# Constructing Linear Encoders with Good Spectra 

Shengtian Yang, Member, IEEE, Thomas Honold, Member, IEEE, Yan Chen, Member, IEEE, Zhaoyang Zhang, Member, IEEE, Peiliang Qiu, Member, IEEE


#### Abstract

Linear encoders with good joint spectra are suitable candidates for optimal lossless joint source-channel coding (JSCC), where the joint spectrum is a variant of the input-output complete weight distribution and is considered good if it is close to the average joint spectrum of all linear encoders (of the same coding rate). In spite of their existence, little is known on how to construct such encoders in practice. This paper is devoted to their construction. In particular, two families of linear encoders are presented and proved to have good joint spectra. The first family is derived from Gabidulin codes, a class of maximum-rank-distance codes. The second family is constructed using a serial concatenation of an encoder of a low-density parity-check code (as outer encoder) with a low-density generator matrix encoder (as inner encoder). In addition, criteria for good linear encoders are defined for three coding applications: lossless source coding, channel coding, and lossless JSCC. In the framework of the codespectrum approach, these three scenarios correspond to the problems of constructing linear encoders with good kernel spectra, good image spectra, and good joint spectra, respectively. Good joint spectra imply both good kernel spectra and good image spectra, and for every linear encoder having a good kernel (resp., image) spectrum, it is proved that there exists a linear encoder not only with the same kernel (resp., image) but also with a good joint spectrum. Thus a good joint spectrum is the most important feature of a linear encoder.


## Index Terms

Code spectrum, Gabidulin codes, linear codes, linear encoders, low-density generator matrix (LDGM), lowdensity parity-check (LDPC) codes, MacWilliams identities, maximum-rank-distance (MRD) codes.

## I. Introduction

Linear codes, owing to their good structure, are widely applied in the areas of channel coding, source coding, and joint source-channel coding (JSCC). A variety of good linear codes such as Turbo codes [1] and low-density parity-check (LDPC) codes [2], [3] have been constructed for channel coding. In the past decade, the parity-check matrices of good linear codes for channel coding have also been employed as encoders for distributed source coding. They proved good both in theory [4], [5] and practice [6]-[8]. However, for the general case of lossless JSCC (based on linear codes), there is still no mature and complete solution (see [9], [10] and the references therein for background information).

We do not even know how to design an implementable optimal JSCC scheme based on linear encoders for arbitrary sources and channels. For instance, much work on practical designs of lossless JSCC based on linear codes has been done for transmission of specific correlated sources over specific multiple-access channels (e.g., [11]-[16]), but it is still not clear how to construct an implementable optimal lossless JSCC scheme for the general case.

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Fig. 1. The proposed lossless joint source-channel encoding scheme based on linear encoders for multiple-access channels in [9].

The same problem occurs in the case of point-to-point transmission, but since traditional (nonlinear) source coding techniques combined with joint source-channel decoding work well in this case, linear-encoder-based schemes are less important here than in distributed JSCC. One exception is the demand for a simple universal encoder, that is, the encoder does not require any knowledge of the source statistics. For background information on lossless JSCC in the point-to-point case, we refer to [17], [18] and the references therein.

Recently, for lossless transmission of correlated sources over multiple-access channels (MACs), we proposed a general scheme based on linear encoders [9]. It was proved optimal if good linear encoders and good conditional probability distributions are chosen. ${ }^{1}$ Fig. 1 illustrates the mechanism of this encoding scheme (see [9, Sec. III-C]), which can also be formulated as

$$
\Phi(\mathrm{x}):=\varphi\left(\mathrm{x}, \Sigma_{m}\left(F\left(\Sigma_{n}(\mathrm{x})\right)\right)+\bar{Y}^{m}\right) \quad \forall \mathrm{x} \in \mathcal{X}^{n}
$$

Roughly speaking, the scheme consists of two steps. First, the source sequence x is processed by a special kind of random affine mapping, $\left(\Sigma_{m} \circ F \circ \Sigma_{n}\right)(\mathbf{x})+\bar{Y}^{m}$, where $F$ is a random linear encoder from $\mathcal{X}^{n}$ to $\mathcal{Y}^{m}, \Sigma_{m}$ and $\Sigma_{n}$ are uniform random interleavers, $\bar{Y}^{m}$ is a uniform random vector, and all of them are independent. Second, the output of the first step, together with the source sequence, is fed into the quantization map $\varphi$ to yield the final output. The first step is to generate uniformly distributed output with the so-called pairwise-independence property, while the second step is to shape the output so that it is suitable for a given channel. From this, two main issues arise: how can we design a good linear encoder and a good quantization map to fulfill the above two goals, respectively? About the former, we proved in [9] that linear encoders with joint spectra (a variant of input-output complete weight distribution) close to the average joint spectrum over all linear encoders (from $\mathcal{X}^{n}$ to $\mathcal{Y}^{m}$ ) are good candidates. We say that such linear encoders have good joint spectra. Hence, for designing a lossless JSCC scheme in practice, the crucial problem is how to construct linear encoders with good joint spectra. To our knowledge, however, this problem has never been studied before.

In this paper, we shall give a thorough investigation of this problem. Our main tool is the code-spectrum approach [9]. As we shall see, the spectra of a linear encoder, including kernel spectrum, image spectrum, and especially joint spectrum, provide an important characterization of its performance for most applications (see Definitions 3.1-3.3, which form the base of this paper). The rest of the paper is organized as follows.

In Section II, the code-spectrum approach is briefly reviewed. In Section III, three kinds of good linear encoders are defined for lossless source coding, channel coding, and lossless JSCC, respectively. They are called $\delta$-asymptotically good linear source encoders (LSEs), $\delta$-asymptotically good linear channel encoders (LCEs), and $\delta$-asymptotically good linear source-channel encoders (LSCEs), respectively. We show that, under some conditions, good LSCEs are also good as LSEs and LCEs. Thus the problem of constructing good LSCEs (i.e., linear encoders with good joint spectra) is of particular interest and importance.

Based on this observation, in Section IV, we proceed to study the general principles for constructing good LSCEs. In Section IV-A, we provide a family of good LSCEs derived from so-called maximum-rank-distance (MRD) codes. In Section IV-B, we investigate the problem of how to construct a good LSCE with the same kernel (resp., image) as a given good LSE (resp., LCE). In Section IV-C, we propose a general serial concatenation scheme for constructing good LSCEs. In light of this general scheme, in Section V, we turn to the analysis of joint spectra of regular low-density generator matrix (LDGM) encoders. We show that the joint spectra of regular LDGM encoders with appropriate parameters are approximately $\delta$-asymptotically good. Based on this fact, we finally construct a family of good LSCEs by means of a serial concatenation of an inner LDGM encoder and an outer encoder of an LDPC code.

[^1]Some advanced tools of the code-spectrum approach are developed in Section VI in order to prove the results in the preceding sections. Section VII concludes the paper.

Since this paper is highly condensed, we refer the reader to [10] for concrete examples and important remarks. In particular, an example is provided there to show how the binary $[7,4,3]$ Hamming code is beaten by a $[7,4,1]$ linear code in the case of lossless JSCC if the encoder (or generator matrix) is not carefully chosen. Recall that an $[n, k, d]$ linear code over a finite field $F$ is a $k$-dimensional subspace of $F^{n}$ with minimum (Hamming) distance $d$.

We close this section with some basic notations and conventions used throughout the paper. In general, mathematical objects such as real variables and deterministic mappings are denoted by lowercase letters. Conventionally, sets, matrices, and random elements are denoted by capital letters, and alphabets are denoted by script capital letters.

The symbols $\mathbb{Z}, \mathbb{N}, \mathbb{N}_{0}, \mathbb{R}, \mathbb{C}$ denote the ring of integers, the set of positive integers, the set of nonnegative integers, the field of real numbers, and the field of complex numbers, respectively. For a prime power $q>1$ the finite field of order $q$ is denoted by $\mathbb{F}_{q}$. The multiplicative subgroup of nonzero elements of $\mathbb{F}_{q}$ is denoted by $\mathbb{F}_{q}^{\times}$ (and similarly for other fields). The index set $\{1,2, \ldots, n\}$ for $n \in \mathbb{N}$ is denoted by $\mathcal{I}_{n}$.

A sequence (or vector) in $\mathcal{X}^{n}$ is denoted by $\mathrm{x}=x_{1} x_{2} \cdots x_{n}$, with $x_{i}$ denoting the $i$ th component. The length of $\mathbf{x}$ is denoted by $|\mathbf{x}|$. For the $l$-fold repetition of a single symbol $a \in \mathcal{X}$, we write $a^{l}$ for brevity. For any set $A=\left\{a_{1}, a_{2}, \ldots, a_{r}\right\} \subseteq \mathcal{I}_{n}$ with $a_{1}<a_{2}<\cdots<a_{r}$, we define the sequence $\left(x_{i}\right)_{i \in A}$ or $x_{A}$ as $x_{a_{1}} x_{a_{2}} \cdots x_{a_{r}}$.

For any maps $f: \mathcal{X}_{1} \rightarrow \mathcal{Y}_{1}$ and $g: \mathcal{X}_{2} \rightarrow \mathcal{Y}_{2}$, the cartesian product $f \odot g: \mathcal{X}_{1} \times \mathcal{X}_{2} \rightarrow \mathcal{Y}_{1} \times \mathcal{Y}_{2}$ is given by $\left(x_{1}, x_{2}\right) \mapsto\left(f\left(x_{1}\right), g\left(x_{2}\right)\right)$. Given a map $f$ from a finite set $X$ to $\mathbb{N}_{0}$ with $\sum_{x \in X} f(x)=n$, the multinomial coefficient $n!/ \prod_{x \in X}(f(x)!)$ is denoted by $\binom{n}{f}$.
The function $1\{\cdot\}$ is a mapping defined by $1\{$ true $\}=1$ and $1\{$ false $\}=0$. Then the indicator function of a subset $A$ of a set $X$ can be written as $1\{x \in A\}$. For $x \in \mathbb{R},\lceil x\rceil$ denotes the smallest integer $\geq x$. For any real-valued functions $f(n)$ and $g(n)$ with domain $\mathbb{N}$, the statement $f(n)=\Theta(g(n))$ refers to the existence of positive constants $c_{1}$ and $c_{2}$ such that $c_{1} g(n) \leq f(n) \leq c_{2} g(n)$ for sufficiently large $n$.

By default, all vectors are regarded as row vectors. An $m \times n$ matrix is denoted by $\mathbf{M}=\left(\mathrm{M}_{i, j}\right)_{i \in \mathcal{I}_{m}, j \in \mathcal{I}_{n}}$, with $\mathrm{M}_{i, j}$ denoting the $(i, j)$-th entry. The transpose of a matrix $\mathbf{M}$ is denoted by $\mathbf{M}^{\top}$. The set of all $m \times n$-matrices over a field $F$ is denoted by $F^{m \times n}$.

When performing probabilistic analysis, all objects of study are related to a basic probability space $(\Omega, \mathcal{A}, \mathrm{P})$ where $\mathcal{A}$ is a $\sigma$-algebra in $\Omega$ and P is a probability measure on $(\Omega, \mathcal{A})$. For any event $A \in \mathcal{A}, \mathrm{P} A=\mathrm{P}(A)$ is called the probability of $A$. A random element is a measurable mapping of $\Omega$ into some measurable space $\left(\Omega^{\prime}, \mathcal{B}\right)$. In this paper, distinct random elements are assumed to be independent, and cartesian products of the same random sets (or maps) are also regarded as cartesian products of independent copies.

For a (discrete) probability distribution $P$ on $\mathcal{X}$, the entropy $H(P)$ is given by $-\sum_{a \in \mathcal{X}} P(a) \ln P(a)$, with $0 \ln 0:=0$. For probability distributions $P$ and $Q$ on $\mathcal{X}$ with $P \ll Q$ (i.e., $P$ absolutely continuous with respect to $Q$ ), the information divergence $D(P \| Q)$ is given by $\sum_{a \in \mathcal{X}} P(a) \ln (P(a) / Q(a))$.

## II. Basics of the Code-Spectrum Approach

In this section, we briefly introduce the basics of the code-spectrum approach [9], a variant of the weightdistribution approach, e.g., [19], [20].

Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite (additive) abelian groups. A linear encoder is a homomorphism $f: \mathcal{X}^{n} \rightarrow \mathcal{Y}^{m}$. The image $f\left(\mathcal{X}^{n}\right) \subseteq \mathcal{Y}^{m}$ of such an $f$ is a linear code over $\mathcal{Y}$ in the usual sense, i.e., a block code of length $m$ over $\mathcal{Y}$ that forms a subgroup of $\mathcal{Y}^{m} .{ }^{2}$ With each $f$ there are associated three kinds of rates: First, the source transmission rate $R_{s}(f):=n^{-1} \ln \left|f\left(\mathcal{X}^{n}\right)\right|$. Second, the channel transmission rate $R_{c}(f):=m^{-1} \ln \left|f\left(\mathcal{X}^{n}\right)\right|$. Third, the coding rate $R(f):=n / m$. There is a simple relation among these quantities, viz., $R(f) R_{s}(f)=R_{c}(f)$. If $f$ is injective, then $R_{s}(f)=\ln |\mathcal{X}|$ and $R_{c}(f)=R(f) \ln |\mathcal{X}|$; if $f$ is surjective, then $R_{s}(f)=\ln |\mathcal{Y}| / R(f)$ and $R_{c}(f)=\ln |\mathcal{Y}|$.

For any linear encoder $f: \mathcal{X}^{n} \rightarrow \mathcal{Y}^{m}$, there exist uniquely determined homomorphisms $f_{i j}: \mathcal{X} \rightarrow \mathcal{Y}(1 \leq i \leq n$, $1 \leq j \leq m)$ such that $f(\mathbf{x})=\left(\sum_{i=1}^{n} x_{i} f_{i 1}, \ldots, \sum_{i=1}^{n} x_{i} f_{i m}\right)=\mathbf{x M}$, where $x_{i} f_{i j}:=f_{i j}\left(x_{i}\right)$ and $\mathbf{M}:=\left(f_{i j}\right) \in$ $\operatorname{Hom}(\mathcal{X}, \mathcal{Y})^{n \times m}$, with $\operatorname{Hom}(\mathcal{X}, \mathcal{Y})$ denoting the abelian group of all homomorphisms from $\mathcal{X}$ to $\mathcal{Y}$ under map addition. In the special case of a prime field $\mathcal{X}=\mathcal{Y}=\mathbb{F}_{p}$, the usual representation of an $\mathbb{F}_{p}$-linear encoder

[^2]$f: \mathbb{F}_{p}^{n} \rightarrow \mathbb{F}_{p}^{m}$ by its generator matrix $\mathbf{M} \in \mathbb{F}_{p}^{n \times m}$ is recovered, since $\mathbb{F}_{p} \cong \operatorname{End}\left(\mathbb{F}_{p},+\right):=\operatorname{Hom}\left(\mathbb{F}_{p}, \mathbb{F}_{p}\right)$ via $a \mapsto\left(\mu_{a}: \mathbb{F}_{p} \rightarrow \mathbb{F}_{p}, x \mapsto a x\right)$. For a general finite field $\mathbb{F}_{q}$, the generator matrix representation requires an $\mathbb{F}_{q}$-linear encoder and is stronger. Nevertheless, identifying a linear encoder with a generator matrix over an appropriate finite field is still useful in general, since every linear encoder with $\mathcal{X}=\mathcal{Y}=\mathbb{F}_{q}$ has a generator matrix representation over a certain subfield of $\mathbb{F}_{q}$.

A particularly simple class of linear encoders from $\mathcal{X}^{n}$ to $\mathcal{X}^{n}$, including the identity map, is formed by the special group automorphisms defined by $\sigma(\mathbf{x}):=x_{\sigma^{-1}(1)} x_{\sigma^{-1}(2)} \cdots x_{\sigma^{-1}(n)}$ for each permutation $\sigma$ in $\mathrm{S}_{n}$, the group of all permutations of $\mathcal{I}_{n}$. These linear encoders are called coordinate permutations or interleavers. ${ }^{3}$ Considering a random encoder uniformly distributed over all permutations in $\mathrm{S}_{n}$, we obtain a uniform random permutation, denoted $\Sigma_{n}$. We tacitly assume that different random permutations occurring in the same expression are independent, and notation such as $\Sigma_{m}$ and $\Sigma_{n}$ refers to different random permutations even in the case $m=n$.

The type of a sequence x in $\mathcal{X}^{n}$ is the empirical distribution $P_{\mathrm{x}}$ on $\mathcal{X}$ defined by

$$
P_{\mathbf{x}}(a):=\frac{1}{|\mathbf{x}|} \sum_{i=1}^{|\mathbf{x}|} 1\left\{x_{i}=a\right\} .
$$

For a (probability) distribution $P$ on $\mathcal{X}$, the set of sequences of type $P$ in $\mathcal{X}^{n}$ is denoted by $\mathcal{T}_{P}^{n}(\mathcal{X})$ or simply $\mathcal{T}_{P}^{n}$. A distribution $P$ on $\mathcal{X}$ is called a type of sequences in $\mathcal{X}^{n}$ if $\mathcal{T}_{P}^{n} \neq \varnothing$. We denote by $\mathcal{P}_{n}(\mathcal{X})$ (or $\mathcal{P}_{n}$ if the alphabet is clear from the context) the set of all types of sequences in $\mathcal{X}^{n}$. Since the set $\mathcal{P}_{n} \backslash\left\{P_{0^{n}}\right\}$ will be frequently used, we denote it by $\mathcal{P}_{n}^{*}$ in the sequel.

The spectrum of a nonempty set $A \subseteq \mathcal{X}^{n}$ is the empirical distribution $\mathrm{S}_{\mathcal{X}}(A)$ on $\mathcal{P}_{n}(\mathcal{X})$ defined by

$$
\mathrm{S}_{\mathcal{X}}(A)(P):=\frac{\left|\left\{\mathrm{x} \in A: P_{\mathbf{x}}=P\right\}\right|}{|A|} \quad \forall P \in \mathcal{P}_{n}(\mathcal{X})
$$

and for convenience, we write $S(A), \mathrm{S}(A)(P)$, or further $\mathrm{S}_{A}(P)$ provided $P$ refers to an element of $\mathcal{P}_{n}(\mathcal{X})$. In other words, $\mathrm{S}(A)$ is the empirical distribution of types of sequences in $A$. The spectrum of $A$ is closely related to the well-established complete weight distribution of $A$ (see e.g., [21, Ch. 7.7] or [22, Sec. 10]). In fact, both distributions differ only by a scaling factor. Nevertheless, introducing the spectrum as a new, independent concept has its merits, see [10] for a detailed explanation.

Analogously, the joint spectrum $\mathrm{S}_{\mathcal{X} \mathcal{Y}}(B)(P, Q)$ of a nonempty set $B \subseteq \mathcal{X}^{n} \times \mathcal{Y}^{m}$ is the empirical distribution of type pairs $\left(P_{\mathbf{x}}, P_{\mathbf{y}}\right)$ of sequence pairs $(\mathbf{x}, \mathbf{y}) \in B$. By considering the marginal and conditional distributions of $\mathrm{S}_{\mathcal{X} \mathcal{Y}}(B)$, we obtain the marginal spectra $\mathrm{S}_{\mathcal{X}}(B)(P), \mathrm{S}_{\mathcal{Y}}(B)(Q)$ and the conditional spectra $\mathrm{S}_{\mathcal{Y} \mid \mathcal{X}}(B)(Q \mid P)$, $\mathrm{S}_{\mathcal{X} \mid \mathcal{Y}}(B)(P \mid Q)$. These definitions can also be extended to the case of more than two alphabets in the obvious way. For convenience of notation, we sometimes write, e.g., $\mathrm{S}(B)(P, Q), \mathrm{S}(B)(P), \mathrm{S}(B)(Q \mid P)$, or further, $\mathrm{S}_{B}(P, Q)$, $\mathrm{S}_{B}(P), \mathrm{S}_{B}(Q \mid P)$, provided $(P, Q)$ refers to an element of $\mathcal{P}_{n}(\mathcal{X}) \times \mathcal{P}_{m}(\mathcal{Y})$.

Furthermore, for any given function $f: \mathcal{X}^{n} \rightarrow \mathcal{Y}^{m}$, we can define its joint spectrum $\mathrm{S}_{\mathcal{X}}(f)$, forward conditional spectrum $\mathrm{S}_{\mathcal{Y} \mid \mathcal{X}}(f)$, and image spectrum $\mathrm{S}_{\mathcal{Y}}(f)$ as $\mathrm{S}_{\mathcal{X} \mathcal{Y}}(\operatorname{rl}(f)), \mathrm{S}_{\mathcal{Y} \mid \mathcal{X}}(\mathrm{rl}(f))$, and $\mathrm{S}_{\mathcal{Y}}(\mathrm{rl}(f))$, respectively, where $\operatorname{rl}(f):=\left\{(\mathbf{x}, f(\mathbf{x})): \mathbf{x} \in \mathcal{X}^{n}\right\}$ is the graph of $f$. In this case, the forward conditional spectrum $\mathrm{S}(f)(Q \mid P)$ or $\mathrm{S}_{f}(Q \mid P)$ is given by $\mathrm{S}_{f}(P, Q) / \mathrm{S}_{\mathcal{X}^{n}}(P)$. If $f$ is a linear encoder, we further define its kernel spectrum as $\mathrm{S}(\operatorname{ker} f)$, where ker $f:=\left\{\mathbf{x} \in \mathcal{X}^{n}: f(\mathbf{x})=0^{m}\right\}$, and in this case, $\mathrm{S}_{\mathcal{Y}}(f)=\mathrm{S}\left(f\left(\mathcal{X}^{n}\right)\right)$.

It is easy to see that coordinate permutations preserve the type and hence the spectrum. Two sets $A, B \subseteq \mathcal{X}^{n}$ are said to be equivalent (under permutation) if $\sigma(A)=B$ for some $\sigma \in \mathrm{S}_{n}$. Two maps $f, g$ from $\mathcal{X}^{n}$ to $\mathcal{Y}^{m}$ are said to be equivalent (under permutation) if $\sigma^{\prime} \circ f \circ \sigma=g$ for some $\sigma \in \mathrm{S}_{n}$ and $\sigma^{\prime} \in \mathrm{S}_{m}$. The notion of equivalence is extended to random sets and maps in the obvious way.

For a random nonempty set $A \subseteq \mathcal{X}^{n}$, we define $\alpha(A)(P)$ or $\alpha_{A}(P):=\mathrm{E}\left[\mathrm{S}_{A}(P)\right] / \mathrm{S}_{\mathcal{X}^{n}}(P)$. To simplify notation, we shall write, e.g., $\overline{\mathrm{S}}_{A}(P)$ in place of $\mathrm{E}\left[\mathrm{S}_{A}(P)\right]$. Similarly, for a random map $F: \mathcal{X}^{n} \rightarrow \mathcal{Y}^{m}$, we define

$$
\begin{equation*}
\alpha_{F}(P, Q):=\frac{\overline{\mathrm{S}}_{F}(P, Q)}{\mathrm{S}_{\mathcal{X}^{n} \times \mathcal{Y}^{m}}(P, Q)} . \tag{1}
\end{equation*}
$$

The definition of $\alpha$ is essentially a ratio of two spectra. Its purpose is to measure the distance from the spectrum of a set to a "random-like" spectrum (e.g., $\mathrm{S}_{\mathcal{X}^{n}}(P)$ ). The reader may also compare it or its logarithm to the notion

[^3]of information divergence. It is clear that equivalent sets or maps have the same $\alpha$. As we shall see, $\alpha$ plays an important role in characterizing the average behavior of an equivalence class of sets or maps (Proposition 6.2 as well as [9, Proposition 2.4]), and hence it provides a good criterion of code performance for various applications (Section III).

## III. Good Linear Encoders for Source Coding, Channel Coding, and Joint Source-Channel Coding

In classical coding theory, a good linear code typically refers to a set of codewords that has good performance for some family of channels or has a large minimum Hamming distance close to one of the well-known upper bounds such as the sphere packing, Plotkin, or Singleton bound. Such a viewpoint may be sufficient for channel coding, but has its limitations in the context of source coding and JSCC. The main reason is that an approach focusing on linear codes (i.e., the image of encoder map) cannot cover all coding-related properties of linear encoders, and relevant criteria for good linear encoders are generally different for different applications.

In [9, Table I], we have only briefly reviewed the criteria of good linear encoders in terms of spectrum requirements for lossless source coding, channel coding, and lossless JSCC. So in this section, we shall resume this discussion, including the concepts of good linear encoders and the relations among different kinds of good linear encoders. Since our main constructions require the underlying alphabet to be a finite field, we assume from now on (unless stated otherwise) that the alphabet of any linear encoder is the finite field $\mathbb{F}_{q}$, where $q=p^{r}$ and $p$ is prime.

We begin with some concepts related to the asymptotic rate of an encoder sequence. Let $\boldsymbol{F}=\left\{F_{k}: \mathbb{F}_{q}^{n_{k}} \rightarrow\right.$ $\left.\mathbb{F}_{q}^{m_{k}}\right\}_{k=1}^{\infty}$ be a sequence of random linear encoders. If $R_{s}\left(F_{k}\right)$ converges in probability to a constant, we say that the asymptotic source transmission rate $R_{s}(\boldsymbol{F})$ of $\boldsymbol{F}$ is p- $\lim _{k \rightarrow \infty} R_{s}\left(F_{k}\right)$. Analogously, we define the asymptotic channel transmission rate and the asymptotic coding rate of $\boldsymbol{F}$ by $R_{c}(\boldsymbol{F}):=\mathrm{p}-\lim _{k \rightarrow \infty} R_{c}\left(F_{k}\right)$ and $R(\boldsymbol{F}):=$ $\lim _{k \rightarrow \infty} R\left(F_{k}\right)$, respectively. When $R\left(F_{k}\right)$ does not necessarily converge, we define the superior asymptotic coding rate $\bar{R}(\boldsymbol{F})$ and inferior asymptotic coding rate $\underline{R}(\boldsymbol{F})$ by taking the limit superior and limit inferior, respectively. To simplify notation, in the rest of this section, when writing $\boldsymbol{F}$, we always mean a sequence $\left\{F_{k}\right\}_{k=1}^{\infty}$ of random linear encoders $F_{k}: \mathbb{F}_{q}^{n_{k}} \rightarrow \mathbb{F}_{q}^{m_{k}}$. To avoid some degenerate cases, we assume that $\lim _{k \rightarrow \infty}\left|F_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)\right|=\infty$.

Next, we introduce the definitions of good linear encoders for lossless source coding, channel coding, and lossless JSCC, respectively. The rationale behind them is explained in [10] using the ideas of [5], [9], [23]-[25].

Definition 3.1: Let $\boldsymbol{F}$ be a sequence of random linear encoders with the asymptotic source transmission rate $R_{s}(\boldsymbol{F})$. If $\boldsymbol{F}$ satisfies the kernel-spectrum condition:

$$
\begin{equation*}
\limsup _{k \rightarrow \infty} \max _{P \in \mathcal{P}_{n_{k}}^{*}} \frac{1}{n_{k}} \ln \alpha_{\text {ker } F_{k}}(P) \leq \delta \tag{2}
\end{equation*}
$$

then it is called a sequence of $\delta$-asymptotically good linear source encoders (LSEs) or is said to be $\delta$-asymptotically SC-good (where the last "C" stands for "coding").

Definition 3.2: Let $\boldsymbol{F}$ be a sequence of random linear encoders with the asymptotic channel transmission rate $R_{c}(\boldsymbol{F})$. If $\boldsymbol{F}$ satisfies the image-spectrum condition:

$$
\begin{equation*}
\limsup _{k \rightarrow \infty} \max _{Q \in \mathcal{P}_{m_{k}}^{*}} \frac{1}{m_{k}} \ln \alpha_{F_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)}(Q) \leq \delta, \tag{3}
\end{equation*}
$$

then it is called a sequence of $\delta$-asymptotically good linear channel encoders (LCEs) or is said to be $\delta$-asymptotically CC-good.

