# Multi-Group Testing for Items with Real-Valued Status under Standard Arithmetic* 

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#### Abstract

Motivated by applications in molecular biology and genotyping, this paper proposes a novel model of group testing for identifying items with real-valued status by using nonbinary pooling designs under standard arithmetic observation. The purpose is to learn more information of each item to be tested rather than identify only which ones are defectives as was done in conventional group testing. This paper provides several efficiently decodable nonadaptive strategies for the considered problem. The major tool is a new structure called $q$-ary additive $(w, d)$-disjunct matrix, which is related to known structures: the conventional disjunct matrix by Kautz and Singleton [35] and the SQ-disjunct matrix by Emad and Milenkovic [26].


Key words: group testing, pooling design.

## 1 Introduction

A frequently used tool to identify an unknown set of defective (positive) elements out of a large collection of elements by group tests is called Group Testing. In the

[^0]classic group testing, a "group" test can be any "subset" of the given collection and its outcome is binary under Boolean operations: YES or NO. The former indicates that there is a positive element in this test and the latter implies no positive elements. Due to a diversity of its applications, there have been many variants of the classic group testing in the literature. Readers are referred to the book [20] and some recent papers [3, 11, 12, 15, 17, 18, 36, 39] for further information.

Most models in literature consider the elements to be tested with a binary status: \{positive(1), negative(0)\}. In some applications, molecular biology [29], blood testing 37] and drug discovery [38, there can be a third category of elements called inhibitors, anti-bodies and blockers, respectively. The presence of such an element in a test can somehow cancel the effect of positive elements. A model addressing this issue has been intensively studied [7, 8, 9, 10, 11, 22, 29, 32] under the name of group testing with inhibitors (GTI). Two other group testing models, mutually obscuring defectives [16] and multiple access communication with interference [6], were built on the real observation that in chemical testing and communication theory it is usually seen that there exists some reaction when two substances meet in a suitable condition and undesired interference when two channels receive or send a message at the same time. Recently, Chen and Fu [12] combine the above notions and consider the multiple mutually-obscuring positives model (MMOP). In this model, more than three categories of elements (a $k$-ary status) are allowed with an additional assumption that certain obscuring phenomena, but unknown, occur among different categories of positive elements.

Inspired by the inhibition and the interference models, this paper considers a quantitative model that assumes the mutual effect of inhibition and interference can be quantized through analyzing a great amount of data in advance. This paper focuses on the problem where the elements to be tested are in nonbinary status and defines multi-group testing. Namely, a test can be applied to any "multi-subset" of
the given set, where "multi" means every single element is allowed to be taken more than once in a single test. The test matrix is then changed from binary in group testing to nonbinary in multi-group testing. Note that the notion of nonbinary tests is not new and can be found in [14, 27, 26, 34] with nonnegative integer matrices, 13 ] with integer matrices and [4] with no restriction on matrices. Obviously, allowing a number of duplicate copies in a test is meaningless under the assumption of Boolean operations. In the considered model, we shall assume that the outcome rule is linear under standard arithmetic.

A mathematical model can be described roughly as follows: Let $\mathbf{x}=\left(x_{1}, x_{2}, \cdots, x_{n}\right)$ $\in \mathbb{R}^{n}$ be an unknown vector, where $x_{i}$ denotes the status of $i$ th item. A measurement can be applied to any vector $\mathbf{y} \in\{0,1, \cdots\}^{n}$ with an outcome

$$
v_{\mathbf{y}} \equiv f(x, y)=\langle\mathbf{y}, \mathbf{x}\rangle=\sum_{i=1}^{n} y_{i} x_{i}
$$

The goal is to learn the unknown vector $\mathbf{x}$ through measurements in an efficient fashion (less measurements and fast decoding).

An obvious feature under this model is that any measurement which is linearly dependent on some other measurements is useless. The reason is that its outcome can then be simply derived from a linear combination of the outcomes of the others. The other feature is that, without any further information of the unknown vector $\mathbf{x}$, $n$ measurements are necessary in the worst case to learn the unknown vector ( $n$ is clearly sufficient). The reason is that after $k$ measurements $\mathbf{x}$ can be any vector in the $n-k$ dimensional subspace whose outcome is consistent with the $k$ measurements. If $k<n$, such vectors are not unique and thus cannot be determined exactly.

To make the multi-group testing model more interesting and challenging, we shall assume that

1. the unknown vector $\mathbf{x}$ is $d$-sparse, that is, $\mathbf{x}$ contains at most $d$ nonzero entries where $d$ is a constant with $d \ll n$;
2. the entries of $\mathbf{x}$ all belong to a certain set $D \subset \mathbb{R}$ which is a priori knowledge;
3. the number of copies from every single item in a measurement is restricted to $\{0,1, \cdots, q-1\}$, i.e., $\mathbf{y} \in \mathbb{Z}_{q}^{n}$, where the integer $q$ is prescribed.

Notice that the outcome $v_{\mathbf{y}}=\sum_{i=1}^{n} y_{i} x_{i}$ can possibly exceed $q$. Moreover, the cardinality of $D$ must be finite because the decoding algorithms proposed in this paper rely critically on $|D|$.

This paper focuses on nonadaptive strategies where measurements are performed simultaneously and therefore all measurements must be settled in advance. A nonadaptive strategy that uses $t$ measurements can be represented by a $t \times n$ matrix $A=\left[a_{i j}\right]$ with columns as items and rows as measurements, and the value at $a_{i j}$ denotes the number of copies of item $j$ in measurement $i$. The nonadaptive multi-group testing problem can then be converted into the problem: Construct a matrix $A=\left[a_{i j}\right]$ with $a_{i j} \in \mathbb{Z}_{q}$ so that the unknown $d$-sparse vector $\mathbf{x} \in D^{n}$ can be determined exactly and efficiently through the outcome vector $\mathbf{v}=A \mathbf{x}$. Clearly, the conventional additive group testing in [1, 19] is a special case of the multi-group testing with $D=\{0,1\}$ and $q=2$.

## Motivation and related work

Our model naturally arises in several situations. In some applications, such as blood testing, what patients or doctors want to know might be not only a yes-or-no answer but also a more precise index, a standard by which the level of some illness can be judged. The major purpose of relaxing $\mathbf{x}$ from the usual set $\{0,1\}$ to a prescribed set $D$ (can be very large) is that we aim to learn more information that each item carries rather than just determine which items are positive.

Of particular interest is that $D$ is allowed to contain not only positive elements but also negative elements, whose presence is in a sense to cancel the effect of positive elements, as inhibitors in GTI mentioned above. In GTI, the status of an inhibitor
can be viewed as $-\infty$ while the outcome is still binary. Formally speaking, it is to identify an unknown sparse vector $\mathbf{x} \in\{-\infty, 0,1\}^{n}$ by measurements $\mathbf{y} \in\{0,1\}^{n}$ with two possible outcomes: $\left\{\begin{array}{ll}1 & \text { if }\langle\mathbf{y}, \mathbf{x}\rangle \geq 1 ; \\ 0 & \text { if }\langle\mathbf{y}, \mathbf{x}\rangle<1 .\end{array}\right.$ However, the setting that one inhibitor is assumed to be able to cancel positive effect of all positive elements is too powerful to be appropriate in practice. It would be more reasonable that certain weaker cancelation effect exists between inhibitors and positives and can be quantized through analyzing a large amount of data in advance. For instance, if the information that one inhibitor cancels $k$ positives is a priori then it can fit into the framework of our model by setting $\mathbf{x} \in\{-k, 0,1\}^{n}$.

Motivated by applications in genotyping, Emad and Milenkovic [27, 26] proposed the Semi-Quantitative Group Testing (SQGT) which is a nonbinary pooling scheme combining an adder channel and an integer-valued quantizer. The quantizer and the nonbinary $\mathbf{x}$ settings in SQGT and our model, respectively, make a difference between them and no one includes the other. It is worth mentioning that nonbinary pooling designs are used commonly. The use of nonbinary pooling designs is based on the fact that "genotyping methods allow for more precise readings at the output than classical binary detectors" [27] and therefore the amount of samples must be reflected in the readings. It leads to an advantage of performance, i.e., using less measurements in the multiset model than in the set model is to be expected as set is a special case of multiset.

