# On Zero–Rate Error Exponents of Finite–State Channels with Input–Dependent States

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**Abstract**

We derive a single–letter formula for the zero–rate reliability (error exponent) of a finite–state channel whose state variable depends deterministically (and recursively) on past channel inputs, where the code complies with a given channel input constraint. Special attention is then devoted to the important special case of the Gaussian channel with inter-symbol interference (ISI), where more explicit results are obtained.

**Index Terms:** Error exponents, Bhattacharyya distance, expurgated codes, finite–state channels, Markov types.

## 1 Introduction

The concept of the reliability function of a channel is almost as old as information theory itself. The first to show that below capacity, the probability of error decays exponentially with the block length, for a sequence of good codes, was Feinstein [14] in 1955. Already in the same year, Elias [11] derived the random coding bound and the sphere–packing bound, and he observed that they exponentially coincide at high rates, for the cases of the binary symmetric channel (BSC) and the binary erasure channel. Six years later, Fano [13], derived the random coding exponent, $E_{\rm r}(R)$, and heuristically also the sphere–packing bound for the general discrete memoryless channel (DMC). In 1965, Gallager [15] improved on $E_{\rm r}(R)$ at low rates by the idea of expurgation of randomly selected codes. In 1967, Shannon, Gallager, and Berlekamp, published their celebrated two–part paper [24], [25], where they derived the classical lower bounds on the error probability for general DMC's: the sphere–packing bound, the zero–rate bound, and the straight–line bound, that improves on the sphere–packing bound at low rates, using the zero–rate bound.

In the realm of channels with memory, the most popular model dealt with, in this context, has been the model of a finite–state channel (FSC) and some of its special cases. The channel coding theorem for FSC's was proved by Blackwell, Breiman and Thomasian [3] in 1958. The random coding exponent for FSC's was derived by Blackwell [2] in 1961, Yudkin [28] in 1967, and further developed by Gallager in his book [16, Section 5.9], especially for the case where the state is known at the receiver.

Ever since these early days of information theory, there has been a vast amount of continued work around error exponents and reliability functions, most notably, for memoryless channels (both discrete and continuous), but also (albeit, much less) for various models of channels with memory (FSC's included), both in the presence and in the absence of feedback. For the latter category, see, e.g., [1], [5], [6], [12], [18], [19], [22], [23], [27], [29], [30] and references therein, for a non–exhaustive list of relevant works from the last three decades.

In this paper, our focus is on the zero–rate reliability of channels from a subclass of the *FSC's with input–dependent states* (without feedback), namely, finite–state channels where the state variable, which designates the memory of the channel, evolves deterministically in response to past channel inputs, as opposed to the more general channel model, where the state evolves stochastically in response to both past inputs and outputs. For a finite input alphabet, this subclass of FSC's is still general enough to include the important model of the inter-symbol interference (ISI) channel, among some other models.

Our primary motivation for studying the zero–rate reliability for these channels is in order to identify and characterize, by means of single–letter formulas, the relevant distance metrics and the maximum achievable minimum distance between codeword pairs under this metric, in analogy to the Hamming distance for the BSC, the Euclidean distance for the Gaussian memoryless channel, and the Bhattacharyya distance for a general DMC. A secondary motivation is that once the zero–rate reliability is known and the sphere–packing bound is known, at least for some positive rate, one can obtain a simple bound at all rates using the straight line in between, by using a straightforward extension of [24, Theorem 1] (see also [26, Theorem 3.8.1]) to channels with memory. For example, even if the sphere–packing exponent is not available, but the capacity $C$ of the channel is known (or at least we have an upper bound for it), then we know that the sphere–packing bound at rate $C$ vanishes, and we can safely use this theorem to connect the above–mentioned straight line to the point $(C, 0)$ in the plane of reliability vs. rate.[1] This bound can be reasonably good at least for low rates.

Our main result, in this paper, is an exact single–letter characterization of the zero–rate reliability (or the maximum achievable minimum 'distance') for FSC's with input–dependent states, and codes that must conform with a given input constraint. More explicit results are provided in the Gaussian case with inter-symbol interference (ISI), which will be treated in some detail later in the paper.

## 2 Preliminaries

Before addressing FSC's, we begin with some preliminaries on the zero–rate reliability of a DMC. Let us define

$$\mathcal{E}_0^+ \triangleq \lim_{R\downarrow 0}\limsup_{n\to\infty}\left[-\frac{\ln P_e(R,n)}{n}\right] \tag{1}$$

$$\mathcal{E}_0^- \triangleq \lim_{R\downarrow 0}\liminf_{n\to\infty}\left[-\frac{\ln P_e(R,n)}{n}\right], \tag{2}$$

[1] This is supported by the fact for any code of rate just above $C + \lambda$ ($\lambda > 0$), the probability of list–error, for an exponential list size of rate $\lambda$, must be bounded away from zero, as can easily be seen from a simple extension of Fano's inequality to list decoding.

where $P_e(R,n)$ is the minimum probability of error that can be attained, for the given channel, by any block code of length $n$ and rate $R$. Consider a DMC, designated by a matrix of input–output transition probabilities $\{p(y|x),\ x \in \mathcal{X},\ y \in \mathcal{Y}\}$. Here the channel input symbol $x$ takes on values in a finite input alphabet $\mathcal{X}$, whereas the channel output symbol $y$ takes on values in the output alphabet $\mathcal{Y}$, which may either be discrete or continuous.[2] When the channel is fed by a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{X}^n$, it outputs a vector $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathcal{Y}^n$ according to

$$P(\boldsymbol{y}|\boldsymbol{x}) = \prod_{t=1}^{n} p(y_t|x_t). \tag{3}$$

For DMC's whose zero–error capacity vanish, the zero-rate reliability is well–known [24], [25] to be given by[3]

$$\mathcal{E}_0^+ = \mathcal{E}_0^- = E_0 \stackrel{\Delta}{=} \max_{\boldsymbol{q}} \left[\sum_{x,x' \in \mathcal{X}} q(x)q(x')d_{\mathrm{B}}(x,x')\right], \tag{4}$$

where $d_{\mathrm{B}}(x,x')$ is the Bhattacharyya distance function, defined as

$$d_{\mathrm{B}}(x,x') = -\ln\left[\sum_{y\in\mathcal{Y}} \sqrt{p(y|x)p(y|x')}\right], \qquad x, x' \in \mathcal{X}, \tag{5}$$

and the maximum is over all possible probability assignments, $\boldsymbol{q} = \{q(x),\ x \in \mathcal{X}\}$, over the input alphabet. This is the best attainable error exponent for *any* code over $\mathcal{X}$.

In the presence of input constraints, the expression (4) may not be achievable since the optimal codes might violate these constraints. For example, suppose that each codeword in the codebook must satisfy the constraint

$$\sum_{t=1}^{n} \phi(x_t) \le n\Gamma, \tag{6}$$

where $\phi : \mathcal{X} \to \mathrm{I\!R}$ is a given function (e.g., $\phi(x) = x^2$) and $\Gamma$ is a prescribed quantity. At first glance, it may be tempting to guess that the best achievable exponent would then be the same as in (4), except that the maximum over $\boldsymbol{q}$ should be restricted to comply with the corresponding single–letter constraint, that is, $\boldsymbol{q} \in \mathcal{Q}_\Gamma$, where $\mathcal{Q}_\Gamma = \{\boldsymbol{q}:\ \sum_x q(x)\phi(x) \le \Gamma\}$.

