# Subspace Recovery from Structured Union of Subspaces

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Abstract-Lower dimensional signal representation schemes frequently assume that the signal of interest lies in a single vector space. In the context of the recently developed theory of compressive sensing (CS), it is often assumed that the signal of interest is sparse in an orthonormal basis. However, in many practical applications, this requirement may be too restrictive. A generalization of the standard sparsity assumption is that the signal lies in a union of subspaces. Recovery of such signals from a small number of samples has been studied recently in several works. Here, we consider the problem of subspace recovery in which our goal is to identify the subspace (from the union) in which the signal lies using a small number of samples, in the presence of noise. More specifically, we derive performance bounds and conditions under which reliable subspace recovery is guaranteed using maximum likelihood (ML) estimation. We begin by treating general unions and then obtain the results for the special case in which the subspaces have structure leading to block sparsity. In our analysis, we treat both general sampling operators and random sampling matrices. With general unions, we show that under certain conditions, the number of measurements required for reliable subspace recovery in the presence of noise via ML is less than that implied using the restricted isometry property which guarantees signal recovery. In the special case of block sparse signals, we quantify the gain achievable over standard sparsity in subspace recovery. Our results also strengthen existing results on sparse support recovery in the presence of noise under the standard sparsity model.

*Index terms*- Maximum likelihood estimation, union of linear subspaces, subspace recovery, compressive sensing, block sparsity

## I. INTRODUCTION

The compressive sensing (CS) framework has established that a small number of measurements acquired via random projections are sufficient for signal recovery when the signal of interest is sparse in a certain basis. Consider a length-N signal  $\mathbf{x}$  which can be represented in a basis  $\mathbf{V}$  such that  $\mathbf{x} = \mathbf{Vc}$ . The signal  $\mathbf{x}$  is said to be k-sparse in the basis  $\mathbf{V}$  if  $\mathbf{c}$  has only k nonzero coefficients where k is much smaller than N. It has been shown in [1]–[3] that  $\mathcal{O}(k \log(N/k))$  compressive measurements are sufficient to recover  $\mathbf{x}$  when the elements of the measurement matrix are random. Signal recovery can

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be performed via optimization or greedy based approaches. A detailed overview of CS can be found in [4].

There are a variety of applications in which complete signal recovery is not necessary. The problem of sparse support recovery (equivalently sparsity pattern recovery or finding the locations of nonzero coefficients of a sparse signal) arises in a wide variety of areas including source localization [5], [6], sparse approximation [7], subset selection in linear regression [8], [9], estimation of frequency band locations in cognitive radio networks [10]–[12], and signal denoising [13]. In these applications, often finding the sparsity pattern of the signal is more important than approximating the signal itself. Further, in the CS framework, once the sparse support is identified, the signal can be estimated using standard techniques. For the problem of complete sparse signal recovery, there is a significant amount of work in the literature that focuses on deriving recovery guarantees and stability with respect to various  $l_a$ norms of the reconstruction error. However, as pointed out in [14], recovery guarantees derived for sparse signals do not always imply exact recovery of the sparse support. The criteria used for sparse support recovery and exact signal recovery are generally different. Although a signal estimate can be close to the original sparse signal, the estimated signal may have a different support compared to the true signal support [14]. For example, Lasso has been shown to be information theoretically optimal in certain regimes of the signal-to-noise ratio (SNR) for sparse support recovery, while in other regimes of SNR, the Lasso fails with high probability in recovering the sparsity pattern [14], [15]. Thus, investigation of recovery conditions for sparse support at any given SNR is an important problem. Performance limits on reliable recovery of the sparsity pattern have been derived by several authors in recent work exploiting information theoretic tools [14], [16]-[22]. Most of these works focus on deriving necessary and sufficient conditions for reliable sparsity pattern recovery assuming the standard sparsity model.

There are practical scenarios where structured properties of the signal are available. Reduced dimensional signal processing for several signal models which go beyond simple sparsity has been treated in recent literature [23]–[28]. One general model that can describe many structured problems is that of a union of subspaces. In this setting, the signal is known to lie in one out of a possible set of subspaces but the specific subspace chosen is unknown. Examples include wideband spectrum sensing [11], time delay estimation with overlapping echoes [24], [29], [30], and signals having finite rate innovation [31], [32]. Conditions under which stable sampling and recovery is possible in a general union of subspaces model are derived in

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[23]–[26]. However, the problem of recovering the subspace in which the signal lies without completely recovering the signal (or the problem of subspace recovery) has not been treated in this more general setting.

In this paper, our goal is to investigate the problem of subspace recovery in the union of subspaces model with a given sampling operator. We consider subspace recovery based on the optimal ML decoding scheme in the presence of noise. While ML is computationally intractable as the signal dimension increases, the analysis gives a benchmark for the optimal performance that is achievable with any practical algorithm. We derive performance in terms of probability of error of the ML decoder when sampling is performed via an arbitrary linear sampling operator. Based on an upper bound on the probability of error, we derive the minimum number of samples required for asymptotically reliable recovery of subspaces in terms of a SNR measure, the dimension of each subspace in the union and a term which quantifies the dependence or overlap among the subspaces. In the special case where sampling is performed via random projections and the subspaces in the union have a specific structure such that each subspace is a sum of some other  $k_0$  subspaces, we obtain a more explicit expression for the minimum number of measurements. This number depends on the number of underlying subspaces, the dimension of each subspace, and the minimum nonzero block SNR (defined in Section IV.B). We note that the conventional sparsity model is a special case of this structure.

The asymptotic probability of error of the ML decoder for sparse support recovery in the presence of noise for the standard sparsity model was first investigated in [14] followed by several other authors [16], [17], [22]. In [14], sufficient conditions were derived on the number of noisy compressive measurements needed to achieve a vanishing probability of error asymptotically for sparsity pattern recovery while necessary conditions were considered in [17]. The analyses in both [14] and [17] are based on the assumption that the sampling operator is random. Here, we follow a similar path assuming the union of subspaces model. However, there are some key differences between our derivations and that in [14]. First, we treat arbitrary (not necessarily random) sampling operators and assume a general union of subspaces model as opposed to the standard sparsity model. Further, the results in [14] were derived based on weak bounds on the probability of error, thus there is a gap between those results and the number of measurements required for the exact probability of error to vanish asymptotically at finite SNR. We consider tighter bounds on the probability error leading to tighter results.

The rest of the paper is organized as follows. In Section II, the problem of subspace recovery from a union of subspace model is introduced. In Section III, performance limits with ML decoder for subspace recovery in terms of the probability of error are derived with a given linear sampling operator considering a general union of subspaces model. Conditions under which asymptotically reliable subspace recovery in the presence of noise is guaranteed are obtained based on the derived upper bound. The results are extended in Section IV to the setting where structured properties of the subspaces in the union are available. We also derive sufficient conditions for subspace recovery when sampling is performed via random projections. In Section V, we compare our results with some existing results in the literature. Practical algorithms to recover subspaces in the union of subspace model and numerical results to validate the theoretical claims are presented in Section VI.

Throughout the paper, we use the following notation. Arbitrary vectors in a Hilbert space  $\mathcal{H}$ , are denoted by lower case letters, e.g., x. Calligraphic letters, e.g., S, are used to represent subspaces in  $\mathcal{H}$ . Vectors in  $\mathbb{R}^N$  are written in boldface lower case letters, e.g. x. Scalars (in  $\mathbb{R}$ ) are also denoted by lower case letters, e.g., x, when there is no confusion. Matrices are written in boldface upper case letters, e.g., A. Linear operators and a set of basis vectors for a given subspace S are denoted by upper case letters, e.g., A. The notation  $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  means that the random vector  $\mathbf{x}$ is distributed as multivariate Gaussian with mean  $\mu$  and the covariance matrix  $\Sigma$ ;  $x \sim \mathcal{X}_m^2(\lambda)$  denotes that the random variable x is distributed as Chi squared with m degrees of freedom and non centrality parameter  $\lambda$ . (The central Chi squared distribution is denoted by  $\mathcal{X}_m^2$ ). By 0, we denote a vector with appropriate dimension in which all elements are zeros, and  $I_k$  is the identity matrix of size k. The conjugate transpose of a matrix A is denoted by  $A^*$ . Finally,  $||.||_2$ denotes the  $l_2$  norm and |.| is used for both the cardinality (of a set) and the absolute value (of a scalar). Special functions used in the paper are: Gaussian Q-function:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{t^{2}}{2}} dt$$
 (1)

Gamma function:

T

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{2}$$

and modified Bessel function with real arguments:

$$K_{\nu}(x) = \int_{0}^{\infty} e^{-x \cosh t} \cosh(\nu t) dt.$$
(3)

## II. PROBLEM FORMULATION

#### II.A Union of subspaces

As discussed in [23]–[26], there are many practical scenarios where the signals of interest lie in a union of subspaces.

**Definition 1.** Union of subspaces: A signal  $x \in \mathcal{H}$  lies in a union of subspaces if  $x \in \mathcal{X}$  where  $\mathcal{X}$  is defined as

$$\mathcal{X} = \bigcup_{i} S_i \tag{4}$$

and  $S_i$ 's are subspaces of  $\mathcal{H}$  which are assumed to be finite dimensional. A signal  $x \in \mathcal{X}$  if and only if there exists  $i_0$  such that  $x \in S_{i_0}$ .

Let  $V_i = \{v_{im}\}_{m=0}^{k-1}$  be a basis for the finite dimensional subspace  $S_i$  where k is the dimension of  $S_i$  (it is noted that while we assume all subspaces to have the same dimension, the analysis can be easily extended for the case where different subspaces have different dimensions). Then each  $x \in S_i$  can be expressed in terms of a basis expansion

$$x = \sum_{m=0}^{k-1} c_i(m) v_{im}$$

where  $c_i(m)$ 's for  $m = 0, 1, \dots, k-1$  are the coefficients corresponding to the basis  $V_i$ . We assume that the subspaces are distinct (i.e. there are no subspaces such that  $S_i \subseteq S_j$ for  $i \neq j$  in the union (4)) and each subspace  $S_i$  is uniquely determined by the basis  $V_i$ . We denote by  $T < \infty$  the number of subspaces in the union  $\mathcal{X}$ .

#### II.B Structured union of subspaces leading to block sparsity

There are certain scenarios in which the signals can be assumed to lie in more structured union of subspaces as considered in [25], [28], [33]. Suppose that each subspace in the union (4) can be represented as a sum of  $k_0$  (out of L) disjoint subspaces [25], [33]. More specifically,

$$\mathcal{S}_i = \bigoplus_{j \in \Sigma_{k_0}} \mathcal{V}_j \tag{5}$$

where  $\{\mathcal{V}_j\}_{j=0}^{L-1}$ 's are disjoint subspaces, and  $\Sigma_{k_0}$  contains  $k_0$ indices from  $\{0, 1, \dots, L-1\}$ . Let  $d_j = \dim(\mathcal{V}_j)$  and  $N = \sum_{j=0}^{L-1} d_j$ . Then there are  $T = \binom{L}{k_0}$  subspaces in the union. Under this formulation, the dimension of each subspace in (4) is  $k = \sum_{\substack{j \in \Sigma_{k_0} \\ k = k_0 d}} d_j$ . In the special case where  $d_j = d$  for all j,  $k = k_0 d$ .