Recently, group testing has been related to compressed sensing in [2, 4, 5, 30]. Compressed sensing is a signal processing technique for recovering a signal by finding solutions to underdetermined linear systems (more unknown variables than equations), which coincides with the essence of multi-group testing under standard arithmetic. As a consequence, results developed in compressed sensing could benefit our model and vice versa. Although the two problems are in the same framework, to the best of our knowledge, there is no research in sparse signal recovery addressing a
problem with the same setting as our model.

## Our contribution

We give nonadaptive strategies for the multi-group testing problem with general $D$. We note that the one-sided case, i.e., $D$ is nonnegative or nonpositive, is a relatively simple case to handle. The reason is that in this case a zero outcome simply implies that all the items appearing in the measurement are zero, in contrast to the general case, a zero outcome can be produced by a combination of some positive elements and negative elements. Although the main result for the one-sided case has its counterpart for the general case, the technique and complexity are very different.

We propose a new combinatorial structure called $q$-ary additive $(w, d)$-disjunct matrices (will be defined later). Such a structure enables us to solve the general case and decode efficiently. It is new but related to known structures: the well-known binary disjunct matrix introduced by Kautz and Singleton [35] and the SQ-disjunct matrix proposed by Emad and Milenkovic [26]. We have a method to construct $q$ ary additive $(w, d)$-disjunct matrices, but not as strong as we like because it relies critically on the construction of conventional disjunct matrices. Also, we provide two methods by applying the Kronecker product to produce a bigger matrix from a smaller one. Although the resulting matrices cannot be applied to solve the general case, they can solve the one-sided case with efficient decoding algorithms. The value of our constructions is not in its practicality in constructing efficient $q$-ary additive $(w, d)$-disjunct matrices, but rather in calling awareness to the existence of such constructions, so that further research can improve on it.

Our decoding algorithms based on $q$-ary additive $(w, d)$-disjunct matrices are quite efficient. For the one-sided case our strategy has a decoding algorithm in $O(|D| t n)$ time and for the general case it is $O\left(|D| t n^{d+1}\right)$. By contrast, even ignoring the time for multiplications of vectors, in the worst case it takes extremely high time complex-
ity $O\left(\sum_{i=0}^{d}\binom{n}{i}|D|^{i}\right)$ to decode $\mathbf{x}$ by simply applying a straightforward brute-force procedure.

The rest of the paper is organized as follows. Section 2 first introduces notations and major tools and then exploits them to solve the one-sided case of the multi-group testing problem under standard arithmetic. Section 3 deals with the general case. Finally, Section 4 provides three constructions mentioned above.

## 2 The one-sided case

This section starts with a simple but useful lemma.

Lemma 2.1 Let $A$ be a matrix in $\mathbb{R}^{t \times n}$. Given a fixed (unknown) vector $\mathbf{x} \in \mathbb{R}^{n}$, let $\mathbf{v}=A \mathbf{x}$. Let $\mathbf{x}^{\prime}=a \mathbf{x}+\mathbf{b}$ for some $0 \neq a \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^{n}$ (known), and let $\mathbf{v}^{\prime}=A \mathbf{x}^{\prime}$. Then the problem of learning $\mathbf{x}$ from $\mathbf{v}$ is equivalent to the problem of learning $\mathbf{x}^{\prime}$ from $\mathrm{v}^{\prime}$.

Proof. The proof follows immediately from the linear mapping from $\mathbf{v}^{\prime}$ to $\mathbf{v}$, that is, $\mathbf{v}^{\prime}=A \mathbf{x}^{\prime}=A(a \mathbf{x}+\mathbf{b})=a A \mathbf{x}+A \mathbf{b}=a \mathbf{v}+A \mathbf{b}$.

To present our algorithms, we first introduce some notations. Throughout this paper, let $\mathbf{x} \in D^{n}$ be an unknown $d$-sparse vector, $A=\left[a_{i j}\right]$ of size $t \times n$ be the matrix corresponding to the measurements and $\mathbf{v}=A \mathbf{x}$ be the outcome vector. For any vector $\mathbf{y}=\left(y_{1}, \cdots, y_{n}\right)$, denote by $\|\mathbf{y}\|_{0} \equiv\left|\left\{y_{j}: y_{j} \neq 0\right\}\right|$ the $l_{0}$ norm (or sparsity) of $\mathbf{y}$. Given any vector $\mathbf{y}=\left(y_{1}, \cdots, y_{t}\right)$, for each $j \in[n]$ define

$$
t_{j}(\mathbf{y})=\left|\left\{i: a_{i j}>y_{i}\right\}\right| .
$$

For any vector $\mathbf{y}=\left(y_{1}, y_{2}, \cdots, y_{n}\right)$ and $\delta \in \mathbb{R}$, define $\mathbf{s}^{\mathbf{y},=\delta}=\left(s_{1}^{\mathbf{y},=\delta}, s_{2}^{\mathbf{y},=\delta}, \cdots, s_{n}^{\mathbf{y},=\delta}\right)$ where

$$
s_{j}^{\mathbf{y},=\delta}=\left\{\begin{array}{cc}
1 & \text { if } y_{j}=\delta \\
0 & \text { otherwise }
\end{array}\right.
$$

For convenience, we shall use $s_{j}$ to denote $s_{j}^{\mathbf{y},=\delta}$ if no confusion occurs without specified superscripts. Likewise, we define $\mathbf{s}^{\mathbf{y}, \geq \delta}$ and $\mathbf{s}^{\mathbf{y}, \leq \delta}$ by replacing $y_{j}=\delta$ with $y_{j} \geq \delta$ and $y_{j} \leq \delta$, respectively. For any two vectors $\mathbf{x}$ and $\mathbf{y}$ of the same dimension, denote by $\mathbf{x} \succeq \mathbf{y}$ if $x_{i} \geq y_{i}$ for all $i$ and by $\mathbf{x} \nsucceq \mathbf{y}$ otherwise.

Consider a fixed $q$-ary matrix $M=\left[m_{i j}\right]$ of size $t \times n$. For any vector $\mathbf{y}$ of length $n$, we define the syndrome vector of $\mathbf{y}$ in $M$ by $\phi_{M}(\mathbf{y})=\left(\phi_{1}(\mathbf{y}), \phi_{2}(\mathbf{y}), \cdots, \phi_{t}(\mathbf{y})\right)$, where

$$
\phi_{j}(\mathbf{y})=\sum_{i=1}^{n} y_{i} \cdot m_{j i} .
$$

For any two $n$-vectors $\mathbf{y}_{0}$ and $\mathbf{y}_{1}$, we say their syndromes $\phi_{M}\left(\mathbf{y}_{0}\right), \phi_{M}\left(\mathbf{y}_{1}\right)$ are different, denoted by $\phi_{M}\left(\mathbf{y}_{0}\right) \neq \phi_{M}\left(\mathbf{y}_{1}\right)$, if and only if there exists some $j \in[t]$ such that $\phi_{j}\left(\mathbf{y}_{0}\right) \neq \phi_{j}\left(\mathbf{y}_{1}\right)$.

Definition 1 Let $M=\left[m_{i j}\right]$ of size $t \times n$ be a q-ary matrix. We say $M$ is additive $(D, d)$-separable if

$$
\phi_{M}\left(\mathbf{y}_{0}\right) \neq \phi_{M}\left(\mathbf{y}_{1}\right)
$$

for any two d-sparse vectors $\mathbf{y}_{0}, \mathbf{y}_{1} \in D^{n}$ with $\mathbf{y}_{0} \neq \mathbf{y}_{1}$.

