

A Framework of Constructions of Minimal Storage Regenerating Codes with the Optimal Access/Update Property

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Abstract—In this paper, we present a generic framework for constructing systematic minimum storage regenerating codes with two parity nodes based on the invariant subspace technique. Codes constructed in our framework not only contain some best known codes as special cases, but also include some new codes with key properties such as the optimal access property and the optimal update property. In particular, for a given storage capacity of an individual node, one of the new codes has the largest number of systematic nodes and two of the new codes have the largest number of systematic nodes with the optimal update property.

Index Terms—Distributed storage, high rate, invariant subspace, MSR code, optimal access, optimal update.

I. INTRODUCTION

DISTRIBUTED storage systems with high reliability have wide applications in large data centers, peer-to-peer storage systems such as OceanStore [14], Total Recall [1], DHash++ [7], and storage in wireless networks. To ensure reliability, the redundancy is crucial for these systems. A popular option to add redundancy is to employ erasure codes which can efficiently store data and protect against node failures. Examples of several distributed storage systems that employ erasure codes are Facebook’s coded Hadoop, Google Colossus and Microsoft Azure [10].

Recently, a new class of erasure codes for distributed storage systems called *minimum storage regenerating* (MSR) codes was introduced in [8]. Consider a file of size $\mathcal{M} = k\alpha$ symbols stored across a distributed storage system with n nodes, each keeping α symbols, that deploys an MSR code by storing the source data on the first k nodes, called *systematic nodes*, and mixtures of the source data on the other $n - k$ nodes, termed

parity nodes. To provide reliability, MSR codes must possess two abilities:

- (a) Reconstruction ability: In particular, an MSR code has the *MDS property* that any k out of the n nodes suffice to reconstruct the whole source data.
- (b) Repair ability: In practical distributed storage systems, the most common failure is failure of a single node. For this scenario, to maintain redundancy one has to repair the failed node by downloading $\beta \leq \alpha$ symbols from each of any $d \geq k$ surviving nodes. The *repair bandwidth* γ is defined as the amount of data downloaded during the repair procedure, i.e., $\gamma = d\beta$. In [8], MSR codes are shown to have the optimal repair property for the following values:

$$(\alpha, \gamma) = \left(\frac{\mathcal{M}}{k}, \frac{\mathcal{M}d}{k(d-k+1)} \right). \quad (1)$$

Up to now, constructions of MSR codes have attracted a lot of attention [2], [4], [5], [6], [12], [13], [15], [16], [17], [18], [20], [21]. However, many constructions have strict constraints on the parameters n, k, d . For example, $d \geq 2k - 2$ in [13], [15], [16], [17], which corresponds to *low rate* (i.e., $\frac{k}{n} \leq \frac{1}{2}$) regime. For *high rate* (i.e., $\frac{k}{n} \geq \frac{1}{2}$) regime, most known constructions are built on the concept of interference alignment, which was originally introduced in the context of wireless communication networks [11], [3], and was later exploited for distributed storage systems [21].

In contrast to other known constructions of high rate MSR codes, the Zigzag code proposed by Tamo *et al.* [18] is an MSR code exhibiting two additional interesting properties: the optimal access property and the optimal update property, which either does not need computing during the download phase of repair or minimizes the reading/writing during update. The Zigzag code works for arbitrary parameters n, k and $d = n - 1$, and requires a small finite field size q , for example, $q = 3$ for $n - k = 2$. It seems that the only shortcoming of the Zigzag code is the storage α of individual nodes, i.e., $\alpha = r^{k-1}$ grows sharply with the increase of k where $r = n - k$. In parallel to [18], the construction of the Zigzag code has also been discovered by Cadambe *et al.* in [5] via subspace interference alignment. In [20], Wang *et al.* presented another MSR code, named long MDS code, that increases the number k of the systematic nodes to nearly three times that of the Zigzag code but still maintains two parity nodes and the same node capacity α . However, a larger finite field size is required and none of the systematic nodes possess

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the optimal access property and the optimal update property simultaneously.

In the literature, there are mainly two repair types: exact repair and functional repair. Compared with the latter, exact repair is preferred since it does not incur additional significant system overhead by regenerating the exact replicas of the lost data in the failed node [9]. Unfortunately, except for the one in [12], all the known MSR codes of high rate, including the aforementioned Zigzag code and long MDS code, can only exactly repair all the systematic nodes optimally with respect to the bound in (1), whereas repair the parity node trivially by downloading the whole original file from all the systematic nodes. For simplicity, throughout this paper we say that such MSR codes possess the optimal repair property and omit that the property is only valid for systematic nodes. It should be noted that this kind of code is acceptable for a practical system due to two aspects: (1) The number of parity nodes is quite smaller compared to that of systematic nodes; (2) The failures of systematic nodes and parity nodes are different since the omission of some raw information would affect the information access time for the former [18].

In this paper, we focus on high rate MSR codes. Obviously, high rate implies a large value of k for fixed n . When $k = n - 1$, the repair bandwidth is the highest, i.e., $\gamma = \mathcal{M}$ by (1). Then, when $k = n - 2 > 1$ and $d = n - 1$ (which can reduce the repair bandwidth since γ is a decreasing function of d in (1)), MSR codes are of great interest because they can achieve the highest rate $\frac{k}{k+2}$ for $\gamma = (k+1)\alpha/2 < \mathcal{M}$. Thus, it is very desirable to construct MSR codes with two parity nodes for arbitrary number of systematic nodes k .

The main contribution of this paper is to present a simple but generic framework to construct MSR codes with two parity nodes based on the invariant subspace technique. Our construction not only contains the modified Zigzag code (the code obtained from the Zigzag code [18] by deleting its first node), and the long MDS code [20] as special cases, but also generates some new MSR codes. Specifically, based on the modified Zigzag code with m systematic nodes, we can obtain three new MSR codes by adding $2m$ or m more systematic nodes. When adding $2m$ more systematic nodes without the optimal access property and the optimal update property, we can construct new code \mathcal{C}_1 over a finite field of size $q \geq 2m + 1$. When adding m more systematic nodes, we can make a choice of either a smaller finite field or new nodes having the optimal update complexity. For the former, the finite field size can be reduced to $q \geq m + 1$, which results in new code \mathcal{C}_2 . For the latter, the resulting new code \mathcal{C}_3 still requires a finite field size $q \geq 2m + 1$. In addition, another new code \mathcal{C}_4 which has the same number of systematic nodes and requires the same size of finite field as those of \mathcal{C}_2 can be derived. All the systematic nodes of \mathcal{C}_4 have the optimal update property but none of them have the optimal access property. In this sense, we provide four code constructions with different parameters that allows for trading-off between the size of the finite field and the number of systematic nodes (with the optimal access/update property). In particular, given an α , the code \mathcal{C}_1 has the largest number of systematic nodes, while \mathcal{C}_3 and \mathcal{C}_4 have the largest number of systematic nodes with the

optimal update property. For comparison, the parameters of the new codes, the Zigzag code, and the long MDS code are listed in Table I.

The rest of this paper is organized as follows. Section II gives preliminaries about the necessary and sufficient conditions for an erasure code with two parity nodes to be an MSR code, and presents the special partition for a given basis. Section III proposes the generic construction, by which some known codes are reinterpreted and four new MSR codes with the optimal access/update property are derived. Finally, Section IV draws concluding remarks.

II. PRELIMINARIES

Let q be a prime power, \mathbf{F}_q be the finite field with q elements, and \mathbf{F}_q^l be the vector space of dimension l over \mathbf{F}_q . For simplicity, throughout this paper we do not specifically distinguish the vector space spanned by row vectors or column vectors if the context is clear.

Assume that a file of size $\mathcal{M} = k\alpha$ denoted by the column vector $f \in \mathbf{F}_q^{k\alpha}$ is partitioned in k parts $f = [f_1^T f_2^T \cdots f_k^T]^T$, each of size α , where T denotes the transpose operator. We encode f using an $(n = k + 2, k)$ MSR code \mathcal{C} and store it across k systematic and two parity storage nodes. Precisely, the first k (systematic) nodes store the file parts f_1, f_2, \dots, f_k in an uncoded form respectively, and the parity nodes store linear combinations of f_1, f_2, \dots, f_k . Without loss of generality, it is assumed that the nodes $k + 1$ and $k + 2$ respectively store $f_{k+1} = f_1 + f_2 + \cdots + f_k$ and $f_{k+2} = \sum_{i=1}^k A_i f_i$ for some $\alpha \times \alpha$ matrices A_1, \dots, A_k over \mathbf{F}_q , where the matrix A_i is called the *coding matrix* for the i th systematic node, $1 \leq i \leq k$. Table II illustrates the structure of a $(k + 2, k)$ MSR code.

TABLE II
STRUCTURE OF A $(k + 2, k)$ MSR CODE

Systematic node	Systematic data
1	f_1
\vdots	\vdots
k	f_k
Parity node	Parity data
1	$f_{k+1} = f_1 + \cdots + f_k$
2	$f_{k+2} = A_1 f_1 + \cdots + A_k f_k$

Note that reconstruction of the original file demands that (i) A_i is invertible when connecting nodes belong to the set $\{1, 2, \dots, k + 1\} \setminus \{i\}$ (or $\{1, 2, \dots, k, k + 2\} \setminus \{i\}$), for any $1 \leq i \leq k$ and (ii) $A_i - A_j$ is invertible when connecting nodes belong to the set $\{1, 2, \dots, k + 2\} \setminus \{i, j\}$, for any $1 \leq i \neq j \leq k$. In other words, the MSR code \mathcal{C} with the MDS property requires [20]

R1. A_i and $A_i - A_j$ are all invertible for any $1 \leq j \neq i \leq k$.

As mentioned in the last section, d is assumed to be $n - 1$ for minimizing the repair bandwidth. Then in order to repair a failed node, only half of data is downloaded from each surviving node. When a systematic node i fails, we download data $S_{i,j} f_j$ from node $j \neq i$ using an $\frac{\alpha}{2} \times \alpha$ matrix $S_{i,j}$ of rank $\frac{\alpha}{2}$, where $S_{i,j}$ is referred to as the *repair matrix* of the j th node for the i th systematic node. To simplify the repair strategy, we

TABLE I

COMPARISON BETWEEN THE NEW CODES AND SOME KNOWN CODES WITH TWO PARITY NODES AND $\alpha = 2^m$, WHERE k, k_A, k_U AND $k_{A\&U}$ DENOTE THE NUMBER OF SYSTEMATIC NODES, THE NUMBER OF SYSTEMATIC NODES WITH THE OPTIMAL ACCESS PROPERTY, THE NUMBER OF SYSTEMATIC NODES WITH THE OPTIMAL UPDATE PROPERTY AND THE NUMBER OF SYSTEMATIC NODES WITH BOTH THE OPTIMAL ACCESS PROPERTY AND THE OPTIMAL UPDATE PROPERTY RESPECTIVELY, AND q DENOTES THE SIZE OF THE FINITE FIELD REQUIRED.

