Linear codes with two or three weights from weakly regular bent functions

Chunming Tang, Nian Li, Yanfeng Qi, Zhengchun Zhou, Tor Helleseth

Abstract

Linear codes with few weights have applications in consumer electronics, communication, data storage system, secret sharing, authentication codes, association schemes, and strongly regular graphs. This paper first generalizes the method of constructing two-weight and three-weight linear codes of Ding et al. [6] and Zhou et al. [27] to general weakly regular bent functions and determines the weight distributions of these linear codes. It solves the open problem of Ding et al. [6]. Further, this paper constructs new linear codes with two or three weights and presents the weight distributions of these codes. They contains some optimal codes meeting certain bound on linear codes.

Index Terms

Linear codes, weight distribution, weakly regular bent functions, cyclotomic fields, secret sharing schemes

I. INTRODUCTION

Throughout this paper, let p be an odd prime and $q = p^m$, where m is a positive integer. An [n, k, d] code C is a k-dimension subspace of \mathbb{F}_p^n with minimum Hamming distance d. Let A_i be the number of codewords with Hamming weight i in C. The polynomial $1 + A_1z + \cdots + A_nz^n$ is called the weight enumerator of C and $(1, A_1, \cdots, A_n)$ called the weight distribution of C. The minimum distance d determines the error correcting capability of C. The weight distribution contains important information for estimating the probability of error detection and correction. Hence, the weight distribution attracts much attention in coding theory and much work focus on the determination of the weight distributions of linear codes. Let t be the number of nonzero A_i in the weight distribution. Then the code C is called a t-weight code. Linear codes can be applied in consumer electronics, communication and data storage system. Linear codes with few weights are of important in secret sharing [3], [25], authentication codes [10], association schemes [1] and strongly regular graphs [2].

Let $F(x) \in \mathbb{F}_q[x]$ and $f(x) = \operatorname{Tr}_1^m(F(x))$, where Tr_1^m is the trace function from \mathbb{F}_q to \mathbb{F}_p . Let $D = \{x \in \mathbb{F}_q^{\times} : f(x) = 0\}$. Denote n = #(D) and $D = \{d_1, d_2, \dots, d_n\}$. Then a linear code of length n defined over \mathbb{F}_p is

$$\mathcal{C}_D = \{ (\mathrm{Tr}_1^m(\beta d_1), \mathrm{Tr}_1^m(\beta d_2), \cdots, \mathrm{Tr}_1^m(\beta d_n)) : \beta \in \mathbb{F}_q \},\$$

where D is called the defining set of C_D .

Note that by the choice of D many linear codes can be constructed [7], [8], [9]. Ding et al. [5], [6] and Zhou et al. [27] constructed some classes of two-weight and three-weight linear codes. Ding et al. [6] presented the weight distribution of C_D for the case $F(x) = x^2$ and proposed an open problem on how to determine the weight distribution of C_D for general planar functions F(x). Zhou et al. [27] gave the weight distribution of C_D for quadratic planar functions F(x).

In this paper, we consider linear codes with two or three weights from weakly regular bent functions. First, we generalize the method of constructing two-weight and three-weight linear codes of Ding et al. [6] and Zhou et al. [27] to general weakly regular bent functions. The weight distributions of these linear codes are determined by the theory of cyclotomic fields. And we solve the open problem of Ding et al. [6]. Further, by choosing the defining sets different from that of Ding et al. [6] and Zhou et al. [27], we construct new linear codes with two or three weights and present the weight distributions of these codes. The weight distributions of linear codes constructed in this paper are completely determined by the sign of the Walsh transform of weakly regular bent functions.

This paper is organized as follows: Section II introduces cyclotomic fields, weakly regular bent functions and exponential sums. Section III generalizes the method of Ding et al. [6] and Zhou et al. [27] to general weakly regular bent functions and determines the weight distributions of these linear codes. Section IV constructs new classes of two-weight and three-weight linear codes. Section V determines the sign of the Walsh transform of some weakly regular bent functions. Section VI makes a conclusion.

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II. PRELIMINARIES

In this section, we state some basic facts on cyclotomic fields, weakly regular bent functions and exponential sums. These results will be used in the rest of the paper for linear codes with few weights. First some notations are given. Let \mathbb{Z} be the rational integer ring and Q the rational field. Let η be the quadratic character of \mathbb{F}_q^{\times} such that $\eta(a) = \begin{cases} 1, & a^{(q-1)/2} = 1; \\ -1, & a^{(q-1)/2} = -1. \end{cases}$ $(a \in \mathbb{F}_q^{\times})$. Let $\left(\frac{a}{p}\right)$ be the Legendre symbol for $1 \le a \le p-1$. Let $p^* = \left(\frac{-1}{p}\right)p = (-1)^{(p-1)/2}p$ and $\zeta_p = e^{\frac{2\pi\sqrt{-1}}{p}}$ be the primitive *p*-th root of unity *p*-th root of unity.

A. Cyclotomic field $Q(\zeta_n)$

Some results on cyclotomic field $Q(\zeta_p)$ [18] are given in the following lemma.

Lemma 2.1: (i) The ring of integers in $K = Q(\zeta_p)$ is $\mathcal{O}_K = \mathbb{Z}(\zeta_p)$ and $\{\zeta_p^i : 1 \le i \le p-1\}$ is an integral basis of \mathcal{O}_K , where $\zeta_p = e^{\frac{2\pi\sqrt{-1}}{p}}$ is the primitive *p*-th root of unity. (ii) The field extension K/Q is Galois of degree p-1 and the Galois group $Gal(K/Q) = \{\sigma_a : a \in (\mathbb{Z}/p\mathbb{Z})^{\times}\}$, where the

automorphism σ_a of K is defined by $\sigma_a(\zeta_p) = \zeta_p^a$.

(iii) The field K has a unique quadratic subfield $L = Q(\sqrt{p^*})$ where $p^* = \left(\frac{-1}{p}\right)p = (-1)^{(p-1)/2}p$, where $\left(\frac{a}{p}\right)$ is the Legendre symbol for $1 \le a \le p-1$. For $1 \le a \le p-1$, $\sigma_a(\sqrt{p^*}) = (\frac{a}{p})\sqrt{p^*}$. Therefore, the Galois group Gal(L/Q) is $\{1, \sigma_\gamma\}$, where γ is any quadratic nonresidue in \mathbb{F}_p .

B. Weakly regular bent functions

Let f(x) be a function from \mathbb{F}_q to \mathbb{F}_p $(q = p^m)$, the Walsh transform of f is defined by

$$\mathcal{W}_f(\beta) := \sum_{x \in \mathbb{F}_q} \zeta_p^{f(x) + \operatorname{Tr}_1^m(\beta x)},$$

where $\zeta_p = e^{2\pi\sqrt{-1}/p}$ is the primitive *p*-th root of unity, $\operatorname{Tr}_1^m(x) = \sum_{i=0}^{m-1} x^{p^i}$ is the trace function from \mathbb{F}_q to \mathbb{F}_p , and $\beta \in \mathbb{F}_q$. The inverse Walsh transform of such f(x) gives

$$\zeta_p^{f(x)} = \frac{1}{p^m} \sum_{\beta \in \mathbb{F}_q} \mathcal{W}_f(\beta) \zeta_p^{-\operatorname{Tr}_1^m(\beta x)}.$$
(1)

The function f(x) is a *p*-ary bent functions, if $|\mathcal{W}_f(\beta)| = p^{\frac{m}{2}}$ for any $\beta \in \mathbb{F}_q$. A bent function f(x) is regular if there exists some p-ary function $f^*(x)$ satisfying $\mathcal{W}_f(\beta) = p^{\frac{m}{2}} \zeta_p^{f^*(\beta)}$ for any $\beta \in \mathbb{F}_q$. A bent function f(x) is weakly regular if there exists a complex u with unit magnitude satisfying that $\mathcal{W}_f(\beta) = up^{\frac{m}{2}} \zeta_p^{f^*(\beta)}$ for some function $f^*(x)$. Such function $f^*(x)$ is called the dual of f(x). From [15], [17], a weakly regular bent function f(x) satisfies that

$$\mathcal{W}_f(\beta) = \varepsilon \sqrt{p^*}^m \zeta_p^{f^*(\beta)},\tag{2}$$

where $\varepsilon = \pm 1$ is called the sign of the Walsh Transform of f(x) and $p^* = \left(\frac{-1}{p}\right)p$. From Equation (1), for the weakly regular bent function f(x), we have $\sum_{\beta \in \mathbb{F}_q} \zeta_p^{f^*(\beta) - \operatorname{Tr}_1^m(\beta x)} = \varepsilon p^m \zeta_p^{f(x)} / \sqrt{p^*}^m$. Note that $p^m = \left(\frac{-1}{p}\right)^m \sqrt{p^*}^{2m}$, we have

$$\mathcal{W}_{f^*}(-x) = \left(\frac{-1}{p}\right)^m \varepsilon \sqrt{p^*}^m \zeta_p^{f(x)}, x \in \mathbb{F}_q.$$
(3)

The dual of a weakly regular bent function is also weakly regular bent and $f^{**}(x) = f(-x)$. The sign of the Walsh transform of f^* is $\left(\frac{-1}{n}\right)^m \varepsilon$. Some results on weakly regular bent functions can be found in [11], [13], [15], [16], [17], [21].

The construction of bent functions is an interesting and hot research topic. A class of bent functions is derived from planar functions. A function mapping from $\mathbb{F}_q \mapsto \mathbb{F}_p$ is a planar function, if for any $a \in \mathbb{F}_q^{\times}$ and $b \in \mathbb{F}_q$, $\#\{x : f(x+a) - f(x) = x\}$ $b\} = 1$. A simple example of planar functions is the square function $F(x) = x^2$. Almost known planar functions are quadratic functions, i.e., $F(x) = \sum_{0 \le i \le j \le m-1} a_{ij} x^{p^i + p^j}$, which are corresponding to semi-fields. Coulter and Matthews [4] introduced a class of non-quadratic planar functions $F(x) = x^{(3^k+1)/2}$, where p = 3, k is odd, and (m,k) = 1. The derived p-ary functions $f(x) = \text{Tr}_1^m(\beta F(x))$ for any $\beta \in \mathbb{F}_q^{\times}$ from these known planar functions are all weakly regular bent functions. It is still an open problem whether the derived p-ary function from any a planar function is a weakly regular bent function. It is often difficult to determine the sign of the Walsh transform of weakly regular bent functions.

