Cut-Set Bounds on Network Information Flow

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Abstract

Explicit characterization of the capacity region of communication networks is a long standing problem. While it is known that network coding can outperform routing and replication, the set of feasible rates is not known in general. Characterizing the network coding capacity region requires determination of the set of all entropic vectors. Furthermore, computing the explicitly known linear programming bound is infeasible in practice due to an exponential growth in complexity as a function of network size. This paper focuses on the fundamental problems of characterization and computation of outer bounds for multi-source multi-sink networks. Starting from the known local functional dependencies induced by the communications network, we introduce the notion of irreducible sets, which characterize implied functional dependencies. We provide recursions for computation of all maximal irreducible sets. These sets act as information-theoretic bottlenecks, and provide an easily computable outer bound for networks with correlated sources. We extend the notion of irreducible sets (and resulting outer bound) for networks with independent sources. We compare our bounds with existing bounds in the literature. We find that our new bounds are the best among the known graph theoretic bounds for networks with correlated sources and for networks with independent sources.

I. INTRODUCTION

The network coding approach introduced in [2], [3] generalizes routing by allowing intermediate nodes to forward coded combinations of all received data packets. This yields many benefits that are by now well documented [4]–[7]. One fundamental open problem is to characterize the capacity region and the classes of codes that achieve capacity. The single session multicast problem is well understood. Its capacity region is characterized by max-flow/min-cut bounds and linear codes are optimal [3].

Significant complications arise in more general scenarios, involving multiple sessions. A computable characterization of the capacity region is still unknown. One approach is to develop bounds as the intersection of a set of linear constraints (specified by the network topology and sink demands) and the set of entropy functions Γ^* (inner bound), or its closure $\overline{\Gamma^*}$ (outer bound) [4], [8], [9]. An exact expression for the capacity region does exist, again in terms of Γ^* [10]. Unfortunately, this expression, or even the bounds [4], [8], [9] cannot be computed in practice, due to the lack of an explicit characterization of the set of entropy functions for three or more random variables. The difficulties arising from the structure of Γ^* are not simply an artifact of the way the capacity region and bounds are written. It has been shown that the problem of determining the capacity region for multi-source network coding is completely equivalent to characterization of Γ^* [11].

One way to resolve this difficulty is via relaxation of the bound, replacing the set of entropy functions with the set of polymatroids Γ (which has a finite, polyhedral characterization). This results in a *geometric bound* that is in principle computable using linear programming [8]. In practice however, the number of variables and constraints in this linear program both increase exponentially with the number of links in the network. This prevents numerical computation for any meaningful case of interest. An alternative approach is to seek *graphical bounds* based on functional dependence properties and cut sets in graphs derived from the original communications network.

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The more difficult problems of characterization and computation of bounds for networks with correlated sources has received less attention than networks with independent sources. For a few special cases, necessary and sufficient conditions for reliable transmission have been found. In particular, it was recently showed [12] that the minimum cut is a necessary and sufficient condition for reliable transmission of multiple correlated sources to *all* sinks. This result includes the necessary and sufficient condition [13], [14] for networks in which every source is demanded by single sink as a special case. However, the correlated source problem is an uncharted area in general. A related important problem is that of separation of distributed source coding and network coding. It has been shown [15] that separation holds for two-source two-sink networks. However it has also been shown by example that separation fails for two-source three-sink and three-source two-sink networks.

In this paper we develop new outer bounds for the capacity region of general multicast networks with correlated sources. We further develop the main concepts to also give tighter bounds for networks with independent sources. The main idea of these bounds is to find subsets of random variables in the network that act as *information blockers*, or *information-theoretic cut sets*. These are sets of variables that determine all other variables in the network. We develop the properties of these sets, which leads to recursive algorithms for their enumeration. These algorithms can be thought of as operating on a specially constructed functional dependence graph that encodes the *local* functional dependencies imposed by encoding and decoding constraints.

A. Organization and Main Results

Section II provides required background, including a review of regions in the entropy space. These regions are used to describe a family of geometric bounds on the capacity region for network coding. We also describe existing graphical bounds. Section III presents main results of the paper. In Section III-A, we generalize the concept of a functional dependence graph (FDG), Definition 7, to handle polymatroidal variables (a wider class of objects than random variables). This gives us a single framework that supports both geometric and graphical bounds. Following on from this, we introduce the notion of irreducible sets and maximal irreducible sets for functional dependence graphs, which are our key ingredients for characterization and computation of capacity bounds. Recursive algorithm finding all maximal irreducible sets for cyclic FDGss is developed using the structural properties of maximal irreducible sets. In Section III-B, we describe construction of a cyclic FDG, called network FDG, from a given multi-source multi-sink network. Maximal irreducible sets in a network FDG are information bottlenecks, and provide Theorem 2¹ which outer bounds the capacity region for networks with correlated sources. It is established that Theorem 2 is the best known graph theoretic bound for multi-source multi-sink networks with correlated sources. In Section III-C we adapt our approach to take advantage of the additional constraints introduced when sources are mutually independent. This results in an improved bound, Theorem 3. In Appendix A we give an algorithm to enumerate all maximal irreducible sets for acyclic FDGs. In Section IV, we compare our new bounds with previously known results: cut-set bound [16], network sharing bound [17], the notion of information dominance [18] and progressive *d*-separating edge-set bound [19].

B. Notation

Sets will be denoted with calligraphic typeface, e.g. \mathcal{X} . Set complement is denoted by the superscript \mathcal{X}^c (where the universal set will be clear from context). Set subscripts identify the set of objects indexed by the subscript: $X_{\mathcal{A}} = \{X_a, a \in \mathcal{A}\}$. Collections of sets are denoted in bold, e.g., \mathcal{A} . The power set $2^{\mathcal{X}}$ is the collection of all subsets of \mathcal{X} . Where no confusion will arise, set union will be denoted by juxtaposition, $\mathcal{A} \cup \mathcal{B} = \mathcal{AB}$, and singletons will be written without braces.

¹A simpler version of this bound was presented at IEEE International Symposium on Information Theory, Seoul, South Korea, June/July 2009 [1]

II. BACKGROUND

A. Poymatroids

We start with a brief review on classes of polymatroids which are used to derived a framework to characterize outer bounds on the network coding capacity region. The framework will also enable us to understand the connection between some geometric bounds and graphical bounds. As we shall see, some of these graphical bounds can be interpreted as relaxations of geometric bounds.

Let \mathcal{X} be a set of *n* variables and *h* be a real-valued function $h : 2^{\mathcal{X}} \to \mathbb{R}$ such that $h(\emptyset) = 0$. Each function *h* can also be viewed as a column vector in \mathbb{R}^{2^n} (or in $\mathbb{R}^{2^{n-1}}$ knowing that $h(\emptyset)$ is always 0) often called the entropy space [20].

Definition 1 (Polymatroidal function or polymatroids): A function $h : 2^{\mathcal{X}} \mapsto \mathbb{R}$ is polymatroidal if it satisfies the following polymatroid axioms (1)-(3) for all disjoint $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathcal{X}$.

$$h(\emptyset) = 0 \tag{1}$$

$$h(\mathcal{B}|\mathcal{A}) \triangleq h(\mathcal{A} \cup \mathcal{B}) - h(\mathcal{A}) \ge 0$$
⁽²⁾

$$I_h(\mathcal{A}; \mathcal{B}|\mathcal{C}) \triangleq h(\mathcal{A} \cup \mathcal{C}) + h(\mathcal{B} \cup \mathcal{C}) - h(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}) - h(\mathcal{C}) \ge 0.$$
(3)

The set \mathcal{X} is called the *ground set* of the polymatroid h.

Definition 2 (Entropic function): A function h is called *entropic* if there exists a set of n discrete random variables $(X_v : v \in \mathcal{X})$ such that

$$h(\mathcal{A}) = H(X_v : v \in \mathcal{A})$$

for all $\mathcal{A} \subseteq \mathcal{X}$. Here, $H(\cdot)$ is the Shannon entropy function.

It is well known that all entropic functions are polymatroids. In the context of entropy functions, those polymatroid axioms are equivalent to the *basic*, or *Shannon-type* inequalities [21]. For these reasons, an element in a ground set of a polymatroid may also be called "variable". Note that the chain rule for polymatroids also directly follows from the definition of $h(\cdot|\cdot)$. Functional dependency and independence in polymatroids can also be similarly defined as in random variables. Specifically, with respect to a polymatroid h,

1) a subset of variables A is a function of another subset of variables B if

$$h(\mathcal{A}|\mathcal{B}) = 0,$$

2) a subset of variables A is conditionally independent of another subset of variables B given C if

$$I_h(\mathcal{A};\mathcal{B}|\mathcal{C})=0$$

Definition 3 (Almost entropic function): A function h is almost entropic if there exists a sequence of entropic functions $H^{(k)}$ such that $\lim_{k\to\infty} H^{(k)} = h$.

Let Γ , Γ^* and $\overline{\Gamma}^*$ be respectively the set of all polymatroidal, entropic and almost entropic functions. It is clear that

$$\Gamma^* \subseteq \bar{\Gamma}^* \subseteq \Gamma. \tag{4}$$

In general, the region Γ^* is not closed and hence $\overline{\Gamma}^*$ strictly contains Γ^* . While $\overline{\Gamma}^*$ is convex [22], it is still extremely hard to characterize $\overline{\Gamma}^*$ (and hence also Γ^*). In fact, $\overline{\Gamma}^*$ is not even a polyhedron for n > 3 [23]. On the contrary, its outer bound Γ is a much simpler polyhedron in the non-negative orthant $\mathbb{R}^{2^n-1}_+$ and in fact is the intersection of

$$m = n + \binom{n}{2} 2^{n-2} \tag{5}$$

half spaces induced by the following elemental inequalities [20]

$$h(A|\mathcal{X} \setminus \{A\}) \ge 0 \tag{6}$$

$$I_h(A; B|\mathcal{C}) \ge 0 \tag{7}$$

where $A, B \in \mathcal{X}$ and $\mathcal{C} \subseteq \mathcal{X} \setminus \{A, B\}$.

B. Network Coding

Let the directed acyclic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ serve as a simplified model of a communication network with error-free point-to-point communication links. We use tail(e) and head(e) to respectively denote tail and the head of the directed edge e. For nodes u, v and edge e, we write $u \to e$ as a shorthand for u = tail(e) and $e \to v$ for v = head(e). Also, for $e, f \in \mathcal{E}$, we write $e \to f$ if head(e) = tail(f). A *path* in a directed graph is a sequence of nodes $v_1, ..., v_n$ such that there exists edges $e_1, ..., e_{n-1}$ with $tail(e_i) = v_i$ and $head(e_i) = v_{i+1}$. Such a path is said to have length n - 1. Node v_n is *reachable* from node v_1 if there exist a path from node v_1 to v_n . Furthermore, node v_n is *connected* to v_1 if there exist nodes $v_1, ..., v_n$ and edges $e_1, ..., e_{n-1}$ with $head(e_i) = v_i$ and $tail(e_i) = v_{i+1}$, and/or $tail(e_i) = v_i$ and $head(e_i) = v_{i+1}$. In other words, v_n is connected to v_1 if the two nodes are connected, by ignoring the direction of the edges.

Let S be an index set for multicast sessions and $\{Y_s : s \in S\}$ be the set of sources. The source s is available at the set of nodes a(s) and is demanded by multiple sink nodes $b(s) \subseteq V$. We call the tuple (a, b) the connection requirement.

In this paper, we assume that the sources are i.i.d. sequences

$$\{(Y_s^n, s \in \mathcal{S}), n = 1, 2, \dots, \}$$

so that copies of $(Y_s^n, s \in S)$ generated at different time n will be independent of each other. However, within the same time instance n, the sources $(Y_s^n, s \in S)$ may be correlated among different sources. In the special case when $(Y_s^n, s \in S)$ is also mutually independent, we will say the sources are independent. Also, the distribution of $(Y_s^n, s \in S)$ and hence entropies of any subset of sources are assumed to be known. For notation simplicity, we will use $(Y_s, s \in S)$ to denote a generic copy of the sources at any particular time instance.