	New code C_1	New code C_2	New code C_3	New code C_4	The Zigzag code [18]	The Long MDS code [20]
k	$3m$	$2m$	$2m$	$2m$	$m+1$	$3m$
k_A	m	m	m	0	$m+1$	$2m$
k_U	m	m	$2m$	$2m$	$m+1$	m
$k_{A\&U}$	m	m	m	0	$m+1$	0
q	$\geq 2m+1$	$\geq m+1$	$\geq 2m+1$	$\geq m+1$	3	$\geq 2m+1$

assume $S_{i,j} = S_i$ for all $1 \leq i \leq k, 1 \leq j \neq i \leq k+2$. Then during the repair process of a node i , one downloads $S_i f_j$ from each node $1 \leq j \neq i \leq k+2$, and eventually gets the following system of linear equations

$$\begin{pmatrix} S_i f_{k+1} \\ S_i f_{k+2} \end{pmatrix} = \underbrace{\begin{pmatrix} S_i \\ S_i A_i \end{pmatrix}}_{\text{useful data}} f_i + \sum_{j=1, j \neq i}^k \underbrace{\begin{pmatrix} S_i \\ S_i A_j \end{pmatrix}}_{\text{interference by } f_j} f_j.$$

Remark 1. A $(k+2, k)$ MSR code with $f_{k+1} = f_1 + f_2 + \dots + f_k$ and $S_{i,j} = S_i$ can be viewed as a kind of canonical form [5], [12], [18], [19], [20]. Firstly, if $f_{k+1} = B_1 f_1 + B_2 f_2 + \dots + B_k f_k$ for some nonsingular $\alpha * \alpha$ matrices $B_j, 1 \leq j \leq k$, then the code can be equivalently converted to the following code

Systematic node	Systematic data
1	f'_1
\vdots	\vdots
k	f'_k
Parity node	Parity data
1	$f'_{k+1} = f'_1 + \dots + f'_k$
2	$f'_{k+2} = A'_1 f'_1 + \dots + A'_k f'_k$

where $f'_i = B_i f_i$ and $A'_i = A_i B_i^{-1}$ for any $1 \leq i \leq k$ by using repair matrices $S'_{i,j} = S_{i,j} B_j^{-1}, 1 \leq j \neq i \leq k, S'_{i,k+1} = S_{i,k+1}$ and $S'_{i,k+2} = S_{i,k+2}$. Secondly, as shown in [19], such a $(k+2, k)$ MSR code can be transformed to a $(k+1, k-1)$ MSR code in canonical form. Thus we only consider MSR codes in canonical form since the difference between the numbers of their nodes $k+2$ and $k+1$ is negligible.

Then, the optimal repair property needs to cancel all the interference terms by R2 and then recover the original data f_i by R3 [20]:

$$\text{R2. } \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_j \end{pmatrix} \right) = \frac{\alpha}{2} \text{ for any } 1 \leq j \neq i \leq k.$$

$$\text{R3. } \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_i \end{pmatrix} \right) = \alpha \text{ for all } 1 \leq i \leq k.$$

The repair procedure firstly computes $S_i f_j, 1 \leq j \neq i \leq k+2$, and then transmits the result to the newcomer storage node. A systematic node is said to have the *optimal access property* if the computation within the surviving nodes is not required during the repair procedure [20]. For some applications such as data centers, the access to information is more costly than the transmission, which may cause a

bottleneck if the amount of the former is larger than that of latter [19]. Hence, an MSR code with more systematic nodes possessing the optimal access property is preferred. It is easy to verify that the i th systematic node with the optimal access property requires

R4. Each row of S_i has only one nonzero element, which equals to 1.

In addition, when a symbol in a systematic node is rewritten, if only the symbol itself and one symbol at each parity node need an update, then the systematic node is said to have the *optimal update property*, which achieves the minimum reading/writing during writing of information [18]. Therefore, an MSR code with more systematic nodes possessing the optimal update property is desired especially in a system where updates are frequent. In fact, the i th systematic node with the optimal update property is equivalent to that every parity element is a linear combination of exactly one element from the i th systematic node, i.e.,

R5. Each column of A_i has only one nonzero element.

Usually, it is favorable for a code to have more systematic nodes for a given α . Recall that the number k of systematic nodes of the Zigzag code is much less than that of the long MDS code. In this paper, we therefore mainly aim at increasing k of the Zigzag code. According to R1, R4 and R5, however, a systematic node has the optimal update property if and only if its coding matrix A_i is either a diagonal matrix or product of a diagonal matrix and a permutation matrix; a systematic node has the optimal access property if and only if its repair matrix S_i is an $\frac{\alpha}{2} \times \alpha$ submatrix of an $\alpha \times \alpha$ permutation matrix. The number of distinct such matrices satisfying R2 and R3 appears to be greatly limited. In [18], [19], it is shown that the largest number of systematic nodes of an MSR code with the optimal access property (resp. both the optimal access property and the optimal update property) is $2 \log_2 \alpha$ (resp. $\log_2 \alpha + 1$).

In what follows, we introduce two useful tools: invariant subspaces and partition sets, which enable us to construct our generic coding matrices and repair matrices satisfying R2 and R3.

A. Invariant subspaces

In this subsection, we determine the coding matrices by using invariant subspaces.

For a matrix A , denote by $\text{span}(A)$ the vector space spanned by its rows, obviously $\dim(\text{span}(A)) = \text{rank}(A)$. Recall that S_i is a matrix of rank $\frac{\alpha}{2}$. Then, R2 implies that $\text{span}(S_i A_j) \subseteq$

$\text{span}(S_i)$. Moreover, it follows from R1 that A_j is of full rank α and consequently we have $\text{rank}(S_i A_j) = \text{rank}(S_i)$. Hence, $\dim(\text{span}(S_i A_j)) = \dim(\text{span}(S_i))$, i.e.,

$$\text{span}(S_i A_j) = \text{span}(S_i) \quad (2)$$

which indicates that $\text{span}(S_i)$ is an *invariant subspace* of vector space $\text{span}(A_j) = \mathbf{F}_q^\alpha$ with respect to the linear transformation T defined by

$$T(x) = xA_j, \quad \text{for any } x \in \mathbf{F}_q^\alpha. \quad (3)$$

Firstly let us look at a simple example. Let $S = \begin{pmatrix} e_0 \\ e_1 \end{pmatrix}$ where e_0, e_1 are two arbitrary row vectors of length α over \mathbf{F}_q , and they are linearly independent. Then by (2), $\text{span}(S)$ is an invariant subspace of $\text{span}(A)$ with respect to $T : x \mapsto xA$ if and only if

$$\begin{pmatrix} e_0 \\ e_1 \end{pmatrix} A = \begin{pmatrix} ae_0 + be_1 \\ ce_0 + de_1 \end{pmatrix} \quad \text{and } ad \neq bc, \quad a, b, c, d \in \mathbf{F}_q.$$

In details, there are 7 cases as below:

- Case 1: $b = c = 0$ and $a, d \neq 0$,
- Case 2: $a = d = 0$ and $b, c \neq 0$,
- Case 3: $b = 0$ and $a, c, d \neq 0$,
- Case 4: $a = 0$ and $b, c, d \neq 0$,
- Case 5: $a, b, c, d \neq 0$ and $ad \neq bc$,
- Case 6: $c = 0$ and $a, b, d \neq 0$,
- Case 7: $d = 0$ and $a, b, c \neq 0$.

Note that if we interchange e_0 with e_1 , Case 3 (respectively, 4) will become Case 6 (respectively, 7). Besides, the coding matrix corresponding to Case 5 is a summation of two coding matrices corresponding to Cases 3 and 4, which would cause higher update complexity for its corresponding systematic node than that for the latter two. Therefore, we mainly consider Cases 1-4. Specifically, we say that the pair (e_0, e_1) with respect to A is

- type I if $\begin{pmatrix} e_0 \\ e_1 \end{pmatrix} A = \begin{pmatrix} ae_0 \\ de_1 \end{pmatrix}$,
- type II if $\begin{pmatrix} e_0 \\ e_1 \end{pmatrix} A = \begin{pmatrix} be_1 \\ ce_0 \end{pmatrix}$,
- type III if $\begin{pmatrix} e_0 \\ e_1 \end{pmatrix} A = \begin{pmatrix} ae_0 \\ ce_0 + de_1 \end{pmatrix}$,
- type IV if $\begin{pmatrix} e_0 \\ e_1 \end{pmatrix} A = \begin{pmatrix} be_1 \\ ce_0 + de_1 \end{pmatrix}$.

