About Adaptive Coding on Countable Alphabets: Max-Stable Envelope Classes

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Abstract—In this paper, we study the problem of lossless universal source coding for stationary memoryless sources on countably infinite alphabets. This task is generally not achievable without restricting the class of sources over which universality is desired. Building on our prior work, we propose natural families of sources characterized by a common dominating envelope. We particularly emphasize the notion of adaptivity, which is the ability to perform as well as an oracle knowing the envelope, without actually knowing it. This is closely related to the notion of hierarchical universal source coding, but with the important difference that families of envelope classes are not discretely indexed and not necessarily nested.

Our contribution is to extend the classes of envelopes over which adaptive universal source coding is possible, namely by including max-stable (heavy-tailed) envelopes which are excellent models in many applications, such as natural language modeling. We derive a minimax lower bound on the redundancy of any code on such envelope classes, including an oracle that knows the envelope. We then propose a constructive code that does not use knowledge of the envelope. The code is computationally efficient and is structured to use an Expanding Threshold for Auto-Censoring, and we therefore dub it the ETAC-code. We prove that the ETAC-code achieves the lower bound on the minimax redundancy within a factor logarithmic in the sequence length, and can be therefore qualified as a near-adaptive code over families of heavy-tailed envelopes. For finite and lighttailed envelopes the penalty is even less, and the same code follows closely previous results that explicitly made the lighttailed assumption. Our technical results are founded on methods from regular variation theory and concentration of measure.

Keywords: countable alphabets; redundancy; adaptive compression; minimax;

I. INTRODUCTION

The problem we address here is that of *coding* a finite sequence of symbols $x_{1:n} = x_1, ..., x_n$, taking values in an (at most) countably infinite *alphabet* \mathcal{X} . A *lossless binary source code* (or *code* for short) is a one-to-one map from finite sequences of symbols in \mathcal{X} of any possible length n to finite sequences of binary $\{0, 1\}$ symbols.

We model sequences as being generated by a *source*, defined as a probability measure $\mathbb{P} \in \mathfrak{M}_1(\mathcal{X}^{\mathbb{N}})$ on the set of infinite sequences of symbols from \mathcal{X} . We work primarily, for a given n, with the finite restriction \mathbb{P}_n of this probability measure. That is, $\mathbb{P}_n(x_{1:n})$ is the probability of the first n symbols of the random sequence, written $X_{1:n} = X_1, ..., X_n$, being equal to $x_{1:n}$. We focus entirely on *stationary memoryless* sources, where $\mathbb{P} = \mathbb{P}_1^{\mathbb{N}}$ is a product measure for some $\mathbb{P}_1 \in \mathfrak{M}_1(\mathcal{X})$ called the marginal, itself a probability measure on \mathcal{X} . A stationary memoryless source therefore generates independent and identically distributed sequences of symbols. Given a source, the task of source coding is to minimize the *expected* codelength:

$$\mathbb{E}[\ell(X_{1:n})] = \sum_{x_{1:n} \in \mathcal{X}^n} \mathbb{P}_n(x_{1:n})\ell(x_{1:n})$$

By the source coding theorem, the Shannon entropy of the source

$$H(\mathbb{P}_n) = -\sum_{x_{1:n} \in \mathcal{X}^n} \mathbb{P}_n(x_{1:n}) \log \mathbb{P}_n(x_{1:n})$$

is a lower bound to the expected codelength of any lossless binary code. (Here and throughout the paper, log denotes the base-2 logarithm). Therefore, one way to measure the performance of any particular code is by its *expected redundancy*, defined as the excess expected length $\mathbb{E}[\ell(X_{1:n})] - H(\mathbb{P}_n)$. This is meaningful when $\frac{1}{n}H(\mathbb{P}_n) = H(\mathbb{P}_1) < \infty$, which we assume to be the case throughout.

In this paper, in addition to having to deal with infinite alphabets, we are particularly interested in coding that performs well over a *source class* Λ , with a common alphabet \mathcal{X} , defined as a collection of various probability distributions \mathbb{P} on $\mathcal{X}^{\mathbb{N}}$. We write Λ_n to denote the restriction $\{\mathbb{P}_n : \mathbb{P} \in \Lambda\}$ of Λ to distributions on the first *n* symbols. We now move on to elaborate the classical notion of universality with respect to a source class and then the notion of adaptivity with respect to collections of source classes. We first pass through some more basics about source coding, and we end with a summary of our contributions and an outline the structure of the paper. We use the introduction as a means to introduce all the main notation used in the rest of the paper.

A. Universal source coding

A code is uniquely decodable if any concatenation of codewords can be parsed into codewords in a unique way. The Kraft-McMillan inequality asserts that for a uniquely decodable code over \mathcal{X}^* , the codelength map $x_{1:n} \mapsto \ell(x_{1:n})$ satisfies $\sum_n \sum_{x_{1:n} \in \mathcal{X}^n} 2^{-\ell(x_{1:n})} \leq 1$, and that conversely, given codelengths that satisfy such an inequality, there exists a corresponding uniquely decodable code. The Kraft-McMillan inequality also establishes a deeper correspondence, one between codes over \mathcal{X}^n and probability distributions over \mathcal{X}^n : (after normalization) $x_{1:n} \mapsto 2^{-\ell(x_{1:n})}$ defines a

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probability distribution over \mathcal{X}^n , conversely, arithmetic coding (Rissanen and Langdon, 1979; Cover and Thomas, 1991) allows us to design uniquely decodable codes from any probability distribution over \mathcal{X}^n . Therefore we may refer to an arbitrary probability distribution $Q_n \in \mathfrak{M}_1(\mathcal{X}^n)$ as a *coding distribution* (Cover and Thomas, 1991).

The correspondence between uniquely decodable codes and probability distributions allows us to describe redundancy as a statistical risk. Indeed, the expected code length of a coding distribution Q_n is $\mathbb{E}_{\mathbb{P}} \left[-\log Q_n(X_{1:n}) \right]$, its expected redundancy, when the source is \mathbb{P} , is the Kullback-Leibler divergence (or relative entropy) between \mathbb{P}_n and Q_n :

$$D(\mathbb{P}_n, Q_n) = \sum_{x_{1:n} \in \mathcal{X}^n} \mathbb{P}_n(x_{1:n}) \log \frac{\mathbb{P}_n(x_{1:n})}{Q_n(x_{1:n})}$$
$$= \mathbb{E}_{\mathbb{P}_n} \left[\log \frac{\mathbb{P}_n(X_{1:n})}{Q_n(X_{1:n})} \right].$$

Some authors also call it the *cumulative entropy risk* (Haussler and Opper, 1997).

The theoretically optimal coding probabilities are given by the source \mathbb{P}_n itself. And by using methods such as arithmetic coding, codes corresponding to \mathbb{P}_n can be designed to have a redundancy that remains bounded by 1 for all n.

In universal coding, one attempts to construct a coding distribution Q_n that achieves low redundancy across an entire source class Λ , without knowing in advance which $\mathbb{P} \in \Lambda$ is actually generating the sequence. Such a construction is called coding with respect to Λ .

To assess a code with respect to a source class, we may adopt one of many perspectives for gauging performance. Here, we study the *maximal redundancy* defined as:

$$R^+(Q_n, \Lambda_n) = \sup_{\mathbb{P} \in \Lambda} D(\mathbb{P}_n, Q_n)$$

which is essentially as high as the redundancy could grow if \mathbb{P} is chosen adversarially at every *n*. Studying this is a way to capture our complete lack of information about which distribution generates the sequence.

The maximal redundancy establishes a uniform rate at which the redundancy grows. The infimum of $R^+(Q_n, \Lambda_n)$ over all Q_n , is called the *minimax redundancy* of Λ :

$$R^{+}(\Lambda_{n}) = \inf_{Q_{n} \in \mathfrak{M}_{1}(\mathcal{X}^{n})} R^{+}(Q_{n}, \Lambda_{n}).$$

The minimax redundancy is a property of the source class Λ and represents the best a code could hope for in terms of a guaranteed expected redundancy over the class Λ .

Universal coding schemes such as the mixture codes developed for memoryless or Markov sources over finite alphabets (Krichevsky and Trofimov, 1981; Shtarkov, 1987; Ryabko, 1990, 1984; Xie and Barron, 2000; Barron et al., 1998; Xie and Barron, 1997; Clarke and Barron, 1994, 1990; Willems, 1998; Gassiat, 2014) have small and well understood maximal redundancies. In the simplest setting, that is when considering stationary memoryless sources over a finite alphabet with size d, the minimax redundancy scales like $(d - 1)/2 \log n$ as the sequence length n tends to infinity. In the language of statistics, classes of sources that can be parametrized by compact subsets of Euclidean spaces are said to be *parametric*. The techniques advocated in the aforementioned references are said to *asymptotically achieve the minimax risk* over the source class in this parametric setting. This is a step beyond strong universality, since the best redundancy decay rate is achieved, and is the notion we strive for in the paper.

B. Adaptive source coding

Although adaptivity became a major topic in mathematical statistics in the beginning of the early 1990's (see Bickel et al., 1998; Tsybakov, 2004; Korostelev and Korosteleva, 2011, and references therein), the expression adaptive coding barely shows up in articles dedicated to lossless source coding. Source coding research has been mostly concerned with universality. As we have outlined, given a possibly very large collection of sources, a universal code attempts to minimize redundancy, that is the difference between the expected codeword length and the expected codeword length that would be achieved by a code tailored to the source. Adaptive coding considers a more general setting: given a collection of source classes, such that for each class, a good universal coder is available, is it possible to engineer a single coding method that performs well over all classes in the collection?

In the parlance of mathematical statistics, an estimator is said to be *adaptive* over a collection of models or classes if it achieves or at least approaches the minimax risk over all models simultaneously. If we slice the collection of sources of unbounded finite Markov order over a finite alphabet into classes of sources of Markov order $k \in \mathbb{N}$, it is well known that Lempel-Ziv coders are not adaptive (see Jacquet et al., 2001; Jacquet and Szpankowski, 1995; Louchard and Szpankowski, 1997; Savari, 1997; Szpankowski, 2001; Gassiat, 2014, and references therein), whereas adaptivity can be achieved over classes of sources of Markov order $k \in \mathbb{N}$ as shown by Ryabko (1984) who calls adaptivity twice-universality. Such adaptivity is also sometimes called hierarchical universality (Merhav and Feder, 1998), which refers mostly to nested discretely indexed classes. Context-tree weighting is also adaptive (Catoni, 2004), and we mention this text individually because it is one of the very few on compression that adopts explicitly the adaptive estimation perspective.

There are in fact different flavors of adaptivity in the statistics literature. The textbooks (Korostelev and Korosteleva, 2011; Tsybakov, 2004; Bickel, Klaassen, Ritov, and Wellner, 1998) define more or less stringent notions of asymptotic adaptivity in the minimax sense. We tune these notions in order to accommodate the context of data compression. Let $(\Lambda(\mu))$ be a collection of source classes indexed by $\mu \in \mathcal{M}$, where \mathcal{M} is not a necessarily nested or even a discrete set of indices. A sequence $(Q_n)_n$ of coding probabilities is said to be *asymptotically adaptive* with respect to a collection $(\Lambda(\mu))_{\mu \in \mathcal{M}}$ of source classes if for all $\mu \in \mathcal{M}$:

 $\mathbf{D}^{\perp}(\mathbf{O}, \mathbf{I}, \mathbf{I})$

$$R^+(Q_n, \Lambda_n(\mu)) = \sup_{\mathbb{P} \in \Lambda(\mu)} D(\mathbb{P}_n, Q_n) \le (1 + o_\mu(1))R^+(\Lambda_n(\mu))$$
(1)

as *n* tends to infinity. If the inequality (1) holds with a factor other than $(1 + o_{\mu}(1))$ (that may depend on μ) larger than 1 to the right, then we say that there is adaptivity *within* this factor. Note that Q_n cannot depend on μ or else the problem is simply one of universality. Bontemps, Boucheron, and Gassiat (2014) describe such an asymptotically adaptive compression scheme for a non-trivial (though restrictive) collection of nonparametric source classes. In order to develop such an adaptive compression technique, it is useful to know the minimax redundancy in each class.

For large collections of massive source classes like the ones we will be handling in this paper, minimax redundancy may itself grow much faster than any such logarithm. In this case, the logarithmic factor may be meaningfully ignored. In order to accommodate this situation, we present a less stringent criterion of adaptivity. We introduce the following terminology: we call $(Q_n)_n$ asymptotically near-adaptive with respect to a collection $(\Lambda(\mu))_{\mu \in \mathcal{M}}$ of source classes if for all $\mu \in \mathcal{M}$:

$$R^+(Q_n, \Lambda_n(\mu)) \le O_\mu(\log n) R^+(\Lambda_n(\mu)), \qquad (2)$$

where the constants involved in the $O_{\mu}(\log n)$ term may depend on the source class $\Lambda(\mu)$. Therefore near-adaptivity is adaptivity within a logarithmic factor in the sequence length.

Adaptive source coding raises several challenges: explicit model selection as in (Barron, Birgé, and Massart, 1999) and source parameter estimation as in two-steps coding schemes (Rissanen, 1984; Barron, Rissanen, and Yu, 1998) should be avoided so as to make online coding and decoding possible; coding and decoding should be computationally efficient and if possible feasible in linear time. Therefore, in addition to striving to achieve the impressive success of Bayesian coding schemes with respect to parametric classes of sources as demonstrated in the aforementioned papers by Barron *et al.*, we also strive to produce efficient near-adaptive universal codes for large collections of source classes.

C. Contributions and organization of the paper

Unlike the mostly finite-alphabet results referred to so far, this paper is concerned with adaptive coding over a countably infinite alphabet \mathcal{X} (say the set of positive integers \mathbb{N}_+ or the set of integers \mathbb{N}) as described for example in Kieffer (1978); Gyorfi et al. (1993); Foster et al. (2002); Orlitsky and Santhanam (2004); Ryabko et al. (2008); Boucheron et al. (2009); Garivier (2009); Bontemps (2011); Gassiat (2014); Bontemps et al. (2014). This does not preclude the finite-alphabet case, which becomes a special instance.

When coping with a countably infinite alphabet, even if the source statistics is known, establishing the existence of optimal codes is non-trivial (Linder et al., 1997). More importantly, when we consider universal coding over the class of sources on countably infinite alphabets, even weak universality is not achievable. This was developed in a sequence of papers, starting with early negative results on stationary sources by Kieffer (1978), and then also established for memoryless sources by Gyorfi, Pali, and van der Meulen (1993; 1994). More recently, delicate asymptotic results for coding over large finite alphabets with unknown size have started to appear (Orlitsky and Santhanam, 2004; Szpankowski and Weinberger, 2012; Yang and Barron, 2013), balancing various finite alphabet sizes and sequence lengths.

