Rate Region of the Vector Gaussian CEO Problem with the Trace Distortion Constraint

Yinfei XU, *Student Member, IEEE,* and Qiao WANG, *Member, IEEE*

Abstract—We establish a new extremal inequality, which is further leveraged to give a complete characterization of the rate region of the vector Gaussian CEO problem with the trace distortion constraint. The proof of this extremal inequality hinges on a careful analysis of the Karush-Kuhn-Tucker necessary conditions for the non-convex optimization problem associated with the Berger-Tung scheme, which enables us to integrate the perturbation argument by Wang and Chen with the distortion projection method by Rahman and Wagner.

Index Terms—CEO problem, distributed source coding, extremal inequality, indirect source, lossy source coding, mean square error, rate region, vector Gaussian source.

I. INTRODUCTION

THE CEO problem, which is a special case of multi-terminal source coding, was first investigated by Berger, Zhang and Viswanathan [1]. Oohama [2] determined the asymptotic sum-rate-distortion function of the scalar Gaussian CEO problem via an ingenious application of the entropy power inequality. A complete characterization of the rate region of the scalar Gaussian CEO was obtained in [3] and [4]. However, extending this result to the vector case is not straightforward due to the fact that the entropy power inequality is not necessarily tight in this setting. Tavilder and Viswanath [5] derived a lower bound on the sum rate of the vector Gaussian CEO problem by partially replacing the entropy power inequality with the worst additive noise lemma. An explicit lower bound on the weighted sum rate of the two-terminal vector Gaussian CEO problem can be found in [6]. Of particular relevance here is the work by Wang and Chen [7], [8], where they derived an outer bound on the rate region of the vector Gaussian CEO problem by establishing a certain extremal inequality; essentially the same result was obtained independently by Ekrem and Ulukus via exploiting the relation between Fisher information matrix and MMSE (minimum mean square error) [9]. The extremal inequality in [7], [8] is a variant of the Liu-Viswanath inequality [10], which is in turn inspired by the seminal work of Weingarten, Steinberg and Shamai [11] on the characterization of the capacity region of the MIMO Gaussian broadcast channel.

However, the outer bound induced by the Wang-Chen extremal inequality is in general not tight. Our main result is a strengthened extremal inequality for the special case where the covariance distortion constraint is replaced with the trace distortion constraint. It turns out that this new extremal inequality yields a complete characterization of the rate region

The authors are with School of Information Science and Engineering, Southeast University, Nanjing 210096, China. Email: {yinfeixu, qiaowang}@seu.edu.cn

of the vector Gaussian CEO problem for this special case. The perturbation argument, which is widely used for establishing extremal inequalities, appears to be insufficient for our purpose. For this reason, we develop a spectral decomposition method, which can be effectively incorporated into the perturbation argument to obtain the desired inequality. It is worth mentioning that our spectral decomposition method is partly motivated by the *distortion projection* technique developed by Rahman and Wagner [12], [13] for the vector Gaussian one-help-one problem (see also [14] for a direct proof based on the perturbation method).

The rest of this paper is organized as follows. In Section II, we present the formulation of the vector Gaussian CEO problem under the trace distortion constraint and the corresponding Berger-Tung upper bound on the weighted sum rate. In Section III, we revisit some mathematical preliminaries which will be used frequently in our proof. In Section IV, we prove certain properties of the spectral decomposition of the mean squared error matrix of the Berger-Tung scheme based on a carefully analysis of the KKT conditions of an associated non-convex optimization problem. In Section V, we establish a new extremal inequality by considering projections into subspaces specified by the spectral decomposition result in the previous section, which is further leveraged to characterize the rate region of the vector Gaussian CEO problem with the trace distortion constraint. Finally, we conclude this paper in SectionVI.

II. PROBLEM STATEMENT AND THE MAIN RESULT

The system model of the vector Gaussian CEO problem is depicted in Figure 1. Let $\{\mathbf{X}(t)\}_{t=1}^{\infty}$ be an $m \times 1$-dimensional i.i.d. vector-valued sequence, where each $\mathbf{X}(t), t = 1, 2, \ldots$ is a Gaussian random vector with mean zero and covariance $\mathbf{K} \succ 0$. For $i = 1, 2, \ldots, L$, let

$$\mathbf{Y}_i(t) = \mathbf{X}(t) + \mathbf{N}_i(t), \qquad i = 1, 2, \ldots, L$$

where $\mathbf{N}_i(t), t = 1, 2, \ldots$ are i.i.d. Gaussian random $m \times 1$-dimensional vectors independent of $\{\mathbf{X}(t)\}_{t=1}^{\infty}$ with mean zero and covariance $\mathbf{\Sigma}_i \succ 0$. The noise processes $\{\mathbf{N}_i(t)\}_{t=1}^{\infty}, i = 1, 2, \ldots, L$, are mutually independent. For $i = 1, 2, \cdots, L$, encoder i computes $C_i = \phi_i^n(\mathbf{Y}_i^n)$ based on its noisy observation $\mathbf{Y}_i^n = \{\mathbf{Y}_i(1), \cdots, \mathbf{Y}_i(n)\}$ using encoding function

$$\phi_i^n : \mathcal{R}^{m \times n} \mapsto \mathcal{M}_i^n = \{1, \cdots, 2^{nR_i}\}$$

and sends C_i to the decoder. Upon receiving $C_1, C_2, \ldots, C_L$, the decoder computes $\hat{\mathbf{X}}^n = \{\hat{\mathbf{X}}(1), \cdots, \hat{\mathbf{X}}(n)\} = \varphi^n(C_1, \cdots, C_L)$, which is an estimate of the remove source $\mathbf{X}^n = \{\mathbf{X}(1), \cdots, \mathbf{X}(n)\}$, using decoding function

$$\varphi^n : \mathcal{M}_1^n \times \ldots \times \mathcal{M}_L^n \mapsto \mathcal{R}^{m \times n}.$$
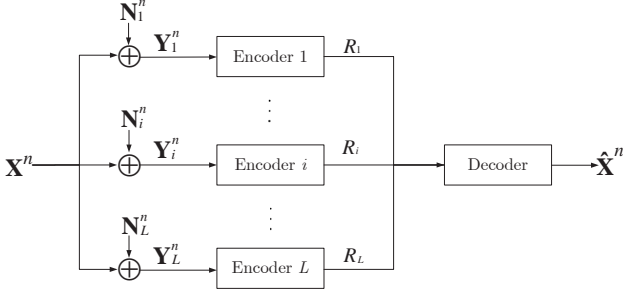
Fig. 1. The vector Gaussian CEO problem with trace constraint $\mathrm{tr}\{\mathrm{cov}(\hat{\mathbf{X}} - \mathbf{X})\} \le d$.

Throughout the paper, we adopt the trace distortion constraint. Specifically, a rate tuple $(R_1, \ldots, R_L, d)$ is said to be achievable subject to the trace distortion constraint d if there exist encoding functions $\phi_1^n, \ldots, \phi_L^n$ and decoding function φ^n such that

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[\mathrm{tr}(\mathrm{cov}(\mathbf{X}(t) - \hat{\mathbf{X}}(t))) \right] \le d,$$

where $\mathbf{x}_j(t)$ and $\hat{\mathbf{x}}_j(t)$ represents the *j-th* component of random vectors $\mathbf{X}(t) \triangleq (\mathbf{x}_1(t), \ldots, \mathbf{x}_m(t))^T$ and $\hat{\mathbf{X}}(t) \triangleq (\hat{\mathbf{x}}_1(t), \ldots, \hat{\mathbf{x}}_m(t))^T$ respectively. The rate region $\mathcal{R}(d)$ is the closure of all achievable rate tuples $(R_1, \cdots, R_L)$ subject to the trace distortion constraint d.

Since the rate region is convex, it can be characterized by its supporting hyper-planes. As a consequence, it suffices to solve the following optimization problem

$$R(d) = \inf_{(R_1, \ldots, R_L) \in \mathcal{R}(d)} \sum_{i=1}^{L} \mu_i R_i$$

for $\mu_i \ge 0, i = 1, \ldots, L$; moreover, there is no loss of generality in assuming $\mu_1 \ge \cdots \ge \mu_L \ge 0$. Note that if $\mu_L = 0$, then one can reduce the L-terminal problem to the $(L-1)$-terminal problem by providing $\mathbf{Y}_L^n$ directly to the decoder and the first $L-1$ encoders. For this reason, we shall focus on the case $\mu_1 \ge \cdots \ge \mu_L > 0$ in the rest of this paper.

It is clear that $R(d) = \infty$ when $d \le \mathrm{tr}\{(\mathbf{K}^{-1} + \sum_{i=1}^{L} \boldsymbol{\Sigma}_i)^{-1}\}$, and $R(d) = 0$ when $d \ge \mathrm{tr}\{\mathbf{K}\}$. Henceforth the only case

$$\mathrm{tr}\{(\mathbf{K}^{-1} + \sum_{i=1}^{L} \boldsymbol{\Sigma}_i)^{-1}\} < d < \mathrm{tr}\{\mathbf{K}\} \tag{1}$$

needs to be considered.