Let \mathcal{RF} be a set of *p*-ary weakly regular bent functions with the following conditions:

(i) f(0) = 0,

(ii) There exists an integer h such that (h-1, p-1) = 1 and $f(ax) = a^h f(x)$ for any $a \in \mathbb{F}_p^{\times}$ and $x \in \mathbb{F}_q$.

Note that \mathcal{RF} contains almost known weakly regular bent functions. We will discuss properties of functions in \mathcal{RF} .

Lemma 2.2: Let $a_i, b_i \in \mathbb{Z}$ $(0 \le i \le p-1)$ such that $\sum_{i=0}^{p-1} a_i \equiv \sum_{i=0}^{p-1} b_i \mod 2$ and $\sum_{i=0}^{p-1} a_i \zeta_p^i = \sum_{i=0}^{p-1} b_i \zeta_p^i$. Then $a_i \equiv b_i \mod 2$.

Proof: From $\sum_{i=0}^{p-1} a_i \zeta_p^i = \sum_{i=0}^{p-1} b_i \zeta_p^i$, we have

$$\sum_{i=0}^{p-1} (a_i - b_i)\zeta_p^i = 0.$$

The minimal polynomial of ζ_p over Q is $1 + x + \cdots + x^{p-1}$. Then we have

$$a_0 - b_0 = a_1 - b_1 = \dots = a_{p-1} - b_{p-1} = \lambda,$$

where $\lambda \in \mathbb{Z}$. Hence,

$$p\lambda = (\sum_{i=0}^{p-1} a_i) - (\sum_{i=0}^{p-1} b_i), \lambda = \frac{1}{p} (\sum_{i=0}^{p-1} a_i - \sum_{i=0}^{p-1} b_i)$$

From $\sum_{i=0}^{p-1} a_i \equiv \sum_{i=0}^{p-1} b_i \mod 2$, we have

$$\lambda \equiv 0 \mod 2.$$

Hence, $a_i \equiv b_i \mod 2$.

Lemma 2.3: Let p be an odd prime, $p^* = \left(\frac{-1}{p}\right)p$, and $a \in \mathbb{F}_q^{\times}$. Then $\sum_{x \in \mathbb{F}_q} \zeta_p^{\operatorname{Tr}_1^m(ax^2)} = (-1)^{m-1} \eta(a) \sqrt{p^*}^m$. *Proof:* This lemma can be found in [24, Theorem 5.15 and Theorem 5.33].

Proposition 2.4: If $f(x) \in \mathcal{RF}$, then $f^*(0) = 0$.

Proof: From $f(x) \in \mathcal{RF}$, there exists an integer h satisfying (h-1, p-1) = 1 and $f(ax) = a^h f(x)$. Since p-1 is even, h is obviously even. Then we have f(-x) = f(x). Let $\{P_+, P_-\}$ be a partition of \mathbb{F}_q^{\times} such that $P_- = \{-x : x \in P_+\}$. Let $C_i = \#\{x \in P_+ : f(x) = i\}$, then

$$\sum_{x \in \mathbb{F}_q} \zeta_p^{f(x)} = 1 + 2 \sum_{i=0}^{p-1} C_i \zeta_p^i = \varepsilon \sqrt{p^*}^m \zeta_p^{f^*(0)}$$

If m is even, then $\varepsilon \sqrt{p^*}^m \in \mathbb{Z}$ and

$$1 + 2\sum_{i=0}^{p-1} C_i \equiv \varepsilon \sqrt{p^*}^m \equiv 1 \mod 2.$$

From Lemma 2.2, we have $f^*(0) = 0$.

If m is odd, from Lemma 2.3,

$$1 + 2\sum_{i=0}^{p-1} C_i \zeta_p^i = \varepsilon \sqrt{p^*}^{m-1} \zeta_p^{f^*(0)} (1 + 2\sum_{s \in \mathbb{F}_p^{\times 2}} \zeta_p^s)$$

Hence, we have

$$1 + 2\sum_{i=0}^{p-1} C_i \equiv \varepsilon \sqrt{p^*}^{m-1} (1 + 2\frac{p-1}{2}) \equiv 1 \mod 2$$

From Lemma 2.2, we also have $f^*(0) = 0$.

Hence, this proposition follows.

Proposition 2.5: If $f(x) \in \mathcal{RF}$, then $f^*(x) \in \mathcal{RF}$.

Proof: If f(x) is weakly regular bent, then $f^*(x)$ is also weakly regular bent. From Proposition 2.4, $f^*(0) = 0$. Hence, we just need to prove that $f^*(x)$ satisfies condition (ii) in the definition of \mathcal{RF} .

For $\forall a \in \mathbb{F}_p^{\times}$ and $\beta \in \mathbb{F}_q$, we have

$$\varepsilon\sqrt{p^*}^m\zeta_p^{f^*(a\beta)} = \sum_{x\in\mathbb{F}_q}\zeta_p^{f(x)+a\mathrm{Tr}_1^m(\beta x)} = \sum_{x\in\mathbb{F}_q}\zeta_p^{f(a^lx)+a\mathrm{Tr}_1^m(\beta a^lx)},$$

where l satisfies that $l(h-1) \equiv 1 \mod (p-1)$. Then

$$\varepsilon \sqrt{p^*}^m \zeta_p^{f^*(a\beta)} = \sum_{x \in \mathbb{F}_q} \zeta_p^{a^{hl} f(x) + a^{l+1} \operatorname{Tr}_1^m(\beta x)}$$

Note that $a^{hl} = a^{l+1}$. Then we have

$$\varepsilon\sqrt{p^*}^m\zeta_p^{f^*(a\beta)} = \sum_{x\in\mathbb{F}_q} \zeta_p^{a^{l+1}(f(x)+\operatorname{Tr}_1^m(\beta x))} = \sigma_{a^{l+1}}(\varepsilon\sqrt{p^*}^m\zeta_p^{f^*(\beta)}) = \varepsilon\sqrt{p^*}^m\zeta_p^{a^{l+1}f^*(\beta)}$$

Hence, for $\forall \beta \in \mathbb{F}_q$, $f^*(a\beta) = a^{l+1}f^*(\beta)$ and $(l+1-1, p-1) \equiv 1$. Hence, $f^*(x) \in \mathcal{RF}$.

Remark By Equation (3), if the sign of the Walsh transform of f(x) is ε , then the sign of the Walsh transform of $f^*(x)$ is $\left(\frac{-1}{n}\right)^m \varepsilon.$

C. Exponential sums from weakly regular bent functions

For determining parameters and weight distributions of linear codes from weakly regular bent functions, some results on exponential sums from weakly regular bent functions in \mathcal{RF} are given.

Lemma 2.6: (Theorem 5.13 [24]) Let p be an odd prime and $p^* = \left(\frac{-1}{p}\right)p$. Then $\sum_{x \in \mathbb{F}_p^{\times}} \left(\frac{x}{p}\right)\zeta_p^x = \sqrt{p^*}$. Lemma 2.7: Let f(x) be a p-ary function from \mathbb{F}_q to \mathbb{F}_p with $\mathcal{W}_f(0) = \varepsilon \sqrt{p^*}^m$, where $\varepsilon \in \{1, -1\}$ and $p^* = \left(\frac{-1}{p}\right)p$. Let $N_f(a) = \#\{x \in \mathbb{F}_q : f(x) = a\}$. Then we have

(1) If m is even, then

$$N_f(a) = \begin{cases} p^{m-1} + \varepsilon(p-1) \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2}, & a = 0; \\ p^{m-1} - \varepsilon \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2}, & a \in \mathbb{F}_p^{\times}. \end{cases}$$

(2) If m is odd, then

$$N_f(a) = \begin{cases} p^{m-1}, & a = 0;\\ p^{m-1} + \varepsilon \sqrt{p^*}^{m-1}, & a \in \mathbb{F}_p^{\times 2};\\ p^{m-1} - \varepsilon \sqrt{p^*}^{m-1}, & a \in \mathbb{F}_p^{\times} \backslash \mathbb{F}_p^{\times 2}. \end{cases}$$

Proof: From $\mathcal{W}_f(0) = \varepsilon \sqrt{p^*}^m$, we have

$$\varepsilon \sqrt{p^*}^m = \sum_{a \in \mathbb{F}_p} N_f(a) \zeta_p^a.$$

(1) If m is even, $\varepsilon \sqrt{p^*}^m \in \mathbb{Z}$, then

$$N_f(0) - \varepsilon \sqrt{p^*}^m + \sum_{a \in \mathbb{F}_p^{\times}} N_f(a) \zeta_p^a = 0$$

Since the minimal polynomial of ζ_p over Q is $1 + x + \cdots + x^{p-1}$, we have

$$N_f(0) - \varepsilon \sqrt{p^*}^m = N_f(a), a \in \mathbb{F}_p^{\times}$$

From $\sum_{a \in \mathbb{F}_n} N_f(a) = q$, we have

$$pN_f(1) = q - \varepsilon \sqrt{p^*}^m.$$

Hence,

$$N_f(a) = \begin{cases} p^{m-1} + \varepsilon(p-1) \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2}, & a = 0; \\ p^{m-1} - \varepsilon \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2}, & a \in \mathbb{F}_p^{\times}. \end{cases}$$

(2) If m is odd, from Lemma 2.3, we have

$$\sum_{x \in \mathbb{F}_p} \zeta_p^{x^2} = \sqrt{p^*}.$$

Further,

$$\sum_{a \in \mathbb{F}_p} N_f(a) \zeta_p^a = \varepsilon \sqrt{p^*}^{m-1} (\zeta_p^0 + \sum_{s \in \mathbb{F}_p^{\times 2}} 2\zeta_p^s).$$

And we have

$$(N_f(0) - \varepsilon \sqrt{p^*}^{m-1})\zeta_p^0 + \sum_{s \in \mathbb{F}_p^{\times 2}} (N_f(s) - 2\varepsilon \sqrt{p^*}^{m-1}) + \sum_{t \in \mathbb{F}_p^{\times} \setminus \mathbb{F}_p^{\times 2}} N_f(t)\zeta_p^t = 0.$$

The minimal polynomial of ζ_p over Q is $1 + x + \cdots + x^{p-1}$. Then we have

$$N_f(0) - \varepsilon \sqrt{p^*}^{m-1} = N_f(s) - 2\varepsilon \sqrt{p^*}^{m-1} = N_f(t),$$

where $s \in \mathbb{F}_p^{\times 2}$ and $t \in \mathbb{F}_p^{\times} \setminus \mathbb{F}_p^{\times 2}$. Note that $\sum_{a \in \mathbb{F}_p} N_f(a) = q$, we have

$$N_f(t) = p^{m-1} - \varepsilon \sqrt{p^*}^{m-1},$$

$$N_f(0) = p^{m-1},$$

$$N_f(s) = p^{m-1} + \varepsilon \sqrt{p^*}^{m-1},$$

where $t \in \mathbb{F}_p^{\times} \backslash \mathbb{F}_p^{\times 2}$ and $s \in \mathbb{F}_p^{\times 2}$.