Now taking  $V_j$  as a basis for  $\mathcal{V}_j$ , a signal in the union can be written as

$$x = \sum_{j \in \Sigma_{k_0}} V_j \mathbf{c}_j \tag{6}$$

where  $\mathbf{c}_j = [c_j(0), \cdots, c_j(d_j - 1)]^T \in \mathbb{R}^{d_j}$  is a  $d_j \times 1$  coefficient vector corresponding to the basis  $V_j$ . It is worth mentioning that we use the same notation  $V_j$  to denote a basis of the subspace  $S_j$  in (4) for  $j = 0, 1, \cdots, T - 1$  (when discussing the general union of subspaces model) and also to denote a basis of the subspace  $\mathcal{V}_j$  in (5) for  $j = 0, 1, \cdots, L-1$  (when discussing the structured union of subspace model). Let V be a matrix constructed by concatenating  $V_i$ 's column wise, such that  $V = [V_0|V_1|\cdots|V_{L-1}]$  and  $\mathbf{c}$  be a  $N \times 1$  vector with  $\mathbf{c} = [\mathbf{c}_0^T|\cdots|\mathbf{c}_{L-1}^T]^T$ . As defined in [25], the vector  $\mathbf{c} \in \mathbb{R}^N$  is called block k-sparse over  $\mathcal{I} = \{d_0, d_1, \cdots, d_{L-1}\}$  if all the elements in  $\mathbf{c}_i$  are zeros for all but  $k_0$  indices where  $N = \sum_{j=0}^{L-1} d_j$ . In this paper, we assume  $d_j = d$  for all j so that N = Ld.

The standard sparsity model used in the CS literature is a special case of this structured union of subspaces model when d = 1. In the standard CS framework,  $x = \mathbf{x}$  is a length-N signal vector which is k-sparse in an N-dimensional orthonormal basis V so that  $\mathbf{x}$  can be represented as  $\mathbf{x} = \mathbf{V}\mathbf{c}$ with c having only  $k \ll N$  significant coefficients. This fits our framework when d = 1 and  $V_j$  is chosen as the j-th column vector of the orthonormal basis V for  $j = 0, 1, \dots, N-1$ . In this case, we have L = N and there are  $T = {N \choose k}$  subspaces in the union.

#### II.C Observation model: Linear sampling

Consider a sampling operator via a bounded linear mapping of a signal x that lies in an ambient Hilbert space  $\mathcal{H}$ . Let the linear sampling operator A be specified by a set of unique sampling vectors  $\{a_m\}_{m=0}^{M-1}$ . With these notations, noisy samples are given by,

$$\mathbf{y} = A\mathbf{x} + \mathbf{w} \tag{7}$$

where **y** is the  $M \times 1$  measurement vector, and the *m*-th element of the vector Ax is given by,  $(Ax)_m = \langle x, a_m \rangle$  for  $m = 0, 1, \dots, M - 1$  where  $\langle ., . \rangle$  denotes the inner product. The noise vector **w** is assumed to be Gaussian with mean **0** and covariance matrix  $\sigma_w^2 \mathbf{I}_M$ .

When  $x \in S_i$  for some *i* in the model (4), the vector of samples can be equivalently represented in the form of a matrix vector multiplication,

$$\mathbf{y} = \mathbf{B}_i \mathbf{c}_i + \mathbf{w} \tag{8}$$

where

$$\mathbf{B}_{i} = AV_{i} = \begin{pmatrix} \langle a_{0}, v_{i0} \rangle & \langle a_{0}, v_{i1} \rangle & \cdots & \langle a_{0}, v_{i(k-1)} \rangle \\ \langle a_{1}, v_{i0} \rangle & \langle a_{1}, v_{i1} \rangle & \cdots & \langle a_{1}, v_{i(k-1)} \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle a_{M-1}, v_{i0} \rangle & \langle a_{M-1}, v_{i1} \rangle & \cdots & \langle a_{M-1}, v_{i(k-1)} \rangle \end{pmatrix}$$

and  $\mathbf{c}_i = [c_i(0) \ c_i(1) \ \cdots \ c_i(k-1)]^T$  is the coefficient vector with respect to the basis  $V_i$ . Further, let  $\mathbf{b}_{im}$  denote the *m*th column vector of the matrix  $\mathbf{B}_i$  for  $m = 0, 1, \cdots, k-1$ and  $i = 0, 1, \cdots, T-1$ . We assume that the linear sampling operator A is a one-to-one mapping between  $\mathcal{X}$  and  $A\mathcal{X}$ . Since  $\{v_{i0}, \cdots, v_{i(k-1)}\}$  is a set of linearly independent basis vectors, then  $\{\mathbf{b}_{i0}, \cdots, \mathbf{b}_{i(k-1)}\}$  are also linearly independent for each  $i = 0, 1, \cdots, T-1$ . It is worth noting that, while this one-to-one condition ensures uniqueness, stronger conditions are required to recover x in a stable manner as discussed in [24].

## II.D Subspace recovery from the union of subspaces model

As discussed in the Introduction, there are applications where it is sufficient to recover the subspace in which the signal of interest lies from the union of subspaces model (4) instead of complete signal recovery. Moreover, if there is a procedure to correctly identify the subspace with vanishing probability of error, then the signal x can be reconstructed with a small  $l_2$  norm error using standard techniques. However, the other way would not be always true, i.e., if an algorithm developed for complete signal recovery is used for subspace recovery, it may not give an equivalent performance guarantee. This is because, even if such an estimate of the signal may be close to the true signal with respect to the considered performance metric (e.g., l2 norm error), the subspace in which the estimated signal lies may be different from the true subspace. This can happen especially when the SNR is not sufficiently large. Thus, investigating the problem of subspace recovery is important and is the main focus of this paper.

The problem of subspace recovery is to identify the subspace in which the signal x lies. The estimated subspace,  $\hat{S}$ ,

via any recovery scheme can be expressed in the following form:

$$\hat{\mathcal{S}} = \zeta(\mathbf{y}) \tag{9}$$

where  $\zeta(\cdot)$  is a mapping from the observation vector **y** to an estimated subspace  $\hat{S} \in \{S_0, \dots, S_{T-1}\}$ . The performance metric used to evaluate the quality of the estimate (9) is taken as the average probability of error defined as

$$P_e = \sum_{\mathcal{S}} Pr(\zeta(\mathbf{y}) \neq \mathcal{S}|\mathcal{S}) Pr(\mathcal{S})$$
(10)

for a given recovery scheme  $\zeta(\mathbf{y})$ . We say that the mapping  $\zeta(\mathbf{y})$  is capable of providing *asymptotically reliable* subspace recovery if  $P_e \to 0$  as  $M \to \infty$ . In this paper, we consider subspace recovery via the ML estimation. Our goal is to address the following issues.

- Performance of the ML estimation scheme in terms of the probability of error in recovering the subspaces from the union of subspaces model (4) in the presence of noise. We are also interested in conditions under which asymptotically reliable subspace recovery in the union is guaranteed with a given sampling operator.
- How much gain in terms of the number of samples required for subspace recovery can be achieved if further information on structures is available for the subspaces in (4) compared to the case when no additional structured information is available (i.e. compared to the standard sparsity model used in CS).
- Illustration of the performance gap between the ML estimation and computationally tractable algorithms for subspace recovery from the union of subspaces model at finite SNR.

The main results of the paper can be summarized as follows. With the general union of subspaces model as defined in (4), and for a given sampling operator, the minimum number of samples required for asymptotically reliable recovery of subspaces in the presence of noise is

$$M > k + \frac{\eta_3}{f(SNR)} \log(\bar{T}_0) \tag{11}$$

where k is the dimension of each subspace, f(SNR) is a measure of the minimum SNR of the sampled signal projected onto the null space of any subspace in the union,  $\overline{T}_0$  is a measure of the number of subspaces in the union with maximum dependence where  $\overline{T}_0 \leq T$  (formal definitions of all these terms are given in Section III), and  $\eta_3$  is a constant. We simplify (11) for the special case where each subspace in the union (4) can be expressed as a sum of  $k_0$  subspaces out of L where each such subspace is d-dimensional such that  $k = k_0 d$ . Then, the problem of subspace recovery reduces to the problem of block sparsity pattern recovery. Further, assuming that the sampling operator is represented by random projections, the number of samples required for asymptotically reliable block sparsity pattern recovery is given by

$$M > k + \frac{\eta_4}{BSNR_{\min}} \log(L - k_0) \tag{12}$$

where  $BSNR_{\min}$  is the minimum nonzero block SNR and  $\eta_4$  is a constant. When d = 1 and L = N where N is the signal

dimension, the block sparsity model reduces to the standard sparsity model. Then, our result shows that

$$M > k + \frac{\eta_2}{CSNR_{\min}} \log(N - k) \tag{13}$$

measurements are required for reliable sparsity pattern recovery where  $CSNR_{\min}(\leq \frac{BSNR_{\min}}{d})$  is the minimum component SNR of the signal. Thus, from (12) and (13), we observe that the number of measurements required for asymptotically reliable subspace recovery beyond the sparsity index (i.e., M - k) reduces approximately d times with a block sparsity model (so that  $k = k_0 d$ ) compared to the standard sparsity model. A detailed comparison between our results and existing results in the literature is given in Section V.

## III. SUBSPACE RECOVERY WITH GENERAL UNIONS

The problem of finding the true subspace from the union (4) based on the observation model (8) via the ML estimation becomes finding the index  $\hat{i}$  such that,

$$\hat{i} = \underset{i=0,\cdots,T-1}{\operatorname{arg\,max}} p(\mathbf{y}|\mathbf{B}_i).$$

When  $x \in S_i$  in (4) for some *i*, and using the observation model (8), we have  $p(\mathbf{y}|\mathbf{B}_i) = \mathcal{N}(\mathbf{B}_i\mathbf{c}_i, \sigma_w^2\mathbf{I}_M)$ . The signal *x* is assumed to be deterministic but unknown. Thus, when  $x \in S_i$ , the coefficient vector  $\mathbf{c}_i$  with respect to a given basis  $\mathbf{B}_i$  is unknown. Assuming that each  $\mathbf{B}_i$  has rank *k* for  $i = 0, \dots, T - 1$ , the ML estimate of  $\mathbf{c}_i$  such that  $p(\mathbf{y}|\mathbf{B}_i)$  is maximized can be found as,  $\hat{\mathbf{c}}_i = (\mathbf{B}_i^*\mathbf{B}_i)^{-1}\mathbf{B}_i^*\mathbf{y}$ . This results in

$$\log(\max_{\mathbf{c}_i} p(\mathbf{y}|\mathbf{B}_i)) = \log\left(\frac{1}{(2\pi\sigma_w^2)^{M/2}}\right) - \frac{1}{2\sigma_w^2} ||\mathbf{y} - \mathbf{P}_i \mathbf{y}||_2^2$$
$$= \log\left(\frac{1}{(2\pi\sigma_w^2)^{M/2}}\right) - \frac{1}{2\sigma_w^2} ||\mathbf{P}_i^{\perp} \mathbf{y}||_2^2$$

where  $\mathbf{P}_i = \mathbf{B}_i (\mathbf{B}_i^* \mathbf{B}_i)^{-1} \mathbf{B}_i^*$  is the orthogonal projector onto the span of  $\{\mathbf{b}_{im}\}_{m=0}^{k-1}$  and  $\mathbf{P}_i^{\perp} = \mathbf{I} - \mathbf{P}_i$ . Thus, the estimated index of the subspace by the ML estimation is,

$$\hat{i} = \underset{i=0,\cdots,T-1}{\operatorname{arg\,min}} ||\mathbf{P}_i^{\perp}\mathbf{y}||_2^2.$$
(14)