By evaluating the standard Berger-Tung scheme, one can readily show that

$$R(d) \le R^{BT}(d),$$

where

$$R^{BT}(d) = \min_{(\mathbf{B}_1, \cdots, \mathbf{B}_L)} \sum_{i=1}^{L-1} \frac{\mu_i - \mu_{i+1}}{2} \log \frac{|\mathbf{K}^{-1} + \sum_{j=1}^{L} \mathbf{B}_j|}{|\mathbf{K}^{-1} + \sum_{j=i+1}^{L} \mathbf{B}_j|}$$

$$+ \sum_{i=1}^{L} \frac{\mu_i}{2} \log \frac{|\boldsymbol{\Sigma}_i^{-1}|}{|\boldsymbol{\Sigma}_i^{-1} - \mathbf{B}_i|} + \frac{\mu_L}{2} \log \frac{|\mathbf{K}^{-1} + \sum_{j=1}^{L} \mathbf{B}_j|}{|\mathbf{K}^{-1}|}. \tag{2}$$

The minimization in (2) is over $(\mathbf{B}_1, \ldots, \mathbf{B}_L)$ subject to constraints

$$\mathrm{tr} \left\{ \left(\mathbf{K}^{-1} + \sum_{i=1}^{L} \mathbf{B}_i \right)^{-1} \right\} \le d, \tag{3}$$

$$\boldsymbol{\Sigma}_i^{-1} \succeq \mathbf{B}_i \succeq 0, \qquad i = 1, \ldots, L.$$

The main result of this paper is the following theorem.

Theorem 1: For any $\mu_1 \ge \cdots \ge \mu_L > 0$ and $d \in (\mathrm{tr}\{(\mathbf{K}^{-1} + \sum_{i=1}^{L} \boldsymbol{\Sigma}_i)^{-1}\}, \mathrm{tr}(\mathbf{K}))$,

$$R(d) = R^{BT}(d).$$

The rest of this paper is devoted to the proof of the converse part of the theorem, i.e.,

$$R(d) \ge R^{BT}(d).$$

III. MATHEMATICAL PRELIMINARIES

We first review some basic properties of conditional Fisher Information Matrix and MSE (mean square error).

Definition 1: Let $(\mathbf{X}, U)$ be a pair of jointly distributed random vectors with differentiable conditional probability density function $f(\mathbf{x}|u)$. The vector-valued score function is defined as

$$\nabla \log f(\mathbf{x}|u) = \left[\frac{\partial \log f(\mathbf{x}|u)}{\partial x_1}, \cdots, \frac{\partial \log f(\mathbf{x}|u)}{\partial x_m} \right]^T.$$

The conditional Fisher Information of $\mathbf{X}$ respect to U is given by

$$J(\mathbf{X}|U) = \mathbb{E} \left[(\nabla \log f(\mathbf{x}|u)) \cdot (\nabla \log f(\mathbf{x}|u))^T \right].$$

Lemma 1 (Cramér–Rao Lower Bound): Let $(\mathbf{X}, U)$ be a pair of jointly distributed random vectors. Assuming that the conditional covariance matrix $\mathrm{cov}(\mathbf{X}|U) \succ \mathbf{0}$, then

$$J(\mathbf{X}|U)^{-1} \preceq \mathrm{cov}(\mathbf{X}|U). \tag{4}$$

One can refer to the proof in [10, Appendix II].

Lemma 2 (Complementary Identity): Let $(\mathbf{X}, \mathbf{N}, U)$ be a tuple of jointly distributed random vectors. If $\mathbf{N}$ follows a Gaussian distribution $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$, and it is independent with $(\mathbf{X}, U)$, then

$$J(\mathbf{X} + \mathbf{N}|U) + \boldsymbol{\Sigma}^{-1} \mathrm{cov}(\mathbf{X}|\mathbf{X} + \mathbf{N}, U) \boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}^{-1} \tag{5}$$

The proof of this complementary identity can be found in [15, Corollary 1].

Lemma 3 (de Bruijn's Identity): Let $(\mathbf{X}, U)$ be a pair of jointly distributed random vectors, and $\mathbf{N} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ be a standard Gaussian random vector, which is independent of $(\mathbf{X}, U)$, then

$$\frac{d}{d\gamma} h(\mathbf{X} + \sqrt{\gamma}\mathbf{N}|U) = \frac{1}{2} \mathrm{tr} \left\{ J(\mathbf{X} + \sqrt{\gamma}\mathbf{N}|U) \boldsymbol{\Sigma} \right\}. \tag{6}$$

This lemma is the conditional version of [16, Theorem 14].

Replacing the variable γ by $1/\gamma$ in de Bruijn's identity and using the complementary identity in lemma 2, one can obtain the following result via simple algebraic manipulations.

Corollary 1:

$$\frac{d}{d\gamma}h(\sqrt{\gamma}\mathbf{X}+\mathbf{N}|U)=\frac{1}{2}\operatorname{tr}\left\{\boldsymbol{\Sigma}^{-1}\operatorname{cov}(\mathbf{X}|\sqrt{\gamma}\mathbf{X}+\mathbf{N})\right\}. \quad (7)$$

Lemma 4 (Data Processing Inequality): Let $(\mathbf{X},U,V)$ be a tuple of jointly distributed random vectors, and $U,V,\mathbf{X}$ form a Markov chain. *i.e.* $U\rightarrow V\rightarrow\mathbf{X}$, then

$$J(\mathbf{X}|U)\preceq J(\mathbf{X}|V). \quad (8)$$

The proof follows easily by the chain rule of Fisher information matrix [17, Lemma 1].

Lemma 5 (Fisher Information Inequality): Let $(\mathbf{X},\mathbf{Y},U)$ be a tuple of jointly distributed random vectors. Assume that $\mathbf{X}$ and $\mathbf{Y}$ be conditionally independent given U, then for any $\gamma\in(0,1)$,

$$J(\sqrt{1-\gamma}\mathbf{X}+\sqrt{\gamma}\mathbf{Y}|U)\preceq(1-\gamma)J(\mathbf{X}|U)+\gamma J(\mathbf{Y}|U). \quad (9)$$

This is an equivalent form of matrix Fisher information inequality. One can refer to [18, Proposition 3] for a detailed discussion.

Lemma 6: Let $(\mathbf{X},U)$ be a pair of jointly distributed random vectors, and $\mathbf{N}\sim\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma})$ be a standard Gaussian random vector, which is independent of $(\mathbf{X},U)$, then for any $\gamma\in(0,1)$, we have

$$\operatorname{cov}(\mathbf{X}|\mathbf{X}+\mathbf{N},U)\preceq\gamma^2\operatorname{cov}(\mathbf{X}|U)+(1-\gamma)^2\boldsymbol{\Sigma}. \quad (10)$$

The proof is left in Appendix A.

IV. PROPERTIES OF $R^{BT}(d)$

In this section, we study the KKT (Karush–Kuhn–Tucker) conditions for the optimization problem $R^{BT}(d)$ and establish some basic properties of the subspaces induced by the eigen-decomposition of the MSE (mean square error) matrix. These properties play a key role in the proof of the converse theorem for the vector Gaussian CEO problem under the trace distortion constraint.

A. KKT Conditions

It is easy to observe that the objective function of the optimization problem $R^{BT}(d)$ goes to infinity as $|\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i|\rightarrow0$ for any $i=1,\ldots,L$. Hence the constraints $\mathbf{B}_i\preceq\boldsymbol{\Sigma}_i^{-1}$, $i=1,\ldots,L$ are not active.

The Lagrangian of the optimization problem $R^{BT}(d)$ is given by

$$\frac{\mu_1}{2}\log\left|\mathbf{K}^{-1}+\sum_{j=1}^{L}\mathbf{B}_j\right|-\sum_{i=1}^{L-1}\frac{\mu_i-\mu_{i+1}}{2}\log\left|\mathbf{K}^{-1}+\sum_{j=i+1}^{L}\mathbf{B}_j\right|$$

$$-\sum_{i=1}^{L}\frac{\mu_i}{2}\log|\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i|+\sum_{i=1}^{L}\frac{\mu_i}{2}\log|\boldsymbol{\Sigma}_i^{-1}|-\frac{\mu_L}{2}\log|\mathbf{K}^{-1}|$$

$$-\sum_{i=1}^{L}\operatorname{tr}(\mathbf{B}_i\boldsymbol{\Psi}_i)+\lambda\left(\operatorname{tr}\left(\left(\mathbf{K}^{-1}+\sum_{j=1}^{L}\mathbf{B}_j\right)^{-1}\right)-d\right),$$

where matrices $\boldsymbol{\Psi}_i,i=1,\ldots,L$ and scalar λ are Lagrange multipliers. Let $\mathbf{B}_1^*,\ldots,\mathbf{B}_L^*$ be the optimal solution of $R^{BT}(d)$. Define

$$\mathbf{C}_i=\left(\mathbf{K}^{-1}+\sum_{j=i}^{L}\mathbf{B}_j^*\right)^{-1},\quad i=1,2,\cdots,L. \quad (11)$$

The KKT conditions for the optimization problem $R^{BT}(d)$ are given by

$$\frac{\mu_1}{2}\mathbf{C}_1+\frac{\mu_1}{2}\left(\boldsymbol{\Sigma}_1^{-1}-\mathbf{B}_1^*\right)^{-1}-\boldsymbol{\Psi}_1-\lambda\mathbf{C}_1^2=0; \quad (12)$$

$$\frac{\mu_1}{2}\mathbf{C}_1+\frac{\mu_k}{2}\left(\boldsymbol{\Sigma}_k^{-1}-\mathbf{B}_k^*\right)^{-1}$$
$$-\sum_{i=1}^{k-1}\frac{\mu_i-\mu_{i+1}}{2}\mathbf{C}_{i+1}-\boldsymbol{\Psi}_k-\lambda\mathbf{C}_1^2=0;$$
$$k=2,\ldots,L; \quad (13)$$

$$\mathbf{B}_k^*\boldsymbol{\Psi}_k=0,\quad k=1,\ldots,L; \quad (14)$$
$$\lambda\left(\operatorname{tr}(\mathbf{C}_1)-d\right)=0; \quad (15)$$
$$\boldsymbol{\Psi}_k\succeq0,\quad k=1,\ldots,L;\quad \lambda\geq0. \quad (16)$$

Notice that the optimization problem $R^{BT}(d)$ is not convex; therefore, the constraint qualifications need to be examined in order to show the existence of Lagrange multipliers $\boldsymbol{\Psi}_i,i=1,\ldots,L$ and λ satisfying the KKT conditions. These technical details are relegated to Appendix B. Here we just point out the following implication of the KKT conditions.