Hence, this lemma follows.

Lemma 2.8: Let $f(x) \in \mathcal{RF}$ with the sign ε of the Walsh transform. Then we have (1) If m is even, then

$$N_{f^*}(a) = \begin{cases} p^{m-1} + \varepsilon(p-1) \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2}, & a = 0; \\ p^{m-1} - \varepsilon \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2}, & a \in \mathbb{F}_p^{\times}. \end{cases}$$

(2) If m is odd, then

$$N_{f^*}(a) = \begin{cases} p^{m-1}, & a = 0;\\ p^{m-1} + \varepsilon \left(\frac{-1}{p}\right) \sqrt{p^*}^{m-1}, & a \in \mathbb{F}_p^{\times 2};\\ p^{m-1} - \varepsilon \left(\frac{-1}{p}\right) \sqrt{p^*}^{m-1}, & a \in \mathbb{F}_p^{\times} \backslash \mathbb{F}_p^{\times 2}. \end{cases}$$

Proof: As $f(x) \in \mathcal{RF}$, f(0) = 0. From Equation (3), we have

$$\mathcal{W}_{f^*}(0) = \left(\frac{-1}{p}\right)^m \varepsilon \sqrt{p^*}^m$$

Hence, from Lemma 2.7, the lemma follows.

Lemma 2.9: Let f(x) be a p-ary function with $W_f(0) = \varepsilon \sqrt{p^*}^m$, Then

$$\sum_{y \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{yf(x)} = \begin{cases} \varepsilon(p-1)\sqrt{p^{*}}^m, & m \text{ is even}; \\ 0, & m \text{ is odd.} \end{cases}$$

Proof:

$$\sum_{y \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{yf(x)} = \sum_{y \in \mathbb{F}_p^{\times}} \sigma_y(\sum_{x \in \mathbb{F}_q} \zeta_p^{f(x)}) = \sum_{y \in \mathbb{F}_p^{\times}} \sigma_y(\varepsilon \sqrt{p^*}^m) = \varepsilon \sqrt{p^*}^m \sum_{y \in \mathbb{F}_p^{\times}} \left(\frac{y}{p}\right)^m$$

If *m* is even, then $\sum_{y \in \mathbb{F}_p^{\times}} \left(\frac{y}{p}\right)^m = p - 1$, that is, $\sum_{y \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{yf(x)} = \varepsilon(p - 1)\sqrt{p^*}^m$. If *m* is odd, then $\sum_{y \in \mathbb{F}_p^{\times}} \left(\frac{y}{p}\right)^m = \sum_{y \in \mathbb{F}_p^{\times}} \left(\frac{y}{p}\right) = 0$ and $\sum_{y \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{yf(x)} = 0$. Hence, this lemma follows.

Lemma 2.10: Let $\beta \in \mathbb{F}_q^{\times}$ and $f(x) \in \mathcal{RF}$ with the sign ε of the Walsh transfrom. (1) If m is even, then

$$\sum_{y,z\in\mathbb{F}_p^{\times}}\sum_{x\in\mathbb{F}_q}\zeta_p^{yf(x)+z\operatorname{Tr}_1^m(\beta x)} = \begin{cases} \varepsilon(p-1)^2\sqrt{p^{*m}}, & f^*(\beta)=0;\\ -\varepsilon(p-1)\sqrt{p^{*m}}, & f^*(\beta)\neq 0. \end{cases}$$

(2) If m is odd, then

$$\sum_{y,z\in\mathbb{F}_p^{\times}}\sum_{x\in\mathbb{F}_q}\zeta_p^{yf(x)+z\operatorname{Tr}_1^m(\beta x)} = \begin{cases} 0, & f^*(\beta) = 0;\\ \varepsilon(\frac{f^*(\beta)}{p})(\frac{-1}{p})^{(m+1)/2}(p-1)p^{(m+1)/2}, & f^*(\beta) \neq 0. \end{cases}$$

Proof: Let $A = \sum_{y,z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{yf(x)+z\operatorname{Tr}_1^m(\beta x)}$. Let l be an integer satisfying $l(h-1) \equiv 1 \mod p-1$. Since $x \longmapsto (\frac{z}{y})^l x$ is a permutation of \mathbb{F}_q , then

$$A = \sum_{y,z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{yf((\frac{z}{y})^l x) + z \operatorname{Tr}_1^m(\beta(\frac{z}{y})^l x)}.$$

From Condition (ii) in the definition of \mathcal{RF} , we have

$$A = \sum_{y,z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{y(\frac{z}{y})^{hl} f(x) + z(\frac{z}{y})^l \operatorname{Tr}_1^m(\beta x)}$$

Note that $y(\frac{z}{y})^{hl} = y(\frac{z}{y})^{l+1} = z(\frac{z}{y})^l$. Then

$$A = \sum_{y,z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{(f(x) + \operatorname{Tr}_1^m(\beta x))z(\frac{z}{y})^l}$$

 $\sim m$

Since (l, p-1) = 1 and $y \longmapsto z(\frac{z}{y})^l$ is a permutation of \mathbb{F}_p^{\times} , then

$$A = \sum_{y \in \mathbb{F}_p^{\times}} \sum_{z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{(f(x) + \operatorname{Tr}_1^m(\beta x))y}$$

= $(p-1) \sum_{y \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{(f(x) + \operatorname{Tr}_1^m(\beta x))y}$
= $(p-1) \sum_{y \in \mathbb{F}_p^{\times}} \sigma_y (\sum_{x \in \mathbb{F}_q} \zeta_p^{f(x) + \operatorname{Tr}_1^m(\beta x)})$
= $(p-1) \sum_{y \in \mathbb{F}_p^{\times}} \sigma_y (\mathcal{W}_f(\beta)).$

From f(0) = 0,

$$A = (p-1) \sum_{y \in \mathbb{F}_p^{\times}} \sigma_y(\varepsilon \sqrt{p^*}^m \zeta_p^{f^*(\beta)}).$$

From Lemma 2.1, we have

$$A = \varepsilon(p-1)\sqrt{p^*}^m \sum_{y \in \mathbb{F}_p^{\times}} \eta^m(y)\zeta_p^{yf^*(\beta)}.$$

When m is even, then

$$A = \varepsilon(p-1) \sqrt{p^*}^m \sum_{y \in \mathbb{F}_p^{\times}} \zeta_p^{yf^*(\beta)}$$

 $A = \varepsilon (p-1)^2 \sqrt{p^*}^m.$

If $f^*(\beta) = 0$, then

If $f^*(\beta) \neq 0$, then

$$A = \varepsilon(p-1)\sqrt{p^*}^m \sigma_{f^*(\beta)} (\sum_{y \in \mathbb{F}_p^\times} \zeta_p^y).$$

From $\sum_{y \in \mathbb{F}_p^{\times}} \zeta_p^y = -1$, we have

$$A = -\varepsilon(p-1)\sqrt{p^*}^m.$$

When m is odd, then

$$A = \varepsilon(p-1)\sqrt{p^*}^m \sum_{y \in \mathbb{F}_p^{\times}} \left(\frac{y}{p}\right) \zeta_p^{yf^*(\beta)}.$$

If $f^*(\beta) = 0$, then

$$A = \varepsilon(p-1)\sqrt{p^*}^m \sum_{y \in \mathbb{F}_p^\times} \left(\frac{y}{p}\right).$$

From $\sum_{y \in \mathbb{F}_p^{\times}} \left(\frac{y}{p} \right) = 0$, we have A = 0. If $f^*(\beta) \neq 0$, then

$$A = \varepsilon(p-1)\sqrt{p^*}^m \sigma_{f^*(\beta)} \left(\sum_{y \in \mathbb{F}_p^\times} \left(\frac{y}{p}\right) \zeta_p^y\right).$$

From Lemma 2.1, we have

$$A = \varepsilon(p-1)\sqrt{p^*}^m \left(\frac{f^*(\beta)}{p}\right)\sqrt{p^*} = \varepsilon\left(\frac{f^*(\beta)}{p}\right) \left(\frac{-1}{p}\right)^{(m+1)/2} (p-1)p^{(m+1)/2}$$

Hence, this lemma follows.

Lemma 2.11: Let $\beta \in \mathbb{F}_q^{\times}$ and $f(x) \in \mathcal{RF}$ with the sign ε of the Walsh transfrom. Let

$$N_{f,\beta} = \#\{x \in \mathbb{F}_q : f(x) = 0, \operatorname{Tr}_1^m(\beta x) = 0\}$$

(1) If m is even, then

$$N_{f,\beta} = \begin{cases} p^{m-2} + \varepsilon \left(\frac{-1}{p}\right)^{m/2} (p-1) p^{(m-2)/2}, & f^*(\beta) = 0; \\ p^{m-2}, & f^*(\beta) \neq 0 \end{cases}$$

(2) If m is odd, then

$$N_{f,\beta} = \begin{cases} p^{m-2}, & f^*(\beta) = 0; \\ p^{m-2} + \varepsilon \left(\frac{f^*(\beta)}{p}\right) \eta^{(m+1)/2} (-1)(p-1)p^{(m-3)/2}, & f^*(\beta) \neq 0. \end{cases}$$

 $\textit{Proof: From } \sum_{x \in \mathbb{F}_p} \zeta_p^{\mathrm{Tr}_1^m(ax)} = \left\{ \begin{array}{ll} 0, & a \in \mathbb{F}_p^{\times}; \\ p, & a = 0, \end{array} \right. \text{ we have }$

$$N_{f,\beta} = p^{-2} \sum_{x \in \mathbb{F}_q} (\sum_{y \in \mathbb{F}_p} \zeta_p^{yf(x)}) (\sum_{z \in \mathbb{F}_p} \zeta_p^{z \operatorname{Tr}_1^m(\beta x)})$$

$$= p^{-2} \sum_{y,z \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_q} \zeta_p^{yf(x) + z \operatorname{Tr}_1^m(\beta x)}$$

$$= p^{-2} \sum_{x \in \mathbb{F}_q} \zeta_p^{0f(x) + 0 \operatorname{Tr}_1^m(\beta x)} + p^{-2} \sum_{z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{0f(x) + z \operatorname{Tr}_1^m(\beta x)}$$

$$+ p^{-2} \sum_{y \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{yf(x) + 0 \operatorname{Tr}_1^m(\beta x)} + p^{-2} \sum_{y,z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{yf(x) + z \operatorname{Tr}_1^m(\beta x)}$$

$$= p^{m-2} + p^{-2} \sum_{y \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{yf(x)} + p^{-2} \sum_{y,z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{yf(x) + z \operatorname{Tr}_1^m(\beta x)}.$$

When m is even, from Lemma 2.9 and Lemma 2.10, If $f^*(\beta) = 0$, then

$$N_{f,\beta} = p^{m-2} + p^{-2} (\varepsilon(p-1)\sqrt{p^*}^m + \varepsilon(p-1)^2 \sqrt{p^*}^m)$$

= $p^{m-2} + p^{-2} \varepsilon(p-1) \sqrt{p^*}^m p$
= $p^{m-2} + \varepsilon(p-1) \sqrt{p^*}^m p^{-1}$
= $p^{m-2} + \varepsilon \left(\frac{-1}{p}\right)^{m/2} (p-1) p^{(m-2)/2}.$

If $f^*(\beta) \neq 0$, then

$$N_{f,\beta} = p^{m-2}$$

When m is odd, from Lemma 2.9 and Lemma 2.10, If $f^*(\beta) = 0$, then

$$N_{f,\beta} = p^{m-2}.$$

If $f^*(\beta) \neq 0$, then

$$N_{f,\beta} = p^{m-2} + \varepsilon \left(\frac{f^*(\beta)}{p}\right) \left(\frac{-1}{p}\right)^{(m+1)/2} (p-1)p^{(m-3)/2}.$$

Hence, this lemma follows.