By definition, it is easily seen that additive ( $D, d$ )-separability is a sufficient and necessary condition for the considered problem. Moreover, a $q$-ary additive $(D, d)$ separable matrix with $q=2$ and $D=\{0,1\}$ reduces to a $d$-detecting matrix in [?]. Although separability provides a solution to identify the unknown vector $\mathbf{x}$, it suffers from lack of efficient algorithms for decoding.

Disjunct matrices were first studied by Kautz and Singleton 35] under the name of zero-false-drop codes, and also known as cover-free families [28] or superimposed codes [23]. A binary matrix is called $d$-disjunct if it satisfies the property: for any fixed column and other $d$ columns, there exists a row such that the designated column is 1 and all the $d$ columns are 0 . Disjunct matrices have been intensively studied for fifty years. Of particular note is that a $d$-disjunct matrix of size $t \times n$ can identify
up to $d$ defectives with efficient decoding complexity $O(t n)$. Recently, the decoding complexity has been further improved based on other combinatorial structures. Several algorithms with sublinear decoding complexity (in $n$ ) were proposed [31, 33, ?]. It is known [19, 23, 24] that a $d$-disjunct matrix of $n$ columns has an upper bound $O\left(d^{2} \log n\right)$ and a lower bound $\Omega\left(d^{2} \log n / \log d\right)$ on the number of rows. There are many constructions attaining the best known upper bound $O\left(d^{2} \log n\right)$ (see [25, 20]). Next, we define a new family of disjunct matrices that can be applied to solve the multi-group testing problem with efficient decoding algorithms.

Definition 2 Let $q, t, n$ and $d$ be positive integers and $w>0$. A q-ary matrix $M=$ $\left[m_{i j}\right]$ of size $t \times n$ is called additive $(w, d)$-disjunct if for any $\mathbf{s} \in\left\{\left(s_{1}, s_{2}, \cdots, s_{n}\right) \in\right.$ $\left.\{0,1\}^{n}: \sum_{j=1}^{n} s_{j} \leq d\right\}$ and for each $k \in[n]$ such that $s_{k}=0$, there exists an $i \in[t]$ such that

$$
m_{i k}>w \sum_{j=1}^{n} m_{i j} s_{j}
$$

The additive disjunct matrices can be related to some known structures. A conventional $d$-disjunct matrix is a binary $(q=2)$ additive $(w, d)$-disjunct matrix for any $w \geq 1$. In particular, when $w \geq q-1$, any $q$-ary additive $(w, d)$-disjunct matrix can be converted simply to a binary $d$-disjunct matrix by replacing every non-zero entry with 1. As a consequence, we have the following bound.

Proposition 2.1 Let $g(n, w, d, q)$ denote the minimum $t$ such that a $t \times n$-ary additive $(w, d)$-disjunct matrix exists. Then $g(n, w, d, q)=O\left(d^{2} \log n\right)$ for any $w \geq 1$ and $g(n, w, d, q)=\Omega\left(d^{2} \log n / \log d\right)$ when $w \geq q-1$.

Another is the SQ-disjunct code defined by Emad and Milenkovic [26]. When $w=1$, a $q$-ary additive $(w, d)$-disjunct matrix is reduced to a special case of the $[q ; Q ; \eta ;(1: d) ; e]$-SQ-disjunct code with $Q=d$, the thresholds $\eta=[0,1, \cdots, d]^{T}$ and $e=0$ (error-free). Thus, several useful constructions in [27] can be applied immediately for constructing $q$-ary additive $(w, d)$-disjunct matrices with $w=1$.

Note that in Definition 2 $w$ is assumed only to be positive and needs not to be $w \geq 1$, which is indeed the case throughout this paper. The case $0<w<1$ seems strange but has its own interest in combinatorial structure. For instance, when $\frac{1}{2} \leq w<1$ a binary $(w, d)$-disjunct matrix is equivalent to a matrix satisfying the property that for any fixed column and $d$ other columns there exists a row such that the designated column has a 1 and the $d$ columns have at least $d-10$ 's. In view of this, we believe that the additive disjunct matrices with $0<w<1$ might have other potential applications.

Next, we study the one-sided case, i.e., elements in $D$ are either all nonnegative or all nonpositive. By symmetry, we may and shall assume that $D=\left\{c_{0}=0, c_{1}, \cdots, c_{m}\right\}$ where $0<c_{1}<c_{2}<\cdots<c_{m}$.

Theorem 2.2 Let $A$ be a q-ary additive $(w, d)$-disjunct matrix of size $t \times n$ with $w=\max _{1 \leq k \leq m}\left\{\frac{c_{m}-c_{k-1}}{c_{k}-c_{k-1}}\right\}$. Then $A$ is additive $(D, d)$-separable.

Proof. Consider any two fixed $d$-sparse vectors $\mathbf{y}, \mathbf{y}^{\prime} \in D^{n}$ with $\mathbf{y} \neq \mathbf{y}^{\prime}$. There exist some $j$ 's $\in[n]$ such that $y_{j} \neq y_{j}^{\prime}$. Let $y_{j}$ be the smallest value among all those $j$ 's and by symmetry we may assume that $y_{j}<y_{j}^{\prime}$. Without loss of generality, we assume that $y_{j}=c_{g-1}$ and therefore $y_{j}^{\prime} \geq c_{g}$. To prove the theorem, it suffices to show that $A \mathbf{y}^{\prime} \neq$ $A \mathbf{y}$, or equivalently $A \cdot\left(y_{1}^{\prime}, \cdots, y_{j}^{\prime}-y_{j}, \cdots, y_{n}^{\prime}\right)^{T} \neq A \cdot\left(y_{1}, \cdots, y_{j-1}, 0, y_{j+1}, \cdots, y_{n}\right)^{T}$. Let $\mathbf{y}_{j}$ denote the vector $\mathbf{y}$ subject to the $j$-th position, i.e., $\left(0, \cdots, 0, y_{j}, 0, \cdots, 0\right)$. The above inequality can be rewritten as

$$
\begin{equation*}
A\left(\mathbf{y}^{\prime}-\mathbf{y}_{j}\right) \neq A\left(\mathbf{y}-\mathbf{y}_{j}\right) \tag{1}
\end{equation*}
$$

Consider the vector $\mathbf{s}^{\mathbf{y}-\mathbf{y}_{j}, \geq c_{1}}$, which is $d$-sparse as is $\mathbf{y}$. Notice that $s_{j}^{\mathbf{y}-\mathbf{y}_{j}, \geq c_{1}}=0$. By definition of additive $(w, d)$-disjunctness, there exists some $t^{*} \in[t]$ such that $a_{t^{*} j}>w \sum_{i=1}^{n} a_{t^{*} i} s_{i}^{\mathbf{y}-\mathbf{y}_{j}, \geq c_{1}}$. Since the minimality of $y_{j}$, we have $\left[\mathbf{y}-\mathbf{y}^{\prime}\right]_{i} \leq c_{m}-c_{g-1}$
for all $i$.

$$
\begin{aligned}
\left\|\left[A\left(\mathbf{y}^{\prime}-\mathbf{y}_{j}\right)-A\left(\mathbf{y}-\mathbf{y}_{j}\right)\right]_{t^{*}}\right\| & \geq\left(y_{j}^{\prime}-y_{j}\right) a_{t^{*} j}-\left(c_{m}-c_{g-1}\right) \sum_{i=1}^{n} a_{t^{*} i} s_{i}^{\mathbf{y}-\mathbf{y}_{j}, \geq c_{1}} \\
& >\left[\left(y_{j}^{\prime}-y_{j}\right) w-\left(c_{m}-c_{g-1}\right)\right] \sum_{i=1}^{n} a_{t^{*} i} s_{i}^{\mathbf{y}-\mathbf{y}_{j}, \geq c_{1}} \\
& \geq\left[\left(c_{g}-c_{g-1}\right) w-\left(c_{m}-c_{g-1}\right)\right] \sum_{i=1}^{n} a_{t^{*} i} i_{i}^{\mathbf{y}-\mathbf{y}_{j}, \geq c_{1}}>0
\end{aligned}
$$

where the last inequality holds as $w=\max _{1 \leq k \leq m}\left\{\frac{c_{m}-c_{k-1}}{c_{k}-c_{k-1}}\right\}$. This proves (11) and therefore concludes the theorem.

