= \sum_{l=-\infty}^{\infty} (\mathcal{R}_h * q_k * q_i^{-*})(lT_q) \exp(j2\pi lf), \end{aligned}$$

which immediately leads to

$$(\mathbf{F}_{hq})_{k,i} = \sum_{l=-\infty}^{\infty} Q_k(f + lf_q) |H(f + lf_q)|^2 Q_i^*(f + lf_q).$$

This allows us to express \mathbf{F}_{hq} as

$$\mathbf{F}_{hq} = \mathbf{F}_q \mathbf{F}_h \mathbf{F}_h^* \mathbf{F}_q^*. \quad (49)$$

Similarly, the equality $\mathbf{F}_{qq} = \mathbf{F}_q \mathbf{F}_q^*$ is then an immediate consequence of (49) by setting $\mathbf{F}_h = \mathbf{I}$.

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