Corollary 2: For $d\in(\operatorname{tr}\{(\mathbf{K}^{-1}+\sum_{i=1}^{L}\boldsymbol{\Sigma}_i)^{-1}\},\operatorname{tr}(\mathbf{K}))$, we have

$$\operatorname{tr}(\mathbf{C}_1)=d. \quad (17)$$

Proof: According to the complementary slackness condition (15), for the purpose of proving (17), it suffices to show $\lambda\neq0$. If $\lambda=0$, then it follows by (12) that $\boldsymbol{\Psi}_1\succ0$, which, together with the complementary slackness condition $\mathbf{B}_1^*\boldsymbol{\Psi}_1=0$ in (14), implies $\mathbf{B}_1^*=0$. Substituting $\mathbf{B}_1^*=0$ into the first equation in (13) gives $\boldsymbol{\Psi}_2\succ0$. Along this way, we may inductively obtain $\mathbf{B}_1^*=\mathbf{B}_2^*=\ldots=\mathbf{B}_L^*=0$, which, in view of (3), implies $\operatorname{tr}(\mathbf{K})\leq d$. This leads to a contradiction with the assumption $d<\operatorname{tr}\{\mathbf{K}\}$. Thus (17) is proved. ∎

B. Spectral-decomposition of MSE

Since the mean square error matrix $\mathbf{C}_1=(\mathbf{K}^{-1}+\sum_{j=1}^{L}\mathbf{B}_j^*)^{-1}$ of the Berger-Tung scheme is positive definite, we can write its spectral representation as below:

$$\mathbf{C}_1=\sum_{n=1}^{m}d_n\boldsymbol{e}_n\boldsymbol{e}_n^T, \quad (18)$$

where the positive real numbers $d_n,n=1,\cdots,m$ stand for the eigenvalues, and $\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_m\in\mathbb{R}^m$ are the corresponding normalized eigenvectors which form an orthogonal basis.

It follows readily from (18) that

$$\mathbf{C}_1^2=\sum_{n=1}^{m}d_n^2\boldsymbol{e}_n\boldsymbol{e}_n^T. \quad (19)$$

In what follows, we denote

$$\boldsymbol{\Delta}_i \triangleq \frac{\mu_i}{2}(\boldsymbol{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1} - \boldsymbol{\Psi}_i. \tag{20}$$

By the matrix identity in KKT conditions (12), we see that

$$\boldsymbol{\Delta}_1 = \lambda\mathbf{C}_1^2 - \frac{\mu_1}{2}\mathbf{C}_1,$$

Substituting (18) and (19) into the above equation leads to the following spectral representation of $\boldsymbol{\Delta}_1$:

$$\boldsymbol{\Delta}_1 = \sum_{n=1}^{m}\left(\lambda d_n^2 - \mu_1\frac{d_n}{2}\right)\boldsymbol{e}_n\boldsymbol{e}_n^T. \tag{21}$$

Now we divide the vector space $\mathbb{R}^m$ into two orthogonal subspaces according to the sign of the eigenvalues $\lambda d_n^2 - \mu_1 d_n/2, n = 1, 2, \ldots, m$. We may define $m \times n_1$ matrix $\mathbf{U}_1 \triangleq (\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n_1})$ in which the eigenvectors $\boldsymbol{e}_n, n = 1, 2, \ldots, n_1$, correspond to the positive eigenvalues. Similarly we may define $m \times (m - n_1)$ matrix $\mathbf{V}_1 \triangleq (\boldsymbol{e}_{n_1+1}, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_m)$, in which the eigenvectors $\boldsymbol{e}_n, n = n_1 + 1, n_1 + 2, \ldots, m$, correspond to non-positive eigenvalues. It can be verified that

$$\mathbf{U}_1^T\boldsymbol{\Delta}_1\mathbf{U}_1 \succ 0, \quad \mathbf{V}_1^T\boldsymbol{\Delta}_1\mathbf{V}_1 \preceq 0, \quad \mathbf{U}_1^T\boldsymbol{\Delta}_1\mathbf{V}_1 = 0; \tag{22}$$
$$\mathbf{U}_1^T\mathbf{C}_1\mathbf{U}_1 \succeq 0, \quad \mathbf{V}_1^T\mathbf{C}_1\mathbf{V}_1 \succeq 0, \quad \mathbf{U}_1^T\mathbf{C}_1\mathbf{V}_1 = 0. \tag{23}$$

At this stage we may rewrite the spectral decomposition of $\boldsymbol{\Delta}_1$ and $\mathbf{C}_1 = (\mathbf{K}^{-1} + \sum_{j=1}^{L}\mathbf{B}_j^*)^{-1}$ according to the positivity/non-positivity structure of eigenspaces as below:

$$\boldsymbol{\Delta}_1 = \mathbf{U}_1\mathbf{U}_1^T\boldsymbol{\Delta}_1\mathbf{U}_1\mathbf{U}_1^T + \mathbf{V}_1\mathbf{V}_1^T\boldsymbol{\Delta}_1\mathbf{V}_1\mathbf{V}_1^T, \tag{24}$$
$$\mathbf{C}_1 = \mathbf{U}_1\mathbf{U}_1^T\mathbf{C}_1\mathbf{U}_1\mathbf{U}_1^T + \mathbf{V}_1\mathbf{V}_1^T\mathbf{C}_1\mathbf{V}_1\mathbf{V}_1^T. \tag{25}$$

Since $\mathbf{V}_1^T\boldsymbol{\Delta}_1\mathbf{V}_1 \preceq 0$, we have

$$\mathbf{V}_1^T\boldsymbol{\Psi}_1\mathbf{V}_1 \succeq \frac{\mu_1}{2}\mathbf{V}_1^T(\boldsymbol{\Sigma}_1^{-1} - \mathbf{B}_1^*)^{-1}\mathbf{V}_1 \succ 0,$$

which means that the subspace spanned by the column vectors of $\mathbf{V}_1$ belongs to the image space of $\boldsymbol{\Psi}_1$, *i.e.*, $\mathbf{V}_1 \subseteq \text{Im}(\boldsymbol{\Psi}_1)$. Thus by the complementary slackness conditions (14) in KKT conditions, we have $\mathbf{B}_1^*\boldsymbol{\Psi}_1 = 0$; as a consequence, the kernel space of $\mathbf{B}_1^*$ contains the image space of $\boldsymbol{\Psi}_1$, *i.e.*, $\text{Ker}(\mathbf{B}_1^*) \supseteq \text{Im}(\boldsymbol{\Psi}_1)$, which implies

$$\mathbf{B}_1^*\mathbf{V}_1 = 0. \tag{26}$$

Henceforth, according to the definition of $\mathbf{V}_1$, we have

$$\begin{aligned}0 &= \mathbf{B}_1^*\mathbf{V}_1\text{diag}(d_{n_1+1}, d_{n_1+2}, \ldots, d_m)\\ &= \mathbf{B}_1^*(\boldsymbol{e}_{n_1+1}, \boldsymbol{e}_{n_1+2}, \ldots, \boldsymbol{e}_m)\text{diag}(d_{n_1+1}, d_{n_1+2}, \ldots, d_m)\\ &= \mathbf{B}_1^*(d_{n_1+1}\boldsymbol{e}_{n_1+1}, d_{n_1+2}\boldsymbol{e}_{n_1+2}, \ldots, d_m\boldsymbol{e}_m)\\ &= \mathbf{B}_1^*\mathbf{C}_1(\boldsymbol{e}_{n_1+1}, \boldsymbol{e}_{n_1+2}, \ldots, \boldsymbol{e}_m)\\ &= \mathbf{B}_1^*\mathbf{C}_1\mathbf{V}_1. \end{aligned} \tag{27}$$

Left-multiplying with $\mathbf{C}_2 = (\mathbf{K}^{-1} + \sum_{j=2}^{L}\mathbf{B}_j^*)^{-1}$ at both sides of (27) yields

$$\begin{aligned}0 &= \mathbf{C}_2\mathbf{B}_1^*\mathbf{C}_1\mathbf{V}_1\\ &= (\mathbf{K}^{-1} + \sum_{j=2}^{L}\mathbf{B}_j^*)^{-1}\mathbf{B}_1^*(\mathbf{K}^{-1} + \sum_{j=1}^{L}\mathbf{B}_j^*)^{-1}\mathbf{V}_1 \end{aligned}$$

$$= (\mathbf{K}^{-1} + \sum_{j=2}^{L}\mathbf{B}_j^*)^{-1}\mathbf{V}_1 - (\mathbf{K}^{-1} + \sum_{j=1}^{L}\mathbf{B}_j^*)^{-1}\mathbf{V}_1$$
$$= \mathbf{C}_2\mathbf{V}_1 - \mathbf{C}_1\mathbf{V}_1, \tag{28}$$

which implies that

$$\mathbf{V}_1^T\mathbf{C}_1\mathbf{V}_1 = \mathbf{V}_1^T\mathbf{C}_2\mathbf{V}_1. \tag{29}$$