For a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ subject to a connection requirement a and b, a deterministic network code (of block length n) is a collection of source and edge random variables $(Y_s^{[n]}, s \in \mathcal{S}, U_e^{(n)}, e \in \mathcal{E})$ where $U_e^{(n)}$ is the message transmitted on the edge $e \in \mathcal{E}$ and $Y_s^{[n]}$ is the block of source symbols (Y_s^1, \ldots, Y_s^n) . Unlike $Y_s^{[n]}$ which is a collection of n i.i.d. random variable, the superscript (n) in $U_e^{(n)}$ is only used to indicate the block length of the code. It does not mean that $U_e^{(n)}$ is a collection of n i.i.d. random variables.

Clearly, these random variables $(Y_s^{[n]}, s \in S, U_e^{(n)}, e \in \mathcal{E})$ cannot be arbitrarily but must satisfy some constraint. In particular, it is required that 1) an edge random variable must be a function of incident edge random variables and source random variables, and 2) for any $s \in S$, a sink node $v \in b(s)$ must be able to reconstruct the demanded source. More precisely, we have the following definition.

Definition 4 (Network code): A network code $\phi_{\mathcal{G}}^{(n)} = \{\phi_e^{(n)}, \phi_{u,s}^{(n)}\}$ of block length n is described by a set of local encoding functions $\phi_e^{(n)}, e \in \mathcal{E}$ and decoding functions $\phi_{u,s}^{(n)}, u \in b(s), s \in \mathcal{S}$

$$\phi_e^{(n)} : \prod_{j \in \mathcal{S}: j \to e} \mathcal{Y}_j^{[n]} \times \prod_{f \in \mathcal{E}: f \to e} \mathcal{U}_f^{(n)} \longmapsto \mathcal{U}_e^{(n)},$$

$$\phi_{u,s}^{(n)} : \prod_{j \in \mathcal{S}: j \to u} \mathcal{Y}_j^{[n]} \times \prod_{f \in \mathcal{E}: f \to u} \mathcal{U}_f^{(n)} \longmapsto \mathcal{Y}_s^{[n]}.$$

Here, the alphabets of the block of source random variables $Y_s^{[n]}$ and edge random variables $U_e^{(n)}$ are denoted by $\mathcal{Y}_s^{[n]}$ and $\mathcal{U}_e^{(n)}$ respectively.

Remark 1: With respect to a given network code, the joint distribution for the set of all source and edge random variables $(Y_s^{[n]}, U_e^{(n)}, s \in S, e \in \mathcal{E})$ will become well-defined. Furthermore, for any $e \in \mathcal{E}$, one can construct a global encoding function such that

$$U_e^{(n)} = \tilde{\phi}_e^{(n)}(Y_s^{[n]}, s \in \mathcal{S})$$
(8)

Definition 5 (Achievability): An edge capacity tuple $\mathbf{c} = (c_e : e \in \mathcal{E}) \in \mathbb{R}^{|\mathcal{E}|}_+$ is called *achievable* if there exists a sequence of network codes

$$\phi_{\mathcal{G}}^{(n)} = \{\phi_e^{(n)}, \phi_{u,s}^{(n)}, e \in \mathcal{E}, s \in \mathcal{S}, u \in b(s)\}$$

(and also the corresponding induced source and edges random variables $(Y_s^{[n]}, U_e^{(n)}, s \in S, e \in \mathcal{E})$) such that

$$\limsup_{n \to \infty} \log_2 \left| \mathcal{U}_e^{(n)} \right| / n \le c_e$$
$$\limsup_{n \to \infty} \Pr\left\{ \phi_{u,s} \left(Y_j^{[n]}, U_f^{(n)} : f \to u, j \to u \right) \neq Y_s^{[n]} \right\} = 0$$

for all $e \in \mathcal{E}$ and $u \in b(s)$.

Remark 2: When sources are correlated, it is natural to assume a fixed joint distribution of the sources. In that case, the network coding capacity region \mathcal{R} is the set of all achievable edge capacity tuples that support the transmission of the sources. When sources are independent, only the entropies but not the joint distribution matter (as one can always compress the sources independently before transmission). Therefore, as in some existing literature, one may instead focus on finding the set of source rates or entropies that a network can transport, subject to a fixed edge capacity tuple.

C. Network Coding Bounds

Definition 6 below provides a standard framework to formulate "geometric" bounds on the set of achievable edge capacity tuples (denoted by \mathcal{R}).

Definition 6: Consider any network coding problem (with an underlying network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and connection requirement (a, b)). For any non-empty subset Δ of polymatroids on the ground set $\mathcal{X} = (Y_s, U_e, s \in \mathcal{S}, e \in \mathcal{E})$, let $\mathcal{R}(\Delta)$ be the set of tuples $(c_e, e \in \mathcal{E}) \in \mathbb{R}^{|\mathcal{E}|}$ for which there exists $h \in \Delta$ satisfying

$$h\left(Y_s:s\in\mathcal{A}\right) - H\left(Y_s:s\in\mathcal{A}\right) = 0, \ \mathcal{A}\subseteq\mathcal{S}$$
(9)

$$h\left(U_e \mid Y_j, j \to e, \ U_f, f \to e\right) = 0, \ e \in \mathcal{E}$$

$$\tag{10}$$

$$h(Y_s \mid Y_j, j \to u, \ U_f, f \to u) = 0, \ u \in b(s), s \in \mathcal{S}$$

$$(11)$$

$$h\left(U_e\right) \le c_e, \ e \in \mathcal{E} \tag{12}$$

Remark 3: Note that, Y_s in (9) can be viewed as a generic source random variable and also as an element in the ground set \mathcal{X} .

Here we can identify constraints due to source correlation (9), network coding (10), decoding (11), edge capacity (12). Each of these constraints defines a region of polymatroids

$$\mathcal{C}_1 \triangleq \{h : h \text{ satisfies (9)}\}$$
(13)

$$\mathcal{C}_2 \triangleq \{h : h \text{ satisfies (10)}\}$$
(14)

$$\mathcal{C}_3 \triangleq \{h : h \text{ satisfies (11)}\}$$
(15)

$$\mathcal{C}_4 \triangleq \{h : h \text{ satisfies (12)}\}.$$
(16)

When sources are independent, i.e., $h(Y_s : s \in S) = \sum_{s \in S} h(Y_s)$, $\mathcal{R}(\Gamma^*)$ and $\mathcal{R}(\overline{\Gamma}^*)$ are respectively inner and outer bounds for \mathcal{R} [8, Chapter 15]. In Yan et al. [10], an exact characterization of \mathcal{R} for multi-source multi-sink network coding was also obtained.

When sources are correlated, using arguments similar to those used in the proof of [8, Theorem 15.9], one can prove that $\mathcal{R}(\bar{\Gamma}^*)$ is still an outer bound for \mathcal{R} . Note that in the bound $\mathcal{R}(\bar{\Gamma}^*)$, only the joint entropies of the sources but not their joint probability distribution are used to derive the bound. Therefore, one can tighten the bound by incorporating additional information about the joint distribution in characterizing bounds (see [24], [25] and [26]).

Since entropy functions and almost entropic functions are polymatroidal (4) and the regions $\overline{\Gamma}^*$, Γ , C_1 , C_2 , C_3 , C_4 are closed and convex, it follows that $\mathcal{R}(\Gamma)$ is an outer bound for the set of achievable rates. The relation of these capacity bounds is summarized below.

$$\mathcal{R} \subseteq \mathcal{R}(\bar{\Gamma}^*) \subseteq \mathcal{R}(\Gamma) \tag{17}$$

Weighted sum-rate bounds induced by $\mathcal{R}(\Gamma)$ can in principle be computed using linear programming. One practical difficulty with numerical computation of such bounds is that the number of variables and the number of constraints due to Γ both increase exponentially with $|\mathcal{S}| + |\mathcal{E}|$ (refer to (5)). Attempts to simplify these bounds using direct application of Fourier-Motzkin [27] may prove fruitless. In [28], the authors have proposed a graph based approach to simplify the bound by exploiting the abundant set of functional dependencies in a network coding problem.

In addition to above bounds, there are also many "graphical" bounds (i.e., bounds that rely on a graph representation of the network coding system) in existing literatures. We will review and compare these bounds, such as cut-set bound [16], network sharing bound [17] and progressive *d*-separating edge-set bound [19] in Section IV.

III. MAIN RESULTS

The main results of this paper are graphical bounds for networks with correlated or independent sources. In Section III-A we will define a functional dependence graph, which represents a set of local functional dependencies between polymatroidal variables. Our definition extends [29] to accommodate cycles containing source nodes, and polymatroidal variables in place of random variables. This section also provides the main technical ingredients for our new bounds. In particular, we describe a test for functional dependence, and give a basic result relating local and global dependence. Section III-B describes our new bound for general multicast networks with correlated sources, based on the implications of local functional dependence. Section III-C considers source independence implications to further strengthen the proposed bound.

The main ingredient of most graph based outer bounds is the following theorem: Theorem 1 (Bottleneck Bound): Let $\mathcal{B} = \{U_{\mathcal{A}}, Y_{\mathcal{W}^c}\}$ be a set such that

$$h(\mathcal{B}) = h(Y_s, s \in \mathcal{S}) \tag{18}$$

for any polymatroid $h \in C_1 \cap C_2 \cap C_3 \cap C_4$. Then,

$$\sum_{e \in \mathcal{A}} c_e \ge H\left(Y_{\mathcal{W}} \mid Y_{\mathcal{W}^c}\right). \tag{19}$$

Proof: Notice that

$$H (Y_{\mathcal{W}} | Y_{\mathcal{W}^{c}}) = h (Y_{\mathcal{W}} | Y_{\mathcal{W}^{c}})$$

= $h (Y_{\mathcal{S}}) - h (Y_{\mathcal{W}^{c}})$
= $h (\mathcal{B}) - h (Y_{\mathcal{W}^{c}})$
= $h (U_{\mathcal{A}} | Y_{\mathcal{W}^{c}})$
 $\leq \sum_{e \in \mathcal{A}} h(U_{e})$
 $\leq \sum_{e \in \mathcal{A}} c_{e}$

and the theorem is proved.

As a consequence, one may identify various subsets \mathcal{B} satisfying (18) and use them to derive bounds for the network coding rate region. The question however is how to find such bottleneck subsets. Finding all bottlenecks can be a very challenging and computing intensive task. In the remaining of the section, we will derive various graph based technique to find such bottlenecks.

A. Functional Dependence Graphs

Definition 7 (Functional Dependence Graph): Let Δ be a set of polymatroids on a ground set $\mathcal{X} = \{X_1, \ldots, X_N\}$. A directed graph $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ is called a *functional dependence graph* for Δ if and only if for all $i = 1, 2, \ldots, N$

$$h(X_i \mid X_j : (j,i) \in \mathcal{E}^*) = 0, \forall h \in \Delta$$
(20)

Alternatively, a function h is said to satisfy the FDG $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ if it satisfies (20). An FDG is called *cyclic* if every node is a member of a directed cycle.

Definition 7 is more general than the FDG of [29, Chapter 2]: Firstly, in our definition there is no distinction between source and non-source random variables. The graph simply characterizes functional dependence between variables. In fact, our definition admits cyclic directed graphs with cycles containing source nodes, and there may be no nodes with in-degree zero (which are source nodes in [29]). We also do not require independence between sources (when they exist), which is implied by the acyclic constraint in [29]. Our definition admits functions h with *additional* functional dependence relationships that are not represented by the graph. It only specifies a certain set of conditional functions which must be zero. Our definition holds for a wider class of objects (variables in polymatroids) rather than only random variables. Clearly an FDG in the sense of [29] satisfies the conditions of Definition 7, but the converse is not true. For clarity, a functional dependence graph (FDG) is defined according to our Definition 7.

Definition 7 specifies an FDG in terms of local dependence structure. Given such local dependence constraints, it is of great interest to determine all implied functional dependence relations. In other words, given an FDG, we wish to find all sets A and B such that h(B|A) = 0 for all h satisfying the FDG.

Definition 8 (\mathcal{A} determines \mathcal{B}): Consider a directed graph $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$. For any sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, we say that \mathcal{A} determines \mathcal{B} (with respect to Procedure A) if there are no elements of \mathcal{B} remaining after the following procedure:

Procedure A:

Remove all the edges outgoing from the nodes in A and subsequently remove all nodes and edge with no incoming edges and nodes respectively.