The probability of error of the ML estimation is given by,

$$= Pr(\mathbf{B}_{estimated} \neq \mathbf{B}_{true}) = \sum_{i} Pr(\hat{i} \neq i | \mathbf{B}_{i}) Pr(\mathbf{B}_{i})$$
  
$$\leq \sum_{i} \sum_{j \neq i} Pr(\hat{i} = i | \mathbf{B} = \mathbf{B}_{j}) Pr(\mathbf{B} = \mathbf{B}_{j})$$
(15)

where  $Pr(\hat{i} = i | \mathbf{B} = \mathbf{B}_j)$  is the probability of selecting  $S_i$ when the true subspace is  $S_j$ . Since the ML estimation decides the subspace  $S_i$  over  $S_j$  when  $||\mathbf{P}_i^{\perp}\mathbf{y}||_2^2 - ||\mathbf{P}_j^{\perp}\mathbf{y}||_2^2 < 0$ ,  $Pr(\hat{i} = i | \mathbf{B} = \mathbf{B}_j)$  is given by

$$Pr(\hat{i} = i|\mathbf{B} = \mathbf{B}_j) = Pr(||\mathbf{P}_i^{\perp}\mathbf{y}||_2^2 - ||\mathbf{P}_j^{\perp}\mathbf{y}||_2^2 < 0)$$

for  $i \neq j$ .

 $P_e$ 

Let  $\Delta_{ij}(\mathbf{y}) = ||\mathbf{P}_i^{\perp}\mathbf{y}||_2^2 - ||\mathbf{P}_j^{\perp}\mathbf{y}||_2^2$  for  $i \neq j$ . When the true subspace is  $\mathcal{S}_j$  so that  $Ax = \mathbf{B}_j\mathbf{c}_j$ , we have  $||\mathbf{P}_j^{\perp}\mathbf{y}||_2^2 = ||\mathbf{P}_j^{\perp}\mathbf{w}||_2^2$  and

$$\mathbf{P}_{i}^{\perp}\mathbf{y} = \mathbf{P}_{i}^{\perp}Ax + \mathbf{P}_{i}^{\perp}\mathbf{w}$$
$$= \mathbf{P}_{i}^{\perp}\mathbf{B}_{j}\mathbf{c}_{j} + \mathbf{P}_{i}^{\perp}\mathbf{w} = \mathbf{P}_{i}^{\perp}\mathbf{B}_{j\setminus i}\mathbf{c}_{j\setminus i} + \mathbf{P}_{i}^{\perp}\mathbf{w}(16)$$

where  $\mathbf{B}_{j\setminus i}\mathbf{c}_{j\setminus i} = \sum_{\mathbf{b}_{jm}\notin\mathcal{R}(\mathbf{B}_i)} \mathbf{b}_{jm}c_j(m)$  and  $\mathcal{R}(\mathbf{A})$  denotes the range space of the matrix  $\mathbf{A}$ . More specifically, the  $M \times l$ matrix  $\mathbf{B}_{j\setminus i}$  contains the columns of  $\mathbf{B}_j$  which are not in the range space of the matrix  $\mathbf{B}_i$  where l is the number of columns in  $\mathbf{B}_{j\setminus i}$ . The  $l \times 1$  vector  $\mathbf{c}_{j\setminus i}$  contains the elements of  $\mathbf{c}_j$  corresponding to the column vectors in  $\mathbf{B}_{j\setminus i}$ .

We conclude that, the decision statistic for selecting  $S_i$  over  $S_j$  is given by  $\Delta_{ij}(\mathbf{y}) = ||\mathbf{P}_i^{\perp}(\mathbf{B}_{j\setminus i}\mathbf{c}_{j\setminus i} + \mathbf{w})||_2^2 - ||\mathbf{P}_j^{\perp}\mathbf{w}||_2^2$ and  $Pr(\Delta_{ij}(\mathbf{y}) < 0) = Pr\left(\frac{||\mathbf{P}_i^{\perp}(\mathbf{B}_{j\setminus i}\mathbf{c}_{j\setminus i} + \mathbf{w})||_2^2}{||\mathbf{P}_j^{\perp}\mathbf{w}||_2^2} < 1\right)$ . When  $\mathbf{B}_j$  is given, the random variable  $g_1 = ||\mathbf{P}_i^{\perp}(\mathbf{B}_{j\setminus i}\mathbf{c}_{j\setminus i} + \mathbf{w})||_2^2/\sigma_w^2$  is a non-central Chi squared random variable with M - k degrees of freedom and non-centrality parameter  $||\mathbf{P}_i^{\perp}(\mathbf{B}_{j\setminus i}\mathbf{c}_{j\setminus i})||_2^2/\sigma_w^2$ . The random variable  $g_2 = ||\mathbf{P}_j^{\perp}\mathbf{w}||_2^2/\sigma_w^2$  is a (central) Chi-squared random variable with M - k degrees of freedom. The two random variables  $g_1$  and  $g_2$  are, in general, correlated and the computation of the exact value of  $Pr(\Delta_{ij}(\mathbf{y}) < 0)$  is difficult. In the following we find an upper bound for the quantity  $Pr(\Delta_{ij}(\mathbf{y}) < 0)$  following techniques similar to those proposed in [14].

## III.A Upper bound on $Pr(\Delta_{ij}(\mathbf{y}) < 0)$

For clarity, we introduce the following notation. Let  $W_{j\setminus i}$  be the set consisting of column indices of  $\mathbf{B}_j$  such that  $\mathbf{b}_{jm} \notin \mathcal{R}(\mathbf{B}_i)$  for  $m = 0, 1, \dots, k-1$  and  $i \neq j$   $(i, j, = 0, 1, \dots, T-1)$ . We then have that  $|W_{j\setminus i}| = l$  where l can take values from  $1, 2, \dots, k$ .

**Lemma 1.** Assume that the sampling operator A is known. For any given signal  $x \in S_j$  so that  $Ax = \mathbf{B}_j \mathbf{c}_j$ , the probability of error in selecting the subspace  $S_i$  over  $S_j$ ,  $Pr(\Delta_{ij}(\mathbf{y}) < 0)$ , is upper bounded by,

$$Pr(\Delta_{ij}(\mathbf{y}) < 0) \le Q\left(\frac{1}{2}(1 - 2\eta_0)\sqrt{\lambda_{j\setminus i}}\right) + \Psi\left(l, \lambda_{j\setminus i}\right)$$
(17)

where  $\lambda_{j\setminus i} = \frac{1}{\sigma_w^2} ||\mathbf{P}_i^{\perp} \mathbf{B}_{j\setminus i} \mathbf{c}_{j\setminus i}||_2^2$ ,  $\Psi(l, \lambda_{j\setminus i}) = \frac{\sqrt{2}}{2^l \Gamma(l/2)} (\eta_0 \lambda_{j\setminus i})^{l/2-1/2} K_{l/2-1/2} \left(\frac{\eta_0 \lambda_{j\setminus i}}{2}\right)$ , Q(x) is the Gaussian Q function (1),  $\Gamma(x)$  is the Gamma function (2),  $K_{\nu}(x)$  is the modified Bessel function (3), and  $0 < \eta_0 < \frac{1}{2}$ .

**Theorem 1.** Assuming that the true subspace is chosen uniformly at random from T subspaces in the union (4), the average probability of error of the ML estimation for subspace recovery is upper bounded by,

$$P_e \leq \frac{1}{T} \sum_{i=0}^{T-1} \sum_{j=0}^{T-1} Q\left(\frac{1}{2}(1-2\eta_0)\sqrt{\lambda_{j\setminus i}}\right) + \Psi\left(l,\lambda_{j\setminus i}\right) \quad (18)$$

where  $\lambda_{j\setminus i}$ ,  $\eta_0$ , Q,  $\Psi$  are as defined in Lemma 1.

*Proof:* The proof follows from Lemma 1 and (15). In general, the subspaces  $S_i$  and  $S_j$  can overlap; i.e. there can be elements in  $S_j$  which are also in  $S_i$ . However, one subspace can not lie in another subspace entirely; i.e. all subspaces  $S_i$ 's are distinct for  $i = 0, 1, \dots, T-1$ . As defined before,  $W_{j\setminus i}$  contains the column indices of  $\mathbf{B}_j$  which are not in  $\mathcal{R}(\mathbf{B}_i)$  and  $|W_{i\setminus i}| = l$  for any  $i \neq j$  where l takes

values from  $1, 2, \dots, k$ . As l increases, the overlap of the two subspaces decreases resulting in more separable subspaces. In the special case where  $S_j$  and  $S_i$  do not intersect at all, we have l = k. Thus, l can be considered as a measure of overlap between any two subspaces  $S_j$  and  $S_i$  for  $i \neq j$  in the union (4). For given l, the probability  $Pr(\Delta_{ij}(\mathbf{y}) < 0)$ in (17) monotonically decreases as  $\lambda_{j\setminus i}$ , defined in Lemma 1, increases. This implies that when  $\lambda_{j\setminus i}$  is large, the probability of selecting  $S_i$  as the true subspace (given that the true subspace is  $S_j$ ) decreases. In other words,  $\lambda_{j\setminus i}$ , is used to characterize the error in selecting the subspace  $S_i$  over  $S_j$  for  $i \neq j$  (or how distinguishable the subspace  $S_i$  is with respect to  $S_j$ ) when the true subspace is  $S_j$ . It is, therefore, of interest to further investigate the quantity  $\lambda_{j\setminus i}$ .

#### III.B Evaluation of $\lambda_{j\setminus i}$

For any given signal  $x \in S_j$ , as defined in Lemma 1,  $\lambda_{j \setminus i}$  is given by,

$$\lambda_{j\backslash i} = \frac{1}{\sigma_w^2} ||\mathbf{P}_i^{\perp} A x||_2^2 = \frac{1}{\sigma_w^2} ||\mathbf{P}_i^{\perp} \mathbf{B}_{j\backslash i} \mathbf{c}_{j\backslash i}||_2^2.$$

When the true subspace is assumed to be  $S_j$ , the quantity  $||\mathbf{P}_i^{\perp}\mathbf{B}_{j\backslash i}\mathbf{c}_{j\backslash i}||_2^2 (= ||\mathbf{P}_i^{\perp}\mathbf{B}_j\mathbf{c}_j||_2^2 = ||\mathbf{P}_i^{\perp}Ax||_2^2)$  denotes the energy of the sampled signal Ax projected onto the null space of  $\mathbf{B}_i$ ; i.e., the energy of the sampled signal which is unaccounted for by  $S_i$  for  $i \neq j$ . Therefore, when  $||\mathbf{P}_i^{\perp}\mathbf{B}_{j\backslash i}\mathbf{c}_{j\backslash i}||_2^2$  is large, the probability that the subspace  $S_i$  is selected as the true subspace becomes small. Further, if  $S_j \subseteq S_i$  for any  $S_i$ , we have  $||\mathbf{P}_i^{\perp}\mathbf{B}_{j\backslash i}\mathbf{c}_{j\backslash i}||_2^2 = 0$ . However, this cannot happen based on our assumption that there is no subspace in the union which completely overlaps another. Thus,  $\lambda_{j\backslash i} > 0$ .