Definition 3.3: Let $\boldsymbol{F}$ be a sequence of random linear encoders. If $\boldsymbol{F}$ satisfies the joint-spectrum condition:

$$
\begin{equation*}
\limsup _{k \rightarrow \infty} \max _{P \in \mathcal{P}_{n_{k}}^{*}, Q \in \mathcal{P}_{m_{k}}} \frac{1}{n_{k}} \ln \alpha_{F_{k}}(P, Q) \leq \delta, \tag{4}
\end{equation*}
$$

then it is called a sequence of $\delta$-asymptotically good linear source-channel encoders (LSCEs) or is said to be $\delta$-asymptotically SCC-good.

When $\delta=0$, we use the simplified term "asymptotically good"; when talking about $\delta$-asymptotically good LSEs (resp., LCEs), we tacitly assume that their asymptotic source (resp., channel) transmission rates exist. Since the conditions for $\delta$-asymptotically good LSEs or LCEs only depend on the kernel or image of the linear encoders involved, we also introduce the concepts of equivalence in the sense of LSE or LCE. Linear encoders $f: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$
and $g: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$ are said to be $S C$-equivalent if their kernels are equivalent, and $C C$-equivalent if their images are equivalent. For convenience, we define the function

$$
\rho(F):=\max _{P \in \mathcal{P}_{n}^{*}, Q \in \mathcal{P}_{m}} \frac{1}{n} \ln \alpha_{F}(P, Q)
$$

of a random linear encoder $F: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$. Then condition (4) can be rewritten as lim $\sup _{k \rightarrow \infty} \rho\left(F_{k}\right) \leq \delta$.
The following propositions relate these three kinds of good linear encoders to each other.
Proposition 3.4: For a sequence $\boldsymbol{F}$ of $\delta$-asymptotically good LSEs, there exists a sequence $\boldsymbol{G}=\left\{G_{k}\right\}_{k=1}^{\infty}$ of $\delta$-asymptotically good LCEs $G_{k}: \mathbb{F}_{q}^{l_{k}} \rightarrow \mathbb{F}_{q}^{n_{k}}$ such that $G_{k}\left(\mathbb{F}_{q}^{l_{k}}\right)=\operatorname{ker} F_{k}$ for all $k \in \mathbb{N}$.

Proposition 3.5: For a sequence $\boldsymbol{F}$ of $\delta$-asymptotically good LCEs, there exists a sequence $\boldsymbol{G}=\left\{G_{k}\right\}_{k=1}^{\infty}$ of $\delta$-asymptotically good LSEs $G_{k}: \mathbb{F}_{q}^{m_{k}} \rightarrow \mathbb{F}_{q}^{l_{k}}$ such that $\operatorname{ker} G_{k}=F_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)$ for all $k \in \mathbb{N}$.

Proposition 3.6: Let $\boldsymbol{F}$ be a sequence of $\delta$-asymptotically good LSCEs. Then $\boldsymbol{F}$ is $\delta$-asymptotically SC-good whenever its asymptotic source transmission rate exists, and $\boldsymbol{F}$ is $\delta \bar{R}(\boldsymbol{F})$-asymptotically CC-good whenever its asymptotic channel transmission rate exists.

These relations are all depicted in Fig. 2. Among them, the relations (Proposition 3.6) that good LSCEs are good


Fig. 2. The relations among different kinds of good linear encoders
LSEs and LCEs deserve much more attention. This fact indicates the fundamental role of good LSCEs, and allows us to concentrate on only one problem, namely, constructing linear encoders with good joint spectra. So in the following sections, we shall investigate this problem in depth. Note, however, that there are two arrows missing in Fig. 2, one from LSE to LSCE and the other from LCE to LSCE. This naturally leads to the following question: Are $\delta$-asymptotically good LSEs (resp., LCEs) $\delta$-asymptotically SCC-good, or can we construct $\delta$-asymptotically good LSCEs which are SC-equivalent (resp., CC-equivalent) to given $\delta$-asymptotically good LSEs (resp., LCEs)?

As $\delta$ is close to zero, it is clear that $\delta$-asymptotically good LSEs (or LCEs) may not be $\delta$-asymptotically SCCgood. [5, Theorem 4] shows that there exist asymptotically good LSEs in the ensemble of low-density parity-check matrices. However, [9, Corollary 4.2] proves that these encoders are not asymptotically SCC-good because the matrices are sparse. It is also well known that for any linear code there exists a CC-equivalent systematic linear encoder. According to [9, Corollary 4.1], high-rate systematic linear encoders are not asymptotically SCC-good. So we can always find asymptotically good systematic LCEs which are not asymptotically SCC-good.

It remains yet to decide whether, and if so, how we can construct $\delta$-asymptotically good LSCEs which are SC-equivalent (resp., CC-equivalent) to given $\delta$-asymptotically good LSEs (resp., LCEs). The answer is positive and will be given in Section IV.

We close this section with an easy result.
Proposition 3.7: Let $\left\{\boldsymbol{F}_{i}=\left\{F_{i, k}\right\}_{k=1}^{\infty}\right\}_{i=1}^{\infty}$ be a family of sequences of random linear encoders $F_{i, k}: \mathbb{F}_{q}^{n_{k}} \rightarrow \mathbb{F}_{q}^{m_{k}}$. Suppose that $\boldsymbol{F}_{i}$ is $\delta_{i}$-asymptotically SCC-good where $\delta_{i}$ is nonincreasing in $i$ and converges to $\delta$ as $i \rightarrow \infty$. Define the random linear encoder $G_{i, k}:=F_{i_{0}, k}$ where $i_{0}:=\arg \min _{1 \leq j \leq i} \rho\left(F_{j, k}\right)$. Then $\left\{G_{k, k}\right\}_{k=1}^{\infty}$ is $\delta$-asymptotically SCC-good.

The proofs of this section are given in Appendix A.

## IV. General Principles for Constructing Good Linear Source-Channel Encoders

## A. A Class of SCC-Good Encoders Derived from Certain Maximum-Rank-Distance Codes

Recall that in [9, Sec. III], a random linear encoder $F: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$ was said to be good for JSCC if it satisfies

$$
\begin{equation*}
\alpha_{F}(P, Q)=1 \tag{5}
\end{equation*}
$$

for all $P \in \mathcal{P}_{n}^{*}$ and $Q \in \mathcal{P}_{m}$. To distinguish good linear encoders for JSCC from good linear encoders in other contexts, we say that $F$ is SCC-good. It is clear that SCC-good linear encoders are asymptotically SCC-good. By [9, Proposition 2.4], we also have an alternative condition:

$$
\begin{equation*}
\mathrm{P}\{\tilde{F}(\mathbf{x})=\mathbf{y}\}=q^{-m} \quad \forall \mathbf{x} \in \mathbb{F}_{q}^{n} \backslash\left\{0^{n}\right\}, \mathbf{y} \in \mathbb{F}_{q}^{m} . \tag{6}
\end{equation*}
$$

Let $F_{n, m}^{\mathrm{RLC}}: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$ be a random linear encoder uniformly distributed over $\mathbb{F}_{q}^{n \times m}$. Obviously, $F_{n, m}^{\mathrm{RLC}}$ is SCC$\operatorname{good}\left(\left[9\right.\right.$, Proposition 2.5]) but in some sense trivial, since its distribution has support $\mathbb{F}_{q}^{n \times m}$, the set of all linear encoders $f: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$. In this subsection we provide further examples of SCC-good random linear encoders with support size much smaller than $\left|\mathbb{F}_{q}^{n \times m}\right|=q^{m n}$. These are derived from so-called maximum-rank-distance (MRD) codes, and have the stronger property

$$
\begin{equation*}
\mathrm{P}\{F(\mathbf{x})=\mathbf{y}\}=q^{-m} \quad \forall \mathbf{x} \in \mathbb{F}_{q}^{n} \backslash\left\{0^{n}\right\}, \mathbf{y} \in \mathbb{F}_{q}^{m} \tag{7}
\end{equation*}
$$

which one might call "SCC-good before symmetrization".
We first give a brief review of MRD codes. Let $n, m, k$ be positive integers with $k \leq \min \{n, m\}$. Write $n^{\prime}=\max \{n, m\}, m^{\prime}=\min \{n, m\}$ (so that $\left(n^{\prime}, m^{\prime}\right)$ equals $(n, m)$ or $(m, n)$ ). An ( $n, m, k$ ) maximum-rankdistance (MRD) code over $\mathbb{F}_{q}$ is a set $\mathcal{C}$ of $q^{k n^{\prime}}$ matrices in $\mathbb{F}_{q}^{n \times m}$ with minimum rank distance $\mathrm{d}_{\mathrm{R}}(\mathcal{C}):=$ $\min _{\mathbf{A}, \mathbf{B} \in \mathcal{C}: \mathbf{A} \neq \mathbf{B}} \operatorname{rank}(\mathbf{A}-\mathbf{B})=m^{\prime}-k+1$. MRD codes are optimal for the rank distance on $\mathbb{F}_{q}^{n \times m}$ in the same way as maximum distance separable (MDS) codes are optimal for the Hamming distance, as the following well-known argument shows: Let $\mathcal{C} \subseteq \mathbb{F}_{q}^{n \times m}$ have minimum rank distance $d, 1 \leq d \leq m^{\prime}$. We may view $\mathcal{C}$ as a code of length $m$ over $\mathbb{F}_{q}^{n}$ (the columns of $\mathbf{A} \in \mathbb{F}_{q}^{n \times m}$ being the "entries" of a codeword). As such $\mathcal{C}$ has Hamming distance $\geq d$ (since a matrix of rank $d$ must have at least $d$ nonzero columns). Hence $|\mathcal{C}| \leq q^{n(m-d+1)}$ by the Singleton bound. By transposing we also have $|\mathcal{C}| \leq q^{m(n-d+1)}$, so that

$$
\begin{equation*}
|\mathcal{C}| \leq \min \left\{q^{n(m-d+1)}, q^{m(n-d+1)}\right\}=q^{n^{\prime}\left(m^{\prime}-d+1\right)} . \tag{8}
\end{equation*}
$$

The codes for which the Singleton-like bound (8) is sharp are exactly the ( $n, m, k$ ) MRD codes with $k=m^{\prime}-d+1$.
MRD codes have maximum cardinality among all $n \times m$ matrix codes over $\mathbb{F}_{q}$ with fixed minimum rank distance. They were introduced in [26] under the name "Singleton system" and investigated further in [27], [28]. As shown in [26]-[28], linear $(n, m, k)$ MRD codes over $\mathbb{F}_{q}$ (i.e. MRD codes that are $\mathbb{F}_{q}$-subspaces of $\mathbb{F}_{q}^{n \times m}$ ) exist for all $n, m, k$ with $1 \leq k \leq m^{\prime}$. The standard construction uses $q$-analogues of Reed-Solomon codes, which are defined as follows: Assuming $n \geq m$ for a moment, let $x_{1}, \ldots, x_{m} \in \mathbb{F}_{q^{n}}$ be linearly independent over $\mathbb{F}_{q}$. For $1 \leq k \leq m$ let $C_{k}$ be the linear $[m, k]$ code over $\mathbb{F}_{q^{n}}$ having generator matrix

$$
\mathbf{G}_{k}=\left(\begin{array}{cccc}
x_{1} & x_{2} & \ldots & x_{m}  \tag{9}\\
x_{1}^{q} & x_{2}^{q} & \ldots & x_{m}^{q} \\
\vdots & \vdots & & \vdots \\
x_{1}^{q^{k-1}} & x_{2}^{q^{k-1}} & \ldots & x_{m}^{q^{k-1}}
\end{array}\right) .
$$

Replace each codeword $\mathbf{c}=\left(c_{1}, \ldots, c_{m}\right) \in C_{k}$ by the $n \times m$ matrix $\mathbf{C}$ having as columns the coordinate vectors of $c_{j}$ with respect to some fixed basis of $\mathbb{F}_{q^{n}}$ over $\mathbb{F}_{q}$. The set $\mathcal{C}_{k}$ of all $q^{n k}$ matrices $\mathbf{C}$ obtained in this way forms a linear $(n, m, k)$ MRD code over $\mathbb{F}_{q}$. The restriction $n \geq m$ is not essential, since transposing each matrix of an $(n, m, k)$ MRD code yields an $(m, n, k)$ MRD code (due to the fact that $\mathbf{A} \mapsto \mathbf{A}^{T}$ preserves the rank distance). We shall follow [29] and call the codes $\mathcal{C}_{k} \subseteq \mathbb{F}_{q}^{n \times m}(1 \leq k \leq m \leq n)$, as well as their transposes, Gabidulin codes.

For our construction of SCC-good random linear encoders we need the following property of Gabidulin codes.
Proposition 4.1: Suppose $\mathcal{C} \subseteq \mathbb{F}_{q}^{n \times m}$ is a Gabidulin code. Then $\{\mathbf{x A}: \mathbf{A} \in \mathcal{C}\}=\mathbb{F}_{q}^{m}$ for every $\mathbf{x} \in \mathbb{F}_{q}^{n} \backslash\left\{0^{n}\right\}$, and similarly $\left\{\mathbf{A y}^{T}: \mathbf{A} \in \mathcal{C}\right\}=\mathbb{F}_{q}^{n}$ for every $\mathbf{y} \in \mathbb{F}_{q}^{m} \backslash\left\{0^{m}\right\}$.

Theorem 4.2: Let $\mathcal{C} \subseteq \mathbb{F}_{q}^{n \times m}$ be a Gabidulin code and $F: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$ a random linear encoder uniformly distributed over $\mathcal{C}$. Then $F$ is SCC-good (before symmetrization).

Theorem 4.2 provides us with SCC-good random linear encoders $F: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$ of support size as small as $q^{n^{\prime}}=q^{\max \{n, m\}}$, realized by an $(n, m, 1)$ MRD code, which has minimum rank distance $d=m^{\prime}=\min \{n, m\}$ (the full-rank case). ${ }^{4}$

As an application of Theorem 4.2, the next example gives us some insights into the general form of a random linear encoder that is SCC-good before symmetrization.

[^4]Example 4.3: Let $m=n=q=2$ and $\mathbb{F}_{4}=\mathbb{F}_{2}[\alpha]$ with $\alpha^{2}=\alpha+1$. It is clear that $\{1, \alpha\}$ is a basis of $\mathbb{F}_{4}$. By definition of Gabidulin codes, we may consider the generator matrix $\mathbf{G}_{1}=\left(\begin{array}{ll}1 & \alpha\end{array}\right)$. Then the Gabidulin code $\mathcal{C}_{1}$ is

$$
\left\{\left(\begin{array}{ll}
0 & 0
\end{array}\right),\left(\begin{array}{ll}
1 & \alpha
\end{array}\right),(\alpha \quad 1+\alpha),(1+\alpha 1)\right\}
$$

or

$$
\left\{\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right),\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right),\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right),\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)\right\}
$$

Note that $\mathcal{C}_{1}$ is isomorphic to $\mathbb{F}_{4}$. By Theorem 4.2, the random linear encoder uniformly distributed over $\mathcal{C}_{1}$, which we also denote by $\mathcal{C}_{1}$, is SCC-good. Interchanging the columns of each matrix in $\mathcal{C}_{1}$ gives another SCC-good random linear encoder

$$
\mathcal{C}_{2}=\left\{\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right),\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right),\left(\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right),\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right)\right\} .
$$

The cosets of $\mathcal{C}_{1}$, such as $\mathcal{C}_{3}=\mathcal{C}_{1}+\left(\begin{array}{ll}0 & 1 \\ 0 & 1\end{array}\right)$, are also SCC-good. Note that $\mathcal{C}_{3}$ contains only one matrix of full rank. Finally, let $\mathcal{C}:=\mathcal{C}_{I}$, where $I$ is an arbitrary random variable taking values in $\mathcal{I}_{3}$. It is clear that $\mathcal{C}$ is SCC-good.

The proofs of this subsection are omitted since Proposition 4.1 and Theorem 4.2 are special cases of [30, Lemma 2.4 and Theorem 2.5], respectively.

## B. Constructing Good LSCEs Based on Good LSEs or LCEs

Condition (6) is a very strong condition, which in fact reflects the property of the alphabet. Combining this with the injectivity property of mappings, we say that an abelian group $\mathcal{X}$ is super good if there exists a sequence $\left\{F_{n}\right\}_{n=1}^{\infty}$ of SCC-good random linear encoders $F_{n}: \mathcal{X}^{n} \rightarrow \mathcal{X}^{n}$ such that

$$
\begin{equation*}
\inf _{n \in \mathbb{N}} \mathrm{P}\left\{\left|\operatorname{ker} F_{n}\right|=1\right\}>0 \tag{10}
\end{equation*}
$$

We shall now prove that a finite abelian group is super good if and only if it is elementary abelian, i.e. isomorphic to $\mathbb{Z}_{p}^{s}$ for some prime $p$ and some integer $s \geq 0$. The nontrivial elementary abelian $p$-groups are exactly the additive groups of the finite fields $\mathbb{F}_{q}$ with $q=p^{r}$. Hence choosing $\mathbb{F}_{q}$ as alphabets incurs no essential loss of generality, and we shall later on keep the assumption that the alphabet is $\mathbb{F}_{q}{ }^{5}$

The following fact [32] shows that all elementary abelian groups are super good.

$$
\begin{equation*}
\mathrm{P}\left\{\left|\operatorname{ker} F_{n, n}^{\mathrm{RLC}}\right|=1\right\}=\prod_{i=1}^{n}\left(1-q^{-i}\right)>K_{q}, \tag{11}
\end{equation*}
$$

where $K_{q}:=\prod_{i=1}^{\infty}\left(1-q^{-i}\right)>1-q^{-1}-q^{-2}$ by Euler's pentagonal number theorem [33].
The next theorem shows that, conversely, a super good finite abelian group is necessarily elementary abelian.
Theorem 4.4: For nontrivial abelian groups $\mathcal{X}$ and $\mathcal{Y}$, if there exists an SCC-good random linear encoder $F$ : $\mathcal{X}^{n} \rightarrow \mathcal{Y}^{m}$, then $\mathcal{X}$ and $\mathcal{Y}$ are elementary abelian $p$-groups for the same prime $p$ (provided that $m, n \geq 1$ ).

Next let us investigate the relation between conditions (6) and (10) for elementary abelian groups.
Theorem 4.5: Let $\mathcal{X}$ be an elementary abelian group of order $q=p^{r}$. Then for every SCC-good random linear encoder $F: \mathcal{X}^{n} \rightarrow \mathcal{X}^{n}$ the following bound holds:

$$
\mathrm{P}\{|\operatorname{ker} F|=1\} \geq \frac{p-2+q^{-n}}{p-1}
$$

An immediate consequence follows.
Corollary 4.6: Let $\mathcal{X}$ be an elementary abelian $p$-group for some prime $p>2$. Then there exists a positive constant $c(|\mathcal{X}|)$ (which may be taken as $c(|\mathcal{X}|)=1-\frac{1}{p-1}$ ) such that $\mathrm{P}\{|\operatorname{ker} F|=1\} \geq c(|\mathcal{X}|)$ for every SCC-good random linear encoder $F: \mathcal{X}^{n} \rightarrow \mathcal{X}^{n}$.

However, the conclusion of Corollary 4.6 does not hold for $p=2$.
Proposition 4.7: If $\mathcal{X}$ is an elementary abelian 2-group, there exists a sequence $\left\{F_{n}\right\}_{n=1}^{\infty}$ of SCC-good random linear encoders $F_{n}: \mathcal{X}^{n} \rightarrow \mathcal{X}^{n}$ such that $\lim _{n \rightarrow \infty} \mathrm{P}\left\{\left|\operatorname{ker} F_{n}\right|=1\right\}=0$.

[^5]With the preparations above, let us investigate the problem of constructing $\delta$-asymptotically good LSCEs which are SC-equivalent (resp., CC-equivalent) to given $\delta$-asymptotically good LSEs (resp., LCEs). The next theorem provides a method for constructing SC-equivalent or CC-equivalent linear encoders.

Theorem 4.8: Let $F: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$ be a random linear encoder. The linear encoder $G_{1}:=F_{m, m}^{\mathrm{RLC}} \circ F$ satisfies

$$
\begin{equation*}
\mathrm{P}\left\{\operatorname{ker} G_{1}=\operatorname{ker} F\right\}>K_{q} \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
\overline{\mathrm{S}}_{G_{1}}(Q \mid P) \leq \mathrm{S}_{\mathbb{F}_{q}^{m}}(Q)+1\left\{Q=P_{0^{m}}\right\} \overline{\mathrm{S}}_{F}\left(P_{0^{m}} \mid P\right) \quad \forall P \in \mathcal{P}_{n}^{*}, Q \in \mathcal{P}_{m} \tag{13}
\end{equation*}
$$

The linear encoder $G_{2}:=F \circ F_{n, n}^{\mathrm{RLC}}$ satisfies

$$
\begin{equation*}
\mathrm{P}\left\{G_{2}\left(\mathbb{F}_{q}^{n}\right)=F\left(\mathbb{F}_{q}^{n}\right)\right\}>K_{q} \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
\overline{\mathrm{S}}_{G_{2}}(Q \mid P)=\overline{\mathrm{S}}_{F\left(\mathbb{F}_{q}^{n}\right)}(Q) \quad \forall P \in \mathcal{P}_{n}^{*}, Q \in \mathcal{P}_{m} . \tag{15}
\end{equation*}
$$

Based on Theorem 4.8, we have the following answer to the problem:
Theorem 4.9: Let $\boldsymbol{f}=\left\{f_{k}\right\}_{k=1}^{\infty}$ be a sequence of linear encoders $f_{k}: \mathbb{F}_{q}^{n_{k}} \rightarrow \mathbb{F}_{q}^{m_{k}}$. If $\boldsymbol{f}$ is a sequence of $\delta$ asymptotically good LSEs such that $R_{c}(\boldsymbol{f})=\ln q$, then there exists a sequence $\left\{g_{1, k}\right\}_{k=1}^{\infty}$ of $\delta$-asymptotically good LSCEs $g_{1, k}: \mathbb{F}_{q}^{n_{k}} \rightarrow \mathbb{F}_{q}^{m_{k}}$ such that $g_{1, k}$ is SC-equivalent to $f_{k}$ for each $k \in \mathbb{N}$. Analogously, if $\boldsymbol{f}$ is a sequence of $\delta$-asymptotically good injective LCEs, then there exists a sequence $\left\{g_{2, k}\right\}_{k=1}^{\infty}$ of $\delta / R(\boldsymbol{f})$-asymptotically good injective LSCEs $g_{2, k}: \mathbb{F}_{q}^{n_{k}} \rightarrow \mathbb{F}_{q}^{m_{k}}$ such that $g_{2, k}$ is CC-equivalent to $f_{k}$ for each $k \in \mathbb{N}$.

Theorem 4.9 is a fundamental result, which not only claims the existence of SC-equivalent (or CC-equivalent) $\delta$-asymptotically good LSCEs but also paves the way for constructing such good LSCEs by concatenating rate- 1 linear encoders. Since rate-1 linear codes (e.g., the "accumulate" code) are frequently used to construct good LCEs (e.g., [19], [34], [35]), we believe that finding good rate-1 LSCEs is an issue deserving further consideration.

The proof of Corollary 4.6 are omitted, while the proofs of the other results are given in Appendix B.

## C. A General Scheme for Constructing Good LSCEs

Theorems 4.2 and 4.9 do give possible ways for constructing asymptotically good LSCEs. However, such constructions are somewhat difficult to implement in practice, because the generator matrices of $F_{n, n}^{\mathrm{RLC}}$ and random linear encoders derived from Gabidulin codes are not sparse. Thus, our next question is how to construct $\delta$-asymptotically good LSCEs based on sparse matrices so that known iterative decoding procedures have low complexity. For such purposes, in this subsection, we shall present a general scheme for constructing $\delta$-asymptotically good LSCEs.

Let $\boldsymbol{F}=\left\{F_{k}\right\}_{k=1}^{\infty}$ be a sequence of random linear encoders $F_{k}: \mathbb{F}_{q}^{n_{k}} \rightarrow \mathbb{F}_{q}^{m_{k}}$. We say that $\boldsymbol{F}$ is $\delta$-asymptotically SCC-good relative to the sequence $\left\{A_{k}\right\}_{k=1}^{\infty}$ of subsets $A_{k} \subseteq \mathcal{P}_{n_{k}}^{*}$ if

$$
\limsup _{k \rightarrow \infty} \max _{P \in A_{k}, Q \in \mathcal{P}_{m_{k}}} \frac{1}{n_{k}} \ln \alpha_{F_{k}}(P, Q) \leq \delta .
$$

Clearly, this is a generalization of $\delta$-asymptotically good LSCEs, and may be regarded as an approximate version of $\delta$-asymptotically good LSCEs when $A_{k}$ is a proper subset of $\mathcal{P}_{n_{k}}^{*}$. The next theorem shows that $\delta$-asymptotically good LSCEs based on these linear encoders may be constructed by serial concatenations.

Theorem 4.10: Let $\left\{G_{k}\right\}_{k=1}^{\infty}$ be a sequence of random linear encoders $G_{k}: \mathbb{F}_{q}^{m_{k}} \rightarrow \mathbb{F}_{q}^{l_{k}}$ that is $\delta$-asymptotically SCC-good relative to the sequence $\left\{A_{k}\right\}_{k=1}^{\infty}$ of sets $A_{k} \subseteq \mathcal{P}_{m_{k}}^{*}$. If there is a sequence $\boldsymbol{F}=\left\{F_{k}\right\}_{k=1}^{\infty}$ of random linear encoders $F_{k}: \mathbb{F}_{q}^{n_{k}} \rightarrow \mathbb{F}_{q}^{m_{k}}$ such that

$$
\begin{equation*}
F_{k}\left(\mathbb{F}_{q}^{n_{k}} \backslash\left\{0^{n_{k}}\right\}\right) \subseteq \bigcup_{P \in A_{k}} \mathcal{T}_{P}^{m_{k}}, \tag{16}
\end{equation*}
$$

then

$$
\limsup _{k \rightarrow \infty} \rho\left(G_{k} \circ \Sigma_{m_{k}} \circ F_{k}\right) \leq \frac{\delta}{\underline{R}(\boldsymbol{F})} .
$$

Remark 4.11: Take, for example,

$$
\begin{equation*}
A_{k}=\left\{P \in \mathcal{P}_{m_{k}}: P(0) \in[0,1-\gamma]\right\} \tag{17}
\end{equation*}
$$

for some $\gamma \in(0,1)$. Then Theorem 4.10 shows that we can construct asymptotically good LSCEs using a serial concatenation scheme, where the inner encoder is approximately $\delta$-asymptotically SCC-good and the outer code has large minimum distance. As we know (see [2], [25], [36], [37], etc.), there exist good LDPC codes over finite fields such that (16) is met for an appropriate $\gamma$, so the problem is reduced to finding a sequence of linear encoders that is $\delta$-asymptotically good relative to a sequence of sets such as (17). In the next section, we shall find such candidates in a family of encoders called LDGM encoders.

Since an injective linear encoder $F_{k}$ always satisfies condition (16) with $A_{k}=\mathcal{P}_{n_{k}}^{*}$, we immediately obtain the following corollary from Theorem 4.10.