We will use $\mathcal{A} \longrightarrow_{\mathcal{A}} \mathcal{B}$ to denote that \mathcal{A} determines \mathcal{B} .

Definition 9 (Blanket): For a given set \mathcal{A} , let $\mu_A(\mathcal{A}) \subseteq \mathcal{X}$ be the set of nodes deleted by the procedure of Definition 8 together with the nodes in \mathcal{A} . We will call $\mu_A(\mathcal{A})$ the *blanket* of \mathcal{A} (with respect to Procedure A).

Clearly $\mu_A(\mathcal{A})$ is the largest set of nodes with $\mathcal{A} \longrightarrow_A \mu_A(\mathcal{A})$. To this end, define for $X_i \in \mathcal{X}$

$$\pi(X_i) = \{X_j \in \mathcal{V} : (X_j, X_i) \in \mathcal{E}^*\}$$
(21)

to be the set of parents of node X_i . Where it does not cause confusion, we will abuse notation and identify variables and nodes in the FDG, e.g. (20) will be written $h(X_i | \pi(X_i)) = 0$ or simply $h(i | \pi(i)) = 0$.

Lemma 1 (Grandparent lemma): Let $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ be an FDG for a polymatroid h. For any $j \in \mathcal{V}$ with $i \in \pi(j) \neq \emptyset$

$$h(j \mid \pi(i), \pi(j) \setminus i) = 0.$$
(22)

Proof: By hypothesis, $h(j \mid \pi(j)) = 0$ for any $j \in \mathcal{V}$. Furthermore, note that for any $h \in \Gamma$, conditioning cannot increase the function h^2 and hence $h(j \mid \pi(j), \mathcal{A}) = 0$ for any $\mathcal{A} \subseteq \mathcal{X}$. Now using

²This is a direct consequence of submodularity (3).

this property, and the chain rule for polymatroids,

$$\begin{aligned} 0 &= h(j \mid \pi(j)) \\ &= h(j \mid \pi(j), \pi(i)) \\ &= h(j, \pi(j), \pi(i)) - h(\pi(j), \pi(i)) \\ &= h(j, \pi(j) \setminus i, \pi(i)) - h(\pi(j), \pi(i)) \\ &= h(j, \pi(j) \setminus i, \pi(i)) - h(\pi(j) \setminus i, \pi(i)) \\ &= h(j \mid \pi(i), \pi(j) \setminus i). \end{aligned}$$

We emphasize that in the proof of Lemma 1, we have only used the submodular property of polymatroids, together with the hypothesized local dependence structure specified by the FDG.

Lemma 2: Let $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ be an FDG for a polymatroid h. Then for disjoint subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$,

$$\mathcal{A} \longrightarrow_{A} \mathcal{B} \implies h(\mathcal{B} \mid \mathcal{A}) = 0.$$
⁽²³⁾

Proof: Let $\mathcal{A} \longrightarrow_{\mathcal{A}} \mathcal{B}$. Then, by Definition 8 there must exist directed paths from some nodes in \mathcal{A} to each node in \mathcal{B} , and there must not exist a directed path to any node in \mathcal{B} which does not also intersect \mathcal{A} . In other words, apart from the paths from nodes in \mathcal{A} and their sub-paths, any other path leading to \mathcal{B} must have an element of \mathcal{A} as its member. Recursively invoking Lemma 1, the lemma is proved.

Definition 10 (Irreducible set): A set of nodes \mathcal{B} is irreducible (with respect to Procedure A) if there is no $\mathcal{A} \subseteq \mathcal{B}$ such that $\mathcal{A} \longrightarrow_{\mathcal{A}} \mathcal{B}$. Furthermore, an irreducible set \mathcal{A} is maximal if $\mu_{\mathcal{A}}(\mathcal{A}) = \mathcal{X}$.

Remark 4: In this paper, we are mainly interested in cyclic FDGs to characterize cut-set bounds on network capacity. However, for other applications, acyclic FDGs may also be of interest. In Appendix A we define maximal irreducible sets for acyclic network and give an algorithm to compute them. For cyclic graphs, every subset of a maximal irreducible set is irreducible. In contrast to acyclic graphs the converse is not true, that is, there can be irreducible sets that are not maximal and are not subsets of any maximal irreducible set.

Corollary 1: If \mathcal{A} and \mathcal{B} are both maximal irreducible sets, then $h(\mathcal{A}) = h(\mathcal{B}) = h(\mathcal{X})$ for any polymatorids satisfying the FDG $(\mathcal{X}, \mathcal{E}^*)$.

Proof: By Definition 10, $\mu_A(\mathcal{A}) = \mu_A(\mathcal{B}) = \mathcal{X}$. Invoking Lemma 2, $h(\mathcal{A}) = h(\mathcal{B}) = h(\mathcal{X})$.

As we shall see, the corollary, together with Theorem 1, can be used to derive capacity bounds for network coding. Therefore, we are interested in finding every maximal irreducible set. This may be accomplished via **AllMaxSetsC**($\mathcal{G}_{\mathcal{N}}^*$, {}) in Algorithm 1, which recursively finds all maximal irreducible sets. In the algorithm, the graph $\mathcal{G}_{\mathcal{N}}^* = (\mathcal{X}_{\mathcal{N}}, \mathcal{E}_{\mathcal{N}}^*)$, where $\mathcal{N} = \{1, \ldots, |\mathcal{X}|\}$, is isomorphic to $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ via some bijection $\sigma : \mathcal{X} \mapsto \mathcal{X}_{\mathcal{N}}$ and hence $(u, v) \in \mathcal{E}^*$ iff $(\sigma(u), \sigma(v)) \in \mathcal{E}_{\mathcal{N}}^*$. For set $\mathcal{A} \subseteq \mathcal{N}$ we define $\mathcal{A}' = \{i \in \mathcal{N} : i > j, \forall j \in \mathcal{A}\}$.

Algorithm 1 AllMaxSetsC($\mathcal{G}_{\mathcal{N}}^*, \mathcal{A}$)

Require: $\mathcal{G}^*_{\mathcal{N}} = (\mathcal{N}, \mathcal{E}^*_{\mathcal{N}}), \mathcal{A} \subseteq \mathcal{N}$ 1: if $i \notin \mu_A(\mathcal{A}^c \setminus \{i\}), \forall i \in \mathcal{A}^c$ then Output \mathcal{A}^c 2: 3: else for all $i \in \mathcal{A}'$ do 4: if $i \in \mu_A \left(\mathcal{A}' \setminus \{i\} \right)$ then 5: Output AllMaxSetsC($\mathcal{G}_{\mathcal{N}}^*, \mathcal{A} \cup \{i\}$) 6: end if 7: end for 8: 9: end if

The actual number of operations (or the time complexity) to execute the function call depends on the topology of the FDG. The recursion tree is described in Figure 1. We make the following observations: (1) the leaf nodes of the recursion tree (such nodes are represented within circles) are subsets containing $|\mathcal{X}|$ and/or complement of maximal irreducible sets (denote by \mathcal{B} a maximal irreducible set and by \mathcal{M} the set of all such sets), (2) any leaf node which is not a complement of any $\mathcal{B} \in \mathcal{M}$ is a subset of some $\mathcal{B}^c, \mathcal{B} \in \mathcal{M}$ and (3) each node of the recursion tree represents a unique set. Hence the number of nodes are upper bounded by the cardinality of the set $\cup_{\mathcal{B} \in \mathcal{M}} 2^{\mathcal{B}^c}$. Using the union bound, the total number of calls of the function AllMaxSetsC($\mathcal{G}^*_{\mathcal{N}}, \{\}$) can be upper bounded by

$$\sum_{\mathcal{B}\in\mathcal{M}} 2^{|\mathcal{X}|-|\mathcal{B}|}.$$

Remark 5: Due to the recursive nature, the algorithm is easy to implement. The number of recursive calls can be further reduced, for example, by providing all cut-sets separating subsets of sources and corresponding sinks and using complement of the cut-sets as input to Algorithm 1 while replacing \mathcal{A}' by \mathcal{A}^c (this is important for input other than {}). We will see in Section IV that the maximal irreducible sets are subsets of such cut-sets.



Fig. 1. Recursion tree, \mathcal{B} is any maximal irreducible set.

B. A Bound for Network with Correlated Sources

So far, we have defined functional dependence graphs, developed some of their properties, and given algorithm for finding all maximal irreducible sets. In order to apply these results to find bounds on network coding capacity, we need to construct FDGs from multi-source communications networks with multicast constraints.

Definition 11 (Network FDG): For a given network coding problem (defined by the network topology \mathcal{G} and connection requirement (a, b)), its induced network FDG is a directed graph $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ defined as follows

• The set of nodes \mathcal{X} is equal to

$$\{U_e, e \in \mathcal{E}\} \cup \{Y_s, s \in \mathcal{S}\} \cup \{\dot{Y}_s^i, s \in \mathcal{S}, i \in b(s)\},\$$

- (A, B) is a directed edge in \mathcal{E}^* if it satisfies one of the following conditions
 - 1) $A = U_e, B = U_f \text{ and } e \to f;$
 - 2) $A = Y_s, B = U_f$ and $s \to f$;

 - 3) $A = U_e, B = \hat{Y}_s^i, i \in b(s) \text{ and } e \to i;$ 4) $A = Y_\ell, B = \hat{Y}_s^i, i \in b(s) \text{ and } i \in a(\ell);$ 5) $A = \hat{Y}_s^i, B = Y_s, \text{ and } i \in b(s).$

Remark 6: In the above definition, the physical meaning of \hat{Y}_s^i is the decoded estimates of Y_s at the sink node $i \in b(s)$. Note that the decoding constraints (11) require that $Y_s = \hat{Y}_s^i, i \in b(s)$ for all $s \in S$.

Example 1 (Network FDG of the butterfly network): Figure 2(a) shows the well-known butterfly network and Figure 2(b) shows its network FDG. Nodes are labeled with node numbers and variables. Edges in the network FDG represent dependencies due to encoding and decoding requirements.



Fig. 2. The butterfly network (a) and its network FDG (b).

In network FDGs, there are nodes for auxiliary variables which represent decoding estimate and are the same as the source variables demanded at the sink. Accordingly, the following procedure finds functional dependency in network FDG taking multicasting into consideration.

Definition 12 (Procedure B): Consider a network FDG as defined in Definition 11. For any sets $\mathcal{A}, \mathcal{B} \subseteq$ \mathcal{X} , we say \mathcal{A} determines \mathcal{B} (with respect to Procedure B) if there are no elements of \mathcal{B} remaining after the following procedure:

Procedure B:

- 1) Remove all edges outgoing from nodes in A and subsequently remove all nodes and edges with no incoming edges and nodes respectively.
- 2) If any \hat{Y}_s^i is removed, (a) remove all \hat{Y}_s^i for $i \in \{1, \dots, |b(s)|\}$ and (b) subsequently remove all edges and nodes with no incoming edges and nodes, go to Step 2. Else terminate.

We will use $\mathcal{A} \longrightarrow_{B} \mathcal{B}$ to denote that \mathcal{A} determines \mathcal{B} with respect to Procedure B.

As before, concepts such as blanket and irreducibility can be similarly defined with respect to Procedure B. Specifically, for a given set \mathcal{A} , its blanket (with respect to Procedure B) is denoted by $\mu_B(\mathcal{A})$ and is defined as the largest set of nodes with $\mathcal{A} \longrightarrow_B \mu_B(\mathcal{A})$. A set of nodes \mathcal{B} is called *irreducible* (with respect to Procedure B) if there is no $\mathcal{A} \subseteq \mathcal{B}$ such that $\mathcal{A} \longrightarrow_B \mathcal{B}$. An irreducible set \mathcal{A} is *maximal* if $\mathcal{X} \setminus \mu_B(\mathcal{A}) = \emptyset$. In addition, if \mathcal{A} and \mathcal{B} are maximal irreducible sets, then

$$h(\mathcal{A}) = h(\mathcal{B})$$

for all polymatroid h satisfying the network FDG.

Furthermore, the recursion described earlier in Algorithm 1 can also be used to find maximal irreducible sets for multi-source multi-sink networks with correlated sources, replacing $\mu_A(\cdot)$ by $\mu_B(\cdot)$.