The above theorem makes an attempt to show that the designated matrix $A$ satisfies the separability property so that the unknown $d$-sparse vector $\mathbf{x}$ can be successfully identified. However, even ignoring the time for multiplications of vectors, it takes extremely high time complexity $O\left(\sum_{i=0}^{d}\binom{n}{i} m^{i}\right)$ to decode $\mathbf{x}$ by simply applying a straightforward brute-force procedure based on the separability property. For what follows, we exploit a more powerful property, disjunctness, of $A$ to quickly identify the unknown vector $\mathbf{x}$. Next, the focus is on decoding complexity.

Lemma 2.3 Suppose $\mathbf{x} \in D^{n}$ is an unknown d-sparse vector. Let $A$ be a q-ary additive $(w, d)$-disjunct matrix of size $t \times n$ with $w \geq \frac{c_{m}}{c_{1}}$ and let $\mathbf{v}=A \mathbf{x}$ be the outcome vector. Then $\mathbf{s}^{\mathbf{x}, \geq c_{1}}$ can be identified from $\mathbf{v}$.

Proof. By Lemma 2.1, this problem can be reduced to the problem of learning the unknown sparse vector $\mathbf{x}^{\prime}=\left(x_{1}^{\prime}, x_{2}^{\prime}, \cdots, x_{n}^{\prime}\right) \in\left\{0,1, \frac{c_{2}}{c_{1}}, \cdots, \frac{c_{m}}{c_{1}}\right\}^{n}$ from the outcome vector $\mathbf{v}^{\prime}=\frac{1}{c_{1}} \mathbf{v}$.

For each $k$ such that $x_{k}^{\prime} \neq 0$, for all $i \in[t]$ we have $a_{i k} \leq \sum_{c=1}^{c_{m} / c_{1}}\left(c \sum_{\left\{j: x_{j}^{\prime}=c\right\}} a_{i j}\right)=v_{i}^{\prime}$. Accordingly, $t_{k}\left(\mathbf{v}^{\prime}\right)=0$ whenever $x_{k}^{\prime} \neq 0$.

Since the vector $\mathbf{x}$ is $d$-sparse, $\sum_{j=1}^{n} s_{j}^{\mathbf{x}^{\prime}, \geq 1} \leq d$. For each $k$ such that $x_{k}^{\prime}=0$, by definition of the $q$-ary additive $(w, d)$-disjunct matrix, there exists $i \in[t]$ such that

$$
a_{i k}>w \sum_{j=1}^{n} a_{i j} s_{j}^{\mathbf{x}^{\prime}, \geq 1} \geq \sum_{j=1}^{n} a_{i j} x_{j}^{\prime}=v_{i}^{\prime}
$$

This implies that $t_{k}\left(\mathbf{v}^{\prime}\right)>0$ if $x_{k}^{\prime}=0$. By the above discussion, we can identify the vector $\mathbf{s}^{\mathbf{x}^{\prime}, \geq 1}$ through the counting function $t_{k}\left(\mathbf{v}^{\prime}\right)$. As a consequence, $\mathbf{s}^{\mathbf{x}, \geq c_{1}}$ can be identified too.

We now analyze the time complexity for the decoding algorithm corresponding to Lemma 2.3, For each $j \in[n]$, it takes $t$ operations for computing the value $t_{j}(\cdot)$. Therefore, the decoding complexity is $O(t n)$.

Corollary 2.4 Let $\mathbf{x} \in\left\{c_{0}=0, c_{1}, \cdots, c_{m}\right\}^{n}$ be an unknown $d$-sparse vector, where $0<c_{1}<c_{2}<\cdots<c_{m}$. Let $w=\max _{1 \leq k \leq m}\left\{\frac{c_{m}-c_{k-1}}{c_{k}-c_{k-1}}\right\}$. Then any $q$-ary additive $(w, d)$-disjunct matrix of size $t \times n$ can be used to identify $\mathbf{x}$ with $O(|D| t n)$ decoding complexity.

Proof. The corollary follows by applying Lemma 2.3 repeatedly ( $m$ times). The precise process is as follows. Let $\mathbf{x}^{1}=\frac{1}{c_{1}} \mathbf{x}$ and $\mathbf{v}^{1}=\frac{1}{c_{1}} \mathbf{v}$ (here we shall use $\mathbf{x}^{i}$ and $\mathbf{v}^{i}$ to denote the updated vectors in the $i$ th round). Since $w \geq \frac{c_{m}}{c_{1}}$, by Lemma 2.3 we know that $\mathbf{s}^{\mathbf{x}^{1}, \geq 1}$ can be identified from $\mathbf{v}^{1}$ successfully.

For $k=2, \cdots, m-1$, define recursively that

$$
\mathbf{v}^{k}=\frac{1}{\frac{c_{k}-c_{k-2}}{c_{k-1}-c_{k-2}}-1}\left(\mathbf{v}^{k-1}-A \mathbf{s}^{\mathbf{x}^{k-1}, \geq 1}\right) \text { and } \mathbf{x}^{k}=\frac{1}{\frac{c_{k}-c_{k-2}}{c_{k-1}-c_{k-2}}-1}\left(\mathbf{x}^{k-1}-\mathbf{s}^{\mathbf{x}^{k-1}, \geq 1}\right)
$$

It is easily verified that $\mathbf{v}^{k}=A \mathbf{x}^{k}$ for all $k$. Note that $\mathbf{x}^{k} \in\left\{0,1, \frac{c_{k+1}-c_{k-1}}{c_{k}-c_{k-1}}, \cdots, \frac{c_{m}-c_{k-1}}{c_{k}-c_{k-1}}\right\}^{n}$ is a $d$-sparse vector and $w \geq \frac{c_{m}-c_{k-1}}{c_{k}-c_{k-1}}$ for each $k=1, \cdots, m$. Applying Lemma 2.3 repeatedly, $\mathbf{s}^{\mathbf{x}^{k}, \geq 1}$ can be identified for all $k=1, \cdots, m$. Consequently, the unknown vector $\mathbf{x}$ can be identified as $\mathbf{x}=\sum_{i=1}^{m}\left(c_{i}-c_{i-1}\right) \mathbf{s}^{\mathbf{x}^{i}, \geq 1}=\sum_{i=1}^{m} c_{i} \mathbf{s}^{\mathbf{x},=c_{i}}$. This completes the proof.

## 3 The general case

In this section, we turn our attention to the general case that elements in $D$ are neither all nonnegative nor all nonpositive. Throughout this section, we shall assume
$D=\left\{z_{m_{2}}, \cdots, z_{1}, 0, c_{1}, \cdots, c_{m_{1}}\right\}$, where $z_{m_{2}}<\cdots<z_{1}<0<c_{1}<\cdots<c_{m_{1}}$. For any vector $\mathbf{v}, \mathbf{r} \in\{0,1\}^{n}$ and $h \in \mathbb{R}$, define

$$
\mathbf{v}^{h \mathbf{r}} \equiv \mathbf{v}+h A \mathbf{r} .
$$

For any vector $\mathbf{v}, d \in \mathbb{N}$ and $h \in \mathbb{R}$, define

$$
t_{j}^{*}(\mathbf{v}, h, d) \equiv \min _{\mathbf{r} \in\{0,1\}^{n},\|\mathbf{r}\|_{0} \leq d} t_{j}\left(\mathbf{v}^{h \mathbf{r}}\right) \text { for } j=1, \cdots, n
$$

Theorem 3.1 Let $\mathbf{x} \in D^{n}$ be an unknown $d$-sparse vector and $w \geq \frac{c_{m_{1}}-z_{m_{2}}}{c_{1}}$. Then any $q$-ary additive ( $w, 2 d$ )-disjunct matrix of size $t \times n$ can be used to identify $\mathbf{s}^{\mathbf{x}, \geq c_{1}}$.