[2] We proceed hereafter under the assumption that $\mathcal{Y}$ is a discrete alphabet, but with the understanding that in the continuous alphabet case, all probability distributions over $\mathcal{Y}$ are replaced by densities, and accordingly, all summations over $\mathcal{Y}$ should be replaced by integrals.

[3] The zero–rate reliability is more commonly denoted by $E_{\mathrm{ex}}(0)$, as it is identified with the expurgated error exponent at rate zero. However, since we consider here zero–rate codes only, we will use the more convenient notation $E_0$, with no risk of confusion with customary notation concerning the Gallager function and random coding exponents, as these quantities will not be addressed in this paper.

It turns out, however, that this is indeed true for some channels, but not in general. In certain cases, one can do better. The point is that the functional

$$E_0(\boldsymbol{q}) \stackrel{\Delta}{=} \sum_{x,x'\in\mathcal{X}} q(x)q(x')d_{\mathrm{B}}(x,x') \tag{7}$$

may not, in general, be concave in $\boldsymbol{q}$. This depends on the given (symmetric) matrix of Bhattacharyya distances, $D = \{d_{\mathrm{B}}(x,x')\}_{x,x'\in\mathcal{X}}$, which in turn, depends solely on the channel and the input alphabet. If $D$ is such that $E_0(\boldsymbol{q}) = \boldsymbol{q}^T D\boldsymbol{q}$ ($\boldsymbol{q}$ being thought of as a column vector), is concave, then $E_0(\boldsymbol{q})$ is the best exponent achievable for codes with codebooks of composition[4] and hence $\max_{\boldsymbol{q}\in\mathcal{Q}_\Gamma} E_0(\boldsymbol{q})$ is indeed the best achievable exponent under the aforementioned input constraint. If, however, $E_0(\boldsymbol{q})$ is not concave, one can improve by taking the upper concave envelope (UCE) of $E_0(\boldsymbol{q})$ (see [8, p. 191, Problem 21]). Accordingly, let us denote

$$\overline{E}_0(\boldsymbol{q}) = \mathrm{UCE}\{E_0(\boldsymbol{q})\} \stackrel{\Delta}{=} \max_{\{(\boldsymbol{w},\boldsymbol{V}):\ \sum_u w(u)v(x|u)=q(x)\ \forall x\in\mathcal{X}\}} \sum_{u\in\mathcal{U}} w(u)\sum_{x,x'} v(x|u)v(x'|u)d_{\mathrm{B}}(x,x'), \tag{8}$$

where $\boldsymbol{w} = \{w(u),\ u\in\mathcal{U}\}$ is a probability vector of a (time–sharing) variable $u$, whose alphabet size $|\mathcal{U}|$ need not exceed $|\mathcal{X}|$ (as can easily be shown using the Carathéodory theorem [8, p. 310, Lemma 3.4]), and $\boldsymbol{V} = \{v(x|u),\ u\in\mathcal{U},\ x\in\mathcal{X}\}$ is a matrix of transition probabilities of $x$ given $u$. The input constraint is then accommodated for $\overline{E}_0(\boldsymbol{q})$, that is, the best attainable exponent is $\max_{\boldsymbol{q}\in\mathcal{Q}_\Gamma} \overline{E}_0(\boldsymbol{q})$.

When $E_0(\cdot)$ is concave, the operator UCE$\{\cdot\}$ is, of course, redundant, so it is instructive to know when is this the case. The concavity of $E_0(\boldsymbol{q})$ over the simplex can easily be checked as follows. Without loss of generality, let $\mathcal{X} = \{1,2,\ldots,K\}$, $K = |\mathcal{X}|$. On substituting $q(K) = 1-\sum_{x<K} q(x)$ into the quadratic form $E_0(\boldsymbol{q}) = \boldsymbol{q}^T D\boldsymbol{q}$, one ends up with the reduced quadratic form $\tilde{\boldsymbol{q}}^T\tilde{D}\tilde{\boldsymbol{q}}$, where $\tilde{\boldsymbol{q}} = \{q(x),\ x = 1,2,\ldots,K-1\}$ and $\tilde{D}$ is a $(K-1)\times(K-1)$ whose $(x,x')$–th entry is $d_{\mathrm{B}}(x,x') - d_{\mathrm{B}}(x,K) - d_{\mathrm{B}}(K,x')$, $x,x'\in\{1,2,\ldots,K-1\}$. Thus, $E_0(\boldsymbol{q})$ is concave iff $\tilde{D}$ is negative semi–define, or equivalently, iff $-\tilde{D} = \{d_{\mathrm{B}}(x,K) + d_{\mathrm{B}}(K,x') - d_{\mathrm{B}}(x,x')\}$ is positive semi–definite. We henceforth denote by $\mathcal{D}(K)$ the class of matrices $\{D\}$ for which $\tilde{D}$ is negative semi–definite.[5]

It should be pointed out that for some rather important special cases, $D\in\mathcal{D}(K)$ and hence $E_0(\boldsymbol{q}) = \boldsymbol{q}^T D\boldsymbol{q}$ is concave on the simplex. For example, if $d_{\mathrm{B}}(x,x')$ is (proportional to) the Hamming

[4]The composition $\boldsymbol{q}$ of a fixed composition code is the empirical distribution of each one of the codewords.

[5]Of course, the choice of the letter $x = K$ as the one with the special stature here is completely arbitrary.

distance (which is always the case, for example, when $K = 2$), then $-\tilde{D}$ is a matrix whose all diagonal elements are 2 and all off–diagonal elements are 1. The eigenvalues of this matrix are 0 and $K$ (the former, with multiplicity of $K-2$) and hence it is positive semi–definite. As another example, if $d_{\mathrm{B}}(x, x')$ is (proportional to) the square–error, $(x - x')^2$, which is the case when the channel is Gaussian, then the $(x, x')$–th element of $-\tilde{D}$ is $(x-K)^2 + (x'-K)^2 - (x-x')^2 = 2(x-K)(x'-K)$, which is obviously positive semi–define, with eigenvalues $2\sum_{x<K}(x-K)^2 > 0$ and 0 (the latter, with multiplicity $K-2$). Thus, for the Gaussian channel, $E_0(\boldsymbol{q})$ is also concave on the simplex. On the other hand, one can easily find channels for which $D$ is not in $\mathcal{D}(K)$ and then $E_0(\boldsymbol{q})$ is not concave.

## 3 Main Result

Consider the following model of the FSC with an input–dependent state, which is defined as follows:

$$P(\boldsymbol{y}|\boldsymbol{x}) = \prod_{t=1}^{n} p(y_t|x_t, s_t), \tag{9}$$

where the state $s_t \in \mathcal{S}$ evolves recursively, in response to the channel input, according to

$$s_{t+1} = f(s_t, x_t), \quad t = 1, 2, \ldots, n-1, \tag{10}$$

$f : \mathcal{S} \times \mathcal{X} \to \mathcal{S}$ being a given *next–state function* and $s_1$ is an arbitrary initial state. It is assumed that the set of states $\mathcal{S}$ has a finite cardinality, that is, $S = |\mathcal{S}| < \infty$, hence the qualifier "finite–state".