Example 2 (Butterfly network): The maximal irreducible sets for the butterfly network in Figure 2(a) are

$$\{1,2\},\{1,5\},\{1,7\},\{1,8\},\{2,4\},\{2,7\},\{2,9\},\{3,4,5\}, \\ \{3,4,8\},\{3,7\},\{3,8,9\},\{4,5,6\},\{5,6,9\},\{6,7\},\{6,8,9\}.$$

Lemma 3: Consider a network FDG as defined in Definition 11. Suppose h is a polymatroid on the ground set $(Y_s, U_e, s \in \mathcal{S}, e \in \mathcal{E})$, satisfying (10) and (11). Then, one can extend h to a polymatroid h' on the ground set

$$\mathcal{X} = \{U_e, e \in \mathcal{E}\} \cup \{Y_s, s \in \mathcal{S}\} \cup \{\hat{Y}_s^i, s \in \mathcal{S}, i \in b(s)\},\$$

such that h' satisfies the network FDG.

Proof: The construction of h' is as follows. For any subset A of X, let

$$\theta \triangleq \{s \in \mathcal{S} : Y_s \notin \mathcal{A} \text{ and } Y_s^i \notin \mathcal{A}, \forall i \in b(s)\}$$

and

$$\delta \triangleq \{ e \in \mathcal{E} : U_e \in \mathcal{A} \}$$

Define

$$h'(\mathcal{A}) = h(U_e, \ e \in \delta, Y_s, \ s \notin \theta).$$
(25)

It can then be verified directly that h' satisfies the network FDG, Definition 11.

We can now state our first main result, an easily computable outer bound for the capacity region of a network coding system.

Theorem 2 (Functional Dependence Bound): Consider a network coding problem and its induced network FDG $(\mathcal{X}, \mathcal{E}^*)$. If $\mathcal{B} = \{U_{\mathcal{A}}, Y_{\mathcal{W}^c}\}$ is a maximal irreducible set (with respect to Procedure B) in $(\mathcal{X}, \mathcal{E}^*)$ and $(c_e, e \in \mathcal{E})$ is achievable, then

$$\sum_{e \in \mathcal{A}} c_e \ge H\left(Y_{\mathcal{W}} \mid Y_{\mathcal{W}^c}\right).$$
(26)

In the special case when sources are independent, then inequality (26) is reduced to

$$\sum_{e \in \mathcal{A}} c_e \ge \sum_{s \in \mathcal{W}} H\left(Y_s\right).$$
(27)

Proof: Let h be a polymatroid in $C_1 \cap C_2 \cap C_3 \cap C_4$. Then by Lemma 3, we can extend h to a polymatroid h' over the ground set \mathcal{X} satisfying the network FDG. Suppose \mathcal{B} is a maximal irreducible set. Then

$$h(\mathcal{B}) = h'(\mathcal{B}) = h(Y_s, s \in \mathcal{S}).$$

Then by Theorem 1, the result follows.

Let \mathcal{M}_B be the set of all maximal irreducible set $\{U_A, Y_{W^c}\}$ with respect to Procedures B and let

$$\triangleq \bigcap_{\{U_{\mathcal{A}}, Y_{\mathcal{W}^c}\} \in \mathcal{M}_B} \left\{ (c_e, e \in \mathcal{E}) : \sum_{e \in \mathcal{A}} c_e \ge H\left(Y_{\mathcal{W}} | Y_{\mathcal{W}^c}\right) \right\}.$$
(28)

Example 3 (Butterfly network): The functional dependence bound for the butterfly network of Figure 2(a), with correlated sources Y_1 and Y_2 is as follows (using the maximal irreducible sets in Example 2).

$$\{c_2, c_5, c_7\} \ge h (Y_1 | Y_2)$$

$$\{c_3, c_5, c_6\} \ge h (Y_2 | Y_1)$$

$$\{c_1 + c_5, c_4 + c_5, c_1 + c_2 + c_3, c_1 + c_2 + c_6, c_1 + c_6 + c_7, c_2 + c_3 + c_4, c_2 + c_3 + c_4, c_3 + c_4 + c_7, c_4 + c_6 + c_7\} \ge h (Y_1, Y_2)$$

If the sources Y_1 and Y_2 are instead independent, we obtain

$$h(Y_1) \le \{c_2, c_5, c_7\} \tag{29}$$

$$h(Y_2) \le \{c_3, c_5, c_6\} \tag{30}$$

$$h(Y_1) + h(Y_2) \le \{c_1 + c_5, c_4 + c_5, c_i + c_j : i \in \{2, 5, 7\}, j \in \{3, 5, 6\}\}$$
(31)

Note that the first two bounds $c_1 + c_5$, $c_4 + c_5$ on the sum rate in (31) follow from the maximal irreducible sets $\{3,7\}, \{6,7\}$ described in Example 2. The last nine bounds are consequences of the individual rate bounds in (29) and (30).

Remark 7: For single source multicast networks, the bound in Theorem 2 will be reduced to the max-flow bound [8, Theorem 11.3] and hence is tight. Summarizing (17) and Theorem 2, we have

$$\mathcal{R} \subseteq \mathcal{R}(\Gamma^*) \subseteq \mathcal{R}(\Gamma) \subseteq \mathcal{R}_{FD}.$$
(32)

The capacity region for the special case of multicast networks in which all correlated sources are demanded by all sinks was established by Han [12] using a simple cut-set based characterization. The cut-sets used by Han [12] are in fact the maximal irreducible sets, yielding the following corollary.

Corollary 2 (When every sink node demands all sources): For multicast networks in which all correlated sources are demanded by all sinks, $\mathcal{R} = \mathcal{R}_{FD}$.

C. When Sources are Independent

In this subsection, we further consider the special case when sources are independent. Unlike the case when sources are correlated, the problem of characterizing graphical bounds for networks with independent sources has been well investigated [2], [16]–[19], [30]. A source independence constraint may imply additional functional dependencies beyond those implied by the network coding and decoding constraints alone. These additional functional dependencies may in turn be used to improve of our characterization of the set of achievable rate region.

To understand the new bound, we first begin with a review of some basic graph concepts. The d-separation criterion [31] is a tool to infer certain conditional independence relationships amongst a set of random variables where (some of) their local conditional independence relations are represented by a Bayesian Network (directed acyclic graph). It has also been shown that the d-separation criterion is valid for finding certain conditional independence in cyclic functional dependence graphs [29] (see Definition 14). The fd-separation criterion [29] is an extension of d-separation finding certain conditional independence relationships in FDGs. In this section, we generalize this result by showing that fd-separation can be used to find conditional independence relationships for polymatroidal variables represented by an FDG.

Definition 13 (Ancestral graph): Consider a directed graph $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ induced by a network coding problem. For any subset $\mathcal{A} \subseteq \{Y_s, s \in \mathcal{S}, U_e, e \in \mathcal{E}\}$, let $An(\mathcal{A})$ denote the set of all nodes in $\{Y_s, s \in \mathcal{S}, U_e, e \in \mathcal{E}\}$ such that for every node $u \in An(\mathcal{A})$, there is a directed path from u to some node v in \mathcal{A} in the subgraph $\overline{\mathcal{G}}^* \triangleq \mathcal{G}^* \setminus \{e : e \to Y_s, s \in \mathcal{S}\}$. The ancestral graph with respect to \mathcal{A} (denoted by $\mathcal{G}^*_{An(\mathcal{A})}$) is a subgraph of \mathcal{G}^* consisting of nodes $\mathcal{A} \cup \operatorname{An}(\mathcal{A})$ and edges $e \in \mathcal{E}^*$ such that $\operatorname{head}(e), \operatorname{tail}(e) \in \mathcal{A} \cup \operatorname{An}(\mathcal{A})$.

Definition 14 (d-separation): A set C d-separates A and B in a network FDG \mathcal{G}^* if the nodes in A and the nodes in \mathcal{B} are disconnected in what remains of $\mathcal{G}^*_{An(\mathcal{A},\mathcal{B},\mathcal{C})}$ after removing all edges outgoing from nodes in C.

Definition 15 (fd-separation [29]): Let \mathcal{G}^* be a network FDG. A set \mathcal{C} fd-separates \mathcal{A} and \mathcal{B} in \mathcal{G}^* if the nodes in \mathcal{A} and the nodes in \mathcal{B} are disconnected in what remains of $\mathcal{G}^*_{An(\mathcal{A},\mathcal{B},\mathcal{C})}$ after removing all edges outgoing from nodes in \mathcal{C} and subsequently, recursively removing all edges that have no source nodes as ancestors.

Now we show that *fd*-separation is valid for polymatroidal variables represented by the subgraph $\overline{\mathcal{G}}^*$ of network FDG (Definition 11). First, note that the subgraph $\overline{\mathcal{G}}^*$ of network FDG is a functional dependence graph in the sense of [29] (with random variables replaced by polymatroidal variables) since the vertices in $\overline{\mathcal{G}}^*$ represent source and edge variables, the edges in $\overline{\mathcal{G}}^*$ represent functional dependencies between the variables and the vertices representing the source variables have no incoming edges.

Lemma 4: If the subset of nodes C *fd*-separates A and B in the subgraph $\overline{\mathcal{G}}^*$ of a network FDG \mathcal{G} for h, then $I_h(A; \mathcal{B} \mid C) = 0$.

Proof: By Definition 15, $\mu_A(\mathcal{C})$ (see Definition 9) *d*-separates \mathcal{A} and \mathcal{B} in $\overline{\mathcal{G}}^*$. But, *d*-separation is implied by the *semi-graphoid* axioms (see [31, Chapter 3]) which are also satisfied by polymatroidal variables. Hence, if $\mu_A(\mathcal{C})$ *d*-separates \mathcal{A} and \mathcal{B} in $\overline{\mathcal{G}}^*$ then $I_h(\mathcal{A}; \mathcal{B} \mid \mu_A(\mathcal{C})) = 0$. By Lemma 2, $h(\mu_A(\mathcal{C})) = h(\mathcal{C})$ and hence

$$h\left(\mathcal{A}\mu_{A}(\mathcal{C})\right) + h\left(\mathcal{B}\mu_{A}(\mathcal{C})\right) - h\left(\mu_{A}(\mathcal{C})\right) - h\left(\mathcal{A}\mathcal{B}\mu_{A}(\mathcal{C})\right) = 0$$

implies

$$I_{h}(\mathcal{A};\mathcal{B} \mid \mathcal{C}) = h(\mathcal{AC}) + h(\mathcal{BC}) - h(\mathcal{C}) - h(\mathcal{ABC}) = 0.$$

In the following, we will give a tighter graphical bound for networks when sources are independent. We will follow a similar approach used to derive Theorem 1 by finding maximal irreducible sets induced by *fd*-separation in subgraph $\overline{\mathcal{G}}^*$ of network FDG.

Definition 16 (Procedure C): Consider a network FDG as defined in Definition 11. For any sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, we say \mathcal{A} determines \mathcal{B} (with respect to Procedure C) if there are no elements of \mathcal{B} remaining after the following procedure:

Procedure C:

- Remove all edges outgoing from A and subsequently recursively remove all nodes and edges with no incoming edges and nodes respectively and all nodes and edges with no source nodes as ancestors. Call the resulting graph \$\tilde{G}^*\$.
 If there exists any \$Y_s\$ disconnected from any \$\tilde{Y}_s^i\$ in \$\tilde{G}^*_{An(Y_s, \tilde{Y}_s^i, A)}\$ then from \$\tilde{G}^*\$ (a) remove
- 2) If there exists any Y_s disconnected from any Y_s^i in $\mathcal{G}^*_{An(Y_s, \hat{Y}_s^i, \mathcal{A})}$ then from \mathcal{G}^* (a) remove \hat{Y}_s^i for all $i \in \{1, ..., |b(s)|\}$ and (b) subsequently recursively remove all nodes and edges with no incoming edges and nodes respectively. Call the resulting graph $\tilde{\mathcal{G}}^*$, go to Step 2. Else terminate.

We will use $\mathcal{A} \longrightarrow_C \mathcal{B}$ to denote that \mathcal{A} determines \mathcal{B} with respect to Procedure C.