In view of (29), $\boldsymbol{e}_{n_1+1}, \boldsymbol{e}_{n_1+2}, \ldots, \boldsymbol{e}_m$ are also the eigenvectors of matrix $\mathbf{C}_2 = (\mathbf{K}^{-1} + \sum_{j=2}^{L}\mathbf{B}_j^*)^{-1}$ with the eigenvalues being $d_{n_1+1}, d_{n_1+2}, \ldots, d_m$. On the other hand, we can conclude that

$$\mathbf{U}_1^T\mathbf{C}_2\mathbf{V}_1 = \mathbf{U}_1^T\mathbf{C}_1\mathbf{V}_1 = 0. \tag{30}$$

Subtracting (12) from the first equation in KKT conditions (13) and invoking (20) gives

$$\boldsymbol{\Delta}_2 = \frac{\mu_1 - \mu_2}{2}\mathbf{C}_2 + \boldsymbol{\Delta}_1. \tag{31}$$

Combining equations (30) and (31) with $\mathbf{U}_1^T\boldsymbol{\Delta}_1\mathbf{V}_1 = 0$, we see

$$\mathbf{U}_1^T\boldsymbol{\Delta}_2\mathbf{V}_1 = 0. \tag{32}$$

Thus we may give matrix $\boldsymbol{\Delta}_2$ the following spectral representation:

$$\boldsymbol{\Delta}_2 = \mathbf{U}_1\mathbf{U}_1^T\boldsymbol{\Delta}_2\mathbf{U}_1\mathbf{U}_1^T + \mathbf{V}_1\mathbf{V}_1^T\boldsymbol{\Delta}_2\mathbf{V}_1\mathbf{V}_1^T. \tag{33}$$

From equation (31), we have $\boldsymbol{\Delta}_2 \succ \boldsymbol{\Delta}_1$ and consequently

$$\mathbf{U}_1^T\boldsymbol{\Delta}_2\mathbf{U}_1 \succ \mathbf{U}_1^T\boldsymbol{\Delta}_1\mathbf{U}_1 \succ 0.$$

On the other hand,

$$\begin{aligned}&\mathbf{V}_1\mathbf{V}_1^T\boldsymbol{\Delta}_2\mathbf{V}_1\mathbf{V}_1^T\\ =&\frac{\mu_1 - \mu_2}{2}\mathbf{V}_1\mathbf{V}_1^T\mathbf{C}_2\mathbf{V}_1\mathbf{V}_1^T + \mathbf{V}_1\mathbf{V}_1^T\boldsymbol{\Delta}_1\mathbf{V}_1\mathbf{V}_1^T\\ =&\sum_{n=n_1+1}^{m}\frac{\mu_1 - \mu_2}{2}d_n\boldsymbol{e}_n\boldsymbol{e}_n^T + \left(\lambda d_n^2 - \frac{\mu_1}{2}d_n\right)\boldsymbol{e}_n\boldsymbol{e}_n^T\\ =&\sum_{n=n_1+1}^{m}\left(\lambda d_n^2 - \mu_2\frac{d_n}{2}\right)\boldsymbol{e}_n\boldsymbol{e}_n^T. \end{aligned} \tag{34}$$

Now we are at the same situation as treating equation (21), and correspondingly the refined spectral representation of matrix $\boldsymbol{\Delta}_2$ can be obtained through a procedure similar to that for $\boldsymbol{\Delta}_1$. Here we may divide the subspace spanned by the column vector of $\mathbf{V}_1$ into two orthogonal subspaces, according to the sign of $\boldsymbol{\Delta}_2$'s eigenvalues $\lambda d_n^2 - \mu_2 d_n/2$, $n = n_1 + 1, n_1 + 2, \ldots, m$. Specifically, we partition the matrix $\mathbf{V}_1$ into a $m \times (n_2 - n_1)$ matrix $\mathbf{W}_1 \triangleq (\boldsymbol{e}_{n_1+1}, \boldsymbol{e}_{n_1+2}, \ldots, \boldsymbol{e}_{n_2})$ and a $m \times (m - n_2)$ matrix $\mathbf{V}_2 \triangleq (\boldsymbol{e}_{n_2+1}, \boldsymbol{e}_{n_2+2}, \ldots, \boldsymbol{e}_m)$, in which n_2 represents the critical number such that

$$\lambda d_n^2 - \mu_2 d_n/2 > 0, \quad n_1 < n \le n_2$$
$$\lambda d_n^2 - \mu_2 d_n/2 \le 0, \quad n_2 < n \le m.$$

On the other hand, combining $\mathbf{U}_1$ and $\mathbf{W}_1$ will form a new $m \times n_2$ matrix $\mathbf{U}_2 \triangleq (\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n_2})$. It is straightforward to verify that

$$\mathbf{W}_1^T\boldsymbol{\Delta}_2\mathbf{W}_1 \succ 0; \quad \mathbf{V}_2^T\boldsymbol{\Delta}_2\mathbf{V}_2 \preceq 0; \quad \mathbf{W}_1^T\boldsymbol{\Delta}_2\mathbf{V}_2 = 0; \tag{35}$$

$$\mathbf{W}_1^T \mathbf{C}_2 \mathbf{W}_1 \succeq 0; \quad \mathbf{V}_2^T \mathbf{C}_2 \mathbf{V}_2 \succeq 0; \quad \mathbf{W}_1^T \mathbf{C}_2 \mathbf{V}_2 = 0. \quad (36)$$

We can further refine the spectral decomposition form of $\boldsymbol{\Delta}_2$ and $\mathbf{C}_2$:

$$\begin{aligned}
\boldsymbol{\Delta}_2 &= \mathbf{U}_1 \mathbf{U}_1^T \boldsymbol{\Delta}_2 \mathbf{U}_1 \mathbf{U}_1^T + \mathbf{V}_1 \mathbf{V}_1^T \boldsymbol{\Delta}_2 \mathbf{V}_1 \mathbf{V}_1^T \\
&= \mathbf{U}_1 \mathbf{U}_1^T \boldsymbol{\Delta}_2 \mathbf{U}_1 \mathbf{U}_1^T + \mathbf{W}_1 \mathbf{W}_1^T \boldsymbol{\Delta}_2 \mathbf{W}_1 \mathbf{W}_1^T \\
&\quad + \mathbf{V}_2 \mathbf{V}_2^T \boldsymbol{\Delta}_2 \mathbf{V}_2 \mathbf{V}_2^T
\end{aligned} \quad (37)$$

$$\begin{aligned}
\mathbf{C}_2 &= \mathbf{U}_2 \mathbf{U}_2^T \mathbf{C}_2 \mathbf{U}_2 \mathbf{U}_2^T + \mathbf{V}_2 \mathbf{V}_2^T \mathbf{C}_2 \mathbf{V}_2 \mathbf{V}_2^T \\
&= \mathbf{U}_1 \mathbf{U}_1^T \mathbf{C}_2 \mathbf{U}_1 \mathbf{U}_1^T + \mathbf{W}_1 \mathbf{W}_1^T \mathbf{C}_2 \mathbf{W}_1 \mathbf{W}_1^T + \mathbf{V}_2 \mathbf{V}_2^T \mathbf{C}_2 \mathbf{V}_2 \mathbf{V}_2^T
\end{aligned} \quad (38)$$

Following the similar steps as in the derivation of (26), we obtain

$$\mathbf{B}_2^* \mathbf{V}_2 = 0. \quad (39)$$

$$\mathbf{V}_2^T \mathbf{C}_2 \mathbf{V}_2 = \mathbf{V}_2^T \mathbf{C}_3 \mathbf{V}_2. \quad (40)$$

Repeating this procedure L times yields the following theorem.