Lemma 2.12: Let $f(x) \in \mathcal{RF}$, then

$$\sum_{y \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{y^2 f(x)} = \varepsilon(p-1) \sqrt{p^*}^m,$$

where ε is the sign of the Walsh transform of f(x). *Proof:* Let $A = \sum_{y \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{y^2 f(x)}$. From Lemma 2.1 and $f^*(0) = 0$, we have

$$A = \sum_{y \in F_p^{\times}} \sigma_{y^2}(W_f(0)) = \sum_{y \in F_p^{\times}} \sigma_{y^2}(\varepsilon \sqrt{p^*}^m \zeta_p^{f^*(0)}) = \varepsilon(p-1) \sqrt{p^*}^m.$$

Hence, this lemma follows.

Lemma 2.13: Let $f(x) \in \mathcal{RF}$ and $\beta \in \mathbb{F}_q^{\times}$. Then

$$\sum_{y,z\in\mathbb{F}_p^{\times}}\sum_{x\in\mathbb{F}_q}\zeta_p^{y^2f(x)+z\operatorname{Tr}_1^m(\beta x)} = \begin{cases} \varepsilon(p-1)^2\sqrt{p^{*m}}, & f^*(\beta)=0;\\ \varepsilon(p-1)\sqrt{p^{*m}}(\sqrt{p^*}-1), & f^*(\beta)\in\mathbb{F}_p^{\times 2};\\ -\varepsilon(p-1)\sqrt{p^{*m}}(\sqrt{p^*}+1), & f^*(\beta)\in\mathbb{F}_p^{\times 2}\backslash\mathbb{F}_p^{\times 2} \end{cases}$$

Proof: Let $A = \sum_{y,z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{y^2 f(x) + z \operatorname{Tr}_1^m(\beta x)}$. Let l be an integer such that $l(h-1) \equiv 1 \mod (p-1)$. Then

$$A = \sum_{y,z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{y^2 f((\frac{z}{y^2})^l x) + z \operatorname{Tr}_1^m(\beta(\frac{z}{y^2})^l x)}$$
$$= \sum_{y,z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{y^2(\frac{z}{y^2})^{hl} f(x) + z(\frac{z}{y^2})^l \operatorname{Tr}_1^m(\beta x)}.$$

Since $y^2(\frac{z}{y^2})^{hl} = y^2(\frac{z}{y^2})^{l+1} = z(\frac{z}{y^2})^l$,

$$A = \sum_{\substack{y,z \in \mathbb{F}_p^{\times} \\ y,z \in \mathbb{F}_p^{\times}}} \sigma_{z(\frac{z}{y^2})^l} (\sum_{x \in \mathbb{F}_q} \zeta_p^{f(x) + \operatorname{Tr}_1^m(\beta x)})$$
$$= \sum_{\substack{y,z \in \mathbb{F}_p^{\times} \\ y,z \in \mathbb{F}_p^{\times}}} \sigma_{z(\frac{z}{y^2})^l} (\mathcal{W}_f(\beta)).$$

As y runs through \mathbb{F}_p^{\times} , $(\frac{z}{y^2})^l$ will run through $\mathbb{F}_p^{\times 2}$ and every value in $\mathbb{F}_p^{\times 2}$ is taken twice. Then

$$A = \sum_{z \in \mathbb{F}_p^{\times}} \sum_{y \in \mathbb{F}_p^{\times}} \sigma_{y^2}(\mathcal{W}_f(\beta))$$

= $(p-1) \sum_{y \in \mathbb{F}_p^{\times}} \sigma_{y^2}(\mathcal{W}_f(\beta))$
= $(p-1) \sum_{y \in \mathbb{F}_p^{\times}} \sigma_{y^2}(\varepsilon \sqrt{p^*}^m \zeta_p^{f^*(\beta)})$
= $\varepsilon (p-1) \sqrt{p^*}^m \sum_{y \in \mathbb{F}_p^{\times}} \sigma_{y^2}(\zeta_p^{f^*(\beta)})$
= $\varepsilon (p-1) \sqrt{p^*}^m \sum_{y \in \mathbb{F}_p^{\times}} \zeta_p^{y^2 f^*(\beta)}.$

Hence,

$$A = \varepsilon(p-1)\sqrt{p^*}^m \sum_{y \in \mathbb{F}_p} \zeta_p^{y^2 f^*(\beta)} - \varepsilon(p-1)\sqrt{p^*}^m$$

 $A = \varepsilon (p-1)^2 \sqrt{p^*}^m.$

From Lemma 2.3, if $f^*(\beta) = 0$, then

If
$$f^*(\beta) \in \mathbb{F}_p^{\times 2}$$
, then

$$A = \varepsilon(p-1)\sqrt{p^*}^m(\sqrt{p^*}-1).$$

If $f^*(\beta) \in \mathbb{F}_p^{\times} \backslash \mathbb{F}_p^{\times 2}$, then

$$A = -\varepsilon(p-1)\sqrt{p^*}^m(\sqrt{p^*}+1).$$

Hence, this lemma follows. Lemma 2.14: Let $f(x) \in \mathcal{RF}$ and $\beta \in \mathbb{F}_q^{\times}$. Let

$$N_{sq,\beta} = \#\{x \in \mathbb{F}_q : f(x) \in \mathbb{F}_p^{\times 2}, \operatorname{Tr}_1^m(\beta x) = 0\},\$$
$$N_{nsq,\beta} = \#\{x \in \mathbb{F}_q : f(x) \in \mathbb{F}_p^{\times} \setminus \mathbb{F}_p^{\times 2}, \operatorname{Tr}_1^m(\beta x) = 0\}.$$

(1) If m is even, then

$$N_{sq,\beta} = \begin{cases} \frac{p-1}{2} [p^{m-2} - \varepsilon(\frac{-1}{p})^{m/2} p^{(m-2)/2}], & f^*(\beta) = 0 \text{ or } f^*(\beta) \in \mathbb{F}_p^{\times} \backslash \mathbb{F}_p^{\times 2}; \\ \frac{p-1}{2} [p^{m-2} + \varepsilon(\frac{-1}{p})^{m/2} p^{(m-2)/2}], & f^*(\beta) \in \mathbb{F}_p^{\times 2}. \end{cases}$$
$$N_{nsq,\beta} = \begin{cases} \frac{p-1}{2} [p^{m-2} - \varepsilon(\frac{-1}{p})^{m/2} p^{(m-2)/2}], & f^*(\beta) = 0 \text{ or } f^*(\beta) \in \mathbb{F}_p^{\times 2}; \\ \frac{p-1}{2} [p^{m-2} + \varepsilon(\frac{-1}{p})^{m/2} p^{(m-2)/2}], & f^*(\beta) \in \mathbb{F}_p^{\times} \backslash \mathbb{F}_p^{\times 2}. \end{cases}$$

(2) If m is odd, then

$$N_{sq,\beta} = \begin{cases} \frac{p-1}{2} [p^{m-2} + \varepsilon \sqrt{p*}^{m-1}], & f^*(\beta) = 0;\\ \frac{p-1}{2} [p^{m-2} - \varepsilon \sqrt{p*}^{m-3}], & f^*(\beta) \in \mathbb{F}_p^{\times 2};\\ \frac{p-1}{2} [p^{m-2} + \varepsilon \sqrt{p*}^{m-3}], & f^*(\beta) \in \mathbb{F}_p^{\times} \backslash \mathbb{F}_p^{\times 2}. \end{cases}$$

$$N_{nsq,\beta} = \begin{cases} \frac{p-1}{2} [p^{m-2} - \varepsilon \sqrt{p*^{m-1}}], & f^*(\beta) = 0;\\ \frac{p-1}{2} [p^{m-2} - \varepsilon \sqrt{p*^{m-3}}], & f^*(\beta) \in \mathbb{F}_p^{\times 2};\\ \frac{p-1}{2} [p^{m-2} + \varepsilon \sqrt{p*^{m-3}}], & f^*(\beta) \in \mathbb{F}_p^{\times} \backslash \mathbb{F}_p^{\times 2} \end{cases}$$

Proof: Let $A = \sum_{x \in \mathbb{F}_q} (\sum_{y \in \mathbb{F}_p} \zeta_p^{y^2 f(x)}) (\sum_{z \in \mathbb{F}_p} \zeta_p^{z \operatorname{Tr}_1^m(\beta x)})$. Note that

$$\sum_{z \in \mathbb{F}_p} \zeta_p^{z \operatorname{Tr}_1^m(\beta x)} = \begin{cases} p, & \operatorname{Tr}_1^m(\beta x) = 0; \\ 0, & \operatorname{Tr}_1^m(\beta x) \neq 0. \end{cases}$$

and

$$\sum_{y \in \mathbb{F}_p} \zeta_p^{y^2 f(x)} = \begin{cases} p, & f(x) = 0;\\ \sqrt{p^*}, & f(x) \in \mathbb{F}_p^{\times 2};\\ -\sqrt{p^*}, & f(x) \in \mathbb{F}_p^{\times} \setminus \mathbb{F}_p^{\times 2} \end{cases}$$