Note that Step 2 of Definition 16 uses *fd*-separation. The concepts for blanket, irreducibility are similarly defined with respect to Procedure C. Specifically, for a given set \mathcal{A} , its blanket (with respect to Procedure C) is denoted by $\mu_C(\mathcal{A})$ and is defined as the largest set of nodes with $\mathcal{A} \longrightarrow_C \mu_C(\mathcal{A})$. A set of nodes \mathcal{B} is called *irreducible* (with respect to Procedure C) if there is no $\mathcal{A} \subseteq \mathcal{B}$ such that $\mathcal{A} \longrightarrow_C \mathcal{B}$. An irreducible set \mathcal{A} is *maximal* if $\mu_C(\mathcal{A}) = \mathcal{X}$. In addition, if \mathcal{A} and \mathcal{B} are maximal irreducible sets, then

$$h(\mathcal{A}) = h(\mathcal{B}).$$

Furthermore, the recursion described earlier in Algorithm 1 can now be used to find maximal irreducible sets for multi-source multi-sink networks with independent sources, replacing $\mu_A(\cdot)$ by $\mu_C(\cdot)$.

Corollary 3: For any given network FDG $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$,

$$\mu_B(\mathcal{A}) \subseteq \mu_C(\mathcal{A}), \forall \mathcal{A} \subseteq \mathcal{X}.$$

We remark that, there may exist some $\mathcal{A} \subseteq \mathcal{V}$ in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ such that $\mu_B(\mathcal{A}) \subsetneq \mu_C(\mathcal{A})$ (refer to Example 4 in which $\mu_B(\{4,5\}) \subsetneq \mu_C(\{4,5\}) = \mathcal{V}$).

Lemma 5: If $\mathcal{A}, \mathcal{B} \subseteq \{Y_s, s \in \mathcal{S}, U_e, e \in \mathcal{E}\}$ and $\mathcal{A} \longrightarrow_C \mathcal{B}$ in a network FDG \mathcal{G}^* , then $h(\mathcal{B} \mid \mathcal{A}) = 0$. Proof: Suppose $\mathcal{A} \longrightarrow_C \mathcal{B}$. Let \mathcal{U} be the set of variables removed by Step 1 in Definition 16. Then by Lemma 2, $h(\mathcal{U} \mid \mathcal{A}) = 0$. Now, let \mathcal{Y} be the set of all nodes representing the estimates $Y_s^i, i \in b(s), s \in \mathcal{W} \subseteq \mathcal{S}$ removed by Step 2(a) in Definition 16. Then, by the definition of *fd*-separation in the subgraph $\overline{\mathcal{G}}^*$ and Lemma 4, $I_h(Y_s; \hat{Y}_s^i \mid \mathcal{A}) = 0, s \in \mathcal{W}, i \in b(s)$. But the decoding constraints (11) imply $\hat{Y}_s^i = Y_s$ then for $s \in \mathcal{W}$,

$$h(Y_s \mid \mathcal{A}) = h(Y_s^i \mid \mathcal{A}) = 0 \Rightarrow h(\mathcal{W} \mid \mathcal{A}) = 0$$

Let \mathcal{Z} be the set of all variables removed by Step 2(b) in Definition 16. Then by Lemma 2,

$$h(\mathcal{Z} \mid \mathcal{A}) = 0$$

Since $\mu_C(\mathcal{A}) = \mathcal{U} \cup \mathcal{Y} \cup \mathcal{W} \cup \mathcal{Z}$,

$$\mathcal{A} \longrightarrow_C \mathcal{B} \Rightarrow \mathcal{B} \subseteq \mathcal{U} \cup \mathcal{W} \cup \mathcal{Z}$$

and hence

$$h(\mathcal{UWZ} \mid \mathcal{A}) = 0 \Rightarrow h(\mathcal{B} \mid \mathcal{A}) = 0.$$

Example 4 (Butterfly Network, Independent Sources): Figure 3 shows the subgraph $\overline{\mathcal{G}}$ for network FDG (Figure 2(b)) of the butterfly network (Figure 2(a)). The independent source maximal irreducible sets are

$$\begin{split} \{1,2\}, \{1,5\}, \{1,7\}, \{1,8\}, \{2,4\}, \{2,7\}, \{2,9\}, \{3,7\}, \\ \{4,5\}, \{4,7\}, \{4,8\}, \{5,7\}, \{5,9\}, \{6,7\}, \{3,8,9\}, \{6,8,9\}. \end{split}$$

The sets {4,5}, {4,7}, {4,8}, {5,7}, {5,9} are new maximal irreducible sets found by replacing $\mu_A(\cdot)$ by $\mu_C(\cdot)$ in Algorithm 1. Source independence is an essential ingredient to find these new maximal irreducible sets. However independence is not necessary to find the other maximal irreducible sets. Also note that the maximal irreducible sets {3,4,5}, {4,5,6}, {3,4,8}, {5,6,9} previously found by Algorithm 1 with $\mu_A(\cdot)$ are further reduced to {4,5}, {4,8}, {5,9} using source independence via $\mu_C(\cdot)$.



Fig. 3. Subgraph $\overline{\mathcal{G}}^*$ of network FDG for the butterfly network.

It may be of theoretical interest to know which functional dependencies are implied by local encoding/decoding functions and which involve source independence. Algorithm 1 with $\mu_B(\cdot)$ and $\mu_C(\cdot)$ can be used to answer this question. Our main result for independent sources is as follows (the proof is similar to Theorem 1).

Theorem 3 (Functional dependence bound, independence contraints): Consider a network coding problem with independent sources and its induced network FDG $(\mathcal{X}, \mathcal{E}^*)$. If $\mathcal{B} = \{U_{\mathcal{A}}, Y_{\mathcal{W}^c}\}$ is a maximal irreducible set (with respect to Procedure C) in $(\mathcal{X}, \mathcal{E}^*)$ and $(c_e, e \in \mathcal{E})$ is achievable, then

$$\sum_{e \in \mathcal{A}} c_e \ge \sum_{s \in \mathcal{W}} H\left(Y_s\right).$$

Let \mathcal{M}_C be the set of all maximal irreducible set $\{U_A, Y_{\mathcal{W}^c}\}$ with respect to Procedure C and let

$$\triangleq \bigcap_{\{U_{\mathcal{A}}, Y_{\mathcal{W}^c}\}\in\mathcal{M}_C} \left\{ (c_e, e\in\mathcal{E}) : \sum_{e\in\mathcal{A}} c_e \ge \sum_{s\in\mathcal{W}} H\left(Y_s\right) \right\}.$$
(33)

Corollary 4: When sources are independent,

$$\mathcal{R}_{FD}^{\perp} \subseteq \mathcal{R}_{FD} \tag{34}$$

and there exists a network for which the inclusion is strict.

 $\mathcal{R}_{FD}^{\perp} \subseteq \mathcal{R}_{FD}$ follows from Corollary 3 and Theorems 1 and 3. Strict inclusion is demonstrated in Example 7 in Section IV.

IV. COMPARISON

We now compare our bounds \mathcal{R}_{FD} and \mathcal{R}_{FD}^{\perp} with some known bounds. It should be noted that a comparison of all these known bounds does not seem to have been previously performed in the literature. This is in part due to the different forms of the bounds. In contrast, our unifying framework enables us to complete this comparison. In addition to establishing the comparative strength of the bounds, the comparison may provide insight into the essential technical ingredients for characterization of the bounds and hence helps answer why one bound is better than (or similar to) another. For comparison purpose we assume that a(s) are singletons for all $s \in S$.

A. Cut-Set Bound

The cut-set bound [16, Theorem 15.10.1] is an outer bound on the capacity region of general multiterminal communication networks. For a subset of sessions $\mathcal{W} \subseteq \mathcal{S}$, let $\mathcal{T}_{\mathcal{W}} = \{\mathcal{T}_{\mathcal{W}} \subseteq \mathcal{V} : a(s) \in \mathcal{T}, b(s) \cap \mathcal{T}^c \neq \emptyset, \forall s \in \mathcal{W}\}$ be the collection of all subsets of nodes $\mathcal{T}_{\mathcal{W}}$ such that these source sessions are available to nodes in $\mathcal{T}_{\mathcal{W}}$, and at least one node in the complement $\mathcal{T}_{\mathcal{W}}^c$ demands each session. Further define $\mathcal{E}(\mathcal{T}_{\mathcal{W}}) = \{e \in \mathcal{E} : tail(e) \in \mathcal{T}_{\mathcal{W}}, head(e) \in \mathcal{T}_{\mathcal{W}}^c\}$ as the cutset of edges separating $\mathcal{T}_{\mathcal{W}}$ and $\mathcal{T}_{\mathcal{W}}^c$. For our case of interest, networks consist of error free point-to-point links, and the cut-set bound reduces to the following simple upper bound [30], which is identical to the max-flow bound of [2] (see also [21]).

Theorem 4: For a network of error free point-to-point channels, if $c_e, e \in \mathcal{E}$ is achievable, then

$$\sum_{s \in \mathcal{W}} H(Y_s) \le \sum_{e \in \mathcal{E}(\mathcal{T}_{\mathcal{W}})} c_e.$$
(35)

Define the corresponding outer bound region,

$$\triangleq \bigcap_{\mathcal{W}\subseteq\mathcal{S},\mathcal{E}(\mathcal{T}_{\mathcal{W}})} \left\{ (c_e : e \in \mathcal{E}) : \sum_{s \in \mathcal{W}} H(Y_s) \le \sum_{e \in \mathcal{E}(\mathcal{T}_{\mathcal{W}})} c_e \right\}.$$
(36)

Now, we compare the cut-set bound with functional dependence bound (Theorem 1). In the proof of the cut-set bound [16, Theorem 15.10.1] the decoding constraints are only loosely enforced. The source messages $y_s : s \in W \subseteq S$ transmitted from nodes in \mathcal{T}_W to nodes in \mathcal{T}_W^c can be decoded from symbols received at nodes in \mathcal{T}_W^c and other source messages $y_s : s \in W^c$, i.e., $Y_W = f(\{U_e : head(e) \in \mathcal{T}_W, tail(e) \in \mathcal{T}_W^c\}, Y_{W^c})$. This is a kind of joint decoding, potentially with extra side information, and hence does not enforce the decoding constraints independently at each sink (this will be clear from Example 5). To simplify notations we consider unicast network.

Theorem 5: $\mathcal{R}_{FD} \subseteq \mathcal{R}_{CS}$ and the inclusion can be strict.

Proof: For $Y_{\mathcal{W}}$ available at some nodes in $\mathcal{T}_{\mathcal{W}} \subseteq \mathcal{V}$, let $\mathcal{A} = \{e : \text{head}(e) \in \mathcal{T}_{\mathcal{W}}, \text{tail}(e) \in \mathcal{T}_{\mathcal{W}}^c\}$ be any cut-set defining \mathcal{R}_{CS} . Then to prove $\mathcal{R}_{FD} \subseteq \mathcal{R}_{CS}$ it is sufficient to prove that in the network FDG $Y_{\mathcal{W}} \subseteq \mu_B(\mathcal{A}, Y_{\mathcal{W}^c})$. Consider paths from $Y_s, s \in \mathcal{S}$ to \hat{Y}_s . Note that in the network FDG of the given network, every path from nodes (representing sources of) $Y_s : s \in \mathcal{W}$ to nodes (representing sinks) $\hat{Y}_s : s \in \mathcal{W}$ in $\mathcal{T}_{\mathcal{W}}^c$ intersects some nodes (representing the edges) in \mathcal{A} . Then by Definitions 8 and 9,

$$Y_{\mathcal{W}} \subseteq \mu_B(\mathcal{A}, Y_{\mathcal{W}^c}). \tag{37}$$

This is because other paths to sink nodes $\hat{Y}_s : s \in \mathcal{W}$ in $\mathcal{T}^c_{\mathcal{W}}$ can only be from nodes $Y_s : s \in \mathcal{W}^c$. Hence there are no other paths from $Y_s, s \in \mathcal{S}$ to sink nodes of $Y_s : s \in \mathcal{W}$ in $\mathcal{T}^c_{\mathcal{W}}$ except those intersecting \mathcal{A} and those containing nodes in $Y_s : s \in \mathcal{W}^c$. By (37) and Theorem 1, $\mathcal{R}_{FD} \subseteq \mathcal{R}_{CS}$. Example 5 below shows $\mathcal{R}_{FD} \subsetneq \mathcal{R}_{CS}$ for the butterfly network.