Corollary 4.12: Let $\left\{G_{k}\right\}_{k=1}^{\infty}$ be a sequence of $\delta$-asymptotically good random LSCEs $G_{k}: \mathbb{F}_{q}^{m_{k}} \rightarrow \mathbb{F}_{q}^{l_{k}}$ and $\boldsymbol{F}=\left\{F_{k}\right\}_{k=1}^{\infty}$ a sequence of injective random linear encoders $F_{k}: \mathbb{F}_{q}^{n_{k}} \rightarrow \mathbb{F}_{q}^{m_{k}}$. Then

$$
\limsup _{k \rightarrow \infty} \rho\left(G_{k} \circ \Sigma_{m_{k}} \circ F_{k}\right) \leq \frac{\delta}{\underline{R}(\boldsymbol{F})} .
$$

In the same vein, we have:
Proposition 4.13: Let $\left\{F_{k}\right\}_{k=1}^{\infty}$ be a sequence of $\delta$-asymptotically good random LSCEs $F_{k}: \mathbb{F}_{q}^{n_{k}} \rightarrow \mathbb{F}_{q}^{m_{k}}$ and $\left\{G_{k}\right\}_{k=1}^{\infty}$ a sequence of surjective random linear encoders $G_{k}: \mathbb{F}_{q}^{m_{k}} \rightarrow \mathbb{F}_{q}^{l_{k}}$. Then

$$
\limsup _{k \rightarrow \infty} \rho\left(G_{k} \circ \Sigma_{m_{k}} \circ F_{k}\right) \leq \delta
$$

The above two results tell us that any linear encoder, if serially concatenated with an outer injective linear encoder or an inner surjective linear encoder, will not have worse performance in terms of condition (4). Maybe, if we are lucky, some linear encoders with better joint spectra can be constructed in this way. Note that a nonsingular rate-1 linear encoder is both injective and surjective, so adding rate-1 linear encoders into a serial concatenation scheme is never a bad idea for constructing good LSCEs. But certainly, the addition of rate-1 encoders may have a negative impact on the decoding performance.

The proof of Corollary 4.12 is omitted, while the proofs of the other results are given in Appendix C.

## V. An Explicit Construction Based on Sparse Matrices

In light of Remark 4.11, we proceed to investigate the joint spectra of regular low-density generator matrix (LDGM) encoders.

First we define three basic linear encoders: A single symbol repetition encoder $f_{c}^{\mathrm{REP}}: \mathbb{F}_{q} \rightarrow \mathbb{F}_{q}^{c}$ is given by $x \mapsto(x, \ldots, x)$, where $c \in \mathbb{N}$. A single symbol check encoder $f_{d}^{\text {CHK }}: \mathbb{F}_{q}^{d} \rightarrow \mathbb{F}_{q}$ is given by $\mathrm{x} \mapsto \sum_{i=1}^{d} x_{i}$, where $d \in \mathbb{N}$. The sum can be abbreviated as $\mathbf{x}_{\oplus}$, and in the sequel, we shall use this kind of abbreviation to denote the sum of all components of a sequence. A random multiplier encoder $F^{\mathrm{RM}}: \mathbb{F}_{q} \rightarrow \mathbb{F}_{q}$ is given by $x \mapsto C x$ where $C$ is uniformly distributed over $\mathbb{F}_{q}^{\times}$.

Let $c, n$, and $d$ be positive integers such that $d$ divides $c n$. A random regular LDGM encoder $F_{c, d, n}^{\mathrm{LD}}: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{c n / d}$ is defined by

$$
\begin{equation*}
F_{c, d, n}^{\mathrm{LD}}:=f_{d, c n}^{\mathrm{CHK}} \circ F_{c n}^{\mathrm{RM}} \circ \Sigma_{c n} \circ f_{c, n}^{\mathrm{REP}}, \tag{18}
\end{equation*}
$$

where $f_{c, n}^{\mathrm{REP}}:=\bigodot_{i=1}^{n} f_{c}^{\mathrm{REP}}, f_{d, n}^{\mathrm{CHK}}:=\bigodot_{i=1}^{n} f_{d}^{\mathrm{CHK}}, F_{n}^{\mathrm{RM}}:=\bigodot_{i=1}^{n} F^{\mathrm{RM}}$.
To calculate the joint spectrum of $F_{c, d, n}^{\mathrm{LD}}$, we first need to calculate the joint spectra of its constituent encoders. We note that definition (18) can be rewritten as

$$
\begin{aligned}
F_{c, d, n}^{\mathrm{LD}} & \stackrel{\mathrm{~d}}{=} F_{d, c n / d}^{\mathrm{CHK}} \circ \Sigma_{c n} \circ f_{c, n}^{\mathrm{REP}} \\
& \stackrel{\mathrm{~d}}{=} f_{d, c n}^{c H K} \circ \Sigma_{c n} \circ F_{c, n}^{\mathrm{REP}},
\end{aligned}
$$

where the symbol $\stackrel{\mathrm{d}}{=}$ means that the random elements at both sides have the same probability distribution, and $F_{c, n}^{\mathrm{REP}}:=F_{c n}^{\mathrm{RM}} \circ f_{c, n}^{\mathrm{REP}} \circ F_{n}^{\mathrm{RM}}, F_{d, n}^{\mathrm{CHK}}:=F_{n}^{\mathrm{RM}} \circ f_{d, n}^{\mathrm{CHK}} \circ F_{d n}^{\mathrm{RM}}$. Thus it suffices to calculate the joint spectra of $f_{c, n}^{\mathrm{REP}}$ and $F_{d, n}^{\mathrm{CHK}}$.

Proposition 5.1:

$$
\begin{equation*}
\mathcal{G}_{f_{c}^{\mathrm{REP}}}(\mathbf{u}, \mathbf{v})=\frac{1}{q} \sum_{a \in \mathbb{F}_{q}} u_{a} v_{a}^{c}, \tag{19}
\end{equation*}
$$

$$
\begin{gather*}
\overline{\mathcal{G}}_{F_{c}^{\mathrm{REP}}}(\mathbf{u}, \mathbf{v})=\frac{1}{q} u_{0} v_{0}^{c}+\frac{1}{q}\left(\mathbf{u}_{\oplus}-u_{0}\right)\left(\frac{\mathbf{v}_{\oplus}-v_{0}}{q-1}\right)^{c},  \tag{20}\\
\mathcal{G}_{f_{c, n}^{\mathrm{REP}}}(\mathbf{u}, \mathbf{v})=\sum_{P \in \mathcal{P}_{n}} \mathrm{~S}_{\mathbb{F}_{q}^{n}}(P) \mathbf{u}^{n P} \mathbf{v}^{n c P}  \tag{21}\\
\mathrm{~S}_{f_{c, n}^{\mathrm{REPP}}}(P, Q)=\mathrm{S}_{\mathbb{F}_{q}^{n}}(P) 1\{P=Q\}  \tag{22}\\
\mathrm{S}_{f_{c, n}^{\mathrm{REP}}}(Q \mid P)=1\{Q=P\} \tag{23}
\end{gather*}
$$

## Proposition 5.2:

$$
\begin{gather*}
\overline{\mathcal{G}}_{F_{d}^{\text {снк }}}(\mathbf{u}, \mathbf{v})=\frac{1}{q^{d+1}}\left[\left(\mathbf{u}_{\oplus}\right)^{d} \mathbf{v}_{\oplus}+\left(\frac{q u_{0}-\mathbf{u}_{\oplus}}{q-1}\right)^{d}\left(q v_{0}-\mathbf{v}_{\oplus}\right)\right],  \tag{24}\\
\overline{\mathrm{S}}_{F_{d, n}^{\text {cнк }}}(P, Q)=\left[\mathbf{u}^{d n P}\right]\left(g_{d, n}^{(1)}(\mathbf{u}, Q)\right),  \tag{25}\\
\overline{\mathrm{S}}_{F_{d, n}^{c н н}}(P, Q) \leq g_{d, n}^{(2)}(O, P, Q) \quad \forall O \in \mathcal{P}_{d n} \text { with } P \ll O,  \tag{26}\\
\frac{1}{n} \ln \alpha_{F_{d, n}^{c н к}}(P, Q) \leq \delta_{d}(P(0), Q(0))+d \Delta_{d n}(P), \tag{27}
\end{gather*}
$$

where $\left[\mathbf{u}^{\mathbf{n}}\right](f)$ denotes the coefficient of monomial $\mathbf{u}^{\mathbf{n}}$ in the polynomial $f$,

$$
\begin{align*}
g_{d, n}^{(1)}(\mathbf{u}, Q):= & \frac{\binom{n}{n Q}}{q^{n(d+1)}}\left[\left(\mathbf{u}_{\oplus}\right)^{d}+(q-1)\left(\frac{q u_{0}-\mathbf{u}_{\oplus}}{q-1}\right)^{d}\right]^{n Q(0)} \\
\times & {\left[\left(\mathbf{u}_{\oplus}\right)^{d}-\left(\frac{q u_{0}-\mathbf{u}_{\oplus}}{q-1}\right)^{d}\right]^{n(1-Q(0))}, }  \tag{28}\\
g_{d, n}^{(2)}(O, P, Q):= & \frac{\binom{n}{n Q}}{q^{n(d+1)} O^{d n P}}\left[1+(q-1)\left(\frac{q O(0)-1}{q-1}\right)^{d}\right]^{n Q(0)} \\
\times & {\left[1-\left(\frac{q O(0)-1}{q-1}\right)^{d}\right]^{n(1-Q(0))}, }  \tag{29}\\
\delta_{d}(x, y):= & \inf _{0<\hat{x}<1} \delta_{d}(x, \hat{x}, y),  \tag{30}\\
\delta_{d}(x, \hat{x}, y):= & d D(x \| \hat{x})+J_{d}(\hat{x}, y),  \tag{31}\\
D(x \| y):= & D((x, 1-x) \|(y, 1-y)), \\
J_{d}(x, y):= & y \ln \left[1+(q-1)\left(\frac{q x-1}{q-1}\right)^{d}\right] \\
& +(1-y) \ln \left[1-\left(\frac{q x-1}{q-1}\right)^{d}\right]  \tag{32}\\
\Delta_{n}(P):= & H(P)-\frac{1}{n} \ln \binom{n}{n P} . \tag{33}
\end{align*}
$$

In the formulas above, we have used the notion of spectrum generating function, a tool to be introduced in Section VI. From Propositions 5.1 and 5.2 we obtain a tight upper bound on the joint-spectrum performance of $F_{c, d, n}^{\mathrm{LD}} .{ }^{6}$

[^6]Theorem 5.3:

$$
\begin{equation*}
\overline{\mathrm{S}}_{F_{c, d, n} \mathrm{D}}(Q \mid P)=\overline{\mathrm{S}}_{F_{d, c n / d}^{c h \mu / d}}(Q \mid P) \tag{34}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{n} \ln \alpha_{F_{c,,, n}^{\mathrm{LD}}}(P, Q) \leq \frac{c}{d} \delta_{d}(P(0), Q(0))+c \Delta_{c n}(P) \tag{35}
\end{equation*}
$$

where $\delta_{d}(x, y)$ is defined by (30) and $\Delta_{n}(P)$ is defined by (33).
Our next theorem, which is based on the results above, shows that for any $\delta>0$, regular LDGM encoders with $d$ large enough are approximately $\delta$-asymptotically SCC-good.

Theorem 5.4: Let $\mathcal{N}$ consist of all positive integers $n$ such that $d$ divides $c n$. Let $\left\{F_{c, d, n}^{\mathrm{LD}}\right\}_{n \in \mathcal{N}}$ be a sequence of random regular LDGM encoders whose coding rate is $r_{0}=d / c$. Let $\left\{A_{n}\right\}_{n \in \mathcal{N}}$ be a sequence of sets $A_{n} \subseteq \mathcal{P}_{n}^{*}$. Then

$$
\limsup _{n \in \mathcal{N}: n \rightarrow \infty} \max _{P \in A_{n}, Q \in \mathcal{P}_{c n / d}} \frac{1}{n} \ln \alpha_{F_{c, d, n}^{\llcorner } \stackrel{ }{\circ}(P, Q) \leq \rho_{0}, ~}^{\text {, }}
$$

where

$$
\rho_{0}:=\lim _{n \in \mathcal{N}: n \rightarrow \infty} \max _{P \in A_{n}} \frac{1}{r_{0}} \ln \left[1+(q-1)\left|\frac{q P(0)-1}{q-1}\right|^{d}\right] .
$$

If

$$
A_{n}=\left\{P \in \mathcal{P}_{n}: P(0) \in\left[\frac{1}{q}-\gamma_{1}, \frac{1}{q}+\gamma_{2}\right]\right\}
$$

where $\gamma_{1} \in(0,1 / q] \backslash\left\{\frac{1}{2}\right\}$ and $\gamma_{2} \in(0,1-1 / q)$, then

$$
\rho_{0}=\frac{1}{r_{0}} \ln \left[1+(q-1)\left(\frac{q \gamma}{q-1}\right)^{d}\right]
$$

where $\gamma:=\max \left\{\gamma_{1}, \gamma_{2}\right\}$. For any $\delta>0$, define

$$
d_{0}(\gamma, \delta):=\left\lceil\frac{\ln \left[\left(e^{r_{0} \delta}-1\right) /(q-1)\right]}{\ln [q \gamma /(q-1)]}\right\rceil .
$$

Then we have $\rho_{0} \leq \delta$ for all $d \geq d_{0}(\gamma, \delta)$.
Theorem 5.4 together with Theorem 4.10 and Remark 4.11 shows that, for any $\delta>0$, we can construct $\delta$ asymptotically good LSCEs by serially concatenating an inner LDGM encoder and an outer encoder of a linear code with large minimum distance. In particular, we may use encoders of LDPC codes as outer encoders. Furthermore, Proposition 3.7 shows that we can find a sequence of asymptotically good LSCEs in a family of sequences of $\delta_{i^{-}}$asymptotically good LSCEs, where $\delta_{i}$ is decreasing in $i$ and converges to zero as $i \rightarrow \infty$. An analogous construction was proposed by Hsu in his Ph.D. dissertation [39], but his purpose was only to find good channel codes and only a rate-1 LDGM encoder was employed as an inner encoder in his construction. A similar construction was proposed by Wainwright and Martinian [40], who proved that such a construction is optimal for channel coding or lossy source coding with side information.

The next example shows how to determine the parameters of the inner LDGM encoder when designing such encoders.

Example 5.5: Let $\delta=0.05$ and $f_{n}: \mathbb{F}_{2}^{n} \rightarrow \mathbb{F}_{2}^{5 n}$ an injective linear encoder. Suppose that the normalized weight of all nonzero codewords of $f_{n}\left(\mathbb{F}_{2}^{n}\right)$ ranges from 0.05 to 0.95 . We shall design a linear encoder $H_{n}: \mathbb{F}_{2}^{n} \rightarrow \mathbb{F}_{2}^{2 n}$ that is $\delta$-asymptotically SCC-good. Let $H_{n}=G_{n} \circ \Sigma_{5 n} \circ f_{n}$ where $G_{n}$ is a random regular LDGM encoder over $\mathbb{F}_{2}$. It is clear that the coding rate of $G_{n}$ must be $\frac{5}{2}$. Using Theorem 5.4 with $q=2, r_{0}=\frac{5}{2}$, and $\gamma_{1}=\gamma_{2}=0.45$, we have $d_{0}\left(\gamma, \delta R\left(f_{n}\right)\right)=35$. Then we may choose LDGM encoders with $c=14$ and $d=35$, so that $\rho_{0} \leq 0.01$, and therefore $\limsup \sin _{n \rightarrow \infty} \rho\left(H_{n}\right) \leq \rho_{0} / R\left(f_{n}\right)=0.05$.

Remark 5.6: Although Theorem 5.4, as well as Example 5.5, shows that LDGM encoders with $d$ large enough are good candidates for inner encoders, this conclusion may not be true for nonideal decoding algorithms. It is well known that the minimum distance of a typical regular LDPC code increases with the column weight of its defining parity-check matrix (e.g., [37]). However, in practice, its performance under the belief propagation (BP) decoding algorithm decreases as the column weight increases (e.g., [41]). For this reason researchers tend to employ
irregular LDPC codes in order to achieve better performance. To some extent, the design of irregular LDPC codes is a compromise between ideal decoding performance and iterative decoding convergence. Similarly, we cannot expect a boost in performance by simply increasing the parameter $d$ of a regular LDGM encoder, and we may also need to consider irregular LDGM encoders, i.e., a class of sparse generator matrices whose row or column weights are not uniform. In this direction some significant work has been done in practice. One example is the "LDGM code" in [42], which is defined as a serial concatenation of two systematic LDGM codes, i.e., two irregular LDGM encoders. These two codes share the same systematic bits, so that the BP decoding algorithm converges easily. It has been shown by simulation that this kind of encoder is good for lossless JSCC [12].

So far, we have presented two families of good linear encoders, one based on Gabidulin codes (Section IV-A) and the other based on LDGM encoders. A comparison between these two families seems necessary. At first glance, it seems that the family based on Gabidulin codes is better than the family based on LDGM codes, since the former is SCC-good while the latter is only asymptotically SCC-good. However, this is not the whole truth, because there is no single linear encoder that is SCC-good. The proposition that follows makes this fact precise. Consequently, in terms of code spectrum, the two families have almost the same performance. On the other hand, in terms of decoding complexities, the family based on LDGM encoders is more competitive, since LDGM encoders as well as LDPC codes are characterized by sparse matrices so that low-complexity iterative decoding algorithms can be employed.

Proposition 5.7: For any linear encoder $f: \mathcal{X}^{n} \rightarrow \mathcal{Y}^{m}$ with $|\mathcal{X}| \geq 2$,

$$
\begin{aligned}
\max _{\substack{P \in \mathcal{P}_{n}^{*}(\mathcal{X}) \\
Q \in \mathcal{P}_{m}(\mathcal{Y})}} \alpha_{f}(P, Q) & \geq \frac{|\mathcal{Y}|^{m}}{\max _{Q \in \mathcal{P}_{m}(\mathcal{Y})}\binom{m}{m Q}} \\
& =\Theta\left(m^{\frac{|\mathcal{Y}|-1}{2}}\right) .
\end{aligned}
$$

The proofs of this section are given in Appendix D.

## VI. Advanced Toolbox of the Code-Spectrum Approach

In this section, we introduce some advanced tools required to prove the results in Sections III-V. In particular, we shall establish tools for serial and parallel concatenations of linear encoders and the MacWilliams identities on the duals of linear encoders. These results are not new in nature, but serve our purpose of providing a concise mathematical treatment and completing the code-spectrum approach. Their proofs are left to the reader as exercises, or can be found in [10].

## A. Spectra with Coordinate Partitions

In this subsection, we introduce a generalization of spectra, viz., spectra of sets with coordinate partitions.
Let $A$ be a subset of $\mathbb{F}_{q}^{n}$, with coordinate set $\mathcal{I}_{n}$. Given a partition $\mathcal{U}$ of $\mathcal{I}_{n}$, we define the $\mathcal{U}$-type $P_{\mathbf{x}}^{\mathcal{U}}$ of $\mathbf{x} \in \mathbb{F}_{q}^{n}$ as

$$
P_{\mathbf{x}}^{\mathcal{U}}=\left(P_{\mathbf{x}}^{U}\right)_{U \in \mathcal{U}}:=\left(P_{x_{U}}\right)_{U \in \mathcal{U}} .
$$

By $\mathcal{P}_{\mathcal{U}}$ we mean the set of all $\mathcal{U}$-types of vectors in $\mathbb{F}_{q}^{n}$, so that $\mathcal{P}_{\mathcal{U}}=\prod_{U \in \mathcal{U}} \mathcal{P}_{|U|}$. A $\mathcal{U}$-type in $\mathcal{P}_{\mathcal{U}}$ is written in the form $P^{\mathcal{U}}:=\left(P^{U}\right)_{U \in \mathcal{U}}$. For a $\mathcal{U}$-type $P^{\mathcal{U}}$, the set of vectors of $\mathcal{U}$-type $P^{\mathcal{U}}$ in $\mathbb{F}_{q}^{n}$ is denoted by $\mathcal{T}_{P^{U}}$. In the sequel, when given $P_{\mathbf{x}}^{\mathcal{U}}$ or $P^{\mathcal{U}}$, we shall slightly abuse the notations $P_{\mathbf{x}}^{\mathcal{V}}$ and $P^{\mathcal{V}}$ for any subset $\mathcal{V}$ of $\mathcal{U}$ to represent part of their components.

Based on the $\mathcal{U}$-type, we define the $\mathcal{U}$-spectrum $\mathrm{S}_{\mathbb{F}_{q}^{u}}(A)$ of a nonempty set $A \subseteq \mathbb{F}_{q}^{n}$ as the empirical distribution of $\mathcal{U}$-types of sequences in $A$, i.e.,

$$
\mathrm{S}_{\mathbb{F}_{q}^{u}}(A)\left(P^{\mathcal{U}}\right):=\frac{\left|\left\{\mathbf{x} \in A: P_{\mathbf{x}}^{\mathcal{U}}=P^{\mathcal{U}}\right\}\right|}{|A|} .
$$

The $\mathcal{U}$-spectrum is in fact a variant of the joint spectrum. ${ }^{7}$ When $\mathcal{U}=\left\{\mathcal{I}_{n}\right\}$, it reduces to the ordinary spectrum. Otherwise it provides more information than the ordinary spectrum, and in the extreme case $\mathcal{U}=\{\{1\},\{2\}, \ldots,\{n\}\}$

[^7]the spectrum $\mathrm{S}_{\mathbb{F}_{q}^{u}}(A)$ determines $A$ uniquely. Since $P^{\mathcal{U}}$ clearly refers to an element of $\mathcal{P}_{\mathcal{U}}$, we sometimes write $\mathrm{S}(A)\left(P^{\mathcal{U}}\right)$ or further $\mathrm{S}_{A}\left(P^{\mathcal{U}}\right)$ in place of $\mathrm{S}_{\mathbb{F}_{q}}(A)\left(\mathcal{P}^{\mathcal{U}}\right)$ for convenience.

Example 6.1: Consider the linear code of Example E. 1 with coordinate partition $\{\{1,2,3\},\{4,5,6,7\}\}$. Its spectrum, written as a two-dimensional array, is

| $w_{1} \backslash w_{2}$ | 0 | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $\frac{1}{16}$ |  |  | $\frac{1}{16}$ |  |
| 1 |  |  | $\frac{3}{16}$ | $\frac{3}{16}$ |  |
| 2 |  | $\frac{3}{16}$ | $\frac{3}{16}$ |  |  |
| 3 |  | $\frac{1}{16}$ |  |  | $\frac{1}{16}$ |

where $w_{1}$ and $w_{2}$ correspond to $\{1,2,3\}$ and $\{4,5,6,7\}$, respectively.
Analogous to ordinary spectra, we further define the marginal and conditional spectra with respect to a proper subset $\mathcal{V}$ of $\mathcal{U}$, denoting them by $\mathrm{S}_{\mathbb{F}_{q}^{\nu}}(A)\left(P^{\mathcal{V}}\right)$ and $\mathrm{S}_{\mathbb{F}_{q}^{u / \nu} \mathbb{F}_{q}^{\mathcal{V}}}(A)\left(P^{\mathcal{U}} \backslash \mathcal{V} \mid P^{\mathcal{V}}\right)$, respectively. We also define the $(\mathcal{U}, \mathcal{V})$ spectrum of a map $f: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$ as $\mathrm{S}_{\mathbb{F}_{q} \mathbb{F}_{q}^{\mathcal{\nu}}}(\operatorname{rl}(f))$ where $\mathcal{U}^{q}$ and $\mathcal{V}$ are partitions of $\mathcal{I}_{n}$ and $\mathcal{I}_{m}$, respectively.

For ease of notation, when we explicitly write $A \subseteq \prod_{i=1}^{s} \mathbb{F}_{q}^{n_{i}}$ with $\sum_{i=1}^{s} n_{i}=n$, we tacitly assume that the default coordinate partition is

$$
\begin{aligned}
\mathcal{U}_{0} & :=\left\{U_{1}, U_{2}, \ldots, U_{s}\right\} \\
& =\left\{\left\{1, \ldots, n_{1}\right\}, \ldots,\left\{\sum_{i=1}^{s-1} n_{i}+1, \ldots, n\right\}\right\} .
\end{aligned}
$$

Thus the default spectrum of $A$ is $\mathrm{S}_{\mathbb{R}_{q}}^{u_{0}}(A)$ and is denoted by $\mathrm{S}_{\mathbb{F}_{q}^{s}}(A)$ (or $\mathrm{S}(A)$ for $s=1$ ).
To further explore the properties of $\mathcal{U}$-spectra, we first take a closer look at the $\mathcal{U}$-type. Recall that any two vectors $\mathbf{x}, \mathbf{x}^{\prime} \in \mathbb{F}_{q}^{n}$ have the same type if and only if $\sigma(\mathbf{x})=\mathbf{x}^{\prime}$ for some $\sigma \in S_{n}$. Since the type is a special case of the $\mathcal{U}$-type, it is natural to ask which permutations in $\mathrm{S}_{n}$ preserve the $\mathcal{U}$-type. A moment's thought shows that the $\mathcal{U}$-type is preserved by any permutation that maps each member of $\mathcal{U}$ onto itself. We denote the set of all such permutations by $\mathrm{S}_{\mathcal{U}}$, which forms a subgroup of $\mathrm{S}_{n}$ isomorphic to $\prod_{U \in \mathcal{U}} \mathrm{~S}_{|U|}$. Now considering a random permutation uniformly distributed over $S_{\mathcal{U}}$, we obtain a generalization of $\Sigma_{n}$, which is denoted by $\Sigma_{\mathcal{U}}$ and is called a uniform random permutation with respect to $\mathcal{U}$. We are now ready to state a fundamental result about $\mathcal{U}$-spectra.

Proposition 6.2: Let $\mathcal{U}$ be a partition of $\mathcal{I}_{n}$. For any $\mathrm{x} \in \mathbb{F}_{q}^{n}$ and any random nonempty set $A \subseteq \mathbb{F}_{q}^{n}$,

$$
\begin{equation*}
\mathrm{E}\left[\frac{1\left\{\mathbf{x} \in \Sigma_{\mathcal{U}}(A)\right\}}{|A|}\right]=q^{-n} \alpha_{A}\left(P_{\mathbf{x}}^{\mathcal{U}}\right) \tag{36}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha_{A}\left(P^{\mathcal{U}}\right):=\frac{\bar{S}_{A}\left(P^{\mathcal{U}}\right)}{\mathrm{S}_{\mathbb{F}_{q}^{n}}\left(P^{\mathcal{U}}\right)} . \tag{37}
\end{equation*}
$$

Moreover, for any proper subset $\mathcal{V}$ of $\mathcal{U}$ and any $Q^{\mathcal{U}} \in \mathcal{P} \mathcal{U}$ with $Q^{\mathcal{V}}=P_{\mathbf{x}}^{\mathcal{V}}$, we have

$$
\begin{equation*}
\mathrm{E}\left[\frac{\left|B \cap \Sigma_{\mathcal{U}}(A)\right|}{|A|}\right]=\frac{\overline{\mathrm{S}}_{A}\left(Q^{\mathcal{U}}\right)}{\prod_{V \in \mathcal{V}} q^{V \mid \mathrm{S}_{\mathbb{F}_{q}^{|V|}}\left(Q^{V}\right)}}, \tag{38}
\end{equation*}
$$

where $B:=\left\{\mathbf{y} \in \mathcal{T}_{Q^{u}}: y_{V}=x_{V}\right.$ for all $\left.V \in \mathcal{V}\right\}$.
Remark 6.3: Identity (36) can be rewritten as

$$
\mathrm{P}\left\{\mathbf{x} \in \Sigma_{\mathcal{U}}(A)\right\}=q^{-n}|A| \alpha_{A}\left(P_{\mathbf{x}}^{\mathcal{U}}\right)
$$

whenever $|A|$ is a constant. For a random function $F: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$, if we let $A=\operatorname{rl}(F) \subseteq \mathbb{F}_{q}^{n} \times \mathbb{F}_{q}^{m}$ (which implies the default partition $\left\{\mathcal{I}_{n}, \mathcal{I}_{m}^{\prime}\right\}^{8}$ ) and note that $|A|=q^{n}$, then [9, Proposition 2.4] (for $\mathcal{X}=\mathcal{Y}=\mathbb{F}_{q}$ ) follows as a special case. In general, for a random function $F: \prod_{i=1}^{s} \mathbb{F}_{q}^{n_{i}} \rightarrow \prod_{i=1}^{t} \mathbb{F}_{q}^{m_{i}}$, we may consider the default coordinate

[^8]partitions $\mathcal{U}_{0}$ and $\mathcal{V}_{0}$ of its domain and range, respectively, and define $\tilde{F}:=\Sigma_{\mathcal{V}_{0}} \circ F \circ \Sigma_{\mathcal{U}_{0}}$. Then Proposition 6.2 yields a generalization of [9, Proposition 2.4], that is,
$$
\mathrm{P}\{\tilde{F}(\mathbf{x})=\mathbf{y}\}=q^{-m} \alpha_{F}\left(P_{\mathbf{x}}^{\mathcal{U}_{0}}, P_{\mathbf{y}}^{\mathcal{V}_{0}}\right),
$$
where $m:=\sum_{i=1}^{t} m_{i}$ and $\alpha_{F}\left(P^{\mathcal{U}_{0}}, Q^{\mathcal{V}_{0}}\right):=\alpha_{\mathrm{rl}(F)}\left(P^{\mathcal{U}_{0}}, Q^{\mathcal{V}_{0}}\right)$.