This general difficulty prompted Boucheron, Garivier, and Gassiat (2009) to first study the redundancy of specific memoryless source classes, namely classes defined by an envelope function. Offline coding techniques for the collection of source classes defined by algebraically vanishing envelopes were introduced in (Boucheron, Garivier, and Gassiat, 2009). Bontemps (2011) designed and analyzed the AC-code (auto-censuring code). The AC-code has a straightforward structure, it uses a sequence of Krichevsky-Trofimov coders with progressively growing alphabets determined by a threshold that is the maximum of all symbols seen so far: the $i^{\rm th}$ symbol is either encoded using Krichevsky-Trofimov mixture encoding for alphabet $\{0, \ldots, \max(x_1, \ldots, x_{i-1})\}$, or Elias penultimate encoding if it is the new maximum. Bontemps proved that this simple code is *adaptive* over the union of classes of sources with exponentially decreasing envelopes. Bontemps, Boucheron, and Gassiat (2014) revisited and simplified Bontemps's techniques, and proved moreover that the AC-code is actually adaptive in the sense of (1) over all classes of sources defined by sub-exponentially decreasing envelopes, that is, envelopes with non-decreasing hazard rate. The AC-code achieves two unexpected benefits: on a practical front it is an online encoding and decoding technique, and on a theoretical front it shows that an effective threshold can be constructed driven by data. The analysis of the AC-code in (Bontemps, Boucheron, and Gassiat, 2014) does not depend on the precise shape of the envelope but strongly benefits from the insights of extreme value theory (EVT) (Falk, Husler, and Reiss, 2011; de Haan and Ferreira, 2006; Beirlant, Goegebeur, Teugels, and Segers, 2004; Resnick, 1987) as the minimax redundancy rate of the classes investigated therein asymptotically depends on the slow variation property of the quantile function of the envelope distribution.

A major question that was left open in this work, however, was: is the same adaptivity possible with the much larger class of heavy-tailed envelopes? These envelopes occur often in practice, and are a distinctive property of text and natural language, domains where compression is used extensively. In this paper, we answer this question in the affirmative.

In Section II, we properly define such heavy-tailed envelope classes. Using the language of EVT, these correspond to *Fréchet max-stable* distributions, and are best expressed using the notion of regular variation. In Section III, we give the construction of the ETAC-code, short for expanding threshold auto-censoring code. This is a new computationally efficient code, which builds on the same principle of the AC-code, but uses a new data-driven threshold that expands near the tail of the distribution rather than staying at the maximum. Its thresholding strategy can be summarized in the following way: symbols that are larger than the current threshold tend to be rare for that sequence length and they are encoded using a fixed naive encoder, whereas smaller symbols tend to occur

more frequently and they are encoded using the asymptotically maximin Krichevsky-Trofimov encoders tailored to the effective alphabet defined by symbols smaller than the threshold.

In Section IV, we provide an overview of our main results. The major contribution is the treatment of heavy-tailed envelopes:

(**Theorem 4**). Over the collection of Fréchet max-stable envelope source classes, the ETAC-code asymptotically achieves the lower bound on the minimax redundancy within a factor logarithmic in the sequence length, and can be therefore qualified as near-adaptive.

We also show that for finite and light-tailed envelopes investigated in (Bontemps et al., 2014), the same code recovers the adaptivity properties of the AC-code, up to an even slower (roughly $\log \log n$) factor. These results require a lower bound on the minimax redundancy for Fréchet max-stable envelope classes, which we give in Section V, and a detailed analysis of the ETAC-code, which we perform in Section VI.

The proof techniques combine traditional approaches from information theory (Ryabko, 1984; Foster et al., 2002; Györfi et al., 1994; Gyorfi et al., 1993; Gassiat, 2014), regular variation arguments (Karlin, 1967; Bingham et al., 1989; Ohannessian and Dahleh, 2012a,b; Ben-Hamou et al., 2014), as well as concentration inequalities (Boucheron et al., 2013). We collect most of this technical content and proofs within the Appendix.

II. ENVELOPE CLASSES

We start with the basic definition of an envelope source class.

Definition 1 (ENVELOPE SOURCE CLASSES). Let f be a mapping from \mathbb{N}_+ to (0,1], with $1 < \sum_{j \in \mathbb{N}_+} f(j) < \infty$. The envelope class $\Lambda(f)$ defined by the function f is the collection of stationary memoryless sources with first marginal distribution dominated by f:

$$\Lambda(f) = \left\{ \mathbb{P} : \forall j \in \mathbb{N}_+, \ \mathbb{P}_1\{j\} \le f(j) , \\ and \ \mathbb{P} \text{ is stationary and memoryless.} \right\}.$$

Envelope classes can be associated with a cumulative distribution, which we call the *envelope distribution*, defined as follows.

Definition 2 (ENVELOPE DISTRIBUTION). Let f be an envelope function. The associated envelope distribution F is defined as

$$F(k) = 1 - \sum_{j > k} f(j)$$

for all k such that $\sum_{j>k} f(j) < 1$, and 0 otherwise. The tail envelope function is then defined as the survival function $\overline{F} = 1 - F$.

Note that the associated probability mass function is equal to F at $\max\{k: \sum_{j\geq k} f(j) \geq 1\}$ and does fall below f. It coincides with f beyond that point and is zero before it.

It is convenient to define a continuous version of the envelope distribution, as follows:

Definition 3 (SMOOTHED ENVELOPE DISTRIBUTION). Let f be an envelope function, and let F be its associated envelope distribution. The smoothed envelope distribution F_c is a cumulative distribution function on \mathbb{R}_+ such that:

- (i) F_c coincides with F on \mathbb{N} , and
- (ii) F_c has a continuous derivative on \mathbb{R}_+ , this derivative is positive at all x such that $F(|x|) \in (0, 1)$..

Since F_c is effectively an extension of F, we allow ourselves to use the F notation to also refer to the smoothed envelope distribution, and mostly avoid the explicit *c*-subscript notation.

For intuition, note that if Y_c is distributed according to the smoothed envelope distribution F_c then $Y = \lceil Y_c \rceil$ is distributed according to the envelope distribution F. We do not elaborate on the existence of the smoothed envelope distribution, as explicit constructions may be given by various interpolation methods. We simply remark that point (ii) in Definition 3 is feasible since envelopes are taken to be strictly positive in Definition 1.

A. Tail Properties and Regular Variation

In general, we would like to accommodate a large variety of models, yet we do not want models that are too exotic, for both theoretical reasons of tractability and practical reasons of plausibility. With this motivation, we focus on envelope functions that enjoy favorable tail properties. We do this by using the machinery of regular variation and extreme value theory (EVT for short). We start with an implicit tail property, but then give a portemanteau theorem that makes explicit various equivalences.

Definition 4 (MAXIMUM DOMAIN OF ATTRACTION). A (smoothed) distribution function F over \mathbb{R} belongs to a maximum domain of attraction (MDA) if there exist sequences $(B_n)_n$ and $(A_n)_n$ with $A_n > 0$ and a non-degenerate distribution function GEV such that $\lim_{n\to\infty} F^n (A_n x + B_n) =$ GEV(x) for all $x \in \mathbb{R}$ where GEV is continuous, or equivalently if the sequence of distribution functions of $(\max(X_1, \ldots, X_n) - B_n)/A_n$ converges pointwise to GEV at every point where GEV is continuous.

There is in fact much more to belonging to a maximum domain of attraction than this weak (in-law) convergence of rescaled and recentered maxima, and some of this is relevant to adaptive compression as we shall see. Therefore we elaborate more aspects of this property, starting with the fundamental theorem of EVT (see de Haan and Ferreira, 2006, Chapter I). For this, we need to define the following quantities. For all $\gamma \in \mathbb{R}$, let $\text{GEV}_{\gamma}(x) = \exp\left(-(1+\gamma x)^{-1/\gamma}\right)$ for x such that $1+\gamma x > 0$ (with the convention $\text{GEV}_0(x) = \exp(-\exp(-x))$). Given a continuous strictly increasing distribution function F, let the function $U: [1, \infty) \to \mathbb{R}$ be a shorthand for the (1 - 1/t)-quantile of F, that is:

$$U(t) = F^{-1} \left(1 - 1/t \right) = \overline{F}^{-1} (1/t).$$
(3)

Theorem 1 (FUNDAMENTAL THEOREM OF EXTREME VALUE THEORY). Given a distribution function F on \mathbb{R} , the following properties are equivalent:

- (i) F belongs to a maximum domain of attraction.
- (ii) There exist $\gamma \in \mathbb{R}$ and a positive scaling sequence $(A_n)_n$ such that

$$\lim_{n \to \infty} F^n \left(A_n x + U(n) \right) = \operatorname{GEV}_{\gamma}(x)$$

for all x such that $1 + \gamma x \ge 0$. This is abbreviated by $F \in MDA(\gamma)$.

(iii) Conditional excess distributions converge weakly toward a generalized Pareto distribution, that is, there exist $\gamma \in \mathbb{R}$ and a positive scaling function σ such that for all x > 0

$$\lim_{t \neq U(\infty)} \frac{\overline{F}(t + \sigma(t)x)}{\overline{F}(t)} = -\log \operatorname{GEV}_{\gamma}(x)$$

(iv) The function U has the extended regular variation property, that is, there exists a non-negative measurable function a on $(1, \infty)$ such that for all x > 0, $\lim_{t \neq \infty} \frac{U(tx) - U(t)}{a(t)}$ exists. In that case there exists $\gamma \in \mathbb{R}$ such that the limit is $\int_{1}^{x} y^{\gamma-1} dy$. This is abbreviated by $U \in \text{ERV}(\gamma)$.

Clause (ii) is known as the Fisher-Tippett-Gnedenko Theorem, clause (iii) as the Balkema-de Haan Theorem, and clause (iv) is due to de Haan. The extreme value index γ in clauses (ii), (iii), and (iv) is the same. The auxiliary function a in (iv) and σ in clause (iii) may be related by choosing $\sigma(t)$ as $a(1/\overline{F}(t))$.

We see therefore that, up to shifting and scaling, the limiting distributions GEV of Definition 4 can only be of the form GEV_{γ}, called *generalized extreme value* distributions. Random variables which have limiting distribution GEV_{γ} produce others who do too under the max operation, which is why such distributions are called *max-stable*. These distributions are known as *Fréchet* when $\gamma > 0$, *Gumbel* when $\gamma = 0$, or *Weibull* when $\gamma < 0$.

Note that in this text, we mostly consider envelopes with unbounded support, that is we assume that $U(\infty) = \lim_{t\to\infty} U(t)$ is infinite. In particular, this means that $\gamma \ge 0$, and we only deal with Fréchet and Gumbel limits. In this framework, U has the more basic *regular variation* property: it is regularly varying with index γ , which we write as $U \in RV(\gamma)$. That is, we have:

$$\forall x > 0, \lim_{t \to +\infty} \frac{U(tx)}{U(t)} = x^{\gamma}.$$

Moreover, if $\gamma > 0$, we can choose γU as the auxiliary function a witnessing $U \in \text{ERV}(\gamma)$ in clause (iv) above. See Appendix A for more on regular variation.

B. Max-Stable Envelope Classes

We are now in position to define the tail properties that we assume for our envelopes. In particular, the smoothed envelope distributions we are interested in belong to some maximum domain of attraction (MDA).

Definition 5 (MAX-STABLE ENVELOPE CLASSES). The envelope class $\Lambda(f)$ with corresponding smoothed envelope distribution function F is said to be a max-stable envelope

class if $F \in MDA(\gamma)$ for some $\gamma \ge 0$. F is said to be a Fréchet (heavy-tailed) envelope if $\gamma > 0$ and to be a Gumbel (light-tailed) envelope if $\gamma = 0$.

Bontemps (2011) and Bontemps et al. (2014) considered a strict subset of the set of Gumbel envelopes. In this paper we consider such envelopes more generally, but more fundamentally, we also include the class of Fréchet envelopes. Fréchet envelopes generalize pure power-law envelopes investigated in (Boucheron et al., 2009). Indeed, assuming that the class admits a Fréchet envelope is equivalent to assuming that the smoothed envelope distribution F is such that \overline{F} is regularly varying with index $-1/\gamma$ ($\overline{F} \in RV(-1/\gamma)$):

$$\forall y \le 1, \quad \lim_{t \to +\infty} \frac{\overline{F}(ty)}{\overline{F}(t)} = y^{-1/\gamma}$$

This amounts to there existing a slowly varying function L, that is $L \in \mathrm{RV}(0)$ (see Appendix A), such that $\overline{F}(x) = x^{-1/\gamma}L(x)$.

The max-stability assumption in the definition of this class of sources is instrumental in both the derivation of the minimax redundancy lower bound and the derivation of the upper bound on the redundancy of the ETAC-code.

III. THE ETAC-CODE

To motivate the construction of the new code, we recall the following theorem from (Boucheron, Garivier, and Gassiat, 2009), which provides an upper-bound on the minimax redundancy of envelope classes and suggests a general design principle for adaptive coding over a collection of envelope classes.

Theorem 2 (MINIMAX REDUNDANCY UPPER BOUND). If $\Lambda(f)$ is an envelope class of memoryless sources, with the tail envelope function \overline{F} then:

$$R^+(\Lambda_n) \le \inf_{u:u \le n} \left[n\overline{F}(u)\log e + \frac{u-1}{2}\log n \right] + 2.$$

Boucheron, Garivier, and Gassiat (2009) also describe settings where this redundancy upper bound is matched by a corresponding lower bound (possibly within a factor of $\log n$). According to Theorem 2, a threshold u_n should be chosen so as to balance the two terms in the upper bound to have the same growth rate. In particular, the rule of thumb that is evident is to choose u_n such that both of these terms are equal: $n\overline{F}(u_n)\log e \approx (u_n-1)/2 \log n$. In the ideal scenario where the envelope distribution is known, this rule of thumb may be combined with known techniques to obtain a code achieving the redundancy upper bound described by Theorem 2. Namely, these techniques consist of arithmetic coding under the envelope distribution in order to encode symbols larger than the threshold u_n , and to encode the sequence of symbols smaller than the threshold u_n using a Krichevsky-Trofimov mixture for alphabet $\{0, 1, \ldots, u_n\}$ (see Boucheron, Garivier, and Gassiat, 2009, for details).