For the direct part of our coding theorem below, it is further assumed that the finite–state machine $f$ is irreducible, namely, for every pair of states $s, s' \in \mathcal{S}$, there exists a finite string $x_1, x_2, \ldots, x_\ell \in \mathcal{X}$ ($\ell \leq S$) that leads the machine from state $s$ to state $s'$. Moreover, we assume that the finite–state machine formed by two independent copies of $f$, that is, the finite–state machine $(s_{t+1}, s'_{t+1}) = (f(s_t, x_t), f(s'_t, x'_t))$, is irreducible as well. For convenience, we henceforth refer to this assumption as *double irreducibility*.

For the converse part, we need a different assumption: we assume that there exists a state $\sigma \in \mathcal{S}$ and a positive integer $r$ such that for every $s \in \mathcal{S}$, there exists a path of length $r$, $x_1, x_2, \ldots, x_r$, that takes the finite–state machine from state $s$ to state $\sigma$ (note that by this definition, $r$ should be independent of $s$). We henceforth refer to this assumption as *uniform approachability*. For example, if $\sigma$ has a self–transition, this assumption is clearly satisfied.

Before we present our main theorem, we first make a few simple observations. Without loss of generality, we will take it for granted that the current state $s_t$ contains the full information for recovery of $x_{t-1}$, that is, there exists a deterministic function $g : \mathcal{S} \to \mathcal{X}$ such that

$$g(s_t) = x_{t-1}. \tag{11}$$

To justify the phrase "without loss of generality", we note that for any given channel of the form (9) and any given next–state function $f$, one can always artificially add the conditioning on $x_{t-1}$ in each factor on the right–hand side (r.h.s.) of eq. (9), that is, represent the model as

$$P(\boldsymbol{y}|\boldsymbol{x}) = \prod_{t=1}^{n} p(y_t|x_t, x_{t-1}, s_t), \quad t = 1, 2, \ldots, n-1, \tag{12}$$

with some arbitrary definition of $x_0 \in \mathcal{X}$, and then re–define the state as $\sigma_t = (s_t, x_{t-1})$. Having done this, we are back to the form (9), where: (i) $s_t$ is replaced by $\sigma_t$, (ii) $\sigma_t$ evolves recursively in response to $\{x_t\}$, using its own next–state function, and (iii) $x_{t-1}$ is recoverable from $\sigma_t$ simply because it includes $x_{t-1}$ as a component.[6]

Once the assumption (11) has been accepted, we have the following simple equalities:

$$p(y_t|x_t, s_t) = p(y_t|x_t, s_t, s_{t+1}) = p(y_t|s_t, s_{t+1}), \tag{13}$$

where the first equality is due to the fact that $s_{t+1}$ is uniquely determined by $x_t$ and $s_t$ (using $f$), and the second equality is because in the presence of $s_{t+1}$, the conditioning on $x_t$ is redundant since $x_t$ is determined by $s_{t+1}$ (using $g$). The mapping between $(x_t, s_t)$ and $(s_t, s_{t+1})$ is obviously one–to–one. Thus, instead of modeling the channel by the parameters $\{p(y|x, s),\ x \in \mathcal{X},\ s \in \mathcal{S}\}$, one might as well model it by the parameters $\{p(y|s, s_+),\ s, s_+ \in \mathcal{S}\}$, and think of the state sequence as the channel input. Note that, in this parametrization, not all $S^2$ state pairs $(s, s_+)$ are necessarily feasible, but only those that are related by the equation

$$s_+ = f(s, g(s_+)), \tag{14}$$

[6] A simple important special case where the assumption $x_{t-1} = g(s_t)$ is trivially satisfied, even without this modification, is the case where $s_t = (x_{t-k}, x_{t-k+1}, \ldots, x_{t-1})$ ($k$ – positive integer), which is simply a shift register fed by $\{x_t\}$. This is the relevant case for the ISI channel with a finite impulse response. In this case, the corresponding finite–state machine also satisfies the double irreducibility assumption and the uniform approachability assumption (for example, the zero–state as a self–transition).

in view of eqs. (10) and (11). The number $L$ of feasible pairs $\{(s,s_+):\ s_+ = f(s,g(s_+))\}$ cannot exceed $K\cdot S$, where $K$ denotes the size of the input alphabet $\mathcal{X}$, as before. An FSC with input–dependent states is, therefore, completely defined by the functions $f$ and $g$, and the parameters $\{p(y|s,s_+)\}$. Accordingly, we shall henceforth denote an FSC by the notation $[\{p(y|s,s_+)\},f,g]$. Let us denote the Bhattacharyya distance between two state pairs, $(s,s_+)$ and $(s',s_+')$ by

$$d_{\mathrm{B}}(s,s_+;s',s_+') = -\ln\left[\sum_{y\in\mathcal{Y}}\sqrt{p(y|s,s_+)p(y|s',s_+')}\right]. \tag{15}$$

The matrix $D$ of all Bhattacharyya distances (15) is, of course, of dimension $L\times L$.

We now redefine $\mathcal{Q}_\Gamma$ to be the class of joint distributions $\{q(s,s_+)\}$ of state pairs that satisfy the following conditions:

1. For every state pair $(s,s_+)$: $q(s,s_+)>0$ implies $s_+ = f(s,g(s_+))$.
2. $\boldsymbol{q}$ has equal marginals, i.e., $\sum_{\tilde{s}\in\mathcal{S}} q(s,\tilde{s}) = \sum_{\tilde{s}\in\mathcal{S}} q(\tilde{s},s) \stackrel{\Delta}{=} \pi(s)$ for every $s\in\mathcal{S}$.
3. All states in $\mathcal{S}_+ \stackrel{\Delta}{=} \{s:\ \pi(s)>0\}$ are fully connected, i.e., for every $s,s'\in\mathcal{S}_+$, there exists a path $s=s_1\to s_2\to\ldots\to s_m = s'$ (with $m\le|\mathcal{S}_+|$), such that $q(s_i,s_{i+1})>0$ for all $i=1,2,\ldots,m-1$.
4. The marginal $\boldsymbol{\pi} = \{\pi(s),\ s\in\mathcal{S}_+\}$ satisfies the input constraint $\sum_{s\in\mathcal{S}_+}\pi(s)\phi[g(s)]\le\Gamma$.

Consider again the definitions of $\mathcal{E}_0^+$ and $\mathcal{E}_0^-$ as in (1), but this time, with an FSC, rather than a DMC, in mind. Also, $\mathcal{E}_0^+(\Gamma)$ and $\mathcal{E}_0^-(\Gamma)$ will be defined in the same way, except that here, $P_e(n,R)$ is redefined as the minimum error probability across all codes that satisfy the input constraint (6) for each codeword. Accordingly, our new definition of $E_0(\boldsymbol{q})$ is

$$E_0(\boldsymbol{q}) \stackrel{\Delta}{=} \sum_{s,s_+,s',s_+'} q(s,s_+)q(s',s_+')d_{\mathrm{B}}(s,s_+;s',s_+'), \tag{16}$$

and once again, $\overline{E}_0(\boldsymbol{q})$ is the UCE of $E_0(\boldsymbol{q})$. Considering the analogous extension of the r.h.s. of eq. (8), here the time–sharing variable $u$ should take on values in an alphabet whose size need not exceed $L$. We are now ready to state our main theorem.