Proof. By Lemma 2.1, this problem is equivalent to the problem of learning $\mathbf{x} \in$ $\left\{\frac{z_{m_{2}}}{c_{1}}, \cdots, \frac{z_{1}}{c_{1}}, 0,1, \frac{c_{2}}{c_{1}}, \cdots, \frac{c_{m_{1}}}{c_{1}}\right\}^{n}$. Let $\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right), A$ be a $q$-ary additive $(w, 2 d)$ disjunct matrix, $w \geq \frac{c_{m_{1}}-z_{m_{2}}}{c_{1}}$, of size $t \times n$ and $\mathbf{v}=A \mathbf{x}$ be the outcome vector corresponding to $\mathbf{x}$. It suffices to identify $\mathbf{s}^{\mathbf{x}, \geq 1}$.

Consider any vector $\mathbf{r}(\mathbf{x})=\left(r_{1}, r_{2}, \cdots, r_{n}\right) \in\{0,1\}^{n}$ with $\|\mathbf{r}(\mathbf{x})\|_{0}=d$ such that $\mathbf{r}(\mathbf{x}) \succeq \mathbf{s}^{\mathbf{x},<0}$. For each $k$ with $x_{k} \geq 1$, for all $i \in[t]$ we have

$$
\begin{array}{rlr}
v_{i}^{\frac{-z_{m_{2}}}{c_{1}} \mathbf{r}(\mathbf{x})} & =v_{i}-\frac{z_{m_{2}}}{c_{1}} \sum_{j=1}^{n} a_{i j} r_{j} \quad\left(\text { since } \mathbf{v}^{\frac{-z_{m_{2}}}{c_{1}} \mathbf{r}(\mathbf{x})}=\mathbf{v}-\frac{z_{m_{2}}}{c_{1}} A \mathbf{r}(\mathbf{x})\right) \\
& \geq \sum_{\left\{j: x_{j} \geq 1\right\}} a_{i j}+\frac{z_{m_{2}}}{c_{1}} \sum_{\left\{j: x_{j}<0\right\}} a_{i j}-\frac{z_{m_{2}}}{c_{1}} \sum_{j=1}^{n} a_{i j} r_{j} \quad\left(\text { since } \mathbf{r}(\mathbf{x}) \succeq \mathbf{s}^{\mathbf{x},<0}\right) \\
& \geq \sum_{\left\{j: x_{j} \geq 1\right\}} a_{i j} \geq a_{i k} .
\end{array}
$$

By definition, $t_{k}\left(\mathbf{v}^{-\frac{z_{m_{2}}}{c_{1}} \mathbf{r}(\mathbf{x})}\right)=0$ and therefore $t_{k}^{*}\left(\mathbf{v},-\frac{z_{m_{2}}}{c_{1}}, d\right)=0$ if $x_{k} \geq 1$.
Consider the case $x_{k}<1$. For any arbitrary vector $\mathbf{r} \in\{0,1\}^{n}$ with $\|\mathbf{r}\|_{0} \leq d$, consider the unknown vector $\mathbf{s}=\left(s_{1}, s_{2}, \cdots, s_{n}\right)$ where

$$
s_{\ell}= \begin{cases}1 & \text { if } x_{\ell} \geq 1 \text { or } r_{\ell}=1 \\ 0 & \text { otherwise }\end{cases}
$$

As $\|\mathbf{r}\|_{0} \leq d$ and $\|\mathbf{x}\|_{0} \leq d$, we have $\|\mathbf{s}\|_{0} \leq 2 d$. Since $A$ is a $q$-ary additive $(w, 2 d)$ disjunct matrix where $w \geq \frac{c_{m_{1}}-z_{m_{2}}}{c_{1}}$, there exists $i \in[t]$ such that

$$
\begin{array}{rlrl}
a_{i k} & >\frac{c_{m_{1}}-z_{m_{2}}}{c_{1}} \sum_{j=1}^{n} a_{i j} s_{j} & & \text { ( by definition of disjunctness) } \\
& =\frac{c_{m_{1}}}{c_{1}} \sum_{j=1}^{n} a_{i j} s_{j}-\frac{z_{m_{2}}}{c_{1}} \sum_{j=1}^{n} a_{i j} s_{j} & & \\
& \geq \sum_{j=1}^{n} a_{i j} x_{j}-\frac{z_{m_{2}}}{c_{1}} \sum_{j=1}^{n} a_{i j} s_{j} & (\text { since } \mathbf{s} \succeq \mathbf{x}) \\
& \geq v_{i}-\frac{z_{m_{2}}}{c_{1}} \sum_{j=1}^{n} a_{i j} s_{j} & & \\
& \geq v_{i}-\frac{z_{m_{2}}}{c_{1}} \sum_{j=1}^{n} a_{i j} r_{j} & & \\
& =v_{i}^{-\frac{z_{m_{2}}}{c_{1}} \mathbf{r}} . & \text { since } \left.\mathbf{s} \succeq \mathbf{r}(\mathbf{x}) \text { and } \frac{z_{m_{2}}}{c_{1}}<0\right)
\end{array}
$$

This implies $t_{k}\left(\mathbf{v}^{-\frac{z_{m_{2}}}{c_{1}} \mathbf{r}}\right)>0$ for any $\mathbf{r} \in\{0,1\}^{n}$ with $\|\mathbf{r}\|_{0} \leq d$. Hence, $t_{k}^{*}\left(\mathbf{v},-\frac{z_{m_{2}}}{c_{1}}, d\right)>$ 0 if $x_{k}<1$.

Therefore, by the above discussion, one can determine whether $x_{k} \geq 1$ through the counting function $t_{k}^{*}\left(\mathbf{v},-\frac{z_{m_{2}}}{c_{1}}, d\right)$.

We now analyze the time complexity for the decoding algorithm corresponding to Lemma 3.1. For each $j \in[n]$, it takes $t\binom{n}{d}$ operations for computing the value $t_{j}^{*}(\cdot)$. Therefore, the decoding complexity is $O\left(t n^{d+1}\right)$.

Combining Corollary 2.4 and Theorem 3.1, we have the following result.

Theorem 3.2 Let $\mathbf{x} \in D^{n}$ be an unknown d-sparse vector and let

$$
w \geq \max \left\{\left\{\left.\frac{c_{m_{1}}-c_{i}-z_{m_{2}}}{c_{i+1}-c_{i}} \right\rvert\, 0 \leq i \leq m_{1}-1\right\} \bigcup\left\{\left.\frac{-z_{m_{2}}+z_{i-1}}{-z_{i}+z_{i-1}} \right\rvert\, 1 \leq i \leq m_{2}\right\}\right\}
$$

Then any $q$-ary additive $(w, 2 d)$-disjunct matrix of size $t \times n$ can be used to identify $\mathbf{x}$ with $O\left(|D| t n^{d+1}\right)$ decoding complexity.