Then, we have

$$A = N_{f,\beta}p^{2} + N_{sq,\beta}(\sqrt{p^{*}})p + N_{nsq,\beta}(-\sqrt{p^{*}})p = N_{f,\beta}p^{2} + (N_{sq,\beta} - N_{nsq,\beta})p\sqrt{p^{*}},$$
(4)

where $N_{f,\beta} = \{x \in \mathbb{F}_q : f(x) = 0, \operatorname{Tr}_1^m(\beta x) = 0\}$. Further, we have

$$\begin{split} A &= \sum_{z,y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_q} \zeta_p^{y^2 f(x) + z \operatorname{Tr}_1^m(\beta x)} \\ &= q + \sum_{z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{z \operatorname{Tr}_1^m(\beta x)} + \sum_{y \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{y^2 f(x)} + \sum_{y,z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{y^2 f(x) + z \operatorname{Tr}_1^m(\beta x)} \\ &= q + \sum_{y \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{y^2 f(x)} + \sum_{y,z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{y^2 f(x) + z \operatorname{Tr}_1^m(\beta x)}. \end{split}$$

From Lemma 2.12, we have

$$A = q + \varepsilon (p-1)\sqrt{p^*}^m + \sum_{y,z \in \mathbb{F}_p^{\times}} \sum_{x \in \mathbb{F}_q} \zeta_p^{y^2 f(x) + z \operatorname{Tr}_1^m(\beta x)}$$

When m is even, from Lemma 2.13, if $f^*(\beta) = 0$, then

$$A = q + \varepsilon (p-1)\sqrt{p^*}^m + \varepsilon (p-1)^2 \sqrt{p^*}^m = q + \varepsilon (p-1)p \sqrt{p^*}^m.$$

From Equation (4) and $N_{f,\beta} + N_{sq,\beta} + N_{nsq,\beta} = p^{m-1}$,

$$N_{sq,\beta} = N_{nsq,\beta} = \frac{p-1}{2} \left[p^{m-2} - \varepsilon \left(\frac{-1}{p} \right)^{m/2} p^{(m-2)/2} \right].$$

If $f^*(\beta) \in \mathbb{F}_p^{\times 2}$, then

$$A = q + \varepsilon(p-1)\sqrt{p^{*}}^{m} + \varepsilon(p-1)\sqrt{p^{*}}^{m}(\sqrt{p^{*}}-1) = q + \varepsilon(p-1)\sqrt{p^{*}}^{m+1}.$$

From Equation (4), we have

$$N_{sq,\beta} = \frac{p-1}{2} [p^{m-2} + \varepsilon \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2}],$$

$$N_{nsq,\beta} = \frac{p-1}{2} [p^{m-2} - \varepsilon \eta^{m/2} (-1) p^{(m-2)/2}].$$

If $f^*(\beta) \in \mathbb{F}_p^{\times} \backslash \mathbb{F}_p^{\times 2}$, then

$$A = q + \varepsilon(p-1)\sqrt{p^{*}}^{m} - \varepsilon(p-1)\sqrt{p^{*}}^{m}(\sqrt{p^{*}} + 1) = q - \varepsilon(p-1)\sqrt{p^{*}}^{m+1}.$$

From Equation (4), we have

$$N_{sq,\beta} = \frac{p-1}{2} [p^{m-2} - \varepsilon \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2}],$$
$$N_{nsq,\beta} = \frac{p-1}{2} [p^{m-2} + \varepsilon \eta^{m/2} (-1) p^{(m-2)/2}].$$

When m is odd, from Lemma 2.13,

if $f^*(\beta) = 0$, then

$$A = q + \varepsilon (p-1)\sqrt{p^*}^m + \varepsilon (p-1)^2 \sqrt{p^*}^m = q + \varepsilon (p-1)p \sqrt{p^*}^m.$$

From Equation (4), we have

$$N_{sq,\beta} = \frac{p-1}{2} [p^{m-2} + \varepsilon \sqrt{p^*}^{m-1}],$$
$$N_{nsq,\beta} = \frac{p-1}{2} [p^{m-2} - \varepsilon \sqrt{p^*}^{m-1}]$$

If $f^*(\beta) \in \mathbb{F}_p^{\times 2}$, then

$$A = q + \varepsilon (p-1)\sqrt{p^*}^{m+1}.$$

From Equation (4), we have

$$N_{sq,\beta} = N_{nsq,\beta} = \frac{p-1}{2} [p^{m-2} - \varepsilon \sqrt{p^*}^{m-3}]$$

If $f^*(\beta) \in \mathbb{F}_n^{\times} \setminus \mathbb{F}_n^{\times 2}$, then

$$A = q - \varepsilon (p - 1) \sqrt{p^*}^{m+1}.$$

From Equation (4), we have

$$N_{sq,\beta} = N_{nsq,\beta} = \frac{p-1}{2} [p^{m-2} + \varepsilon \sqrt{p^*}^{m-3}].$$

Hence, this lemma follows.

III. THE WEIGHT DISTRIBUTIONS OF LINEAR CODES FROM WEAKLY REGULAR BENT FUNCTIONS

In this section, we generalize the construction method of linear codes with few weights by Ding et al.[5], [6] to general weakly regular bent functions and determine the weight distributions of the corresponding linear codes.

Let f(x) be a *p*-ary function mapping from \mathbb{F}_q to \mathbb{F}_p . Define

$$D_f = \{ x \in \mathbb{F}_a^\times : f(x) = 0 \}.$$

Let $n = \#(D_f)$ and $D_f = \{d_1, d_2, \dots, d_n\}$. A linear code defined from D_f is

$$\mathcal{C}_{D_f} = \{ \mathbf{c}_\beta : \beta \in \mathbb{F}_q \}$$

where $\mathbf{c}_{\beta} = (\operatorname{Tr}_{1}^{m}(\beta d_{1}), \operatorname{Tr}_{1}^{m}(\beta d_{2}), \cdots, \operatorname{Tr}_{1}^{m}(\beta d_{n}))$. For a general function f(x), it is difficult to determine the weight distribution of $\mathcal{C}_{D_{f}}$. However, for some special function f(x), this problem can be solved. Ding et al. [6] determined the weight distribution of $\mathcal{C}_{D_{f}}$ for $f(x) = \operatorname{Tr}_{1}^{m}(x^{2})$ and proposed an open problem on how to determine the weight distribution of $\mathcal{C}_{D_{f}}$ for general planar functions F(x), where $f(x) = \operatorname{Tr}_{1}^{m}(F(x))$. Zhou et al. [27] solved the weight distribution of $\mathcal{C}_{D_{f}}$ for the case that F(x) is a quadratic planar function. We generalize their work to weakly regular bent functions and solve the open problem proposed by Ding et al. [6]. Our results on linear codes $\mathcal{C}_{D_{f}}$ from weakly regular bent functions in \mathcal{RF} are listed in the following theorems and corollaries.

Theorem 3.1: Let m be even and $f(x) \in \mathcal{RF}$ with the sign ε of the Walsh transform. Then \mathcal{C}_{D_f} is a two-weight linear code with parameters $[p^{m-1} - 1 + \varepsilon(p-1)p^{(m-2)/2}, m]$, whose weight distribution is listed in Table I.

TABLE I THE WEIGHT DISTRIBUTION OF \mathcal{C}_{D_f} for even m

Weight	Mutilipicity
0	1
$(p-1)p^{m-2}$	$p^{m-1} - 1 + \varepsilon \left(\frac{-1}{p}\right)^{m/2} (p-1)p^{(m-2)/2}$
$(p-1)[p^{m-2}+\varepsilon(\frac{-1}{p})^{m/2}p^{(m-2)/2}]$	$(p-1)[p^{m-1}-\varepsilon(\frac{-1}{p})^{m/2}p^{(m-2)/2}]$

Proof: From Lemma 2.7 and Lemma 2.11, this theorem follows.

Theorem 3.2: Let m be odd and $f(x) \in \mathcal{RF}$. Then \mathcal{C}_{D_f} is a three-weight linear code with parameters $[p^{m-1}-1,m]$, whose weight distribution is listed in Table II.

Proof: From Lemma 2.7 and Lemma 2.11, this theorem follows.

Let $f(x) \in \mathcal{RF}$. For any $a \in \mathbb{F}_p^{\times}$ and $x \in \mathbb{F}_q$, f(x) = 0 if and only if $f(ax) = a^h f(x) = 0$. Then we can select a subset \overline{D}_f of D_f such that $\bigcup_{a \in \mathbb{F}_p^{\times}} a\overline{D}_f$ is just a partition of D_f . Hence, the code \mathcal{C}_{D_f} can be punctured into a shorter linear codes $\mathcal{C}_{\overline{D}_f}$, whose parameters and the weight distributions are given in the following two corollaries.

Corollary 3.3: Let m be even and $f(x) \in \mathcal{RF}$ with the sign ε of the Walsh transform. Then $\mathcal{C}_{\overline{D}_f}$ is a two-weight linear code with parameters $[\frac{p^{m-1}-1}{p-1} + \varepsilon p^{(m-2)/2}, m]$, whose weight distribution is listed in Table III.

TABLE II THE WEIGHT DISTRIBUTION OF \mathcal{C}_{D_f} for odd m

Weight	Mutilipicity
0	1
$(p-1)p^{m-2}$	$p^{m-1} - 1$
$(p-1)(p^{m-2}-p^{(m-3)/2})$	$\frac{p-1}{2}(p^{m-1}+p^{(m-1)/2})$
$(p-1)(p^{m-2}+p^{(m-3)/2})$	$\frac{p-1}{2}(p^{m-1}-p^{(m-1)/2})$

TABLE III THE WEIGHT DISTRIBUTION OF $\mathcal{C}_{\overline{D}_f}$ for even m

Weight	Mutilipicity
0	1
p^{m-2}	$p^{m-1} - 1 + \varepsilon \left(\frac{-1}{p}\right)^{m/2} (p-1)p^{(m-2)/2}$
$p^{m-2} + \varepsilon \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2}$	$(p-1)[p^{m-1}-\varepsilon(\frac{-1}{p})^{m/2}p^{(m-2)/2}]$

Proof: From Theorem 3.1, this corollary follows.

Corollary 3.4: Let m be odd and $f(x) \in \mathcal{RF}$. Then $\mathcal{C}_{\overline{D}_f}$ is a three-weight linear code with parameters $[(p^{m-1}-1)/(p-1), m]$, whose weight distribution is listed in Table IV.

Proof: From Theorem 3.2, this corollary follows.