**Theorem 1** *Consider the FSC $[\{p(y|s,s^+)\},f,g]$, with the input constraint (6). If the uniform approachability assumption is met,*

$$\mathcal{E}_0^+(\Gamma) \le \max_{\boldsymbol{q}\in\mathcal{Q}_\Gamma}\overline{E}_0(\boldsymbol{q}). \tag{17}$$

*If $f$ is doubly irreducible,*

$$\mathcal{E}_0^-(\Gamma) \geq \max_{\boldsymbol{q}\in\mathcal{Q}_\Gamma} \overline{E}_0(\boldsymbol{q}). \tag{18}$$

*Consequently, if both assumptions hold,*

$$\mathcal{E}_0^+(\Gamma) = \mathcal{E}_0^-(\Gamma) = \max_{\boldsymbol{q}\in\mathcal{Q}_\Gamma} \overline{E}_0(\boldsymbol{q}). \tag{19}$$

The remaining part of this section is devoted to the proof of Theorem 1.

*Proof.* The proof is divided into two parts – the direct part, asserting that

$$\mathcal{E}_0^-(\Gamma) \geq \max_{\boldsymbol{q}\in\mathcal{Q}_\Gamma} \overline{E}_0(\boldsymbol{q}), \tag{20}$$

and the converse part, which tells that

$$\mathcal{E}_0^+(\Gamma) \leq \max_{\boldsymbol{q}\in\mathcal{Q}_\Gamma} \overline{E}_0(\boldsymbol{q}). \tag{21}$$

Beginning with the direct part, to fix ideas, consider first the case where $E_0(\boldsymbol{q})$ is concave and then $\overline{E}_0(\boldsymbol{q}) = E_0(\boldsymbol{q})$. Let $\boldsymbol{q}^*$ be an[7] achiever of the $\max_{\boldsymbol{q}\in\mathcal{Q}_\Gamma} E_0(\boldsymbol{q})$. For convenience, let us assume[8] that $q^*(s,s_+) \geq q_{\min} > 0$ for all state pairs for which $s_+ = f(s,g(s_+))$, thus $\mathcal{S}_+ = \mathcal{S}$. Consider an oriented multi–graph $G$ having a total of $n$ arcs (edges) and $|\mathcal{S}_+|$ vertices, labeled by the members of $\mathcal{S}_+$. For every ordered pair $(s,s_+)$, let $G$ contain[9] $nq^*(s,s_+)$ arcs stemming from vertex $s$ and ending at vertex $s_+$.

From the construction in [9, p. 433], we learn that given such a directed multigraph $G$, there exist (exponential many) state sequences of length $n$, $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$, with $s_1 = f(s_n, g(s_1))$, that are identified with various Eulerian circuits[10] on $G$. In other words, there exist many sequences $\boldsymbol{s}$ with the property that the number of transitions from $s_t = s$ to $s_{t\oplus 1} = s_+$ is exactly $nq^*(s,s_+)$, where $\oplus$ denotes addition modulo $n$, that is, we adopt the cyclic convention that $s_n$ is followed by $s_1$ (hence the requirement $s_1 = f(s_n, g(s_1))$). The validity of this statement is based on properties of $G$ that are guaranteed by the definition of the class $\mathcal{Q}_\Gamma$ to which $\boldsymbol{q}^*$ belongs (see, in particular,

[7] We refer to *an* achiever, rather than *the* achiever, because for a general matrix $D$, the maximum may be achieved by more than one distribution $\boldsymbol{q}$.

[8] If this is not the case, one can slightly alter $\boldsymbol{q}^*$ with an arbitrarily small degradation in $E_0(\boldsymbol{q})$.

[9] We are assuming, without essential loss of generality, that $\{nq^*(s,s_+)\}$ are all integers. If this is not the case, $q^*(s,s_+)$ can be approximated arbitrarily closely, for large $n$, by rational numbers with denominator $n$.

[10] An Eulerian circuit is a walk on a graph, starting an ending at the same vertex, where each arc is used exactly once.

properties 3 and 4 in [9, p. 433], which are reflected in items 2 and 3 in the definition of $\mathcal{Q}_\Gamma$). For convenience, we make the convention that the initial state $s_1$ is always a certain fixed member $\sigma$ of $\mathcal{S}$.

Let $\mathcal{T}_n(\boldsymbol{q}^*)$ be the set of all state sequences $\{\boldsymbol{s}\}$ with the properties described in the previous paragraph, that is, the so called *Markov type* associated with $\boldsymbol{q}^*$ (see, e.g., [9], [7, Subsection VII.A] and references therein). Let $M$ be a fixed (independent of $n$) positive integer and consider an independent random selection of $2M-1$ members from $\mathcal{T}_n(\boldsymbol{q}^*)$, each one under the uniform distribution across $\mathcal{T}_n(\boldsymbol{q}^*)$, i.e.,

$$\Pi(\boldsymbol{s}) = \begin{cases} \frac{1}{|\mathcal{T}_n(\boldsymbol{q}^*)|} & \boldsymbol{s} \in \mathcal{T}_n(\boldsymbol{q}^*) \\ 0 & \text{elsewhere} \end{cases} \tag{22}$$

Let $\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_{2M-1}$ be the resulting randomly chosen state sequences. We can think of this collection as a random code for the channel

$$P(\boldsymbol{y}|\boldsymbol{s}) \stackrel{\triangle}{=} \prod_{t=1}^{n} p(y_t|s_t, s_{t\oplus 1}). \tag{23}$$

We next apply an expurgation process (see, e.g., [16, Subsection 5.7], [26, Subsection 3.3]), which guarantees that there exists a sub-code of size $M$ for which each each codeword contributes a conditional error probability that does not exceed $(2\overline{P_{\mathrm{e|m}}^{1/\rho}})^\rho$, where $\rho$ is an arbitrary positive real, and $\overline{P_{\mathrm{e|m}}^{1/\rho}}$ is the expectation of $P_{\mathrm{e|m}}^{1/\rho}$ under the above defined ensemble. Therefore, within this sub-code,

$$\max_{1\le m\le M} P_{\mathrm{e}|m} \le \left\{4M\sum_{\boldsymbol{s},\boldsymbol{s}'}\Pi(\boldsymbol{s})\Pi(\boldsymbol{s}')\left[\sum_{\boldsymbol{y}}\sqrt{P(\boldsymbol{y}|\boldsymbol{s})P(\boldsymbol{y}|\boldsymbol{s}')}\right]^{1/\rho}\right\}^\rho, \tag{24}$$

and consequently,

$$\begin{aligned} &\limsup_{n\to\infty}\frac{\ln\left[\max_{1\le m\le M} P_{\mathrm{e}|m}\right]}{n} \\ \le\ & \liminf_{\rho\to\infty}\limsup_{n\to\infty}\frac{1}{n}\ln\left(\left\{4M\sum_{\boldsymbol{s},\boldsymbol{s}'}\Pi(\boldsymbol{s})\Pi(\boldsymbol{s}')\exp\left[-\frac{1}{\rho}\sum_{t=1}^{n} d_{\mathrm{B}}(s_t,s_{t\oplus1};s_t',s_{t\oplus1}')\right]\right\}^\rho\right) \\ \stackrel{\text{(a)}}{\le}\ & -\sum_{s,s_+,s',s_+'} q^*(s,s_+)q^*(s',s_+')d_{\mathrm{B}}(s,s_+;s',s_+') \\ =\ & -E_0(\boldsymbol{q}^*) = -\max_{\boldsymbol{q}\in\mathcal{Q}_\Gamma} E_0(\boldsymbol{q}), \end{aligned} \tag{25}$$

where the inequality marked by (a) is justified by using the method of types for Markov chains ([7, Subsection VII.A] and references therein), and on the basis of the double irreducibility assumption (see Appendix for the details).