Let the eigendecomposition of  $\mathbf{P}_i^{\perp}$  be  $\mathbf{P}_i^{\perp} = \mathbf{Q}_i \mathbf{\Lambda}_i \mathbf{Q}_i^T$ where  $\mathbf{Q}_i$  is a unitary matrix consisting of eigenvectors of  $\mathbf{P}_i^{\perp}$ and  $\mathbf{\Lambda}_i$  is a diagonal matrix in which the diagonal elements represent eigenvalues of  $\mathbf{P}_i^{\perp}$  which are M - k ones and kzeros. Then, for given l,

$$\lambda_{j\backslash i} = \frac{1}{\sigma_w^2} ||\mathbf{P}_i^{\perp} \mathbf{B}_{j\backslash i} \mathbf{c}_{j\backslash i}||_2^2 = \sum_{m \in \mathcal{Q}_i} \alpha_{m,i}^2(l) \ge (M-k) \alpha_{\min,l}^2$$
(19)

where  $\alpha_{m,i}(l) = \frac{1}{\sigma_w} \langle \mathbf{q}_{m,i}, \mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i} \rangle$  for given  $l, \mathbf{q}_{m,i}$  is the m-th eigenvector of  $\mathbf{P}_i^{\perp}$ ,  $\mathcal{Q}_i$  is the set containing indices corresponding to nonzero eigenvalues where  $|\mathcal{Q}_i| = M - k$  and  $\alpha_{\min,l} = \min_{i:j \neq i} |\alpha_{m,i}(l)|$ .

and  $\alpha_{\min,l} = \min_{i;i \neq j} |\alpha_{m,i}(l)|$ . Note that  $(M-k)\alpha_{\min,l}^2$  is a measure of the minimum SNR of the sampled signal, Ax, projected onto the null space of any subspace  $S_i$  for  $i \neq j$ ,  $i = 0, 1, \dots, T-1$  such that  $|\mathcal{W}_{j\setminus i}| = l$  given that the true subspace in which the signal lies is  $S_j$ .

For a given subspace  $S_j$ , define  $T_j(l)$  to be the number of subspaces  $S_i$  such that  $|W_{j\setminus i}| = l$ . With these notations, the probability of error in (18) can be further upper bounded by,

$$P_{e} \leq \frac{1}{T} \sum_{j=0}^{T-1} \sum_{l=1}^{k} T_{j}(l) \left( Q\left(\frac{1}{2}(1-2\eta_{0})\sqrt{(M-k)\alpha_{\min,l}^{2}}\right) + \Psi\left(l, (M-k)\alpha_{\min,l}^{2}\right) \right)$$
(20)

where  $\Psi\left(l, (M-k)\alpha_{\min,l}^2\right) = \frac{\sqrt{2}}{2^l\Gamma(l/2)}(\eta_0(M-k)\alpha_{\min,l}^2)^{l/2-1/2}K_{l/2-1/2}(\eta_0(M-k)\alpha_{\min,l}^2/2).$  To obtain

(20) we used the facts that Q(x) is monotonically non increasing in x and  $\Psi(s, x)$  is monotonically non increasing in x for given s when x > 0. The quantity  $T_j(l)$  is a measure of the overlap between  $S_j$  and any subspace  $S_i$  for  $i \neq j, i = 0, 1, \dots, T-1$ . To compute  $T_j(l)$  explicitly, the specific structures of the subspaces should be known. For example, in the standard sparsity model used in CS in which the union in (4) consists of  $T = \binom{N}{k}$  subspaces from an orthonormal basis V of dimension N, there are  $\binom{k}{l}\binom{N-k}{l}$  number of sets such that  $|\mathcal{W}_{j\setminus i}| = l$ , thus  $T_j(l) = \binom{k}{l}\binom{N-k}{l}$ . In that particular case,  $T_j(l)$  is the same for all  $j = 0, 1, \dots, T-1$ . To further upper bound (20), we let

Then,

$$P_{e} \leq \sum_{l=1}^{k} T_{0}(l) \left( Q \left( \frac{1}{2} (1 - 2\eta_{0}) \sqrt{(M - k) \alpha_{\min,l}^{2}} \right) + \Psi \left( l, (M - k) \alpha_{\min,l}^{2} \right) \right).$$
(22)

 $T_0(l) = \max_{j=0,1,\cdots,T-1} T_j(l).$ 

**Theorem 2.** Let  $\alpha_{\min,l}^2$  and  $T_0(l)$  be as defined in (19) and (21), respectively. Suppose that sampling is performed via a sampling operator A. Then  $P_e$  in (22) vanishes asymptotically (i.e.,  $\lim_{(M-k)\to\infty} P_e \to 0$ ) if the following condition is satisfied:

$$M > k + \max\{M_1, M_2\}$$

where

$$M_{1} = \max_{l=1,\dots,k} \left\{ f_{1}(l) \\ = \frac{8}{(1-2\eta_{0})^{2}\alpha_{\min,l}^{2}} \left\{ \log(T_{0}(l)) + \log(1/2) \right\} \right\} (23)$$

$$M_{2} = \max_{l=1,\dots,k} \{f_{2}(l) \\ = \frac{2(k/2 + r_{0} - 1)}{r_{0}\eta_{0}\alpha_{\min,l}^{2}} \left\{ \log(T_{0}(l)) + \log\left(\frac{2b_{0}}{\sqrt{\pi}}\right) \right\} (24)$$

with  $0 < \eta_0 < 1/2$ ,  $b_0 = \frac{\sqrt{2\pi}}{4}$  and  $r_0 > 0$ .

Proof: See Appendix B.

Let  $l_i \in \{1, \dots, k\}$  be the value of l which maximizes  $f_i(l)$  as defined in Theorem 2 for i = 1, 2. For  $M_2$ , it can be verified that we can find constants  $\eta_0$  and  $r_0$  in the defined regimes such that  $\frac{8}{(1-2\eta_0)^2} > \frac{2(k/2+r_0-1)}{r_0\eta_0}$  if k is fairly small. Then the dominant factor of  $M_1$  and  $M_2$  can be written in the form of  $\frac{\eta_3}{\bar{\alpha}_{\min}^2} \log(\bar{T}_0)$  where  $\bar{\alpha}_{\min}^2$  and  $\bar{T}_0$  are the corresponding values of  $\alpha_{\min,l}^2$  and  $T_0(l)$  when  $l = l_0$  for  $l_0 \in \{l_1, l_2\}$  and  $\eta_3$  is an appropriate constant. Since, most of the scenarios we are interested in are for the case where k is sufficiently small, we get the minimum number of samples required for reliable subspace recovery as

$$M \ge k + \frac{\eta_3}{\bar{\alpha}_{\min}^2} \log(\bar{T}_0).$$
<sup>(25)</sup>

It is further noted that  $T_0(l) \leq T$  for all l and thus  $\overline{T}_0 \leq T$ where T is the total number of subspaces in the union (4).

## III.C Random sampling

(21)

Next, we consider the special case where the sampling operator is a  $M \times N$  matrix in which the elements are realizations of a random variable (e.g. Gaussian). Then we have  $\mathbf{B}_i = \mathbf{AV}_i$  in (8) where  $\mathbf{A}$  is the random sampling matrix and  $\mathbf{V}_i = [\mathbf{v}_{i0}|\cdots|\mathbf{v}_{i(k-1)}]$  is the  $N \times k$  matrix in which columns consist of the basis vectors of the subspace  $S_i$  for  $i = 0, 1, \cdots, T - 1$ . The only term which depends on the sampling operator in the expression for the upper bound on the probability of error in (18) is  $\lambda_{j\setminus i}$ . When the sampling operator is a random projection matrix,  $\lambda_{j\setminus i}$  can be evaluated as follows.

**Proposition 1.** Consider that the sampling matrix **A** consists of elements drawn from a Gaussian ensemble with mean zero and variance 1. When M - k is sufficiently large, we may approximate  $\lambda_{j\setminus i}$  as

$$\lambda_{j\setminus i} \to \frac{1}{\sigma_w^2} (M-k) || \sum_{m \in \mathcal{W}_{j\setminus i}} \mathbf{v}_{jm} c_j(m) ||_2^2$$

where as defined before,  $W_{j\setminus i}$   $(l = |W_{j\setminus i}|)$  denotes the set consisting of indices of basis vectors in  $S_j$  which are not in  $S_i$ .

Proof: See Appendix C. The quantity  $\sum_{m \in W_{j\setminus i}} \mathbf{v}_{jm} c_j(m)$  is the portion of the original signal **x** that is unaccounted for by the subspace  $S_i$ when the true subspace is  $S_j$  for  $j \neq i$ . Let  $\tilde{\alpha}_{\min,l}^2 = \frac{1}{\sigma_w^2} \min_{i,j,j\neq i} || \sum_{m \in W_{j\setminus i}} \mathbf{v}_{jm} c_j(m)||_2^2$  be the minimum (over  $i, j = 0, 1, \dots, T-1$ ) SNR of the original signal **x** which is unaccounted for by the subspace  $S_i$  when the true subspace is  $S_j$  such that  $|W_{j\setminus i}| = l$  for  $j \neq i$ . Then, with random sampling, the upper bound on the probability of error of the ML estimation in (18) reduces to (22) after replacing  $\alpha_{\min,l}^2$ in (22) by  $\tilde{\alpha}_{\min,l}^2$ . It is worth mentioning that  $\alpha_{\min,l}^2$  in (22) is a measure of SNR after sampling while  $\tilde{\alpha}_{\min,l}^2$  is a measure of SNR before sampling the signal.

## IV. SUBSPACE RECOVERY FROM STRUCTURED UNION OF SUBSPACES

In this section, we simplify the results obtained in Section III when the subspaces in the union (4) have structured properties leading to block sparsity.

#### **IV.A Block sparsity**

With the block sparsity model as discussed in Subsection II.B, the observation vector **y** can be written in the form of

$$\mathbf{y} = AV\mathbf{c} + \mathbf{w} = \mathbf{B}\mathbf{c} + \mathbf{w} \tag{26}$$

where  $\mathbf{B} = AV$  is a  $M \times N$  matrix,  $V = [V_0|V_1|\cdots|V_{L-1}]$ is as defined in Subsection II.B and **c** has L blocks (of size d each) in which all but  $k_0$  blocks are zeros; i.e., **c** is a block  $k_0$ -sparse vector. Further letting  $\mathbf{B}[i] = AV_i$  be a  $M \times d$  matrix, we can represent **B** as a concatenation of column blocks  $\mathbf{B}[i]$  for  $i = 0, 1, \cdots, L - 1$ . With this specific structure, the subspace recovery problem reduces to finding the indices of blocks in **c** such that the elements inside that block are nonzero, i.e., the problem of finding the block sparsity pattern. In addition to the structured union of subspaces model considered here in which the block sparsity pattern is observed, there are other instances where block sparsity arises such as in multiband signals [34], and in measurements of gene expression levels [27] [35].