Proof. By Theorem 3.1, $\mathbf{s}^{\mathbf{x}, \geq c_{1}}$ can be identified since $w \geq \frac{c_{m_{1}}-z_{m_{2}}}{c_{1}}$. Let $\mathbf{x}^{1}=\mathbf{x}-$ $c_{1} \mathbf{s}^{\mathbf{x}, \geq c_{1}}$ and for $i=2, \cdots, m_{1}-1$ define recursively $\mathbf{x}^{i}=\mathbf{x}^{i-1}-\left(c_{i}-c_{i-1}\right) \mathbf{s}^{\mathbf{x}, \geq c_{i}}$. Note
that $\mathbf{x}^{i} \in\left\{z_{m_{2}}, \cdots, z_{1}, 0, c_{i+1}-c_{i}, \cdots, c_{m_{1}}-c_{i}\right\}^{n}$ and $\left\|\mathbf{x}^{i}\right\|_{0} \leq d$ for $i=1, \cdots, m_{1}-1$. With $w \geq \frac{c_{m_{1}}-c_{i}-z_{m_{2}}}{c_{i+1}-c_{i}}$, applying Theorem 3.1 repeatedly, one can successfully identify $\mathbf{s}^{\mathbf{x}^{i}, \geq c_{i+1}-c_{i}}$ (or equivalently $\mathbf{s}^{\mathbf{x}, \geq c_{i+1}}$ ) for $i=1, \cdots, m_{1}-1$. As a result, one can identify $\mathbf{s}^{\mathbf{x},=c_{i}}=\mathbf{s}^{\mathbf{x}, \geq c_{i}}-\mathbf{s}^{\mathbf{x}, \geq c_{i+1}}$ for all $i=1, \cdots, m_{1}$ (for well-defineness let $c_{m_{1}+1}=\infty$ ).

Next, let $\mathbf{x}^{\prime}=\left(\sum_{i=1}^{m_{1}} c_{i} \mathbf{s}^{\mathbf{x},=c_{i}}-\mathbf{x}\right) \in\left\{0,-z_{1}, \cdots,-z_{m_{2}}\right\}^{n}$; hence $\left\|\mathbf{x}^{\prime}\right\|_{0} \leq d$. As $w \geq \max _{1 \leq i \leq m_{2}}\left\{\frac{-z_{m_{2}}+z_{i-1}}{-z_{i}+z_{i-1}}\right\}$, by Corollary 2.4, $\mathrm{x}^{\prime}$ can be identified. This completes the proof.

Note that the bound on $w$ in Theorem 3.2 is not necessary the best choice. One might obtain a better bound by first applying the transformation method introduced in Section 2. The following demonstrates such an example.

Example 1 Let $\mathbf{x}^{1} \in\{-2,0,1,4\}^{n}$ with $\left\|\mathbf{x}^{1}\right\|_{0} \leq d$ and let $\mathbf{x}^{2}=-\mathbf{x}^{1} \in\{-4,-1,0,2\}^{n}$. By Lemma 2.1, we know that the problem of learning $\mathbf{x}^{1}$ is equivalent to the problem of learning $\mathbf{x}^{2}$. However, to apply Theorem 3.2 to $\mathbf{x}^{2}$, we have to require $w^{2} \geq$ $\max \left\{\frac{2-(-4)}{2-0}, \frac{4-0}{1-0}, \frac{4-1}{4-1}\right\}=4$ which is smaller than $6=\max \left\{\frac{4-(-2)}{1-0}, \frac{4-1-(-2)}{4-1}, \frac{2-0}{2-0}\right\} \leq w^{1}$, required by simply applying Theorem 3.2 to $\mathbf{x}^{1}$.

## 4 Constructions for multi-group testing

This section proposes three constructions for the multi-group testing problem. The first one is to construct a $q$-ary additive $(w, d)$-disjunct matrix from a conventional binary disjunct matrix by deleting some rows.

Theorem 4.1 Let $A$ be a q-ary additive ( $w, d$ )-disjunct matrix of size $t \times n$ and $w \leq \frac{q-1}{d}$. If $A$ has $d+2$ rows $R_{1}, \cdots, R_{d+2}$ pairwise disjoint with entries in $\{0,1\}$, then $A^{\prime}$ obtained from $A$ by deleting $R_{d+2}$ and replacing $R_{k}$ with $R_{k}^{\prime}=(w d+1) R_{k}+R_{d+2}$ for $k=1, \cdots, d+1$ is a q-ary additive $(w, d)$-disjunct matrix of size $(t-1) \times n$.

Proof. For convenience, represent $A$ using row indices $[t]$ and column indices $[n]$ and without loss of generality assume $R_{1}, \cdots, R_{d+2}$ be the first $d+2$ rows, i.e.,
indexed from 1 to $d+2$. Let $A^{\prime}$ be the obtained matrix (using the same indices with $A$ ). Obviously, every entry in $A^{\prime}$ is at most $q$. Consider fixed $j \in[n]$ and $j_{1}, \cdots, j_{d} \in[n] \backslash\{j\}$. Since $A$ is $q$-ary additive $(w, d)$-disjunct, there exists an $i \in[t]$ such that $a_{i j} \geq w \sum_{k=1}^{d} a_{i j_{k}}$. There are only three cases as follows.
If $i \notin\{1, \cdots, d+2\}$, then row $i$ is in $A^{\prime}$ and $a_{i j}^{\prime}=a_{i j} \geq w \sum_{k=1}^{d} a_{i j_{k}}=w \sum_{k=1}^{d} a_{i j_{k}}^{\prime}$, as desired.

If $i \in\{1, \cdots, d+1\}$, then row $i$ is in $A^{\prime}, a_{i j}=1$ and $a_{i j_{k}}=0$ for $k=1 \cdots, d$. Therefore, $a_{i j}^{\prime}=(w d+1)>w \sum_{k=1}^{d} 1 \geq w \sum_{k=1}^{d} a_{i j_{k}}^{\prime}$ where the last inequality holds for $a_{i j_{k}}^{\prime} \leq 1$.
If $i=d+2$, then row $i$ is not in $A^{\prime}$. We need to find another row $i^{\prime}$ in $A^{\prime}$ with the desired property. In this case, $a_{d+2, j}=1$ and $a_{d+2, j_{k}}=0$ for $k=1 \cdots, d$. Since $R_{1}, \cdots, R_{d+1}$ are pairwise disjoint, at least one of them, say $i^{\prime}$, has all 0 entries at the columns $j_{k}$ 's for $k=1, \cdots, d$, i.e., $a_{i^{\prime} j_{k}}=0$ for $k=1, \cdots, d$. Since $a_{d+2, j}=1$ and therefore $a_{i^{\prime} j}=0$, we have $a_{i^{\prime} j}^{\prime}=1$. Thus, $a_{i^{\prime} j}^{\prime}>w \sum_{k=1}^{d} a_{i^{\prime} j_{k}}^{\prime}=0$.

Since $j, j_{1}, \cdots, j_{d}$ are chosen arbitrarily, the proof is complete.
A binary matrix $A$ is called transversal if its rows can be divided into disjoint families such that rows in each family are disjoint. We say a family of size $b$ if it has $b$ rows. Denote $f_{b}(A)$ as the number of disjoint families of size at least $b$ in the matrix A.

## Example 2 Let

$$
A=\left(\begin{array}{llllllllllllllll}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\hline 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)_{12 \times 16} .
$$

It is easily verified that $A$ is 2-disjunct and transversal as it can be divided into 3 disjoint families. Further, $f_{4}(A)=3$.

Corollary 4.2 Let $A$ be a transversal d-disjunct matrix of size $t \times n$. There exists $a$ $q$-ary additive $(w, d)$-disjunct matrix of size $t^{\prime} \times n$ with $q \geq w d+2$ and $t^{\prime}=t-f_{d+2}(A)$.

Proof. The Corollary follows immediately from Theorem 4.1.

In [21], Du et al proved that there exists a transversal $d$-disjunct matrix $A$ of size $t \times n$ with $t=(2+o(1))\left[\frac{d \log n}{\log (d \log n)}\right]^{2}$ and $f_{d+2}(A)=\frac{d \log n}{\log (d \log n)}$. As a result, we have the following.

Corollary 4.3 Let $q \geq w d+2$. There exists a $q$-ary additive $(w, d)$-disjunct matrix of size $t \times n$ with $t=(2+o(1))\left[\frac{d \log n}{\log (d \log n)}\right]^{2}-\frac{d \log n}{\log (d \log n)}$.