IV. NEW TWO-WEIGHT AND THREE-WEIGHT LINEAR CODES FROM WEAKLY REGULAR BENT FUNCTIONS

In this section, by choosing defining sets different from that in Section III, we construct new linear codes with two or three weights and determine their weight distributions. Define

$$D_{f,nsq} = \{ x \in \mathbb{F}_q : f(x) \in \mathbb{F}_p^{\times} \setminus \mathbb{F}_p^{\times 2} \},\$$
$$D_{f,sq} = \{ x \in \mathbb{F}_q : f(x) \in \mathbb{F}_p^{\times 2} \},\$$

where $f(x) \in \mathcal{RF}$. With the similar definition of \mathcal{C}_{D_f} , we can define $\mathcal{C}_{D_{f,nsq}}$ and $\mathcal{C}_{D_{f,sq}}$ corresponding to the defining sets $D_{f,nsq}$ and $D_{f,sq}$ respectively.

Theorem 4.1: Let m be even. Let $f(x) \in \mathcal{RF}$ with the sign ε of the Walsh transform. Then $\mathcal{C}_{D_{f,nsq}}$ and $\mathcal{C}_{D_{f,sq}}$ are twoweight linear codes with the same parameters $\left[\frac{p-1}{2}\left(p^{m-1}-\varepsilon\left(\frac{-1}{p}\right)^{m/2}p^{(m-2)/2}\right),m\right]$ and the same weight distribution in Table V.

Proof: From Lemma 2.7 and Lemma 2.14, this theorem follows.

Example Let p = 3, m = 6, and $f(x) = \text{Tr}_1^6(w^7 x^{210})$, where w is a primitive element of \mathbb{F}_{3^6} . The sign of the Walsh transform of f(x) is $\varepsilon = 1$. Then $\mathcal{C}_{D_{f,nsq}}$ and $\mathcal{C}_{D_{f,sq}}$ in Theorem 4.1 have the same parameters [252, 6, 162] and the same weight enumerator $1 + 476z^{162} + 252z^{180}$, which is verified by the Magma program.

Example Let p = 3, m = 6, and $f(x) = \text{Tr}_1^6(x^{10})$. The sign of the Walsh transform of f(x) is $\varepsilon = -1$. Then $\mathcal{C}_{D_{f,nsq}}$ and $\mathcal{C}_{D_{f,sq}}$ in Theorem 4.1 have the same parameters [234, 6, 144] and the same weight enumerator $1 + 234z^{144} + 494z^{162}$, which is verified by the Magma program.

Example Let p = 5, m = 6, and $f(x) = \text{Tr}_1^6(x^{26})$. The sign of the Walsh transform of f(x) is $\varepsilon = -1$. Then $\mathcal{C}_{D_{f,nsq}}$ and $\mathcal{C}_{D_{f,sq}}$ in Theorem 4.1 have the same parameters [6300, 6, 5000] and the same weight enumerator $1 + 9324z^{5000} + 6300z^{5100}$, which is verified by the Magma program.

TABLE IV THE WEIGHT DISTRIBUTION OF $\mathcal{C}_{\overline{D}_f}$ for odd m

Weight	Mutilipicity
0	1
p^{m-2}	$p^{m-1} - 1$
$p^{m-2} - p^{(m-3)/2}$	$\frac{p-1}{2}(p^{m-1}+p^{(m-1)/2})$
$p^{m-2} + p^{(m-3)/2}$	$\frac{p-1}{2}(p^{m-1}-p^{(m-1)/2})$

TABLE V The weight distribution of $\mathcal{C}_{D_{f,nsq}}$ and $\mathcal{C}_{D_{f,sq}}$ for even m

Weight	Mulitiplicity		
0	1		
$\frac{(p-1)^2}{2}p^{m-2}$	$\frac{p+1}{2}p^{m-1} + \frac{p-1}{2}\varepsilon(\frac{-1}{p})^{m/2}p^{(m-2)/2} - 1$		
$\frac{p-1}{2}[(p-1)p^{m-2} - 2\varepsilon \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2}]$	$\frac{p-1}{2}p^{m-1} - \frac{p-1}{2}\varepsilon(\frac{-1}{p})^{m/2}p^{(m-2)/2}$		

Example Let p = 5, m = 6, and $f(x) = \text{Tr}_1^6(wx^{26})$, where w is a primitive element of \mathbb{F}_{5^6} . The sign of the Walsh transform of f(x) is $\varepsilon = 1$. Then $\mathcal{C}_{D_{f,nsq}}$ and $\mathcal{C}_{D_{f,sq}}$ in Theorem 4.1 have the same parameters [6200, 6, 4900] and the same weight enumerator $1 + 6200z^{4900} + 9424z^{5000}$, which is verified by the Magma program.

Theorem 4.2: Let m be odd and $f(x) \in \mathcal{RF}$ with the sign ε of the Walsh transform. Then $\mathcal{C}_{D_{f,nsq}}$ is a three-weight linear code with parameters $\left[\frac{p-1}{2}(p^{m-1}-\varepsilon\sqrt{p^{*}}^{m-1}),m\right]$ and the weight distribution in Table VI.

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Weight	Mulitiplicity	
0	1	
$\frac{(p-1)^2}{2}p^{m-2}$	$p^{m-1} - 1$	
$\frac{p-1}{2}[(p-1)p^{m-2} + \varepsilon(1-p^*)\sqrt{p^*}^{m-3}]$	$\frac{p-1}{2}\left[p^{m-1} + \varepsilon\left(\frac{-1}{p}\right)\sqrt{p^*}^{m-1}\right]$	
$\frac{p-1}{2}[(p-1)p^{m-2} - \varepsilon(1+p^*)\sqrt{p^*}^{m-3}]$	$\frac{p-1}{2}\left[p^{m-1}-\varepsilon\left(\frac{-1}{p}\right)\sqrt{p^{*}}^{m-1}\right]$	

TABLE VI THE WEIGHT DISTRIBUTION OF $C_{D_{f,nsa}}$ FOR ODD m

Proof: From Lemma 2.7 and Lemma 2.14, this theorem follows.

Example Let p = 5, m = 5, and $f(x) = \text{Tr}_1^5(x^2)$. The sign of the Walsh transform of f(x) is $\varepsilon = 1$. Then the code $C_{D_{f,nsq}}$ has parameters [1200, 5, 940] and weight enumerator $1 + 1200z^{940} + 1300z^{960} + 624x^{1000}$, which is verified by the Magma program.

Example Let p = 3, m = 5, and $f(x) = \text{Tr}_1^5(wx^2)$, where w is a primitive element of \mathbb{F}_{3^5} . The sign of the Walsh transform of f(x) is $\varepsilon = -1$. Then the code $\mathcal{C}_{D_{f,nsq}}$ in Theorem 4.2 has parameters [60, 5, 40] and weight enumerator $1 + 24z^{40} + 40z^{48} + 60z^{52}$, which is verified by the Magma program.

Theorem 4.3: Let m be odd and $f(x) \in \mathcal{RF}$ with the sign ε of the Walsh transform. Then $\mathcal{C}_{D_{f,sq}}$ is a three-weight linear code with parameters $[\frac{p-1}{2}(p^{m-1} + \varepsilon \sqrt{p^*}^{m-1}), m]$ and the weight distribution in Table VII.

TABLE VII

The weight distribution of $\mathcal{C}_{D_{f,sq}}$ for odd m		
Weight	Mulitiplicity	
0	1	
$\frac{(p-1)^2}{2}p^{m-2}$	$p^{m-1} - 1$	
$\frac{p-1}{2}[(p-1)p^{m-2} + \varepsilon(1+p^*)\sqrt{p^*}^{m-3}]$	$\frac{p-1}{2}\left[p^{m-1} + \varepsilon\left(\frac{-1}{p}\right)\sqrt{p^*}^{m-1}\right]$	
$\frac{p-1}{2}[(p-1)p^{m-2} + \varepsilon(p^*-1)\sqrt{p^*}^{m-3}]$	$\frac{p-1}{2}\left[p^{m-1}-\varepsilon\left(\frac{-1}{n}\right)\sqrt{p^*}^{m-1}\right]$	

Proof: From Lemma 2.7 and Lemma 2.14, this theorem follows.

Example Let p = 5, m = 5, and $f(x) = \text{Tr}_1^5(x^2)$. The sign of the Walsh transform of f(x) is $\varepsilon = 1$. Then the code $C_{D_{f,sq}}$ in Theorem 4.3 has parameters [1300, 5, 1000] and weight enumerator $1 + 624z^{1000} + 1200z^{1040} + 1300z^{1060}$, which is verified by the Magma program.

Example Let p = 3, m = 5, and $f(x) = \text{Tr}_1^5(wx^2)$, where w is a primitive element of \mathbb{F}_{3^5} . The sign of the Walsh transform of f(x) is $\varepsilon = -1$. Then the code $\mathcal{C}_{D_{f,sq}}$ in Theorem 4.3 has parameters [40, 5, 28] and weight enumerator $1+40z^{28}+60z^{32}+24z^{40}$, which is verified by the Magma program.

Let $f(x) \in \mathcal{RF}$. There exists an integer h such that (h-1, p-1) = 1 and $f(ax) = a^h f(x)$ for any $a \in \mathbb{F}_p^{\times}$ and $x \in \mathbb{F}_q$. Note that h is even. Therefore, f(ax) is a quadratic residue (quadratic nonresidue) in \mathbb{F}_p^{\times} if and only if f(x) is a quadratic residue (quadratic nonresidue) in \mathbb{F}_p^{\times} . With the similar definition of \overline{D}_f , we define $\overline{D}_{f,nsq}$ as a subset of $D_{f,nsq}$ such that $\bigcup_{a \in \mathbb{F}_p^{\times}} a\overline{D}_{f,nsq}$ is just a partition of $D_{f,nsq}$. And we similarly define $\overline{D}_{f,sq}$. Hence, we can construct the corresponding linear codes $\mathcal{C}_{\overline{D}_{f,nsq}}$ and $\mathcal{C}_{\overline{D}_{f,sq}}$, whose parameters and weight distributions are given in the following three corollaries.

Corollary 4.4: Let m be even and $f(x) \in \mathcal{RF}$ with the sign ε of the Walsh transform. Then $\mathcal{C}_{\overline{D}_{f,nsq}}$ and $\mathcal{C}_{\overline{D}_{f,sq}}$ are twoweight linear codes with the same parameters $\left[\frac{1}{2}(p^{m-1} - \varepsilon(\frac{-1}{p})^{m/2}p^{(m-2)/2}), m\right]$ and the same weight distribution in Table VIII.