## B. Encoders and Conditional Probability Distributions

In this subsection, we show that any encoder may be regarded as a conditional probability distribution. Such a viewpoint is very helpful in calculating the spectrum of a complex encoder composed of many simple encoders.
Proposition 6.4: For any random function $F: \prod_{i=1}^{s} \mathbb{F}_{q}^{n_{i}} \rightarrow \prod_{i=1}^{t} \mathbb{F}_{q}^{m_{i}}$,

$$
\begin{equation*}
\mathrm{P}\left\{F^{\sim}(\mathbf{x}) \in \mathcal{T}_{Q^{\nu_{0}}}\right\}=\overline{\mathrm{S}}_{F}\left(Q^{\mathcal{V}_{0}} \mid P_{\mathbf{x}}^{\mathcal{U}_{0}}\right) \tag{39}
\end{equation*}
$$

for all $\mathrm{x} \in \prod_{i=1}^{s} \mathbb{F}_{q}^{n_{i}}$ and $Q^{\mathcal{V}_{0}} \in \mathcal{P}_{\mathcal{V}_{0}}$, where $F^{\sim}:=F \circ \Sigma_{\mathcal{U}_{0}}$, and $\mathcal{U}_{0}$ and $\mathcal{V}_{0}$ are the default coordinate partitions.
Remark 6.5: Identity (39) can also be rewritten as

$$
\mathrm{P}\left\{F^{\sim}(\mathbf{x}) \in \mathcal{T}_{Q^{\nu_{0}}} \mid \mathbf{x} \in \mathcal{T}_{P u_{0}}\right\}=\overline{\mathrm{S}}_{F}\left(Q^{\mathcal{V}_{0}} \mid P^{\mathcal{U}_{0}}\right)
$$

which clearly indicates that the average forward conditional spectrum $\overline{\mathrm{S}}_{F}\left(Q^{\mathcal{V}_{0}} \mid P^{\mathcal{U}_{0}}\right)$ may be regarded as the transition probability from $P^{\mathcal{U}_{0}}$ to $Q^{\mathcal{V}_{0}}$ under $F^{\sim}$. This fundamental observation implies that coding modules like $F^{\sim}$ or $\tilde{F}$ (instead of $F$ ) should be regarded as basic units in a coding system, and that the serial concatenation of such units may behave like the serial concatenation of conditional probability distributions. The following proposition proves this speculation.

Proposition 6.6: For any two random functions $F: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$ and $G: \mathbb{F}_{q}^{m} \rightarrow \mathbb{F}_{q}^{l}$,

$$
\overline{\mathrm{S}}_{G \circ \Sigma_{m} \circ F}(Q \mid O)=\sum_{P \in \mathcal{P}_{m}} \overline{\mathrm{~S}}_{F}(P \mid O) \overline{\mathrm{S}}_{G}(Q \mid P),
$$

where $O \in \mathcal{P}_{n}$ and $Q \in \mathcal{P}_{l}$.

## C. Spectrum Generating Functions

In Section VI-B, we introduced a method for calculating the spectra of serial concatenations of encoders. In this subsection, we proceed to investigate another important combination of encoders, viz. parallel concatenations. To cope with problems involving concatenations (cartesian products) of sequences, we shall introduce the approach of spectrum generating functions.

At first, we need some additional terminology for partitions to simplify the treatment of spectrum generating functions. Associated with any partition $\mathcal{U}$ of a set $S$ is the mapping $\pi_{\mathcal{U}}: S \rightarrow \mathcal{U}$ that maps $s \in S$ to the member of $\mathcal{U}$ containing $s$. A partition $\mathcal{V}$ of $S$ is a refinement of $\mathcal{U}$ if and only if there is a (unique) mapping $\psi: \mathcal{V} \rightarrow \mathcal{U}$ such that $\pi_{\mathcal{U}}=\psi \circ \pi_{\mathcal{V}}$.

Now let $A$ be a nonempty subset of $\mathbb{F}_{q}^{n}$ and $\mathcal{U}$ a partition of $\mathcal{I}_{n}$. The $\mathcal{U}$-spectrum generating function of $A$ is a polynomial in $q|\mathcal{U}|$ indeterminates, whose coefficients form the $\mathcal{U}$-spectrum of $A$. As an element of $\mathbb{C}\left[u_{U, a} ; U \in\right.$ $\left.\mathcal{U}, a \in \mathbb{F}_{q}\right]$ (which denotes the ring of polynomials in the indeterminates $u_{U, a}$ and with coefficients in $\mathbb{C}$ ), it can be defined as

$$
\begin{aligned}
\mathcal{G}_{\mathbb{F}_{q}^{u}}(A)\left(\mathbf{u}_{\mathcal{U}}\right): & =\frac{1}{|A|} \sum_{\mathbf{x} \in A} \prod_{i=1}^{n} u_{\pi_{\mathcal{U}}(i), x_{i}} \\
& =\sum_{P^{u} \in \mathcal{P}_{\mathcal{U}}}\left(\mathrm{S}_{A}\left(P^{\mathcal{U}}\right) \prod_{U \in \mathcal{U}} \prod_{a \in \mathbb{F}_{q}} u_{U, a}^{|U| P^{U}(a)}\right),
\end{aligned}
$$

where $\mathbf{u}_{\mathcal{U}}:=\left(\mathbf{u}_{U}\right)_{U \in \mathcal{U}}=\left(u_{U, a}\right)_{U \in \mathcal{U}, a \in \mathbb{F}_{q}}$. For convenience, we sometimes write $\mathcal{G}_{\mathbb{F}_{q}}(A)$ or $\mathcal{G}(A)\left(\mathbf{u}_{\mathcal{U}}\right)$, or further $\mathcal{G}_{A}\left(\mathbf{u}_{\mathcal{U}}\right)$ (since $\mathcal{U}$ conveys all necessary information), and write $\mathbf{u}^{\mathbf{v}}$ in place of $\prod_{i \in \mathcal{I}} u_{i}^{v_{i}}$ for any sequences $\mathbf{u}, \mathbf{v}$ with the same coordinate set $\mathcal{I}$ (whenever the product makes sense). Thus the product $\prod_{a \in \mathbb{F}_{q}} u_{U, a}^{|U| P^{U}(a)}$ is rewritten as $\mathbf{u}_{U}^{|U| P^{U}}$. As is done for $\mathcal{U}$-spectra, we write $\mathcal{G}_{\mathbb{F}_{q}^{s}}(A)$ (or $\mathcal{G}(A)$ for $s=1$ ) when $\mathcal{U}$ is the default coordinate partition.

Example 6.7: Let us compute the spectrum generating function of the linear encoder in Example E.1. Its input coordinate set is $\{1,2,3,4\}$ and its output coordinate set is $\{1,2,3,4,5,6,7\}$, so its spectrum generating function with respect to the default coordinate partition is

$$
\begin{equation*}
\frac{1}{16}\left[u_{0}^{4} v_{0}^{7}+u_{0}^{3} u_{1}\left(3 v_{0}^{4} v_{1}^{3}+v_{0}^{3} v_{1}^{4}\right)+u_{0}^{2} u_{1}^{2}\left(2 v_{0}^{4} v_{1}^{3}+3 v_{0}^{3} v_{1}^{4}+v_{1}^{7}\right)+u_{0} u_{1}^{3}\left(v_{0}^{4} v_{1}^{3}+3 v_{0}^{3} v_{1}^{4}\right)+u_{1}^{4} v_{0}^{4} v_{1}^{3}\right] \tag{40}
\end{equation*}
$$

where $u_{i}$ and $v_{i}(i=0,1)$ correspond to the symbol $i$ in the input and output alphabets, respectively. If we replace the default output partition by

$$
\{A, B\}=\{\{1,2,3\},\{4,5,6,7\}\} \quad(\text { cf. Example 6.1) }
$$

then we obtain

$$
\begin{align*}
& \frac{1}{16}\left\{u_{0}^{4} a_{0}^{3} b_{0}^{4}+u_{0}^{3} u_{1}\left[a_{0}^{2} a_{1} b_{0}^{2} b_{1}^{2}+a_{0} a_{1}^{2}\left(2 b_{0}^{3} b_{1}+b_{0}^{2} b_{1}^{2}\right)\right]\right. \\
& +u_{0}^{2} u_{1}^{2}\left[\left(a_{0}^{3}+a_{0}^{2} a_{1}\right) b_{0} b_{1}^{3}+a_{0} a_{1}^{2}\left(b_{0}^{3} b_{1}+b_{0}^{2} b_{1}^{2}\right)+a_{1}^{3}\left(b_{0}^{3} b_{1}+b_{1}^{4}\right)\right] \\
& \left.\left.+u_{0} u_{1}^{3}\left[a_{0}^{2} a_{1}\left(b_{0}^{2} b_{1}^{2}+2 b_{0} b_{1}^{3}\right)+a_{0} a_{1}^{2} b_{0}^{2} b_{1}^{2}\right)\right]+u_{1}^{4} a_{0}^{2} a_{1} b_{0}^{2} b_{1}^{2}\right\}, \tag{41}
\end{align*}
$$

where $a_{i}$ and $b_{i}$ correspond to the partial coordinate sets $A$ and $B$, respectively.
The relation between spectrum generating functions for different partitions is well described by a special substitution homomorphism, which we shall define now. Let $\psi$ be a map of $\mathcal{U}$ onto $\mathcal{V}$. It induces a mapping from $\mathbb{C}\left[u_{U, a} ; U \in \mathcal{U}, a \in \mathbb{F}_{q}\right]$ to $\mathbb{C}\left[v_{V, a} ; V \in \mathcal{V}, a \in \mathbb{F}_{q}\right]$ given by

$$
f\left(\left(u_{U, a}\right)_{U \in \mathcal{U}, a \in \mathbb{F}_{q}}\right) \mapsto f\left(\left(v_{\psi(U), a}\right)_{U \in \mathcal{U}, a \in \mathbb{F}_{q}}\right),
$$

which is a substitution homomorphism by [43, Corollary 5.6]. Intuitively $\psi$ does nothing but substitutes each indeterminate $u_{U, a}$ with $v_{\psi(U), a}$.

Proposition 6.8: Suppose $\mathcal{U}$ and $\mathcal{V}$ are two partitions of $\mathcal{I}_{n}$. If $\mathcal{U}$ is a refinement of $\mathcal{V}$ and $\psi: \mathcal{U} \rightarrow \mathcal{V}$ is the map such that $\pi_{\mathcal{V}}=\psi \circ \pi_{\mathcal{U}}$, then $\psi$ maps $\mathcal{G}_{\mathbb{F}_{q}^{u}}(A)$ to $\mathcal{G}_{\mathbb{F}_{q}^{\nu}}(A)$ for $A \subseteq \mathbb{F}_{q}^{n}$.

Example 6.9: Let us apply Proposition 6.8 to Example 6.7. It is easy to see that the map from the partition $\{A, B\}$ to the default output partition $\left\{\mathcal{I}_{7}\right\}$ is the constant map $\psi(x)=\mathcal{I}_{7}$, and therefore (40) follows from (41) with substitutions $a_{i}, b_{i} \mapsto v_{i}$.

Another notable fact is multiplicativity of the spectrum generating function with respect to the cartesian product of sets.

Proposition 6.10: For any sets $A_{i} \subseteq \mathbb{F}_{q}^{n_{i}}$ where $1 \leq i \leq s$,

$$
\mathcal{G}_{\prod_{i=1}^{s} A_{i}}\left(\mathbf{u}_{\mathcal{I}_{s}}\right)=\prod_{i=1}^{s} \mathcal{G}_{A_{i}}\left(\mathbf{u}_{i}\right)
$$

From Propositions 6.8 and 6.10, three corollaries follow.
Corollary 6.11: For any sets $A_{1} \subseteq \mathbb{F}_{q}^{n_{1}}$ and $A_{2} \subseteq \mathbb{F}_{q}^{n_{2}}$,

$$
\mathcal{G}_{A_{1} \times A_{2}}(\mathbf{u})=\mathcal{G}_{A_{1}}(\mathbf{u}) \cdot \mathcal{G}_{A_{2}}(\mathbf{u}) .
$$

Corollary 6.12:

$$
\mathcal{G}_{\mathbb{F}_{q}^{n}}(\mathbf{u})=\left[\mathcal{G}_{\mathbb{F}_{q}}(\mathbf{u})\right]^{n}=\left(\frac{\sum_{a \in \mathbb{F}_{q}} u_{a}}{q}\right)^{n} .
$$

Corollary 6.13: For any two maps $f_{1}: \mathbb{F}_{q}^{n_{1}} \rightarrow \mathbb{F}_{q}^{m_{1}}$ and $f_{2}: \mathbb{F}_{q}^{n_{2}} \rightarrow \mathbb{F}_{q}^{m_{2}}$,

$$
\mathcal{G}_{f_{1} \odot f_{2}}(\mathbf{u}, \mathbf{v})=\mathcal{G}_{f_{1}}(\mathbf{u}, \mathbf{v}) \cdot \mathcal{G}_{f_{2}}(\mathbf{u}, \mathbf{v}),
$$

where $f_{1} \odot f_{2}$ is understood as a map from $\mathbb{F}_{q}^{n_{1}+n_{2}}$ to $\mathbb{F}_{q}^{m_{1}+m_{2}}$.
Note that Corollary 6.12 is an easy consequence of Corollary 6.11 , and that Corollary 6.13 is the desired tool for computing the spectra of parallel concatenations of linear encoders.

When $A \subseteq \mathcal{X}^{n}$ is random, its associated spectrum generating function is also random. To analyze a random polynomial, we consider its expectation. For a random polynomial $F: \Omega \rightarrow \mathbb{C}\left[u_{a} ; a \in A\right]$ with finite image, its expectation can be defined as

$$
\mathrm{E}[F]:=\sum_{f \in F(\Omega)} \mathrm{P}\{F=f\} f
$$

by using the $\mathbb{C}$-algebra structure of the polynomial ring. ${ }^{9}$ Analogous to ordinary expectations, expectations of random polynomials have the following properties: For any random polynomials $F_{1}$ and $F_{2}$ over $\mathbb{C}\left[u_{a} ; a \in A\right]$,

$$
\mathrm{E}\left[F_{1}+F_{2}\right]=\mathrm{E}\left[F_{1}\right]+\mathrm{E}\left[F_{2}\right] .
$$

If $F_{1}$ and $F_{2}$ are independent, then

$$
\mathrm{E}\left[F_{1} F_{2}\right]=\mathrm{E}\left[F_{1}\right] \mathrm{E}\left[F_{2}\right],
$$

which also implies $\mathrm{E}\left[f_{1} F_{2}\right]=f_{1} \mathrm{E}\left[F_{2}\right]$. Using these properties, we also have $\mathrm{E}[F]=\sum_{\mathbf{n} \in \mathbb{N}_{0}^{A}} \mathrm{E}\left[\left[\mathbf{u}^{\mathbf{n}}\right](F)\right] \cdot \mathbf{u}^{\mathbf{n}}$. The next proposition states an important property of expectations of random spectrum generating functions. To simplify the notation, we shall write, e.g., $\overline{\mathcal{G}}_{\mathbb{F}_{q}^{u}}(A)$ in place of $\mathrm{E}\left[\mathcal{G}_{\mathbb{F}_{q}^{u}}(A)\right]$.

Proposition 6.14: Let $\mathcal{U}$ be a partition of $\mathcal{I}_{n}$ and $\left\{F_{\underline{U}}{ }^{q}: \mathbb{F}_{q} \rightarrow \mathbb{F}_{q}\right\}_{U \in \mathcal{U}}$ be a collection of random bijective mappings, which induces a substitution homomorphism $\bar{F}: \mathbb{C}\left[u_{U, a} ; U \in \mathcal{U}, a \in \mathbb{F}_{q}\right] \rightarrow \mathbb{C}\left[u_{U, a} ; U \in \mathcal{V}, a \in \mathbb{F}_{q}\right]$ given by

$$
f\left(\left(u_{U, a}\right)_{U \in \mathcal{U}, a \in \mathbb{F}_{q}}\right) \mapsto f\left(\left(\mathrm{E}\left[u_{U, F_{U}(a)}\right]\right)_{U \in \mathcal{U}, a \in \mathbb{F}_{q}}\right)
$$

and a random map $F: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{n}$ given by

$$
\mathbf{x} \mapsto F^{(1)}\left(x_{1}\right) F^{(2)}\left(x_{2}\right) \cdots F^{(n)}\left(x_{n}\right),
$$

where $F^{(i)}$ is an independent copy of $F_{\pi_{\mathcal{u}}(i)}$. Then $\overline{\mathcal{G}}_{\mathbb{F}_{q}^{u}}(F(A))=\bar{F}\left(\overline{\mathcal{G}}_{\mathbb{F}_{q}^{u}}(A)\right)$ for any random nonempty set $A \subseteq \mathbb{F}_{q}^{n}$.

## D. MacWilliams Identities

One of the most famous results in coding theory are the MacWilliams identities [44], relating the weight enumerator of a linear code to that of its dual code. In this subsection, we shall introduce the MacWilliams identities in the framework of the code-spectrum approach. This may be regarded as a combination of the results in [45]-[47].

The dual $A^{\perp}$ of a linear code $A \subseteq \mathbb{F}_{q}^{n}$ is the orthogonal set $\left\{\mathbf{x} \in \mathbb{F}_{q}^{n}: \mathbf{x z}^{\top}=0\right.$ for all $\left.\mathbf{z} \in A\right\}$. Clearly, $A^{\perp}$ is a subspace of $\mathbb{F}_{q}^{n}$ as well. The next theorem shows the relation between $A^{\perp}$ and $A$ in terms of spectrum generating functions.

Theorem 6.15: Let $A$ be a subspace of $\mathbb{F}_{q}^{n}$ and $\mathcal{U}$ a partition of $\mathcal{I}_{n}$. Then

$$
\mathcal{G}_{A^{\perp}}\left(\mathbf{u}_{\mathcal{U}}\right)=\frac{1}{\left|A^{\perp}\right|} \mathcal{G}_{A}\left(\left(\mathbf{u}_{U} \mathbf{M}\right)_{U \in \mathcal{U}}\right)
$$

where $\mathbf{M}$ is the $q \times q$ matrix (indexed by the elements of $\mathbb{F}_{q}$ ) defined by

$$
\begin{equation*}
\mathrm{M}_{a_{1}, a_{2}}=\chi\left(a_{1} a_{2}\right) \quad \forall a_{1}, a_{2} \in \mathbb{F}_{q}, \tag{42}
\end{equation*}
$$

using the "generating" character $\chi(x):=e^{2 \pi i \operatorname{Tr}(x) / p}$ with $\operatorname{Tr}(x):=x+x^{p}+\cdots+x^{p^{r-1}}$.
Remark 6.16: Note that $x \mapsto \operatorname{Tr}(x)$, the absolute trace of $\mathbb{F}_{q}$, is $\mathbb{F}_{p}$-linear, and hence $\chi(x)$ is a homomorphism from the additive group of $\mathbb{F}_{q}$ to the multiplicative group $\mathbb{C}^{\times}$(a so-called additive character of $\mathbb{F}_{q}$ ). It is easy to see that $\sum_{x \in \mathbb{F}_{q}} \chi(a x)=0$ for $a \neq 0$, so that $q^{-\frac{1}{2}} \mathbf{M}$ is a symmetric unitary matrix. In particular,

$$
\begin{equation*}
\sum_{x \in \mathbb{F}_{q}} \mathrm{M}_{a, x}=\sum_{x \in \mathbb{F}_{q}} \mathrm{M}_{x, a}=q 1\{a=0\} \tag{43}
\end{equation*}
$$

One important application of Theorem 6.15 is calculating the spectrum of a linear encoder $\mathbf{x}=\mathbf{y A}^{\top}$ when the spectrum of $\mathbf{y}=\mathbf{x A}$ is known. The next theorem gives the details.

[^9]Theorem 6.17: Let $\mathbf{A}$ be an $n \times m$ matrix over $\mathbb{F}_{q}$. Define the linear encoders $f: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$ and $g: \mathbb{F}_{q}^{m} \rightarrow \mathbb{F}_{q}^{n}$ by $f(\mathbf{x}):=\mathbf{x A}$ and $g(\mathbf{y}):=\mathbf{y} \mathbf{A}^{\top}$, respectively. Let $\mathcal{U}$ be a partition of $\mathcal{I}_{n}$ and $\mathcal{V}$ a partition of $\mathcal{I}_{m}$. Then

$$
\mathcal{G}_{\mathbb{F}_{q}^{\nu} \mathbb{F}_{q}^{u}}(-g)\left(\mathbf{v} \mathcal{V}, \mathbf{u}_{\mathcal{U}}\right)=\frac{1}{q^{m}} \mathcal{G}_{\mathbb{F}_{q} \mathbb{F}_{q}^{\nu}}(f)\left(\left(\mathbf{u}_{U} \mathbf{M}\right)_{U \in \mathcal{U}},\left(\mathbf{v}_{V} \mathbf{M}\right)_{V \in \mathcal{V}}\right),
$$

where $M$ is defined by (42).

## VII. Conclusion

In this paper, we present some general principles and schemes for constructing linear encoders with good joint spectra:

- In Section IV-A, we provide a family of SCC-good random linear encoders derived from Gabidulin MRD codes.
- In Section IV-B, it is proved in Theorem 4.9 that we can construct $\delta$-asymptotically good LSCEs which are SC-equivalent (resp., CC-equivalent) to given $\delta$-asymptotically good LSEs (resp., LCEs).
- In Section IV-C, we propose in Theorem 4.10 a general serial concatenation scheme for constructing good LSCEs.
- In Section V, the joint spectrum of a regular LDGM encoder is analyzed. By means of Theorem 5.4, we show that regular LDGM encoders with appropriate parameters are approximately $\delta$-asymptotically SCC-good. Based on this analysis, we finally present a serial concatenation scheme with one encoder of an LDPC code as outer encoder and one LDGM encoder as inner encoder, and prove it to be asymptotically SCC-good.
In addition, we define in Section III three code-spectrum criteria for good linear encoders, so that all important coding issues are subsumed under one single research problem: Constructing linear encoders with good spectra. Through investigating the relations among these criteria, we find that a good joint spectrum is the most important feature of a linear encoder.

The main ideas of this paper formed during the period from 2007 to 2008 . Since then, there have been many advances in coding theory, two of them deserving particular attention. One is spatial coupling [48], a fundamental mechanism that helps increase the BP threshold of a new ensemble of codes to the MAP threshold of its underlying ensemble. In fact, this technique has already been used for a long time in the design of LDPC convolutional codes [49], and its excellent iterative decoding performance is well known, e.g. from [50]. Clearly, combining this technique with the LDGM-based scheme (in Sec. V) seems a promising way for designing good coding schemes in practice. For example, we may serially concatenate an outer encoder of a quasi-cyclic LDPC code (e.g., [51]) with an inner spatially-coupled regular LDGM encoder. The other advance are polar codes [52], which constitute the first known code construction that approaches capacity within a gap $\epsilon>0$ with delay and complexity both depending polynomially on $1 / \epsilon$ [53], [54]. However, the minimum distance of a polar code is only a sublinear function of the block length. It is unknown if there exists a fundamental trade-off among minimum distance, decoding complexity, gap to capacity, etc. Regardless of whether such a law exists, it is valuable in practice to think of the "sublinear" counterpart of linear encoders with good joint spectra, that is, we may allow $\lim _{k \rightarrow \infty} \min _{\mathbf{x} \neq 0^{n_{k}}}\left(H\left(P_{\mathbf{x}}\right) R\left(f_{k}\right)+\right.$ $\left.H\left(P_{f_{k}(\mathbf{x})}\right)\right)=0$ (cf. [10] and [9, Theorem 4.1]).

## Appendix A <br> Proofs of Results in Section III

Proof of Proposition 3.4: Let $\mathbf{G}_{k}$ be an $l_{k} \times n_{k}$ generator matrix that yields ker $F_{k}$. Thus $G_{k}(\mathbf{x}):=\mathbf{x} \mathbf{G}_{k}$ is the linear encoder desired.

Proof of Proposition 3.5: Let $\mathbf{H}_{k}$ be an $l_{k} \times m_{k}$ parity-check matrix that yields $F_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)$. Thus $G_{k}(\mathbf{x}):=\mathbf{x H}_{k}^{\top}$ is the linear encoder desired.

Proof of Proposition 3.6: It follows from (4) that

$$
\begin{equation*}
\max _{P \in \mathcal{P}_{n_{k}}^{*}, Q \in \mathcal{P}_{m_{k}}} \frac{1}{n_{k}} \ln \alpha_{F_{k}}(P, Q) \leq \delta+\epsilon \tag{44}
\end{equation*}
$$

for any $\epsilon>0$ and sufficiently large $k$. Then it follows that

$$
\begin{aligned}
& \max _{P \in \mathcal{P}_{n_{k}}^{*}} \frac{1}{n_{k}} \ln \alpha_{\text {ker } F_{k}}(P) \stackrel{(\mathrm{a})}{=} \max _{\mathrm{x}: \mathbf{x} \neq 0^{n_{k}}} \frac{1}{n_{k}} \ln \left(\mathrm{E}\left[\frac{q^{n_{k}} 1\left\{\mathbf{x} \in \Sigma_{n_{k}}\left(\operatorname{ker} F_{k}\right)\right\}}{\left|\operatorname{ker} F_{k}\right|}\right]\right) \\
& \leq \max _{\mathbf{x}: \mathbf{x} \neq 0^{n_{k}}} \frac{1}{n_{k}} \ln \left(q^{m_{k}} \mathrm{P}\left\{F_{k}^{\sim}(\mathbf{x})=0^{m_{k}}\right\}\right) \\
&=\max _{\mathrm{x}: \mathbf{x} \neq 0^{n_{k}}} \frac{1}{n_{k}} \ln \left(q^{m_{k}} \mathrm{P}\left\{\tilde{F}_{k}(\mathbf{x})=0^{m_{k}}\right\}\right) \\
& \stackrel{(\mathrm{b})}{=} \max _{P \in \mathcal{P}_{n_{k}}^{*}} \frac{1}{n_{k}} \ln \alpha_{F_{k}}\left(P, P_{0^{m_{k}}}\right) \\
& \stackrel{(\mathrm{c})}{\leq} \delta+\epsilon
\end{aligned}
$$

for sufficiently large $k$, where (a) follows from Proposition 6.2, (b) from [9, Proposition 2.4], and (c) follows from (44). Since $\epsilon$ is arbitrary, we conclude that $\boldsymbol{F}$ is $\delta$-asymptotically SC-good.