Define the support set of the block sparse signal c as

$$\mathcal{U} := \{i \in \{0, 1, \cdots, L-1\} | \mathbf{c}_i \neq 0\}$$

which consists of the indices of the subspaces in the sum in (6) or the indices of the nonzero blocks in c. With the above formulation, there are  $T = {L \choose k_0}$  such support sets and the *j*-th support set is denoted by  $U_j$  for  $j = 0, 1, \dots, T-1$ .

Given that the true block support set is  $U_j$ , the measurement vector in (26) can be written as,

$$\mathbf{y} = \bar{\mathbf{B}}_j \bar{\mathbf{c}}_j + \mathbf{w}$$

where  $\bar{\mathbf{B}}_j = A\bar{V}_j$ ,  $\bar{V}_j = [V_{u_j^0}|\cdots|V_{u_j^{k_0-1}}]$  and  $u_j^m$  denotes the *m*-th index in the set  $\mathcal{U}_j$  for  $m = 0, 1, \cdots, k_0 - 1$ . Similar interpretation holds for the vector  $\bar{\mathbf{c}}_j$ . To compute the minimum number of samples required for asymptotically reliable subspace recovery with this structured union of subspaces model based on ML estimation, we can follow a similar approach as in Theorem 2 with appropriate notation changes. In this case, we can explicitly find  $T_0(l)$  required in Theorem 2. More specifically, for given l, there are  $\binom{k_0}{l}\binom{L-k_0}{l}$ number of sets such that  $|\mathcal{U}_j \setminus i| = l$  for any given  $\mathcal{U}_j$ . Then  $T_j(l) = T_0(l) = \binom{k_0}{l}\binom{L-k_0}{l}$ . In the next section, we extend the analysis to the case where the sampling operator is represented by random projections.

#### IV.B Sampling via random projections

We assume that the signal of interest x is a  $N \times 1$  vector and the sampling operator is a  $M \times N$  matrix with random elements. Further, assume that the  $N \times N$  basis matrix V defined in Section II.B is orthonormal.

When the sampling operator is a  $M \times N$  random matrix **A**, the block sparse observation model in (26), can be rewritten as,

$$\mathbf{y} = \mathbf{B}\mathbf{c} + \mathbf{w} \tag{27}$$

where  $\mathbf{B} = \mathbf{AV}$ ,  $\mathbf{V}$  is a  $N \times N$  orthonormal matrix,  $\mathbf{c}$  is a block sparse signal with  $k_0$  nonzero blocks each of length d and elements in  $\mathbf{A}$  are drawn from a random ensemble.

Compared to the analysis in Subsection III.C with general unions when the sampling operator is a random projection matrix, with the block sparsity model, we can further simplify the expression obtained for  $\lambda_{j\setminus i}$  in Proposition 1. We define the minimum nonzero block SNR as follows:

**Definition 2.** The minimum nonzero block SNR is defined as  $BSNR_{min} = \min_{m \in \mathcal{U}} \frac{||\mathbf{c}_m||_2^2}{\sigma_w^2}$  where  $\mathcal{U}$  is the set containing the indices corresponding to nonzero blocks of the block sparse signal as defined in Section IV.A.

**Proposition 2.** Let  $BSNR_{min}$  be the minimum nonzero block SNR of a block sparse signal. When the matrix A consists

of elements drawn from a Gaussian ensemble with mean zero and variance 1, for any  $U_j$  and  $U_i$  with  $l = |U_{j\setminus i}|$  we have,

$$\lambda_{j\setminus i} = \frac{1}{\sigma_w^2} (M - k_0 d) \sum_{m=0}^{l-1} ||\mathbf{V}_{u_{j\setminus i}^m} \mathbf{c}_{u_{j\setminus i}^m}||_2^2 \ge (M - k_0 d) l \text{BSNR}_{\min}$$

where  $u_{j\setminus i}^m$  denotes the *m*-th index of the set  $\mathcal{U}_{j\setminus i}$  which contains the indices of the subspaces in  $\mathcal{U}_j$  which are not in  $\mathcal{U}_i$ .

*Proof:* Proof follows from Proposition 1 and the following results:

$$\begin{aligned} ||\sum_{m=0}^{l-1} \mathbf{V}_{u_{j\backslash i}^{m}} \mathbf{c}_{u_{j\backslash i}^{m}} ||_{2}^{2} &= \langle \sum_{m=0}^{l-1} \mathbf{V}_{u_{j\backslash i}^{m}} \mathbf{c}_{u_{j\backslash i}^{m}}, \sum_{m=0}^{l-1} \mathbf{V}_{u_{j\backslash i}^{m}} \mathbf{c}_{u_{j\backslash i}^{m}} \rangle \\ &= \sum_{m=0}^{l-1} \langle \mathbf{V}_{u_{j\backslash i}^{m}} \mathbf{c}_{u_{j\backslash i}^{m}}, \mathbf{V}_{u_{j\backslash i}^{m}} \mathbf{c}_{u_{j\backslash i}^{m}} \rangle \\ &+ \sum_{m \neq t} \langle \mathbf{V}_{u_{j\backslash i}^{m}} \mathbf{c}_{u_{j\backslash i}^{m}}, \mathbf{V}_{u_{j\backslash i}^{t}} \mathbf{c}_{u_{j\backslash i}^{t}} \rangle \\ &= \sum_{m=0}^{l-1} ||\mathbf{V}_{u_{j\backslash i}^{m}} \mathbf{c}_{u_{j\backslash i}^{m}} ||_{2}^{2} \end{aligned}$$
(28)

where the last equality is due to the fact that the columns of V are orthogonal. Then (28) is lower bounded by,

$$||\sum_{m=0}^{l-1} \mathbf{V}_{u_{j\setminus i}^m} \mathbf{c}_{u_{j\setminus i}^m} ||_2^2 \ge \sigma_w^2 l \text{BSNR}_{\text{min}}$$

which completes the proof.

**Corollary 1.** When the sampling operator is a random projection matrix where the elements are drawn from a Gaussian ensemble with mean zero and the variance 1, the upper bound on the probability of error of the ML estimation in (18) for block sparsity pattern recovery reduces to,

$$P_{e} \leq \sum_{l=1}^{k_{0}} {\binom{k_{0}}{l}} {\binom{L-k_{0}}{l}}$$

$$\left(Q\left(\frac{1}{2}(1-2\eta_{0})\sqrt{(M-k)l\text{BSNR}_{\min}}\right) + \Psi\left(l,\text{BSNR}_{\min}\right)^{2}\right)$$
where  $k_{0} = k/d, \Psi\left(l,\text{BSNR}_{\min}\right) = \frac{\sqrt{2}}{2^{l}\Gamma(l/2)}(\eta_{0}(M-k_{0}d)l\text{BSNR}_{\min})^{l/2-1/2}$ 

$$K_{l/2-1/2}(\eta_0(M-k_0d))l\text{BSNR}_{\min}/2)$$
 and  $0 < \eta_0 < 1/2$ .

Next, we investigate sufficient conditions which state how the number of samples M scales with the other parameters  $(L, k_0, d, \text{BSNR}_{\min})$  to ensure that the probability of error in (29) vanishes asymptotically with the block sparse model (27).

**Lemma 2.** When (M - k)BSNR<sub>min</sub>  $\rightarrow \infty$ , the probability of error of the ML estimation (29) in recovering the block sparsity pattern vanishes asymptotically if the following conditions are satisfied:

$$M > k + \max\{\bar{M}_1, \bar{M}_2\}$$
 (30)

where

$$\bar{M}_1 = \frac{16}{BSNR_{\min}(1-2\eta_0)^2} \left(\log(L-k_0) + \log\left(\frac{e}{\sqrt{2}}\right)\right) (31)$$

$$\bar{M}_{2} = \frac{4(k_{0}/2 + r_{0} - 1)}{\eta_{0}r_{0}\mathrm{BSNR_{min}}} \left\{ \log(L - k_{0}) + \frac{1}{2}\log\left(\frac{2b_{0}e^{2}}{\sqrt{\pi}}\right) \right\} (32)$$
with  $0 < \eta_{0} < \frac{1}{2}, r_{0} > 0$  and  $b_{0} = \frac{\sqrt{2\pi}}{4}$  are constants.

*Proof:* Proof follows from Theorem 2 and using the relations, that  $\binom{k_0}{l} \leq \binom{L-k_0}{l}$  for  $k_0 \leq L/2$ , and  $\log(\binom{L-k_0}{l}) \leq l \log\left(\frac{e(L-k_0)}{l}\right)$ .

From Lemma 2, we can write the minimum number of random samples required for reliable block sparsity pattern recovery asymptotically in the form of  $\mathcal{O}(k + \frac{\eta_4}{BSNR_{\min}} \log(L-k_0))$ for some constant  $\eta_4$  in the case where  $k_0$  is sufficiently small.

**Remarks 1.** When  $BSNR_{\min} \rightarrow \infty$ , M > k measurements are sufficient for asymptotically reliable block sparsity pattern recovery with ML estimation.

#### IV.C Revisiting the standard sparsity model

In the standard sparsity model considered widely in the CS literature, the subspaces in the union (4) are assumed to be k-dimensional subspaces of an orthonormal basis. This is a special case of the block sparse model when d = 1. To have a fair comparison to the performance of the ML estimation in the presence of noise with the standard sparsity model and block sparsity model, we introduce further notations. Define the min- $||\mathbf{c}_{m}(i)||_{2}^{2}$ imum component SNR,  $\text{CSNR}_{\min} = \min_{\substack{m \in \mathcal{U}, i=0, \cdots, d-1}} \prod_{\substack{m \in$ so that  $BSNR_{\min} \ge dCSNR_{\min}$ . Then, when the sampling is performed via random projections, the probability of error of the ML estimation with the standard sparsity model is upper bounded as in (33) where  $\Psi(l, \text{CSNR}_{\min})$  is as defined in Corollary 1. With these notations, the probability of error of the ML estimation with block sparsity model (29) can be rewritten as in (34) where L = N/d and  $k_0 = k/d$ as defined previously. By obtaining the conditions under which  $P_e$  in (33) and (34) vanishes asymptotically, it can be shown that the dominant part of the required number of random samples for reliable subspace recovery asymptotically in the presence of noise can be expressed in the form of  $\mathcal{O}(k + \frac{1}{d} \frac{\hat{\eta}_1}{CSNR_{\min}} \log(L - k_0))$  with block sparsity model and  $\mathcal{O}(k + \frac{\hat{\eta}_2}{CSNR_{\min}}(\log(N-k)))$  with the standard sparsity model where  $\hat{\eta}_1$  and  $\hat{\eta}_2$  are positive constants. Thus, when the signal **x** exhibits block sparsity pattern with  $k = k_0 d$  where k is the total number of non zero coefficients of the sparse signal,  $k_0$ is the number of blocks and d is the block size, the required number of random samples beyond k (i.e. in terms of M-k) is reduced by approximately a factor of d compared to that with the standard sparsity model. Note that the above analysis is for the worst case, i.e. the upper bounds on the probability of error are obtained considering the minimum block/component SNR. The actual number of measurements required for reliable subspace recovery can be less than that predicted in Lemma 2.