Finally, let $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_M\}$ be a sub–code with the property $\max_{1 \le m \le M} P_{\mathrm{e}|m} \le e^{-n[E_0(\boldsymbol{q}^*) - o(n)]}$ (where the indices $1, 2, \ldots, M$ are after possible relabeling). Then each $n$-tuple $\boldsymbol{s}_m = (s_{m,1}, \ldots, s_{m,n})$, $m = 1, 2. \ldots, M$, uniquely determines a corresponding codeword $\boldsymbol{x}_m = (x_{m,1}, \ldots, x_{m,n})$ according to $x_{m,t} = g(s_{m,t\oplus 1})$, $t = 1, 2, \ldots, n$, which obviously satisfies the input constraint (6), and so, the actual code for the given channel is $\mathcal{C} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M\}$. This completes the proof of the direct part the for case where $E_0(\boldsymbol{q})$ is concave.

To complete the proof of the direct part for the general case, we repeat the very same construction, but now, we combine it with time sharing. In particular, consider the more explicit form of $\overline{E}_0(\boldsymbol{q}^*)$ as

$$\overline{E}_0(\boldsymbol{q}^*) = \max_{\boldsymbol{w}, \boldsymbol{V}} \sum_{u \in \mathcal{U}} w(u) E_0[v(\cdot, \cdot|u)], \tag{26}$$

where $\boldsymbol{w} = \{w(u),\ u \in \mathcal{U}\}$ is a probability assignment on $u$, $\boldsymbol{V} = \{v(s, s_+|u),\ s, s_+ \in \mathcal{S},\ s_+ = f(s, g(s_+)),\ u \in \mathcal{U}\}$ is a set of probability assignments on state pairs given $u$, and the maximum is over all pairs $\{(\boldsymbol{w}, \boldsymbol{V})\}$ such that $\sum_{u \in \mathcal{U}} w(u) V(s, s_+|u) = q^*(s, s_+)$. Let $\boldsymbol{w}^*$ and $\boldsymbol{V}^*$ be achievers of the maximum on the r.h.s. of (26). For each codeword, the block of length $n$ is divided into $|\mathcal{U}|$ segments, each one of length $nw^*(u)$, labeled by $u \in \mathcal{U}$. Specifically, for every $m = 1, 2, \ldots, M$, we proceed as follows. For every $u \in \mathcal{U}$, select, independently at random, a member from $\mathcal{T}_{nw^*(u)}[v^*(\cdot, \cdot|u)]$, as the $u$–th segment of the state sequence associated with codeword, which is concatenated to all previous segments. Now, after expurgation of such randomly selected code, a straightforward extension of the derivation in (24) and (25) would yield an error exponent of $\sum_{u \in \mathcal{U}} w^*(u) E_0[v^*(\cdot, \cdot|u)] = \overline{E}_0(\boldsymbol{q}^*)$. Note that there is no need to worry about tailoring consecutive segments of the state sequence, because by our convention, all segments begin and end at state $\sigma$. This completes the proof of the direct part.

Moving on to the converse part, let $\mathcal{C}$ be an arbitrary rate–$\epsilon$ code ($\epsilon > 0$, infinitesimally small) of length $n$, that satisfies the input constraint (6) for each codeword. Consider the transformation of each codeword $\boldsymbol{x}_m$ in $\mathcal{C}$ into a state sequence $\boldsymbol{s}_m$, according to the recursion $s_{m,t+1} = f(s_{m,t}, x_{m,t})$, $t = 1, 2, \ldots, n-1$, $m = 1, 2, \ldots, M = e^{n\epsilon}$, where $s_{m,1} = \sigma$, which is a uniformly approachable state, and all codewords are extended (if needed) to be of length $n' = n + r$ (complying with the same recursion also for $t = n, n+1, \ldots, n'-1$), such that $f(s_{n'}, x_{n'}) = \sigma$, which is possible by the uniform approachability assumption. This extension of the codewords can only decrease the probability of error, so any lower bound on the error probability of the modified code is also a lower bound for the original code. The price of this extension is a possible increase in the average cost, but by no more than $r \cdot \max_x \phi(x)/n \stackrel{\Delta}{=} c/n$, which is vanishing as $n$ grows without bound, since $r$ depends only on $f$, but not on $n$.

For the sake of convenience, we denote the new block length by $n$ again, rather than $n'$. Consider now the resulting collection of state sequences, $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_M\}$, which can be considered as a code for the channel (23). Obviously, each $\boldsymbol{s}_m$ belongs to some Markov type $\mathcal{T}_n(\boldsymbol{q})$ where $\boldsymbol{q} \in \mathcal{Q}_{\Gamma+c/n}$. Since the number of distinct Markov types cannot exceed $(n+1)^{S^2}$, then at least $(n+1)^{-S^2} e^{n\epsilon}$ 'codewords' must belong to the same Markov type $\mathcal{T}_n(\boldsymbol{q})$. Obviously, the probability of error of the original given code (after the extension) cannot be smaller than $(n+1)^{-S^2}$ times the probability of error of the smaller code $\mathcal{C}' = |\mathcal{C} \cap \mathcal{T}_n(\boldsymbol{q})|$. Thus, any upper bound on the error exponent of $\mathcal{C}'$ is also an upper bound on the error exponent of the original code, and so, from this point onward we may assume that all codewords are of the same Markov type $\mathcal{T}_n(\boldsymbol{q})$, $\boldsymbol{q} \in \mathcal{Q}_{\Gamma+c/n}$.