Now we extend the analysis to the general case. From now on, let $\{e_0, \dots, e_{2^m-1}\}$ be the standard basis of \mathbf{F}_q^α where $\alpha = 2^m$, i.e., i th basis vector

$$e_i = (0, \dots, 0, 1, 0, \dots, 0), \quad 0 \leq i \leq 2^m - 1,$$

with only the i th entry being nonzero. Divide the basis into 2^{m-1} pairs, i.e.,

$$(e_{i_1}, e_{j_1}), (e_{i_2}, e_{j_2}), \dots, (e_{i_{2^{m-1}}}, e_{j_{2^{m-1}}}), \quad (4)$$

where $0 \leq i_1 < i_2 < \dots < i_{2^{m-1}} \leq 2^m - 1$, $0 \leq j_1 < j_2 < \dots < j_{2^{m-1}} \leq 2^m - 1$ and $i_s \neq j_t$ for any $1 \leq s, t \leq$

2^{m-1} . For simplicity, assume that any pair forms an invariant subspace of \mathbf{F}_q^α with respect to T and all the pairs are of the same type, i.e.,

$$\begin{pmatrix} e_{i_1} \\ \vdots \\ e_{i_{2^{m-1}}} \\ \vdots \\ e_{j_{2^{m-1}}} \end{pmatrix} A = \begin{pmatrix} a_{i_1} e_{i_1} + b_{j_1} e_{j_1} \\ \vdots \\ a_{i_{2^{m-1}}} e_{i_{2^{m-1}}} + b_{j_{2^{m-1}}} e_{j_{2^{m-1}}} \\ \vdots \\ c_{i_1} e_{i_1} + d_{j_1} e_{j_1} \\ \vdots \\ c_{i_{2^{m-1}}} e_{i_{2^{m-1}}} + d_{j_{2^{m-1}}} e_{j_{2^{m-1}}} \end{pmatrix}$$

where a_i, b_i, c_i and d_i are some constants, then the coding matrix A can be uniquely determined. Accordingly, we call A type I, II, III, IV coding matrix respectively. By convenience, write

$$\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} A = \begin{pmatrix} aV_0 + bV_1 \\ cV_0 + dV_1 \end{pmatrix}$$

where a, b, c and d can be coefficients in \mathbf{F}_q or $\frac{\alpha}{2} \times \frac{\alpha}{2}$ diagonal matrices over \mathbf{F}_q and

$$V_0 = \begin{pmatrix} e_{i_1} \\ \vdots \\ e_{i_{2^{m-1}}} \end{pmatrix}, \quad V_1 = \begin{pmatrix} e_{j_1} \\ \vdots \\ e_{j_{2^{m-1}}} \end{pmatrix}, \quad (5)$$

and still use V_0 and V_1 to represent their corresponding sets $\{e_{i_1}, e_{i_2}, \dots, e_{i_{2^{m-1}}}\}$ and $\{e_{j_1}, e_{j_2}, \dots, e_{j_{2^{m-1}}}\}$ respectively in the following sections if the context is clear.

B. Partition of the basis $\{e_0, \dots, e_{2^m-1}\}$

In this subsection, we present a class of partition sets of the basis of \mathbf{F}_q^α to obtain V_0 and V_1 in (5), which had been used in [20], and will be crucial to our constructions as well.

Assume that there are m partition sets of the basis of \mathbf{F}_q^α as follows

$$\{e_0, e_1, \dots, e_{2^m-1}\} = V_{1,0} \cup V_{1,1} = \dots = V_{m,0} \cup V_{m,1} \quad (6)$$

such that

$$|V_{i_1, j_1} \cap V_{i_2, j_2} \cap \dots \cap V_{i_l, j_l}| = 2^{m-l} \quad (7)$$

for any $1 \leq i_1 < i_2 < \dots < i_l \leq m$, $j_t = 0, 1$, $1 \leq t \leq l \leq m$. It should be noted that (7) is useful when designing the code satisfying R2 and R3. Clearly, $|V_{i_1, j_1} \cap V_{i_2, j_2} \cap \dots \cap V_{i_m, j_m}| = 1$ for any $j_1, j_2, \dots, j_m \in \{0, 1\}$ by (7). Without loss of generality, we can set

$$\{e_j\} = \{e_{(j_1, j_2, \dots, j_m)}\} = V_{1, j_1} \cap V_{2, j_2} \cap \dots \cap V_{m, j_m},$$

where (j_1, j_2, \dots, j_m) is the binary expansion of the integer j . Recursively applying (7) to $l = m - 1, \dots, 1$, we then get

$$V_{i,t} = \{e_j | j_i = t\} \quad (8)$$

for $1 \leq i \leq m$ and $t = 0, 1$. Table III gives two examples of the set partitions that satisfy (6) and (7).

Based on the m partition sets in (8), define

$$V_{i+sm,t} = V_{i,t}, \quad i = 1, 2, \dots, m, \quad s \in \mathbf{N}^*, \quad t = 0, 1. \quad (9)$$

TABLE III

(A) AND (B) DENOTE THE m SET PARTITIONS OF V THAT SATISFY (6) AND (7) FOR $m = 2$ AND $m = 3$, RESPECTIVELY.

i	1	2	\bar{i}	1	2
$V_{i,0}$	e_0	e_0	$V_{i,1}$	e_2	e_1
	e_1	e_2		e_3	e_3

(A)

i	1	2	3	\bar{i}	1	2	3
$V_{i,0}$	e_0	e_0	e_0	$V_{i,1}$	e_4	e_2	e_1
	e_1	e_1	e_2		e_5	e_3	e_3
	e_2	e_4	e_4		e_6	e_6	e_5
	e_3	e_5	e_6		e_7	e_7	e_7

(B)

For any $1 \leq i_1, i_2 \leq sm$ and $i_1 \not\equiv i_2 \pmod{m}$, define $V_{i_1, i_2, j_1, j_2} = V_{i_2, i_1, j_2, j_1} = V_{i_1, j_1} \cap V_{i_2, j_2}$ for $j_1, j_2 = 0, 1$. Then

$$\begin{aligned} V_{i_1, j_1} &= (V_{i_1, j_1} \cap V_{i_2, 0}) \cup (V_{i_1, j_1} \cap V_{i_2, 1}) \\ &= V_{i_1, i_2, j_1, 0} \cup V_{i_1, i_2, j_1, 1}, \end{aligned} \quad (10)$$

and thus we have the following results, which will be frequently used in the sequel.

Lemma 1. For any $i, j \geq 1$ and $i \not\equiv j \pmod{m}$, we have

(i)

$$\begin{aligned} &\text{rank}(A_i - A_j) \\ &= \text{rank} \left(\begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} (A_i - A_j) \right) \\ &= \text{rank} \left(\begin{pmatrix} V_{i,j,0,0} \\ V_{i,j,0,1} \\ V_{i,j,1,0} \\ V_{i,j,1,1} \end{pmatrix} (A_i - A_j) \right), \end{aligned}$$

(ii)

$$\begin{aligned} &\text{rank} \left(\begin{pmatrix} V_{i,0} + u_i V_{i,1} \\ (V_{i,0} + u_i V_{i,1}) A_j \end{pmatrix} \right) \\ &= \text{rank} \left(\begin{pmatrix} V_{i,j,0,0} + u_i V_{i,j,1,0} \\ V_{i,j,0,1} + u_i V_{i,j,1,1} \\ (V_{i,j,0,0} + u_i V_{i,j,1,0}) A_j \\ (V_{i,j,0,1} + u_i V_{i,j,1,1}) A_j \end{pmatrix} \right) \end{aligned}$$

where $u_i \in \mathbf{F}_q$.

Proof: The proof is given in Appendix. ■

III. GENERIC CONSTRUCTION OF CODES WITH 2 PARITY NODES

In this section, we construct MSR codes with parameters $n = k + 2$ and $k = tm$, where t, m are some integers and $\alpha = 2^m$, with the coding matrices being the types defined in subsection 2.1.

Generic Construction: The $(n = k + 2, k)$ code \mathcal{C} has $\alpha \times \alpha$ coding matrices A_i and $\frac{\alpha}{2} \times \alpha$ repair matrices S_i for $1 \leq i \leq k$, such that

- 1) $\begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} A_i = \begin{pmatrix} a_i V_{i,0} + b_i V_{i,1} \\ c_i V_{i,0} + d_i V_{i,1} \end{pmatrix}$ for $1 \leq i \leq k$,
- 2) $S_i = V_{i,1}$ or $V_{i,0} + t_i V_{i,1}$ for $1 \leq i \leq k$,

where a_i, b_i, c_i, d_i and t_i can be coefficients in \mathbf{F}_q or $\frac{\alpha}{2} \times \frac{\alpha}{2}$ diagonal matrices over \mathbf{F}_q such that

$$\begin{pmatrix} a_i V_{i,0} + b_i V_{i,1} \\ c_i V_{i,0} + d_i V_{i,1} \end{pmatrix}$$

is invertible for $1 \leq i \leq k$.

As for Generic Construction, we have the following proposition.

Proposition 1. For a $(k + 2, k)$ MSR code generated by the generic construction,

- (i) $S_i \neq S_j$ for any $1 \leq i \neq j \leq k$;
- (ii) There do not exist four repair matrices $S_{j_1}, S_{j_2}, S_{j_3}$ and S_{j_4} such that $S_{j_l} = V_{i,0} + t_l V_{i,1}$, $1 \leq l \leq 3$, and $S_{j_4} = V_{i,1}$ or $V_{i,0} + t_4 V_{i,1}$, for an integer $1 \leq i \leq m$ where j_1, j_2, j_3, j_4 are four distinct integers in $\{1, \dots, k\}$ and t_1, t_2, t_3, t_4 are four distinct elements or matrices over \mathbf{F}_q ;

Proof: The proof is given in Appendix. ■

According to Proposition 1, in a $(k + 2, k)$ MSR code generated by the generic construction, there are at most three repair matrices of the form $S_i = V_{i,1}$ or $V_{i,0} + t_i V_{i,1}$, each appearing at most once, for any given $1 \leq i \leq m$, i.e., the number of systematic nodes is bounded by $k \leq 3m$. In the following, through choosing some appropriate coding matrices in our framework, several $(k + 2, k)$ MSR codes, $k \leq 3m$, with the optimal access property and/or the optimal update property are obtained. This generates not only the known constructions such as the Zigzag code (except for one node) [18] and the long MDS code [20], but also some new codes.

A. Reinterpretation of known constructions

Based on coding matrices of type II, construct an $(n = k + 2, k = m)$ code by

- $\begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} A_i = \begin{pmatrix} \Lambda_{i,1} V_{i,1} \\ \Lambda_{i,0} V_{i,0} \end{pmatrix}$ for $1 \leq i \leq m$,
- $S_i = V_{i,0}$, for $1 \leq i \leq m$,

where $\Lambda_{i,0}$ and $\Lambda_{i,1}$ are $\frac{\alpha}{2} \times \frac{\alpha}{2}$ diagonal matrices over \mathbf{F}_q . In fact, it is a modification of the Zigzag code by deleting its first node [18]. The modified Zigzag code has almost the same properties as that of the Zigzag code, i.e., all the systematic nodes of the modified Zigzag code possess both the optimal access property and the optimal update property.