#### V. COMPARISON WITH EXISTING RESULTS

## V.A Existing results for support recovery with the standard sparsity model

The most related existing work on deriving sufficient conditions for the ML estimation to succeed in the presence of noise

2) with the standard sparsity model is presented in [14]. There, taking the canonical basis as the sparsifying basis, the results are derived based on the following bound on the probability of error:

$$P_e \le \sum_{l=1}^{k} \binom{k}{l} \binom{N-k}{l} 4 \exp\left\{-\frac{(M-k)lCSNR_{\min}}{64(lCSNR_{\min}+8)}\right\}.$$
 (35)

When  $CSNR_{\min} \to \infty$ , it can be easily seen that this upper bound is bounded away from zero (i.e. it is bounded by  $4e^{-(M-k)/64}\left(\binom{Ld}{k}-1\right) > 0$ ). Based on the upper bound (35), it was shown in [14] that

$$M > k + (\eta_1 + 2048) \max\left\{\tilde{M}_1 = \log\left(\binom{N-k}{k}\right), \\ \tilde{M}_2 = \frac{\log(N-k)}{CSNR_{\min}}\right\}$$
(36)

measurements are required for asymptotically reliable sparsity pattern recovery where  $\eta_1$  is a constant (which is different from the one used earlier in the paper). When the minimum component SNR,  $CSNR_{\min} \rightarrow \infty$ , the ML estimation requires  $k + (\eta_1 + 2048)k \log((N - k)/k)$  measurements for asymptotically reliable recovery, which is much larger than k. However, as shown in [17], [36], when the measurement noise power is negligible (or in the no noise case), the exhaustive search decoder is capable of recovering the sparsity pattern with M = k + 1 measurements with high probability. Thus, the limits predicted by the existing results in the literature for sparsity pattern recovery in terms of the minimum number of measurements show a gap with what is actually required. When d = 1, V is the standard canonical basis, and A is a random Gaussian matrix, the structured union of subspaces model considered in Section IV (specifically the equation (27)) is the same as the model considered in [14]. Our results show that when  $CSNR_{\min} \rightarrow \infty$ , the upper bound on the probability of error in (33) vanishes with the standard sparsity model when M > k. More specifically, when  $CSNR_{\min} \rightarrow \infty$ , our results imply that  $\mathcal{O}(k)$  measurements are sufficient for asymptotically reliable sparsity pattern recovery with the ML estimation which is intuitive. Further, at finite  $CSNR_{min}$ , when  $M_2$  dominates  $M_1$  in (36) the lower bound in [14] has the same scaling with respect to L, k, d and  $CSNR_{\min}$  to that is obtained in this paper with the standard sparsity model.

## V.B Existing results for signal recovery with union of subspaces

The problem of stable recovery of signals that lie in a union of subspaces model is addressed in [23]–[26], [28]. In these works, the main focuss is to derive sufficient conditions that ensure reliable recovery of the complete signals while in this paper, our focus is only in identifying the low dimensional subspace in which the signal lies. Nevertheless, it is interesting to compare the results since it will provide insights into identifying the regions of the parameters (L, k, SNR, etc..) that ensure asymptotically reliable subspace recovery using the existing algorithms developed for exact signal recovery.

The following result is shown in [26].

$$P_e \le \sum_{l=1}^{k} \binom{k}{l} \binom{N-k}{l} \left( Q\left(\frac{1}{2}(1-2\eta_0)\sqrt{(M-k)l\text{CSNR}_{\min}}\right) + \Psi\left(l,\text{CSNR}_{\min}\right) \right)$$
(33)

$$P_e \le \sum_{l=1}^{k_0} \binom{k_0}{l} \binom{L-k_0}{l} \left( Q\left(\frac{1}{2}(1-2\eta_0)\sqrt{(M-k)ld\mathrm{CSNR_{min}}}\right) + \Psi\left(l,d\mathrm{CSNR_{min}}\right) \right)$$
(34)

**Theorem 3** ([26]). For any given t > 0, if

1

$$M > \frac{2}{c\delta} \left( \log(2T) + k \log\left(\frac{12}{\delta}\right) + t \right)$$
(37)

then, the matrix A in (7) satisfies the restricted isometry property (RIP) with the restricted isometry constant  $\delta$  (for formal definition of RIP readers may refer to [26]).

In [27], the authors derived the sufficient conditions for complete signal recovery in the block sparsity model. When the samples are acquired via random projections (elements in **A** are Gaussian) with the notations used in Section IV.B, the minimum number of samples required for the sampling matrix to satisfy block RIP with high probability is given by (from Theorem 3 and [27])

$$M \ge \frac{36}{7\delta} \left( \log \left( 2 \binom{L}{k_0} \right) + k \log \left( \frac{12}{\delta} \right) + t \right)$$
(38)

for some t > 0 and  $0 < \delta < 1$  is the restricted isometry constant. This is roughly in the order of  $\tilde{\eta}_1 k + \tilde{\eta}_2 k_0 \log(L/k_0)$ for some positive constants  $\tilde{\eta}_1$  and  $\tilde{\eta}_2$ . Thus, block sparse signals can be reliably recovered using computationally tractable algorithms (e.g. extension of BP - mixed  $l_2/l_1$  norm recovery algorithms) with  $\tilde{\eta}_1 k + \tilde{\eta}_2 k_0 \log(L/k_0)$  measurements when there is no noise. In the presence of noise, the BP based algorithm developed in [25] is shown to be robust so that the norm of the recovery error is bounded by the noise level. As shown in Section IV.B, it requires roughly the order of  $k + (\eta_4/BSNR_{\min})\log(L-k_0)$  measurements (when  $k_0$  is fairly small) for reliable block sparsity pattern recovery with ML estimation. Here, the second term is significant at finite  $BSNR_{\min}$  while it vanishes when  $BSNR_{\min} \rightarrow \infty$ . At finite  $BSNR_{\min}$ , when  $k_0$  is sublinear w.r.t. L, it can be shown that  $k_0 \log(L/k_0) >> \log(L-k_0)$ . Thus, in that region of  $k_0$ , the relevant scaling obtained in (38) is larger than what is required by the optimal ML estimation derived in this paper at finite  $BSNR_{min}$ . The exact difference between them depends on the value of  $BSNR_{\min}$  and the relevant constants.

#### VI. NUMERICAL RESULTS

Several computationally tractable algorithms for sparsity pattern recovery with standard sparsity have been derived and discussed quite extensively in the literature. Extensions of such algorithms for model based or structured CS have also been considered in several recent works. For example, extensions of CoSamp and iterative hard thresholding algorithms for model based CS were considered in [23]. Extensions of OMP algorithm for block sparsity pattern recovery (BOMP) were considered in [27], [37] while [25], [38], [39] considered the Group Lasso algorithm for block sparse signal recovery.

Our goal in this section is to validate the tightness of the derived upper bounds on the probability of error of the ML estimation and provide numerical results to illustrate the performance gap when employing practical algorithms for subspace recovery. Simulating the ML algorithm is difficult due to its high computational complexity in the high dimensions. Nevertheless, we show the performance for reasonably sized signal dimensions and samples just to demonstrate the tightness of the probability of error bound. For the structured union of subspaces model considered in Section IV.A, the problem reduces to recovering the block sparsity pattern of a block sparse signal. The performance of the ML algorithm is compared to block-OMP as proposed in [27] which is provided in Algorithm 1 where the set  $\hat{\mathcal{U}}$  contains the estimated indices of the nonzero blocks of a block sparse signal.

## Algorithm 1 Block-OMP (B-OMP) for block sparsity pattern recovery

Input: y, B,  $k_0$ 

- 1) Initialize t = 1,  $\hat{\mathcal{U}}(0) = \emptyset$ , residual vector  $\mathbf{r}_0 = \mathbf{y}$
- 2) Find the index  $\lambda(t)$  such that  $\lambda(t) = \underset{i=0,\cdots,L-1}{\arg \max} ||\mathbf{B}[i]^*\mathbf{r}_{t-1}||_2$
- 3) Set  $\hat{\mathcal{U}}(t) = \hat{\mathcal{U}}(t-1) \cup \{\lambda(t)\}$
- 4) Compute the projection operator  $\mathbf{P}(t) = \mathbf{B}(\hat{\mathcal{U}}(t)) \left( \mathbf{B}(\hat{\mathcal{U}}(t))^T \mathbf{B}(\hat{\mathcal{U}}(t)) \right)^{-1} \mathbf{B}(\hat{\mathcal{U}}(t))^T$ . Update the residual vector:  $\mathbf{r}_t = (\mathbf{I} \mathbf{P}(t))\mathbf{y}$  (note:  $\mathbf{B}(\hat{\mathcal{U}}(t))$ ) denotes the submatrix of **B** in which columns are taken from **B** corresponding to the indices in  $\hat{\mathcal{U}}(t)$ )
- 5) Increment t = t+1 and go to step 2 if  $t \le k_0$ , otherwise, stop and set  $\hat{\mathcal{U}} = \hat{\mathcal{U}}(t-1)$

Results in Figures 1 and 2, are based on the special structure as considered in (5) for subspaces leading to block sparsity and the sampling operator is assumed to be a random matrix in which elements are drawn from a Gaussian ensemble with mean zero and variance 1. Further, we let  $N \times N$  matrix V be the standard canonical basis. In Fig 1 (a), the exact probability of error of the ML estimation (obtained via simulation) and the upper bound on the probability of error derived in (29) vs M/N are shown. In the block sparsity model, we let N = 50, d = 2, L = 25,  $BSNR_{min} = 13dB$  and three different plots correspond to  $k_0 = 3, 4, 5$ . In Fig. 1 (b), we let d = 1 (i.e. the standard sparsity model) so that the upper bound on the probability of error reduces to (33). We also



(b)  $N = L = 50, d = 1, CSNR_{\min} = 10dB$ 

Fig. 1. Exact probability of error and the derived upper bound on the probability of error of the ML recovery for block sparsity pattern recovery

let  $CSNR_{min} = 10dB$  and different curves correspond to different values of k in Fig.1 (b). The exact probability of error of the ML estimation is obtained via Monte Carlo simulations with  $10^5$  runs. In the upper bounds (29) and (33), we let  $\eta_0 = 1/4$ . It can be seen from Fig. 1(a) and 1(b) that the derived upper bound on the probability of error is fairly a tight bound on the exact probability of error especially as M/Nincreases and the tightness is more significant in Fig. 1(a). It should be noted that for d = 2, we have  $k = k_0 d$ , thus the total number of non zero coefficients is larger in Fig. 1(a) than that with d = 1 in Fig. 1(b). Thus, it is seen that derived upper bound becomes tighter as k increases. It is also worth mentioning that the derived upper bound on the probability of error in [14] with the standard sparsity model (as in (35)) is bounded away from 1 for the selected parameter values mentioned above.

In Fig. 2, the performance of the block sparsity pattern recovery with ML and B-OMP algorithms is shown when  $BSNR_{min}$  varies. In Fig. 2, we let  $k_0 = 5$ , L = 25, d = 2 and N = 50. For B-OMP,  $10^4$  runs are performed for a given projection matrix and averaged over 100 runs. In Fig. 2, the ratio between the minimum and maximum block SNR in both cases considered is set at 1.825. As observed in Fig. 1, from



Fig. 2. Performance of the ML estimation and the B-OMP algorithm for block sparsity pattern recovery; L = 25,  $k_0 = 5$ , d = 2, and thus k = 10, N = 50

Fig. 2 it can be seen that the derived upper bound on the probability of error of the ML estimation is fairly closer to the exact probability error obtained via Monte Carlo simulations, especially as  $BSNR_{min}$  increases. Further, for a given finite  $BSNR_{min}$ , there seems to be a considerable performance gap between the B-OMP and the ML estimation. That is the price to pay for the computational complexity of the ML estimation vs the computationally efficient B-OMP algorithm.