Also by (44), we have

$$
\begin{equation*}
\overline{\mathrm{S}}_{F_{k}}(P, Q) \leq e^{n_{k}(\delta+\epsilon)} \mathrm{S}_{\mathbb{F}_{q}^{n_{k}} \times \mathbb{F}_{q}^{m_{k}}}(P, Q) \quad \forall P \in \mathcal{P}_{n_{k}}^{*}, Q \in \mathcal{P}_{m_{k}} \tag{45}
\end{equation*}
$$

for sufficiently large $k$. Then for any $Q \in \mathcal{P}_{m_{k}}^{*}$, it follows that

$$
\begin{aligned}
\overline{\mathrm{S}}_{F_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)}(Q) & \stackrel{(\mathrm{a})}{=} \overline{\mathrm{S}}_{F_{k}}(Q) \\
& =\sum_{P \in \mathcal{P}_{n_{k}}} \overline{\mathrm{~S}}_{F_{k}}(P, Q) \\
& \stackrel{(\mathrm{b})}{=} \sum_{P \in \mathcal{P}_{n_{k}}^{*}} \overline{\mathrm{~S}}_{F_{k}}(P, Q) \\
& \stackrel{(\mathrm{c})}{\leq} \sum_{P \in \mathcal{P}_{n_{k}}^{*}} e^{n_{k}(\delta+\epsilon)} \mathrm{S}_{\mathbb{F}_{q}^{n_{k}} \times \mathbb{F}_{q}^{m_{k}}}(P, Q) \\
& \leq e^{n_{k}(\delta+\epsilon)} \mathrm{S}_{\mathbb{F}_{q}^{m_{k}}}(Q)
\end{aligned}
$$

for sufficiently large $k$, where (a) follows from the linear property of $F_{k}$, (b) from $Q \neq P_{0^{m_{k}}}$, and (c) follows from (45). Hence we have

$$
\begin{aligned}
\limsup _{k \rightarrow \infty} \max _{Q \in \mathcal{P}_{m_{k}}} \frac{1}{m_{k}} \ln \alpha_{F_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)}(Q) & =\limsup _{k \rightarrow \infty} \max _{Q \in \mathcal{P}_{m_{k}}} \frac{1}{m_{k}} \ln \frac{\overline{\mathrm{~S}}_{F_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)}(Q)}{\mathrm{S}_{\mathbb{F}_{q}^{m_{k}}}(Q)} \\
& \leq \limsup _{k \rightarrow \infty} \frac{1}{m_{k}} \ln e^{n_{k}(\delta+\epsilon)} \\
& =(\delta+\epsilon) \bar{R}(\boldsymbol{F}) .
\end{aligned}
$$

Because $\epsilon$ is arbitrary, $\boldsymbol{F}$ is $\delta \bar{R}(\boldsymbol{F})$-asymptotically CC-good.
Proof of Proposition 3.7: Note that

$$
\limsup _{k \rightarrow \infty} \rho\left(G_{k, k}\right) \leq \limsup _{k \rightarrow \infty} \rho\left(G_{i, k}\right) \leq \delta_{i} \quad \forall i \in \mathbb{N}
$$

and hence $\limsup \sup _{k \rightarrow \infty} \rho\left(G_{k, k}\right) \leq \inf _{i \in \mathbb{N}} \delta_{i}=\delta$.

## Appendix B

## Proofs of Results in Section IV-B

Proof of Theorem 4.4: Condition (6) implies that for every pair $\mathrm{x} \in \mathcal{X}^{n} \backslash\left\{0^{n}\right\}, \mathrm{y} \in \mathcal{Y}^{m}$ there exists at least one linear encoder $f: \mathcal{X}^{n} \rightarrow \mathcal{Y}^{m}$ satisfying $f(\mathbf{x})=\mathbf{y} .{ }^{10}$ Since $m, n \geq 1$, this in turn implies, for every pair $x \in \mathcal{X} \backslash\{0\}, y \in \mathcal{Y}$, the existence of at least one group homomorphism $h: \mathcal{X} \rightarrow \mathcal{Y}$ satisfying $h(x)=y$.
${ }^{10}$ The existence of such an $f$ depends only on the types of $\mathbf{x}$ and $\mathbf{y}$, since $f$ is linear iff $\sigma \circ f \circ \pi$ is linear for any permutations $\pi \in \mathrm{S}_{n}$, $\sigma \in \mathrm{S}_{m}$.

If $p$ is a prime dividing $|\mathcal{X}|$, there exists $x \in \mathcal{X}$ of order $p$. Then, by the condition above, every $y \in \mathcal{Y}$ must have order 1 or $p$, so $\mathcal{Y} \cong \mathbb{Z}_{p}^{s}$ for some $s$. If $|\mathcal{X}|$ had a prime divisor $q \neq p$, then $\mathcal{Y} \cong \mathbb{Z}_{p}^{s} \cong \mathbb{Z}_{q}^{t}$ and so $|\mathcal{Y}|=1$, a contradiction. Thus, $\mathcal{X}$ and $\mathcal{Y}$ must be $p$-groups for the same prime $p$, and $\mathcal{Y}$ must be elementary abelian. Finally, if $\mathcal{X}$ contained an element $x$ of order $p^{2}$, we would have $p x \neq 0$ but $h(p x)=p h(x)=0$ for any group homomorphism $h: \mathcal{X} \rightarrow \mathcal{Y}$. This implies again $|\mathcal{Y}|=1$ and concludes the proof.

To prove Theorem 4.5, we need the following lemma.
Lemma B.1: Let $F: \mathcal{X}^{n} \rightarrow \mathcal{Y}^{m}$ be a random linear encoder. If $F$ is SCC-good, then

$$
\begin{equation*}
\mathrm{E}[|\operatorname{ker} F|]=1+|\mathcal{Y}|^{-m}\left(|\mathcal{X}|^{n}-1\right) . \tag{46}
\end{equation*}
$$

Proof:

$$
\begin{aligned}
\mathrm{E}[|\operatorname{ker} F|] & =\mathrm{E}[|\operatorname{ker} \tilde{F}|] \\
& =\mathrm{E}\left[\sum_{\mathbf{x} \in \mathcal{X}^{n}} 1\left\{\tilde{F}(\mathbf{x})=0^{m}\right\}\right] \\
& =1+\sum_{\mathbf{x} \in \mathcal{X}^{n} \backslash\left\{0^{n}\right\}} \mathrm{P}\left\{\tilde{F}(\mathbf{x})=0^{m}\right\} \\
& \stackrel{(a)}{=} 1+|\mathcal{Y}|^{-m}\left(|\mathcal{X}|^{n}-1\right),
\end{aligned}
$$

where (a) follows from (6).
Proof of Theorem 4.5: By Lagrange's theorem, $|\operatorname{ker} F|$ can take only values in $\left\{1, p, \ldots, p^{r n}\right\}$. Hence, using Lemma B.1, we obtain

$$
\begin{aligned}
1+\frac{q^{n}-1}{q^{n}} & =\mathrm{E}[|\operatorname{ker} F|] \\
& \geq \mathrm{P}\{|\operatorname{ker} F|=1\}+p \cdot(1-\mathrm{P}\{|\operatorname{ker} F|=1\})
\end{aligned}
$$

Solving for $\mathrm{P}\{|\operatorname{ker} F|=1\}$ gives the stated inequality.
Proof of Proposition 4.7: Suppose $\mathcal{X} \cong \mathbb{Z}_{2}^{s}$, so that $\mathcal{X}^{n} \cong \mathbb{Z}_{2}^{n s} \cong\left(\mathbb{F}_{2}^{n s},+\right)$ for all $n \in \mathbb{N}$. Let $F_{n}: \mathbb{F}_{2}^{n s} \rightarrow \mathbb{F}_{2}^{n s}$ be the random linear encoder derived from a binary ( $n s, n s, 2$ ) Gabidulin MRD code $\mathcal{C}$ in accordance with Theorem 4.2. By definition, the code $\mathcal{C}$ consists of $2^{2 n s}$ matrices $\mathbf{A} \in \mathbb{F}_{2}^{n s \times n s}$ with $\operatorname{rank}(\mathbf{A}) \in\{0, n s-1, n s\}$, and by the rank distribution of MRD codes ([26, Theorem 5.6] or [27, Theorem 5]), there are $2\left(2^{n s}-1\right)$ matrices of rank $n s$ in $\mathcal{C}$, so that

$$
\lim _{n \rightarrow \infty} \mathrm{P}\left\{\left|\operatorname{ker} F_{n}\right|=1\right\}=\lim _{n \rightarrow \infty} \frac{2\left(2^{n s}-1\right)}{2^{2 n s}}=0 .
$$

Since $F_{n}$ is SCC-good, this proves the proposition.
Proof of Theorem 4.8: Inequalities (12) and (14) follow immediately from (11), so our task is to evaluate the average conditional spectra of $G_{1}$ and $G_{2}$.

For any $P \in \mathcal{P}_{n}^{*}$ and $Q \in \mathcal{P}_{m}$,

$$
\begin{aligned}
& \overline{\mathrm{S}}_{G_{1}}(Q \mid P) \stackrel{(\mathrm{a})}{=} \sum_{O \in \mathcal{P}_{m}} \overline{\mathrm{~S}}_{F}(O \mid P) \overline{\mathrm{S}}_{F_{m, m}^{\text {RLC }}}(Q \mid O) \\
&= \sum_{O \in \mathcal{P}_{m}^{*}} \overline{\mathrm{~S}}_{F}(O \mid P) \overline{\mathrm{S}}_{F_{m, m}^{\text {RLC }}}(Q \mid O) \\
&+\overline{\mathrm{S}}_{F}\left(P_{0^{m}} \mid P\right) \overline{\mathrm{S}}_{F_{m, m}^{\text {RLC }}}\left(Q \mid P_{0^{m}}\right) \\
& \stackrel{(\mathrm{b})}{=} \mathrm{S}_{\mathbb{F}_{q}^{m}}(Q) \sum_{O \in \mathcal{P}_{m}^{*}} \overline{\mathrm{~S}}_{F}(O \mid P) \\
&+1\left\{Q=P_{0^{m}}\right\} \overline{\mathrm{S}}_{F}\left(P_{0^{m}} \mid P\right) \\
& \leq \mathrm{S}_{\mathbb{F}_{q}^{m}}(Q)+1\left\{Q=P_{0^{m}}\right\} \overline{\mathrm{S}}_{F}\left(P_{0^{m}} \mid P\right)
\end{aligned}
$$

where (a) follows from Proposition 6.6 and (b) follows from (5). This concludes (13).

Analogously, for any $P \in \mathcal{P}_{n}^{*}$ and $Q \in \mathcal{P}_{m}$,

$$
\begin{aligned}
\overline{\mathrm{S}}_{G_{2}}(Q \mid P) & \stackrel{(\mathrm{a})}{=} \sum_{O \in \mathcal{P}_{n}} \overline{\mathrm{~S}}_{F_{n, n}^{\mathrm{RlLC}}}(O \mid P) \overline{\mathrm{S}}_{F}(Q \mid O) \\
& \stackrel{(\mathrm{b})}{=} \sum_{O \in \mathcal{P}_{n}} \mathrm{~S}_{\mathbb{F}_{q}^{n}}(O) \overline{\mathrm{S}}_{F}(Q \mid O) \\
& =\sum_{O \in \mathcal{P}_{n}} \overline{\mathrm{~S}}_{F}(O, Q) \\
& =\overline{\mathrm{S}}_{F}(Q) \\
& =\overline{\mathrm{S}}_{F\left(\mathbb{F}_{q}^{n}\right)}(Q)
\end{aligned}
$$

where (a) follows from Proposition 6.6 and (b) follows from (5). This concludes (15) and hence completes the proof.

Proof of Theorem 4.9: For the first statement, recall that the source transmission rate of $\delta$-asymptotically good LSEs must converge. Then for any $\epsilon>0$, since $\boldsymbol{f}$ is $\delta$-asymptotically SC-good and $R_{c}(\boldsymbol{f})=\ln q$, we have

$$
\begin{equation*}
\alpha_{\text {ker } f_{k}}(P) \leq e^{n_{k}(\delta+\epsilon)} \quad \forall P \in \mathcal{P}_{n_{k}}^{*} \tag{47}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)\right| \geq q^{m_{k}} e^{-m_{k} \epsilon} \geq q^{m_{k}} e^{-2 n_{k} \epsilon / R(\boldsymbol{f})} \tag{48}
\end{equation*}
$$

for sufficiently large $k$. Define $G_{1, k}:=F_{m_{k}, m_{k}}^{\mathrm{RLC}} \circ f_{k}$. It follows from Theorem 4.8 that

$$
\begin{equation*}
\mathrm{P}\left\{\operatorname{ker} G_{1, k}=\operatorname{ker} f_{k}\right\}>K_{q} \tag{49}
\end{equation*}
$$

and

$$
\overline{\mathrm{S}}_{G_{1, k}}(Q \mid P) \leq \mathrm{S}_{\mathbb{F}_{q}^{m_{k}}}(Q)+1\left\{Q=P_{0^{m_{k}}}\right\} \mathrm{S}_{f_{k}}\left(P_{0^{m_{k}}} \mid P\right) \quad \forall P \in \mathcal{P}_{n_{k}}^{*}, Q \in \mathcal{P}_{m_{k}}
$$

Hence for any $P \in \mathcal{P}_{n_{k}}^{*}$ and $Q \in \mathcal{P}_{m_{k}}$,

$$
\begin{aligned}
\alpha_{G_{1, k}}(P, Q) & =\frac{\overline{\mathrm{S}}_{G_{1, k}}(Q \mid P)}{\mathrm{S}_{\mathbb{F}_{q}^{m_{k}}(Q)}} \\
& \leq 1+1\left\{Q=P_{0^{m_{k}}}\right\} \alpha_{f_{k}}\left(P, P_{0^{m_{k}}}\right) \\
& \stackrel{(\mathrm{a})}{=} 1+1\left\{Q=P_{0^{m_{k}}}\right\} q^{m_{k}} \mathrm{P}\left\{f_{k}^{\sim}(\mathbf{x})=0^{m_{k}}\right\} \\
& =1+1\left\{Q=P_{0^{m_{k}}}\right\} q^{m_{k}} \mathrm{P}\left\{\mathbf{x} \in \Sigma_{n_{k}}\left(\operatorname{ker} f_{k}\right)\right\} \\
& \stackrel{(\mathrm{b})}{=} 1+1\left\{Q=P_{0^{m_{k}}}\right\} q^{m_{k}} \frac{\left|\operatorname{ker} f_{k}\right|}{q^{n_{k}}} \alpha_{\operatorname{ker} f_{k}}(P) \\
& =1+1\left\{Q=P_{0^{m_{k}}}\right\} \frac{q^{m_{k}}}{\left|f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)\right|} \alpha_{\operatorname{ker} f_{k}}(P) \\
& \stackrel{(\mathrm{c})}{\leq} 1+1\left\{Q=P_{0^{m_{k}}}\right\} e^{2 n_{k} \epsilon / R(\boldsymbol{f})} \alpha_{\text {ker } f_{k}}(P) \\
& \stackrel{(\mathrm{d})}{\leq} e^{n_{k}(\delta+2 \epsilon+2 \epsilon / R(\boldsymbol{f}))}
\end{aligned}
$$

for sufficiently large $k$, where (a) follows from [9, Proposition 2.4] and x is a vector of type $P$, (b) from Proposition 6.2, (c) from (48), and (d) follows from (47). Thus for sufficiently large $k$,

$$
\rho\left(G_{1, k}\right) \leq \delta+2 \epsilon+\frac{2 \epsilon}{R(\boldsymbol{f})}
$$

Define the random linear encoder $G_{1, k}^{\prime}$ as $G_{1, k}$ given the event $A_{k}:=\left\{\operatorname{ker} G_{1, k}=\operatorname{ker} f_{k}\right\}$. Then it follows that for sufficiently large $k$,

$$
\begin{aligned}
\rho\left(G_{1, k}^{\prime}\right) & \leq \rho\left(G_{1, k}\right)-\frac{1}{n_{k}} \ln \mathrm{P}\left(A_{k}\right) \\
& \leq \delta+2 \epsilon+\frac{2 \epsilon}{R(\boldsymbol{f})}-\frac{1}{n_{k}} \ln \mathrm{P}\left(A_{k}\right) \\
& \leq \delta+3 \epsilon+\frac{2 \epsilon}{R(\boldsymbol{f})},
\end{aligned}
$$

where (a) follows from (49). Since $\epsilon$ is arbitrary, $\left\{G_{1, k}^{\prime}\right\}_{k=1}^{\infty}$ is a sequence of $\delta$-asymptotically good LSCEs such that $\operatorname{ker} G_{1, k}^{\prime}=\operatorname{ker} f_{k}$. By [9, Proposition 4.1], we conclude that there exists a sequence $\left\{g_{1, k}\right\}_{k=1}^{\infty}$ of $\delta$-asymptotically good LSCEs $g_{1, k}: \mathbb{F}_{q}^{n_{k}} \rightarrow \mathbb{F}_{q}^{m_{k}}$ such that $g_{1, k}$ is SC-equivalent to $f_{k}$ for each $k \in \mathbb{N}$.

The proof of the second statement is analogous. Let $\epsilon>0$ be given. Since $f$ is $\delta$-asymptotically CC-good,

$$
\begin{equation*}
\alpha_{f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)}(Q) \leq e^{m_{k}(\delta+\epsilon)} \quad \forall Q \in \mathcal{P}_{m_{k}}^{*} \tag{50}
\end{equation*}
$$

for sufficiently large $k$. Define $G_{2, k}:=f_{k} \circ F_{n_{k}, n_{k}}^{\mathrm{RLL}}$. Then it follows from Theorem 4.8 that

$$
\begin{equation*}
\mathrm{P}\left\{G_{2, k}\left(\mathbb{F}_{q}^{n_{k}}\right)=f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)\right\}>K_{q} \tag{51}
\end{equation*}
$$

and

$$
\overline{\mathrm{S}}_{G_{2, k}}(Q \mid P)=\mathrm{S}_{f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)}(Q) \quad \forall P \in \mathcal{P}_{n_{k}}^{*}, Q \in \mathcal{P}_{m_{k}}
$$

Hence for any $P \in \mathcal{P}_{n_{k}}^{*}$ and $Q \in \mathcal{P}_{m_{k}}$,

$$
\begin{aligned}
\alpha_{G_{2, k}}(P, Q) & =\frac{\overline{\mathrm{S}}_{G_{2, k}}(Q \mid P)}{\mathrm{S}_{\mathbb{F}_{q}^{m_{k}}}(Q)} \\
& =\frac{\mathrm{S}_{f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)}(Q)}{\mathrm{S}_{\mathbb{F}_{q}^{m_{k}}}(Q)} \\
& =\alpha_{f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)}(Q)
\end{aligned}
$$

for sufficiently large $k$. Define the random linear encoder $G_{2, k}^{\prime}$ as $G_{2, k}$ given the event $B_{k}:=\left\{G_{2, k}\left(\mathbb{F}_{q}^{n_{k}}\right)=\right.$ $\left.f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)\right\}$. Then it follows that for any $P \in \mathcal{P}_{n_{k}}^{*}$ and $Q \in \mathcal{P}_{m_{k}}^{*}$,

$$
\alpha_{G_{2, k}^{\prime}}(P, Q) \leq \frac{\alpha_{f\left(\mathbb{F}_{q}^{n_{k}}\right)}(Q)}{\mathrm{P}\left(B_{k}\right)} \stackrel{(\mathrm{a})}{\leq} e^{m_{k}(\delta+2 \epsilon)}
$$

for sufficiently large $k$, where (a) follows from (50) and (51). Since $f_{k}$ is injective, we have

$$
\alpha_{G_{2, k}^{\prime}}\left(P, P_{0^{m_{k}}}\right)=0 \quad \forall P \in \mathcal{P}_{n_{k}}^{*}
$$

and $R\left(f_{k}\right)=R_{c}\left(f_{k}\right) / \ln q$ converges as $k \rightarrow \infty$. Therefore,

$$
\rho\left(G_{2, k}^{\prime}\right) \leq \frac{\delta+2 \epsilon}{R(\boldsymbol{f})}+\epsilon
$$

for sufficiently large $k$. Since $\epsilon$ is arbitrary, $\left\{G_{2, k}^{\prime}\right\}_{k=1}^{\infty}$ is a sequence of $\delta / R(\boldsymbol{f})$-asymptotically good LSCEs such that $G_{2, k}^{\prime}\left(\mathbb{F}_{q}^{n_{k}}\right)=f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)$. By [9, Proposition 4.1], we conclude that there exists a sequence $\left\{g_{2, k}\right\}_{k=1}^{\infty}$ of $\delta / R(\boldsymbol{f})$ asymptotically good LSCEs $g_{2, k}: \mathcal{X}^{n_{k}} \rightarrow \mathcal{Y}^{m_{k}}$ such that $g_{2, k}$ is CC-equivalent to $f_{k}$ for each $k \in \mathbb{N}$.

## Appendix C

Proofs of Results in Section IV-C
Proof of Theorem 4.10: For any $\epsilon>0$, since $G_{k}$ is $\delta$-asymptotically SCC-good relative to $A_{k}$, we have

$$
\begin{equation*}
\overline{\mathrm{S}}_{G_{k}}(Q \mid P) \leq e^{m_{k}(\delta+\epsilon)} \mathrm{S}_{\mathbb{F}_{q}^{l_{k}}}(Q) \quad \forall P \in A_{k}, Q \in P_{l_{k}} \tag{52}
\end{equation*}
$$

for sufficiently large $k$. Then for all $O \in \mathcal{P}_{n_{k}}^{*}$ and $Q \in P_{l_{k}}$,

$$
\begin{aligned}
\overline{\mathrm{S}}_{G_{k} \circ \Sigma_{m_{k}} \circ F_{k}}(Q \mid O) & \stackrel{(\mathrm{a})}{=} \sum_{P \in \mathcal{P}_{m_{k}}} \overline{\mathrm{~S}}_{F_{k}}(P \mid O) \overline{\mathrm{S}}_{G_{k}}(Q \mid P) \\
& \stackrel{(\mathrm{b})}{=} \sum_{P \in A_{k}} \overline{\mathrm{~S}}_{F_{k}}(P \mid O) \overline{\mathrm{S}}_{G_{k}}(Q \mid P) \\
& \stackrel{(\mathrm{c})}{\leq} \sum_{P \in A_{k}} e^{m_{k}(\delta+\epsilon)} \mathrm{S}_{\mathbb{F}_{q}^{l_{k}}}(Q) \overline{\mathrm{S}}_{F_{k}}(P \mid O) \\
& \leq e^{m_{k}(\delta+\epsilon)} \mathrm{S}_{\mathbb{F}_{q}^{l_{k}}}(Q)
\end{aligned}
$$

for sufficiently large $k$, where (a) follows from Proposition 6.6, (b) from condition (16), and (c) follows from (52). Therefore, for sufficiently large $k$,

$$
\rho\left(G_{k} \circ \Sigma_{m_{k}} \circ F_{k}\right) \leq \frac{\delta+\epsilon}{\underline{R}(\boldsymbol{F})}+\epsilon .
$$

Since $\epsilon$ is arbitrary, this establishes the theorem.
Proof of Proposition 4.13: For any $\epsilon>0$, since $F_{k}$ is $\delta$-asymptotically SCC-good, we have

$$
\begin{equation*}
\overline{\mathrm{S}}_{F_{k}}(P \mid O) \leq e^{n_{k}(\delta+\epsilon)} \mathrm{S}_{\mathbb{F}_{q}^{m_{k}}}(P) \quad \forall O \in P_{n_{k}}^{*}, P \in P_{m_{k}} \tag{53}
\end{equation*}
$$

for sufficiently large $k$. Then for all $O \in \mathcal{P}_{n_{k}}^{*}$ and $Q \in P_{l_{k}}$,

$$
\begin{aligned}
\overline{\mathrm{S}}_{G_{k} \circ \Sigma_{m_{k}} \circ F_{k}}(Q \mid O) & \stackrel{(\mathrm{a})}{=} \sum_{P \in \mathcal{P}_{m_{k}}} \overline{\mathrm{~S}}_{F_{k}}(P \mid O) \overline{\mathrm{S}}_{G_{k}}(Q \mid P) \\
& \stackrel{(\mathrm{b})}{\leq} \sum_{P \in \mathcal{P}_{m_{k}}} e^{n_{k}(\delta+\epsilon)} \mathrm{S}_{\mathbb{F}_{q}^{m_{k}}}(P) \overline{\mathrm{S}}_{G_{k}}(Q \mid P) \\
& =e^{n_{k}(\delta+\epsilon)} \sum_{P \in \mathcal{P}_{m_{k}}} \overline{\mathrm{~S}}_{G_{k}}(P, Q) \\
& =e^{n_{k}(\delta+\epsilon)} \overline{\mathrm{S}}_{G_{k}}(Q) \\
& =e^{n_{k}(\delta+\epsilon)} \overline{\mathrm{S}}_{G_{k}\left(\mathbb{F}_{q}^{m_{k}}\right)}(Q) \\
& \stackrel{(\mathrm{c})}{=} e^{n_{k}(\delta+\epsilon)} \mathrm{S}_{\mathbb{F}_{q}^{t_{k}}}(Q)
\end{aligned}
$$

for sufficiently large $k$, where (a) follows from Proposition 6.6, (b) from (53), and (c) follows from the surjectivity of $G_{k}$. Therefore, $\rho\left(G_{k} \circ \Sigma_{m_{k}} \circ F_{k}\right) \leq \delta+\epsilon$ for sufficiently large $k$. This concludes the proof, because $\epsilon$ is arbitrary.

## Appendix D

Proofs of Results in Section V
Proof of Proposition 5.1: The identity (19) holds clearly. This together with Proposition 6.14 gives (20). From (19) and Corollary 6.13, it further follows that

$$
\begin{aligned}
\mathcal{G}_{f_{\varepsilon, n}^{\mathrm{REP}}}(\mathbf{u}, \mathbf{v}) & =\left(\frac{1}{q} \sum_{a \in \mathbb{F}_{q}} u_{a} v_{a}^{c}\right)^{n} \\
& =\sum_{P \in \mathcal{P}_{n}} \mathrm{~S}_{\mathbb{F}_{q}^{n}}(P) \mathbf{u}^{n P} \mathbf{v}^{n c P} .
\end{aligned}
$$

This proves (21), and then identities (22) and (23) follows.
Proof of Proposition 5.2: Note that the generator matrix of $F_{d}^{\mathrm{CHK}}$ is the transpose of the generator matrix of $F_{d}^{\text {REP }}$. Then by Theorem 6.17, it follows that

$$
\begin{aligned}
\overline{\mathcal{G}}_{-F_{d}^{\text {cнк }}}(\mathbf{u}, \mathbf{v}) & =\frac{1}{q^{d}} \overline{\mathcal{G}}_{F_{d}^{\text {ReP }}}(\hat{\mathbf{v}}, \hat{\mathbf{u}}) \\
& \stackrel{(\mathrm{a})}{=} \frac{1}{q^{d+1}}\left[\hat{u}_{0}^{d} \hat{v}_{0}+\left(\frac{\hat{\mathbf{u}}_{\oplus}-\hat{u}_{0}}{q-1}\right)^{d}\left(\hat{\mathbf{v}}_{\oplus}-\hat{v}_{0}\right)\right] \\
& \stackrel{(\mathrm{b})}{=} \frac{1}{q^{d+1}}\left[\left(\mathbf{u}_{\oplus}\right)^{d} \mathbf{v}_{\oplus}+\left(\frac{q u_{0}-\mathbf{u}_{\oplus}}{q-1}\right)^{d}\left(q v_{0}-\mathbf{v}_{\oplus}\right)\right]
\end{aligned}
$$

where $\hat{\mathbf{u}}=\mathbf{u M}$ and $\hat{\mathbf{v}}=\mathbf{v M}$, (a) follows from Proposition 5.1, and (b) follows from property (43). This together with Proposition 6.14 concludes (24).