Theorem 2: In $\mathbb{R}^m$, there exist three sets of column orthogonal matrices[1] : $\{\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_L\}$, $\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_L\}$, $\{\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{L-1}\}$, such that the following properties hold:

1) [Spectrum of $\mathbf{C}_i$]

$$\mathbf{C}_i = \mathbf{U}_i \mathbf{U}_i^T \mathbf{C}_i \mathbf{U}_i \mathbf{U}_i^T + \mathbf{V}_i \mathbf{V}_i^T \mathbf{C}_i \mathbf{V}_i \mathbf{V}_i^T, \ i = 1, \ldots, L. \quad (41)$$

2) [Spectrum of $\boldsymbol{\Delta}_i$]

$$\boldsymbol{\Delta}_1 = \mathbf{U}_1 \mathbf{U}_1^T \boldsymbol{\Delta}_1 \mathbf{U}_1 \mathbf{U}_1^T + \mathbf{V}_1 \mathbf{V}_1^T \boldsymbol{\Delta}_1 \mathbf{V}_1 \mathbf{V}_1^T, \quad (42)$$

$$\begin{aligned}
\boldsymbol{\Delta}_{i+1} &= \mathbf{U}_i \mathbf{U}_i^T \boldsymbol{\Delta}_{i+1} \mathbf{U}_i \mathbf{U}_i^T + \mathbf{V}_i \mathbf{V}_i^T \boldsymbol{\Delta}_{i+1} \mathbf{V}_i \mathbf{V}_i^T \\
&= \mathbf{U}_i \mathbf{U}_i^T \boldsymbol{\Delta}_{i+1} \mathbf{U}_i \mathbf{U}_i^T + \mathbf{W}_i \mathbf{W}_i^T \boldsymbol{\Delta}_{i+1} \mathbf{W}_i \mathbf{W}_i^T \\
&\quad + \mathbf{V}_{i+1} \mathbf{V}_{i+1}^T \boldsymbol{\Delta}_{i+1} \mathbf{V}_{i+1} \mathbf{V}_{i+1}^T, \\
&\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, L-1.
\end{aligned} \quad (43)$$

3) [Positive/Negative definiteness]

$$\begin{aligned}
\mathbf{U}_i^T \boldsymbol{\Delta}_i \mathbf{U}_i &\succ 0, & i &= 1, \ldots, L; \\
\mathbf{W}_i^T \boldsymbol{\Delta}_{i+1} \mathbf{W}_i &\succ 0, & i &= 1, \ldots, L-1; \\
\mathbf{V}_i^T \boldsymbol{\Delta}_i \mathbf{V}_i &\preceq 0, & i &= 1, \ldots, L.
\end{aligned} \quad (44)$$

4) [Orthogonality] For any $1 \le i \le L$,

$$\mathbf{B}_i^* \mathbf{V}_i = 0. \quad (45)$$

[1] One $m \times n$ dimensional ($m \ge n$) matrix $\mathbf{A}$ is called column orthogonal iff $\mathbf{A}^T \mathbf{A} = \mathbf{I}$.

V. CONVERSE

In this section we establish a new extremal inequality, which is further leveraged to give a complete characterization of the rate region of the vector Gaussian CEO problem with the trace distortion constraint. However, it appears difficult to give a direct proof of this extremal inequality using the perturbation method. To overcome this difficulty, we project the mean square error matrix of the Berger-Tung scheme into its eigenspaces, and estimate each term of the extremal inequality in its respective subspace. This approach is partly inspired by the work of Rahman and Wagner on the vector Gaussian one-help-one problem [13].

A. Extremal Inequality

Theorem 3: Let $\mathbf{B}_1^*, \ldots, \mathbf{B}_L^*$ be the optimal solution of $R^{BT}(d)$. For any random variables $(M_1, \ldots, M_L, Q)$ jointly distributed with $(\mathbf{X}, \mathbf{Y}_1, \ldots, \mathbf{Y}_L)$ such that

$$\begin{aligned}
&p(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_L, m_1, \ldots, m_L, q) \\
&= p(\mathbf{x}) p(q) \prod_{i=1}^{L} p(\mathbf{y}_i | \mathbf{x}) p(m_i | \mathbf{y}_i, q),
\end{aligned} \quad (46)$$

and

$$\begin{aligned}
&\sum_{n=1}^{m} \mathbb{E} \left[(\mathbf{x}_n - \mathbb{E}[\mathbf{x}_n | M_1, \ldots, M_L])^2 \right] \\
&= \operatorname{tr} \{ \operatorname{cov}(\mathbf{X} | M_1, \ldots, M_L) \} \\
&\le d,
\end{aligned} \quad (47)$$

we have

$$\begin{aligned}
&\sum_{i=1}^{L-1} (\mu_i - \mu_{i+1}) h(\mathbf{X} | M_{i+1}, \ldots, M_L) \\
&\quad - \mu_1 h(\mathbf{X} | M_1, \ldots, M_L) - \sum_{i=1}^{L} \mu_i h(\mathbf{Y}_i | \mathbf{X}, M_i, Q) \\
&\ge \sum_{i=1}^{L-1} \frac{\mu_i - \mu_{i+1}}{2} \log |(2\pi e) \mathbf{C}_{i+1}| - \frac{\mu_1}{2} \log |(2\pi e) \mathbf{C}_1| \\
&\quad - \sum_{i=1}^{L} \frac{\mu_i}{2} \log |(2\pi e)(\boldsymbol{\Sigma}_i - \boldsymbol{\Sigma}_i \mathbf{B}_i^* \boldsymbol{\Sigma}_i)|.
\end{aligned} \quad (48)$$

Note that

$$\begin{aligned}
&h\left(\left[\mathbf{U}_i, \ (\boldsymbol{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1} \mathbf{V}_i \right]^T \boldsymbol{\Sigma}_i^{-1} \mathbf{Y}_i | \mathbf{X}, M_i, Q \right) \\
&\le h(\mathbf{U}_i^T \boldsymbol{\Sigma}_i^{-1} \mathbf{Y}_i | \mathbf{X}, M_i, Q) \\
&\quad + h(\mathbf{V}_i^T (\boldsymbol{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1} \boldsymbol{\Sigma}_i^{-1} \mathbf{Y}_i | \mathbf{X}, M_i, Q).
\end{aligned}$$

On the other hand, following by the matrix equality,

$$\begin{aligned}
&(2\pi e) \begin{pmatrix} \mathbf{U}_i^T \\ \mathbf{V}_i^T (\boldsymbol{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1} \end{pmatrix} \boldsymbol{\Sigma}_i^{-1} (\boldsymbol{\Sigma}_i - \boldsymbol{\Sigma}_i \mathbf{B}_i^* \boldsymbol{\Sigma}_i) \boldsymbol{\Sigma}_i^{-1} \\
&\quad \cdot \left(\mathbf{U}_i \ \ (\boldsymbol{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1} \mathbf{V}_i \right) \\
&= \begin{pmatrix} (2\pi e) \mathbf{U}_i^T (\boldsymbol{\Sigma}_i^{-1} - \mathbf{B}_i^*) \mathbf{U}_i & 0 \\ 0 & (2\pi e) \mathbf{V}_i^T (\boldsymbol{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1} \mathbf{V}_i \end{pmatrix}.
\end{aligned}$$

By Taking logarithm for the determinant of matrix to both sides, we have

$$\frac{1}{2}\log|(2\pi e)(\mathbf{\Sigma}_i - \mathbf{\Sigma}_i\mathbf{B}_i^*\mathbf{\Sigma}_i)| + \log|\mathbf{\Sigma}^{-1}|$$
$$+ \log|\left[\mathbf{U}_i,\ (\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1}\mathbf{V}_i\right]|$$
$$= \frac{1}{2}\log|(2\pi e)\mathbf{U}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)\mathbf{U}_i|$$
$$+ \frac{1}{2}\log|(2\pi e)\mathbf{V}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1}\mathbf{V}_i|.$$

Therefore, it suffices to prove

$$\sum_{i=1}^{L-1}(\mu_i - \mu_{i+1})h(\mathbf{X}|M_{i+1},\ldots,M_L)$$
$$- \mu_1 h(\mathbf{X}|M_1,\ldots,M_L)$$
$$- \sum_{i=1}^{L}\mu_i h(\mathbf{U}_i^T\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i|\mathbf{X},M_i,Q)$$
$$- \sum_{i=1}^{L}h(\mathbf{V}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1}\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i|\mathbf{X},M_i,Q)$$
$$\geq \sum_{i=1}^{L-1}\frac{\mu_i - \mu_{i+1}}{2}\log|(2\pi e)\mathbf{C}_{i+1}| - \frac{\mu_1}{2}\log|(2\pi e)\mathbf{C}_1|$$
$$- \sum_{i=1}^{L}\frac{\mu_i}{2}\log|(2\pi e)\mathbf{U}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)\mathbf{U}_i|$$
$$- \sum_{i=1}^{L}\frac{\mu_i}{2}\log|(2\pi e)\mathbf{V}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1}\mathbf{V}_i| \quad (49)$$

To the end of proving inequality (49), we define $2L$ mutually independent zero mean Gaussian distributed random vectors $\mathbf{X}_{\{1,\ldots,L\}}^G, \mathbf{X}_{\{2,\ldots,L\}}^G, \ldots, \mathbf{X}_{\{L\}}^G$ and $\mathbf{N}_1^G, \mathbf{N}_2^G, \ldots, \mathbf{N}_L^G$, which are independent of $(\mathbf{X}, \mathbf{Y}_1, \ldots, \mathbf{Y}_L, M_1, \ldots, M_L, Q)$. Here their distributions are

$$\mathbf{X}_{\{i,\ldots,L\}}^G \sim \mathcal{N}\left(0, (\mathbf{K}^{-1} + \mathbf{B}_i^* + \cdots + \mathbf{B}_L^*)^{-1}\right), \quad i = 1,\ldots,L;$$
$$\mathbf{N}_i^G \sim \mathcal{N}\left(0, (\mathbf{\Sigma}_i - \mathbf{\Sigma}_i\mathbf{B}_i^*\mathbf{\Sigma}_i)\right), \quad i = 1,\ldots,L.$$

Following [10], [14], we use the *covariance preserved transform* proposed by Dembo *et al.* in [16]. Specifically, for any $\gamma \in (0,1)$, define

$$\mathbf{X}_{i,\gamma} = \sqrt{1-\gamma}\mathbf{X} + \sqrt{\gamma}\mathbf{X}_{\{i,\ldots,L\}}^G, \quad i = 1,\ldots,L;$$
$$\mathbf{Y}_{i,\gamma} = \sqrt{1-\gamma}\mathbf{Y}_i + \sqrt{\gamma}\mathbf{N}_i^G, \quad i = 1,\ldots,L. \quad (50)$$

Consider the functional

$$g(\gamma) = \sum_{i=1}^{L-1}(\mu_i - \mu_{i+1})h(\mathbf{X}_{i+1,\gamma}|M_{i+1},\ldots,M_L)$$
$$- \mu_1 h(\mathbf{X}_{1,\gamma}|M_1,\ldots,M_L)$$
$$- \sum_{i=1}^{L}\mu_i h(\mathbf{U}_i^T\mathbf{\Sigma_i}^{-1}\mathbf{Y}_{i,\gamma}|\mathbf{X},M_i,Q)$$
$$- \sum_{i=1}^{L}\mu_i h(\mathbf{V}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1}\mathbf{\Sigma}_i^{-1}\mathbf{Y}_{i,\gamma}|\mathbf{X},M_i,Q).$$

The following lemma is needed for evaluating the derivative of $g(\gamma)$ with respect to γ.