Through a combination of coding matrices of types I, III and VI, the long MDS code [20] can also be constructed by

- $\begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} A_i = \begin{cases} \begin{pmatrix} \lambda_{i,0} V_{i,0} + k_i V_{i,1} \\ \lambda_{i,1} V_{i,1} \end{pmatrix}, & \text{if } 1 \leq i \leq m \\ \begin{pmatrix} \lambda_{i,0} V_{i,0} \\ \lambda_{i,1} V_{i,1} + k_i V_{i,0} \end{pmatrix}, & \text{if } m + 1 \leq i \leq 2m \\ \begin{pmatrix} \lambda_{i,0} V_{i,0} \\ \lambda_{i,1} V_{i,1} \end{pmatrix}, & \text{if } 2m + 1 \leq i \leq 3m \end{cases}$

$$\bullet S_i = \begin{cases} V_{i,0}, & \text{if } 1 \leq i \leq m \\ V_{i,1}, & \text{if } m+1 \leq i \leq 2m \\ V_{i,0} + V_{i,1}, & \text{if } 2m+1 \leq i \leq 3m \end{cases}$$

where $\lambda_{i,0}, \lambda_{i,1} \in \mathbf{F}_q^*$, $k_j = \lambda_{j,0} - \lambda_{j,1}$ and $k_{j+m} = \lambda_{j+m,1} - \lambda_{j+m,0}$ for all $1 \leq i \leq k$ and $1 \leq j \leq m$.

Moreover, it is possible to choose $\Lambda_{i,s}$ and $\lambda_{j,s}$ respectively in the constructions of the modified Zigzag code and the long MDS code [20] such that the conditions R1-R5 are satisfied.

B. New code \mathcal{C}_1

Using the coding matrices of types II and III, we construct the first new code.

Construction 1. The $(n = k + 2, k = 3m)$ code \mathcal{C}_1 has $\alpha \times \alpha$ coding matrices A_i and $\frac{\alpha}{2} \times \alpha$ repair matrices S_i for $1 \leq i \leq k$, such that

$$1) \begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} A_i = \begin{cases} \begin{pmatrix} \lambda_{i,1} V_{i,1} \\ \lambda_{i,0} V_{i,0} \end{pmatrix}, & \text{if } 1 \leq i \leq m \\ \begin{pmatrix} \lambda_{i,0} V_{i,0} \\ \lambda_{i,1} V_{i,1} + k_{i-m} V_{i,0} \end{pmatrix}, & \text{if } m+1 \leq i \leq 3m \end{cases}$$

$$2) S_i = \begin{cases} V_{i,0}, & \text{if } 1 \leq i \leq m \\ V_{i,0} + t_{i-m} V_{i,1}, & \text{if } m+1 \leq i \leq 3m \end{cases}$$

where $\lambda_{i,0}, \lambda_{i,1}, k_j, t_j \in \mathbf{F}_q^*$ for all $1 \leq i \leq k$ and $1 \leq j \leq 2m$.

Theorem 1. \mathcal{C}_1 is a code with the MDS property if and only if

- (i) $\lambda_{i,0} \lambda_{i,1} \neq \lambda_{j,0} \lambda_{j,1}$ for any $1 \leq i \neq j \leq m$,
- (ii) $\begin{cases} \lambda_{i,s} \neq \lambda_{j,s}, & \text{if } j = i + m \\ \lambda_{i,s} \neq \lambda_{j,t}, & \text{if } j \neq i + m \end{cases}$ for any $m+1 \leq i < j \leq 3m$ and $s, t = 0, 1$,
- (iii) $\begin{cases} \lambda_{i,1}(\lambda_{i,0} - k_{j-m}) \neq \lambda_{j,0} \lambda_{j,1}, & \text{if } j = i + m, i + 2m \\ \lambda_{i,0} \lambda_{i,1} \neq \lambda_{j,0}^2, \lambda_{j,1}^2, & \text{otherwise} \end{cases}$ for any $1 \leq i \leq m$ and $m+1 \leq j \leq 3m$.

Proof: The proof is given in Appendix. ■

Theorem 2. \mathcal{C}_1 is a code with the optimal repair property if and only if

- (i) $\lambda_{i,1} = t_i^2 \lambda_{i,0}$ and $t_i = -t_{i+m}$ for all $1 \leq i \leq m$,
- (ii) $\lambda_{i,1} = \lambda_{i,0} + t_i k_{i-m}$ and $\lambda_{i+m,1} = \lambda_{i+m,0} + t_{i-m} k_i$ for all $m+1 \leq i \leq 2m$,
- (iii) \mathbf{F}_q is of odd characteristic.

Proof: The proof is given in Appendix. ■

Theorem 3. The first m systematic nodes of \mathcal{C}_1 have both the optimal access property and the optimal update property.

Proof: The proof is given in Appendix. ■

According to item (ii) of Theorem 1 and items (ii) and (iii) of Theorem 2 (which indicate $\lambda_{i,0} \neq \lambda_{i,1}$ for any $m+1 \leq i \leq 3m$), a finite field \mathbf{F}_q of odd characteristic with at least $2m$ pairwise distinct nonzero elements is necessary to ensure the code \mathcal{C}_1 to be an MSR code. In the following theorem, a class of concrete coefficients for code \mathcal{C}_1 is given.

Theorem 4. The code \mathcal{C}_1 in Construction 1 is an MSR code if

$$k_i = k_{i+m} = -2\gamma^i, \lambda_{i,0} = \lambda_{i,1} = \lambda_{i+m,0} = \lambda_{i+2m,1} = \gamma^i, \\ \lambda_{i+m,1} = \lambda_{i+2m,0} = -\gamma^i, t_i = -1, t_{i+m} = 1$$

for $1 \leq i \leq m$, where γ is a primitive element of finite field \mathbf{F}_q of odd characteristic with $q \geq 2m+1$. In particular, $q = \min\{p^i \geq 2m+1 | p \text{ is an odd prime, } i \geq 1\}$ is the optimal alphabet size for \mathcal{C}_1 to be an MSR code.

Proof: The proof is given in Appendix. ■

Remark 2. For a given storage capacity $\alpha = 2^m$ per node, our code \mathcal{C}_1 and the long MDS code in [20] have the biggest size $3m$ among all the MSR codes with high rate. Unlike the long MDS code, \mathcal{C}_1 has m systematic nodes possessing the optimal access property and the optimal update property simultaneously. However, \mathcal{C}_1 may require a larger alphabet size than that of the long MDS code in certain situations since only the finite field of odd characteristic is feasible for the construction of \mathcal{C}_1 .

Finally, an illustrative example of code \mathcal{C}_1 is given.

Example 1. For $m = 2$, the coding matrices and repair matrices of the code \mathcal{C}_1 are as follows:

$$A_1 = \begin{pmatrix} 2e_2 \\ 2e_3 \\ 2e_0 \\ 2e_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 4e_1 \\ 4e_0 \\ 4e_3 \\ 4e_2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2e_0 \\ 2e_1 \\ e_0 + 3e_2 \\ e_1 + 3e_3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 4e_0 \\ 2e_0 + e_1 \\ 4e_2 \\ 2e_2 + e_3 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 3e_0 \\ 3e_1 \\ e_0 + 2e_2 \\ e_1 + 2e_3 \end{pmatrix}, \quad A_6 = \begin{pmatrix} e_0 \\ 2e_0 + 4e_1 \\ e_2 \\ 2e_2 + 4e_3 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} e_0 \\ e_1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} e_0 \\ e_2 \end{pmatrix}, \quad S_3 = \begin{pmatrix} e_0 - e_2 \\ e_1 - e_3 \end{pmatrix},$$

$$S_4 = \begin{pmatrix} e_0 - e_1 \\ e_2 - e_3 \end{pmatrix}, \quad S_5 = \begin{pmatrix} e_0 + e_2 \\ e_1 + e_3 \end{pmatrix}, \quad S_6 = \begin{pmatrix} e_0 + e_1 \\ e_2 + e_3 \end{pmatrix}$$

where 2 is chosen as a primitive element of \mathbf{F}_5 and all the calculations are done over \mathbf{F}_5 . It can be easily verified that R1-R3 hold and R4-R5 hold for $1 \leq i \leq m$, which are consistent with Theorems 4 and 3, respectively.

C. New code \mathcal{C}_2

Deleting the last m systematic nodes in \mathcal{C}_1 , we can get the second new code.

Construction 2. The $(n = k + 2, k = 2m)$ code \mathcal{C}_2 has $\alpha \times \alpha$ coding matrices A_i and $\frac{\alpha}{2} \times \alpha$ repair matrices S_i for $1 \leq i \leq k$, such that

$$1) \begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} A_i$$

$$\begin{aligned}
&= \begin{cases} \begin{pmatrix} \lambda_{i,1}V_{i,1} \\ \lambda_{i,0}V_{i,0} \end{pmatrix}, & \text{if } 1 \leq i \leq m \\ \begin{pmatrix} \lambda_{i,0}V_{i,0} \\ \lambda_{i,1}V_{i,1} + k_{i-m}V_{i,0} \end{pmatrix}, & \text{if } m+1 \leq i \leq 2m \end{cases} \\
2) \quad &\begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} A_i \\
&= \begin{cases} \begin{pmatrix} \lambda_{i,1}V_{i,1} \\ \lambda_{i,0}V_{i,0} \end{pmatrix}, & \text{if } 1 \leq i \leq m \\ \begin{pmatrix} \lambda_{i,0}V_{i,0} \\ \lambda_{i,1}V_{i,1} + k_{i-m}V_{i,0} \end{pmatrix}, & \text{if } m+1 \leq i \leq 2m \end{cases} \\
3) \quad S_i &= \begin{cases} V_{i,0}, & \text{if } 1 \leq i \leq m \\ V_{i,0} + t_{i-m}V_{i,1}, & \text{if } m+1 \leq i \leq 2m \end{cases}
\end{aligned}$$

where $\lambda_{i,0}, \lambda_{i,1}, k_j, t_j \in \mathbf{F}_q^*$ for all $1 \leq i \leq k$ and $1 \leq j \leq m$.

Hereafter we state the results of \mathcal{C}_2 without proofs since they are included in those given in the last subsection.

Theorem 5. \mathcal{C}_2 is a code with the MDS property if and only if

- (i) $\lambda_{i,0}\lambda_{i,1} \neq \lambda_{j,0}\lambda_{j,1}$ for any $1 \leq i \neq j \leq m$,
- (ii) $\lambda_{i,s} \neq \lambda_{j,t}$ for any $m+1 \leq i \neq j \leq 2m$ and $s, t = 0, 1$,
- (iii) $\begin{cases} \lambda_{i,1}(\lambda_{i,0} - k_i) \neq \lambda_{j,0}\lambda_{j,1}, & \text{if } j = i + m \\ \lambda_{i,0}\lambda_{i,1} \neq \lambda_{j,0}^2, \lambda_{j,1}^2, & \text{if } j \neq i + m \end{cases}$ for any $1 \leq i \leq m$ and $m+1 \leq j \leq 2m$.