When the envelope distribution is not known and we strive for adaptivity, one is tempted to replace it with an empirical counterpart. Interestingly, the AC-code, which does operate without knowledge of the envelope, *does not* choose the threshold suggested by Theorem 2 and uses instead the maximum. The ETAC-code which we propose here does get closer to this principle: by dropping the constants and the $\log n$ term, we choose a threshold m_n such that $n\overline{F}(m_n) \approx m_n$. Since such a threshold obviously depends on the source class, we construct a threshold from the data to empirically mimic m_n .

A. Construction of the ETAC-Code

The ETAC encoder (Algorithm 1) uses an arithmetic encoder (Rissanen and Langdon, 1979) and a penultimate Elias encoder (Elias, 1975) as subroutines. Its input is a message $x_{1:n}$, that is, a string of positive integers. First 0 is appended at the end of the message. Then the ETAC encoder scans the message by iterating over indices $1, \ldots, n + 1$ (lines 5–19).

Throughout the iterations, the algorithm maintains a priority queue PQ (a simple binary heap (Cormen et al., 2001) is enough). At iteration corresponding to index $i \in \{1, \ldots, n + 1\}$, the priority queue represents the censorship set $C_i \subset \mathcal{X}$ to be specified, but which only depends on the past symbols from $x_{1:i-1}$, and not on the entire sequence. The current censorship set C_i consists of symbols not smaller than a threshold τ .

Algorithm 1 ETAC encoder
Require: $x_{1:n}$, a sequence of positive integers
1: Append 0 at the end of message $x_{1:n}$
2: Initialize priority queue $PQ \leftarrow \{x_1\}$ and threshold $\tau \leftarrow$
x_1
3: Initialize counters (empty dictionary)
4: Initialize the arithmetic encoder using counters
5: for $i \in 1, \ldots, \operatorname{length}(x_{1:n}0)$ do
6: $j \leftarrow x_i$
7: if $0 < j \le \tau$ then
8: feed arithmetic encoder with j
9: emit the output of arithmetic encoder if any
10: $n^j \leftarrow n^j + 1$
11: else
12: feed arithmetic encoder with 0
{ this forces the arithmetic encoder to output the
whole encoding of the current substring }
13: emit the output of the arithmetic encoder
14: feed the Elias encoder with $\max(1, j - \tau + 1)$
15: emit the output of the Elias encoder
16: update or initialize n^j
17: update PQ and τ
18: end if
19: end for

If the current symbol $x_i > \tau$, or if $x_i = 0$, the arithmetic encoder is fed with a 0 which acts as a terminating symbol. This forces the arithmetic encoder to output the total encoding of the portion of the message that followed the previously censored symbol. This arithmetic encoding is then emitted by the ETAC encoder. Then $y := \max(x_i - \tau + 1, 1)$ is fed to the Elias penultimate encoder (Elias, 1975). The latter delivers a self-delimited binary encoding of y using $2\ell(\ell(y)) + \ell(y)$ bits where $\ell(z) = \lfloor \log_2(\max(z, 1)) \rfloor + 1$ is the length of the binary encoding of integer z. Then the ETAC encoder emits the Elias encoding. If $x_i = 0$ and only in this case, the input to the Elias encoder is 1, this signals the end of the message. The queue, the threshold and counters are updated. The censored symbol is inserted in the queue. If the second smallest symbol in the queue is strictly smaller than the queue size, then the smallest element of the queue is popped. Note that there is no need for iterating the popping process as the size of the priority queue is non-decreasing (in the sequel, the random size of PQ after scanning n symbols is denoted by M_n , this random variable is formally defined by Equation 4, its properties, including monotonicty are discussed afterwards).

The number of elementary operations required by queue maintenance is proportional to the logarithm of the size of the queue. As the expected total number of symbols inserted in the queue is sub-linear, the total expected computational cost of the queue maintenance is sub-linear.

After reading the i^{th} symbol from the message, the alphabet used by the arithmetic encoder is $0, \ldots, \tau$, the state of the arithmetic encoder is a function of the counts $n_i^0 = 0, n_i^1, \ldots, n_i^{\tau}$. Counts may be handled using map or dictionary data structures provided by modern programming languages.

From a bird-eye viewpoint, the scanning process creates a *censored* sequence $\tilde{x}_{1:n}$ such that every symbol that happens to be in a censorship set is replaced by the special 0 symbol:

$$\widetilde{x}_i = x_i \times \mathbf{1}\{x_i \notin \mathcal{C}_i\}$$

The sequence of censored symbols defines a parsing of the message into substrings of uncensored symbols that are terminated by 0. Each substring is encoded by the arithmetic encoder provided with incrementally updated sequences of probability vectors (thanks to the counters $n^j, j \leq \max_{k \leq i}(x_k)$). The ETAC encoder interleaves the outputs of the arithmetic encoder and the outputs of the penultimate Elias encoder. Encoding is performed in an incremental way, even though arithmetic coding may require buffering (Shayevitz et al., 2006).

Let N be the total number of redacted symbols, in the text i_1, \ldots, i_n denote the sequence of indexes of redacted symbols. Even though the ETAC encoder does not produce explicitly these two strings, in the text we will call $C_{\rm M}$ the concatenation of the arithmetic codewords corresponding to the encoding of $\tilde{x}_{1:n+1}$, and $C_{\rm E}$ the concatenation of the codewords produced by the Elias encoding of the subsequence $x_{i_{1:N}}$ of redacted symbols.

By construction, the input $y_{1:n}$ of the ETAC decoder (Algorithm 2) is a binary sequence that can be parsed into a unique sequence of self-delimited codewords originating alternatively from the arithmetic encoder and from the Elias encoder. The functioning of the decoder mirrors the functioning of the encoder. While scanning the input, each time it decodes a symbol, the decoder updates the appropriate counters and maintains a priority queue representing the current censorship set.

Algorithm 2 ETAC decoder

Require: $y_{1:n}$ a binary string produced by Algorithm 1
1: $i \leftarrow 1$
2: initialize counters
3: initialize $PQ \leftarrow \varnothing, \tau \leftarrow 1$
4: state \leftarrow arithmetic
5: while $i \leq n$ do
6: if state = Elias then
7: feed the Elias decoder with y_i
8: if y_i terminates an Elias codeword then
9: emit the output of the Elias decoder
10: update counters, τ and PQ
11: state \leftarrow arithmetic
12: end if
13: else
14: feed the arithmetic decoder with y_i
15: if the arithmetic decoder outputs symbols then
16: emit the output of the arithmetic decoder
17: update counters, τ and PQ
18: end if
19: if y_i terminates an arithmetic codeword then
20: state \leftarrow Elias
21: end if
22: end if
23: end while

In words, the current threshold τ is the smallest symbol in the priority queue. The queue contains the M largest symbols that have been scanned from the message so far (that is from $x_{1:i}$), and by construction either $\tau = x_{M,i} \leq M$ or M = i and $\tau = x_{i,i}$.

We now give the details of the censorship set, the encoding of the censored sequence $\tilde{x}_{1:n}$, and the encoding of the redacted symbols $x_{i_{1:N}}$. Our constructions use the *order* statistics $x_{k,i}$, $k = 1, \ldots, i$, defined for various values of $i \in \{1, \cdots, n\}$ as a non-increasing rearrangement of the symbols in the truncated sequence $x_{1:i}$:

$$\min x_{1:i} = x_{i,i} \le \dots \le x_{1,i} = \max x_{1:i}.$$

The censorship sets C_i are constructed as follows. We do not censor for i = 1, that is we have $C_1 = \emptyset$. Then, for every i > 1, we censor as follows:

$$\mathcal{C}_i = \left\{ j \in \mathbb{N} : j > M_{i-1} \right\},\$$

where the (empirical) *threshold* sequence $(M_i)_{i \in 1:n}$, defined as

$$M_i = \min(i, \{k : x_{k,i} \le k\}),$$
(4)

is a sequence of integers such that at each $i: 1 \le i \le n$, M_i can be computed from $x_{1:i}$, staying consistent with the past side-information hypothesis.

From the definition of the thresholds M_i , note the important fact that both these and the corresponding order statistics $x_{M_i,i}$ are non-decreasing. In fact, as we shall see, M_i and $x_{M_i,i}$ are roughly equivalent and are the empirical version of the thresholds m_n suggested by Theorem 2 and which we subsequently define in Equation (5). Therefore we are effectively using as threshold the value $x_{M_{i-1},i-1}$, the M_{i-1} - th order statistic at step i - 1, that is the M_{i-1} -th largest symbol among the first i - 1 symbols. The name "expanding threshold" is used to contrast with the AC-code (Bontemps, 2011) which chooses the 1-st order statistic (the maximum) as the threshold, whereas here the censure zone "expands" to higher order statistics (smaller than the maximum).

The censored sequence $\tilde{x}_{1:n}$ is encoded into the string $C_{\rm M}$ as follows. We start by appending an extra 0 at the end of the original censored sequence, to signal the termination of the input. We therefore in fact encode $\tilde{x}_{1:n}0$ into $C_{\rm M}$. We do this by performing a progressive arithmetic coding (Rissanen and Langdon, 1979) using coding probabilities $Q^{n+1}(\tilde{x}_{1:n}0)$ given by:

$$\tilde{Q}^{n+1}(\tilde{x}_{1:n}0) = \tilde{Q}_{n+1}(0 \mid x_{1:n}) \prod_{i=0}^{n-1} \tilde{Q}_{i+1}(\tilde{x}_{i+1} \mid x_{1:i})$$

where the predictive probabilities \tilde{Q}_{i+1} are a variant of Krichevsky-Trofimov mixtures,

$$\tilde{Q}_{i+1}\left(\tilde{X}_{i+1} = j \mid X_{1:i} = x_{1:i}\right) = \frac{n_i^j + \frac{1}{2}}{i + \frac{M_i + 1}{2}}$$

The n_i^j notation refers to the number of occurrences of symbol j among the first i symbols (in $x_{1:i}$), with the convention that $n_i^0 = 0$ for all i. What these coding probabilities represent, in effect, is a mixture code consisting of progressively enlarging the alphabet based on the thresholds to include symbols $\{0, 1, \dots, M_i\}$, and feeding an arithmetic coder with Krichevsky-Trofimov mixtures over this growing alphabet. Thanks to M_i being determined by the data, the enlargement of the alphabet is performed online and is driven by the order statistics of the symbols seen so far.

The subsequence $x_{i_{1:N}}$ of redacted symbols is encoded into the string $C_{\rm E}$ as follows. Instead of encoding the symbol values directly, we encode excesses over the thresholds, which are known under the past side-information hypothesis: for each $i \in i_{1:N}$, we encode $x_i - M_{i-1} + 1$ using Elias penultimate coding (Elias, 1975), where the +1 is added to make sure these values are strictly greater than 1. The extra 0 initially appended to the message yields the only 1 that is fed to the arithmetic encoder, it unambiguously signals to the decoder that the 0 symbol decoded from $C_{\rm M}$ is in fact the termination signal. This ensures that the overall code is instantaneously decodable, and that it therefore corresponds to an implicit coding probability Q_n .

B. The Exact Thresholds

Now that we have constructed the ETAC-code, let us revisit and compare with Theorem 2. Recall that the thresholding scheme suggested by this theorem uses as threshold an integer u_n such that $n\overline{F}(u_n) \log e \approx (u_n-1)/2 \log n$. How well does the ETAC-code heed this rule of thumb?

Since the encoding is done sequentially, the threshold is adjusted as-we-go, but let us focus primarily on the final threshold M_n as defined by Equation (4) for i = n, and the corresponding order statistic $X_{M_n,n}$. From this, it seems that the ETAC-code uses an apparently suboptimal threshold,

by empirically defining M_n as the smallest integer k such that $X_{k,n} \leq k$. Yet, we can make two observations based on this: the first is that $M_n \approx X_{M_n,n}$ and the second is that $\overline{F}_n(X_{M_n,n}) \approx X_{M_n,n}/n$, where \overline{F}_n denotes the empirical cumulative distribution function.

We would therefore expect the behavior of both M_n and $X_{M_n,n}$ to closely follow that of their "exact" counterpart, that is the value m_n which gives $\overline{F}(m_n) \approx m_n/n$. Therefore the rule of thumb of Theorem 2 is indeed followed up to the constant and logarithmic terms. To analyze the ETAC-code, it is thus important to study this exact threshold and understand its asymptotic properties. To make this more convenient, instead of using the envelope distribution and working with integer n, we can use the *smoothed* envelope distribution F and define for all positive real t:

$$m(t) = \{x : \overline{F}(x) = x/t\} = \{y : U(t/y) = y\}$$
(5)

and in particular: $m_n \equiv m(n)$. Note that, by construction, we always have that m_n is non-decreasing, bounded from above by n, and satisfies $\overline{F}(m_n) = m_n/n$ and $U(n/m_n) = m_n$.

Using this threshold, Theorem 2 then gives us a minimax redundancy upper bound:

$$R^{+}(\Lambda_{n}) \leq m_{n} \log e + \frac{m_{n} - 1}{2} \log n + 2$$

$$\leq m_{n} \log n + 2 \quad \text{(for } n \geq 8) \tag{6}$$

when Λ is a max-stable envelope class with corresponding $U \in \text{ERV}(\gamma)$ with $\gamma \geq 0$.

Though m(t) is defined in an implicit way, its most relevant properties can be established with little effort thanks to the notion of De Bruijn conjugacy (see Appendix A), which plays an important role in the asymptotic inversion of regularly varying functions. The asymptotic behaviors of functions mand U (and therefore F) are connected by the following lemma. Namely, m inherits the regular variation property of U, and the decomposition of m and U (as products of a slowly varying function and a power function) are related.

Lemma 3 (PROPERTIES OF THE EXACT THRESHOLD). Assume that $U : [1, \infty) \longrightarrow \mathbb{R}_+$ is increasing to infinity, is continuously differentiable and that $U \in \text{RV}(\gamma)$ with $\gamma \ge 0$ (satisfied under Definition 3). Let $m: (U(1), \infty) \rightarrow \mathbb{R}_+$ be defined as in Equation (5). Then m satisfies:

- (i) m is well-defined and increasing;
- (ii) *m* is continuously differentiable;
- (iii) $m(t) \longrightarrow \infty$ and $t/m(t) \longrightarrow \infty$ as $t \longrightarrow \infty$;
- (iv) *m* is regularly varying with index $\gamma/(\gamma + 1)$ ($m \in \mathbb{RV}(\gamma/(1+\gamma))$). Moreover, if $U(t) = t^{\gamma}L(t)$ where *L* is slowly varying, then letting $L_1(t) = L(t^{1/(1+\gamma)})$ and L_1^* be a De Bruijn conjugate of L_1 ,

$$m(t) \sim t^{\gamma/(\gamma+1)} L_1^*(t)^{-1/(1+\gamma)}$$
 as $t \to \infty$.