Lemma 7: For the afore-defined $\mathbf{X}_{i,\gamma}$ and $\mathbf{Y}_{i,\gamma}$, we have
1)

$$2(1-\gamma)\frac{d}{d\gamma}h(\mathbf{X}_{i,\gamma}|M_i,\ldots,M_L)$$
$$= \mathrm{tr}\left\{\mathbf{C}_i\left(J(\mathbf{X}_{i,\gamma}|M_i,\ldots,M_L) - \mathbf{C}_i^{-1}\right)\right\} \quad (51)$$

2)

$$2(1-\gamma)\frac{d}{d\gamma}h(\mathbf{U}_i^T\mathbf{\Sigma}_i^{-1}\mathbf{Y}_{i,\gamma}|\mathbf{X},M_i,Q)$$
$$\geq \mathrm{tr}\left\{\mathbf{U}_i\mathbf{U}_i^T - \mathbf{U}_i\mathbf{U}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1}\mathbf{U}_i\mathbf{U}_i^T\right.$$
$$\left.\cdot\mathbf{\Sigma}_i^{-1}\mathrm{cov}(\mathbf{Y}_{i,\gamma}|\mathbf{X},M_i,Q)\mathbf{\Sigma}_i^{-1}\right\} \quad (52)$$

3)

$$2(1-\gamma)\frac{d}{d\gamma}h(\mathbf{V}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1}\mathbf{\Sigma}_i^{-1}\mathbf{Y}_{i,\gamma}|\mathbf{X},M_i,Q)$$
$$\geq \mathrm{tr}\left\{\mathbf{V}_i\mathbf{V}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1}\mathbf{V}_i\mathbf{V}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)\right.$$
$$\left.-\mathbf{V}_i\mathbf{V}_i^T\right\} \quad (53)$$

Proof:
1) Using de Bruijn's identity (6) in Lemma 3 and taking $\gamma' = \gamma/(1-\gamma)$, we obtain

$$\frac{d}{d\gamma}h(\mathbf{X}_{i,\gamma}|M_i,\ldots,M_L)$$
$$= \frac{d}{d\gamma}\left\{h(\mathbf{X} + \sqrt{\frac{\gamma}{1-\gamma}}\mathbf{X}_{\{i,\ldots,L\}}^G|M_i,\ldots,M_L)\right.$$
$$\left.+ n\log(1-\gamma)\right\}$$
$$= \frac{1}{2}\mathrm{tr}\left\{\frac{1}{(1-\gamma)^2}J(\mathbf{X} + \sqrt{\frac{\gamma}{1-\gamma}}\mathbf{X}_{\{i,\ldots,L\}}^G|M_i,\ldots,M_L)\right.$$
$$\left.\cdot\mathbf{C}_i - \frac{1}{1-\gamma}\mathbf{I}\right\}. \quad (54)$$

Multiplying both sides with $2(1-\gamma)$ yields

$$2(1-\gamma)\frac{d}{d\gamma}h(\mathbf{X}_{i,\gamma}|M_i,\ldots,M_L)$$
$$= \mathrm{tr}\left\{J(\sqrt{1-\gamma}\mathbf{X} + \sqrt{\gamma}\mathbf{X}_{\{i,\ldots,L\}}^G|M_i,\ldots,M_L)\mathbf{C}_i - \mathbf{I}\right\}$$
$$= \mathrm{tr}\left\{J(\mathbf{X}_{i,\gamma}|M_i,\ldots,M_L)\mathbf{C}_i - \mathbf{I}\right\}$$
$$= \mathrm{tr}\left\{\mathbf{C}_i\left(J(\mathbf{X}_{i,\gamma}|M_i,\ldots,M_L) - \mathbf{C}_i^{-1}\right)\right\}. \quad (55)$$

2) Using the alternative form of de Bruijn's identity (7) in Corollary 1 and taking $\gamma' = (1-\gamma)/\gamma$, we obtain inequality (56) at the top of next page.
In (56), inequality (a) follows from Lemma 6. Multiplying both sides of (56) $2(1-\gamma)$ gives

$$2(1-\gamma)\frac{d}{d\gamma}h(\mathbf{U}_i^T\mathbf{\Sigma}_i^{-1}\mathbf{Y}_{i,\gamma}|\mathbf{X},M_i,Q)$$
$$\geq \mathrm{tr}\left(\mathbf{I} - \left(\mathbf{U}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)\mathbf{U}_i\right)^{-1}\right.$$
$$\mathrm{cov}(\sqrt{1-\gamma}\mathbf{U}_i^T\mathbf{\Sigma}_i^{-1}\mathbf{Y}_i + \sqrt{\gamma}\mathbf{U}_i^T\mathbf{\Sigma}_i^{-1}\mathbf{N}_i^G|\mathbf{X},M_i,Q)\right\}$$
$$\overset{(a)}{\geq} \mathrm{tr}\left\{\mathbf{U}_i^T\mathbf{U}_i - \mathbf{U}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1}\mathbf{U}_i\right.$$
$$\left.\cdot\mathbf{U}_i^T\mathbf{\Sigma}_i^{-1}\mathrm{cov}(\mathbf{Y}_{i,\gamma}|\mathbf{X},M_i,Q)\mathbf{\Sigma}_i^{-1}\mathbf{U}_i\right\}$$
$$= \mathrm{tr}\left\{\mathbf{U}_i\mathbf{U}_i^T - \mathbf{U}_i\mathbf{U}_i^T(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1}\mathbf{U}_i\mathbf{U}_i^T\right.$$

$$\frac{d}{d\gamma}h(\mathbf{U}_i^T\boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_{i,\gamma}|\mathbf{X},M_i,Q)$$

$$=\frac{d}{d\gamma}\left\{h(\sqrt{\frac{1-\gamma}{\gamma}}\mathbf{U}_i^T\boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_i+\mathbf{U}_i^T\boldsymbol{\Sigma}_i^{-1}\mathbf{N}_i^G|\mathbf{X},M_i,Q)+n_i\log\gamma\right\}$$

$$=\frac{1}{2}\operatorname{tr}\left\{\frac{1}{\gamma}\mathbf{I}-\frac{1}{\gamma^2}\left(\mathbf{U}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)\mathbf{U}_i\right)^{-1}\operatorname{cov}(\mathbf{U}_i^T\boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_i|\sqrt{\frac{1-\gamma}{\gamma}}\mathbf{U}_i^T\boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_i+\mathbf{U}_i^T\boldsymbol{\Sigma}_i^{-1}\mathbf{N}_i^G,\mathbf{X},M_i,Q)\right\}$$

$$=\frac{1}{2}\operatorname{tr}\left\{\frac{1}{\gamma}\mathbf{I}-\frac{1}{\gamma^2(1-\gamma)}\left(\mathbf{U}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)\mathbf{U}_i\right)^{-1}\operatorname{cov}(\sqrt{1-\gamma}\mathbf{U}_i^T\boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_i|\sqrt{1-\gamma}\mathbf{U}_i^T\boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_i+\sqrt{\gamma}\mathbf{U}_i^T\boldsymbol{\Sigma}_i^{-1}\mathbf{N}_i^G,\mathbf{X},M_i,Q)\right\}$$

$$\overset{(a)}{\geq}\frac{1}{2}\operatorname{tr}\left\{\frac{1}{\gamma}\mathbf{I}-\frac{1}{\gamma^2(1-\gamma)}\left(\mathbf{U}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)\mathbf{U}_i\right)^{-1}\left(\gamma^2\operatorname{cov}(\sqrt{1-\gamma}\mathbf{U}_i^T\boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_i|\mathbf{X},M_i,Q)+\gamma(1-\gamma)^2\mathbf{U}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)\mathbf{U}_i\right)\right\}.\tag{56}$$

$$\frac{d}{d\gamma}h(\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_{i,\gamma}|\mathbf{X},M_i,Q)$$

$$=\frac{d}{d\gamma}\left\{h(\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_i+\sqrt{\frac{\gamma}{1-\gamma}}\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\boldsymbol{\Sigma}_i^{-1}\mathbf{N}_i^G|\mathbf{X},M_i,Q)+(n-n_i)\log\gamma\right\}$$

$$=\frac{1}{2}\operatorname{tr}\left\{\frac{1}{(1-\gamma)^2}J(\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_i+\sqrt{\frac{\gamma}{1-\gamma}}\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\boldsymbol{\Sigma}_i^{-1}\mathbf{N}_i^G|\mathbf{X},M_i,Q)\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\mathbf{V}_i-\frac{1}{1-\gamma}\mathbf{I}\right\}$$