By Corollary 6.13, we further have

$$
\begin{aligned}
\overline{\mathcal{G}}_{F_{d, n}^{c, n k}}(\mathbf{u}, \mathbf{v})= & \frac{1}{q^{n(d+1)}}\left[\left(\mathbf{u}_{\oplus}\right)^{d} \mathbf{v}_{\oplus}+\left(\frac{q u_{0}-\mathbf{u}_{\oplus}}{q-1}\right)^{d}\left(q v_{0}-\mathbf{v}_{\oplus}\right)\right]^{n} \\
= & \frac{1}{q^{n(d+1)}}\left\{\left[\left(\mathbf{u}_{\oplus}\right)^{d}+(q-1)\left(\frac{q u_{0}-\mathbf{u}_{\oplus}}{q-1}\right)^{d}\right]_{0}\right. \\
& \left.+\left[\left(\mathbf{u}_{\oplus}\right)^{d}-\left(\frac{q u_{0}-\mathbf{u}_{\oplus}}{q-1}\right)^{d}\right]\left(\mathbf{v}_{\oplus}-v_{0}\right)\right\}^{n} \\
= & \frac{1}{q^{n(d+1)}} \sum_{Q \in \mathcal{P}_{n}}\left\{\binom{n}{n Q} \mathbf{v}^{n Q}\right. \\
& \times\left[\left(\mathbf{u}_{\oplus}\right)^{d}+(q-1)\left(\frac{q u_{0}-\mathbf{u}_{\oplus}}{q-1}\right)^{d}\right]^{n Q(0)} \\
& \left.\times\left[\left(\mathbf{u}_{\oplus}\right)^{d}-\left(\frac{q u_{0}-\mathbf{u}_{\oplus}}{q-1}\right)^{d}\right]^{n(1-Q(0))}\right\} .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\overline{\mathrm{S}}_{F_{d, n}^{c \text { нн }}}(P, Q) & =\left[\mathbf{u}^{d n P} \mathbf{v}^{n Q}\right]\left(\overline{\mathcal{G}}_{F_{d, n}^{c \text { нн }}}(\mathbf{u}, \mathbf{v})\right) \\
& =\left[\mathbf{u}^{d n P}\right]\left(g_{d, n}^{(1)}(\mathbf{u}, Q)\right)
\end{aligned}
$$

where $g_{d, n}^{(1)}(\mathbf{u}, Q)$ is defined by (28). This proves (25).
Since $g_{d, n}^{(1)}(\mathbf{u}, Q)$ is a polynomial with nonnegative coefficients, $\left[\mathbf{u}^{d n P}\right]\left(g_{d, n}^{(1)}(\mathbf{u}, Q)\right)$ can be bounded above by

$$
\left[\mathbf{u}^{d n P}\right]\left(g_{d, n}^{(1)}(\mathbf{u}, Q)\right) \leq \frac{g_{d, n}^{(1)}(O, Q)}{O^{d n P}}=g_{d, n}^{(2)}(O, P, Q)
$$

where $O$ is an arbitrary type in $\mathcal{P}_{d n}$ such that $P \ll O$, and $g_{d, n}^{(2)}(O, P, Q)$ is defined by (29). This gives (26).

Finally, let us estimate $\alpha_{F_{d, n}^{\text {снн }}}(P, Q)$.

$$
\begin{aligned}
\alpha_{F_{d, n}^{c+k}}(P, Q) \leq & \frac{g_{d, n}^{(2)}(O, P, Q)}{\mathrm{S}_{\mathbb{F}_{q}^{d n} \times \mathbb{F}_{q}^{n}}(P, Q)} \\
= & \frac{1}{\binom{d n}{d n P} O^{d n P}}\left[1+(q-1)\left(\frac{q O(0)-1}{q-1}\right)^{d}\right]^{n Q(0)} \\
& \times\left[1-\left(\frac{q O(0)-1}{q-1}\right)^{d}\right]^{n(1-Q(0))} \\
= & \frac{e^{d n H(P)} P^{d n P}}{\left({ }_{d n}^{d n}\right) O^{d n P}}\left[1+(q-1)\left(\frac{q O(0)-1}{q-1}\right)^{d}\right]^{n Q(0)} \\
& \times\left[1-\left(\frac{q O(0)-1}{q-1}\right)^{d}\right]^{n(1-Q(0))} \\
= & e^{d n \Delta_{d n}(P)} e^{d n D(P \| O)} \\
& \times\left[1+(q-1)\left(\frac{q O(0)-1}{q-1}\right)^{d}\right]^{n Q(0)} \\
& \times\left[1-\left(\frac{q O(0)-1}{q-1}\right)^{d}\right]^{n(1-Q(0))}
\end{aligned}
$$

Note that $D(P \| O) \geq D(P(0) \| O(0))$ with equality if and only if $P(a)(1-O(0))=O(a)(1-P(0))$ for all $a \neq 0$, and thus we obtain a minimized upper bound (27).

Proof of Theorem 5.3: By the definition of $F_{c, d, n}^{\mathrm{LD}}$ and Proposition 6.6, it follows that

$$
\begin{aligned}
\overline{\mathrm{S}}_{F_{c, d, n}}^{\mathrm{LD}}(Q \mid P) & =\sum_{O \in \mathcal{P}_{\mathcal{c n}}} \overline{\mathrm{S}}_{f_{c, n}^{\mathrm{REP}}}(O \mid P) \overline{\mathrm{S}}_{F_{d, c n / d}^{\mathrm{CHK}}}(Q \mid O) \\
& \stackrel{(\mathrm{a})}{=} \sum_{O \in \mathcal{P}_{c n}} 1\{O=P\} \overline{\mathrm{S}}_{F_{d, c n / d}^{\mathrm{cHK}}}(Q \mid O) \\
& =\overline{\mathrm{S}}_{F_{d, c n}^{\mathrm{CH} / d}}(Q \mid P),
\end{aligned}
$$

where (a) follows from (23). This proves (34).
Furthermore, we have

$$
\begin{aligned}
\frac{1}{n} \ln \alpha_{F_{c, d, n}}^{\mathrm{LD}}(P, Q) & =\frac{1}{n} \ln \frac{\overline{\mathrm{~S}}_{F_{c, d, n}^{\mathrm{LD}}}(Q \mid P)}{\mathrm{S}_{\mathbb{F}_{q}^{\text {cn/d }}}(Q)} \\
& \stackrel{(\mathrm{a})}{=} \frac{1}{n} \ln \frac{\overline{\mathrm{~S}}_{F_{d, c n / d}^{c h k}}(Q \mid P)}{\mathrm{S}_{\mathbb{F}_{q}^{c n / d}}(Q)} \\
& =\frac{1}{n} \ln \alpha_{F_{d, c h / d}^{\text {chk }}}(P, Q) \\
& \stackrel{(\mathrm{b})}{\leq} \frac{c}{d} \delta_{d}(P(0), Q(0))+c \Delta_{c n}(P),
\end{aligned}
$$

where (a) follows from (34) and (b) follows from (27). This concludes (35) and hence completes the proof.
To prove Theorem 5.4, we need the following lemma.
Lemma D.1: For all $x, y \in[0,1]$,

$$
\delta_{d}(x, y) \leq J_{d}(x, y) \leq \ln \left[1+(q y-1)\left(\frac{q x-1}{q-1}\right)^{d}\right]
$$

where $\delta_{d}(x, y)$ and $J_{d}(x, y)$ are defined by (30) and (32), respectively.

Proof: When $x \in(0,1)$ and $y \in[0,1]$, the first inequality clearly holds by taking $\hat{x}=x$ in (30). If however $x=0$, then

$$
\lim _{\hat{x} \rightarrow 0} \delta_{d}(0, \hat{x}, y)=\lim _{\hat{x} \rightarrow 0}\left(d \ln \frac{1}{1-\hat{x}}+J_{d}(\hat{x}, y)\right)=J_{d}(0, y) .
$$

Hence $\delta_{d}(0, y) \leq J_{d}(0, y)$. A similar argument also applies to the case of $x=1$. The second inequality follows from Jensen's inequality.

Proof of Theorem 5.4: By Theorem 5.3, Lemma D.1, and the condition $r_{0}=d / c$, it follows that

$$
\begin{aligned}
& \frac{1}{n} \ln \alpha_{F_{c, d, n}}^{\llcorner\mathrm{D}} \\
&(P, Q)
\end{aligned} \leq \frac{1}{r_{0}} \ln \left[1+(q Q(0)-1)\left(\frac{q P(0)-1}{q-1}\right)^{d}\right]+c \Delta_{c n}(P)
$$

where (a) follows from the strict increasing property of $\ln x, Q(0) \in[0,1]$, and the inequality

$$
\binom{n}{n P} \geq \frac{1}{(n+1)^{q}} e^{n H(P)} \quad(\text { see [55, Lemma 2.3]). }
$$

Note that $\lim _{n \rightarrow \infty} q \ln (c n+1) / n=0$. All conclusions of the theorem follow immediately.
Proof of Proposition 5.7:

$$
\begin{aligned}
\max _{\substack{P \in \mathcal{P}_{n}^{*}(\mathcal{X}), Q \in \mathcal{P}_{m}(\mathcal{Y})}} \alpha_{f}(P, Q) & \geq \max _{Q \in \mathcal{P}_{m}(\mathcal{Y})} \frac{\mathrm{S}_{f}\left(Q \mid P_{a^{n}}\right)}{\mathrm{S}_{\mathcal{Y}_{m}^{m}}(Q)} \\
& \geq \frac{\max _{Q \in \mathcal{P}_{m}(\mathcal{Y})} \mathrm{S}_{f}\left(Q \mid P_{a^{n}}\right)}{\max _{Q \in \mathcal{P}_{m}(\mathcal{Y})} \mathrm{S}_{\mathcal{Y}^{m}}(Q)} \\
& =\frac{|\mathcal{Y}|^{m}}{\max _{Q \in \mathcal{P}_{m}(\mathcal{Y})}\left({ }_{m Q}^{m}\right)} \\
& \stackrel{(\mathrm{a})}{=} \frac{|\mathcal{Y}|^{m}}{\Theta\left(m^{-\frac{|\mathcal{Y}|-1}{2}}|\mathcal{Y}|^{m}\right)} \\
& =\Theta\left(m^{\frac{|\mathcal{Y}|-1}{2}}\right)
\end{aligned}
$$

where $a \in \mathcal{X} \backslash\{0\}$ and (a) follows from Stirling's approximation.

## Appendix E

## Omitted Material of Section II

Example E.1: We start with the binary $[7,4,3]$ Hamming code, the smallest non-trivial perfect code. Here $\mathcal{X}=$ $\mathcal{Y}=\mathbb{F}_{2}, n=4, m=7$. The encoder $f: \mathbb{F}_{2}^{4} \rightarrow \mathbb{F}_{2}^{7}$ can be taken as $f(\mathbf{x})=\mathbf{x G}_{1}$ with

$$
\mathbf{G}_{1}=\left(\begin{array}{ccccccc}
1 & 1 & 0 & 1 & 0 & 0 & 0  \tag{54}\\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1
\end{array}\right)
$$

(We have replaced the last row of the more common cyclic generating matrix by the complement of the third row.) This particular choice of $\mathbf{G}_{1}$ ensures that (1111111) encodes a message of weight 2 . The input-output weight distribution of $f$, counting the number of message-codeword pairs ( $\mathbf{x}, f(\mathbf{x})$ ) having fixed weight pair $\left(w_{1}, w_{2}\right) \in\{0,1,2,3,4\} \times\{0,1,2,3,4,5,6,7\}$, is given by the following array (with zero entries omitted):

| $w_{1} \backslash w_{2}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 |  |  |  |  |  |  |  |
| 1 |  |  |  | 3 | 1 |  |  |  |
| 2 |  |  | 2 | 3 |  |  | 1 |  |
| 3 |  |  |  | 1 | 3 |  |  |  |
| 4 |  |  |  | 1 |  |  |  |  |
|  |  |  |  |  |  |  |  |  |

Up to the normalizing factor $\frac{1}{16}$ this is also the spectrum of $f$ (since we are in the binary case). The function $\alpha_{f}(P, Q)$ or, equivalently, $\alpha_{f}\left(w_{1}, w_{2}\right)$ is obtained by dividing each entry of this array by the corresponding number $\binom{4}{w_{1}}\binom{7}{w_{2}}$ (total number of pairs $(\mathbf{x}, \mathbf{y}) \in \mathbb{F}_{2}^{4} \times \mathbb{F}_{2}^{7}$ having weight pair $\left.\left(w_{1}, w_{2}\right)\right)$ and scaling by $2^{11} / 2^{4}=128$. The numbers $\alpha_{f}\left(w_{1}, w_{2}\right)$ are shown in the following table:

| $w_{1} \backslash w_{2}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 128 |  |  |  |  |  |  |  |
| 1 |  |  |  | $\frac{96}{35}$ | $\frac{32}{35}$ |  |  |  |
| 2 |  |  |  | $\frac{128}{105}$ | $\frac{192}{105}$ |  |  | $\frac{64}{3}$ |
| 3 |  |  |  | $\frac{32}{35}$ | $\frac{96}{35}$ |  |  |  |
| 4 |  |  |  | $\frac{128}{35}$ |  |  |  |  |

The encoder $f$ has been chosen in such a way that it minimizes the maximum of $\alpha_{f}\left(w_{1}, w_{2}\right)$, taken over all $\left(w_{1}, w_{2}\right)$ with $w_{1} \neq 0$. The corresponding maximum is $\alpha_{f}(2,7)=\frac{64}{3}$.

As we shall see later, the Hamming code considered in Example E. 1 is not a good channel code in the sense of (3), because it contains the all-one codeword. This also implies that its associated encoder $f$ cannot have a small maximum of $\alpha_{f}$ (over all $\left(w_{1}, w_{2}\right)$ with $w_{1} \neq 0$ ). Indeed, the optimal encoder $f$ in Example E. 1 has $\max _{w_{1} \neq 0, w_{2}} \alpha_{f}\left(w_{1}, w_{2}\right)=\frac{64}{3}$, which is far from the lower bound

$$
\frac{2^{n}}{\max _{0 \leq k \leq n}\binom{n}{k}}=\frac{2^{7}}{\binom{7}{3}}=\frac{128}{35} \quad \text { (see Proposition 5.7) }
$$

for binary linear [7,4] codes. However, this lower bound can be achieved by choosing a different code, as our next example shows.

Example E.2: We extend the binary $[7,3,4]$ simplex code (even-weight subcode of the Hamming code) by a word of weight 1 to a linear $[7,4,1]$ code $C$. The weight distribution of $C$ is then $A_{0}=A_{1}=1, A_{2}=0, A_{3}=4$, $A_{4}=7, A_{5}=3, A_{6}=A_{7}=0$. The encoder $f: \mathbb{F}_{2}^{4} \rightarrow \mathbb{F}_{2}^{7}$ is chosen in such a way that the four codewords of small and large weight (weights 1 and 5 ) encode words of weight 2 . This can be done, since these four codewords are linearly dependent. For example, we can choose $f(\mathbf{x})=\mathbf{x G}_{2}$ with

$$
\mathbf{G}_{2}=\left(\begin{array}{lllllll}
1 & 1 & 1 & 1 & 0 & 0 & 0  \tag{55}\\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}\right)
$$

The input-output weight distribution of $f$ is

| $w_{1} \backslash w_{2}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 |  |  |  |  |  |  |  |
| 1 |  |  |  | 2 | 2 |  |  |  |
| 2 |  | 1 |  |  | 2 | 3 |  |  |
| 3 |  |  |  | 2 | 2 |  |  |  |
| 4 |  |  |  | 1 |  |  |  |  |

and the maximum of $\alpha_{f}$ over all $\left(w_{1}, w_{2}\right)$ with $w_{1} \neq 0$ is

$$
\alpha_{f}(4,4)=\frac{1 / 2^{4}}{\binom{4}{4}\binom{7}{4} / 2^{11}}=\frac{128}{35},
$$

meeting the lower bound as asserted. Further values close to the lower bound are $\alpha_{f}(2,1)=\alpha_{f}(2,5)=\frac{64}{21}$.
From the perspective of traditional coding theory, it is absurd to state that a linear code of minimum distance one is better than a perfect linear code of minimum distance three (and otherwise the same parameters). This is mainly because the code length of our examples is too short. In fact, as length goes to infinity, any linear code that has a linear encoder achieving the lower bound of Proposition 5.7 (or up to an exponentially negligible factor) satisfies the asymptotic Gilbert-Varshamov (GV) bound (see [9, Remark 4.1] and Theorem F.6). Moreover, as proven in [9],
these linear encoders are universal for all sources and channels, although the decoder may be dependent on the source and channel. We shall dig into this issue in Section F, where we show that encoder (55) is in fact better than encoder (54) in some sense.

## Appendix F

## Omitted Material of Section III

For better understanding of the definitions of good linear encoders, let us review the original requirements of good linear encoders for lossless source coding, channel coding, and lossless JSCC, respectively. ${ }^{11}$

Lossless source coding [5]: A sequence $\boldsymbol{F}$ of random linear encoders with the asymptotic source transmission rate $R_{s}(\boldsymbol{F})$ is said to be $\delta$-asymptotically good for lossless source coding if for any $\epsilon>0$ there exists a sequence of events $A_{k} \in \mathcal{A}$ such that for sufficiently large $k$,

$$
\begin{gather*}
\mathrm{P}\left(A_{k}\right) \geq 1-\epsilon,  \tag{56}\\
\left|R_{s}\left(F_{k}\right)-R_{s}(\boldsymbol{F})\right| \leq \epsilon \quad \forall \omega \in A_{k},  \tag{57}\\
\max _{\mathbf{x}, \hat{\mathbf{x}}: \mathbf{x} \neq \hat{\mathbf{x}}} \frac{1}{n_{k}} \ln \mathrm{P}\left\{F_{k}(\mathbf{x})=F_{k}(\hat{\mathbf{x}}) \mid A_{k}\right\} \leq-R_{s}(\boldsymbol{F})+\delta+\epsilon . \tag{58}
\end{gather*}
$$

The use of event $A_{k}$ is to exclude some encoders with unwanted rates or some bad encoders that may have a major impact on the average performance. In coding theory such a technique is called "expurgating code ensembles". Since $F_{k}$ is linear, condition (58) is equivalent to

$$
\begin{equation*}
\max _{\mathbf{x}: \mathbf{x} \neq \mathbf{0}} \frac{1}{n_{k}} \ln \mathrm{P}\left\{\mathbf{x} \in \operatorname{ker} F_{k} \mid A_{k}\right\} \leq-R_{s}(\boldsymbol{F})+\delta+\epsilon \tag{59}
\end{equation*}
$$

Channel coding [23]-[25]: A sequence $\boldsymbol{F}$ of random linear encoders with the asymptotic channel transmission rate $R_{C}(\boldsymbol{F})$ is said to be $\delta$-asymptotically good for channel coding if for any $\epsilon>0$ there exists a sequence of events $A_{k} \in \mathcal{A}$ such that for sufficiently large $k$,

$$
\begin{gather*}
\mathrm{P}\left(A_{k}\right) \geq 1-\epsilon,  \tag{60}\\
\left|R_{c}\left(F_{k}\right)-R_{c}(\boldsymbol{F})\right| \leq \epsilon \quad \forall \omega \in A_{k},  \tag{61}\\
\max _{\mathbf{y}, \hat{\mathbf{y}}: \mathbf{y} \neq \hat{\mathbf{y}}} \frac{1}{m_{k}} \ln \left(\frac{\mathrm{P}\left\{\mathbf{y} \in \mathcal{C}_{F_{k}}, \hat{\mathbf{y}} \in \mathcal{C}_{F_{k}} \mid A_{k}\right\}}{\mathrm{P}\left\{\mathbf{y} \in \mathcal{C}_{F_{k}} \mid A_{k}\right\} \mathrm{P}\left\{\hat{\mathbf{y}} \in \mathcal{C}_{F_{k}} \mid A_{k}\right\}}\right) \leq \delta+\epsilon, \tag{62}
\end{gather*}
$$

where $\mathcal{C}_{F_{k}}:=F_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)+\bar{Y}^{m_{k}}$, and $\bar{Y}^{m_{k}}$ is a uniform random vector on $\mathbb{F}_{q}^{m_{k}}$. Clearly, for any $f_{k} \in F_{k}(\Omega)$, $\mathrm{P}\left\{\mathbf{y} \in \mathcal{C}_{f_{k}}\right\}=f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right) /\left|q^{m_{k}}\right|$, so it follows from (61) that

$$
\left|\frac{1}{m_{k}} \ln \mathrm{P}\left\{\mathbf{y} \in \mathcal{C}_{F_{k}} \mid A_{k}\right\}+\ln q-R_{c}(\boldsymbol{F})\right| \leq \epsilon
$$

for all $\mathbf{y} \in \mathbb{F}_{q}^{m_{n}}$. Because $F_{k}$ is linear, we also have

$$
\begin{aligned}
\mathrm{P}\left\{\mathbf{y} \in \mathcal{C}_{F_{k}}, \hat{\mathbf{y}} \in \mathcal{C}_{F_{k}} \mid A_{k}\right\} & =\mathrm{P}\left\{\mathbf{y} \in \mathcal{C}_{F_{k}}, \hat{\mathbf{y}}-\mathbf{y} \in F_{k}\left(\mathbb{F}_{q}^{n_{k}}\right) \mid A_{k}\right\} \\
& =\sum_{f_{k}} \mathrm{P}\left\{F_{k}=f_{k} \mid A_{k}\right\} \mathrm{E}\left[1\left\{\mathbf{y} \in \mathcal{C}_{f_{k}}, \hat{\mathbf{y}}-\mathbf{y} \in f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)\right\}\right] \\
& =\sum_{f_{k}} \mathrm{P}\left\{F_{k}=f_{k} \mid A_{k}\right\} \mathrm{P}\left\{\mathbf{y} \in \mathcal{C}_{f_{k}}\right\} 1\left\{\hat{\mathbf{y}}-\mathbf{y} \in f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)\right\} .
\end{aligned}
$$

[^10]This together with (61) gives

$$
\left|\frac{1}{m_{k}} \ln \frac{\mathrm{P}\left\{\mathbf{y} \in \mathcal{C}_{F_{k}}, \hat{\mathbf{y}} \in \mathcal{C}_{F_{k}} \mid A_{k}\right\}}{\mathrm{P}\left\{\hat{\mathbf{y}}-\mathbf{y} \in F_{k}\left(\mathbb{F}_{q}^{n_{k}}\right) \mid A_{k}\right\}}+\ln q-R_{c}(\boldsymbol{F})\right| \leq \epsilon,
$$

so that condition (62) can be rewritten as

$$
\begin{equation*}
\max _{\mathbf{y}: \mathbf{y} \neq \mathbf{0}} \frac{1}{m_{k}} \ln \mathrm{P}\left\{\mathbf{y} \in F_{k}\left(\mathbb{F}_{q}^{n_{k}}\right) \mid A_{k}\right\} \leq R_{c}(\boldsymbol{F})-\ln q+\delta+\epsilon . \tag{63}
\end{equation*}
$$

Lossless JSCC [9]: A sequence $\boldsymbol{F}$ of random linear encoders is said to be $\delta$-asymptotically good for lossless JSCC if

$$
\begin{equation*}
\limsup _{k \rightarrow \infty} \max _{\substack{\mathbf{x}, \mathbf{x}: \mathbf{x} \neq \hat{\mathbf{x}} \\ \mathbf{y}, \hat{\mathbf{y}}}} \frac{1}{n_{k}} \ln \left(\frac{\mathrm{P}\left\{\mathcal{F}_{F_{k}}(\mathbf{x})=\mathbf{y}, \mathcal{F}_{F_{k}}(\hat{\mathbf{x}})=\hat{\mathbf{y}}\right\}}{\mathrm{P}\left\{\mathcal{F}_{F_{k}}(\mathbf{x})=\mathbf{y}\right\} \mathrm{P}\left\{\mathcal{F}_{F_{k}}(\hat{\mathbf{x}})=\hat{\mathbf{y}}\right\}}\right) \leq \delta, \tag{64}
\end{equation*}
$$

where $\mathcal{F}_{F_{k}}(\mathrm{x}):=F_{k}(\mathrm{x})+\bar{Y}^{m_{k}}$ and $\bar{Y}^{m_{k}}$ is a uniform random vector on $\mathbb{F}_{q}^{m_{k}}$. By the arguments in the proof of [9, Proposition 2.6], we have the following alternative condition:

$$
\begin{equation*}
\limsup _{k \rightarrow \infty} \max _{\substack{\mathbf{x}: \mathbf{x} \neq \mathbf{y} \\ \mathbf{y}}} \frac{1}{n_{k}} \ln \left(q^{m_{k}} \mathrm{P}\left\{F_{k}(\mathbf{x})=\mathbf{y}\right\}\right) \leq \delta . \tag{65}
\end{equation*}
$$

The requirements above are fundamental, but are not easy and convenient for use. The next three propositions show that spectra of linear encoders can serve as alternative criteria for good linear encoders, and that the uniform random permutation is a useful tool for constructing good linear encoders.

Proposition F.1: Let $\boldsymbol{F}$ be a sequence of random linear encoders with the asymptotic source transmission rate $R_{s}(\boldsymbol{F})$. If $\boldsymbol{F}$ satisfies the kernel-spectrum condition (2), then the sequence of random linear encoders $F_{k}^{\sim}=F_{k} \circ \Sigma_{n_{k}}$ is $\delta$-asymptotically good for lossless source coding.

Proof: For any $\epsilon>0$, define the sequence of events

$$
A_{k}=\left\{\omega \in \Omega:\left|R_{s}\left(F_{k}^{\sim}\right)-R_{s}(\boldsymbol{F})\right| \leq \frac{\epsilon}{3}\right\} .
$$

It is clear that $\lim _{k \rightarrow \infty} \mathrm{P}\left(A_{k}\right)=1$, so that conditions (56) and (57) hold. Furthermore, we have

$$
\begin{aligned}
\max _{\mathbf{x}: \mathbf{x} \neq 0^{n_{k}}} \frac{1}{n_{k}} \ln \mathrm{P}\left\{\mathbf{x} \in \operatorname{ker} F_{k}^{\sim} \mid A_{k}\right\}= & \max _{\mathbf{x}: \mathbf{x} \neq 0^{n_{k}}} \frac{1}{n_{k}} \ln \mathrm{E}\left[\left.\frac{q^{n_{k}} 1\left\{\mathbf{x} \in \operatorname{ker} F_{k}^{\sim}\right\}}{\left|F_{k}^{\sim}\left(\mathbb{F}_{q}^{n_{k}}\right)\right|\left|\operatorname{ker} F_{k}^{\sim}\right|} \right\rvert\, A_{k}\right] \\
\leq & \max _{\mathbf{x}: \mathbf{x} \neq 0^{n_{k}}} \frac{1}{n_{k}} \ln \mathrm{E}\left[\left.\frac{q^{n_{k}} 1\left\{\mathbf{x} \in \Sigma_{n_{k}}\left(\operatorname{ker} F_{k}\right)\right\}}{e^{n_{k}\left(R_{s}(\boldsymbol{F})-\frac{\varepsilon}{3}\right.}\left|\operatorname{ker} F_{k}\right|} \right\rvert\, A_{k}\right] \\
\leq & -R_{S}(\boldsymbol{F})+\frac{\epsilon}{3} \\
& +\max _{\mathrm{x}: \mathbf{x} \neq 0^{n_{k}}} \frac{1}{n_{k}} \ln \left(\frac{q^{n_{k}}}{\mathrm{P}\left(A_{k}\right)} \mathrm{E}\left[\frac{1\left\{\mathbf{x} \in \Sigma_{n_{k}}\left(\operatorname{ker} F_{k}\right)\right\}}{\left|\operatorname{ker} F_{k}\right|}\right]\right) \\
& \stackrel{(\mathrm{a})}{\leq}-R_{s}(\boldsymbol{F})+\frac{2 \epsilon}{3}+\max _{P \in \mathcal{P}_{n_{k}}} \frac{1}{n_{k}} \ln \alpha_{\text {ker } F_{k}}(P) \\
& \stackrel{(\mathrm{b})}{\leq}-R_{s}(\boldsymbol{F})+\delta+\epsilon
\end{aligned}
$$

for sufficiently large $k$, where (a) follows from Proposition 6.2 and $\lim _{k \rightarrow \infty} \mathrm{P}\left(A_{k}\right)=1$, and (b) follows from (2). This concludes (59) and hence proves the proposition.