Example 3 Let

$$
A^{\prime}=\left(\begin{array}{llllllllllllllll}
5 & 5 & 5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 5 & 5 & 5 & 5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 5 & 5 & 5 & 1 & 1 & 1 & 1 \\
5 & 0 & 0 & 1 & 5 & 0 & 0 & 1 & 5 & 0 & 0 & 1 & 5 & 0 & 0 & 1 \\
0 & 5 & 0 & 1 & 0 & 5 & 0 & 1 & 0 & 5 & 0 & 1 & 0 & 5 & 0 & 1 \\
0 & 0 & 5 & 1 & 0 & 0 & 5 & 1 & 0 & 0 & 5 & 1 & 0 & 0 & 5 & 1 \\
5 & 0 & 0 & 1 & 1 & 5 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 1 & 5 \\
0 & 5 & 0 & 1 & 1 & 0 & 5 & 0 & 0 & 1 & 0 & 5 & 5 & 0 & 1 & 0 \\
0 & 0 & 5 & 1 & 1 & 0 & 0 & 5 & 5 & 1 & 0 & 0 & 0 & 5 & 1 & 0
\end{array}\right)_{9 \times 16}
$$

obtained from $A$ by using operations as in Theorem 4.1. Then $A^{\prime}$ is a 6-ary (2,2)disjunct matrix of size $9 \times 16$.

The following two constructions are based on a special operation of matrices, referred to Kronecker Product. Of particular note is that the resulting matrices are $q$-ary additive $(D, d)$-separable where $D$ is one-sided. Although the resulting matrices do not satisfy the additive disjunctness property, they also admit efficient decoding algorithms.

Definition 3 If $A$ and $B$ are matrices of size $n \times m$ and $s \times t$ respectively, then the Kronecker Product $A \otimes B$ of the two matrices $A$ and $B$ is the $n s \times m t$ matrix whose entries $(A \otimes B)_{i j, k \ell}=A_{i k} B_{j \ell}$ with row indices listed as $11, \cdots, 1 s, 21, \cdots, n 1, \cdots, n s$ and column indices as $11, \cdots, 1 t, 21, \cdots, m 1, \cdots, m t$.

The Kronecker Product of two matrices $A$ and $B$ can also be viewed as

$$
A \otimes B=\left(\begin{array}{cccc}
a_{11} B & a_{12} B & \cdots & a_{1 m} B \\
a_{21} B & a_{22} B & \cdots & a_{2 m} B \\
\vdots & \vdots & \ddots & \vdots \\
a_{n 1} B & a_{n 2} B & \cdots & a_{n m} B
\end{array}\right)
$$

Theorem 4.4 Let $A$ be a matrix of size $t \times n$ that can successfully identify any unknown $d$-sparse vector $\mathbf{y} \in\left\{0, c_{1}, c_{2}, \cdots, c_{m}\right\}^{n}$, where $0<c_{1}<\cdots<c_{m}$. Let $B$ be a binary d-disjunct matrix of size $t^{\prime} \times n^{\prime}$. Then $B \otimes A$ can successfully identify any unknown $d$-sparse vector $\mathbf{x} \in\left\{0, c_{1}, c_{2}, \cdots, c_{m}\right\}^{n n^{\prime}}$.

Proof. Partition the unknown vector $\mathbf{x}$ equally into $n^{\prime}$ pieces $\mathbf{x}_{1}, \mathbf{x}_{2}, \cdots, \mathbf{x}_{n^{\prime}}$ where $\mathbf{x}_{j} \in\left\{0, c_{1}, c_{2}, \cdots, c_{m}\right\}^{n}$ for all $j=1, \cdots, n^{\prime}$. Then

$$
(B \otimes A) \mathbf{x}=\left(\begin{array}{cccc}
b_{11} A & b_{12} A & \cdots & b_{1 n^{\prime}} A \\
b_{21} A & b_{22} A & \cdots & b_{2 n^{\prime}} A \\
\vdots & \vdots & \ddots & \vdots \\
b_{t^{\prime} 1} A & b_{t^{\prime} 2} A & \cdots & b_{t^{\prime} n^{\prime}} A
\end{array}\right)\left(\begin{array}{c}
\mathbf{x}_{1} \\
\mathbf{x}_{2} \\
\vdots \\
\mathbf{x}_{n^{\prime}}
\end{array}\right)=\left(\begin{array}{c}
\mathbf{v}_{1} \\
\mathbf{v}_{2} \\
\vdots \\
\mathbf{v}_{t^{\prime}}
\end{array}\right)
$$

where the outcome vector $\mathbf{v}=\left(\begin{array}{c}\mathbf{v}_{1} \\ \mathbf{v}_{2} \\ \vdots \\ \mathbf{v}_{t^{\prime}}\end{array}\right)$ with each $\mathbf{v}_{i}$ a vector of length $t$. We now want to show that the unknown vector $\mathbf{x}$ can be identified from the outcome vector $\mathbf{v}$.

We first show that one can determine whether $\mathbf{x}_{j}=\mathbf{0}$, where $\mathbf{0}$ denotes a vector whose entries are all zero, for all $j$ through the $d$-disjunctness property. Obviously, there are at most $d \mathbf{x}_{j}$ 's such that $\mathbf{x}_{j} \neq \mathbf{0}$ as $\|\mathbf{x}\|_{0} \leq d$. Let $J$ be such a set consisting of indices of these $\mathbf{x}_{j}$ 's with $\mathbf{x}_{j} \neq \mathbf{0}$. For a fixed $k$ with $\mathbf{x}_{k}=\mathbf{0}$, by the definition of $d$-disjunctness, there exists an $\ell$ such that $b_{\ell k}=1$ and $b_{\ell j}=0$ for all $j \in J$. It follows that $\mathbf{v}_{\ell}=\left(\begin{array}{llll}b_{\ell 1} A & b_{\ell 2} A & \cdots & b_{\ell n^{\prime}} A\end{array}\right)\left(\begin{array}{c}\mathbf{x}_{1} \\ \mathbf{x}_{2} \\ \vdots \\ \mathbf{x}_{n^{\prime}}\end{array}\right)=\left(\begin{array}{c}0 \\ 0 \\ \vdots \\ 0\end{array}\right)$. In contrast, for each $j \in J$, for each $i$ with $b_{i j}=1$ we have $\mathbf{v}_{i}=\left(\begin{array}{llll}b_{i 1} A & b_{i 2} A & \cdots & b_{i n^{\prime}} A\end{array}\right)\left(\begin{array}{c}\mathbf{x}_{1} \\ \mathbf{x}_{2} \\ \vdots \\ \mathbf{x}_{n^{\prime}}\end{array}\right) \neq\left(\begin{array}{c}0 \\ 0 \\ \vdots \\ 0\end{array}\right)$ since $b_{i j} A \mathbf{x}_{j} \neq \mathbf{0}$ and entries in $b_{i j^{\prime}} A \mathbf{x}_{j^{\prime}}$ are all nonnegative. As a result, one can determine whether $\mathbf{x}_{j}$ is $\mathbf{0}$ or not by checking $\mathbf{v}_{i}=\mathbf{0}$ for some $i$ with $b_{i j}=1$.

For each $j$ with $\mathbf{x}_{j} \neq \mathbf{0}$, by the definition of $d$-disjunctness, there exists an $i$ such that $b_{i j}=1$ and $b_{i j^{\prime}}=0$ for all $j^{\prime} \in J \backslash\{j\}$. It follows that

$$
\mathbf{v}_{i}=\left(\begin{array}{llll}
b_{i 1} A & b_{i 2} A & \cdots & b_{i n^{\prime}} A
\end{array}\right)\left(\begin{array}{c}
\mathbf{x}_{1} \\
\mathbf{x}_{2} \\
\vdots \\
\mathbf{x}_{n^{\prime}}
\end{array}\right)=A \mathbf{x}_{j}
$$

Consequently, $\mathbf{x}_{j}$ can be identified from the outcome $\mathbf{v}_{i}$ since $A$ can successfully identify any unknown vector $\mathbf{y} \in\left\{0, c_{1}, c_{2}, \cdots, c_{m}\right\}^{n}$ with $\|\mathbf{y}\|_{0} \leq d$, where $0<c_{1}<$ $\cdots<c_{m}$. By the above discussion, one can successfully identify $\mathbf{x}$ by using the matrix $B \otimes A$.