Theorem 6. \mathcal{C}_2 is a code with the optimal repair property if and only if

- (i) $\lambda_{i,1} = t_i^2 \lambda_{i,0}$ for all $1 \leq i \leq m$,
- (ii) $\lambda_{i,1} \neq \lambda_{i,0} + t_{i-m}k_{i-m}$ for any $m+1 \leq i \leq 2m$.

Theorem 7. The first m systematic nodes of \mathcal{C}_2 have both the optimal access property and the optimal update property.

According to item (i) of Theorem 5 and item (i) of Theorem 6, a finite field \mathbf{F}_q with at least m pairwise distinct nonzero square elements is necessary to ensure the code \mathcal{C}_2 to be an MSR code. Let $q = p^i$ where p is a prime and i is a positive integer. It is well known that all the nonzero elements in \mathbf{F}_q are square elements for $p = 2$ but only half the nonzero elements in \mathbf{F}_q are square elements for $p > 2$. Then, the MSR code \mathcal{C}_2 requires $q \geq m+1$ for $p = 2$ or $q \geq 2m+1$ for $p > 2$. Straightforwardly, there exists a positive integer i such that $q = 2^i$ lies between $m+1$ and $2m$. That is, a finite field of characteristic 2 is more suitable to construct the MSR code \mathcal{C}_2 . In the following theorem, a class of concrete coefficients for code \mathcal{C}_2 is given.

Theorem 8. The code \mathcal{C}_2 in Construction 2 is an MSR code if

$$\lambda_{i,0} = \lambda_{i,1} = \lambda_{i+m,0} = \lambda_{i+m,1} = \gamma^i, \text{ and } t_i = k_i = 1$$

for all $1 \leq i \leq m$, where γ is a primitive element of finite field \mathbf{F}_q of characteristic 2 with $q \geq m+1$. In particular, $q = \min\{2^i \geq m+1 | i \geq 1\}$ is the optimal alphabet size for \mathcal{C}_2 to be an MSR code.

An illustrative example of code \mathcal{C}_2 is given as follows.

Example 2. For $m = 3$, the coding matrices and repair

matrices of the code \mathcal{C}_2 are as follows:

$$A_1 = \begin{pmatrix} \gamma e_4 \\ \gamma e_5 \\ \gamma e_6 \\ \gamma e_7 \\ \gamma e_0 \\ \gamma e_1 \\ \gamma e_2 \\ \gamma e_3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \gamma^2 e_2 \\ \gamma^2 e_3 \\ \gamma^2 e_0 \\ \gamma^2 e_1 \\ \gamma^2 e_6 \\ \gamma^2 e_7 \\ \gamma^2 e_4 \\ \gamma^2 e_5 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} e_1 \\ e_0 \\ e_3 \\ e_2 \\ e_5 \\ e_4 \\ e_7 \\ e_6 \end{pmatrix}, \quad A_4 = \begin{pmatrix} \gamma e_0 \\ \gamma e_1 \\ \gamma e_2 \\ \gamma e_3 \\ \gamma e_4 + e_0 \\ \gamma e_5 + e_1 \\ \gamma e_6 + e_2 \\ \gamma e_7 + e_3 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} \gamma^2 e_0 \\ \gamma^2 e_1 \\ \gamma^2 e_2 + e_0 \\ \gamma^2 e_3 + e_1 \\ \gamma^2 e_4 \\ \gamma^2 e_5 \\ \gamma^2 e_6 + e_4 \\ \gamma^2 e_7 + e_5 \end{pmatrix}, \quad A_6 = \begin{pmatrix} e_0 \\ e_1 + e_0 \\ e_2 \\ e_3 + e_2 \\ e_4 \\ e_5 + e_4 \\ e_6 \\ e_7 + e_6 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad S_2 = \begin{pmatrix} e_0 \\ e_1 \\ e_4 \\ e_5 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} e_0 \\ e_2 \\ e_4 \\ e_6 \end{pmatrix}, \quad S_4 = \begin{pmatrix} e_0 + e_4 \\ e_1 + e_5 \\ e_2 + e_6 \\ e_3 + e_7 \end{pmatrix},$$

$$S_5 = \begin{pmatrix} e_0 + e_2 \\ e_1 + e_3 \\ e_4 + e_6 \\ e_5 + e_7 \end{pmatrix}, \quad S_6 = \begin{pmatrix} e_0 + e_1 \\ e_2 + e_3 \\ e_4 + e_5 \\ e_6 + e_7 \end{pmatrix}$$

where γ is chosen as a primitive element of \mathbf{F}_{2^2} and all the calculations are done over \mathbf{F}_{2^2} . It can be verified that R1-R3 hold and R4-R5 hold for $1 \leq i \leq m$, which are consistent with Theorems 8 and 7, respectively.

D. New Code \mathcal{C}_3

By means of combination of coding matrices of types I and II, we propose the third new code.

Construction 3. The $(n = k+2, k = 2m)$ code \mathcal{C}_3 has $\alpha \times \alpha$ coding matrices A_i and $\frac{\alpha}{2} \times \alpha$ repair matrices S_i for $1 \leq i \leq k$, such that

$$1) \quad \begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} A_i = \begin{cases} \begin{pmatrix} \lambda_{i,1}V_{i,1} \\ \lambda_{i,0}V_{i,0} \end{pmatrix}, & \text{if } 1 \leq i \leq m \\ \begin{pmatrix} \lambda_{i,0}V_{i,0} \\ \lambda_{i,1}V_{i,1} \end{pmatrix}, & \text{if } m+1 \leq i \leq 2m \end{cases}$$

$$2) S_i = \begin{cases} V_{i,0}, & \text{if } 1 \leq i \leq m \\ V_{i,0} + t_{i-m}V_{i,1}, & \text{if } m+1 \leq i \leq 2m \end{cases}$$

where $\lambda_{i,0}, \lambda_{i,1}, t_j \in \mathbf{F}_q^*$ for all $1 \leq i \leq k$ and $1 \leq j \leq m$.

Theorem 9. \mathcal{C}_3 is a code with the MDS property if and only if

- (i) $\lambda_{i,0}\lambda_{i,1} \neq \lambda_{j,0}\lambda_{j,1}$ for any $1 \leq i \neq j \leq m$,
- (ii) $\lambda_{i,s} \neq \lambda_{j,t}$ for any $m+1 \leq i \neq j \leq 2m$ and $s, t = 0$ or 1,
- (iii) $\lambda_{i,0}\lambda_{i,1} \neq \begin{cases} \lambda_{j,0}\lambda_{j,1}, & \text{if } j = i+m \\ \lambda_{j,0}^2, \lambda_{j,1}^2, & \text{if } j \neq i+m \end{cases}$ for any $1 \leq i \leq m$ and $m+1 \leq j \leq 2m$.

Proof: The proof is given in Appendix. ■

Theorem 10. \mathcal{C}_3 is a code with the optimal repair property if and only if

- (i) $\lambda_{i,1} = t_i^2 \lambda_{i,0}$ for all $1 \leq i \leq m$,
- (ii) $\lambda_{i,0} \neq \lambda_{i,1}$ for any $m+1 \leq i \leq 2m$.

Proof: The proof is given in Appendix. ■

Theorem 11. The $2m$ systematic nodes of \mathcal{C}_3 have the optimal update property and the first m nodes have the optimal access property.

According to item (ii) of Theorem 9 and item (ii) of Theorem 10, a finite field \mathbf{F}_q with at least $2m$ pairwise distinct nonzero elements is required to guarantee the code \mathcal{C}_3 to be an MSR code. Specifically, over \mathbf{F}_q with $q \geq 2m+1$, we can give a class of concrete coefficients for code \mathcal{C}_3 as follows.

Theorem 12. The code \mathcal{C}_3 in Construction 3 is an MSR code if

$$\lambda_{i,0} = \lambda_{i,1} = \lambda_{i+m,0} = \gamma^i, \quad \lambda_{i+m,1} = \gamma^{\lfloor \frac{q}{2} \rfloor + i}, \quad \text{and } t_i = 1$$

for all $1 \leq i \leq m$, where γ is a primitive element of \mathbf{F}_q with $q \geq 2m+1$. In particular, $q = \min\{p^i \geq 2m+1 \mid p \text{ is a prime, } i \geq 1\}$ is the optimal alphabet size for \mathcal{C}_3 to be an MSR code.

Proof: The proof is given in Appendix. ■

Finally to illustrate the construction of code \mathcal{C}_3 , we give an example.

Example 3. For $m = 2$, the coding matrices and repair matrices of the code \mathcal{C}_3 are as follows:

$$A_1 = \begin{pmatrix} 2e_2 \\ 2e_3 \\ 2e_0 \\ 2e_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 4e_1 \\ 4e_0 \\ 4e_3 \\ 4e_2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2e_0 \\ 2e_1 \\ 3e_2 \\ 3e_3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 4e_0 \\ e_1 \\ 4e_2 \\ e_3 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} e_0 \\ e_1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} e_0 \\ e_2 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} e_0 + e_2 \\ e_1 + e_3 \end{pmatrix}, \quad S_4 = \begin{pmatrix} e_0 + e_1 \\ e_2 + e_3 \end{pmatrix},$$

where 2 is chosen as a primitive element of \mathbf{F}_5 and all the calculations are done over \mathbf{F}_5 . It can be easily verified that R1-R3 hold and R4 holds for $1 \leq i \leq m$ and R5 holds for $1 \leq i \leq 2m$, which are consistent with Theorems 12 and 11, respectively. Moreover, this example can be illustrated in another way as in Table IV.

TABLE IV

COLUMNS 1, 2, 3, 4 ARE SYSTEMATIC NODES AND COLUMNS R AND Z ARE PARITY NODES. EACH ELEMENT IN COLUMN R IS A LINEAR COMBINATION OF THE SYSTEMATIC ELEMENTS IN THE SAME ROW, WHILE EACH ELEMENT IN COLUMN Z IS A LINEAR COMBINATION OF THE SYSTEMATIC ELEMENTS WITH THE SAME SYMBOL. FOR INSTANCE, THE FIRST ELEMENT IN COLUMN R IS A LINEAR COMBINATION OF THE ELEMENTS IN THE FIRST ROW AND IN COLUMNS 1, 2, 3 AND 4, AND THE ♣ IN COLUMN Z IS A LINEAR COMBINATION OF ALL THE ♣ ELEMENTS IN COLUMNS 1, 2, 3 AND 4.