Now, the channel (23) is obviously memoryless w.r.t. pairs of consecutive states $\{(s_t, s_{t\oplus 1})\}$, and we can therefore invoke the proof of Theorem 4 in [25] for memoryless channels. Combining eqs. (1.12), (1.36), (1.40), (1.42), (1.43) and (1.53) of [25] (with $K$ of [25] being replaced by $L$, in our notation), we learn that

$$-\frac{\ln P_e(\epsilon, n)}{n} \le \frac{1}{M^2} \sum_{t=1}^{n} \sum_{s, s_+, s', s'_+} M_t(s, s_+) M_t(s', s'_+) d_{\mathrm{B}}(s, s_+; s', s'_+) + o(n), \tag{27}$$

where $M_t(s, s_+)$ is the number of codewords in (a subset of) $\mathcal{C}'$ such that $(s_{m,t}, s_{m.t\oplus 1}) = (s, s_+)$ and $o(n)$ is a term that tends to zero as $n \to \infty$. It now readily follows that

$$\begin{aligned}
\mathcal{E}_0^+(\Gamma) &= \lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} \left[-\frac{\ln P_e(\epsilon, n)}{n}\right] && (28)\\
&\le \limsup_{n\to\infty} \left[\frac{1}{n} \sum_{t=1}^{n} E_0\left(\frac{M_t(\cdot,\cdot)}{M}\right)\right] && (29)\\
&\le \limsup_{n\to\infty} \left[\frac{1}{n} \sum_{t=1}^{n} \overline{E}_0\left(\frac{M_t(\cdot,\cdot)}{M}\right)\right] && (30)
\end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{n\to\infty} \overline{E}_0\left(\frac{1}{n}\sum_{t=1}^{n}\frac{M_t(\cdot,\cdot)}{M}\right) && (31)\\
&= \limsup_{n\to\infty} \overline{E}_0(\boldsymbol{q}) && (32)\\
&\leq \limsup_{n\to\infty} \sup_{\boldsymbol{q}\in\mathcal{Q}_{\Gamma+c/n}} \overline{E}_0(\boldsymbol{q}) && (33)\\
&= \sup_{\boldsymbol{q}\in\mathcal{Q}_{\Gamma}} \overline{E}_0(\boldsymbol{q}). && (34)
\end{aligned}
$$

This completes the proof of the converse part, and hence also the proof of Theorem 1.

## 4 The Gaussian Channel with ISI

In this section, we consider the important special case of the Gaussian channel with ISI. Our objective is to provide more explicit results, which are available thanks to the facts that: (i) the finite–state machine $f$ is simple, and more importantly, and (ii) the Bhattacharyya distance is proportional to the Euclidean distance, for which $E_0(\boldsymbol{q})$ is concave, and hence the operator UCE$\{\cdot\}$ becomes redundant.

The Gaussian ISI channel is defined by

$$
y_t = \sum_{i=0}^{k} h_i x_{t-i} + w_t, \tag{35}
$$

where $\{w_t\}$ is Gaussian white noise with zero mean, variance $\sigma^2$, and is independent of the channel input, $\{x_t\}$. Here, $\{h_i\}_{i=0}^{k}$ are the ISI channel coefficients. Obviously, the state of the channel, in this case, is given by the contents a shift register of length $k$, fed by the input, i.e., $s_t = (x_{t-k}, x_{t-k+1}, \ldots, x_{t-1}) \stackrel{\Delta}{=} x_{t-k}^{t-1}$, and the corresponding next–state function $f$ is doubly irreducible and uniformly approachable. The channel input power is limited to $\Gamma$, that is, the input constraint (6) is imposed with the cost function $\phi(x) = x^2$.

First, a straightforward calculation of the Bhattacharyya distance for the Gaussian ISI channel (35) yields

$$
\begin{aligned}
d_{\mathrm{B}}(s_t, s_{t\oplus 1}; \tilde{s}_t, \tilde{s}_{t\oplus 1}) &= d_{\mathrm{B}}(x_{t-k}^{t}, \tilde{x}_{t-k}^{t})\\
&= \frac{1}{8\sigma^2}\left(\sum_{i=0}^{k} h_t x_{t-i} - \sum_{i=0}^{k} h_t \tilde{x}_{t-i}\right)^2. && (36)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E_0(\boldsymbol{q}) &= \frac{1}{8\sigma^2}\sum_{x_0^k,\tilde{x}_0^k} q(x_0^k)q(\tilde{x}_0^k)\left(\sum_{i=0}^k h_i x_{k-i} - \sum_{i=0}^k h_i\tilde{x}_{k-i}\right)^2 && (37)\\
&= \frac{1}{4\sigma^2}\left[\sum_{x_0^k} q(x_0^k)\left(\sum_{i=0}^k h_i x_{k-i}\right)^2 - \left(\sum_{x_0^k} q(x_0^k)\sum_{i=0}^k h_i x_{k-i}\right)^2\right]\\
&= \frac{1}{4\sigma^2}\sum_{x_0^k} q(x_0^k)\left(\sum_{i=0}^k h_i x_{k-i}\right)^2 - \frac{1}{4\sigma^2}\left(\sum_{i=0}^k h_i \sum_{x_{k-i}} q(x_{k-i})x_{k-i}\right)^2\\
&= \frac{1}{4\sigma^2}\sum_{i=0}^k\sum_{j=0}^k h_i h_j \sum_{x_0 x_{|i-j|}} q(x_0, x_{|i-j|})x_0 x_{|i-j|} - \frac{1}{4\sigma^2}\left(\sum_{i=0}^k h_i\sum_{x_0} q(x_0)x_0\right)^2. && (38)
\end{aligned}
$$

The above expression should be maximized subject to a set of constraints that reflect the fact that $q(x_0^k)$ stems from an empirical distribution (of each codeword), i.e., the marginals of $(x_{i_1}, x_{i_2}, \ldots, x_{i_l})$ $(l \leq k)$ depend on the indices $i_1, i_2, \ldots, i_l$ only via the differences $i_2 - i_1, i_3 - i_2, \ldots, i_l - i_{l-1}$. An additional constraint is, of course, the power constraint $\sum_{x_0} q(x_0)x_0^2 \leq \Gamma$. Since the objective function is concave in $\boldsymbol{q}$ and the constraints are linear, this is, in principle, a standard convex programming problem.

It would be insightful to examine now the behavior in the case where $\{x_t\}$ takes on continuous values on the real line. In this case, in the limit of large $n$, the last expression reads, in the frequency domain, as follows:

$$E_0(\boldsymbol{q}) = \frac{1}{4\sigma^2}\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi} S_x(e^{i\omega})|H(e^{i\omega})|^2 \mathrm{d}\omega - \bar{X}^2|H(e^{i0})|^2\right], \tag{39}$$

where $H(e^{i\omega})$ $(i = \sqrt{-1})$ is the frequency response (the Fourier transform) associated with impluse response $\{h_i\}_{i=0}^k$, $S_x(e^{i\omega})$ is power spectrum of an underlying stationary process $\{X_t\}$, and $\bar{X}$ is the DC component of $\{X_t\}$. In other words, we think of the input power spectrum as

$$S_x(e^{i\omega}) = S_x'(e^{i\omega}) + 2\pi\bar{X}\delta(\omega), \qquad -\pi \leq \omega < \pi \tag{40}$$

where $S_x'(e^{i\omega})$ does not include a Dirac delta function at the origin. We can now express the zero–rate exponent as

$$E_0(\boldsymbol{q}) = \frac{1}{4\sigma^2}\cdot\frac{1}{2\pi}\int_{-\pi}^{+\pi} S_x'(e^{i\omega})|H(e^{i\omega})|^2 \mathrm{d}\omega, \tag{41}$$

which should be maximized under the power constraint

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} S_x'(e^{i\omega})\mathrm{d}\omega + \bar{X}^2 \le \Gamma.$$

It is now obvious that any non–zero value of $\bar{X}$ is just a waste, at the expense of the available power, which does not contribute to $E_0(\boldsymbol{q})$, and the best input spectrum is that of a sinusoidal process at the frequency $\omega_0$ that maximizes the amplitude response $|H(e^{i\omega})|$. If $\omega_0 = 0$, this means a DC process, which strictly speaking, contradicts our conclusion that the DC component should vanish. In this case, one can approach the maximum achievable exponent by a sinusoidal waveform of an arbitrarily low frequency, so that the response is close as desired to the maximum. Thus, the maximum achievable exponent when $\mathcal{X} = \mathrm{I\!R}$ is given by

$$\sup_{\boldsymbol{q}\in\mathcal{Q}_\Gamma} E_0(\boldsymbol{q}) = \frac{\Gamma}{4\sigma^2}\cdot\max_{\omega}|H(e^{i\omega})|^2. \tag{42}$$

To create $M$ orthogonal codewords, one can generate each one with a slightly different frequency in the vicinity of $\omega_0$. This is, of course, an upper bound also for any discrete–alphabet input.