Proposition F.2: Let $\boldsymbol{F}$ be a sequence of random linear encoders with the asymptotic channel transmission rate $R_{c}(\boldsymbol{F})$. If $\boldsymbol{F}$ satisfies the image-spectrum condition (3), then the sequence of random linear encoders $\Sigma_{m_{k}} \circ F_{k}$ is $\delta$-asymptotically good for channel coding.

Proof: Use argument similar to that of Proposition F.1.
Proposition F.3: Let $\boldsymbol{F}$ be a sequence of random linear encoders. If $\boldsymbol{F}$ satisfies the joint-spectrum condition (4), then the sequence of random linear encoders $\tilde{F}_{k}=\Sigma_{m_{k}} \circ F_{k} \circ \Sigma_{n_{k}}$ is $\delta$-asymptotically good for lossless JSCC.

Proof: Apply [9, Proposition 2.4].
Remark F.4: It should be noted that conditions (2)-(4) are only sufficient but not necessary. In other words, there may exist other good (random) linear encoders. Note, for example, that condition (3) requires that the average
spectrum of $F_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)$ should be uniformly close to the spectrum of $\mathbb{F}_{q}^{m_{k}}$. This requirement is obviously very strict. For instance, all linear codes containing the all-one vector are excluded by this condition with $\delta=0$. To take into account such cases, we would need to use some sophisticated bounding techniques (see e.g., [25]), which are still not mature or even infeasible (Example F.8) for more complicated cases. So for simplicity of analysis, we choose conditions (2)-(4). At least to some extent, this choice is reasonable. This is because $F_{n_{k}, m_{k}}^{\mathrm{RLC}}$ is asymptotically SCC-good (resp. SC- and CC-good), and hence it can be shown by Markov's inequality that most linear encoders are asymptotically SCC-good (resp. SC- and CC-good).

Remark F.5: Conditions (2)-(4) all apply to sequences of random encoders. For readers not familiar with probabilistic analysis we provide some further explanation: First, a deterministic encoder is a special random encoder, so conditions (2)-(4) also apply to a sequence of deterministic encoders. Second, for example, if a sequence of random linear encoders is $\delta$-asymptotically SCC-good, then there exists a sequence of sample encoders that is $\delta$-asymptotically SCC-good ([9, Proposition 4.1]). The proof of this fact relies on Markov's inequality and the fact that the size of the set over which the maximum (in (4)) is taken is a polynomial function of $m_{k}$ and $n_{k}$. In fact, by the same argument, we can obtain a stronger result:

$$
\lim _{k \rightarrow \infty} \mathrm{P}\left\{\max _{P \in \mathcal{P}_{n_{k}}^{*}, Q \in \mathcal{P}_{m_{k}}} \frac{1}{n_{k}} \ln \frac{\mathrm{~S}_{F_{k}}(P, Q)}{\mathrm{S}_{\mathbb{F}_{q}^{n_{k}} \times \mathbb{F}_{q}^{m_{k}}}(P, Q)} \leq \delta+\epsilon\right\}=1
$$

for any $\epsilon>0$. Third, as we shall see, a typical sample encoder of good random encoders (in the sense of (2)-(4)) has a fundamental property, which is characterized by the so-called entropy weight, where "typical" means that the set of such encoders contains most of the probability mass. Since the proof is again a simple application of Markov's inequality, we will leave it to the reader as an exercise.

For better understanding of these three kinds of good linear encoders, let us take a look from another perspective. We define the entropy weight $H(\mathbf{x})$ of a vector $\mathbf{x} \in \mathbb{F}_{q}^{n}$ by $H(\mathbf{x}):=\ln \binom{n}{n P_{x}}$, and the normalized entropy weight as $h(\mathbf{x})=H(\mathbf{x}) / n$. Recalling the identity

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \ln \binom{n}{n P}=H(P)
$$

(cf. [55, Lemma 2.3]),
we have $h(\mathbf{x}) \approx H\left(P_{\mathbf{x}}\right)$, so $H\left(P_{\mathbf{x}}\right)$ may be used as a substitute for $h(\mathbf{x})$ in the asymptotic sense, and accordingly we call $H\left(P_{\mathbf{x}}\right)$ the asymptotic normalized entropy weight of $\mathbf{x}$. Now suppose $\delta=0$ and $k$ is large enough. From (2) and [9, Proposition 2.1], it follows that a typical linear encoder $f_{k}$ of an asymptotically SC-good $F_{k}$ satisfies

$$
\limsup _{k \rightarrow \infty} \frac{1}{n_{k}} \ln \left|\mathcal{T}_{P}^{n_{k}} \cap \operatorname{ker} f_{k}\right| \leq H(P)-R_{s}(\boldsymbol{F})
$$

where $P \in \mathcal{P}_{n_{k}}^{*}$. This implies that ker $f_{k}$ does not contain nonzero vectors of normalized entropy weight less than $R_{s}(\boldsymbol{F})$. Similarly, it follows from (3) that a typical linear encoder $f_{k}$ of an asymptotically CC-good $F_{k}$ satisfies

$$
\limsup _{k \rightarrow \infty} \frac{1}{m_{k}} \ln \left|\mathcal{T}_{Q}^{m_{k}} \cap f_{k}\left(\mathbb{F}_{q}^{n_{k}}\right)\right| \leq H(Q)+R_{c}(\boldsymbol{F})-\ln q,
$$

where $Q \in \mathcal{P}_{m_{k}}^{*}$. The case of (4) is more complicated (cf. Theorem F.6). It follows that a typical linear encoder $f_{k}$ of an asymptotically SCC-good $F_{k}$ satisfies

$$
\limsup _{k \rightarrow \infty} \frac{1}{n_{k}} \ln \left|\left(\mathcal{T}_{P}^{n_{k}} \times \mathcal{T}_{Q}^{m_{k}}\right) \cap \operatorname{rl}\left(f_{k}\right)\right| \leq H(P)+\frac{H(Q)-\ln q}{\bar{R}(\boldsymbol{F})},
$$

where $P \in \mathcal{P}_{n_{k}}^{*}$ and $Q \in \mathcal{P}_{m_{k}}$. This implies that $\operatorname{rl}\left(f_{k}\right)$ does not contain pairs of low-entropy-weight vectors except $\left(0^{n_{k}}, 0^{m_{k}}\right)$. In other words, all nonzero input vectors of low entropy weight must be mapped to high-entropy-weight vectors. In particular, all small-weight vectors, as well as the all- $x$ vectors for $x \in \mathbb{F}_{q}^{\times}$, must be mapped to vectors of close-to-uniform type and hence (Hamming) weight around $m_{k}(1-1 / q)$.

From the perspective of entropy weight, many linear codes with large minimum distance are not good because they contain vectors of low entropy weight, for example the all-one vector. In fact, entropy weight guides us to a subclass of linear codes with not only large minimum distance but also large minimum entropy distance [56]. At this point, it is appropriate to mention a bound that includes the above three properties.

Theorem F.6 ([9]): For any $r>0$, there is a sequence $\boldsymbol{f}$ of linear encoders $f_{k}: \mathbb{F}_{q}^{n_{k}} \rightarrow \mathbb{F}_{q}^{m_{k}}$ such that $R(\boldsymbol{f})=r$ and

$$
\begin{equation*}
\liminf _{k \rightarrow \infty} \min _{\mathbf{x} \neq 0^{n_{k}}}\left(H\left(P_{\mathbf{x}}\right) R\left(f_{k}\right)+H\left(P_{f_{k}(\mathbf{x})}\right)\right) \geq \ln q \tag{66}
\end{equation*}
$$

where $f_{k}$ is injective for $r \leq 1$ and surjective for $r \geq 1$.
Theorem F. 6 is a special case of [ 9 , Theorem 4.1] for $\delta=0$ (but with some simple improvements), and may be regarded as an extension of the asymptotic GV bound. For $r>1$, we can take $f_{k}(\mathbf{x})=\mathbf{0}^{m_{k}}$ and then get

$$
\liminf _{k \rightarrow \infty} \min _{\mathbf{x} \in \operatorname{ker} f_{k} \backslash\left\{0^{\left.n_{k}\right\}}\right.} H\left(P_{\mathbf{x}}\right) \geq r^{-1} \ln q=R_{s}(\boldsymbol{f}) .
$$

If $r<1$, since $H\left(P_{\mathbf{x}}\right) \leq \ln q$, we have

$$
\liminf _{k \rightarrow \infty} \min _{\mathbf{x} \neq 0^{n_{k}}} H\left(P_{f_{k}(\mathbf{x})}\right) \geq \ln q-r \ln q=\ln q-R_{c}(\boldsymbol{f})
$$

Although the left-hand side of (66) provides a refinement of traditional minimum Hamming distance, it still cannot ensure good coding performance. In fact, condition (4) (resp., (2) and (3)) requires the joint (resp., kernel and image) spectrum of the encoder to be close to the average joint (resp., kernel and image) spectrum of all linear encoders of the same coding rate (cf. Remark F.9). We may call such encoders random-like encoders. Since minimum Hamming weight and minimum entropy weight only focus on one or two specific points of weight distribution or spectrum, linear encoders designed under these criteria cannot be universally good (see Example F.8).

So far, we have extensively discussed criteria of good linear encoders in an abstract manner. A comparison between the linear encoders (54) and (55) in Examples E. 1 and E. 2 will help the reader understand why we care about joint spectra and why about the whole shape of the spectrum. The following example shows that the joint spectrum has such a great impact on the performance of lossless JSCC, that a perfect linear code of minimum distance three may perform worse than a linear code of minimum distance one (and otherwise the same parameters) if the generator matrix is not carefully chosen.

Example F.7: Consider a zero-one binary independent and identically distributed (IID) source with probability $p_{1}$ of symbol 1 and a binary symmetric channel (BSC) with crossover probability $p_{2}$. Further consider a coding scheme that transmits four source symbols by utilizing the channel seven times. The scheme is based on Fig. 1 with the quatization module removed, where the linear encoder used is either $\mathbf{G}_{1}$ or $\mathbf{G}_{2}$ defined by (54) and (55), respectively. Because the source and channel are both IID and the channel is an additive noise channel over $\mathbb{F}_{2}$, the two random interleavers and the random vector module in Fig. 1 can all be omitted. Since the code length is short, we can easily compute the exact decoding error probabilities under maximum a posteriori (MAP) decoding. For further comparison, we also include results for the linear encoders

$$
\mathbf{G}_{3}=\left(\begin{array}{lllllll}
1 & 1 & 0 & 1 & 0 & 0 & 0  \tag{67}\\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)
$$

and

$$
\mathbf{G}_{4}=\left(\begin{array}{ccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0  \tag{68}\\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}\right)
$$

which yield the same linear code as $\mathbf{G}_{1}$ and $\mathbf{G}_{2}$, respectively. Fig. 3 compares the performance of $\mathbf{G}_{i}(1 \leq i \leq 4)$ for $p_{1} \in(0,0.02)$ and $p_{2}=0.16$. For example, the order of their performance at $p=0.008$, from best to worst, is $\mathbf{G}_{2}, \mathbf{G}_{3}, \mathbf{G}_{1}$, and $\mathbf{G}_{4}$. There are two interesting facts to be learned. First, $\mathbf{G}_{3}$ outperforms $\mathbf{G}_{1}$ and $\mathbf{G}_{2}$ outperforms $\mathbf{G}_{4}$. This implies that the choice of generator matrix does have an impact on JSCC performance. Second, $\mathbf{G}_{2}$ beats $\mathbf{G}_{1}$ for all $p_{1} \in(0,0.02)$. This surprising result shows that in JSCC, a perfect code of minimum distance three may perform worse than a code of minimum distance one if the generator matrix is not chosen properly.

In order to explain this phenomenon, we shall introduce the concept of pairwise discrimination, which forms the key idea of lossless JSCC and will now be expressed in an intuitive but less strict manner. Recall the concept of a


Fig. 3. MAP decoding error probability versus $p_{1}$ comparison among $\mathbf{G}_{i}$ for $p_{1} \in(0,0.02)$ and $p_{2}=0.16$.
typical set (cf. [57]). Let $X^{n}=\left(X_{1}, X_{2}, \ldots, X_{n}\right)$ be a random $n$-dimensional vector over $\mathbb{F}_{q}$. The typical set $A_{\epsilon}^{(n)}$ of $X^{n}$ is defined as the set of all vectors $\mathbf{x} \in \mathbb{F}_{q}^{n}$ satisfying

$$
\begin{equation*}
e^{-n\left(h\left(X^{n}\right)+\epsilon\right)} \leq \mathrm{P}_{X^{n}}(\mathbf{x}) \leq e^{-n\left(h\left(X^{n}\right)-\epsilon\right)} \tag{69}
\end{equation*}
$$

where $h\left(X^{n}\right):=H\left(X^{n}\right) / n=-n^{-1} \mathrm{E}\left[\ln \mathrm{P}_{X^{n}}\left(X^{n}\right)\right]$ and $\mathrm{P}_{X^{n}}(\mathrm{x}):=\mathrm{P}\left\{X^{n}=\mathrm{x}\right\}$. Usually, we add some conditions to ensure that $h\left(X^{n}\right)$ converges to the so-called entropy rate as $n \rightarrow \infty$ and that $-n^{-1} \ln \mathrm{P}_{X^{n}}\left(X^{n}\right)$ converges to the entropy rate almost surely. But here, we just borrow the concept and do not rigorously justify every detail. Two distinct $n$-dimensional vectors are considered to be discriminable if at least one of them is not in the typical set $A_{\epsilon}^{(n)}$. In a more intuitive manner, we may think of two distinct vectors indiscriminable if both of them are elements of the high-probability set $B_{\epsilon}^{(n)}:=\left\{\mathbf{x} \in \mathbb{F}_{q}^{n}: \mathrm{P}_{X^{n}}(\mathbf{x}) \geq e^{-n\left(h\left(X^{n}\right)+\epsilon\right)}\right\}$.

Roughly speaking, the art of lossless JSCC is to focus on the most probable source vectors (with a high total probability) and to choose appropriate channel input vectors for them so that all these source vectors, combined with any channel output vector in the high-probability set, are pairwise discriminable.

Keeping this idea in mind, we continue the discussion of Example F.7. We note that the most probable source vectors are the zero vector and all weight-one vectors, whose total probability is $\left(1-p_{1}\right)^{7}+7 p_{1}\left(1-p_{1}\right)^{6} \geq 0.992$ for $p_{1} \in(0,0.02)$. Because the zero vector owns the dominant probability, we only need to pay attention to the pairs consisting of the zero vector and a vector of weight one. Other pairs consisting of two weight-one vectors may be ignored. Therefore, the performance of an injective linear encoder is mainly determined by its output for weight-one input. The output weight distribution of $\mathbf{G}_{i}$ for weight-one input is listed in Table I. In order to make

TABLE I
Output weight distributions of $\mathbf{G}_{i}$ FOR weight-one input

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathbf{G}_{1}$ |  |  |  | 3 | 1 |  |  |  |
| $\mathbf{G}_{2}$ |  |  |  | 2 | 2 |  |  |  |
| $\mathbf{G}_{3}$ |  |  |  | 3 |  |  |  | 1 |
| $\mathbf{G}_{4}$ |  | 1 |  | 2 | 1 |  |  |  |

the zero vector and all weight-one vectors discriminable at channel output, a good strategy is to map these source vectors to channel input vectors as far from each other as possible in terms of Hamming distance (since the channel is a BSC). Therefore, we shall get a boost in performance if we map weight-one vectors to vectors of weight as large as possible. Comparing the output weight distributions of $\mathbf{G}_{i}$ in Table I, especially for weights $\geq 4$, it is easy to figure out that $\mathbf{G}_{2}$ is better than $\mathbf{G}_{1}, \mathbf{G}_{3}$ better than $\mathbf{G}_{1}$, and $\mathbf{G}_{4}$ worse than $\mathbf{G}_{2}$. The comparison between $\mathbf{G}_{2}$
and $\mathbf{G}_{3}$ is slightly more complicated, because one has two vectors of weight 4 while the other has one vector of weight 7 . This explains why $\mathbf{G}_{2}$ and $\mathbf{G}_{3}$ have almost the same performance for small $p$, as shown in Fig. 3.

In Example F. 7 we successfully explained the importance of the joint spectrum of a linear encoder, but in this case choosing the all-one vector as a codeword is not a bad idea, which seems contrary to facts about entropy weight, that is, the all-one vector is of zero entropy weight and hence must be avoided. While the viewpoint of minimum Hamming distance is very appropriate for coding over a BSC, it is not a good measure for designing universal linear encoders. The next example shows that a binary linear code containing the all-one vector may have very bad performance for some special channels, even if it is a perfect code.

Example F.8: For any nonzero $\mathbf{x}_{0} \in \mathbb{F}_{q}^{n}$, define an additive noise channel $J_{\mathbf{x}_{0}}: \mathbb{F}_{2}^{n} \rightarrow \mathbb{F}_{2}^{n}$ by $\mathbf{x} \mapsto \mathbf{x}+\mathbf{N}$, where $\mathbf{N}$ is a random noise with distribution

$$
\mathrm{P}\{\mathbf{N}=\mathbf{x}\}:= \begin{cases}0.5, & \text { for } \mathbf{x}=\mathbf{0} \\ 0.5, & \text { for } \mathbf{x}=\mathbf{x}_{0} \\ 0, & \text { otherwise }\end{cases}
$$

Clearly, the capacity of $J_{\mathbf{x}_{0}}$ is $(n-1) \ln 2$, independent of the choice of $\mathbf{x}_{0}$. Now consider a channel coding scheme based on Fig. 1 with the quantization module removed. It transmits four bits over a vector channel $J_{1^{\ell} 0^{7-\ell}}(\ell=1$, $2, \ldots, 7$ ). The linear encoder used is $\mathbf{G}_{1}$ or $\mathbf{G}_{2}$ defined by (54) and (55), respectively. The interleaver before linear encoder and the random vector module can be omitted because the source is assumed to be uniform and the channel noise is additive. It is easy to figure out the decoding error probability. The trick is to check whether the noise vector $1^{\ell} 0^{7-\ell}$ hits a codeword of the linear code. If it misses, the transmitted information can be decoded successfully; otherwise, the information can only be guessed with error probability $\frac{1}{2}$. Owing to the random interleaver after linear encoder, we should compute the decoding error probability for each possible interleaver and then compute their average. Accordingly, for channel $J_{1^{\ell} 0^{7-\ell}}$, the decoding error probability of a linear code is $n_{\ell} /\left(2\binom{7}{\ell}\right)$ where $n_{\ell}$ is the number of codewords of weight $\ell$. Table II lists the decoding error probability of $\mathbf{G}_{1}$ and $\mathbf{G}_{2}$ for $\ell=1,2$, $\ldots, 7$. Note that for $\ell=7$, the performance of $\mathbf{G}_{1}$ is very bad. This is because there is only one vector of weight

TABLE II
Decoding error probability of $\mathbf{G}_{i}$ FOR $\ell=1,2, \ldots, 7$

|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{G}_{1}$ | 0 | 0 | $\frac{1}{10}$ | $\frac{1}{10}$ | 0 | 0 | $\frac{1}{2}$ |
| $\mathbf{G}_{2}$ | $\frac{1}{14}$ | 0 | $\frac{2}{35}$ | $\frac{1}{10}$ | $\frac{1}{14}$ | 0 | 0 |

7 and hence the random interleaver cannot help the codeword avoid being hit by noise. Note that this issue cannot be resolved by simply increasing the code length, so any binary linear code containing the all-one vector performs bad over channel $J_{1^{n}}$.

The reader may argue that Example F. 8 is too special and that random-like encoders perhaps do not work in certain examples. The fact that follows will show that a random-like encoder defined by (4) is universally good in the asymptotic sense. We continue to utilize the concept of pairwise discrimination as well as typical set in a less strict manner.

Remark F.9: Consider a pair $\left(X^{n}, Y^{m}\right)$ of random vectors and its typical set $A_{\epsilon}^{(n, m)}$. Let $f: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{m}$ be a linear encoder whose joint spectrum is approximately

$$
\begin{equation*}
\mathrm{S}_{\mathbb{F}_{q}^{n}}(P) \mathrm{S}_{\mathbb{F}_{q}^{m}}(Q)=q^{-(m+n)}\binom{n}{n P}\binom{m}{m Q} \tag{4}
\end{equation*}
$$

for $P \in \mathcal{P}_{n}^{*}$ and $Q \in \mathcal{P}_{m}$. It follows from [9, Proposition 2.4] that for any distinct pairs ( $\mathbf{x}, \mathbf{y}$ ) and ( $\left.\mathbf{x}^{\prime}, \mathbf{y}^{\prime}\right)$ in $A_{\epsilon}^{(n, m)}$,

$$
\mathrm{P}\left\{\tilde{f}\left(\mathbf{x}^{\prime}\right)=\mathbf{y}^{\prime} \mid \tilde{f}(\mathbf{x})=\mathbf{y}\right\}=\mathrm{P}\left\{\tilde{f}\left(\mathbf{x}^{\prime}-\mathbf{x}\right)=\mathbf{y}^{\prime}-\mathbf{y} \mid \tilde{f}(\mathbf{x})=\mathbf{y}\right\}= \begin{cases}0, & \text { for } \mathbf{x}=\mathbf{x}^{\prime} \\ q^{-m}, & \text { otherwise }\end{cases}
$$

so that

$$
\begin{equation*}
\mathrm{P}\left\{\left|A_{\epsilon}^{(n, m)} \cap \operatorname{rl}(\tilde{f})\right|>1 \mid \tilde{f}(\mathbf{x})=\mathbf{y}\right\} \leq q^{-m}\left|A_{\epsilon}^{(n, m)}\right| . \tag{70}
\end{equation*}
$$

Note that $\left|A_{\epsilon}^{(m, n)}\right|$ is bounded above by $e^{H\left(X^{n}, Y^{m}\right)+n \epsilon}$ (cf. (69)), so if $m \ln q>H\left(X^{n}, Y^{m}\right)+n \epsilon$, the probability of $\operatorname{rl}(\tilde{f})$ containing other members of $A_{\epsilon}^{(m, n)}$ given $(\mathbf{x}, \tilde{f}(\mathbf{x}))$ being a member of $A_{\epsilon}^{(m, n)}$ is asymptotically negligible. In other words, with high probability, the pair $(\mathbf{x}, \tilde{f}(\mathbf{x}))$ is pairwise discriminable with every other pair in $\operatorname{rl}(\tilde{f})$. Note that (70) does not depend on the probability distribution of $\left(X^{n}, Y^{m}\right)$, but only on their joint entropy.

What is the use of (70)? Imagine we are transmitting source vector $X^{n}$ over a channel and suppose that the current sample drawn from $X^{n}$ is $\mathbf{x}$. If we send $\tilde{f}(\mathbf{x})$ over the channel, then after receiving the channel output $\mathbf{z}$, using knowledge of the source and channel, we get the a posteriori information about the channel input, identified with $Y^{m}=Y^{m}(\mathbf{z})$. Combining it with the a priori knowledge of the source, we obtain a pair of random vectors, $\left(X^{n}, Y^{m}\right)$. In a typical case, the pair $(\mathbf{x}, \tilde{f}(\mathbf{x}))$ must be a member of the typical set $A_{\epsilon}^{(n, m)}$ of $\left(X^{n}, Y^{m}\right)$, so if

$$
\begin{equation*}
m \ln q>H\left(X^{n}, Y^{m}\right)+n \epsilon \tag{71}
\end{equation*}
$$

for some $\epsilon>0$, we can decode successfully with high probability by guessing the unique typical pair. To illustrate this further, let us consider two special cases.

First, we suppose that the channel is noiseless, so that $H\left(X^{n}, Y^{m}\right)=H\left(X^{n}\right)$. Condition (71) then becomes

$$
\frac{m}{n} \ln q>\frac{1}{n} H\left(X^{n}\right)+\epsilon,
$$

a familiar condition for the achievable rate of lossless source coding.
Second, we suppose that the source is uniformly distributed (i.e., channel coding), so we can assume that $H\left(X^{n}, Y^{m}\right)=n \ln q+H\left(Y^{m}\right)$. Condition (71) then becomes

$$
n \ln q<m \ln q-H\left(Y^{m}\right)-n \epsilon
$$

and further,

$$
R_{c}(f) \leq \frac{n}{m} \ln q<\ln q-\frac{1}{m} H\left(Y^{m}\right)-R(f) \epsilon .
$$

Note that $\ln q-\frac{1}{m} H\left(Y^{m}\right)$ has the same form as the capacity formula of those channels whose capacity is achieved by the uniform input probability distribution. In fact, if we add a random vector module and a quantization module (as depicted in Fig. 1) to simulate the capacity-achieving input probability distribution of a given channel, we can eventually obtain a capacity-achieving coding scheme, but we shall not delve further into this because we are interested in its relevance for coding rather than its nature as a problem of information theory.

In Remark F. 9 we showed that an asymptotically SCC-good linear encoder is universally good for JSCC. By a similar argument, we can also show that other kinds of random-like encoders, i.e., those defined by (2) and (3), are also universally good for lossless source coding and channel coding, respectively.