## VII. CONCLUSION

In this paper, we investigated the problem of subspace recovery based on reduced dimensional samples when the signal of interest lies in a union of subspaces. With a given sampling operator, we derived the performance of the optimal ML estimation for subspace recovery in the presence of noise in terms of the probability of error. We further obtained conditions under which asymptotically reliable subspace recovery is guaranteed.

We extended the analysis to a special case of union of subspaces model which reduces to block sparsity. When the samples are obtained via random projections, sufficient conditions required for asymptotically reliable block sparsity pattern recovery with the ML estimation were derived. Performance gain in terms of the minimum number of samples required for asymptotically reliable subspace recovery with the block sparse model was quantified compared to that with the standard sparsity model. Our results further strengthen the existing results for sparsity pattern recovery with the standard sparsity model used in CS framework with random projections. More specifically, our results for sufficient conditions for asymptotically reliable subspace recovery are derived based on a tighter bound on the probability of error of the ML estimation compared to the existing results in the literature with the standard sparsity model. We further discussed and illustrated numerically the performance gap between the ML estimation and the computationally tractable algorithms (e.g. B-OMP) used for subspace recovery with the structured union of subspaces model.

An interesting future direction will be to extend the analysis

with the single node system to a multiple node system in First consider the case where w > 0. Then distributed networks.

## APPENDIX A

#### Proof of Lemma 1

To prove Lemma 1, we consider a similar argument to that considered in [14] with certain differences as noted in the following. As shown in [14], we may write,

$$\Delta_{ij}(\mathbf{y}) = ||\mathbf{P}_i^{\perp}\mathbf{y}||_2^2 - ||\mathbf{P}_i^{\perp}\mathbf{w}||_2^2 + ||\mathbf{P}_i^{\perp}\mathbf{w}||_2^2 - ||\mathbf{P}_j^{\perp}\mathbf{y}||_2^2.$$

For any given  $\delta > 0$ , define the events

$$h_1(\delta) = \left\{ |\frac{||\mathbf{P}_j^{\perp} \mathbf{y}||_2^2 - ||\mathbf{P}_i^{\perp} \mathbf{w}||_2^2}{\sigma_w^2}| \ge \delta \right\}$$
(39)

and

$$h_2(\delta) = \left\{ \frac{||\mathbf{P}_i^{\perp} \mathbf{y}||_2^2 - ||\mathbf{P}_i^{\perp} \mathbf{w}||_2^2}{\sigma_w^2} \le 2\delta \right\}.$$
 (40)

Then  $Pr(\Delta_{ii}(\mathbf{y}) < 0)$  implies that at least one event in (39) and (40) is true. Based on the union bound, we can write

$$Pr(\Delta_{ij}(\mathbf{y}) < 0) \le Pr(h_1(\delta)) + Pr(h_2(\delta)).$$

With the standard sparsity model and assuming that the sampling is performed via random projections, upper bounds on the probabilities  $Pr(h_1(\delta))$  and  $Pr(h_2(\delta))$  are derived in [14]. In contrast, in the following, we derive exact value for  $Pr(h_2(\delta))$  and a tighter bound for  $Pr(h_1(\delta))$  assuming that the sampling operator A is known. Thus, even for the standard sparsity model, the results presented in this paper tightens the results derived in [14].

We first evaluate  $Pr(h_1(\delta))$ . Let  $\Delta_{ij}^1(\mathbf{y}) = \frac{1}{\sigma_w^2}(||\mathbf{P}_j^{\perp}\mathbf{y}||_2^2 - ||\mathbf{P}_i^{\perp}\mathbf{w}||_2^2)$ . Assuming the true subspace is  $S_j, \Delta_{ij}^1(\mathbf{y})$  reduces to  $\Delta_{ij}^1(\mathbf{y}) = \frac{1}{\sigma_{\infty}^2} (||\mathbf{P}_j^{\perp} \mathbf{w}||_2^2 - ||\mathbf{P}_i^{\perp} \mathbf{w}||_2^2)$ . As shown in [14], the random variable  $\Delta_{ij}^1(\mathbf{y})$  can be represented as  $\Delta_{ij}^1(\mathbf{y}) =$  $x_1 - x_2$  where  $x_1$  and  $x_2$  are independent and  $x_1, x_2 \sim \mathcal{X}_l^2$ where l is the cardinality of the set  $\mathcal{W}_{j\setminus i}$  as defined before. With these notations, we can write

$$Pr(h_1(\delta)) = Pr(|x_1 - x_2| \ge \delta) = Pr((x_1 - x_2) \ge \delta) + Pr((x_1 - x_2) < -\delta)$$

The pdf of the random variable  $w = x_1 - x_2$  is symmetric around zero and thus we have,

$$Pr(h_1(\delta)) = 2Pr((x_1 - x_2) \ge \delta).$$

**Proposition 3.** When  $x_1 \sim \mathcal{X}_l^2$  and  $x_2 \sim \mathcal{X}_l^2$ , the random variable  $w = x_1 - x_2$  has the following pdf:

$$= \begin{cases} f_w(w) \\ f_w^+(w) = \frac{w^{\frac{l}{2} - \frac{1}{2}}}{\sqrt{\pi} 2^l \Gamma(l/2)} K_{1/2 - l/2}\left(\frac{w}{2}\right); & \text{if } w \ge 0 \\ f_w^-(w) = \frac{(-w)^{\frac{l}{2} - \frac{1}{2}}}{\sqrt{\pi} 2^l \Gamma(l/2)} K_{1/2 - l/2}\left(\frac{-w}{2}\right); & \text{if } w < 0 \end{cases}$$

where  $K_{\nu}(x)$  is the modified Bessel function.

*Proof:* Since  $x_1$  and  $x_2$  are independent, the pdf of w = $x_1 - x_2$  is given by [40]

$$f_w(w) = \begin{cases} \int_0^\infty f_{x_1}(w+x_2)f_{x_2}(x_2)dx_2; & if \ w \ge 0\\ \int_{-w}^\infty f_{x_1}(w+x_2)f_{x_2}(x_2)dx_2; & if \ w < 0 \end{cases}$$

$$f_w^+(w) = \int_0^\infty \frac{(w+x_2)^{l/2-1}e^{-(w+x_2)/2}}{2^{l/2}\Gamma(l/2)} \frac{x_2^{l/2-1}e^{-x_2/2}}{2^{l/2}\Gamma(l/2)} dx_2$$
  

$$= \frac{e^{-w/2}}{2^l(\Gamma(l/2))^2} \int_0^\infty x_2^{l/2-1}(w+x_2)^{l/2-1}e^{-x_2} dx_2$$
  

$$= \frac{e^{-w/2}}{2^l(\Gamma(l/2))^2} \frac{1}{\sqrt{\pi}} w^{l/2-1/2}e^{w/2}\Gamma(l/2)K_{1/2-l/2}(w/2)$$
  

$$= \frac{w^{l/2-1/2}K_{1/2-l/2}(w/2)}{\sqrt{\pi}2^l\Gamma(l/2)}$$

where  $K_{\nu}(x)$  is the modified Bessel function and the third equality is obtained using the integral result  $\int_0^\infty x^{\nu-1}(x+\beta)^{\nu-1}e^{-\mu x}dx = \frac{1}{\sqrt{\pi}}\left(\frac{\beta}{\mu}\right)^{\nu-1/2}e^{\beta\mu/2}\Gamma(\nu)K_{1/2-\nu}\left(\frac{\beta\mu}{2}\right)$  for  $\mu,\nu>0$  in [41, p. 348].

When w < 0, we have,

$$f_w^-(w) = \frac{e^{-w/2}}{2^l (\Gamma(l/2))^2} \int_{-w}^{\infty} x_2^{l/2-1} (w+x_2)^{l/2-1} e^{-x_2} dx_2.$$
(42)

Letting z = -w where z > 0, (42) can be rewritten as,

$$f_w^-(w) = \frac{e^{z/2}}{2^l (\Gamma(l/2))^2} \int_z^\infty x_2^{l/2-1} (x_2 - z)^{l/2-1} e^{-x_2} dx_2.$$
(43)

Using the integral result,  $\int_u^\infty x^{\nu-1}(x-u)^{\nu-1}e^{-\mu x}dx =$  $\frac{1}{\sqrt{\pi}} \left(\frac{u}{\mu}\right)^{\nu-1/2} e^{-\mu u/2} \Gamma(\nu) K_{\nu-1/2} \left(\frac{\mu u}{2}\right) \text{ in [41, p. 347] and}$ the relation  $K_{\nu}(x) = K_{-\nu}(x)$ , we get  $f_{w}^{-}(w)$  as in (41), completing the proof.

**Proposition 4.** For  $\delta > 0$ , the probability  $Pr(w > \delta)$  is given by,

$$Pr(w > \delta) \leq \frac{\sqrt{2}}{2^{l+1}\Gamma(l/2)} \delta^{l/2-1/2} K_{l/2-1/2}(\delta/2)$$

where  $K_{\nu}(x)$  is the modified Bessel function, and  $\Gamma(.)$  is the Gamma function.

Proof: Based on (41), we have

$$Pr(w > \delta) = \int_{\delta}^{\infty} f_w^+(w) dw$$
  
= 
$$\int_{\delta}^{\infty} \frac{w^{l/2 - 1/2} K_{1/2 - l/2}(w/2)}{\sqrt{\pi} 2^l \Gamma(l/2)} dw.$$
(44)

Using the equivalent integral representation of  $K_{\nu}(az) =$  $\frac{z^{\nu}}{2}\int_{0}^{\infty} e^{-\frac{a}{2}\left(t+\frac{z^{2}}{t}\right)}t^{-\nu-1}dt$  [41, p. 917], we can write the integral in (44) as,

$$Pr(w > \delta) = \frac{1}{\sqrt{\pi}2^{l+1}\Gamma(l/2)} \int_{\delta}^{\infty} \int_{0}^{\infty} e^{-\frac{1}{4}\left(t + \frac{w^{2}}{t}\right)} t^{l/2 - 3/2} dt dt (45)$$

Since 
$$\int_{\delta}^{\infty} e^{-\frac{w^2}{4t}} dw = \sqrt{2\pi} Q\left(\frac{\delta}{\sqrt{2t}}\right)$$
, (45) reduces to,

1

$$Pr(w > \delta) = \frac{\sqrt{2}}{2^{l+1}\Gamma(l/2)} \int_0^\infty e^{-t/4} t^{l/2 - 3/2} Q\left(\frac{\delta}{\sqrt{2t}}\right) dt$$

$$\leq \frac{\sqrt{2}}{2^{l+2}\Gamma(l/2)} \int_{0}^{\infty} t^{l/2-3/2} e^{-t/4-\frac{\delta^{2}}{4t}} dt \qquad (46)$$

$$= \frac{\sqrt{2}}{2^{l+1}\Gamma(l/2)} \delta^{l/2-1/2} K_{l/2-1/2}(\delta/2)$$
(47)

12

where we used the inequality  $Q(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}$  for x > 0, and the relation,  $\int_0^\infty x^{\nu-1} e^{-\beta/x-\gamma x} dx = 2 \left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu(2\sqrt{\beta\gamma})$ for  $\beta > 0$  and  $\gamma > 0$  [41, p. 368] while obtaining (46) and (47), respectively, which completes the proof.