Example 5: For the butterfly network of Figure 2(a), the functional dependence bound of Theorem 1, is strictly tighter than the cut-set bound, Theorem 4. More specifically, the cut-set bound is

$$H(Y_1) \leq \{c_2, c_5, c_7\}$$

$$H(Y_2) \leq \{c_3, c_5, c_6\}$$

$$H(Y_1) + H(Y_2) \leq \{c_1 + c_4 + c_5, c_i + c_j :$$

$$i \in \{2, 5, 7\}, j \in \{3, 5, 6\}\}$$

On the other hand, (38) is tighter using the functional dependence bound (via the maximal irreducible sets $\{3,7\}$ and $\{6,7\}$ corresponding to the sets of variables $\{U_1, U_5\}$ and $\{U_4, U_5\}$ respectively, see Examples 2 and 3).

$$H(Y_1) + H(Y_2) \le \{c_1 + c_5, c_4 + c_5\}$$
(38)

B. Network Sharing Bound

The network sharing bound [17, Theorem 1] is defined for a special type of multiple unicast (each session is demanded at only one sink) networks called |S|-pairs three-layer networks [17] where S is the set of source sessions. Three-layer networks are a network extension of the distributed source coding model of [32]. Each channel $e \in \mathcal{E}$ of finite capacity c_e has direct access to certain source sessions and each sink has access to certain channels.

Definition 17: A three-layer network is a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ characterized by a tuple $(\mathcal{E}', \alpha, \beta)$ such that |a(s)| = |b(s)| = 1 for all $s \in \mathcal{S}$. Here,

- 1) \mathcal{E}' is the set of edges in the middle layer such that $tail(e) \neq head(f)$ for all distinct $e, f \in \mathcal{E}'$. In other words, all the middle layer edges are not directly connected.
- 2) source connection $\alpha : \mathcal{E}' \mapsto 2^{\mathcal{S}}$ specifies the first layer edges, which have the form (a(s), tail(e)) for $s \in \alpha(e)$.
- 3) sink connection $\beta : \mathcal{E}' \mapsto 2^{\mathcal{S}}$ specifies the third layer edges, which have the form (head(e), b(s)) where $s \in \beta(e)$.

To define the network sharing bound, we assume without loss of generality that S is a strict totally ordered set (with the binary order relation \prec). We say $s_{i(\prec)} = k$ if, given the total order \prec , s_i is the kth element in the set with respect to \prec .

Definition 18: For a given three-layer network (see Definition 17), a network sharing edge-set \mathcal{F} with respect to \prec on a subset of sources $\mathcal{W} \subseteq \mathcal{S}$ is the set

$$\mathcal{F}(\mathcal{W}, \prec) \triangleq \{ e : \beta(e) \cap \mathcal{W} \neq \emptyset, \alpha(e) \nsubseteq \mathcal{W}[\beta(e)] \}$$

where $\mathcal{W}[\beta(e)] \triangleq \{s_i \in \mathcal{W} : s_i \prec s_j, s_j \in \beta(e)\}$. In other words, $\mathcal{F}(\mathcal{W}, \prec)$ may be viewed as the set of edges e such that there exists $s \in \alpha(e)$ and $s' \in \beta(e)$ satisfying $s' \prec s$.

Example 6 (2-pairs three-layer butterfly network): Figure 4 shows an example of a three-layer network, where $S = \{s_1, s_2\}$ the sources are located at nodes $a(s_1) = 1$, $a(s_2) = 2$ and demanded at nodes $b(s_1) = 9$, $b(s_2) = 10$. The source and sink connections (shown with dashed edges) are $\alpha(e_1) = \{s_1\}$, $\beta(e_1) = \{s_2\}$, $\alpha(e_2) = \{s_1, s_2\}$, $\beta(e_2) = \{s_1, s_2\}$, $\alpha(e_3) = \{s_2\}$ and $\beta(e_3) = \{s_1\}$.



Fig. 4. 2-pairs three-layer butterfly network.

Theorem 6 (Theorem 1, [17]): Consider a three-layer unicast network $(\mathcal{V}, \mathcal{E})$. If the edge capacity tuple $(c_e, e \in \mathcal{E})$ is achievable, then

$$\sum_{s \in \mathcal{W}} H(Y_s) \le \sum_{e \in \mathcal{F}(\mathcal{W}, \prec)} c_e.$$
(39)

Similar to (36), define the *network sharing region* as the subset of $\mathbb{R}^{|\mathcal{E}|}_+$ such that (39) holds.

$$\mathcal{R}_{NS} \triangleq \bigcap_{\mathcal{W} \subseteq \mathcal{S}, \text{ and total order } \prec} \left\{ (c_e : e \in \mathcal{E}) : (39) \text{ holds} \right\}.$$
(40)

Now, we show that the functional dependence bound, Theorem 1, and the network sharing bound [17], Theorem 6, are identical (restricting attention to three-layer networks). This also proves that Theorem 3 (for independent sources) is better than the network sharing bound.

Theorem 7: For a three-layer network,

$$\mathcal{R}_{NS} = \mathcal{R}_{FD}$$

The proof of the theorem follows from Lemmas 6 and 7 below. Also note that the proof of the network sharing bound uses subadditivity of entropies similar to Theorem 1. For simplicity and clarity, we prove Lemmas 6 and 7 for W = S. With similar proof methods, the following statements can be proved.

- For any W ⊆ S and some order ≺, let U_{F(W,≺)} be a set of variables flowing through network sharing edge-set F(W, ≺) then, ∃B ∈ M : B ⊆ F(W, ≺) ∪ W^c where M is the collection of all maximal irreducible sets of the three layer network.
- 2) For every maximal irreducible set of a given three-layer network there exists an equivalent set $\mathcal{F}(\mathcal{W}, \prec) \cup \mathcal{W}^c$ obtained by some ordering (relation) \prec .

Lemma 6: In a three-layer network, every network sharing edge-set contains a maximal irreducible set not containing any source variables. That is, for any network sharing edge-set $\mathcal{F}(\mathcal{S}, \prec)$,

$$\exists \mathcal{B} \in \mathcal{M} : \mathcal{B} \subseteq U_{\mathcal{F}(\mathcal{S},\prec)}$$

where \mathcal{M} is the collection of all maximal irreducible sets in the FDG of the three-layer network.

Proof: Let $U_{\mathcal{F}(\mathcal{S},\prec)}$ be the set of network sharing edge-set variables obtained via order \prec . Then by the definition of the network sharing bound, Theorem 6,

$$\{e: s_i \in \beta(e), s_{i(\prec)} = 1\} \subseteq \mathcal{F}(\mathcal{S}, \prec)$$

and so

$$Y_{\{s_i:s_{i(\prec)}=1\}} \in \mu_A(U_{\mathcal{F}(\mathcal{S},\prec)}).^3$$

Also note that

$$\{e: s_j \in \beta(e), s_{j(\prec)} = 2\}$$

and so

$$Y_{\{s_j:s_{j(\prec)}=2\}} \in \mu_A(U_{\mathcal{F}(\mathcal{S},\prec)}, Y_{\{s_i:s_{i(\prec)}=1\}}) = \mu_A(U_{\mathcal{F}(\mathcal{S},\prec)}).$$

In general,

$$\{e: s_j \in \beta(e)\} \subseteq \mathcal{F}(\mathcal{S}, \prec) \cup \{e: s_i \in \alpha(e), s_{i(\prec)} < s_{j(\prec)}\}$$

implies

$$Y_{s_j} \in \mu_A(U_{\mathcal{F}}(\mathcal{S}, \prec), Y_{\{s_i:s_{i(\prec)} < s_{j(\prec)}\}}) = \mu_A(U_{\mathcal{F}(\mathcal{S}, \prec)})$$

Therefore, for any network sharing edge-set under some order of source nodes induced by the relation \prec ,

$$\mu_A(U_{\mathcal{F}(\mathcal{S},\prec)}) = U_{\mathcal{E}} \cup Y_{\mathcal{S}}$$

and hence there exists $\mathcal{B} \subseteq U_{\mathcal{F}(\mathcal{S},\prec)}$.

We remark that there could exist maximal irreducible sets which are proper subsets of network sharing edge-sets. On the other hand, the following lemma proves that, for a given maximal irreducible set, we can always find an equivalent network sharing edge-set. The lemma also describes the ordering induced by \prec for which $\mathcal{B} = U_{\mathcal{F}(S,\prec)}$.

Lemma 7: For any maximal irreducible set $U_{\mathcal{F}(S,\prec)}$ not containing any source variables in the FDG of a given three-layer network, there exists the network sharing edge-set $\mathcal{F}(S,\prec)$ obtained via a reordering of the source nodes.

Proof: Let $\mathcal{B} = U_{\mathcal{E}_1}, \mathcal{E}_1 \subseteq \mathcal{E}$ be a maximal irreducible set (not containing any source nodes). By Definition 10, $\mu_A(U_{\mathcal{E}_1}) = U_{\mathcal{E}} \cup Y_{\mathcal{S}}$. Now, (recalling π to be the set of parents (21)) let $\mathcal{S}_1 \triangleq \{s_i : \pi(Y_{s_i}) \subseteq U_{\mathcal{E}_1}\} \subseteq \mathcal{S}$ be a set of sources which are immediate children of nodes in $U_{\mathcal{E}_1}$ and are not children of any other nodes. Also define $\mathcal{E}_2 \triangleq \{e : \pi(U_e) \subseteq Y_{\mathcal{S}_1}, e \notin \mathcal{E}_1\}$. Recursively define the following sets

$$\mathcal{S}_{i} \triangleq \left\{ s : \pi(Y_{s}) \subseteq \bigcup_{j \in \{1, \dots, i\}} U_{\mathcal{E}_{j}}, s \notin \bigcup_{j \in \{1, \dots, i-1\}} \mathcal{S}_{j} \right\}$$
$$\mathcal{E}_{i} \triangleq \left\{ e : \pi(U_{e}) \subseteq \bigcup_{j \in \{1, \dots, i-1\}} Y_{\mathcal{S}_{j}}, e \notin \bigcup_{j \in \{1, \dots, i-1\}} \mathcal{E}_{j} \right\}.$$

³We consider $\mu_A(\cdot)$ since the network is unicast which yields cyclic FDG described in Section III-A.

 $\subseteq \mathcal{F}(\mathcal{S}, \prec) \cup \{ U_e : s_i \in \alpha(e), s_{i(\prec)} = 1 \}$

Note that the nodes in Y_{S_i} have incoming edges only from edges in $U_{\mathcal{E}_j}, j \leq i$ and nodes in $U_{\mathcal{E}_i}$ have incoming edges only from nodes in $Y_{S_j}, j \leq i-1$. Also, for any $i \neq j$, Y_{S_i} and Y_{S_j} , and $U_{\mathcal{B}_i}$ and $U_{\mathcal{B}_j}$ are disjoint. In the three-layer network,

$$\mathcal{E}_{i} = \left\{ e : \alpha(e) \subseteq \{ \mathcal{S}_{j} : j \leq i-1 \} = \mathcal{S}[\beta(e)], \\ e \notin \bigcup_{j \in \{1, \dots, i-1\}} \mathcal{E}_{j} \right\}$$

where i > 1. But, by definition of the network sharing bound, edges $e \in \mathcal{E}_i, i > 1$ will not be included in the network sharing edge-set for any order relation \prec such that $\{s_{i(\prec)} : s_i \in \mathcal{S}_1\} < ... < \{s_{i(\prec)} : s_i \in \mathcal{S}_m\}, \max_{s_i \in \mathcal{S}_m} s_{i(\prec)} = |\mathcal{S}|$ (ordering of sessions within each \mathcal{S}_i is irrelevant). Hence there exists a network sharing edge-set $\mathcal{F}(\mathcal{S}, \prec) \subseteq \mathcal{E}_1$ such that $U_{\mathcal{F}(\mathcal{S}, \prec)} \subseteq U_{\mathcal{E}_1}, U_{\mathcal{E}_1} \in \mathcal{M}$.