	1	2	3	4	R	Z
0	♠	♥	♣	♣		♣
1	◇	♣	♥	♥		♥
2	♣	◇	♠	♠		♠
3	♥	♠	◇	◇		◇

E. New code \mathcal{C}_4

Based on the coding matrices of type II, we can present the fourth new code.

Construction 4. The $(n = k+2, k = 2m)$ code \mathcal{C}_4 has $\alpha \times \alpha$ coding matrices A_i and $\frac{\alpha}{2} \times \alpha$ repair matrices S_i for $1 \leq i \leq k$, such that

$$1) \begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} A_i = \begin{pmatrix} \lambda_{i,1}V_{i,1} \\ \lambda_{i,0}V_{i,0} \end{pmatrix} \text{ for } 1 \leq i \leq k,$$

$$2) S_i = V_{i,0} + t_i V_{i,1} \text{ for } 1 \leq i \leq k,$$

where $\lambda_{i,0}, \lambda_{i,1}, t_i \in \mathbf{F}_q^*$ for all $1 \leq i \leq k$.

Theorem 13. \mathcal{C}_4 is a code with the MDS property if and only if

- (i) $\lambda_{i,0}\lambda_{i,1} \neq \lambda_{j,0}\lambda_{j,1}$ for any $1 \leq i < j \leq k$ and $i \neq j-m$,
- (ii) $\lambda_{i,s} \neq \lambda_{i+m,s}$ for any $1 \leq i \leq m$ and $s = 0, 1$.

Proof: The proof is given in Appendix. ■

Theorem 14. \mathcal{C}_4 is a code with the optimal repair property if and only if

- (i) $\lambda_{i,1} = t_{i+m}^2 \lambda_{i,0}$ and $\lambda_{i+m,1} = t_i^2 \lambda_{i+m,0}$ for all $1 \leq i \leq m$,
- (ii) $\lambda_{i,1} \neq t_i^2 \lambda_{i,0}$ for any $1 \leq i \leq k$.

Proof: The proof is given in Appendix. ■

Theorem 15. The $2m$ systematic nodes of \mathcal{C}_4 have the optimal update property.

According to item (i) of Theorem 13 and item (i) of Theorem 14, a finite field \mathbf{F}_q with at least m pairwise distinct nonzero square elements is necessary to ensure the code \mathcal{C}_4 to be an MSR code. Similar to code \mathcal{C}_2 , we have the following concrete construction for the new code \mathcal{C}_4 .

Theorem 16. The code \mathcal{C}_4 in Construction 4 is an MSR code if

$$\lambda_{i,0} = \gamma^i, \lambda_{i,1} = \gamma^{i+2}, \lambda_{i+m,0} = \lambda_{i+m,1} = \gamma^{i+1}, t_i = 1, t_{i+m} = \gamma$$

for all $1 \leq i \leq m$, where γ is a primitive element of finite field \mathbf{F}_q of characteristic 2 with $q \geq m + 1$. In particular, $q = \min\{2^i \geq m + 1 \mid i \geq 1\}$ is the optimal alphabet size for \mathcal{C}_4 to be an MSR code.

Remark 3. In [18], 2-duplication of the Zigzag code with parameters $(n = k + 2, k = 2m + 2)$ was proposed, which has the similar coding matrices as those of \mathcal{C}_4 . Although the number of systematic nodes of \mathcal{C}_4 is two less than that of 2-duplication of the Zigzag code, it has better repair bandwidth. When repairing a failed systematic node, only half of the data need to be downloaded from each surviving node of \mathcal{C}_4 , while the fraction of the data need to be downloaded from each surviving node of 2-duplication of the Zigzag code is $\frac{m+1}{2m+1}$.

Finally to illustrate the construction of code \mathcal{C}_4 , we give an example.

Example 4. For $m = 2$, the coding matrices and repair matrices of the code \mathcal{C}_4 are as follows:

$$A_1 = \begin{pmatrix} e_2 \\ e_3 \\ \gamma e_0 \\ \gamma e_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \gamma e_1 \\ \gamma^2 e_0 \\ \gamma e_3 \\ \gamma^2 e_2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \gamma^2 e_2 \\ \gamma^2 e_3 \\ \gamma^2 e_0 \\ \gamma^2 e_1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} e_1 \\ e_0 \\ e_3 \\ e_2 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} e_0 + e_2 \\ e_1 + e_3 \end{pmatrix}, \quad S_2 = \begin{pmatrix} e_0 + e_1 \\ e_2 + e_3 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} e_0 + \gamma e_2 \\ e_1 + \gamma e_3 \end{pmatrix}, \quad S_4 = \begin{pmatrix} e_0 + \gamma e_1 \\ e_2 + \gamma e_3 \end{pmatrix}$$

where γ is chosen as a primitive element of \mathbf{F}_{2^2} and all the calculations are done over \mathbf{F}_{2^2} . It can be verified that R1-R3 hold and R5 holds for $1 \leq i \leq 2m$, which are consistent with Theorems 16 and 15, respectively. Moreover, this example can be illustrated in another way as in the following table.

	I	2	3	4	R	Z
0	♠	♥	♠	♥		♣
I	♦	♣	♦	♣		♥
2	♣	♦	♣	♦		♠
3	♥	♠	♥	♠		♦

F. Other new codes

Combined coding matrices of types I and IV (with repair matrices $V_{i,0} + t_i V_{i,1}$ and $V_{i,0}$), types II and IV (with repair matrices $V_{i,0} + t_i V_{i,1}$), types III and IV (with repair matrices $V_{i,0} + t_i V_{i,1}$ and $V_{i,0}$), types III, III and IV (with repair matrices $V_{i,0} + t_i V_{i,1}$, $V_{i,0} + t_{i+m} V_{i,1}$ and $V_{i,0}$), four new MSR codes with $k = 2m$ or $3m$ can be obtained, but the other properties (eg. optimal access, optimal update, the size of the finite fields required) are not as good as the aforementioned new codes \mathcal{C}_1 , \mathcal{C}_2 , \mathcal{C}_3 and \mathcal{C}_4 .

IV. CONCLUDING REMARKS

In this paper, we proposed a simple but generic framework to construct high rate MSR codes with two parity nodes. The framework can not only generate the modified Zigzag code and the long MDS code, but also generate four new MSR codes \mathcal{C}_1 , \mathcal{C}_2 , \mathcal{C}_3 and \mathcal{C}_4 with the optimal access/update property. The optimal sizes of the finite fields required for the four codes were also determined. Notably, by these four new MSR codes, we could get a tradeoff between the size of the finite field and the number of systematic nodes (with the optimal access/update property).

Our construction can be generalized to the $(k+r, k = 3m)$ or $(k+r, k = 2m)$ MSR code with arbitrary $r > 2$ parity nodes for $\alpha = r^m$. For this generalization, we firstly need to partition the basis $\{e_0, e_1, \dots, e_{r^m-1}\}$ of \mathbf{F}_q^α into r subsets V_0, V_1, \dots, V_r with equal sizes. Then, types I-IV coding matrices can be similarly determined based on invariant subspaces of dimension r but with complicated forms. By means of these matrices, we can obtain the generalized codes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and \mathcal{C}_4 with $r > 2$ parity nodes, which still possesses the optimal access/update property. The optimal alphabet size q , however, is difficult to determine and hence will be left for future research.

APPENDIX

Proof of Lemma 1: (i) According to (10), in matrix notation, $V_{i,0}, V_{i,1}$ are equivalent to $\begin{pmatrix} V_{i,j,0,0} \\ V_{i,j,0,1} \end{pmatrix}$ and $\begin{pmatrix} V_{i,j,1,0} \\ V_{i,j,1,1} \end{pmatrix}$ under elementary row transformation, respectively, i.e., $\begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix}$ is equivalent to $\begin{pmatrix} V_{i,j,0,0} \\ V_{i,j,0,1} \\ V_{i,j,1,0} \\ V_{i,j,1,1} \end{pmatrix}$ under elementary row transformation. Thus

$$\text{rank} \left(\begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} \right) = \text{rank} \left(\begin{pmatrix} V_{i,j,0,0} \\ V_{i,j,0,1} \\ V_{i,j,1,0} \\ V_{i,j,1,1} \end{pmatrix} \right).$$

Immediately, the assertion follows from the fact that the matrix $\begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix}$ is of full rank.

(ii) It follows from that the matrix $\begin{pmatrix} V_{i,0} + u_i V_{i,1} \\ (V_{i,0} + u_i V_{i,1}) A_j \end{pmatrix}$ is equivalent to $\begin{pmatrix} V_{i,j,0,0} + u_i V_{i,j,1,0} \\ V_{i,j,0,1} + u_i V_{i,j,1,1} \\ (V_{i,j,0,0} + u_i V_{i,j,1,0}) A_j \\ (V_{i,j,0,1} + u_i V_{i,j,1,1}) A_j \end{pmatrix}$ under elementary row transformation. ■

Proof of Proposition 1: (i) It is obvious otherwise R2 and R3 can not be satisfied simultaneously for repair matrices S_i, S_j and coding matrix A_i .

(ii) If there exist such four repair matrices, according to the generic construction, then the coding matrix A_{j_4} satisfies

$$\begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} A_{j_4} = \begin{pmatrix} aV_{i,0} + bV_{i,1} \\ cV_{i,0} + dV_{i,1} \end{pmatrix}$$

where a, b, c and d can be coefficients in \mathbf{F}_q or $\frac{\alpha}{2} \times \frac{\alpha}{2}$ diagonal matrices over \mathbf{F}_q . Consider

$$\begin{aligned} & \text{rank} \begin{pmatrix} V_{i,0} + t_l V_{i,1} \\ (V_{i,0} + t_l V_{i,1}) A_{j_4} \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} V_{i,0} + t_l V_{i,1} \\ (a + ct_l) V_{i,0} + (b + dt_l) V_{i,1} \end{pmatrix} \\ &= \frac{\alpha}{2}, \quad l = 1, 2, 3. \end{aligned}$$

Then we have that the equation

$$ct^2 + (a - d)t - b = 0$$

has three distinct roots $t = t_1, t_2$ and t_3 , which is possible only if $b = c = 0$ and $a = d \neq 0$, in this case, A_{j_4} is a diagonal matrix, therefore

$$\text{rank} \begin{pmatrix} S_{j_4} \\ S_{j_4} A_{j_4} \end{pmatrix} = \frac{\alpha}{2} \quad \text{for any } S_{j_4}$$

and then R3 can not be satisfied. \blacksquare

Proof of Theorem 1: \mathcal{C}_1 has the MDS property if and only if R1 holds. Obviously, A_i is invertible for all $1 \leq i \leq k$ since $\lambda_{i,0}, \lambda_{i,1} \neq 0$. In what follows, by means of Lemma 1 we establish the necessary and sufficient conditions of $\text{rank}(A_i - A_j) = \alpha$ for any $1 \leq i \neq j \leq k$ in the following three cases.