The proof is given in Appendix C.

As for the empirical thresholds M_n , or equivalently the corresponding order statistic $X_{M_n,n}$, these are random variables that prove well concentrated around their mean or median values. In particular, if the source is close to the envelope distribution, and if the latter belongs to a max-domain of

IV. MAIN RESULTS

We now give the main contributing result of the paper, which is the near-adaptivity of the ETAC-code on the collection of Fréchet (heavy-tailed) max-stable envelope source classes. The components of this result are then presented in detail in the rest of the paper, in terms of the minimax redundancy lower bound in Section V and the analysis of the ETAC-code in Section VI. We also give a somewhat stronger adaptivity of the ETAC-code, when restricted to a sub-class of Gumbel (light-tailed) max-stable envelopes. In particular, the overhead is not logarithmic in the sequence length, but logarithmic in the minimax redundancy, which grows much slower (roughly log log n). Within this setting, this shows that there is no major loss in switching to the new code, which does not explicitly make the light-tailed assumption, from the AC-code of (Bontemps et al., 2014), which does.

A. Near-adaptivity to Fréchet max-stable envelope source classes

Our main result can be stated as:

Theorem 4 (FRECHET NEAR-ADAPTIVITY OF THE ETAC CODE). Let Q_n denote the coding probability defined by the ETAC-code, let Λ be a Fréchet max-stable envelope class with ultimately non-increasing envelope and with corresponding exact threshold sequence $(m_n)_{n \in \mathbb{N}_+}$. We then have that there exists a constant κ_{Λ} (that may depend on Λ) such that:

$$(\kappa_{\Lambda} + o_{\Lambda}(1))m_n \le R^+(\Lambda_n) \le R^+(Q_n, \Lambda_n)$$
$$\le (5/2 + o_{\Lambda}(1))m_n \log n.$$

In particular, the ETAC-code is asymptotically near-adaptive (cf. Equation (2)):

$$R^+(Q_n, \Lambda_n) \le (5/2\kappa_\Lambda + o_\Lambda(1)) \log n \ R^+(\Lambda_n).$$

We provide here a guideline proof of this theorem, which relies on components presented in Sections V and VI, as well as their details in the appendices.

Proof: The lower bound on the minimax redundancy is given by Theorem 8 of Section V. To sketch its proof, note that we first use a maximin Bayes redundancy approach to prove that the minimax redundancy is lower bounded by the number K_n of distinct symbols that appear in the sequence $X_{1:n}$. We then show that for sources belonging to max-stable envelope classes, K_n and M_n (and therefore m_n , by Appendix B) are asymptotically within constant factors of each other.

The upper bound on the redundancy of the ETAC-code follows from the results of Section VI, where we separately analyze the codelengths of the arithmetically mixtureencoded censored sequence $C_{\rm M}$ and the Elias-encoded individual redacted symbols $C_{\rm E}$. We show here how to combine this analysis to provide the upper bound. Assume that we are dealing with a particular source from a max-stable envelope class Λ . For the mixture-encoded censored sequence, Equation (10) bounds the difference between the expected length of $C_{\rm M}$ and the optimal codelength given by the Shannon entropy:

$$\mathbb{E}\left[\ell(C_{\mathsf{M}}) + \log \mathbb{P}_n(X_{1:n})\right] \le (1/2 + o_{\Lambda}(1)) \ m_n \log n$$

Meanwhile, Lemma 11 gives a final breakdown of the expected length of $C_{\rm E}$. In particular, since we are considering only Fréchet distributions, we can use Equation (19):

$$\mathbb{E}[\ell(C_{\rm E})] \le (2 + o_{\Lambda}(1)) \ m_n \log n \,.$$

Merging the two upper bounds leads to:

$$\mathbb{E}\left[\ell(C_{\rm E}) + \ell(C_{\rm M}) + \log \mathbb{P}_n(X_{1:n})\right] \le (5/2 + o_{\Lambda}(1)) \, m_n \log n \, ,$$

and since this bound does not depend on the particular source but only on the source class, we can take a supremum of the left-hand side over the entire class to obtain an upper bound on $R^+(Q_n, \Lambda_n)$.

As a straightforward consequence of Theorem 4 and point (iv) of Lemma 3, we have exact rates of growth of the lower bound on the minimax redundancy and the redundancy of the ETAC-code, in terms of the regular variation properties of the envelope:

Corollary 5. Let Q_n denote the coding probability defined by the ETAC-code, let Λ be a Fréchet max-stable envelope class with ultimately non-increasing envelope and with smoothed envelope distribution in the maximum domain of attraction $MDA(\gamma)$ for some $\gamma > 0$. Then there exist a slowly varying function L_{Λ} and a constant κ_{Λ} (both depending on the envelope that defines Λ), such that:

$$\kappa_{\Lambda} L_{\Lambda}(n) n^{\gamma/(\gamma+1)} \leq R^{+}(\Lambda_{n})$$

$$\leq R^{+}(Q_{n},\Lambda_{n})$$

$$\leq (5/2 + o_{\Lambda}(1)) \log n L_{\Lambda}(n) n^{\gamma/(\gamma+1)}.$$

This corollary is particularly informative, since it shows that both the minimax redundancy and the ETAC-code redundancy grow as powers of n, and therefore the logarithmic factor in the definition of near-adaptivity is not an unreasonable relaxation to the notion of adaptivity in the context of Fréchet maxstable envelope classes. Note also the vanishing per-symbol redundancy, at a rate of roughly $n^{-1/(1+\gamma)}$ which is slower the heavier the tail (the larger γ) is.

B. Near-adaptivity to light-tailed envelope source classes

In this section, we tie the results of this paper with those of Bontemps et al. (2014), where an explicit light-tailed assumption was made. This is the notion of an envelope distribution F that has *non-decreasing hazard rate*, that is it can be associated with a log-convex smoothed tail function. The terminology comes from the notion of *hazard function* whose derivative being non-decreasing is equivalent to this log-convexity condition. It is worth noting that this means that F itself is, almost, log-concave. These distributions are a rich subset of light-tailed distributions. In particular, geometric

envelopes are at the boundary of such distributions, as they exhibit log-linear smoothed tail functions.

contribution of the AC-code The presented in (Bontemps et al., 2014) is that this code is adaptive, in the sense of Equation (1), to the collection of classes with non-decreasing hazard rate. The performance of the AC-code on such classes may be understood in a very intuitive way: the AC-code encodes the n-th symbol in a way that is not more expensive than encoding a symbol from a source on an alphabet of size $U(n) = F^{-1}(1 - 1/n)$, that is with redundancy U(n)/(2n) bits. The AC-code can perform in this way for two reasons: with overwhelming probability, the largest sample in a sequence of length n, is not larger than U(n); on many sources in such a class, with high probability, most of the symbols that are smaller than U(n) do occur in a sequence of length n, there is no penalty in coding as if the actual alphabet were of size U(n).

The ETAC-code does not take such a simplistic approach, it attempts to calibrate the effective alphabet size in a much more cautious way. An intuitive interpretation of the empirical threshold M_n is the following: symbols larger than M_n have low empirical frequency in the sequence, they may be encoded with the general purpose code; symbols smaller than M_n tend to have larger empirical frequency, and on some sources from the envelope classes considered in this paper, a large proportion of the symbols that are smaller than m_n do occur in a typical sequence (this observation is documented in the literature (Archibald, Knopfmacher, and Prodinger, 2006; Grübel and Hitczenko, 2009; Ben-Hamou et al., 2014)). Up to the Elias encoding, the ETAC-code encodes a sequence of length n as if the actual alphabet were of cardinality m_n . The choice of m_n balances the cost of escaping large symbols and the overhead incurred by oversizing the effective alphabet.

On the other hand, Bontemps et al. (2014) establish that for non-decreasing hazard rate envelope classes, $U(t) = F^{-1}(1 - 1/t)$ is not only slowly varying but also enjoys the special property that, according to Bojanic and Seneta (1971), the De Bruijn conjugate U^* of U is asymptotically equivalent to 1/U. By Lemma 3, this in turn implies that $\lim_{t\to\infty} m(t)/U(t) = 1$. Operationally, this means that choosing the threshold as $M_n \approx X_{M_n,n}$ (ETAC-code) or as $X_{1,n}$ (AC-code) asymptotically does not make a difference as far as coding envelope classes defined by such light-tailed envelopes. This entails (see Bontemps et al., 2014) the fact that the minimax redundancy of such classes is asymptotically not smaller than $\log(e) \int_1^n U(x)/(2x) dx \ge U(n) \log(n)/4$.

Therefore, we expect the ETAC-code to perform well, despite its cautious approach. The following theorem establishes precisely that: up to a $\log m_n \approx \log \log n$ factor, the ETAC-code is asymptotically adaptive with respect to envelope classes defined by envelope distributions with non-decreasing hazard rate.

Theorem 6 (NON-DECREASING HAZARD RATE NEAR-ADAP-TIVITY OF THE ETAC CODE). Let Q_n denote the coding probability defined by the ETAC-code, let Λ be an envelope class such that the envelope has the non-decreasing hazard rate property, with corresponding exact threshold sequence $(m_n)_{n\in\mathbb{N}_+}$. We then have that:

$$(1/4 + o_{\Lambda}(1)) \ m_n \log n \le R^+(\Lambda_n)$$

$$\le R^+(Q_n, \Lambda_n)$$

$$\le (2 + o_{\Lambda}(1)) \ m_n \log n \log m_n \,.$$

In particular, the ETAC-code is not only asymptotically nearadaptive (cf. Equation (1), noting that $m_n \leq n$):

$$R^+(Q_n, \Lambda_n) \le (8 + o_\Lambda(1)) \log m_n R^+(\Lambda_n),$$

but furthermore the multiplicative factor is of order $O_{\Lambda}(\log \log n)$.

Proof: The proof of the minimax redundancy lower bound is given in (Bontemps et al., 2014). As for the redundancy upper bound, the only difference with the proof of Theorem 4 is to use the weaker Elias codelength bound given by Equation (18):

$$\mathbb{E}[\ell(C_{\rm E})] \le (2 + o_{\Lambda}(1)) \ m_n \log n \log m_n \ .$$

The near-adaptivity follows immediately. As for the claim that $m_n = O_{\Lambda}(\log n)$, it follows from the fact that $m(t)/U(t) \to 1$ as shown in (Bontemps et al., 2014), recalling that $U(t) = \overline{F}^{-1}(t)$, where \overline{F} has a sub-exponential tail.

Theorems 4 and 6 raise several questions. Between heavytailed envelope functions handled by Theorems 4 and very light tailed envelope functions handled by Theorem 6, lie an intermediate family of envelope functions with slowly varying tail quantile functions $(U(tx)/U(t) \rightarrow 1 \text{ as } t \rightarrow \infty)$ for all x > 0) but with decreasing hazard rate. If we consider sampling from the associated envelope distribution, the literature dedicated to infinite urn schemes (See Karlin, 1967; Gnedin et al., 2007; Ben-Hamou et al., 2014) shows that as ntends to infinity, the number of rare symbols - that are likely to be censored and to enter the priority queue maintained by the ETAC encoder – is not stochastically bounded, but it tends to be negligible with respect to the number of distinct symbols in the sample. The AC-code is not likely to be adaptive with respect to envelope classes defined by such envelope distributions. The minimax redundancy of such envelope classes remains to be determined, and so is the performance of the ETAC code over thoses classes. Indeed, a very natural question raised by the advances reported in the present paper, is the cost of adaptivity in compression against countable alphabets. In density estimation (Lepski, 1992), or tail index estimation (Carpentier and Kim, 2014; Boucheron and Thomas, 2015) for example, there are problems where adaptive estimation suffers a logarithmic loss with respect to minimax risk. We still do not know whether this is the case for adaptive compression against envelope classes.

V. MINIMAX REDUNDANCY OF FRÉCHET ENVELOPE CLASSES

We now lower bound the minimax redundancy with respect to the envelope class $\Lambda(f)$ when the envelope function F is Fréchet. In this section, we make the additional assumption that f is ultimately monotonically non-increasing. This is primarily to make the presentation more transparent when relating the regular variation properties of various functions, namely f, F, and the distribution G that ensues from the Bayes construction.

We use the standard approach of the relationship between minimax and maximin redundancies. In particular, consider a set $\mathcal{P} = \{P_{\theta}, \theta \in \Theta\}$ of memoryless sources over the countable alphabet \mathbb{N}_+ indexed by a parameter space Θ and let π be a (prior) probability measure on Θ . We call (\mathcal{P}, π) a *Bayes model*. If the parameter θ is chosen according to π and subsequently a sequence $X_{1:n}$ of length n is observed from the source P_{θ} , then the *Bayes redundancy* is the mutual information between θ to $X_{1:n}$. Of fundamental importance is the fact (see, for example, Clarke and Barron (1990)) that the minimax redundancy is lower bounded by the Bayes redundancy with respect to any choice of prior probability distribution:

$$I(\boldsymbol{\theta}, X_{1:n}) \leq R^+(\mathcal{P}^n)$$

Moreover, whenever $\mathcal{P} \subset \Lambda$, we have $R^+(\mathcal{P}^n) \leq R^+(\Lambda_n)$, and we can engineer a lower bound to the minimax redundancy by properly choosing the Bayes model (\mathcal{P}, π) . In the remainder of this section we start by doing precisely that, we then bound the resulting mutual information by the expected number of distinct symbols in the sequence, and lastly we relate the growth of the latter to the index of regular variation to establish a lower bound that matches the redundancy of the ETAC-code up to a logarithmic factor.

A. Building a Bayes model

In an appropriate Bayes model, we would like each P_{θ} to be a member of $\Lambda(f)$ in an intuitively 'worst-case' fashion: we want to capture the tail behavior dictated by f. The parameters can then simply 'dither' around this tail.