Remark 8: Although the network sharing bound turns out to be the same as the functional dependence bound for in a three-layer network, the functional dependence bound is not a simple extension of the network sharing bound for more general networks. In fact, the functional dependence bound uses a completely different approach for characterizing bottlenecks such that, given the network coding and decoding constraints, variables flowing in a bottleneck determine all other variables. Also, the network sharing bound is computationally more complex compared to the functional dependence bound in the sense that all possible orderings of the sources need to be considered to find a network sharing edge-set.

Corollary 5: For a three-layer network

$$\mathcal{R}_{FD}^{\perp} \subseteq \mathcal{R}_{NS} \tag{41}$$

where \mathcal{R}_{FD}^{\perp} is the bound (33) for independent sources and \mathcal{R}_{NS} the network sharing region (40). There exists a network for which $\mathcal{R}_{FD}^{\perp} \subsetneq \mathcal{R}_{NS}$.

Proof: A direct consequence of Corollary 4 and Theorem 7.

An important implication of Theorem 7 together with Theorem 1 is that (1) the network sharing bound can be applied to three-layer networks with correlated sources and (2) the source independence constraint is not exploited to characterize network sharing edge-sets.

C. Information Dominance and a New Bound

The notion of information dominance and its graphical characterization was introduced in [18]. *Definition 19:* Given $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, an edge set $\mathcal{A} \subseteq \mathcal{E}$ informationally dominates $\mathcal{B} \subseteq \mathcal{E}$ if for all network codes $\tilde{\phi}$ (8) and $|\mathcal{S}|$ -tuples of messages $\mathbf{x} = (x_1, ..., x_{|\mathcal{S}|})$ and $\mathbf{y} = (y_1, ..., y_{|\mathcal{S}|})$,

$$\tilde{\phi}_{\mathcal{A}}(\mathbf{x}) = \tilde{\phi}_{\mathcal{A}}(\mathbf{y}) \implies \tilde{\phi}_{\mathcal{B}}(\mathbf{x}) = \tilde{\phi}_{\mathcal{B}}(\mathbf{y}).$$

Also define

$$Dom(\mathcal{A}) \triangleq \{e : \mathcal{A} \text{ informationally dominates } e\}.$$
 (42)

Definition 20 ($\mathcal{G}(\text{Dom}(\mathcal{A}), s)$): Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, an edge set $\mathcal{A} \subseteq \mathcal{E}$ and a source session $s \in \mathcal{S}, \mathcal{G}(\text{Dom}(\mathcal{A}), s)$ is the graph obtained by the following manipulation:

• remove edges and nodes that do not have a path to Y_s in \mathcal{G} ,

• remove all edges in \mathcal{A} ,

• remove edges and nodes that are not reachable from a source edge in the remaining graph.

The conditions of the theorem below characterize Dom(A).

Theorem 8 ([18], Theorem 10): For an edge set $\mathcal{A} \subseteq \mathcal{E}$, the set $\text{Dom}(\mathcal{A})$ satisfies the following conditions.

$$\mathcal{A} \subseteq \text{Dom}(\mathcal{A}) \tag{D1}$$

$$Y_s \in \text{Dom}(\mathcal{A}) \iff \hat{Y}_s \in \text{Dom}(\mathcal{A})$$
 (D2)

Every
$$e \in \mathcal{E} \setminus \text{Dom}(\mathcal{A})$$
 is reachable from a source (D3)

 $Y_s \in \mathcal{G}(\text{Dom}(\mathcal{A}), s) \text{ is connected to } \hat{Y}_s, \forall s \in \mathcal{S}$ (D4)

Furthermore, any set \mathcal{B} satisfying these conditions contains $Dom(\mathcal{A})$.

Although the authors give this notion of information dominance in [18], they did not use it to derive an easily computable bound. We now formulate a new bound using information dominance along similar lines to our other bounds and compare this new bound with ours.

According to [18], the well known linear programming bound $\mathcal{R}(\Gamma)$ uses a constraint that can be viewed as a restricted version of information dominance used in the linear programming outer bound defined in [18, Section VIII].

We remark that, for directed acyclic networks, the bound in [18, Section VIII] simply coincides with $\mathcal{R}(\Gamma)$. It uses Γ , C_1 , C_2 , C_3 and C_4 (as used in $\mathcal{R}(\Gamma)$) together with information dominance. However, since information dominance is implied by $C_1 \cap C_2 \cap C_3 \cap C_4 \cap \Gamma$, it does not actually introduce any new constraints. This can be rigorously justified by Corollary 7 proved in this section, since only polymatroid constraints are used, apart from the constraints introduced by network demands, to characterize the set $\mu_C(\cdot)$, and hence $\text{Dom}(\cdot)$ (Theorem 8).

Following our program for developing bounds established in Sections III, we now define maximal information dominating sets and formulate a bound in terms of these sets.

Definition 21 (Maximal Information Dominating Set): For a given network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a set $\mathcal{A} \subseteq \mathcal{E}$ is a maximal information dominating set if $\text{Dom}(\mathcal{A}) = \mathcal{E}$ and no proper subset of \mathcal{A} has the same property.

Lemma 8: For a given network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ the joint entropy of any maximal information dominating set is the same as the joint entropy of all source random variables.

Proof: First note that the set of all source random variables, Y_S , is a maximal dominating set. Let $U_{\mathcal{E}}$ denote set of all edge random variables. Then

$$H(U_{\mathcal{E}}, Y_{\mathcal{S}}) = H(U_{\mathcal{E}} \mid Y_{\mathcal{S}}) + H(Y_{\mathcal{S}}) = H(Y_{\mathcal{S}})$$

Now, let \mathcal{B} be any other maximal information dominating set. Then $H(\{U_{\mathcal{E}}, Y_{\mathcal{S}}\} \setminus \mathcal{B} \mid \mathcal{B}) = 0$ and hence $H(U_{\mathcal{E}}, Y_{\mathcal{S}}) = H(\mathcal{B})$

Theorem 9 (Information Dominance Bound): Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a given network with network coding constraints. Let $\{U_{\mathcal{A}}, Y_{\mathcal{W}^c}\}$ be a maximal information dominating set according to Definition 21. Then

$$\sum_{s \in \mathcal{W}} H(Y_s) \le \sum_{e \in \mathcal{A}} c_e.$$
(43)

Proof: The proof is similar to that in Theorem 1, by invoking Lemma 8 and submodularity. Let \mathcal{I} be the set of all maximal information dominating sets. Define the *information dominance region* as follows.

$$\mathcal{R}_{ID} \triangleq \bigcap_{\{U_{\mathcal{A}}, Y_{\mathcal{W}^c}\} \in \mathcal{I}} \left\{ \left(c_e : e \in \mathcal{E} \right) : (43) \text{ holds} \right\}.$$
(44)

In the following we establish that $Dom(\mathcal{A}) \subseteq \mu_C(\mathcal{A})$. This will lead us to the conclusion that $\mathcal{R}_{FD}^{\perp} \subseteq \mathcal{R}_{ID}$. We will proceed by considering each of the conditions (D1) - (D4) in the definition of information dominance, and relating them to μ_C .

Lemma 9: Let $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ be the network FDG of a given network. Then for any $\mathcal{A} \subseteq \mathcal{X}$, $\mu_C(\mathcal{A})$ satisfies Conditions (D1), (D2) of Dom(\mathcal{A}).

Proof: By Definition 16, the node representing the source variable Y_s is in $\mu_C(\mathcal{A})$ if and only if any of nodes $\hat{Y}_s^i, i \in b(s)$ representing decoding constraints (i.e., estimated source variables) is in $\mu_C(\mathcal{A})$. This is equivalent to the Condition (D2) for Dom(\mathcal{A}). Also note that, by definition, $\mathcal{A} \subseteq \mu_C(\mathcal{A})$.

Definition 22: For a given FDG $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ and a set of nodes $\mu_C(\mathcal{A}) \subseteq \mathcal{X}$, the graph $\mathcal{G}^* \setminus \mu_C(\mathcal{A})$ contains nodes \mathcal{X} and edges $\mathcal{E}^* \setminus \{e : \text{head}(e) \in \mu_C(\mathcal{A})\}.$

Condition (D3) for $\text{Dom}(\cdot)$ requires every node $A \in \mathcal{X} \setminus \mu_C(\mathcal{A})$ to have a directed path from a source node in $\mathcal{G}^* \setminus \mu_C(\mathcal{A})$.

Note that in [18], it is explicitly assumed that there exists a path from some source nodes to every edge of a given network. Without this assumption (D3) may not be satisfied. Therefore we impose the

same restriction to ensure that $\mu_C(\mathcal{A})$ satisfies (D3) (this assumption is used in the proof of Lemma 10 below). It is also assumed in [18] that for every session s there is a path from node a(s) to b(s) in a given $|\mathcal{S}|$ -pair communication network.

Lemma 10: Let $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ be a network FDG. Every node in $\mathcal{X} \setminus \mu_C(\mathcal{A})$ has a directed path from a source node in $\mathcal{G}^* \setminus \mu_C(\mathcal{A})$.

Proof: By assumption (on the network model [18]), every node in the network FDG \mathcal{G}^* has a directed path from some source node. Now we prove that the statement of the lemma is true by contradiction. Assume that there exists a node $A \in \mathcal{X} \setminus \mu_C(\mathcal{A})$ in $\mathcal{G}^* \setminus \mu_C(\mathcal{A})$ which has no directed path from any source node. Then it follows that every path from any source node to the node A in \mathcal{G}^* intersects at least one node from $\mu_C(\mathcal{A})$. Then, $A \in \mu_C(\mathcal{A})$ and hence there cannot exist such a node $A \in \mathcal{G}^* \setminus \mu_C(\mathcal{A})$. \blacksquare *Corollary 6:* Let $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ be a network FDG. Then for any set $\mathcal{A} \in \mathcal{X}, \mu_C(\mathcal{A})$ satisfies Condition

(D3) of $Dom(\mathcal{A})$.

So far we have shown that Conditions (D1) - (D3) of $Dom(\cdot)$ are satisfied by $\mu_C(\cdot)$. Now we show that the Condition (D4) is equivalent to *fd*-separation in network FDG.

Lemma 11: For a given network \mathcal{G} , Y_s is connected to \hat{Y}_s in $\mathcal{G}(\text{Dom}(\mathcal{A}), s)$ if and only if \mathcal{A} does not *fd*-separate Y_s and \hat{Y}_s in the network FDG, i.e., Condition (D4) of $\text{Dom}(\cdot)$ and *fd*-separation are the same.

Proof: By Definition 20, $\mathcal{G}(\text{Dom}(\mathcal{A}), s)$ is a subgraph of the network obtained by 1) considering the ancestral part of Y_s , \hat{Y}_s and then 2) removing edges in \mathcal{A} and subsequently removing all edges which have no path from any source. Now, if the edge representing Y_s incoming to the node a(s) is connected to an edge representing \hat{Y}_s outgoing from any node in b(s) in $\mathcal{G}(\text{Dom}(\mathcal{A}), s)$ then in the network FDG, \mathcal{A} does not *fd*-separate Y_s and \hat{Y}_s . Also, in network FDG, if \mathcal{A} does not *fd*-separate Y_s and \hat{Y}_s in $\mathcal{G}(\text{Dom}(\mathcal{A}), s)$.

This leads us to the following conclusions. By Lemmas 9 and Corollary 6, Conditions (D1) - (D3) are satisfied by our notion of functional dependence in Definition 16. By Lemma 11, Condition (D4) is equivalent to *fd*-separation, which is employed in Definition 16 and hence

Corollary 7:

$$\mu_C(\mathcal{A}) \subseteq \text{Dom}(\mathcal{A}). \tag{45}$$

Corollary 8:

$$\mathcal{R}_{FD}^{\perp} \subseteq \mathcal{R}_{ID}.\tag{46}$$

The corollary follows from Corollary 7 and the fact that the information dominance bound (Theorem 9) and the functional dependence bound for independent sources (Theorem 3), apart from characterization of $Dom(\cdot)$ and $\mu_C(\cdot)$, use the same arguments.

D. Progressive d-Separating Edge-Set Bound

In [19] the authors describe a procedure to determine whether a given set of edges bounds the capacity of the given network. The progressive *d*-separating edge-set (P*d*E) bound uses the concept of fd-separation [29]. The results are given for general cyclic multi-source multi-sink networks with noisy channels.