Case 1: When $1 \leq i \neq j \leq m$,

$$\begin{pmatrix} V_{i,j,0,0} \\ V_{i,j,0,1} \\ V_{i,j,1,0} \\ V_{i,j,1,1} \end{pmatrix} (A_i - A_j) = \begin{pmatrix} \lambda_{i,1} V_{i,j,1,0} - \lambda_{j,1} V_{i,j,0,1} \\ \lambda_{i,1} V_{i,j,1,1} - \lambda_{j,0} V_{i,j,0,0} \\ \lambda_{i,0} V_{i,j,0,0} - \lambda_{j,1} V_{i,j,1,1} \\ \lambda_{i,0} V_{i,j,0,1} - \lambda_{j,0} V_{i,j,1,0} \end{pmatrix}$$

i.e., $\text{rank}(A_i - A_j) = \alpha \Leftrightarrow \lambda_{i,0} \lambda_{i,1} \neq \lambda_{j,0} \lambda_{j,1}$.

Case 2: When $m + 1 \leq i < j \leq 3m$, if $j = i + m$, by (9)

$$\begin{aligned} & \begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} (A_i - A_j) \\ &= \begin{pmatrix} \lambda_{i,0} V_{i,0} \\ \lambda_{i,1} V_{i,1} + k_{i-m} V_{i,0} \end{pmatrix} - \begin{pmatrix} \lambda_{j,0} V_{i,0} \\ \lambda_{j,1} V_{i,1} + k_{j-m} V_{i,0} \end{pmatrix} \\ &= \begin{pmatrix} (\lambda_{i,0} - \lambda_{j,0}) V_{i,0} \\ (\lambda_{i,1} - \lambda_{j,1}) V_{i,1} + (k_{i-m} - k_{j-m}) V_{i,0} \end{pmatrix} \end{aligned}$$

i.e., $\text{rank}(A_i - A_j) = \alpha \Leftrightarrow \lambda_{i,s} \neq \lambda_{j,s}$ for $s = 0, 1$; Otherwise,

$$\begin{aligned} & \begin{pmatrix} V_{i,j,0,0} \\ V_{i,j,0,1} \\ V_{i,j,1,0} \\ V_{i,j,1,1} \end{pmatrix} (A_i - A_j) \\ &= \begin{pmatrix} (\lambda_{i,0} - \lambda_{j,0}) V_{i,j,0,0} \\ (\lambda_{i,0} - \lambda_{j,1}) V_{i,j,0,1} - k_{j-m} V_{i,j,0,0} \\ (\lambda_{i,1} - \lambda_{j,0}) V_{i,j,1,0} + k_{i-m} V_{i,j,0,0} \\ (\lambda_{i,1} - \lambda_{j,1}) V_{i,j,1,1} + k_{i-m} V_{i,j,0,1} - k_{j-m} V_{i,j,1,0} \end{pmatrix} \end{aligned}$$

i.e., $\text{rank}(A_i - A_j) = \alpha \Leftrightarrow \lambda_{i,s} \neq \lambda_{j,t}$ for $s, t = 0, 1$.

Case 3: When $1 \leq i \leq m$ and $m + 1 \leq j \leq 3m$, if $j = i + m$

or $i + 2m$, according to (9)

$$\begin{aligned} & \begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} (A_i - A_j) \\ &= \begin{pmatrix} \lambda_{i,1} V_{i,1} \\ \lambda_{i,0} V_{i,0} \end{pmatrix} - \begin{pmatrix} \lambda_{j,0} V_{i,0} \\ \lambda_{j,1} V_{i,1} + k_{j-m} V_{i,0} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_{i,1} V_{i,1} - \lambda_{j,0} V_{i,0} \\ (\lambda_{i,0} - k_{j-m}) V_{i,0} - \lambda_{j,1} V_{i,1} \end{pmatrix}, \end{aligned}$$

i.e., $\text{rank}(A_i - A_j) = \alpha \Leftrightarrow \lambda_{i,1}(\lambda_{i,0} - k_{j-m}) \neq \lambda_{j,0} \lambda_{j,1}$; Otherwise,

$$\begin{aligned} & \begin{pmatrix} V_{i,j,0,0} \\ V_{i,j,0,1} \\ V_{i,j,1,0} \\ V_{i,j,1,1} \end{pmatrix} (A_i - A_j) \\ &= \begin{pmatrix} \lambda_{i,1} V_{i,j,1,0} - \lambda_{j,0} V_{i,j,0,0} \\ \lambda_{i,1} V_{i,j,1,1} - \lambda_{j,1} V_{i,j,0,1} - k_{j-m} V_{i,j,0,0} \\ \lambda_{i,0} V_{i,j,0,0} - \lambda_{j,0} V_{i,j,1,0} \\ \lambda_{i,0} V_{i,j,0,1} - \lambda_{j,1} V_{i,j,1,1} - k_{j-m} V_{i,j,1,0} \end{pmatrix} \end{aligned}$$

i.e., $\text{rank}(A_i - A_j) = \alpha \Leftrightarrow \lambda_{i,0} \lambda_{i,1} \neq \lambda_{j,0}^2, \lambda_{j,1}^2$. \blacksquare

Proof of Theorem 2: \mathcal{C}_1 is a code with the optimal repair property if and only if R2 and R3 hold. Firstly, by means of Lemma 1 we establish the necessary and sufficient conditions for R2 according to the following three cases.

Case 1: For $1 \leq i \leq m$,

(a) When $1 \leq j \neq i \leq m$,

$$\begin{aligned} & \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_j \end{pmatrix} \right) \\ &= \text{rank} \left(\begin{pmatrix} V_{i,j,0,0} \\ V_{i,j,0,1} \\ V_{i,j,0,0} A_j \\ V_{i,j,0,1} A_j \end{pmatrix} \right) \\ &= \text{rank} \left(\begin{pmatrix} V_{i,j,0,0} \\ V_{i,j,0,1} \\ \lambda_{j,1} V_{i,j,0,1} \\ \lambda_{j,0} V_{i,j,0,0} \end{pmatrix} \right) \\ &= \alpha/2, \end{aligned}$$

(b) When $m + 1 \leq j \leq 3m$, if $j = i + m$ or $i + 2m$, by (9)

$$\begin{aligned} & \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_j \end{pmatrix} \right) \\ &= \text{rank} \left(\begin{pmatrix} V_{i,0} \\ V_{i,0} A_j \end{pmatrix} \right) \\ &= \text{rank} \left(\begin{pmatrix} V_{j,0} \\ \lambda_{j,0} V_{j,0} \end{pmatrix} \right) \\ &= \alpha/2; \end{aligned}$$

Otherwise,

$$\begin{aligned}
& \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_j \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} V_{i,j,0,0} \\ V_{i,j,0,1} \\ V_{i,j,0,0} A_j \\ V_{i,j,0,1} A_j \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} V_{i,j,0,0} \\ V_{i,j,0,1} \\ \lambda_{j,0} V_{i,j,0,0} \\ \lambda_{j,1} V_{i,j,0,1} + k_{j-m} V_{i,j,0,0} \end{pmatrix} \right) \\
&= \alpha/2.
\end{aligned}$$

Case 2: For $m+1 \leq i \leq 2m$,

(a) When $1 \leq j \leq m$, if $j = i - m$, by (9)

$$\begin{aligned}
& \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_j \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} V_{j,0} + t_j V_{j,1} \\ (V_{j,0} + t_j V_{j,1}) A_j \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} V_{j,0} + t_j V_{j,1} \\ \lambda_{j,1} V_{j,1} + t_j \lambda_{j,0} V_{j,0} \end{pmatrix} \right) \\
&= \alpha/2 \\
&\Leftrightarrow \lambda_{j,1} = t_j^2 \lambda_{j,0};
\end{aligned}$$

Otherwise,

$$\begin{aligned}
& \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_j \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} V_{i,j,0,0} + t_{i-m} V_{i,j,1,0} \\ V_{i,j,0,1} + t_{i-m} V_{i,j,1,1} \\ (V_{i,j,0,0} + t_{i-m} V_{i,j,1,0}) A_j \\ (V_{i,j,0,1} + t_{i-m} V_{i,j,1,1}) A_j \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} V_{i,j,0,0} + t_{i-m} V_{i,j,1,0} \\ V_{i,j,0,1} + t_{i-m} V_{i,j,1,1} \\ \lambda_{j,1} (V_{i,j,0,1} + t_{i-m} V_{i,j,1,1}) \\ \lambda_{j,0} (V_{i,j,0,0} + t_{i-m} V_{i,j,1,0}) \end{pmatrix} \right) \\
&= \alpha/2.
\end{aligned}$$

(b) When $m+1 \leq j \neq i \leq 3m$, if $j = i + m$, by (9)

$$\begin{aligned}
& \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_j \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} V_{j,0} + t_{i-m} V_{j,1} \\ (V_{j,0} + t_{i-m} V_{j,1}) A_j \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} V_{j,0} + t_{i-m} V_{j,1} \\ (\lambda_{j,0} + t_{i-m} k_{j-m}) V_{j,0} + t_{i-m} \lambda_{j,1} V_{j,1} \end{pmatrix} \right) \\
&= \alpha/2 \\
&\Leftrightarrow \lambda_{j,1} = \lambda_{j,0} + t_{i-m} k_{j-m} \\
&\Leftrightarrow \lambda_{i+m,1} = \lambda_{i+m,0} + t_{i-m} k_i;
\end{aligned}$$

Otherwise,

$$\begin{aligned}
& \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_j \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} V_{i,j,0,0} + t_{i-m} V_{i,j,1,0} \\ V_{i,j,0,1} + t_{i-m} V_{i,j,1,1} \\ (V_{i,j,0,0} + t_{i-m} V_{i,j,1,0}) A_j \\ (V_{i,j,0,1} + t_{i-m} V_{i,j,1,1}) A_j \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} V_{i,j,0,0} + t_{i-m} V_{i,j,1,0} \\ V_{i,j,0,1} + t_{i-m} V_{i,j,1,1} \\ \lambda_{j,0} (V_{i,j,0,0} + t_{i-m} V_{i,j,1,0}) \\ \lambda_{j,1} (V_{i,j,0,1} + t_{i-m} V_{i,j,1,1}) \\ + k_{j-m} (V_{i,j,0,0} + t_{i-m} V_{i,j,1,0}) \end{pmatrix} \right) \\
&= \alpha/2.
\end{aligned}$$

Case 3: For $2m+1 \leq i \leq 3m$, similarly to that of Case 2,

$$\begin{aligned}
& \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_j \end{pmatrix} \right) = \alpha/2 \text{ for } 1 \leq j \neq i \leq 3m \\
&\Leftrightarrow \lambda_{l,1} = \begin{cases} t_{l+m}^2 \lambda_{l,0}, & \text{if } 1 \leq l \leq m, \\ \lambda_{l,0} + t_l k_{l-m}, & \text{if } m+1 \leq l \leq 2m. \end{cases}
\end{aligned}$$

Combing all the cases above, we have that R2 holds if and only if

$$\begin{aligned}
& \lambda_{i,1} = t_i^2 \lambda_{i,0} \text{ for } 1 \leq i \leq m, \\
& t_i^2 = t_{i+m}^2 \text{ for } 1 \leq i \leq m,
\end{aligned} \tag{11}$$

and

$$\lambda_{i,1} = \lambda_{i,0} + t_i k_{i-m}, \quad \lambda_{i+m,1} = \lambda_{i+m,0} + t_{i-m} k_i \tag{12}$$

for $m+1 \leq i \leq 2m$.