Let $\Theta = \{0,1\}^{\mathbb{N}}$ be the space of all 0-1 sequences. For any such sequence $\theta = (\theta_k)_{k \in \mathbb{N}}$ define $P_{\theta} \in \mathcal{P}$ as, for each $j \in \mathbb{N}_+$:

 $P_{\boldsymbol{\theta}}(j)$

$$= \begin{cases} f(j)/Z & \text{for every } j < j_0\\ \min_{t \in \{0,1\}} f(j_0 + 2k + t) & \text{when } j = j_0 + 2k + \theta_k\\ & \text{for some } k \in \mathbb{N}\\ 0 & \text{when } j = j_0 + 2k + (1 - \theta_k)\\ & \text{for some } k \in \mathbb{N} \end{cases}$$

where

$$Z = \frac{\sum_{j < j_0} f(j)}{1 - \sum_{k \in \mathbb{N}} \min_{t=0,1} f(j_0 + 2k + t)}$$

This construction keeps the probability of the first $j_0 - 1$ symbols constant as θ varies. At and beyond j_0 , it breaks the alphabet in blocks of size 2 indexed by k, assigning the smallest of the two values of f in each block to one or the other symbol, according to the component θ_k of θ corresponding to that block. For j_0 , we can choose any value such that $Z \ge 1$. In particular, since $\sum_{j\ge 1} f(j) > 1$, we can always choose j_0 such that $\sum_{j<j_0} f(j) \ge 1$. It follows that $P_{\theta}(j)$'s as defined are indeed probabilities. Furthermore, for all j we have that $P_{\theta}(j) \le f(j)$, and therefore $\mathcal{P} \subset \Lambda(f)$ as desired. P_{θ} matches one of the values of f within each block in the tail, and is (almost 'worst-case' in this sense.

To complete the model, let the prior π be such that θ is a sequence of independent identically distributed Bernoulli-1/2 random variables. Note that the probability multiset $\{P_{\theta}(j) : j \in \mathbb{N}_+\}$ is the same for all θ . The only difference is in the within-block positioning in the tail, randomized by θ . It is in this sense that the parameters 'dither' the tail behavior.

B. Computing the Bayes redundancy

To proceed with the computation of the Bayes redundancy, we start with an observation: under the posterior distribution, the parameters $\theta_1, \ldots, \theta_k, \ldots$ are still independent. We first provide an intuitive argument. Given the sequence, there are two distinct possibilities per block k: either it is represented or it is not. If it is, then the corresponding parameter θ_k is known deterministically and none of the other parameters influence it. If it is not, then the a posteriori distribution of θ_k is its a priori distribution, because one cannot infer about it from the data, and the other parameters have no influence on this either. Therefore, given the observations, the parameters remain independent.

In order to get a formal proof of independence, it is enough to check that for each $k, \theta_1, \ldots, \theta_k$ are independent under the posterior distribution. A basic result in Bayesian theory asserts that the density of the posterior distribution with respect to the prior distribution is proportional to the likelihood. Given observations X_1, \ldots, X_n , the likelihood at $\theta_1, \ldots, \theta_k$ can be computed using counters $N_j^0 = \sum_{i=1}^n \mathbb{I}_{X_i=j_0+2j}, N_j^1 =$ $\sum_{i=1}^n \mathbb{I}_{X_i=j_0+2j+1}$ and $N_j = N_j^0 + N_j^1 = \max(N_j^0, N_j^1)$. It is proportional to

$$\prod_{j=1}^{k} \left(\mathbb{I}_{N_{j}^{\theta_{j}}=N_{j}} \left(P_{\boldsymbol{\theta}}(j_{0}+2j+\theta_{j}) \right)^{N_{j}} \right) \,.$$

Note that the joint distribution of N_1, \ldots, N_k does not depend on θ , and conditionally on N_1, \ldots, N_k , the counters $(N_j^0)_{j \le k}$ are independent. The likelihood is thus proportional to a product of functions of the θ_j , implying the desired independence.

Using this observation, thanks to the chain rule for mutual information, the Bayes redundancy can be written as

$$I\left(\boldsymbol{\theta}, X_{1:n}\right) = \sum_{k \in \mathbb{N}} I\left(\theta_k, X_{1:n} | \theta_{1:k-1}\right)$$
$$= \sum_{k \in \mathbb{N}} I\left(\theta_k, X_{1:n}\right) \,.$$

By conditioning further on N, as N_k and θ_k are independent, we have for each k:

$$I(\theta_k, X_{1:n})$$

= $I(\theta_k, X_{1:n}|N_k = 0) \mathbb{P}(N_k = 0)$
+ $I(\theta_k, X_{1:n}|N_k = 1) \mathbb{P}(N_k = 1).$

The first term is the case when block k is not represented: conditionally on $N_k = 0$, θ_k and $X_{1:n}$ are independent. Therefore, $I(\theta_k, X_{1:n}|N_k = 0) = 0$. The second term is when block k is represented: then θ_k is known deterministically, i.e. a noiseless binary channel. Therefore, $I(\theta_k, X_{1:n}|N_k = 1) = 1$ (bit). Hence,

$$I(\boldsymbol{\theta}, X_{1:n}) = \sum_{k \in \mathbb{N}} \mathbb{P}(N_k = 1)$$

$$\geq \mathbb{E}[K_n] - j_0 + 1, \qquad (7)$$

where K_n denotes the number of distinct symbols in $X_{1:n}$. The inequality follows from the fact that $\sum_{k \in \mathbb{N}} \mathbb{P}(N_k = 1)$ is the expected number of distinct symbols with values at j_0 or beyond. Just like **N**, the distribution of K_n does not depend on the value of the parameter θ . The expected number of distinct symbols when sampling from a given discrete distribution has been studied in depth in the literature (Gnedin et al., 2007), and we can use the assumptions on the tail behavior of the envelope f to characterize the asymptotic behavior of its expectation.

C. Bounding the minimax redundancy

The probability multiset of Equation (7) can be reindexed (using j' instead of j, to make it clear it's a new indexing) as follows:

$$g(j') = \begin{cases} f(j')/Z & \text{if } j' < j_0 \\ f(2j' - j_0) \land f(2j' - j_0 + 1) & \text{if } j' \ge j_0 . \end{cases}$$
(8)

This new probability mass function on \mathbb{N}_+ corresponds to a cumulative distribution, which we call G. Since the number of distinct symbols in a sequence from G has the same law as that from any source in the Bayes construction, we can use it to study the expectation $\mathbb{E}K_n$. We first show how p and G inherit certain properties from f and F respectively, via the following lemma proved in Appendix C. Recall that m_n is defined as the solution of $\overline{F}(x) = x/n$, where F is the smoothed envelope distribution.

Lemma 7. If f is ultimately monotonically non-decreasing and $F \in MDA(\gamma)$ with $\gamma > 0$, then so are g and Grespectively. Furthermore, if we define $(m'_n)_n$ by $m'_n = \min \{k \in \mathbb{N}_+ : \overline{G}(k) \le k/n\}$, we have that $m_n/m'_n \to 2$, as $n \to \infty$.

The literature on infinite urn schemes, starting with (Karlin, 1967) and surveyed in (Gnedin et al., 2007), describes tight connections between the tail behavior of the sampling distribution and the sequence $(\mathbb{E}K_n)_n$. These results establish asymptotic relationships between $\mathbb{E}K_n$, n, γ and the slowly varying function of G. Our goal here, instead, is to relate $(\mathbb{E}K_n)_n$ and the sequence of exact thresholds $(m_n)_n$. To this effect, we prove a key result in Appendix B, Lemma 17, which effectively bounds K_n from below by m'_n , up to a constant factor. We state it here for clarity:

Lemma (Lemma 17 in Appendix B). Let a distribution Gon \mathbb{N}_+ belong to some MDA $(\gamma), \gamma > 0$, with a probability mass function that is ultimately monotonically non-increasing. Define $m'_n = \min \{k \in \mathbb{N}_+ : \overline{G}(k) \le k/n\}$. Then there exists a constant κ'_{γ} and some n_0 (that may depend on G), such that for all $n \ge n_0$, the expected number of distinct symbols in a sample from G satisfies

$$\kappa_{\gamma}' m_n' \le \mathbb{E} K_n$$

By combining Lemmas 7 and 17, we can assert that there exists a constant κ_{γ} (that depends only on γ) and some n_0 (that may depend on F more generally), such that for all $n \ge n_0$ we have:

$$\kappa_{\gamma} m_n \le \mathbb{E} K_n \,. \tag{9}$$

We are finally in position to combine the Bayes model construction and these asymptotic characterizations to give a lower bound on the minimax redundancy. By combining Equations (7) and (9), and using the minimax-maximin relationship, we have established the following theorem.

Theorem 8 (FRECHET MINIMAX REDUNDANCY LOWER BOUND). Let $\Lambda(f)$ be the envelope class defined by a function f that is ultimately monotonically non-increasing. If the envelope is Fréchet with index $\gamma > 0$ in the sense of Definition 5, and $(m_n)_n$ is defined according to (5), then, for some constant κ_{γ} , for large enough n,

$$R^+(\Lambda_n) \ge \kappa_{\gamma} m_n$$
.

Compare this to the upper bound on this redundancy expressed in Equation (6) which was obtained using Theorem 2. According to recent results (Acharya et al., 2014) obtained in a slight variant of our model, the logarithmic gap between lower and upper bounds for the minimax redundancy is likely to be due to the weakness of Theorem 2 to fully capture the richness of the max-stable envelope classes.

VI. ANALYSIS OF THE ETAC-CODE

We now complete the paper by analyzing the redundancy of the ETAC-code. We start with direct bounds on the codelengths of the two strings comprising the code, the mixture encoding $C_{\rm M}$ and the Elias encoding $C_{\rm E}$, in terms of the data-driven threshold sequences M_n . These need to be related to the exact threshold m_n , to tie the redundancy of the code with the minimax redundancy lower bound and give precise asymptotic growth expressions. For the mixture encoding, the direct bound is sufficient upon using the distribution-free properties of the thresholds given in Appendix B. For the Elias code, further work is needed to place it in the proper form, and most of this section is dedicated to that analysis. The results presented here are combined in their final form in Theorem 4 of Section IV.

A. Codelength of the mixture encoding $C_{\rm M}$

The difference between the length of the progressive mixture encoding of the censored sequence can be compared with the ideal codeword length for the source output (see Lemma 2 and the proof of Theorem 8 in Boucheron et al., 2009, for details):

$$\begin{split} \ell(C_{\mathsf{M}}) &+ \log \mathbb{P}_n(X_{1:n}) \\ &= -\log \operatorname{KT}_{M_n+1}(\widetilde{X}_{1:n}) + \log \mathbb{P}_n(X_{1:n}) \\ &- \log Q_n(\widetilde{X}_{1:n}) + \log \operatorname{KT}_{M_n+1}(\widetilde{X}_{1:n}) \end{split}$$

where κT_{M_n+1} is the Krichevsky-Trofimov mixture coding probability over an alphabet of cardinality M_n+1 . The second part of the equality is non-positive, and it follows that:

$$\ell(C_{\rm M}) + \log \mathbb{P}_n(X_{1:n}) \le \frac{M_n + 1}{2} \log(n) + 2.$$

We can then appeal directly to the distribution-free properties of the thresholds given in Appendix B, to bound $\mathbb{E}[M_n]$ asymptotically by m_n . In particular, if the source belongs to a max-stable envelope class Λ , and (m_n) is the corresponding exact threshold sequence as in Equation (5), we then have:

$$\mathbb{E}\left[\ell(C_{\rm M}) + \log \mathbb{P}_n(X_{1:n})\right] \leq \frac{\mathbb{E}[M_n] + 1}{2} \log(n) + 2$$

$$\leq \frac{m_n + 3\sqrt{m_n} + 4}{2} \log(n) + 2$$

$$\leq (1/2 + o_{\Lambda}(1))m_n \log(n) \quad (10)$$

where the second inequality follows from Lemma 16 and the last inequality holds since m_n grows unbounded with n, by Lemma 3.

B. Codelength of the Elias encoding $C_{\rm E}$

Over light-tailed envelope classes, the contribution of the Elias penultimate encoding of the redacted symbols to the redundancy of the AC-code is asymptotically negligible, relative to the mixture code length and the minimax redundancy (Bontemps et al., 2014). The argument is transparent: when using the AC-code the threshold, which is the maximum, corresponds to a rank within the order statistics that is deterministic, equal to 1, and redacted symbols are just records (excesses over maxima) of an independent sequence of identically distributed random variables. They may be analyzed using the well-established theory of records (see Resnick, 1987). Furthermore, the fact that envelopes have nondecreasing hazard rate considerably simplifies the analysis of extreme order statistics (see Boucheron and Thomas, 2012).

Over max-stable envelope classes, the analysis of the contribution of the Elias encoding to the redundancy of the ETACcode faces new challenges. These stem from the fact that redacted symbols are not records anymore, not even k-th rank records for a deterministic k, as the threshold is determined from the data itself. Moreover, it is not straightforward to transfer properties from a sequence drawn from the smoothed envelope distribution to one drawn from a specific distribution in the envelope class. The details of the approach we follow involve tackling the problem on these fronts.

In what follows let G denote our sampling distribution, which belongs to an envelope class given by $F \in MDA(\gamma)$, $\gamma \geq 0$. As in (Bontemps et al., 2014), the length of the Elias encoding is readily upper bounded as follows.

$$\mathbb{E}[\ell(C_{\mathsf{E}})] \le 2\sum_{i=1}^{n-1} \mathbb{E}\left[\mathbb{I}\{X > M_i\}\left(\log(1+X-M_i)+\rho\right)\right]$$

where we write a generic X instead of X_{i+1} , because X_{i+1} is always independent of M_i . The ρ term is a parametrization choice. It contributes to the sum with a factor of $\mathbb{P}\{X > M_i\} = \mathbb{E}[\overline{G}(M_i)]$. We shortly bound the latter in Lemma 10, and meanwhile place most of our focus on bounding the logarithmic term.

We go through these general steps:

• For each *i*, we condition on $M_i = u$. This reduces the problem to bounding the following 'pointwise' (in the threshold) function from above:

$$\Sigma(u) := \mathbb{E}\left[\mathbb{I}\{X > M_i\}\log(1 + X - M_i)|M_i = u\right]$$

Note that upon conditioning, we lose the dependence on i. The influence of i on the total expectation is only through the distribution of M_i .

- We then take a total expectation E[Σ(M_i)] for each i, and transfer the pointwise bounds. Since we would like to express E[ℓ(C_E)] as a function of the thresholds, we take care to relate the various bounds to E[M_i], and m_i.
- To bound E[l(C_E)], we combine the bounds for various values of *i* in the sum:

$$\mathbb{E}[\ell(C_{\rm E})] \le 2\sum_{i=1}^{n-1} \left(\mathbb{E}[\Sigma(M_i)] + \rho \mathbb{E}[\overline{G}(M_i)] \right).$$
(11)

Although each step corresponds to a simple statement, we list the results as lemmas, to cleanly delineate the proofs. We start with giving a pointwise bound on $\Sigma(u)$.