$$\overset{(a)}{\geq}\frac{1}{2}\operatorname{tr}\left\{\frac{1}{(1-\gamma)^2}J(\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\boldsymbol{\Sigma}_i^{-1}\mathbf{Y}_i+\sqrt{\frac{\gamma}{1-\gamma}}\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\boldsymbol{\Sigma}_i^{-1}\mathbf{N}_i^G|\mathbf{X})\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\mathbf{V}_i-\frac{1}{1-\gamma}\mathbf{I}\right\}$$

$$\overset{(b)}{=}\frac{1}{2(1-\gamma)}\operatorname{tr}\left\{\mathbf{V}_i^T\left((1-\gamma)(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)\boldsymbol{\Sigma}_i(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)+\gamma(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)\right)\mathbf{V}_i\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\mathbf{V}_i-\mathbf{I}\right\}$$

$$\overset{(c)}{=}\frac{1}{2(1-\gamma)}\operatorname{tr}\left\{\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)\mathbf{V}_i\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\mathbf{V}_i-\mathbf{V}_i^T\mathbf{V}_i\right\}\tag{57}$$

$$\cdot\boldsymbol{\Sigma}_i^{-1}\operatorname{cov}(\mathbf{Y}_{i,\gamma}|\mathbf{X},M_i,Q)\boldsymbol{\Sigma}_i^{-1}\Big\},$$

where (a) follows from the simple fact that for any positive definite matrix $\mathbf{A}$ and column orthogonal matrix $\mathbf{P}$,

$$\left(\mathbf{P}^T\mathbf{A}\mathbf{P}\right)^{-1}\preceq\mathbf{P}^T\mathbf{A}^{-1}\mathbf{P}.$$

3) Again using de Bruijn's identity (6) in Lemma 3 and taking $\gamma'=\gamma/(1-\gamma)$, we obtain inequality (57) at the top of next page.

In (57), (a) follows from the data processing inequality of Fisher information matrix in Lemma 4; (b) is due to the fact that $(\mathbf{Y}_i,\mathbf{X}_i)$ and $\mathbf{N}_i^G$ are independently distributed Gaussians; (c) is due to $\mathbf{B}_i^*\mathbf{V}_i=0$ (see Proposition 3 in Theorem 2). By multiplying both sides of (57) with $2(1-\gamma)$, and switching the matrices in the trace operator, we obtain (53) as desired. ∎

Since

$$\operatorname{tr}\left\{\mathbf{V}_i\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\mathbf{V}_i\mathbf{V}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)-\mathbf{V}_i\mathbf{V}_i^T\right\}$$
$$=\operatorname{tr}\left\{\mathbf{U}_i\mathbf{U}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\mathbf{U}_i\mathbf{U}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)-\mathbf{U}_i\mathbf{U}_i^T\right\},\tag{58}$$

it follows by (58) and Lemma 7 that

$$2(1-\gamma)g'(\gamma)$$
$$\leq\sum_{i=1}^{L-1}\operatorname{tr}\left\{(\mu_i-\mu_{i+1})\mathbf{C}_{i+1}\right.$$

$$\left.\cdot\left(J(\mathbf{X}_{i+1,\gamma}|M_{i+1},\ldots,M_L)-\mathbf{C}_{i+1}^{-1}\right)\right\}$$
$$-\operatorname{tr}\left\{\mu_1\mathbf{C}_1\cdot\left(J(\mathbf{X}_{1,\gamma}|M_1,\ldots,M_L)-\mathbf{C}_1^{-1}\right)\right\}$$
$$-\sum_{i=1}^{L}\operatorname{tr}\left\{\mu_i\mathbf{U}_i\mathbf{U}_i^T(\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)^{-1}\mathbf{U}_i\mathbf{U}_i^T\right.$$
$$\left.\cdot\left((\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i^*)-\boldsymbol{\Sigma}_i^{-1}\operatorname{cov}(\mathbf{Y}_{i,\gamma}|\mathbf{X},M_i,Q)\boldsymbol{\Sigma}_i^{-1}\right)\right\}.\tag{59}$$

Notice that when $\gamma=0$, $g(\gamma)$ equals to l.h.s. of extremal inequality (49); when $\gamma=1$, $g(\gamma)$ equals to r.h.s. of extremal inequality (49). We have the following theorem regarding the derivative of $g(\gamma)$ with respect to γ, and its proof is given in the next section.

Theorem 4: We have

$$2(1-\gamma)g'(\gamma)\leq 0.\tag{60}$$

Note that (60) implies the existence of a monotonically decreasing path from $\gamma=0$ to $\gamma=1$, from which the desired extremal inequality follows immediately.

B. Proof of Theorem 4

To prove Theorem 4, we consider the right part of (59). Recall the KKT conditions (12) and (13):

$$\frac{\mu_1}{2}\mathbf{C}_1=\lambda\mathbf{C}_1^2-\boldsymbol{\Delta}_1,$$

$$I_1 = \sum_{i=1}^{L-1} \text{tr} \left\{ 2\mathbf{U}_i \mathbf{U}_i^T (\boldsymbol{\Delta}_{i+1} - \boldsymbol{\Delta}_i) \mathbf{U}_i \mathbf{U}_i^T \Big(J(\mathbf{X}_{i+1,\gamma}|M_{i+1},\ldots,M_L) - \mathbf{C}_{i+1}^{-1} \Big) \right\}$$
$$+ \text{tr} \left\{ 2\mathbf{U}_1 \mathbf{U}_1^T \boldsymbol{\Delta}_1 \mathbf{U}_1 \mathbf{U}_1^T \Big(J(\mathbf{X}_{1,\gamma}|M_1,\ldots,M_L) - \mathbf{C}_1^{-1} \Big) \right\}; \tag{62a}$$

$$I_2 = \sum_{i=1}^{L-1} \text{tr} \left\{ 2\mathbf{V}_i \mathbf{V}_i^T (\boldsymbol{\Delta}_{i+1} - \boldsymbol{\Delta}_i) \mathbf{V}_i \mathbf{V}_i^T \Big(J(\mathbf{X}_{i+1,\gamma}|M_{i+1},\ldots,M_L) - \mathbf{C}_{i+1}^{-1} \Big) \right\}$$
$$+ \text{tr} \left\{ 2\mathbf{V}_1 \mathbf{V}_1^T \boldsymbol{\Delta}_1 \mathbf{V}_1 \mathbf{V}_1^T \Big(J(\mathbf{X}_{1,\gamma}|M_1,\ldots,M_L) - \mathbf{C}_1^{-1} \Big) \right\}; \tag{62b}$$

$$I_3 = - \sum_{i=1}^{L} \text{tr} \left\{ \mu_i \mathbf{U}_i \mathbf{U}_i^T (\boldsymbol{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1} \mathbf{U}_i \mathbf{U}_i^T \Big((\boldsymbol{\Sigma}_i^{-1} - \mathbf{B}_i^*) - \boldsymbol{\Sigma}_i^{-1} \text{cov}(\mathbf{Y}_{i,\gamma}|\mathbf{X}, M_i, Q) \boldsymbol{\Sigma}_i^{-1} \Big) \right\}. \tag{62c}$$

$$I_4 = - 2\lambda \, \text{tr} \left\{ \mathbf{C}_1^2 \Big(J(\mathbf{X}_{1,\gamma}|M_1,\ldots,M_L) - \mathbf{C}_1^{-1} \Big) \right\}. \tag{62d}$$

$$\frac{\mu_i - \mu_{i+1}}{2} \mathbf{C}_{i+1} = \boldsymbol{\Delta}_{i+1} - \boldsymbol{\Delta}_i, \ i = 1,\ldots,L-1.$$

By using the spectral decomposition property 1 of $\mathbf{C}_i = (\mathbf{K}^{-1} + \sum_{j=i}^{L} \mathbf{B}_j^*)^{-1}$, $i = 1,2,\ldots,L$, in Theorem 2, we obtain that

$$2(1-\gamma)g'(\gamma) \le I_1 + I_2 + I_3 + I_4, \tag{61}$$

where the terms in the r.h.s are defined at the top of this page.

In what follows, we estimate the above four terms respectively, starting with I_2.