TABLE VIII THE WEIGHT DISTRIBUTION OF $\mathcal{C}_{\overline{D}_{f,nsq}}$ and $\mathcal{C}_{\overline{D}_{f,sq}}$ for even m

Weight	Mulitiplicity	
0	1	
$\frac{p-1}{2}p^{m-2}$	$\frac{p+1}{2}p^{m-1} + \frac{p-1}{2}\varepsilon(\frac{-1}{p})^{m/2}p^{(m-2)/2} - 1$	
$\frac{1}{2}[(p-1)p^{m-2} - 2\varepsilon \left(\frac{-1}{p}\right)^{m/2} p^{(m-2)/2}]$	$\frac{p-1}{2}p^{m-1} - \frac{p-1}{2}\varepsilon(\frac{-1}{p})^{m/2}p^{(m-2)/2}$	

Proof: From Theorem 4.1, this corollary follows.

Example Let p = 3, m = 4, and $f(x) = \text{Tr}_1^4(x^2)$. The sign of the Walsh transform of f(x) is $\varepsilon = -1$. Then $\mathcal{C}_{\overline{D}_{f,nsq}}$ and $\mathcal{C}_{\overline{D}_{f,sq}}$ in Corollary 4.4 have the same parameters [15, 4, 9] and the same weight enumerator $1 + 50z^9 + 30z^{12}$, which is verified by the Magma program. This code is optimal due to the Griesmer bound.

Example Let p = 3, m = 4, and $f(x) = \text{Tr}_1^4(wx^2)$, where w is a primitive element of \mathbb{F}_{3^4} . The sign of the Walsh transform of f(x) is $\varepsilon = 1$. Then $\mathcal{C}_{\overline{D}_{f,nsq}}$ and $\mathcal{C}_{\overline{D}_{f,sq}}$ in Corollary 4.4 have the same parameters [12, 4, 6] and the same weight enumerator $1 + 24z^6 + 56z^9$, which is verified by the Magma program. This code is optimal due to the Griesmer bound.

Example Let p = 3, m = 6, and $f(x) = \text{Tr}_1^6(wx^2)$, where w is a primitive element of \mathbb{F}_{36} . The sign of the Walsh transform of f(x) is $\varepsilon = 1$. Then $\mathcal{C}_{\overline{D}_{f,nsq}}$ and $\mathcal{C}_{\overline{D}_{f,sq}}$ in Corollary 4.4 have the same parameters [126, 6, 81] and the same weight enumerator $1 + 476z^{81} + 252z^{90}$, which is verified by the Magma program. This code is optimal due to the Griesmer bound.

Example Let p = 5, m = 4, and $f(x) = \text{Tr}_1^4(x^2)$, where w is a primitive element of \mathbb{F}_{5^4} . The sign of the Walsh transform of f(x) is $\varepsilon = -1$. Then $\mathcal{C}_{\overline{D}_{f,nsq}}$ and $\mathcal{C}_{\overline{D}_{f,sq}}$ in Corollary 4.4 have the same parameters [65, 4, 50] and the same weight enumerator $1 + 364z^{50} + 260z^{55}$, which is verified by the Magma program. This code is optimal due to the Griesmer bound.

Example Let p = 5, m = 4, and $f(x) = \text{Tr}_1^4(wx^2)$, where w is a primitive element of \mathbb{F}_{5^4} . The sign of the Walsh transform of f(x) is $\varepsilon = 1$. Then $\mathcal{C}_{\overline{D}_{f,nsq}}$ and $\mathcal{C}_{\overline{D}_{f,sq}}$ in Corollary 4.4 have the same parameters [60, 4, 45] and the same weight enumerator $1 + 240z^{45} + 384z^{50}$, which is verified by the Magma program.

Corollary 4.5: Let m be odd and $f(x) \in \mathcal{RF}$ with the sign ε of the Walsh transform. Then $\mathcal{C}_{\overline{D}_{f,nsq}}$ is a three-weight linear code with parameters $[\frac{1}{2}(p^{m-1} - \varepsilon \sqrt{p^*}^{m-1}), m]$ and the weight distribution in Table IX.

TABLE IX	
The weight distribution of $\mathcal{C}_{\overline{D}_{f,nsq}}$	For odd m

Weight	Mulitiplicity	
0	1	
$\frac{p-1}{2}p^{m-2}$	$p^{m-1} - 1$	
$\frac{1}{2}[(p-1)p^{m-2} + \varepsilon(1-p^*)\sqrt{p^*}^{m-3}]$	$\frac{p-1}{2}\left[p^{m-1} + \varepsilon\left(\frac{-1}{p}\right)\sqrt{p^*}^{m-1}\right]$	
$\frac{1}{2}[(p-1)p^{m-2} - \varepsilon(1+p^*)\sqrt{p^*}^{m-3}]$	$\frac{p-1}{2}[p^{m-1} - \varepsilon(\frac{-1}{p})\sqrt{p^*}^{m-1}]$	

Proof: From Theorem 4.2, this corollary follows.

Example Let p = 3, m = 5, and $f(x) = \text{Tr}_1^5(x^2)$, where w is a primitive element of \mathbb{F}_{3^5} . The sign of the Walsh transform of f(x) is $\varepsilon = 1$. Then $\mathcal{C}_{\overline{D}_{f,nsq}}$ in Corollary 4.5 has parameters [36, 5, 21] and the weight enumerator $1 + 72z^{21} + 90z^{24} + 80z^{27}$, which is verified by the Magma program. This code is almost optimal since the optimal code with length 36 and dimension 5 has minimal weight 22.

Example Let p = 3, m = 5, and $f(x) = \text{Tr}_1^5(wx^2)$, where w is a primitive element of \mathbb{F}_{3^5} . The sign of the Walsh transform of f(x) is $\varepsilon = -1$. Then $\mathcal{C}_{\overline{D}_{f,nsq}}$ in Corollary 4.5 has parameters [45, 5, 27] and the weight enumerator $1 + 80z^{27} + 72z^{30} + 90z^{33}$, which is verified by the Magma program. This code is almost optimal since the optimal code with length 45 and dimension 5 has minimal weight 28.

Corollary 4.6: Let m be odd and $f(x) \in \mathcal{RF}$ with the sign ε of the Walsh transform. Then $\mathcal{C}_{\overline{D}_{f,sq}}$ is a three-weight linear code with parameters $[\frac{1}{2}(p^{m-1} + \varepsilon \sqrt{p^*}^{m-1}), m]$ and the weight distribution in Table X.

TABLE X The weight distribution of $\mathcal{C}_{\overline{D}_{f,sq}}$ for odd m

Weight	Mulitiplicity
0	1
$\frac{p-1}{2}p^{m-2}$	$p^{m-1} - 1$
$\frac{1}{2}[(p-1)p^{m-2} + \varepsilon(1+p^*)\sqrt{p^*}^{m-3}]$	$\frac{p-1}{2}\left[p^{m-1} + \varepsilon\left(\frac{-1}{p}\right)\sqrt{p^*}^{m-1}\right]$
$\frac{1}{2}[(p-1)p^{m-2} + \varepsilon(p^*-1)\sqrt{p^*}^{m-3}]$	$\frac{p-1}{2}\left[p^{m-1} - \varepsilon\left(\frac{-1}{p}\right)\sqrt{p^*}^{m-1}\right]$

Proof: From Theorem 4.3, this corollary follows.

Example Let p = 3, m = 3, and $f(x) = \text{Tr}_1^3(wx^2)$, where w is a primitive element of \mathbb{F}_{3^3} . The sign of the Walsh transform of f(x) is $\varepsilon = -1$. Then $\mathcal{C}_{\overline{D}_{f,sq}}$ in Corollary 4.6 has parameters [6,3,3] and the weight enumerator $1 + 8z^3 + 6z^4 + 12z^5$, which is verified by the Magma program. This code is optimal due to the Griesmer bound.

Example Let p = 5, m = 3, and $f(x) = \text{Tr}_1^3(wx^2)$, where w is a primitive element of \mathbb{F}_{5^3} . The sign of the Walsh transform of f(x) is $\varepsilon = -1$. Then $\mathcal{C}_{\overline{D}_{f,sq}}$ in Corollary 4.6 has parameters [10, 3, 7] and the weight enumerator $1 + 40z^7 + 60z^8 + 24z^{10}$, which is verified by the Magma program. This code is optimal due to the Griesmer bound.

Example Let p = 7, m = 3, and $f(x) = \text{Tr}_1^3(x^2)$, where w is a primitive element of \mathbb{F}_{7^3} . The sign of the Walsh transform of f(x) is $\varepsilon = 1$. Then $\mathcal{C}_{\overline{D}_{f,sq}}$ in Corollary 4.6 has parameters [21, 3, 17] and the weight enumerator $1 + 126z^{17} + 168z^{18} + 48z^{21}$, which is verified by the Magma program. This code is optimal due to the Griesmer bound.

V. THE SIGN OF WALSH TRANSFORM OF SOME WEAKLY REGULAR BENT FUNCTIONS

In this section, we summarizes all known weakly regular bent functions over \mathbb{F}_{p^m} with odd characteristic p in Table XI, and aim at determining the sign of Walsh transform for some known weakly regular bent functions, which give parameters of linear codes from these functions. Quadratic bent functions in Table XI, as a class of weakly regular bent functions, have been used in [27] to construct linear codes. The sign of the Walsh transform of these functions is given in [15, Proposition 1].

Bent Function	m	p	Reference
$\sum_{i=0}^{\lfloor m/2 \rfloor} \operatorname{Tr}_1^m(c_i x^{p^i+1})$	arbitrary	arbitrary	[15], [20], [23], etc
$\int_{i=0}^{p^{k}-1} \operatorname{Tr}_{1}^{m}(c_{i}x^{i(p^{k}-1)}) + \operatorname{Tr}_{1}^{\ell}(\epsilon x^{\frac{p^{m}-1}{e}})$	m = 2k	arbitrary	[15], [19], [22], etc
$\operatorname{Tr}_{1}^{m}(cx^{\frac{3^{m}-1}{4}+3^{k}+1})$	m = 2k	p = 3	[14]
$\mathrm{Tr}_{1}^{m}(x^{p^{3k}+p^{2k}-p^{k}+1}+x^{2})$	m = 4k	arbitrary	[17]
$\operatorname{Tr}_{1}^{m}(cx^{\frac{3^{i}+1}{2}}), i \text{ odd}, \gcd(i,n) = 1$	arbitrary	p = 3	[4]

TABLE XI KNOWN WEAKLY REGULAR BENT FUNCTIONS OVER \mathbb{F}_{p^m} , p odd

A. Linear codes from Dillon type bent functions

In this subsection, for even m = 2k and odd prime p, we consider the Dillon type of bent functions, namely the functions of the form

$$f(x) = \sum_{i=1}^{p^k - 1} \operatorname{Tr}_1^m(c_i x^{i(p^k - 1)}) + \operatorname{Tr}_1^\ell(\delta x^{\frac{p^m - 1}{e}}),$$
(5)

where $e|p^k+1, c_i \in \mathbb{F}_{p^m}$ for $i = 0, 1, \dots, p^k-1, \delta \in \mathbb{F}_p^l$ and l is the smallest positive integer such that l|m and $e|(p^l-1)$.