Lemma 9. Given $\epsilon > 0$ there exists t_0 (which will depend on both ϵ and F), such that for all $t > t_0$, we have:

$$\Sigma(u) \le \overline{G}(u)\log(t) + (\gamma/\ln 2 + \epsilon)\overline{F}(t)$$
(12)

We would now like to take the expectation $\mathbb{E}[\Sigma(M_i)]$. The only term that contributes is $\mathbb{E}[\overline{G}(M_i)]$, which can be bounded as follows.

Lemma 10. We have:

$$\mathbb{E}[\overline{G}(M_i)] \le \frac{\mathbb{E}[M_{i+1}]}{i+1}.$$

As in the derivation of Equation 10 for the mixture codelength, we can now use the concentration properties of M_i given by Lemma 16, to relate it back to m_i . Given $\epsilon > 0$, for large enough i we have:

$$\mathbb{E}[\overline{G}(M_i)] \le (1+\epsilon)\frac{m_{i+1}}{i+1}.$$
(13)

With the choice of $t = m_{i+1}$, Equations (12) and (13) give us that for large enough *i*:

$$\mathbb{E}[\Sigma(M_i)]$$

$$\leq (1+\epsilon)\frac{m_{i+1}}{i+1}\log(t) + (\gamma/\ln 2 + \epsilon)\overline{F}(t)$$

$$\leq (1+\epsilon)\frac{m_{i+1}}{i+1}\log(m_{i+1}) + (\gamma/\ln 2 + \epsilon)\frac{m_{i+1}}{i+1}.$$
(15)

Lastly, by combining these steps via Equation (11), we obtain a master bound on the expected Elias codelength, valid for both Fréchet and Gumbel envelopes, and which we can further specialize in the Fréchet case. We present this in the following lemma.

Lemma 11 (ELIAS CODELENGTH). Given $F \in MDA(\gamma)$, then for all G in the envelope class Λ characterized by F, we have the following bounds for the Elias portion of the ETAC-code: (i) Sum bound:

$$\mathbb{E}[\ell(C_{\rm E})] \le (2 + o_{\Lambda}(1)) \sum_{i=1}^{n} \frac{1}{i} m_i \log m_i \qquad (16)$$

(ii) Integral bound:

$$\mathbb{E}[\ell(C_{\rm E})] \le (2 + o_{\Lambda}(1)) \int_{1}^{n} \frac{1}{t} m(t) \log m(t) \mathrm{d}t.$$
 (17)

(iii) Direct bound (for both Gumbel and Fréchet):

$$\mathbb{E}[\ell(C_{\rm E})] \le (2 + o_{\Lambda}(1))m_n \log n \log m_n.$$
(18)

(iv) If $\gamma > 0$ (only Fréchet):

$$\mathbb{E}[\ell(C_{\rm E})] \le (2 + o_{\Lambda}(1))m_n \log n.$$
(19)

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APPENDIX A REGULAR VARIATION

Useful material concerning regular variation and applications to the analysis of extreme value theory can be found in (Bingham et al., 1989) and in the appendix of (de Haan and Ferreira, 2006). In the current appendix, we gather some of the basic definitions and properties which we need in the paper.

Definition 6 (REGULAR VARIATION). A measurable function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is regularly varying if and only if for all x > 0,

$$\lim_{t \to \infty} \frac{h(tx)}{h(t)}$$

exists as a function of x.

If h is regularly varying, there exists some $\alpha \in \mathbb{R}_+$, such that $\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^{\alpha}$, α is called the regular variation index of h, this is abbreviated as $h \in \text{RV}(\alpha)$. If the regular variation index is 0, i.e. $\forall x : \frac{h(tx)}{h(t)} \to 1$, then h is said to be *slowly varying*.

Definition 7 (EXTENDED REGULAR VARIATION). A measurable function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is said to have the extended regular variation property if and only if for all x, y > 0,

$$\lim_{t \to \infty} \frac{h(tx) - h(t)}{h(ty) - h(t)}$$

exists as a function of x, y.

If *h* has the extended regular variation property, there exists some $\alpha \in \mathbb{R}_+$, such that $\lim_{t\to\infty} \frac{h(tx)-h(t)}{h(ty)-h(t)} = \frac{\int_1^x u^{\alpha-1} du}{\int_1^y u^{\alpha-1} du}$, α is called the extended regular variation index of *f*, this is abbreviated as $f \in \text{ERV}(\alpha)$. If $f \in \text{ERV}(\alpha)$, then there exists an auxiliary function $a \in \text{RV}(\alpha)$, such that

$$\lim_{t \to \infty} \frac{h(tx) - h(t)}{a(t)} = \int_1^x u^{\alpha - 1} \mathrm{d}u$$

For $\alpha > 0$, we have $\text{ERV}(\alpha) = \text{RV}(\alpha)$. If $f \in \text{ERV}(0)$ and $\lim_{t\to\infty} h(t) = \infty$, then $f \in \text{RV}(0)$.

A fundamental result in regular variation theory asserts that the convergence in Definitions 6 and 7 is locally uniform over compact sets. Potter's inequalities (See Bingham et al., 1989; de Haan and Ferreira, 2006, for a proof) provide us with a useful quantitative formulation of this result.

Lemma 12 (POTTER'S INEQUALITIES). Let $f \in \text{RV}(\alpha)$ then for all $\epsilon, \delta > 0$, there exists $t_0(\epsilon, \delta)$ such that for all t, x > 0such that $\min(t, tx) > t_0(\epsilon, \delta)$,

$$\left|\frac{h(tx)}{h(t)} - x^{\alpha}\right| \le \epsilon x^{\alpha} \max(x^{\delta}, x^{-\delta}).$$

Another core result is Karamata's integration theorem (See de Haan and Ferreira, 2006, Theorem B.1.5), which intuitively tells us that regularly varying functions can be integrated roughly like their defining monomials.

Theorem 13 (KARAMATA'S INTEGRATION). Let h be regularly varying with index α . Then:

- There exists $t_0 > 0$ such that h(t) is positive and locally bounded for $t \ge t_0$.
- If $\alpha \geq -1$ then:

$$\lim_{t \to \infty} \frac{th(t)}{\int_{t_0}^t h(s) \mathrm{d}s} = \alpha + 1.$$

- If
$$\alpha \leq -1$$
 and $\int h(s) ds < \infty$, then:

$$\lim_{t \to \infty} \frac{th(t)}{\int_t^\infty h(s) \mathrm{d}s} = -\alpha - 1.$$

A concept that proves very useful when relating various regularly varying functions is that of De Bruijn conjugacy, which is effectively a notion of asymptotic inversion of slowly varying functions.

Theorem 14 (DE BRUIJN CONJUGACY). (Bingham et al., 1989, Proposition 1.5.15) Let $L \in \text{RV}(0)$, then there exists a function $L^* \in \text{RV}(0)$ such that $L^*(x)L(xL^*(x)) \to 1$ and $L(x)L^*(xL(x)) \to 1$ as $x \to \infty$. Any function satisfying these two relations is asymptotically equivalent to L^* . The functions (L, L^*) are said to form a pair of De Bruijn conjugates.

APPENDIX B

PROPERTIES OF THE EXACT AND EMPIRICAL THRESHOLDS

At the heart of the upper bound on the redundancy of the ETAC-code derived in Section VI is an understanding of the connection between the empirical threshold M_n constructed from data coming from a specific distribution in the envelope

class and the exact threshold m_n induced by the envelope distribution. It turns out that these results can be derived without particular assumptions on the distributions, and we present them in this appendix. After this, we tackle the task of relating m_n to the number of distinct symbols K_n , which is the quantity that governs the minimax lower bound presented in Section V. There, we use more closely the max-stability property of the envelope distribution.

A. Distribution-free properties

When analyzing the AC-code of (Bontemps, 2011), one needs to work only with extreme order statistics, that is statistics of constant order, and in particular the threshold there is the maximum (order 1). The fact that the threshold M_n of the ETAC-code is effectively equal to $X_{M_n,n}$, which is an *intermediate order statistic* (that is $M_n \to \infty$ while $M_n/n \to 0$ in probability) rather than an extreme may seem to add difficulty to the analysis of the code. Nevertheless, the fluctuations of M_n around its mean value $\mathbb{E}M_n$ can be bounded in a surprisingly simple way. Moreover, this concentration result does not depend on any assumption regarding the distribution of the sample. The fluctuation bounds only depend on the fact that M_n is a function of independent random variables that does not depend too much on any of them.

When working with random variables that can be expressed as functions of other random variables, self-boundedness is a property that can simplify the derivation of moments bounds and concentration properties. We give here the basic definition of self-bounded random variables (refer to Boucheron, Lugosi, and Massart, 2013, Chapters 3 and 6).

Definition 8 (SELF-BOUNDEDNESS). A non-negative random variable $Z = g(X_1, ..., X_n)$, that is a function of n other variables $X_1, ..., X_n$, is called self-bounded if there exists a collection of measurable functions $(g_i)_{i \le n}$, such that letting $Z_i = g_i(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$, we have

$$0 \leq Z - Z_i \leq 1 \quad \text{for each } i \leq n$$

$$\sum_{i=1}^n (Z - Z_i) \leq Z.$$

The next lemma establishes self-boundedness and uses it to assert that whatever the sampling distribution, M_n has "sub-Poissonian" tails.

Lemma 15. Let $X_{1,n} \ge ... \ge X_{n,n}$ be the order statistics of an i.i.d. sample, let $M_n = \min(n, \inf\{k \colon X_{k,n} \le k\})$, then:

- (i) M_n is a self-bounded random variable, as in Definition 8.
- (*ii*) We have the moment bounds:

$$\operatorname{var}(M_n) \leq \mathbb{E}M_n$$

and for all $\lambda \in \mathbb{R}$,

$$\log \mathbb{E}\left[\mathrm{e}^{\lambda(M_n - \mathbb{E}M_n)}\right] \leq \mathbb{E}M_n\left(\mathrm{e}^{\lambda} - \lambda - 1\right) \,.$$

(iii) For all t > 0, we have:

$$\mathbb{P}\left\{M_n - \mathbb{E}M_n \ge t\right\} \le \exp\left(-\frac{t^2}{2(\mathbb{E}M_n + t/3)}\right),\,$$

and

$$\mathbb{P}\left\{M_n - \mathbb{E}M_n \le -t\right\} \le \exp\left(-\frac{t^2}{2(\mathbb{E}M_n)}\right) \,.$$

Proof of Lemma 15: Let $Z = M_n(X_1, \ldots, X_n)$, and for each $i = 1, \ldots, n$, let

$$Z_i = \inf \{ M_n(X_1, \dots, X_{i-1}, x'_i, X_{i+1}, \dots, X_n) : x'_i \in \mathbb{N} \}$$

As M_n is non-decreasing with respect to the product order on the sample, in order to have

$$M_n(X_1,\ldots,X_{i-1},x'_i,X_{i+1},\ldots,X_n) \le M_n,$$

it is necessary to choose $x'_i \leq X_i$.

If $X_i \leq X_{M_n,n}$, choosing x'_i smaller than X_i does not modify the M_n largest order statistics and the value of M_n . If $X_i > X_{M_n,n}$ by choosing $x'_i < X_{M_n,n}$, we obtain

$$M_n(X_1, \dots, X_{i-1}, x'_i, X_{i+1}, \dots, X_n)$$

=
$$\begin{cases} M_n - 1 & \text{if } X_{M_n, n} \le M_n - 1 \\ M_n & \text{otherwise.} \end{cases}$$

Hence $0 \leq Z - Z_i \leq 1$ for all $1 \leq i \leq n$, and $\sum_{i=1}^{n} (Z - Z_i) \leq M_n$. This establishes (i). Then (ii) and (iii) follow from Corollary 3.7 and Theorem 6.12 in (Boucheron, Lugosi, and Massart, 2013).

As usual, concentration inequalities need to be complemented by bounds on expectations. Fortunately, the expected value of M_n can again be related to m_n without any distributional assumptions.

Lemma 16. Let G be a source that belongs to an envelope class Λ defined by a smoothed distribution F. Recall the definition of the exact threshold sequence m_n as the solution of $\overline{F}(x) = x/n$ and of the threshold sequence $M_n =$ $\min(n, \{k : X_{k,n} \le k\})$, where $X_{1,n} \ge X_{2,n} \ge ... \ge X_{n,n}$ are the order statistics of an n-length sequence from G. Then, for all n, we have:

$$\mathbb{E}M_n \le m_n + 3\sqrt{m_n} + 3.$$

Proof of Lemma 16:

We prove a stronger, two-sided, inequality involving the analog to m_n defined directly for G instead of F:

$$m'_n = \min\left\{k \colon \overline{G}(k) \le k/n\right\}$$

In particular, we show that:

$$m'_n - 3\sqrt{m'_n} - 2 \le \mathbb{E}M_n \le m'_n + 3\sqrt{m'_n} + 3.$$

The assertion of the lemma then follows from the fact that $m'_n \leq m_n$, which is a direct consequence of the fact that if G is in the envelope class defined by F, then $\overline{G} \leq \overline{F}$.

We compare the expectations of M_n and m'_n with the following steps. Let $\text{median}[M_n]$ be a median of the distribution of M_n , that is $\text{median}[M_n]$ satisfies $\mathbb{P}\{M_n \leq \text{median}[M_n]\} \geq 1/2$ and $\mathbb{P}\{M_n \geq \text{median}[M_n]\} \geq 1/2$. If we establish concentration bounds to quantify $\mathbb{P}\{M_n \leq a\}$ and $\mathbb{P}\{M_n \geq b\}$ for suitable a, b > 0 in the tail of M_n , we may choose a and b such that these probabilities drop below 1/2. We can then deduce that:

$$a \leq \operatorname{median}[M_n] \leq b$$

To move from the median to the mean, note that by the Lévy-Mallows inequality and point (ii) of Lemma 15, we have

$$|\text{median}[M_n] - \mathbb{E}M_n| \le \sqrt{\operatorname{var}(M_n)} \le \sqrt{\mathbb{E}M_n},$$

from which we can directly deduce

$$\begin{aligned} & \text{median}[M_n] - \sqrt{\text{median}[M_n]} \\ & \leq \mathbb{E}M_n \\ & \leq \text{median}[M_n] + 1 + \sqrt{\text{median}[M_n]}, \end{aligned}$$

and thus

$$a - \sqrt{b} \le \mathbb{E}M_n \le b + \sqrt{b} + 1. \tag{20}$$

It remains to establish the concentration bounds, and to obtain explicit values for a and b. For this, we compare the events of interest to binomial tails, namely the empirical tail count:

$$n\overline{G}_n(x) = \sum_{i=1}^n \mathbb{I}\{X_i > x\}.$$

Note the following properties:

- (i) nG_n is a non-increasing random function of x and, for fixed x, has a binomial distribution of parameters n and G(x).
- (ii) If $\overline{G}(x) \le q$ and $Z \sim \text{Binomial}(n, q)$, then $\mathbb{P}\{n\overline{G}_n(x) \ge b\} \le \mathbb{P}\{Z \ge b\}$.
- (iii) If $\overline{G}(x) \ge p$ and $Z \sim \text{Binomial}(n, p)$, then $\mathbb{P}\{n\overline{G}_n(x) \le a\} \le \mathbb{P}\{Z \le a\}.$
- (iv) We have $n\overline{G}_n(M_n-1) \ge M_n-1$. It follows that when $x \le M_n \underline{1}$ we have $n\overline{G}_n(x) \ge x$.
- (v) We have $n\overline{G}_n(M_n) \leq M_n$. It follows that, when $x \geq M_n$ we have $n\overline{G}_n(x) \leq x$.