Then, we have

$$Pr(h_1(\delta)) = \frac{\sqrt{2}}{2^l \Gamma(l/2)} \delta^{l/2 - 1/2} K_{l/2 - 1/2}(\delta/2).$$
(48)

Next we compute the quantity  $Pr(h_2(\delta))$ . Let  $\Delta_{ij}^2(\mathbf{y}) =$  $\frac{1}{\sigma^2}(||\mathbf{P}_i^{\perp}\mathbf{y}||_2^2 - ||\mathbf{P}_i^{\perp}\mathbf{w}||_2^2)$ . Then we have,  $\Delta_{ij}^2(\mathbf{y}) = \frac{1}{\sigma^2} (||\mathbf{P}_i^{\perp} \mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i}||_2^2 + 2\mathbf{w}^T \mathbf{P}_i^{\perp} \mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i}).$ 

Since  $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma_w^2 \mathbf{I}_M), \ \Delta_{ij}^2(\mathbf{y})$  is a Gaussian random variable with pdf,

$$\Delta_{ij}^2(\mathbf{y}) \sim \mathcal{N}\left(\frac{1}{\sigma_w^2} ||\mathbf{P}_i^{\perp} \mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i}||_2^2, \frac{4}{\sigma_w^2} ||\mathbf{P}_i^{\perp} \mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i}||_2^2\right).$$

Thus.

ŀ

$$\begin{aligned} \Pr(h_2(\delta)) &= \Pr\left(\Delta_{ij}^2(\mathbf{y}) \le 2\delta\right) \\ &= 1 - Q\left(\frac{2\delta - \frac{1}{\sigma_w^2} ||\mathbf{P}_i^{\perp} \mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i}||_2^2}{\frac{2}{\sigma_w} ||\mathbf{P}_i^{\perp} \mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i}||_2}\right) \\ &= 1 - Q\left(\frac{2\delta - \lambda_{j \setminus i}}{2\sqrt{\lambda_{j \setminus i}}}\right). \end{aligned}$$

Since it is desired to control  $\delta$  such that  $Pr(h_2(\delta)) \leq 1/2$ , we select  $\delta^* = \eta_0 \lambda_{j \setminus i}$  where  $\eta_0 < \frac{1}{2}$ . With this choice  $Pr(h_2(\delta))$ reduces to,

$$Pr(h_2(\delta)) = Q\left(\frac{1}{2}\sqrt{\lambda_{j\setminus i}}(1-2\eta_0)\right)$$

where we used the relation 1 - Q(-x) = Q(x) for x > 0, while  $Pr(h_1(\delta))$  reduces to,

$$Pr(h_1(\delta)) = \frac{\sqrt{2}}{2^l \Gamma(l/2)} (\eta_0 \lambda_{j \setminus i})^{l/2 - 1/2} K_{l/2 - 1/2} (\eta_0 \lambda_{j \setminus i}/2).$$
(49)

#### APPENDIX B

Proof of Theorem 2

To obtain conditions under which the probability of error bound in (22) asymptotically vanishes, we rely on the following corollary.

**Corollary 2.** Let  $T_0(l)$  and  $\alpha_{\min,l}^2$  be as defined in Subsection III.B. The probability of error of the ML estimation in (22) is further upper bounded by

$$P_e \le \sum_{l=1}^{k} T_0(l) \left( \frac{1}{2} e^{-\frac{1}{8}(1-2\eta_0)^2 (M-k)\alpha_{\min,l}^2} + \phi_l \right)$$
(50)

where

$$\phi_l = \frac{\sqrt{2\pi}}{4\Gamma(l/2)} \left(\frac{1}{4}\eta_0(M-k)\alpha_{\min,l}^2\right)^{l/2-1} e^{-\frac{1}{2}\eta_0(M-k)\alpha_{\min,l}^2}$$

*Proof:* Using the Chernoff bound for the Q function where  $Q(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}$ , we can upper bound the term  $Q\left(\frac{1}{2}(1-2\eta_0)\sqrt{(M-k)\alpha_{\min,l}^2}\right)$  as,

$$Q\left(\frac{1}{2}(1-2\eta_0)\sqrt{(M-k)\alpha_{\min,l}^2}\right) \leq \frac{1}{2}e^{-\frac{1}{8}(1-2\eta_0)^2(M-k)\alpha_{\min,l}^2}$$

for  $\eta_0 < \frac{1}{2}$ .

To obtain (51) we used the relation  $K_{\nu}(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z}$  when  $\nu << z$ , completing the proof.

It is further noted that when k is fairly small and  $\alpha_{\min,l}^2$ is sufficiently large, the condition required for (51) is often satisfied. We consider the conditions under which the each term in (50) goes to 0 asymptotically, equivalently logarithm of each term  $\rightarrow -\infty$ . First consider the first term in the summation in (50) for which the logarithm gives,

$$\log T_0(l) + \log(1/2) - \frac{1}{8}(1 - 2\eta_0)^2 (M - k) \alpha_{\min,l}^2$$
  

$$\leq \max_l \{\log(T_0(l)) + \log(1/2) - \frac{1}{8}(1 - 2\eta_0)^2 (M - k) \{\alpha_{\min,l}^2\} \} \to -\infty$$

as  $(M - k) \to \infty$  when  $M > k + M_1$  where  $M_1 = \max_{l=1,\dots,k} \left\{ \frac{8}{(1-2\eta_0)^2 \alpha_{\min,l}^2} \{ \log(T_0(l)) + \log(1/2) \} \right\}$ . Considering the second term in (50), let

$$\Pi_{1} = \log T_{0}(l) + \log \left(\frac{b_{0}}{\Gamma(l/2)}\right) + (l/2 - 1) \log \left(\frac{1}{4}\eta_{0}(M - k)\alpha_{\min,l}^{2}\right) - \frac{1}{2}\eta_{0}(M - k)\alpha_{\min,l}^{2}$$
(52)

where  $b_0 = \frac{\sqrt{2\pi}}{4}$ . When  $\frac{1}{4}\eta_0(M-k)\alpha_{\min,l}^2$  is sufficiently large, we can find  $0 < q_0 < \frac{1}{(k/2-1)}$  such that  $\log\left(\frac{1}{4}\eta_0(M-k)\alpha_{\min,l}^2\right) < q_0\frac{1}{2}\eta_0(M-k)\alpha_{\min,l}^2$ . Then (52) is upper bounded by

$$\Pi_{1} \leq \max_{l=1,\dots,k} \left\{ \log(T_{0}(l)) + \log\left(\frac{b_{0}}{\Gamma(3/2)}\right) - \left(\frac{1}{2}\eta_{0}(M-k)\alpha_{\min,l}^{2}\right)(1-q_{0}(k/2-1)) \right\} = (52)$$

where  $0 < q_0 < \frac{1}{(k/2-1)}$ . We can write  $q_0$  in the form of  $q_0 = \frac{1}{2(k/2+r_0-1)}$  for some  $r_0 > 0$ . Thus, (53) can be rewritten

$$\Pi_2 = \max_{l=1,\cdots,k} \left\{ \log(T_0(l)) + \log\left(\frac{2b_0}{\sqrt{\pi}}\right) - \left(\frac{1}{2}\eta_0(M-k)\alpha_{\min,l}^2\right)\frac{r_0}{r_0+k/2-1} \right\} \to -\infty$$

 $\phi_{l} = \frac{\sqrt{2\pi}}{4\Gamma(l/2)} \left(\frac{1}{4}\eta_{0}(M-k)\alpha_{\min,l}^{2}\right)^{\prime} e^{-\frac{1}{2}\eta_{0}(M-k)\alpha_{\min,l}^{2}} (51)_{\text{as}} (M-k) \to \infty \text{ when } M > k + M_{2} \text{ where } M_{2} = \frac{1}{4} \left(\frac{M-k}{2}\right)^{2} \left(\frac{1}{4}\eta_{0}(M-k)\alpha_{\min,l}^{2}\right)^{2} \left(\frac{$ 

#### APPENDIX C

#### Proof of Proposition 1

We rewrite  $\lambda_{j\setminus i} = \frac{1}{\sigma_w^2} ||\mathbf{P}_i^{\perp} \mathbf{B}_{j\setminus i} \mathbf{c}_{j\setminus i}||_2^2$ . The *t*-th element of the vector  $\mathbf{B}_{j\setminus i} \mathbf{c}_{j\setminus i}$  can be written as  $\langle \mathbf{a}_t, \sum_{m \in \mathcal{W}_{j\setminus i}} \mathbf{v}_{jm} c_j(m) \rangle$ where  $\mathbf{a}_t$ 's are row vectors of  $\mathbf{A}$  for  $t = 0, 1, \cdots, M - 1$ . Assuming that the elements of  $\mathbf{A}$  are independent Gaussian with mean zero and variance 1, it can be easily seen that  $\langle \mathbf{a}_t, \sum_{m \in \mathcal{W}_{j\setminus i}} \mathbf{v}_{jm} c_j(m) \rangle$  is a realization of a Gaussian random variable with mean zero and variance  $||\sum_{m \in \mathcal{W}_{j\setminus i}} \mathbf{v}_{jm} c_j(m)||_2^2$ . Further, the elements of  $\mathbf{B}_{j\setminus i} \mathbf{c}_{j\setminus i}$  are independent of each other since  $\mathbf{a}_t$ 's are independent for  $t = 0, 1, \cdots, M - 1$ . Thus, the random vector  $\mathbf{B}_{j\setminus i} \mathbf{c}_{j\setminus i} \sim$ 

 $t = 0, 1, \cdots, M - 1$ . Thus, the random vector  $\mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i} \sim \mathcal{N}(\mathbf{0}, || \sum_{m \in \mathcal{W}_{j \setminus i}} \mathbf{v}_{jm} c_j(m) ||_2^2 \mathbf{I}_M)$ . With given realizations, con-

sider again the transformation  $\mathbf{Q}_i^T \mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i}$  where  $\mathbf{Q}_i$  is the unitary matrix with eigenvectors of  $\mathbf{P}_i^{\perp}$ . Since the elements in  $\mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i}$  are independent and identically distributed (iid), the unitary transformation does not change the distribution of  $\mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i}$ . Then  $||\mathbf{P}_i^{\perp} \mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i}||_2^2 = ||\Lambda_i \mathbf{Q}_i^T \mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i}||_2^2$  is a sum of M - k iid random variables. Thus when (M - k) is sufficiently large, invoking the law of large numbers, we may approximate  $||\mathbf{P}_i^{\perp} \mathbf{B}_{j \setminus i} \mathbf{c}_{j \setminus i}||_2^2 \to (M - k)|| \sum_{m \in \mathcal{W}_{j \setminus i}} \mathbf{v}_{jm} c_j(m)||_2^2$ 

which completes the proof.

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