Helleseth and Kholosha first investigated the non-binary Dillon type monomial bent function of the form $f(x) = \text{Tr}_1^m(cx^{i(p^k-1)})$ in [15] and proved that such bent functions only exist for p = 3. Later, this result was generalized by Jia et al. by adding a short trace term on $\text{Tr}_1^m(cx^{i(p^k-1)})$ and showed that new binomial bent functions can be found for $p \ge 3$ [19]. In 2013, Li et al. considered a general form of Dillon type of functions defined by Equation (5) and derived more Dillon type of bent functions from carrying out some suitable manipulation on certain exponential sums for any prime p [22].

For a function f(x) defined by Equation (5), it is obviously that f(0) = 0. For any $a \in \mathbb{F}_p^{\times}$, $a^{p-1} = 1$ and $f(ax) = f(x) = a^{p-1}f(x)$ satisfying that ((p-1)-1, p-1) = 1.

It has been proved in [22, Theorem 1] that each bent function of the form (5) is regular bent. Moreover, its Walsh transform value satisfies

$$\mathcal{W}_f(0) = p^k \zeta_p^{f(0)} = (-1)^{\frac{(p-1)m}{4}} \sqrt{p^*}^m.$$

Hence, $f(x) \in \mathcal{RF}$ and the sign ε of the Walsh transform is $(-1)^{\frac{(p-1)m}{4}}$. From results in Section III and IV, the weight distributions of these codes \mathcal{C}_{D_f} , $\mathcal{C}_{\overline{D}_f}$, $\mathcal{C}_{\overline{D}_{f,nsq}}$, $\mathcal{C}_{D_{f,sq}}$ and $\mathcal{C}_{\overline{D}_{f,sq}}$ from the function f(x) defined by Equation (5) can be obtained.

B. Linear codes from Helleseth-Kholosha ternary monomial bent functions

Let m = 2k with k odd, p = 3 and α be a primitive element of \mathbb{F}_{p^m} . Helleseth and Kholosha conjectured in [14], [15] in 2006 that the function

$$f(x) = \operatorname{Tr}_{1}^{m} (cx^{\frac{3^{m-1}}{4} + 3^{k} + 1})$$
(6)

is a weakly regular bent function if $c = \alpha^{\frac{3^k+1}{4}}$, and for any $\beta \in \mathbb{F}_{p^m}$, its Walsh transform is equal to

$$\mathcal{W}_{f}(\beta) = -3^{k} \zeta_{3}^{\pm \operatorname{Tr}_{1}^{k}(\frac{\beta 3^{k}+1}{\alpha(\delta+1)})}, \quad \delta = \alpha^{\frac{3^{m}-1}{4}}.$$
(7)

A partial proof of this conjecture can be found in [14]. The weakly regular bentness of f(x) defined by Equation (6) was first proved by Helleseth et al. in 2009 based on the Stickelberger's theorem [13]. In 2012, Gong et al. proved this conjecture from a different approach and solved the sign problem in the dual function of f(x) by finding the trace representation of the dual function [12].

Since m = 2k, $2|\frac{3^m - 1}{4}$ and $\frac{3^m - 1}{4} + 3^k + 1$ is even. Then f(x) = f(-x). For any $a \in \mathbb{F}_3^{\times}$, $f(ax) = f(x) = a^2 f(x)$ and (2 - 1, 3 - 1) = 1. It is obviously that f(0) = 0. Hence, $f(x) \in \mathcal{RF}$. From Equation (7), $\mathcal{W}_f(0) = -3^k = (-1)^{\frac{m}{2} + 1}\sqrt{3^*}^m$. The sign of Walsh transform of f(x) is $(-1)^{\frac{m}{2} + 1}$. From results in Section III and IV, the weight distributions of these codes $\mathcal{C}_{D_f}, \mathcal{C}_{\overline{D}_{f,nsq}}, \mathcal{C}_{\overline{D}_{f,nsq}}, \mathcal{C}_{D_{f,sq}}$ and $\mathcal{C}_{\overline{D}_{f,sq}}$ from the function f(x) defined by Equation (6) can be obtained.

C. Linear codes from Helleseth-Kholosha p-ary binomial bent functions

An infinite class of weakly regular binomial bent functions was found by Helleseth and Kholosha in [17] in 2010 for an arbitrary odd characteristic as follows. Let m = 4k and p be an odd prime. Then the function

$$f(x) = \operatorname{Tr}_{1}^{m}(x^{p^{3k}+p^{2k}-p^{k}+1}+x^{2})$$
(8)

is a weakly regular bent function. Moreover, for any $\beta \in \mathbb{F}_{p^m}$ its Walsh transform is equal to

$$\mathcal{W}_f(b) = -p^{2k} \zeta_p^{\text{Tr}_1^k(x_0)/4},\tag{9}$$

where x_0 is the unique solution in \mathbb{F}_{p^k} of the polynomial

$$\phi_b(x) := b^{p^{2k}+1} + (b^2 + x)^{\frac{p^{2k}+1}{2}} + b^{p^k(p^{2k}+1)} + (b^2 + x)^{\frac{p^k(p^{2k}+1)}{2}}$$

For a function f(x) defined in Equation (8), f(0) = 0. Note that $(p^{3k} + p^{2k} - p^k + 1, p - 1) = (2, p - 1) = 2$. For any $a \in \mathbb{F}_p^{\times}$, $f(ax) = a^2 f(x)$ and (2 - 1, p - 1) = 1. Hence, $f(x) \in \mathcal{RF}$. From Equation (9), we have

$$\mathcal{W}_f(0) = -p^{2k} = -\sqrt{p^*}^m.$$

The sign of the Walsh transform of f(x) is -1. From results in Section III and IV, the weight distributions of these codes C_{D_f} , $C_{\overline{D}_f}$, $C_{D_{f,nsq}}$, $C_{\overline{D}_{f,nsq}}$, $C_{D_{f,sq}}$, $C_{D_{f,sq}}$ from the function f(x) defined by Equation (8) can be obtained.

D. Linear Codes From Coulter-Matthews Ternary Monomial Bent Functions

It is well known that every component function $\operatorname{Tr}_1^m(c\pi(x)), c \in \mathbb{F}_{p^m}^{\times}$ of a planar function $\pi(x)$ over \mathbb{F}_{p^m} is a bent function. Thus, for any planar function $\pi(x)$ one can obtain that $f(x) = \operatorname{Tr}_1^m(c\pi(x))$ is a bent function for any nonzero $c \in \mathbb{F}_{p^m}$. This is another approach of constructions of bent functions. However, the construction of planar functions is a hard problem and until now there are only few known such functions (see [26] for example for a summary of known constructions). Note that all the known planar functions have the form of $\sum_{i,j=0}^{m-1} c_{ij} x^{p^i+p^j}$ where $c_{ij} \in \mathbb{F}_{p^m}$ with only one exception, namely the Coulter-Matthews class of functions of the form $x^{\frac{3^i+1}{2}}$ where i is odd and (i,m) = 1. Since the quadratic bent functions have been discussed in the previous subsetion, thus we only need consider the Coulter-Matthews class of bent functions of the form

$$f(x) = \operatorname{Tr}_{1}^{m}(cx^{\frac{3^{i+1}}{2}})$$
(10)

in this subsection, where $c\in \mathbb{F}_{p^m}^{\times}, \, i \text{ is odd and } (i,m)=1.$

In 2009, Helleseth et al. in [13] proved that the bent function f(x) defined by Equation (10) is weakly regular bent according to Stickelberger's theorem and they did not discussed its dual. For f(x) defined by Equation (10), f(0) = 0. Since *i* is odd, $2|\frac{3^i+1}{2}$ and f(x) = f(-x). For any $\alpha \in \mathbb{F}_3^{\times}$, $f(ax) = f(x) = a^2 f(x)$ and (2-1,3-1) = 1. Hence, $f(x) \in \mathcal{RF}$. Let $u = (\frac{3^i+1}{2}, 3^m - 1)$, then $u|(3^{2i} - 1, 3^m - 1)$ and $u|3^{(2i,m)} - 1$. From (i,m) = 1, $u|3^{(2,m)} - 1$. Then u|8. From $2|\frac{3^i+1}{2}$, $u \in \{2,4,8\}$. Since *i* is odd, $3^i + 1 = 3^{2\frac{i-1}{2}}3 + 1 \equiv 4 \mod 8$ and $\frac{3^i+1}{2} \equiv 2 \mod 4$. Then $(\frac{3^i+1}{2}, 3^m - 1) = 2$ and

$$\mathcal{W}_{f}(0) = \sum_{x \in \mathbb{F}_{q}} \zeta_{p}^{\operatorname{Tr}_{1}^{m}(cx^{(3^{i}+1)/2})} = \sum_{x \in \mathbb{F}_{q}} \zeta_{p}^{\operatorname{Tr}_{1}^{m}(cx^{2})}.$$

From Lemma 2.3,

$$\mathcal{W}_f(0) = (-1)^{m-1} \eta(c) \sqrt{p^*}^m.$$

The sign ε of Walsh transform of f(x) is $(-1)^{m-1}\eta(c)$. From results in Section III and IV, the weight distributions of these codes C_{D_f} , $C_{\overline{D}_f}$, $C_{D_{f,nsq}}$, $C_{\overline{D}_{f,nsq}}$, $C_{D_{f,sq}}$ and $C_{\overline{D}_{f,sq}}$ from the function f(x) defined by Equation (10) can be obtained.

VI. CONCLUSION

In this paper, we construct linear codes with two or three weights from weakly regular bent functions. We first generalize the constructing method of Ding et al.[6] and Zhou et al. [27] and determine the weight distributions of these linear codes. We solve the open problem proposed by Ding et al. [6].

Further, by choosing defining sets different from Ding et al.[6] and Zhou et al. [27], we construct new linear codes with two weights or three weights from weakly regular bent functions, which contain some optimal linear codes with parameters meeting certain bound on linear codes. The weight distributions of these codes are determined by the sign of the Walsh transfrom of weakly regular bent functions. The two-weight codes in this paper can be used in strongly regular graphs with the method in [2], and the three-weight codes in this paper can give association schemes introduced in [1]. The following work will study how to construct the linear codes with few weights from more general function f(x).

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