The first three properties are evident. The last two make it clear that M_n is effectively an empirical version of m'_n . To establish (iv) $n\overline{G}_n(M_n-1) \ge M_n-1$: all statistics from $X_{1,n}$ to $X_{M_n-1,n}$ are no less than $X_{M_n-1,n}$; the latter is itself greater than $M_n - 1$, by the definition of M_n . To establish (v) $n\overline{G}_n(M_n) \le M_n$: no order statistic beyond $X_{M_n,n}$ can be strictly greater than $X_{M_n,n}$; the latter itself either is no greater than M_n or is so but $M_n = n$, by the definition of M_n ; in both cases the claim remains valid.

Let t > 0. When $M_n \ge m'_n + 1 + t$, we have that $M_n - 1 \ge m'_n + t$ (this is why we need the extra +1). It follows from (iv) that $n\overline{G}_n(m'_n + t) \ge m'_n + t$. Then by the non-increasing property we also have $n\overline{G}_n(m'_n) \le m'_n + t$. By the definition of m'_n , we have $\overline{G}(m'_n) \le m'_n/n$. Let $Z_1 \sim \text{Binomial}(n, m'_n/n)$, then by (ii) we have:

$$\mathbb{P}\{M_n \ge m'_n + 1 + t\} \le \mathbb{P}\{n\overline{G}_n(m'_n) \ge m'_n + t\}$$
$$\le \mathbb{P}\{Z_1 \ge m'_n + t\}.$$

On the other hand, when $M_n \leq m'_n - 1 - t$, it follows from (v) that $n\overline{G}_n(m'_n - 1 - t) \leq m'_n - 1 - t$. (The case $m'_n = 1$ becomes pathological in what follows, but since it allows for any choice of t to yield a vacuous lower bound of the median, we ignore it here.) By the non-increasing property, we also have that $n\overline{G}_n(m'_n - 1) \leq m'_n - 1 - t$. By the definition of m'_n , we have that $\overline{G}(m'_n-1) \ge (m'_n-1)/n$ (this is where we need the extra -1). This time, let $Z_2 \sim \text{Binomial}(n, (m'_n-1)/n)$, then by (iii) we have:

$$\mathbb{P}\{M_n \le m'_n - 1 - t\} \le \mathbb{P}\{n\overline{G}_n(m'_n - 1) \le m'_n - 1 - t\}$$
$$\le \mathbb{P}\{Z_2 \le m'_n - 1 - t\}.$$

We recall now Bernstein's inequalities to bound the tail of binomial distributions. In particular, we have:

$$\mathbb{P}\{Z_1 \ge m'_n + t\} \le \exp\left[-\frac{3}{8}\left(t \wedge \frac{t^2}{m'_n}\right)\right],\$$

and

$$\mathbb{P}\{Z_2 \le m'_n - 1 - t\} \le \exp\left(-\frac{1}{2}\frac{(t+1)^2}{m'_n - 1}\right)$$

It is then easy to verify that the choice of $t = 2\sqrt{m'_n}$ sets both of these bounds below the desired level of $\frac{1}{2}$ for all values of m'_n . Therefore, we can bound median $[M_n]$ by $a = m'_n - 1 - t = m'_n - 2\sqrt{m'_n} - 1$ from below and by $b = m'_n + 1 + t = m'_n + 2\sqrt{m'_n} + 1 = (\sqrt{m'_n} + 1)^2$ from above, and use these quantities in Equation (20) to bound $\mathbb{E}[M_n]$. The constants claimed in the lemma follow immediately.

B. Distribution-dependent properties

We now describe a general connection between M_n and the number of distinct symbols K_n , that is the expected size of the empirical alphabet. From the very definition of M_n , if $M_n < n$, we have $K_n \le 2M_n$. Indeed, as $X_{M_n,n} < M_n$ there are no more than M_n distinct symbols not larger than $X_{M_n,n}$ and there are at most M_n distinct symbols larger than $X_{M_n,n}$. Hence, whatever the sampling distribution,

$$\mathbb{E}K_n \leq 2\mathbb{E}M_n$$
.

As we use $\mathbb{E}K_n$ in the lower bound on minimax redundancy, we actually need an inequality in the other direction. We now establish this under distributional assumptions.

Lemma 17. Let a distribution G on \mathbb{N}_+ belong to some $MDA(\gamma), \gamma > 0$, with a probability mass function that is ultimately monotonically non-increasing. Let $(m'_n)_n$ be defined as $m'_n = \min \{k \in \mathbb{N}_+ : \overline{G}(k) \le k/n\}$. Then there exists a constant κ'_{γ} and some n_0 (that may depend on G), such that for all $n \ge n_0$, the expected number of distinct symbols in a sample from G satisfies

$$\kappa_{\gamma}' m_n' \le \mathbb{E} K_n$$

Proof:

Let g denote the probability mass function corresponding to G. The regular variation property of G then passes in a straightforward way to g via so-called Tauberian theorems. In particular, recalling that we can write $\overline{G}(x) = x^{-1/\gamma}L(x)$, a simple adaptation of Theorem 1.7.2 of Bingham et al. (1989) shows that as $j \to \infty$:

$$g(j) \sim \frac{\overline{G}(j)}{\gamma j}$$

Given $\beta \ge 1$, let $k_n = \beta m'_n$ be a dilation of the threshold m'_n , which we will choose appropriately. Note that as $n \to \infty$,

we also have that both $k_n, m'_n \to \infty$. Given $\epsilon > 0$, we shall choose n_0 (which may depend on all of ϵ , β and G), such that for all $n > n_0$, several assertions hold true. In particular, for all $j \geq m'_n$, we have:

- (i) $g(\underline{j})$ is monotonically non-increasing (by assumption). (ii) $\frac{\overline{G}(\underline{j})}{(1+\epsilon)\gamma j} \leq g(\underline{j}) \leq \frac{(1+\epsilon)\overline{G}(\underline{j})}{\gamma j}$ (by the Tauberian theorem).
- (iii) $\frac{\overline{G}(j)}{j} \geq \frac{1}{(1+\epsilon)} \frac{\overline{G}(j-1)}{j-1}$ (by regular variation limits over compact intervals, cf. Potter's inequalities in Lemma 12).
- $\begin{array}{ll} \text{(iv)} & \overline{G}(\beta m'_n) \leq (1+\epsilon)\beta^{-1/\gamma}\overline{G}(m'_n) \text{ (by regular variation).} \\ \text{(v)} & \frac{\overline{G}(\beta m'_n)}{m'_n} \geq \frac{1}{(1+\epsilon)}\beta^{-1/\gamma}\frac{\overline{G}(m'_n-1)}{m'_n-1} \text{ (by regular variation and similarly to (iii) above).} \end{array}$

Let L_n be the number of distinct symbols in X_1, \ldots, X_n that are larger than k_n . Then, when n is beyond n_0 , we have:

$$\mathbb{E}L_n = \sum_{k>k_n} (1 - (1 - g(k))^n)$$

$$\geq \sum_{k>k_n} ng(k) \left(1 - \frac{ng(k)}{2}\right)$$

$$\geq n\overline{G}(k_n) \left(1 - \frac{ng(k_n)}{2}\right)$$

$$\geq n\overline{G}(k_n) \left(1 - (1 + \epsilon)\frac{n\overline{G}(k_n)}{2\gamma k_n}\right)$$

where the first line is exact, the second step is an approximation, the third and fourth steps use assertions (i) and (ii) respectively.

If $\gamma > 1$, we can simply set $\beta = 1$, which would give us

$$\mathbb{E}L_n \ge n\overline{G}(m'_n) \left(1 - (1+\epsilon) \frac{n\overline{G}(m'_n)}{2\gamma m'_n} \right)$$
$$\ge n \frac{1}{(1+\epsilon)} m'_n \frac{\overline{G}(m'_n-1)}{m'_n - 1} \left(1 - (1+\epsilon) \frac{n\overline{G}(m'_n)}{2\gamma m'_n} \right)$$
$$\ge \frac{m'_n}{1+\epsilon} \left(1 - \frac{1+\epsilon}{2} \right),$$

where the first step is a substitution, the second step uses assertion (iii), and the last step uses the fact that $\gamma \geq 1$ and the definition of m'_n , which implies that $G(m'_n)/m'_n \leq 1/n$ whereas $G(m'_n - 1)/(m'_n - 1) > 1/n$.

If $\gamma < 1$, if we attempt the above we end up with a lower bound that may be negative and thus vacuous. We remedy the problem by choosing β appropriately. We have:

 $\mathbb{E}L_n$

$$\geq n\overline{G}(\beta m'_{n}) \left(1 - (1+\epsilon) \frac{n\overline{G}(\beta m'_{n})}{2\gamma\beta m'_{n}} \right)$$

$$\geq n \frac{1}{(1+\epsilon)} \beta^{-\frac{1}{\gamma}} m'_{n} \frac{\overline{G}(m'_{n-1})}{m'_{n-1}} \left(1 - (1+\epsilon)\beta^{-\frac{1}{\gamma}} \frac{n\overline{G}(m'_{n})}{2\gamma\beta m'_{n}} \right)$$

$$\geq \beta^{-\frac{1}{\gamma}} \frac{m'_{n}}{1+\epsilon} \left(1 - \beta^{-\frac{1}{\gamma}-1} \frac{1+\epsilon}{2\gamma} \right),$$

where now the second step uses assertions (iv) and (v), and the last step uses again the definition of m'_n . Therefore, we may choose $\beta = \gamma^{-\frac{1}{\gamma+1}}$ when $\gamma < 1$ to obtain the same functional form of the lower bound when $\gamma \geq 1$, up to a constant factor. We combine these two cases by letting $k_n = \left(1 \lor \gamma^{-\frac{\gamma}{\gamma+1}}\right)$ and choosing $\epsilon = 1/3$, to obtain:

$$\mathbb{E}L_n \ge \left(1 \wedge \gamma^{\frac{1}{\gamma+1}}\right) \frac{m'_n}{1+\epsilon} \left(1 - \frac{1+\epsilon}{2}\right)$$
$$\ge \frac{1}{4} \left(1 \wedge \gamma^{\frac{1}{\gamma+1}}\right) m'_n.$$

This bound is sufficient to complete the lemma, since $\mathbb{E}K_n \geq \mathbb{E}L_n$. We can try to improve it, by incorporating symbols below m'_n . However, without further assumptions, we cannot do so. One thing we can do, in the $\gamma < 1$ case, is to smooth out this bound by incorporating symbols between m'_n and k_n . Let S_n be the number of distinct symbols $k \leq k_n$ in X_1, \ldots, X_n , with the same choice of k_n and for $n \ge n_0$. We then have:

$$\mathbb{E}S_n = \sum_{k \le k_n} (1 - (1 - g(k))^n) \mathbb{I}_{g(k) > 0}$$

$$\geq \sum_{m'_n < k \le k_n} (1 - (1 - g(k_n))^n)$$

$$\geq (k_n - m'_n) \left(1 - e^{-1/(1 + \epsilon)^2}\right)$$

$$\geq \frac{1}{4} (k_n - m_n).$$

Here, the first line is exact. The second step uses assertion (i) both to bound the probabilities and to insure their positivity (indeed, if g(j) = 0 for some $j \ge m'_n$, then it is so beyond that by monotonicity, which contradicts the regular variation property at infinity). The third step uses an approximation, in addition to assertions (ii) and (v), and the definitions of m'_n and β . The last step is an arbitrary (not necessary the best) choice of ϵ . This bound is zero if $\gamma \geq 1$. If $\gamma < 1$, however, we have:

$$\mathbb{E}S_n \ge \frac{m_n}{4} \left(\gamma^{-\gamma/(\gamma+1)} - 1 \right) \,.$$

Combining L_n and S_n , we can write:

$$\mathbb{E}K_n = \mathbb{E}L_n + \mathbb{E}S_n$$

$$\geq \frac{m_n}{4} \left(1 \wedge \gamma^{1/(\gamma+1)} + \left(\gamma^{-\gamma/(\gamma+1)} - 1\right)_+ \right).$$

APPENDIX C PROOFS OF LEMMAS IN THE MAIN TEXT

Proof of Lemma 3

- (i) For sufficiently large t, U(t/1) 1 > 0, and as $x \mapsto$ U(t/x) - x decreases continuously to $-\infty$ on $[1,\infty)$, there exists some x = m(t) such that U(t/x) - x = 0. Hence, the function m is defined over $(U^{-1}(1), \infty)$. If $U^{-1}(1) < t < t', U(t'/m(t)) > U(t/m(t)) = m(t),$ hence m(t') > m(t).
- (ii) The continuous differentiability of m over $(U^{-1}(1), \infty)$ is a consequence of the implicit function theorem (see (Trench, 2003)). Moreover, the derivative of m satisfies:

$$m'(t) = \frac{U'(t/m(t))}{(t/m(t))U'(t/m(t)) + m(t)}$$

- (iii) Assume on the contrary that m is upper bounded by $B < \infty$, then $U(t/B) \le U(t/m(t)) \le B$ for $t \in (U^{-1}(1), \infty)$. As U(t/B) tends to infinity, we obtain a contradiction. Assume now that t/m(t) is upper bounded by $C < \infty$, then $m(t) = U(t/m(t)) \le C$, we obtain another contradiction.
- (iv) As $U \in \text{ERV}(\gamma), \gamma \ge 0$ and $U(\infty) = \infty, U \in \text{RV}(\gamma)$. Then the function $L(t) = t^{-\gamma}U(t)$ is slowly varying. The definition of m translates into

$$m(t) = \frac{t^{\gamma}}{m(t)^{\gamma}} L\left(\frac{t}{m(t)}\right) \,,$$

or equivalently

$$1 = \frac{t^{\gamma}}{m(t)^{1+\gamma}} L\left(\left(t\frac{t^{\gamma}}{m(t)^{1+\gamma}}\right)^{1/(1+\gamma)}\right)$$

The function $L_1(t) = L(t^{1/(1+\gamma)})$ is slowly varying, hence the function $L_1^*: t \to \frac{t^{\gamma}}{m(t)^{1+\gamma}}$ appears as its De Bruijn conjugate, as such it is a slowly varying function. One line of computation reveals that m is regularly varying with index $\gamma/(\gamma + 1)$ and

$$m(t) \sim t^{\gamma/(\gamma+1)} / (L_1^*(t))^{1/(1+\gamma)}$$
.