It would be interesting now to have also a lower bound on the achievable zero–rate exponent for a given finite–alphabet size $K$. To this end, we will analyze the behavior for a specific class of input signals. When the finite input alphabet corresponds to the $K$ quantization levels of a uniform quantizer $Q(\cdot)$, i.e., $\{\pm(i-1/2)\Delta,\ i=1,2,\ldots,K/2\}$ ($K$ even), and $\Delta$ is reasonably small, it is natural, in view of the above, to consider the quantized sinusoid as an input signal $x_t = Q[A\sin(\omega_0 t+\phi)]$, where $A \le (K-1)\Delta/2$ is chosen to meet the input power constraint, $\sum_t Q^2[A\sin(\omega_0 t+\phi)] \le n\Gamma$. Obviously, the smaller is $\Delta$ (i.e., the larger is $K$ for a given $A$), the smaller is the loss compared to the clean (unquantized) sinusoid. We next examine this loss.

Let $e_t = Q[A\sin(\omega_0 t+\phi)] - A\sin(\omega_0 t+\phi)$ designate the quantization error signal. Then,

$$\begin{aligned}
\Gamma &= \frac{1}{n}\sum_{t=1}^{n} Q^2[A\sin(\omega_0 t+\phi)] &(43)\\
&= \frac{1}{n}\sum_{t=1}^{n}[A\sin(\omega_0 t+\phi)+e_t]^2 &(44)\\
&= \frac{A^2}{2}+\frac{2}{n}\sum_{t=1}^{n} Ae_t\sin(\omega_0 t+\phi)+\frac{1}{n}\sum_{t=1}^{n} e_t^2 &(45)\\
&\to \frac{A^2}{2}+2R_{xe}(0)+R_{ee}(0), &(46)
\end{aligned}$$

where we define

$$R_{ee}(\ell) = \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} e_t e_{t+\ell} \tag{47}$$

and

$$R_{xe}(\ell) = \lim_{n\to\infty} \frac{A}{n} \sum_{t=1}^{n} e_{t+\ell} \sin(\omega_0 t + \phi). \tag{48}$$

Denoting $H_{\max}^2 = |H(e^{i\omega_0})|^2$, we now have:

$$\begin{aligned}
& \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \left[ \sum_{\ell=0}^{k} h_\ell x_{t-\ell} \right]^2 && (49)\\
= \;& \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \left[ \sum_{\ell=0}^{k} h_\ell (A \sin[\omega_0(t-\ell)+\phi] + e_{t-\ell}) \right]^2 && (50)\\
= \;& \frac{A^2}{2} H_{\max}^2 + \sum_{\ell=0}^{k}\sum_{j=0}^{k} h_\ell h_j [R_{xe}(\ell-j) + R_{xe}(j-\ell) + R_{ee}(\ell-j)] && (51)\\
= \;& [\Gamma - 2R_{xe}(0) - R_{ee}(0)] H_{\max}^2 + \sum_{\ell=0}^{k}\sum_{j=0}^{k} h_\ell h_j [R_{xe}(\ell-j) + R_{xe}(j-\ell) + R_{ee}(\ell-j)] && (52)\\
= \;& \Gamma H_{\max}^2 - \Lambda && (53)
\end{aligned}$$

where $\Lambda$ is the loss due to quantization, i.e.,

$$\Lambda = [2R_{xe}(0) + R_{ee}(0)] H_{\max}^2 - \sum_{\ell=0}^{k}\sum_{j=0}^{k} h_\ell h_j [R_{xe}(\ell-j) + R_{xe}(j-\ell) + R_{ee}(\ell-j)]. \tag{54}$$

For the case where $\omega_0$ is irrational, one can find in [17, eqs. (44), (45), (51)] all the relevant joint second order statistics needed here. In particular, for the sinuoidal input under discussion,

$$R_{ee}(\ell) = \sum_{m=-\infty}^{\infty} \varepsilon_m \exp\{2\pi i \ell \lambda_m\} = \sum_{m=-\infty}^{\infty} \varepsilon_m \cos(2\pi\ell\lambda_m), \tag{55}$$

where $\lambda_m = \langle (2m-1)\omega_0/2\pi \rangle$, and

$$\varepsilon_m = \left[ \frac{\Delta}{\pi} \sum_{\ell=1}^{\infty} \frac{J_{2m-1}(2\pi\ell A/\Delta)}{\ell} \right]^2, \tag{56}$$

$J_m(z)$ being the $m$–th coefficient in the Fourier series expansion of the periodic function $\exp(iz\sin s)$, as a function of $s$, and

$$R_{xe}(\ell) = A\Delta \cos(\omega_0 \ell) \sum_{m=1}^{\infty} \frac{J_1(2\pi m A/\Delta)}{m} \triangleq AB\cos(\omega_0\ell). \tag{57}$$

We therefore obtain

$$\begin{aligned}\Lambda &= \left[2AB+\sum_{m=-\infty}^{\infty}\varepsilon_m\right]H_{\max}^2-2AB\sum_{\ell=0}^{k}\sum_{j=0}^{k}h_\ell h_j\cos[\omega_0(\ell-j)]- \\ &\quad \sum_{m=-\infty}^{\infty}\varepsilon_m\sum_{\ell=0}^{k}\sum_{j=0}^{k}h_\ell h_j\cos[2\pi(\ell-j)\lambda_m] \qquad (58)\\ &= \left[2AB+\sum_{m=-\infty}^{\infty}\varepsilon_m\right]H_{\max}^2-2H_{\max}^2AB-\sum_{m=-\infty}^{\infty}\varepsilon_m|H(e^{2\pi i\lambda_m})|^2 \qquad (59)\\ &= \sum_{m=-\infty}^{\infty}\varepsilon_m[H_{\max}^2-|H(e^{2\pi i\lambda_m})|^2]. \qquad (60)\end{aligned}$$

This expression is intuitively appealing: each term is the loss due to spectral term of $\{e_t\}$ that is in a non–optimal frequency (higher order harmonic) $\lambda_m$, where the power gain is $|H(e^{2\pi i\lambda_m})|^2$, rather than the optimal frequency $\omega_0$, where the power gain is $|H(e^{2\pi i\omega_0})|^2 = H_{\max}^2$. Thus, to summarize, the exponent of the finite–alphabet case is upper bounded by $\Gamma H_{\max}^2/(4\sigma^2)$ and lower bounded by $(\Gamma H_{\max}^2-\Lambda)/(4\sigma^2)$, where it should be kept in mind that $\Lambda$ depends on the ratio $A/\Delta \le (K-1)/2$ via $\{\varepsilon_m\}$. In [17, eq. (50)], there is a more explicit expression for $\varepsilon_m$. As $K$ increases, the loss $\Lambda$ decreases, essentially inverse proportionally to $K^2$,

On a related note, in the continuous–time version of the problem, where the channel is an additive white Gaussian noise channel, without bandwidth constraints, but only a peak–power constraint, a binary input $x_t \in \{-\sqrt{\Gamma}, +\sqrt{\Gamma}\}$ is as good as any $x_t \in [-\sqrt{\Gamma}, +\sqrt{\Gamma}]$ since the filter response to the latter can be approximated arbitrarily closely using binary inputs, as is shown in [21]. In other words, when $\{x_t\}$ is not discretized in time, it can be discretized in amplitude even as coursely as in binary quantization without essential loss of optimality.