Secondly, we determine the necessary and sufficient conditions for R3. It is easy to verify that $\text{rank} \left(\begin{pmatrix} S_i \\ S_i A_i \end{pmatrix} \right) = \alpha$ for $1 \leq i \leq m$. For $m+1 \leq i \leq 3m$,

$$\begin{aligned}
& \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_i \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} V_{i,0} + t_{i-m} V_{i,1} \\ (V_{i,0} + t_{i-m} V_{i,1}) A_i \end{pmatrix} \right) \\
&= \text{rank} \left(\begin{pmatrix} V_{i,0} + t_{i-m} V_{i,1} \\ (\lambda_{i,0} + t_{i-m} k_{i-m}) V_{i,0} + t_{i-m} \lambda_{i,1} V_{i,1} \end{pmatrix} \right) \\
&= \alpha \\
&\Leftrightarrow \lambda_{i,1} \neq \lambda_{i,0} + t_{i-m} k_{i-m},
\end{aligned}$$

which together with (12) gives $t_j \neq t_{j+m}$ for any $1 \leq j \leq m$, and further, associated with (11) implies that $t_j = -t_{j+m}$ for all $1 \leq j \leq m$ and \mathbf{F}_q should be a finite field of odd characteristic. This finishes the proof. ■

Proof of Theorem 3: It is easy to verify that R4 and R5 are satisfied for the first m nodes due to (8) and the fact that $\{e_0, \dots, e_{2^m-1}\}$ is the standard basis. ■

Proof of Theorem 4: We only prove item (iii) of Theorem 1 hereafter since the other items of Theorems 1 and 2 can be easily verified.

Given two integers $1 \leq i \leq m$ and $m+1 \leq j \leq 3m$, if $j \equiv i \pmod{m}$, then

$$\lambda_{i,1} (\lambda_{i,0} - k_{j-m}) = \gamma^i (\gamma^i + 2\gamma^i) \neq -\gamma^{2i} = \lambda_{j,0} \lambda_{j,1}$$

since $4\gamma^i \neq 0$; Otherwise, define $j' = j - lm$ where $lm + 1 \leq j \leq (l + 1)m$ for $1 \leq l \leq 2$, i.e., $1 \leq j' \neq i \leq m$, then we have

$$\lambda_{i,0}\lambda_{i,1} = \gamma^{2i} \neq \gamma^{2j'} = \lambda_{j,s}^2 \text{ for } s = 0, 1.$$

Thus, item (iii) of Theorem 1 is satisfied. ■

Proof of Theorem 9: \mathcal{C}_3 has the MDS property if and only if R1 holds. In what follows, we only prove it for the case that $1 \leq i \leq m, m + 1 \leq j \leq 2m$. The other cases can be proven similarly as those of Cases 1-2 in the proof of Theorem 1.

When $1 \leq i \leq m$ and $m + 1 \leq j \leq 2m$, if $j = i + m$, by (9) we have

$$\begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} (A_i - A_j) = \begin{pmatrix} \lambda_{i,1}V_{i,1} - \lambda_{j,0}V_{i,0} \\ \lambda_{i,0}V_{i,0} - \lambda_{j,1}V_{i,1} \end{pmatrix},$$

which together with Lemma 1 gives

$$\text{rank}(A_i - A_j) = \alpha \Leftrightarrow \lambda_{i,1}\lambda_{i,0} \neq \lambda_{j,0}\lambda_{j,1};$$

Otherwise,

$$\begin{pmatrix} V_{i,j,0,0} \\ V_{i,j,0,1} \\ V_{i,j,1,0} \\ V_{i,j,1,1} \end{pmatrix} (A_i - A_j) = \begin{pmatrix} \lambda_{i,1}V_{i,j,1,0} - \lambda_{j,0}V_{i,j,0,0} \\ \lambda_{i,1}V_{i,j,1,1} - \lambda_{j,1}V_{i,j,0,1} \\ \lambda_{i,0}V_{i,j,0,0} - \lambda_{j,0}V_{i,j,1,0} \\ \lambda_{i,0}V_{i,j,0,1} - \lambda_{j,1}V_{i,j,1,1} \end{pmatrix}$$

associated with Lemma 1, which implies that

$$\text{rank}(A_i - A_j) = \alpha \Leftrightarrow \lambda_{i,0}\lambda_{i,1} \neq \lambda_{j,0}^2, \lambda_{j,1}^2.$$

Proof of Theorem 10: \mathcal{C}_3 is a code with the optimal repair property if and only if R2 and R3 hold. For $m + 1 \leq i \leq 2m$, we have

$$\begin{aligned} \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_i \end{pmatrix} \right) &= \text{rank} \left(\begin{pmatrix} V_{i,0} + t_{i-m}V_{i,1} \\ \lambda_{i,0}V_{i,0} + \lambda_{i,1}t_{i-m}V_{i,1} \end{pmatrix} \right) \\ &= \alpha \\ &\Leftrightarrow \lambda_{i,0} \neq \lambda_{i,1}. \end{aligned}$$

The analysis for the remainder cases are omitted herein since they are similar to those of \mathcal{C}_1 . ■

Proof of Theorem 12: Since the other items of Theorems 9 and 10 can be easily satisfied, we only verify item (iii) of Theorem 9 herein.

Given two integers $1 \leq i \leq m$ and $m + 1 \leq j \leq 2m$, define $j' = j - m$. Obviously, $1 \leq j' \leq m$. If $j' = i$, we have $\lambda_{i,0}\lambda_{i,1} = \gamma^{2i} \neq \gamma^{\lfloor \frac{q}{2} \rfloor + 2i} = \lambda_{j,0}\lambda_{j,1}$; Otherwise,

$$\frac{\lambda_{j,0}^2}{\lambda_{i,0}\lambda_{i,1}} = \frac{\gamma^{2j'}}{\gamma^{2i}} = \gamma^{2j'-2i} \neq 1,$$

and

$$\frac{\lambda_{j,1}^2}{\lambda_{i,0}\lambda_{i,1}} = \gamma^{2\lfloor \frac{q}{2} \rfloor + 2j'-2i} = \begin{cases} \gamma^{2j'-2i}, & q \text{ odd} \\ \gamma^{2j'-2i+1}, & q \text{ even} \end{cases} \neq 1$$

where we use the facts that $1 \leq |2j' - 2i + 1| \leq 2m - 1 \leq q - 2$ and $\gamma^l = 1$ if and only if $l \equiv 0 \pmod{q-1}$. Thus, item (iii) of Theorem 9 is satisfied. ■

Proof of Theorem 13: \mathcal{C}_4 has the MDS property if and only if R1 holds.

When $1 \leq i < j \leq k$, if $j \neq i + m$, similarly as Case 1 in the proof of Theorem 1, we have

$$\text{rank}(A_i - A_j) = \alpha \Leftrightarrow \lambda_{i,0}\lambda_{i,1} \neq \lambda_{j,0}\lambda_{j,1};$$

Otherwise, by (9) we have

$$\begin{aligned} \begin{pmatrix} V_{i,0} \\ V_{i,1} \end{pmatrix} (A_i - A_j) &= \begin{pmatrix} \lambda_{i,1}V_{i,1} \\ \lambda_{i,0}V_{i,0} \end{pmatrix} - \begin{pmatrix} \lambda_{j,1}V_{i,1} \\ \lambda_{j,0}V_{i,0} \end{pmatrix} \\ &= \begin{pmatrix} (\lambda_{i,1} - \lambda_{j,1})V_{i,1} \\ (\lambda_{i,0} - \lambda_{j,0})V_{i,0} \end{pmatrix}, \end{aligned}$$

which together with Lemma 1 implies

$$\text{rank}(A_i - A_j) = \alpha \Leftrightarrow \lambda_{i,s} \neq \lambda_{j,s} \text{ for } s = 0, 1. \quad \blacksquare$$

Proof of Theorem 14: \mathcal{C}_4 is a code with the optimal repair property if and only if R2 and R3 hold.

For $1 \leq i \leq k$, we have

$$\begin{aligned} \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_i \end{pmatrix} \right) &= \text{rank} \left(\begin{pmatrix} V_{i,0} + t_i V_{i,1} \\ \lambda_{i,1} V_{i,1} + t_i \lambda_{i,0} V_{i,0} \end{pmatrix} \right) \\ &= \alpha \\ &\Leftrightarrow \lambda_{i,1} \neq t_i^2 \lambda_{i,0}. \end{aligned}$$

Similar to Case 2(a) in the proof of Theorem 2, we can get

$$\begin{aligned} \text{rank} \left(\begin{pmatrix} S_i \\ S_i A_j \end{pmatrix} \right) &= \frac{\alpha}{2} \text{ for any } 1 \leq i \neq j \leq k \\ \Leftrightarrow \lambda_{i,1} &= t_{i+m}^2 \lambda_{i,0}, \lambda_{i+m,1} = t_i^2 \lambda_{i+m,0} \text{ for all } 1 \leq i \leq m. \quad \blacksquare \end{aligned}$$

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