Proof of Lemma 7

Since f is ultimately monotonically non-decreasing, it immediately follows that the same is true for g. We focus therefore on showing that $G \in MDA(\gamma)$. For this, we sandwich \overline{G} by a scaled version of \overline{F} .

Given $\epsilon > 0$, then for k large enough, we have:

$$\overline{G}(k) = \sum_{j'>k} g(j') = \sum_{j'>k} f(2j'-j_0) \wedge f(2j'-j_0+1)$$

$$\leq \sum_{j'>k} \frac{1}{2} \left[f(2j'-j_0) + f(2j'-j_0+1) \right]$$

$$= \frac{1}{2} \overline{F}(2k-j_0) \leq \frac{(1+\epsilon)}{2} \overline{F}(2k), \qquad (21)$$

where we have simply used the fact that the minimum lies below the average and the regular variation property of \overline{F} , with a slack of $1 + \epsilon$.

Since F is regularly varying with index $-1/\gamma$, by a simple adaptation of Theorem 1.7.2 of Bingham et al. (1989) (cf. the proof of Lemma 17 for a full relationship), so is f with index $-1/\gamma - 1$. In particular, it follows from this that $f(j+1)/f(j) \rightarrow 1$. Given $\delta > 0$, we thus have that for j' large enough:

$$\frac{f(2j'-j_0) \wedge f(2j'-j_0+1)}{f(2j'-j_0) + f(2j'-j_0+1)} > \frac{1}{2+\delta}.$$

Using this observation and the same steps above, we have that for j' large enough:

$$\overline{G}(k) > \sum_{j'>k} \frac{1}{2+\delta} \left[f(2j'-j_0) + f(2j'-j_0+1) \right] \\ = \frac{1}{2\sqrt{1+\epsilon}} \overline{F}(2k-j_0) \ge \frac{1}{2(1+\epsilon)} \overline{F}(2k), \quad (22)$$

where we choose the δ of the ratio test appropriately to get $2 + \delta = 2\sqrt{1 + \epsilon}$, to compound its effect with the regular variation slack of $\sqrt{1 + \epsilon}$ given to \overline{F} .

From the sandwiching offered by the two bounds of Equations (21) and (22), it follows immediately that \overline{G} is also regularly varying at infinity with index $-1/\gamma$, and that thus $G \in MDA(\gamma)$.

To compare m'_n to m_n , note that if $k \leq m_{n/(1+\epsilon)}/2$, then since for all $t < m_{n/(1+\epsilon)}$ we have $\overline{F}(t) > \frac{t}{n/(1+\epsilon)}/2$, Equation (22) gives us that $\overline{G}(k) > \frac{1}{2(1+\epsilon)}(1+\epsilon)\frac{2k}{n} = k/n$. It follows that $m'_n > k$ for all $k \leq m_{n/(1+\epsilon)}/2$, and thus $m'_n \geq m_{n/(1+\epsilon)}/2$. By the regular variation property of m_n (see Lemma 3), we have $m_{n/(1+\epsilon)} \sim (1+\epsilon)^{-\frac{\gamma}{\gamma+1}}m_n$. This means that for large enough n, we can pay an additional factor of $1+\epsilon$ to get $m_{n/(1+\epsilon)} > \frac{1}{1+\epsilon}(1+\epsilon)^{-\frac{\gamma}{\gamma+1}}m_n > \frac{1}{(1+\epsilon)^2}m_n$. We thus have, for large enough n:

$$m'_n \ge \frac{1}{2(1+\epsilon)^2}m_n.$$

A bound in the other direction follows similarly.

Proof of Lemma 9

Before we proceed, we give a convenient representation of $\Sigma(u)$ in an integral form. We have:

$$\begin{split} \Sigma(u) &= \frac{1}{\ln 2} \int_{1}^{\infty} \mathbb{I}\{y > u\} \ln(1+y-u) \mathbb{P}(\mathrm{d}y) \\ &= \frac{1}{\ln 2} \int_{1}^{\infty} \mathbb{I}\{y > u\} \int_{1}^{\infty} \frac{\mathbb{I}\{u < x < y\}}{1+x-u} \mathrm{d}x \mathbb{P}(\mathrm{d}y) \\ &= \frac{1}{\ln 2} \int_{1}^{\infty} \mathbb{I}\{x > u\} \frac{1}{1+x-u} \int_{1}^{\infty} \mathbb{I}\{y > x\} \mathbb{P}(\mathrm{d}y) \mathrm{d}x \\ &= \frac{1}{\ln 2} \int_{u}^{\infty} \frac{\overline{G}(x)}{1+x-u} \mathrm{d}x, \end{split}$$
(23)

where we have written an integral form of the logarithm and used Fubini's theorem to swap the integrals.

When G belongs to an envelope class defined by F, we have $\overline{G} \leq \overline{F}$, and therefore we can see from Equation (23) that $\Sigma(u)$ under G is dominated by that under F. In particular, when $F \in \text{MDA}(\gamma)$ with $\gamma > 0$, it admits logarithmic moments, and we trivially see that $\Sigma(u)$ is finite. But what we are really interested in is the decay of $\Sigma(u)$ as u grows.

Equation (23) shows that the decay of $\Sigma(u)$ is governed by the decay of $\overline{G}(u)$ itself, which dominates for small values of x, and is then complemented by the decay of 1/(1 + x - u). We can capture this compromise by splitting the integral at some arbitrary point, say u + t - 1 for some $t \ge 1$. We have:

$$\Sigma(u) = \frac{1}{\ln 2} \int_{u}^{u+t-1} \frac{\overline{G}(x)}{1+x-u} \mathrm{d}x \tag{24}$$

$$+\frac{1}{\ln 2} \int_{u+t-1}^{\infty} \frac{\overline{G}(x)}{1+x-u} \mathrm{d}x \tag{25}$$

$$\leq \overline{G}(u)\log(t) + \frac{1}{\ln 2} \int_{t}^{\infty} \frac{\overline{G}(y)}{y} \mathrm{d}y, \qquad (26)$$

where we have split the integral, bounded \overline{G} in both parts by its largest value, and performed the integration of the first part and a change of variable in the second. For the latter, we proceed by first bounding by the envelope:

$$\int_t^\infty \frac{\overline{G}(y)}{y} \mathrm{d}y \le \int_t^\infty \frac{\overline{F}(y)}{y} \mathrm{d}y.$$

We would now like to invoke Karamata's integration theorem. Let us make the change of variable y = m(z) and set $z_0 = m^{-1}(t)$. By using the property that $\overline{F}(m(z)) = m(z)/z$ and by performing an integration by parts, we get:

$$\int_t^\infty \frac{\overline{F}(y)}{y} \mathrm{d}y = \int_{z_0}^\infty \frac{\mathrm{d}m(z)}{z} = -\frac{m(z_0)}{z_0} + \int_{z_0}^\infty \frac{m(z)}{z^2} \mathrm{d}z.$$

Now note that $\frac{m(z)}{z^2}$ is regularly varying with index $\frac{\gamma}{\gamma+1}-2$, by Lemma 3. By using Karamata's integration theorem, Theorem 13, we find that given $\epsilon > 0$, for large enough t,

$$\int_{z_0}^{\infty} \frac{m(z)}{z^2} \mathrm{d}z \le (\gamma + 1 + \epsilon \ln 2) \frac{m(z_0)}{z_0}.$$

Combining the last three equations together, we have:

$$\int_{t}^{\infty} \frac{G(y)}{y} \mathrm{d}y \le (\gamma + \epsilon \ln 2) \frac{m(z_0)}{z_0},$$

and the claim follows using the fact that $m(z_0)/z_0 = \overline{F}(m(z_0)) = \overline{F}(t)$.

Proof of Lemma 10

Recall that:

$$\mathbb{E}[\overline{G}(M_i)] = \mathbb{P}\{X_{i+1} > M_i\} = \mathbb{E}[\mathbb{I}\{X_{i+1} > M_i\}]$$

We would like to exploit the independence structure (in fact, only the exchangeability aspect of independence). To make this symmetry complete for the event of interest, in what follows we effectively replace M_i by a new threshold, equal to $M_{i+1} - 1$.

Let ς indicate a uniform random permutation of $1, \dots, i+1$ that we inject into the probability space. Note that M_i never decreases and increases at most by 1 at every new sample (see also the property of self-boundedness in the Appendix of the paper). Furthermore M_{i+1} is permutation invariant, as its definition relies only on order statistics. In light of these properties, we can write:

$$\mathbb{P}\{X_{i+1} > M_i\} \le \mathbb{P}\{X_{i+1} > M_{i+1} - 1\}$$

= $\mathbb{P}\{X_{\varsigma(i+1)} > M_{i+1} - 1\}.$

Let us now condition on the values of the samples X_1, \dots, X_{i+1} . This fixes the value of $M_{i+1}-1$, by invariance. The only randomness that remains in the last expression is that due to permutations. Now note that the event $\{X_{\varsigma(i+1)} > M_{i+1} - 1\}$ occurs a fraction of times corresponding to the number of samples strictly larger than $M_{i+1} - 1$, or equivalently greater than or equal to M_{i+1} . Thus:

$$\mathbb{P}\{X_{\varsigma(i+1)} > M_{i+1} - 1 | X_1, \cdots, X_{i+1}\} = \frac{\sum_{j=1}^{i+1} \mathbb{I}\{X_j > M_{i+1} - 1\}}{i+1}.$$

Finally, observe that we have:

$$\sum_{j=1}^{i+1} \mathbb{I}\{X_j > M_{i+1} - 1\} = \max\{0, \ k : X_{k,i+1} \ge M_{i+1}\}$$
$$\leq M_{i+1},$$

where the inequality follows from the fact that all order statistics of rank greater than or equal to M_{i+1} are no greater than M_{i+1} , by the definition $M_{i+1} = \min\{i+1, k: X_{k,i+1} \le k\}$. Therefore, as claimed:

$$\mathbb{E}[\overline{G}(M_i)] \leq \mathbb{E}\left[\mathbb{P}\{X_{\varsigma(i+1)} > M_{i+1} - 1 | X_1, \cdots, X_{i+1}\}\right]$$
$$\leq \frac{\mathbb{E}[M_{i+1}]}{i+1}.$$

Proof of Lemma 11

We have that given ϵ , then beyond some i_0 :

$$\mathbb{E}[\ell(C_{\rm E})] \leq 2 \sum_{i=1}^{n-1} \left(\mathbb{E}[\Sigma(M_i)] + \rho \mathbb{E}[G(M_i)] \right) \\ \leq 2 \sum_{i=1}^{i_0-2} \left(\mathbb{E}[\Sigma(M_i)] + \rho \mathbb{E}[G(M_i)] \right) \\ + \left(2 + \epsilon'\right) \sum_{i=i_0-1}^{n-1} \frac{m_{i+1}}{i+1} \log(m_{i+1}) \\ + \left(\gamma/\ln 2 + \rho + \epsilon'\right) \frac{m_{i+1}}{i+1} \\ \leq \kappa + \left(2 + \epsilon\right) \sum_{i=i_0-1}^{n-1} \frac{m_{i+1} \log(m_{i+1})}{i+1},$$

where we have combined Equations (13) and (15) into Equation (11), and where the adjustment between ϵ and ϵ' is made a priori.

To establish the integral bound, we are at first tempted to assume that $(m_i \ln m_i)/i$ is non-increasing. However, this is not strictly true. The furthest m_{i+1} will move away from m_i is when \overline{F} remains constant (equal to $\frac{m_i}{i}$) between m_i and m_{i+1} . This would mean that $m_{i+1}/(i+1) = \frac{m_i}{i}$, and thus $m_{i+1} \leq (1+1/i)m_i$. From this, we find that:

$$\frac{m_{i+1}\log m_{i+1}}{i+1} \le \frac{m_i\log m_i}{i} + \frac{m_i\log(1+1/i)}{i} \le \frac{m_i\log m_i}{i} + \frac{m_i}{i}.$$

Therefore, the integral can deviate from the sum by at most $\sum \frac{m_i}{i}$, which is of negligible order compared to the magnitude of the sum.

The direct bound follows by noting that we can use Jensen's inequality and the fact that m(n) is non-decreasing, to show:

$$\int_{1}^{n} \frac{1}{t} m(t) \log m(t) dt$$

$$\leq \left(\int_{1}^{n} \frac{1}{t} m(t) dt \right) \log \left(\frac{\int_{1}^{n} \frac{1}{t} m^{2}(t) dt}{\int_{1}^{n} \frac{1}{t} m(t) dt} \right)$$

$$\leq m(n) \log(n) \log m(n).$$

Lastly, to specialize to the Fréchet case, recall (by Lemma 3) that m(t) is $RV_{\gamma/(\gamma+1)}$, therefore we also have that $(m(t) \log m(t))/t$ is $RV_{-1/(\gamma+1)}$. Karamata's integration theorem, Theorem 13, then tells us that given $\epsilon > 0$, there exists a t_0 and $t_1 > t_0$ such that for all $n > t_1$:

$$\frac{m_n \log m_n}{\int_{t_0}^n \frac{1}{t} m(t) \log m(t) dt} \ge (1-\epsilon) \left[\frac{-1}{1+\gamma} + 1\right] = (1-\epsilon) \frac{\gamma}{1+\gamma}.$$
(27)

When $\gamma > 0$, we can therefore combine Equations (17) and (27) to write that there exists a constant κ such that for large enough n:

$$\mathbb{E}[\ell(C_{\rm E})] \leq \kappa + (2+\epsilon)\frac{\gamma+1}{\gamma}m_n\log m_n$$
$$\leq (2+o_{\Lambda}(1))\frac{\gamma+1}{\gamma}m_n\log m_n$$
$$\leq (2+o_{\Lambda}(1))m_n\log n,$$

where for the last expression we have used the regular variation property of $m_n \sim n^{\gamma/(\gamma+1)}L_m(n)$, for some slowly varying function L_m (given in Lemma 3), and the fact that $\log L(n)/\log n \to 0$ for any slowly varying function L.