Note that the decoding complexity of Theorem 4.4 depends on the decoding complexity of the underlying matrix $A$.

Corollary 4.5 Let $\mathbf{x} \in\left\{0, c_{1}, c_{2}, \cdots, c_{m}\right\}^{n n^{\prime}}$ be a d-sparse vector, where $0<c_{1}<$ $\cdots<c_{m}$. Let $A$ be a q-ary additive $(w, d)$-matrix of size $t \times n$ with $w=\max _{1 \leq k \leq m}\left\{\frac{c_{m}-c_{k-1}}{c_{k}-c_{k-1}}\right\}$ as in Corollary 2.4 and let $B$ be a binary d-disjunct matrix of size $t^{\prime} \times n^{\prime}$. Then the $q$-ary matrix $B \otimes A$ can successfully identify the unknown vector $\mathbf{x}$. Furthermore, the decoding complexity is $O\left(t t^{\prime} n n^{\prime}+d m t n\right)$.

Proof. The identification result follows immediately from Corollary 2.4 and Theorem 4.4. The decoding complexity follows by taking $O\left(t t^{\prime} n n^{\prime}\right)$ time in determining $\mathbf{x}_{j}=\mathbf{0}$ or not and then applying the decoding algorithm in Corollary 2.4 to those $\mathbf{x}_{j}$ 's with $\mathbf{x}_{j} \neq \mathbf{0}$ at most $d$ times.

Following the idea in [26] of concatenating several matrices, we obtain the following result: Let $A$ be a binary $d$-disjunct matrix of size $t \times n$. Let $u=c_{m} d$ and $n^{\prime}=\left\lfloor\log _{u}(1+(u-1)(q-1))\right\rfloor$. Construct a $q$-ary matrix $C$ of size $t \times n n^{\prime}$ by concatenating $n^{\prime}$ matrices: $C=\left(\begin{array}{llll}A_{1} & A_{2} & \cdots & A_{n^{\prime}}\end{array}\right)$, where $A_{j}=\left(\sum_{i=0}^{j-1} u^{i}\right) A$ for $1 \leq j \leq n^{\prime}$; equivalently $C=B \otimes A$ where

$$
B=\left(\begin{array}{lllll}
1 & 1+u & 1+u+u^{2} & \cdots & \frac{u^{n^{\prime}}-1}{u-1}
\end{array}\right) .
$$

Theorem 4.6 Let $C$ be the q-ary matrix as defined above. Then $C$ can successfully identify any unknown $d$-sparse vector $\mathbf{x} \in\left\{0, c_{1}, \cdots, c_{m}\right\}^{n n^{\prime}}$ with $0<c_{1}<\cdots<c_{m}$ where all $c_{i}$ 's are positive integers. Moreover, the decoding complexity is $O\left(t n n^{\prime}\right)$.

Proof. Let $\mathbf{v}=C \mathbf{x}$ be the outcome vector. Consider $\mathbf{x}=\left(\begin{array}{llll}\mathbf{x}_{1} & \mathbf{x}_{2} & \cdots & \mathbf{x}_{n^{\prime}}\end{array}\right)$ where $\mathbf{x}_{j}$ 's are vectors of length $n$. We prove this theorem by showing $\mathbf{x}_{n^{\prime}}, \cdots, \mathbf{x}_{1}$ can be identified successfully one by one.

Observe that

$$
\begin{equation*}
\mathbf{v}=\sum_{j=1}^{n^{\prime}} A_{j} \mathbf{x}_{j}=\left(\sum_{j=1}^{n^{\prime}-1} \frac{u^{j}-1}{u-1} A \mathbf{x}_{j}\right)+\frac{u^{n^{\prime}}-1}{u-1} A \mathbf{x}_{n^{\prime}} . \tag{2}
\end{equation*}
$$

Since $\sum_{j=1}^{n^{\prime}-1}\left\|\mathbf{x}_{j}\right\|_{0} \leq d$ and $c_{m}$ is the largest value, each entry of the term $\left(\sum_{j=1}^{n^{\prime}-1} \frac{u^{j}-1}{u-1} A \mathbf{x}_{j}\right)$ in (2) is at most $\left(\frac{u^{n^{\prime}-1}-1}{u-1}\right) c_{m} d<\frac{u^{n^{\prime}}-1}{u-1}$. It follows that, taking the floor of numbers in the vector $\frac{u-1}{u^{n^{\prime}}-1} \mathbf{v}$ componentwisely, we have $\left\lfloor\frac{u-1}{u^{n^{\prime}}-1} \mathbf{v}\right\rfloor=A \mathbf{x}_{n^{\prime}}$. Note that the above equality holds when $c_{i}$ 's are positive integers, as required. This equality implies that $\mathbf{x}_{n^{\prime}}$ can be identified from the outcome $\left\lfloor\frac{u-1}{u^{n^{\prime}}-1} \mathbf{v}\right\rfloor$ as $A$ is a $d$-disjunct matrix, which guarantees the identification of any $d$-sparse vector with entries all nonnegative in $O(t n)$ decoding time.

Assume now that $\mathbf{x}_{n^{\prime}}, \mathbf{x}_{n^{\prime}-1}, \cdots, \mathbf{x}_{k+1}$ have been identified. Define $\mathbf{v}_{k}=\mathbf{v}-$ $\sum_{j=k+1}^{n^{\prime}} A_{j} \mathbf{x}_{j}$. An analogous argument shows that $\left\lfloor\frac{u-1}{u^{k}-1} \mathbf{v}_{k}\right\rfloor=A \mathbf{x}_{k}$. By the $d$-disjunctness property again, one can identify $\mathbf{x}_{k}$ from the outcome $\left\lfloor\frac{u-1}{u^{k}-1} \mathbf{v}_{k}\right\rfloor$. Repeating this process, $\mathbf{x}_{n^{\prime}}, \cdots, \mathbf{x}_{1}$ can be identified successfully one by one, and therefore the total decoding complexity is $O\left(t n n^{\prime}\right)$. This completes the proof.

## Concluding remarks

In this section, three constructions are proposed for three purposes: the general case, the one-sided case, and the one-sided integer case. So, it might make little sense to compare their performances in absolute terms. To conclude this section, we simply list the rate of their performances in comparison with the difficulty of their goals, where the rate of a $t \times n$ matrix $M$ is defined by $R(M)=\lim _{n \rightarrow \infty} n / t$. Table 1 lists the corresponding rates of the three constructions. As shown results for the first two constructions, we can conclude that the more restrictions on $D$, the better rates.

Table 1: Rates of three constructions: (1)general case
(2)one-sided case (3)one-sided integer case

$$
\begin{array}{cl}
\frac{(\log (d \log n))^{2} n}{(d \log n)^{2}} & \text { Cor. 4.3 } \\
\frac{(\log (d \log n))^{2} n}{(d \log n)^{2}} \times \frac{n}{d^{2} \log n} & \text { Cor. 4.5 } \\
\frac{n}{d^{2} \log n} \times\left(\log q-\log c_{m} d\right) & \text { Thm. 4.6 } \tag{2}
\end{array}
$$

Table 2 lists the corresponding decoding complexities for the underlying matrices of the same size $t \times n$.

Table 2: Decoding complexities of three constructions

|  | Corollary [4.3 | Corollary 4.5 | Theorem[4.6] |
| :--- | :---: | :---: | :---: |
| general <br> one-sided <br> one-sided integer | $O\left(\|D\| t n^{d+1}\right)$ | $O(\|D\| t n)$ | $O(\|D\| t n)$ |
|  |  |  | $O(t n)$ |

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