## Appendix

*Justification of Inequality (a) in Equation (25).* We are interested in an exponential upper bound on the expression

$$\left\{\Pi(\boldsymbol{s})\Pi(\boldsymbol{s}')\exp\left[-\frac{1}{\rho}\sum_{t=1}^{n}d_{\mathrm{B}}(s_t,s_{t\oplus 1};s_t',s_{t\oplus 1}')\right]\right\}^{\rho}. \qquad (\mathrm{A.1})$$

Using the method of types for Markov chains, we find that the exponential rate of this quantity is of the exponential order of $\exp\{-nZ(\rho)\}$, where

$$\begin{aligned}Z(\rho) &= \min_{w_{SS_+S'S'_+}} \Big\{\rho[H_w(S_+|S)+H_w(S'_+|S')-H_w(S_+,S'_+|S,S')]- \\ &\quad \sum_{s,s_+,s',s'_+} w_{SS_+S'S'_+}(s,s_+,s',s'_+)d_{\mathrm{B}}(s,s_+;s',s'_+)\Big\}, \end{aligned} \tag{A.2}$$

where $w_{SS_+S'S'_+}$ is a generic joint distribution of a dummy quadruple of random variables $(S,S_+,S',S'_+)$ over $\mathcal{S}^4$, $H_w(\cdot|\cdot)$ are various conditional entropies induced by $w_{SS_+S'S'_+}$, and the weighted divergences are defined in the usual way. We note that since $w_{SS_+S'S'_+}$ is the empirical distribution of two pairs of consecutive states, it must always satisfy the stationarity conditions

$$\sum_{s_1,s_2} w_{SS'}(s_1,s_2)w_{S_+S'_+|SS'}(s_3,s_4|s_1,s_2) = w_{SS'}(s_3,s_4) \quad \forall s_3,s_4. \tag{A.3}$$

Since $\Pi(\boldsymbol{s})$ supports only members in $\mathcal{T}_n(\boldsymbol{q}^*)$, we also note that

$$\sum_{s',s'_+} w_{SS_+S'S'_+}(s,s_+,s',s'_+) = q^*(s,s_+) \tag{A.4}$$

$$\sum_{s,s_+} w_{SS_+S'S'_+}(s,s_+,s',s'_+) = q^*(s',s'_+). \tag{A.5}$$

Now, let is denote

$$\Delta(w_{SS_+S'S'_+}) = H_w(S_+|S)+H_w(S'_+|S')-H_w(S_+,S'_+|S,S') \tag{A.6}$$

and note that $\Delta(w_{SS_+S'S'_+}) \geq 0$ with equality iff $w_{SS_+S'S'_+}$ satisfies $w_{SS_+S'S'_+}(s,s_+,s',s'_+) = w_{SS'}(s,s')q^*(s_+|s)q^*(s'_+|s')$, where $q^*(s_+|s) \stackrel{\Delta}{=} q^*(s,s_+)/\pi^*(s)$. Now, let $w^{\rho}_{SS_+S'S'_+}$ denote the minimizing $w_{SS_+S'S'_+}$ for a given $\rho$. Considering a sequence $\rho_\ell \to \infty$, we have

$$\begin{aligned}\limsup_{\ell\to\infty} Z(\rho_\ell) &= \limsup_{\ell\to\infty} \min_{w_{SS_+S'S'_+}} \Big[\rho_\ell \cdot \Delta(w_{SS_+S'S'_+})- \\ &\quad - \sum_{s,s_+,s',s'_+} w_{SS_+S'S'_+}(s,s_+,s',s'_+)d_{\mathrm{B}}(s,s_+;s',s'_+)\Big] \\ &= \limsup_{\ell\to\infty} \Big[\rho_\ell \cdot \Delta(w^{\rho_\ell}_{SS_+S'S'_+})- \\ &\quad - \sum_{s,s_+,s',s'_+} w^{\rho_\ell}_{SS_+S'S'_+}(s,s_+,s',s'_+)d_{\mathrm{B}}(s,s_+;s',s'_+)\Big]\end{aligned}$$

$$\geq \quad -\liminf_{\ell\to\infty}\left[\sum_{s,s_+,s',s'_+} w^{\rho_\ell}_{SS_+S'S'_+}(s,s_+,s',s'_+)d_{\mathrm{B}}(s,s_+;s',s'_+)\right]. \tag{A.7}$$

As $\ell \to \infty$, there is a subsequence with indices $\{\ell_i\}$ that tends to the limit inferior in the last line of (A.7), and within this subsequence, there is a sub–subsequence for which $w^{\rho_{\ell_i}}_{SS_+S'S'_+}(s,s_+,s',s'_+)$ converges[11] to some limiting distribution of the form of the form $w_\infty(s,s')q^*(s_+|s)q^*(s'_+|s')$, as otherwise, the $\rho\cdot\Delta$ term would tend to infinity, and hence cannot achieve the minimum, which is finite. Thus,

$$\limsup_{\ell\to\infty} Z(\rho_\ell) \geq -\sum_{s,s_+,s',s'_+} w_\infty(s,s')q^*(s_+|s)q^*(s'_+|s')d_{\mathrm{B}}(s,s_+;s',s'_+). \tag{A.8}$$

Now, since $w_\infty(s,s')q^*(s_+|s)q^*(s'_+|s')$ is a limit of empirical distributions of pairs of consecutive states, then, as mentioned in (A.3), it must satisfy

$$\sum_{s,s'} w_\infty(s,s')q^*(s_+|s)q^*(s'_+|s') = w_\infty(s_+,s'_+) \quad \forall\ s_+,s'_+. \tag{A.9}$$

One solution to these equations is obviously $w_\infty(s,s') = \pi^*(s)\pi^*(s')$, but since we have assumed double irreducibility, then the corresponding pair of independent Markov chains has a unique stationary state distribution, which then must be $\pi^*(s)\pi^*(s')$. Thus,

$$\begin{aligned}\limsup_{\ell\to\infty} Z(\rho_\ell) &\geq -\sum_{s,s_+,s',s'_+} \pi^*(s)\pi^*(s')q^*(s_+|s)q^*(s'_+|s')d_{\mathrm{B}}(s,s_+;s',s'_+) \\ &= -\sum_{s,s_+,s',s'_+} q^*(s,s_+)q^*(s',s'_+)d_{\mathrm{B}}(s,s_+;s',s'_+).\end{aligned} \tag{A.10}$$

## References


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