Definition 23 (PdE Procedure): The PdE procedure determines whether a given set of edges \mathcal{A} bounds the capacity of information flow for sources $Y_{\mathcal{W}} \subseteq Y_{\mathcal{S}}$ for some ordering of the elements of \mathcal{W} defined by the relation \prec as follows.

In the functional dependence graph⁴ of the given network, remove all vertices and edges in G
except those encountered when moving backward one or more edges starting from any of the
vertices representing A, {Y_{si(≺)} : s_i ∈ W} and {Ŷ_{si(≺)} : s_i ∈ W}. Further remove edges coming out
of vertices representing A and Y_{W^c} and successively remove edges coming out of vertices and on
cycles that have no incoming edges, excepting source vertices. Set i = 1.

⁴The definition of a functional dependence graph used here is different from that defined in Section III, see [19].

- 2) (Iterations) If $Y_{s_{i(\prec)}}$ is not disconnected (in an undirected sense) from all of its estimates $\hat{Y}_{s_{i(\prec)}}$, then STOP (one has no bound). Else if $Y_{s_{i(\prec)}}$ is disconnected (in an undirected sense) from one of its estimates then: (a) remove the edges coming out of the vertex representing $Y_{s_{i(\prec)}}$. (b) Successively remove edges coming out of vertices and edges coming out of vertices that have no paths from source vertices.
- 3) (Termination and Bound) Increment *i*. If $i \leq |\mathcal{W}|$ go to Step 2. If $i = |\mathcal{W}| + 1$

$$\sum_{s \in \mathcal{W}} H(Y_s) \le \sum_{e \in \mathcal{A}(\mathcal{W}, \prec)} c_e.$$
(47)

where $\mathcal{A}(\mathcal{W}, \prec)$ is referred as a PdE set.

Theorem 10: The progressive d-separating edge-set bound is

$$\sum_{s \in \mathcal{W}} H(Y_s) \le \sum_{e \in \mathcal{A}(\mathcal{W}, \prec)} c_e \tag{48}$$

where $\mathcal{A}(\mathcal{W}, \prec)$ is the collection of subsets of \mathcal{E} that are PdE sets (Definition 23) for \mathcal{W} under the ordering relation \prec .

The progressive d-separating edge-set region is

$$\mathcal{R}_{PdE} \triangleq \bigcap_{\mathcal{W} \subseteq \mathcal{S}, \prec} \left\{ (c_e : e \in \mathcal{E}) : (48) \text{ holds} \right\}.$$
(49)

From the definitions of the network sharing bound and the PdE bound it can be noted that both bounds depend on a choice of source ordering and to compute the tightest bounds all possible orderings have to be considered. Also note that determination of the tightest PdE sets involves exhaustively searching over all subsets of edges for a given source ordering. In contrast, we will use structural properties of functional dependence to efficiently compute all network bottlenecks, namely the maximal irreducible sets.

Theorem 11:

$$\mathcal{R}_{FD}^{\perp} \subseteq \mathcal{R}_{PdE}.$$
(50)

Furthermore, there exists a network such that the inclusion is strict.

Proof: Let \mathcal{A} be a progressive *d*-separating edge-set bounding the rate with respect to $Y_{\mathcal{W}}$ for a given network. Then we prove that $\mu_C(\mathcal{A}, Y_{\mathcal{W}^c}) = \mathcal{X}$ in network FDG. The rest follows from Theorem 3.

First note that the Step 1(a) in Definition 23 considers ancestral part of $\{\mathcal{A}, Y_{\mathcal{W}}, \hat{Y}_{\mathcal{W}}\}\)$ and Step 1(b) removes edges outgoing from nodes in $\mathcal{A}, Y_{\mathcal{W}^c}$ and subsequently removes nodes and edges with no incoming edges and nodes respectively (except for source nodes). Denote the resulting graph by \mathcal{G}' . Step 2 checks connectivity of $Y_s : s \in \mathcal{W}$ and $\hat{Y}_s : s \in \mathcal{W}$ in \mathcal{G}' in iterative manner with respect to some \prec .

In contrast, Definition 16 first removes edges outgoing from nodes in $\mathcal{A}, Y_{\mathcal{W}^c}$ and successively removes nodes and edges with no incoming edges and nodes respectively. In the second stage, it checks connectivity of each $Y_s, s \in \mathcal{W}$ with $\hat{Y}_s, S \in \mathcal{W}$ in $\mathcal{G}^*_{An(\mathcal{A},Y_{\mathcal{W}^c})}$ in iterative manner. But note that

$$\mathcal{G}^*_{An(\mathcal{A},Y_{\mathcal{W}^c},Y_s,\hat{Y}^i_s:i\in b(s))} \subseteq \mathcal{G}', s \in \mathcal{W}.$$

Hence, if $Y_s : s \in \mathcal{W}$ and $\hat{Y}_s : s \in \mathcal{W}$ are disconnected in \mathcal{G}' then they are disconnected in $\mathcal{G}_{An(\mathcal{A},Y_{\mathcal{W}^c})}$. Thus if a progressive *d*-separating edge-set \mathcal{A} bounds the rate of the sources $Y_{\mathcal{W}}$ for a given network then $Y_{\mathcal{W}} \subseteq \mu_C(\mathcal{A})$ which implies $\mu_C(\mathcal{A}, Y_{\mathcal{W}^c}) = \mu_C(\mathcal{A}, Y_{\mathcal{W}}, Y_{\mathcal{W}^c}) = \mathcal{X}$ (since $\mu_C(Y_S) = \mathcal{X}$) and hence $\mu_C(\mathcal{A}, Y_{\mathcal{W}^c}) = \mathcal{X}$ in network FDG. Strict inclusion is demonstrated in the following example.

Example 7: Figure 5 shows a three-layer network. Note that source pairs Y_1, Y_2 and Y_3, Y_4 form two butterfly networks. We show that there exists a maximal irreducible set which is strictly smaller than a PdE set for bounding the sum-rate capacity of all sources.



Fig. 5. A network example.

The sum-rate bound, Theorem 3, for any proper subset of the sources is identical to PdE bound, however, the set $\{U_2, U_3, U_4, U_5\}$ is a maximal irreducible set yielding

$$\sum_{s=1}^{5} H(Y_s) \le c_3 + c_3 + c_4 + c_5.$$

Note that, for $s \in \{1, 2, 3, 4, 5\}$, Y_s and \hat{Y}_s are *fd*-separated by $\{U_2, U_3, U_4, U_5\}$ in the subgraph $\bar{\mathcal{G}}^*_{An(Y_s, \hat{Y}_s, U_2, U_3, U_4, U_5)}$ of network FDG. One can also check from Figure 6 that removing $\{U_2, U_3, U_4, U_5\}$ for PdE bound does not disconnect the sources Y_1, Y_2, Y_3, Y_4 from their respective sinks. Also note that all source variables are in Dom(2, 3, 4, 5) and hence the information dominance bound is also tighter than the PdE bound for the network in Figure 5.



Fig. 6. Removing outgoing edges of $\{U_2, U_3, U_4, U_5\}$ in the network of Figure 5.

Close inspection of Definition 23 reveals that fd-separation is weaker in the PdE bound since it does not consider the ancestral part of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ when using the fd-separation criteria to check $\mathcal{A} \perp \mathcal{B} | \mathcal{C}$. The PdE bound can be therefore strengthened by modifying it to consider the ancestral part of $\{Y_k, \hat{Y}_k, \mathcal{A}\}$. The resulting improved PdE bound would be the same as our bound for independent sources, Theorem 3.

V. CONCLUSION

Explicit characterization and computation of the multi-source network coding capacity region requires determination of the set of all entropic vectors Γ^* , which is known to be an extremely hard problem.

The best known outer bound can in principle be computed using a linear programming approach. In practice this is infeasible due to an exponential growth in the number of constraints and variables with the network size. We extended previous notions of functional dependence graphs to accommodate not only cyclic graphs, but more abstract notions of independence. In particular we considered polymatroidal functions, and demonstrated efficient and systematic methods to find functional dependencies implied by the given local dependencies. This led to one of our main results, which was a new, easily computable outer bound, based on characterization of all implied functional dependencies. We showed that the easily computable functional dependence bound is indeed an outer bound on the capacity region of general multicast networks with correlated sources. We extended the notion of irreducible sets for networks with independent sources and formulated a tighter outer bound for such networks. We compared the tightness of our proposed bounds with other existing bounds. We showed that our proposed bounds improve on the cut-set bound, the network sharing bound, a new bound derived from information dominance, and the PdE bound. Finally, we showed how to make a minor modification of the PdE bound, tightening it to coincide with our bound.

APPENDIX

MAXIMAL IRREDUCIBLE SETS FOR ACYCLIC GRAPHS

In a directed acyclic graph, let An(A) denote the set of ancestral nodes, i.e., for every node $a \in An(A)$, there is a directed path from a to some $b \in A$. Of particular interest are the maximal irreducible sets:

Definition 24: An irreducible set \mathcal{A} is *maximal* in an acyclic FDG $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ if $\mathcal{X} \setminus \mu_A(\mathcal{A}) \setminus An(\mathcal{A}) \triangleq (\mathcal{X} \setminus \mu_A(\mathcal{A})) \setminus An(\mathcal{A}) = \emptyset$, and no proper subset of \mathcal{A} has the same property.

Note that for acyclic graphs, every subset of a maximal irreducible set is irreducible. Irreducible sets can be augmented in the following way.

Corollary 9 (Augmentation): Let $\mathcal{A} \subseteq \mathcal{V}$ in an acyclic FDG $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$. Let $\mathcal{B} = \mathcal{X} \setminus \mu_A(\mathcal{A}) \setminus An(\mathcal{A})$. Then $\mathcal{A} \cup \{b\}$ is irreducible for every $b \in \mathcal{B}$.

This suggests a process of recursive augmentation to find all maximal irreducible sets in an acyclic FDG (a similar process of augmentation was used in [33]). Let \mathcal{G}^* be a topologically sorted⁵ acyclic FDG $\mathcal{G}^* = (\{0, 1, 2, ...\}, \mathcal{E}^*)$. Its maximal irreducible sets can be found recursively via **AllMaxSetsA**($\mathcal{G}^*, \{\}$) in Algorithm 2. In fact, **AllMaxSetsA**($\mathcal{G}^*, \mathcal{A}$) finds all maximal irreducible sets containing \mathcal{A} given that the set \mathcal{A} is an irreducible set and \mathcal{G}^* is finite.

Algorithm 2 AllMaxSetsA(\mathcal{G}, \mathcal{A})Require: $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*), \mathcal{A} \subseteq \mathcal{X}$ 1: $\mathcal{B} \leftarrow \mathcal{X} \setminus \mu_A(\mathcal{A}) \setminus An(\mathcal{A})$ 2: if $\mathcal{B} \neq \emptyset$ then3: Output {AllMaxSetsA($\mathcal{G}^*, \mathcal{A} \cup \{b\}) : b \in \mathcal{B}$ }

4: **else**

5: Output \mathcal{A}

6: **end if**

The actual number of calls of the function AllMaxSetsA(\cdot, \cdot) to compute all maximal irreducible sets depends on the topology of the FDG. For example, for a line FDG $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ with $\mathcal{X} = \{i : 1 \le i \le n\}$ and $\mathcal{E}^* = \{(i, i+1) : 1 \le i < n\}$, the number of times the function AllMaxSetsA(\cdot, \cdot) called is only n+1 (linear in the order of \mathcal{G}^*).

For an acyclic FDG \mathcal{G}^* , let \mathcal{S} denote the set of nodes which do not have any parent nodes. Clearly, \mathcal{S} is a maximal irreducible set. Let \mathcal{S} be the set of nodes without a parent node in a given acyclic FDG $\mathcal{G}^* = (\mathcal{X}, \mathcal{E}^*)$ and let \mathcal{A} be another maximal irreducible set then $h(\mathcal{S}) \ge h(\mathcal{A})$ since $\mu_A(\mathcal{S}) = \mathcal{X}$ and hence $h(\mathcal{S}) = h(\mathcal{X}) \ge h(\mathcal{A})$.

⁵Here, we assume that if there is a directed edge from node i to j, then $i \prec j$ [8, Proposition 11.5].

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