## Appendix G <br> Proofs of Results in Section VI-A

Proof of Proposition 6.2: It is clear that, for any x and $\hat{\mathrm{x}}$ satisfying $P_{\mathbf{x}}^{\mathcal{U}}=P_{\hat{\mathbf{x}}}^{\mathcal{U}}$,

$$
\begin{aligned}
\mathrm{E}\left[\frac{1\left\{\mathrm{x} \in \Sigma_{\mathcal{U}}(A)\right\}}{|A|}\right] & =\mathrm{E}\left[\frac{1\left\{\mathbf{x} \in \Sigma_{\mathcal{U}}(A)\right\}}{\left|\Sigma_{\mathcal{U}}(A)\right|}\right] \\
& \stackrel{(a)}{=} \mathrm{E}\left[\frac{1\left\{\hat{\mathbf{x}} \in \Sigma_{\mathcal{U}}(A)\right\}}{\left|\Sigma_{\mathcal{U}}(A)\right|}\right] \\
& =\mathrm{E}\left[\frac{\left.1 \hat{\mathbf{x}} \in \Sigma_{\mathcal{U}}(A)\right\}}{|A|}\right]
\end{aligned}
$$

where (a) follows from the fact that the distribution of $\Sigma_{\mathcal{U}}(A)$ is invariant under any permutation in $S_{\mathcal{U}}$. Then it follows that

$$
\begin{aligned}
\mathrm{E}\left[\frac{1\left\{\mathbf{x} \in \Sigma_{\mathcal{U}}(A)\right\}}{|A|}\right] & =\frac{1}{\left|\mathcal{T}_{P_{\mathbf{x}}^{u}}\right|} \sum_{\hat{\mathbf{x}} \in \mathcal{T}_{P_{\mathbf{x}}^{u}}} \mathrm{E}\left[\frac{1\left\{\hat{\mathbf{x}} \in \Sigma_{\mathcal{U}}(A)\right\}}{|A|}\right] \\
& =\frac{1}{\prod_{U \in \mathcal{U}}\left(|U| P_{P_{x_{U}}}\right)} \mathrm{E}\left[\frac{\left|A \cap \mathcal{T}_{P_{\mathbf{x}}^{u}}\right|}{|A|}\right] \\
& \stackrel{(\mathrm{a})}{=} q^{-n} \alpha_{A}\left(P_{\mathbf{x}}^{\mathcal{U}}\right)
\end{aligned}
$$

where (a) follows from [9, Proposition 2.1] and the definition of $\mathcal{U}$-spectrum. This proves (36), and identity (38) comes from

$$
\mathrm{E}\left[\frac{\left|B \cap \Sigma_{\mathcal{U}}(A)\right|}{|A|}\right]=\sum_{\mathbf{y} \in B} \mathrm{E}\left[\frac{1\left\{\mathbf{y} \in \Sigma_{\mathcal{U}}(A)\right\}}{|A|}\right]
$$

combined with (36) and [9, Proposition 2.1].

## Appendix H

## Proofs of Results in Section Vi-B

Proof of Proposition 6.4:

$$
\begin{aligned}
\mathrm{P}\left\{F^{\sim}(\mathbf{x}) \in \mathcal{T}_{Q^{\nu_{0}}}\right\} & =\mathrm{P}\left\{\tilde{F}(\mathbf{x}) \in \mathcal{T}_{Q^{\nu_{0}}}\right\} \\
& =\mathrm{E}\left[1\left\{\tilde{F}(\mathbf{x}) \in \mathcal{T}_{Q^{\nu_{0}}}\right\}\right] \\
& =\mathrm{E}\left[\sum_{\mathbf{y} \in \mathcal{T}_{Q^{2}} \nu_{0}} 1\{(\mathbf{x}, \mathbf{y}) \in \operatorname{rl}(\tilde{F})\}\right] \\
& =\mathrm{E}\left[\sum_{\mathbf{y} \in \mathcal{T}_{Q^{\nu_{0}}}} 1\left\{(\mathbf{x}, \mathbf{y}) \in \Sigma_{\mathcal{U}_{0} \cup \mathcal{V}_{0}}(\operatorname{rl}(F))\right\}\right] \\
& =\mathrm{E}\left[\left|\left(\mathbf{x} \times \mathcal{T}_{Q^{\nu_{0}}}\right) \cap \Sigma_{\mathcal{U}_{0} \cup \mathcal{V}_{0}}(\operatorname{rl}(F))\right|\right] \\
& \stackrel{(a)}{=} \frac{\overline{\mathrm{S}}_{F}\left(P_{\mathbf{x}}^{\mathcal{U}_{0}}, Q^{\nu_{0}}\right)}{\prod_{U \in \mathcal{U}_{0}} \mathrm{~S}_{\mathbb{F}_{q}^{U \mid}}\left(P_{x_{U}}\right)} \\
& =\overline{\mathrm{S}}_{F}\left(Q^{\mathcal{V}_{0}} \mid P_{\mathbf{x}}^{\mathcal{U}_{0}}\right)
\end{aligned}
$$

where (a) follows from Proposition 6.2 and the identity $|\mathrm{rl}(F)|=\prod_{i=1}^{s} q^{n_{i}}$.
Proof of Proposition 6.6:

$$
\begin{aligned}
& \overline{\mathrm{S}}_{G \circ \Sigma_{m} \circ F}(Q \mid O) \stackrel{(\mathrm{a})}{=} \mathrm{P}\left\{\left(G^{\sim} \circ F^{\sim}\right)(\mathbf{x}) \in \mathcal{T}_{Q}^{l}\right\} \\
&= \sum_{P \in \mathcal{P}_{m}} \mathrm{P}\left\{F^{\sim}(\mathbf{x}) \in \mathcal{T}_{P}^{m}, G^{\sim}\left(F^{\sim}(\mathbf{x})\right) \in \mathcal{T}_{Q}^{l}\right\} \\
&= \sum_{P \in \mathcal{P}_{m}}\left(\mathrm{P}\left\{F^{\sim}(\mathbf{x}) \in \mathcal{T}_{P}^{m}\right\}\right. \\
&\left.\times \mathrm{P}\left\{G^{\sim}\left(F^{\sim}(\mathbf{x})\right) \in \mathcal{T}_{Q}^{l} \mid F^{\sim}(\mathrm{x}) \in \mathcal{T}_{P}^{m}\right\}\right) \\
& \stackrel{(\mathrm{b})}{=} \sum_{P \in \mathcal{P}_{m}} \overline{\mathrm{~S}}_{F}(P \mid O) \overline{\mathrm{S}}_{G}(Q \mid P),
\end{aligned}
$$

where (a) and (b) follow from Proposition 6.4, and x is an arbitrary sequence such that $P_{\mathbf{x}}=O$.

## Appendix I

## Proofs of Results in Section VI-C

Proof of Proposition 6.8: Since $\pi_{\mathcal{V}}=\psi \circ \pi_{\mathcal{U}}$,

$$
\begin{aligned}
\mathcal{G}_{\mathbb{F}_{\dot{q}}}(A)\left(\mathbf{v}_{\mathcal{V}}\right) & =\frac{1}{|A|} \sum_{\mathbf{x} \in A} \prod_{i=1}^{n} v_{\pi_{\mathcal{V}}(i), x_{i}} \\
& =\frac{1}{|A|} \sum_{\mathbf{x} \in A} \prod_{i=1}^{n} v_{\psi\left(\pi_{\mathcal{U}}(i)\right), x_{i}} \\
& =\psi\left(\mathcal{G}_{\mathbb{F}_{q}^{u}}(A)\left(\mathbf{u}_{\mathcal{U}}\right)\right) .
\end{aligned}
$$

Proof of Proposition 6.10: By definition,

$$
\begin{aligned}
\mathcal{G}_{\prod_{i=1}^{s} A_{i}}\left(\mathbf{u}_{\mathcal{I}_{s}}\right) & =\frac{1}{\left|\prod_{i=1}^{s} A_{i}\right|} \sum_{\mathbf{x} \in \prod_{i=1}^{s} A_{i}} \prod_{i=1}^{s} \mathbf{u}_{U_{i}}^{\left|U_{i}\right| P_{x_{U_{i}}}} \\
& =\frac{1}{\prod_{i=1}^{s}\left|A_{i}\right|} \prod_{i=1}^{s} \sum_{\mathbf{x}_{i} \in A_{i}} \mathbf{u}_{U_{i}}^{\left|U_{i}\right| P_{x_{i}}} \\
& =\prod_{i=1}^{s} \mathcal{G}_{A_{i}}\left(\mathbf{u}_{i}\right)
\end{aligned}
$$

where $\left\{U_{1}, \ldots, U_{s}\right\}$ is the default coordinate partition.
Proof of Corollary 6.11:

$$
\begin{aligned}
\mathcal{G}_{A_{1} \times A_{2}}(\mathbf{u}) & \stackrel{(\mathrm{a})}{=} \mathcal{G}_{A_{1} \times A_{2}}(\mathbf{u}, \mathbf{u}) \\
& \stackrel{(\mathrm{b})}{=} \mathcal{G}_{A_{1}}(\mathbf{u}) \cdot \mathcal{G}_{A_{2}}(\mathbf{u})
\end{aligned}
$$

where (a) follows from Proposition 6.8 and (b) follows from Proposition 6.10.
Proof of Corollary 6.13:

$$
\begin{aligned}
\mathcal{G}_{f_{1} \odot f_{2}}(\mathbf{u}, \mathbf{v}) & \stackrel{(\mathrm{a})}{=} \mathcal{G}_{f_{1} \odot f_{2}}(\mathbf{u}, \mathbf{u}, \mathbf{v}, \mathbf{v}) \\
& \stackrel{(\mathrm{b})}{=} \mathcal{G}_{f_{1}}(\mathbf{u}, \mathbf{v}) \cdot \mathcal{G}_{f_{2}}(\mathbf{u}, \mathbf{v})
\end{aligned}
$$

where (a) follows from Proposition 6.8 with $\operatorname{rl}\left(f_{1} \odot f_{2}\right) \subseteq \mathbb{F}_{q}^{n_{1}+n_{2}} \times \mathbb{F}_{q}^{m_{1}+m_{2}}=\mathbb{F}_{q}^{n_{1}} \times \mathbb{F}_{q}^{n_{2}} \times \mathbb{F}_{q}^{m_{1}} \times \mathbb{F}_{q}^{m_{2}}$, and (b) follows from Proposition 6.10 with $\operatorname{rl}\left(f_{1} \odot f_{2}\right)=\operatorname{rl}\left(f_{1}\right) \times \operatorname{rl}\left(f_{2}\right)$.

Proof of Proposition 6.14: Let $A^{\prime}=F(A)$. Since $F$ is bijective, the generating function $\mathcal{G}_{\mathbb{F}_{q}}\left(A^{\prime}\right)$ can be rewritten as

$$
\begin{aligned}
\mathcal{G}_{\mathbb{F}_{q}^{u}}\left(A^{\prime}\right)\left(\mathbf{u}_{\mathcal{U}}\right) & =\frac{1}{\left|A^{\prime}\right|} \sum_{\mathbf{x}} 1\left\{\mathbf{x} \in A^{\prime}\right\} \prod_{i=1}^{n} u_{\pi_{\mathcal{U}}(i), x_{i}} \\
& =\frac{1}{|A|} \sum_{\mathbf{x}} 1\{\mathbf{x} \in A\} \prod_{i=1}^{n} u_{\pi_{\mathcal{U}}(i), F^{(i)}\left(x_{i}\right)} .
\end{aligned}
$$

Taking expectations on both sides, we obtain

$$
\begin{aligned}
\overline{\mathcal{G}}_{\mathbb{F}_{q}^{u}}\left(A^{\prime}\right)\left(\mathbf{u}_{\mathcal{U}}\right) & =\sum_{\mathbf{x}} \mathrm{E}\left[\frac{1\{\mathbf{x} \in A\}}{|A|}\right] \prod_{i=1}^{n} \mathrm{E}\left[u_{\pi_{\mathcal{U}}(i), F_{\pi_{\mathcal{U}}(i)}\left(x_{i}\right)}\right] \\
& =\overline{\mathcal{G}}_{\mathbb{F}_{q}^{u}}(A)\left(\left(\mathrm{E}\left[u_{U, F_{U}(a)}\right]\right)_{U \in \mathcal{U}, a \in \mathbb{F}_{q}}\right),
\end{aligned}
$$

which is just $\bar{F}\left(\overline{\mathcal{G}}_{\mathbb{F}_{q}^{u}}(A)\right)$.

## Appendix J

## Proofs of Results in Section VI-D

To prove Theorem 6.15, we need two lemmas.
Lemma J. 1 (see e.g., [58]): For a subspace $A$ of $\mathbb{F}_{q}^{n}$,

$$
\begin{equation*}
\frac{1}{|A|} \sum_{\mathbf{x}_{1} \in A} \chi\left(\mathbf{x}_{1} \cdot \mathbf{x}_{2}\right)=1\left\{\mathbf{x}_{2} \in A^{\perp}\right\} \quad \forall \mathbf{x}_{2} \in \mathbb{F}_{q}^{n} \tag{72}
\end{equation*}
$$

The reader is referred to [58, Lemma A.1] for a proof.
Lemma J.2: Let $\mathcal{U}$ be a partition of $\mathcal{I}_{n}$. Then

$$
\sum_{\mathbf{x}_{2} \in \mathbb{F}_{q}^{n}} \chi\left(\mathbf{x}_{1} \cdot \mathbf{x}_{2}\right) \prod_{i=1}^{n} u_{\pi_{\mathcal{U}}(i), x_{2, i}}=\prod_{U \in \mathcal{U}}\left(\mathbf{u}_{U} \mathbf{M}\right)^{|U| P_{\mathbf{x}_{1}}^{U}}
$$

for all $\mathbf{x}_{1} \in \mathbb{F}_{q}^{n}$, where $\mathbf{M}$ is defined by (42).
Proof:

$$
\begin{aligned}
\sum_{\mathbf{x}_{2} \in \mathbb{F}_{q}^{n}} \chi\left(\mathbf{x}_{1} \cdot \mathbf{x}_{2}\right) \prod_{i=1}^{n} u_{\pi \mathcal{U}(i), x_{2, i}} & \stackrel{(a)}{=} \sum_{\mathbf{x}_{2} \in \mathbb{F}_{q}^{n}} \prod_{i=1}^{n} \chi\left(x_{1, i} x_{2, i}\right) u_{\pi \mathcal{U}}(i), x_{2, i} \\
& =\prod_{U \in \mathcal{U}} \prod_{i \in U} \sum_{a_{2} \in \mathbb{F}_{q}} \chi\left(x_{1, i} a_{2}\right) u_{U, a_{2}} \\
& =\prod_{U \in \mathcal{U}} \prod_{a_{1} \in \mathbb{F}_{q}}\left(\sum_{a_{2} \in \mathbb{F}_{q}} \chi\left(a_{1} a_{2}\right) u_{U, a_{2}}\right)^{|U| P_{x_{1}}^{U}\left(a_{1}\right)} \\
& =\prod_{U \in \mathcal{U}}\left(\mathbf{u}_{U} \mathbf{M}\right)^{|U| P_{\mathbf{x}_{1}}^{U}}
\end{aligned}
$$

where (a) follows from $\chi\left(\mathbf{x}_{1} \cdot \mathbf{x}_{2}\right)=\prod_{i=1}^{n} \chi\left(x_{1, i} x_{2, i}\right)$.
Proof of Theorem 6.15:

$$
\begin{aligned}
\mathcal{G}_{A^{\perp}}\left(\mathbf{u}_{\mathcal{U}}\right) & =\frac{1}{\left|A^{\perp}\right|} \sum_{\mathbf{x}_{2} \in \mathbb{F}_{q}^{n}} 1\left\{\mathbf{x}_{2} \in A^{\perp}\right\} \prod_{i=1}^{n} u_{\pi_{\mathcal{U}}(i), x_{2, i}} \\
& \stackrel{(\text { a })}{=} \frac{1}{\left|A^{\perp}\right|} \sum_{\mathbf{x}_{2} \in \mathbb{F}_{q}^{n}} \frac{1}{|A|} \sum_{\mathbf{x}_{1} \in A} \chi\left(\mathbf{x}_{1} \cdot \mathbf{x}_{2}\right) \prod_{i=1}^{n} u_{\pi_{\mathcal{U}}(i), x_{2, i}} \\
& =\frac{1}{|A|\left|A^{\perp}\right|} \sum_{\mathbf{x}_{1} \in A} \sum_{\mathbf{x}_{2} \in \mathbb{F}_{q}^{n}} \chi\left(\mathbf{x}_{1} \cdot \mathbf{x}_{2}\right) \prod_{i=1}^{n} u_{\pi_{\mathcal{U}}(i), x_{2, i}} \\
& \stackrel{(\mathrm{~b})}{=} \frac{1}{|A|\left|A^{\perp}\right|} \sum_{\mathbf{x}_{1} \in A} \prod_{U \in \mathcal{U}}\left(\mathbf{u}_{U} \mathbf{M}\right)^{|U| P_{\mathbf{x}_{1}}^{U}} \\
& =\frac{1}{\left|A^{\perp}\right|} \mathcal{G}_{A}\left(\left(\mathbf{u}_{U} \mathbf{M}\right)_{U \in \mathcal{U}}\right)
\end{aligned}
$$

where (a) follows from Lemma J. 1 and (b) follows from Lemma J.2.
Proof of Theorem 6.17: Define the sets

$$
Z_{1}:=\left\{(\mathbf{x} \mathbf{A}, \mathbf{x}) \in \mathbb{F}_{q}^{m} \times \mathbb{F}_{q}^{n}: \mathbf{x} \in \mathbb{F}_{q}^{n}\right\}
$$

and

$$
Z_{2}:=\left\{\left(\mathbf{y},-\mathbf{y A}^{\top}\right) \in \mathbb{F}_{q}^{m} \times \mathbb{F}_{q}^{n}: \mathbf{y} \in \mathbb{F}_{q}^{m}\right\}
$$

Clearly, for any $\mathbf{z}_{1}=(\mathbf{x A}, \mathbf{x}) \in Z_{1}$ and $\mathbf{z}_{2}=\left(\mathbf{y},-\mathbf{y} \mathbf{A}^{\boldsymbol{\top}}\right) \in Z_{2}$, we have

$$
\begin{aligned}
\mathbf{z}_{1} \cdot \mathbf{z}_{2} & =(\mathbf{x A}) \cdot \mathbf{y}+\mathbf{x} \cdot\left(-\mathbf{y} \mathbf{A}^{\top}\right) \\
& =(\mathbf{x A}) \mathbf{y}^{\top}-\mathbf{x}\left(\mathbf{y} \mathbf{A}^{\top}\right)^{\top} \\
& =\mathbf{x A} \mathbf{y}^{\top}-\mathbf{x A} \mathbf{y}^{\top} \\
& =0
\end{aligned}
$$

which implies $Z_{2} \subseteq Z_{1}^{\perp}$. Note that $\left|Z_{1}\right|\left|Z_{2}\right|=q^{m+n}$. This, together with the identity $\left|Z_{1}\right|\left|Z_{1}^{\perp}\right|=q^{m+n}$, gives $Z_{2}=Z_{1}^{\perp}$.

Then it follows from Theorem 6.15 that

$$
\begin{aligned}
\mathcal{G}_{\mathbb{F}_{q}^{\nu} \mathbb{F}_{q}^{u}}(-g)\left(\mathbf{v}_{\mathcal{V}}, \mathbf{u}_{\mathcal{U}}\right) & =\mathcal{G}_{\mathbb{F}_{\mathscr{Z}}^{\nu} \mathbb{F}_{q}^{u}}\left(Z_{2}\right)\left(\mathbf{v}_{\mathcal{V}}, \mathbf{u}_{\mathcal{U}}\right) \\
& =\mathcal{G}_{\mathbb{F}_{q}^{\nu} \mathbb{F}_{q}^{u}}\left(Z_{1}^{\perp}\right)\left(\mathbf{v}_{\mathcal{V}}, \mathbf{u}_{\mathcal{U}}\right) \\
& =\frac{1}{\left|Z_{1}^{\perp}\right|} \mathcal{G}_{\mathbb{F}_{q}^{\nu} \mathbb{F}_{q}^{u}}\left(Z_{1}\right)\left(\left(\mathbf{v}_{V} \mathbf{M}\right)_{V \in \mathcal{V}},\left(\mathbf{u}_{U} \mathbf{M}\right)_{U \in \mathcal{U}}\right) \\
& =\frac{1}{q^{m}} \mathcal{G}_{\mathbb{F}_{q}^{u} \mathbb{F}_{q}^{\nu}}(f)\left(\left(\mathbf{u}_{U} \mathbf{M}\right)_{U \in \mathcal{U}},\left(\mathbf{v}_{V} \mathbf{M}\right)_{V \in \mathcal{V}}\right)
\end{aligned}
$$

as desired.

## Appendix K <br> Spectrum or Complete Weight Distribution?

As discussed in Section II, now that spectrum is simply the normalization of complete weight distribution, why use it at all? In the nonrandom setting, these two concepts make indeed no difference. If random encoders are involved, however, there is a remarkable difference.

Let $A$ be a random nonempty subset of $\mathbb{F}_{q}^{n}$. Then its average spectrum is $\bar{S}(A)$ while its average complete weight distribution can be expressed as $\mathrm{E}[|A| \mathrm{S}(A)]$. Note that the equation $\mathrm{E}[|A| \mathrm{S}(A)]=\mathrm{E}[|A|] \overline{\mathrm{S}}(A)$ does not hold in general (unless $|A|$ and $\mathrm{S}(A)$ are uncorrelated, e.g., $|A|$ is nonrandom), so there is no simple relation between $\overline{\mathrm{S}}(A)$ and $\mathrm{E}[|A| S(A)]$. In fact, for certain random sets we can obtain an elegant exact formula for the average spectrum, but only a less strict approximate expression for the average complete weight distribution (for example, with the assumption that the vectors are not necessarily distinct [59]). Similarly, there are some cases in which the complete weight distribution is more appropriate.

Example K.1: Consider the random linear encoder $F_{m, n}^{\mathrm{RLC}}$ over $\mathbb{F}_{q}$. By [9, Proposition 2.5], its average joint spectrum is

$$
\overline{\mathrm{S}}_{F_{m, n}^{\mathrm{RLC}}}(P, Q)= \begin{cases}q^{-m} 1\left\{Q=P_{0^{n}}\right\} & P=P_{0^{m}} \\ q^{-m-n}\binom{m}{m P}\binom{n}{n Q} & \text { otherwise }\end{cases}
$$

so the average image spectrum of $F_{m, n}^{\mathrm{RLC}}$, or equivalently, the average spectrum of $C_{1}=F_{m, n}^{\mathrm{RLC}}\left(\mathbb{F}_{q}^{m}\right)$ is

$$
\overline{\mathrm{S}}_{C_{1}}(Q)= \begin{cases}q^{-m}+q^{-n}\left(1-q^{-m}\right) & Q=P_{0^{n}} \\ q^{-n}\left(1-q^{-m}\right)\binom{n}{n Q} & \text { otherwise }\end{cases}
$$

On the other hand, it follows from Proposition 6.2 that the average spectrum of $C_{2}=\operatorname{ker} F_{n, m}^{\mathrm{RLC}}$ is

$$
\overline{\mathrm{S}}_{C_{2}}(P)=\binom{n}{n P} \mathrm{E}\left[\frac{1\left\{\mathbf{x} \in C_{2}\right\}}{\left|C_{2}\right|}\right],
$$

where x is an arbitrary vector of $\mathcal{T}_{P}^{n}$. Since the expectation term on the right-hand side is too complicated, we cannot proceed without resorting to approximation. However, computing instead the average complete weight distribution, we obtain

$$
\begin{aligned}
\mathrm{E}\left[\left|C_{2}\right| \mathrm{S}_{C_{2}}(P)\right] & =\binom{n}{n P} \mathrm{E}\left[1\left\{\mathbf{x} \in C_{2}\right\}\right] \\
& =\binom{n}{n P} P\left\{F_{n, m}^{\mathrm{RLC}}(\mathbf{x})=0^{m}\right\} \\
& = \begin{cases}1 & P=P_{0^{n}} \\
q^{-m}\binom{n}{n P} & \text { otherwise },\end{cases}
\end{aligned}
$$

which is surprisingly simple compared to the spectrum form. A similar situation is encountered when computing the average complete weight distribution of $C_{1}$. In fact, the generator matrix of $F_{n, m}^{\mathrm{RLC}}$ is the transpose of the generator matrix of $F_{m, n}^{\mathrm{RLC}}$, so $C_{1}$ and $C_{2}$, as a pair of dual codes, satisfy

$$
\mathrm{E}\left[\left|C_{2}\right| \mathcal{G}_{C_{2}}(\mathbf{u})\right]=\overline{\mathcal{G}}_{C_{1}}(\mathbf{u M})
$$

and

$$
\mathrm{E}\left[\left|C_{1}\right| \mathcal{G}_{C_{1}}(\mathbf{u})\right]=\overline{\mathcal{G}}_{C_{2}}(\mathbf{u M})
$$

according to Theorem 6.15. These two identities explain why the average spectrum of $C_{1}$ and the average complete weight distribution of $C_{2}$ are easy to compute while the other two quantities do not have simple expressions.

It turns out that most of the cases treated in this paper are more conveniently formulated in the spectrum form. For this reason we have chosen the code-spectrum approach. Furthermore, as a side benefit, the law of serial concatenation of linear encoders can be intuitively put in analogy with the concatenation of conditional probability distributions (see Section VI-B).

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    S. Yang was with the Department of Information Science and Electronic Engineering, Zhejiang University, Hangzhou 310027, China and also with Zhejiang Provincial Key Laboratory of Information Network Technology, Zhejiang University, Hangzhou 310027, China. He now resides at Zhengyuan Xiaoqu 10-2-101, Fengtan Road, Hangzhou 310011, China (e-mail: yangst@codlab.net).
    T. Honold, Z. Zhang, and P. Qiu are with the Department of Information Science and Electronic Engineering, Zhejiang University, Hangzhou 310027, China (e-mail: honold@zju.edu.cn; ning_ming@zju.edu.cn; qiupl@zju.edu.cn).
    Y. Chen was with the Department of Information Science and Electronic Engineering, Zhejiang University, Hangzhou 310027, China. She is now with Huawei Technologies Co., Ltd (Shanghai), Shanghai 201206, China (e-mail: bigbird.chenyan@ huawei.com).

[^1]:    ${ }^{1}$ In [9] a linear encoder is called a "linear code", which in fact conflicts with the traditional meaning of the term "linear code". To avoid such conflicts as well as possible misunderstanding, we use the term "linear encoder" in this paper.

[^2]:    ${ }^{2}$ In the special case $\mathcal{X}=\mathcal{Y}=\left(\mathbb{F}_{q},+\right)$, where this concept refers to linearity over the prime field $\mathbb{F}_{p}$ of $\mathbb{F}_{q}$, we rather speak of additive encoders/codes, so that there is no conflict with the stronger concept of $\mathbb{F}_{q}$-linearity.

[^3]:    ${ }^{3}$ For convenience, we shall slightly abuse the term "permutation" to refer to an induced coordinate permutation as long as the exact meaning is clear from the context.

[^4]:    ${ }^{4}$ It was shown in [30] that Theorem 4.2 actually holds for all MRD codes. Furthermore, it was proved that the minimum support size of an SCC-good-before-symmetrization random linear encoder is exactly $q^{\max \{n, m\}}$, and that a random linear encoder of support size $q^{\max \{n, m\}}$ is SCC-good before symmetrization if and only if it is uniformly distributed over an ( $n, m, 1$ ) MRD code.

[^5]:    ${ }^{5}$ A similar conclusion is valid in classical coding theory. It was shown in [31] that group codes over general groups can be no better than linear codes over finite fields, in terms of Hamming distance.

[^6]:    ${ }^{6}$ The tightness of the bound follows from [38, Theorem 1].

[^7]:    ${ }^{7}$ When $\mathcal{U}$ is the disjoint union of $\mathcal{V}$ and $\mathcal{W}$, we sometimes write $S_{\mathbb{F}_{q} \mathbb{F}_{q} \mathcal{W}}(A)$, which looks more like an ordinary joint spectrum and is used to distinguish between coordinates.

[^8]:    ${ }^{8}$ There is a collision between coordinate sets when we consider the pair $(\mathbf{x}, F(\mathbf{x}))$ as a vector of $\mathbb{F}_{q}^{m+n}$. The trick is to rename the output coordinate set as $\mathcal{I}_{m}^{\prime}=\left\{1^{\prime}, 2^{\prime}, \ldots, m^{\prime}\right\}$, so that the whole coordinate set is $\mathcal{I}_{n} \cup \mathcal{I}_{m}^{\prime}$.

[^9]:    ${ }^{9}$ Since the collection of random polynomials with finite image is enough for our purpose, we shall not discuss the expectation of a general random polynomial.

[^10]:    ${ }^{11}$ This review is merely based on the ideas and results in previous papers. For technical reasons, the requirements we give here are not the same as those in the literature.