Lemma 8: The term I_2 can be upper bounded by

$$I_2 \le I_5 + I_6, \tag{63}$$

where

$$I_5 = \sum_{i=1}^{L-1} \text{tr} \left\{ 2\mathbf{W}_i \mathbf{W}_i^T \boldsymbol{\Delta}_{i+1} \mathbf{W}_i \mathbf{W}_i^T \right.$$
$$\left. \cdot \Big(J(\mathbf{X}_{i+1,\gamma}|M_{i+1},\ldots,M_L) - \mathbf{C}_{i+1}^{-1} \Big) \right\} \tag{64a}$$

$$I_6 = \text{tr} \left\{ 2\mathbf{V}_L \mathbf{V}_L^T \boldsymbol{\Delta}_L \mathbf{V}_L \mathbf{V}_L^T \Big(J(\mathbf{X}_{L,\gamma}|M_L) - \mathbf{C}_L^{-1} \Big) \right\}. \tag{64b}$$

Proof: By Proposition 2 in Theorem 2:

$$\mathbf{V}_i \mathbf{V}_i^T \boldsymbol{\Delta}_{i+1} \mathbf{V}_i \mathbf{V}_i^T$$
$$= \mathbf{W}_i \mathbf{W}_i^T \boldsymbol{\Delta}_{i+1} \mathbf{W}_i \mathbf{W}_i^T + \mathbf{V}_{i+1} \mathbf{V}_{i+1}^T \boldsymbol{\Delta}_{i+1} \mathbf{V}_{i+1} \mathbf{V}_{i+1}^T,$$
$$i = 1,\ldots,L.$$

we can rewrite I_2 as follows:

$$I_2$$
$$= \sum_{i=1}^{L-1} \text{tr} \left\{ 2\mathbf{W}_i \mathbf{W}_i^T \boldsymbol{\Delta}_{i+1} \mathbf{W}_i \mathbf{W}_i^T \right.$$
$$\left. \cdot \Big(J(\mathbf{X}_{i+1,\gamma}|M_{i+1},\ldots,M_L) - \mathbf{C}_{i+1}^{-1} \Big) \right\} \tag{65a}$$
$$+ \text{tr} \left\{ 2\mathbf{V}_L \mathbf{V}_L^T \boldsymbol{\Delta}_L \mathbf{V}_L \mathbf{V}_L^T \Big(J(\mathbf{X}_{L,\gamma}|M_L) - \mathbf{C}_L^{-1} \Big) \right\} \tag{65b}$$
$$+ \sum_{i=1}^{L-1} \text{tr} \left\{ 2\mathbf{V}_i \mathbf{V}_i^T \boldsymbol{\Delta}_i \mathbf{V}_i \mathbf{V}_i^T \Big(J(\mathbf{X}_{i,\gamma}|M_i,\ldots,M_L) - \mathbf{C}_i^{-1} \right.$$

$$\left. - J(\mathbf{X}_{i+1,\gamma}|M_{i+1},\ldots,M_L) + \mathbf{C}_{i+1}^{-1} \Big) \right\} \tag{65c}$$
$$\le I_5 + I_6,$$

where the last inequality is because (65c) is upper bounded by 0 as shown below.

By definition (50),

$$\mathbf{X}_{i,\gamma} = \sqrt{1-\gamma}\mathbf{X} + \sqrt{\gamma}\mathbf{X}_{\{i,\ldots,L\}}^G,$$
$$\mathbf{X}_{i+1,\gamma} = \sqrt{1-\gamma}\mathbf{X} + \sqrt{\gamma}\mathbf{X}_{\{i+1,\ldots,L\}}^G,$$

where the covariance matrices of $\mathbf{X}_{\{i,\ldots,L\}}^G$ and $\mathbf{X}_{\{i+1,\ldots,L\}}^G$ are $\mathbf{C}_i = (\mathbf{K}^{-1} + \sum_{j=i}^{L} \mathbf{B}_j^*)^{-1}$ and $\mathbf{C}_{i+1} = (\mathbf{K}^{-1} + \sum_{j=i+1}^{L} \mathbf{B}_j^*)^{-1}$ respectively. In view of the positive semidefinite partial order

$$\mathbf{C}_i \preceq \mathbf{C}_{i+1},$$

we can assume that

$$\mathbf{X}_{i+1,\gamma} \leftrightarrow \mathbf{X}_{i,\gamma} \leftrightarrow (M_i, M_{i+1},\ldots,M_L) \leftrightarrow (M_{i+1},\ldots,M_L)$$

form a Markov chain. Thus by the data processing inequality in Lemma 4, we have

$$J(\mathbf{X}_{i+1,\gamma}|M_{i+1},\ldots,M_L) \preceq J(\mathbf{X}_{i,\gamma}|M_i, M_{i+1},\ldots,M_L). \tag{66}$$

On the other hand, $\mathbf{B}_i^* \mathbf{V}_i = 0$ (Proposition 4 in Theorem 2) yields that

$$\mathbf{V}_i^T \mathbf{C}_i^{-1} \mathbf{V}_i = \mathbf{V}_i^T \mathbf{C}_{i+1}^{-1} \mathbf{V}_i, \tag{67}$$

and Proposition 3 in Theorem 2 implies that

$$\mathbf{V}_i \mathbf{V}_i^T \boldsymbol{\Delta}_i \mathbf{V}_i \mathbf{V}_i^T \preceq 0. \tag{68}$$

Finally, combining (66), (67) and (68) gives the upper bound (63). ∎

Substituting the upper bound (63) into (61) yields

$$2(1-\gamma)g'(\gamma) \le I_1 + I_5 + I_6 + I_3 + I_4. \tag{69}$$

We now upper bound the first two terms in r.h.s of (69).

Lemma 9: For the terms I_1 and I_5,

$$I_1 + I_5 \le I_7, \tag{70}$$

where

$$I_7 = \sum_{i=1}^{L} \mathrm{tr} \Big\{ 2\mathbf{U}_i \mathbf{U}_i^T \mathbf{\Delta}_i \mathbf{U}_i \mathbf{U}_i^T$$
$$\cdot \Big((\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*) - \mathbf{\Sigma}_i^{-1} \mathrm{cov}(\mathbf{Y}_{i,\gamma}|\mathbf{X}, M_i, Q)\mathbf{\Sigma}_i^{-1} \Big) \Big\}. \tag{71}$$

Proof: It follows from Proposition 3 in Theorem 2 that

$$\mathbf{W}_i^T \mathbf{\Delta}_{i+1} \mathbf{W}_i \succ 0, \qquad i = 1, \ldots, L-1. \tag{72}$$

On the other hand,

$$J(\mathbf{X}_{i+1,\gamma}|M_{i+1}, \ldots, M_L) - \mathbf{C}_{i+1}^{-1}$$
$$\overset{(a)}{\preceq} (1-\gamma)J(\mathbf{X}|M_{i+1}, \ldots, M_L) - (1-\gamma)\mathbf{C}_{i+1}^{-1}$$
$$\overset{(b)}{\preceq} (1-\gamma)\left(\sum_{j=i+1}^{L} \big(\mathbf{\Sigma}_j^{-1} - \mathbf{\Sigma}_j^{-1} \mathrm{cov}(\mathbf{Y}_j|\mathbf{X}, M_j, Q)\mathbf{\Sigma}_j^{-1} \big) \right)$$
$$- (1-\gamma)\big(\mathbf{K}^{-1} + \mathbf{C}_{i+1}^{-1} \big)$$
$$\overset{(c)}{=} \sum_{j=i+1}^{L} (\mathbf{\Sigma}_j^{-1} - \mathbf{B}_j^*) - \mathbf{\Sigma}_j^{-1} \mathrm{cov}(\mathbf{Y}_{j,\gamma}|\mathbf{X}, M_j, Q)\mathbf{\Sigma}_j^{-1}, \tag{73}$$

where (a) follows from the definition of random vector $\{\mathbf{X}_{i+1,\gamma}\}$ and Fisher information inequality in Lemma 5, (b) can proved by using the argument in [9, Section 6.2] (for completeness, we rewrite the proof in [9] in Appendix C), and (c) is due to the definition of random vector $\{\mathbf{Y}_{j,\gamma}\}$.

Finally, we obtain the bound (70) by substituting (72) (73) into I_5 and (73) into I_1 then simplifying it using the relationship (Proposition 2 in Theorem 2):

$$\mathbf{U}_{i+1}\mathbf{U}_{i+1}^T \mathbf{\Delta}_{i+1}\mathbf{U}_i\mathbf{U}_i^T$$
$$= \mathbf{U}_i\mathbf{U}_i^T \mathbf{\Delta}_{i+1}\mathbf{U}_i\mathbf{U}_i^T + \mathbf{W}_i\mathbf{W}_i^T \mathbf{\Delta}_{i+1}\mathbf{W}_i\mathbf{W}_i^T,$$
$$i = 1, \ldots, L.$$

∎

Substituting the upper bound (70) into (69) gives

$$2(1-\gamma)g'(\gamma) \le I_6 + I_7 + I_3 + I_4. \tag{74}$$

We now upper bound each term separately.

Lemma 10: For the first term I_6 in (74),

$$I_6 \le 0. \tag{75}$$

Proof: By data processing inequality in Lemma 4,

$$\mathbf{V}_L^T J(\mathbf{X}_{L,\gamma}|M_L)\mathbf{V}_L - \mathbf{V}_L^T(\mathbf{K}^{-1} + \mathbf{B}_L^*)\mathbf{V}_L$$
$$\succeq \mathbf{V}_L^T J(\mathbf{X}_{L,\gamma})\mathbf{V}_L - \mathbf{V}_L^T(\mathbf{K}^{-1} + \mathbf{B}_L^*)\mathbf{V}_L$$
$$= \mathbf{V}_L^T J(\sqrt{1-\gamma}\mathbf{X} + \sqrt{\gamma}\mathbf{X}_L^G)\mathbf{V}_L - \mathbf{V}_L^T(\mathbf{K}^{-1} + \mathbf{B}_L^*)\mathbf{V}_L$$
$$= \mathbf{V}_L^T((1-\gamma)\mathbf{K} + \gamma(\mathbf{K}^{-1} + \mathbf{B}_L^*)^{-1})^{-1}\mathbf{V}_L$$
$$\quad - \mathbf{V}_L^T(\mathbf{K}^{-1} + \mathbf{B}_L^*)\mathbf{V}_L$$
$$= \mathbf{V}_L^T \mathbf{K}^{-1}(\mathbf{K}^{-1} + (1-\gamma)\mathbf{B}_L^*)^{-1}(\mathbf{K}^{-1} + \mathbf{B}_L^*)\mathbf{V}_L$$
$$\quad - \mathbf{V}_L^T(\mathbf{K}^{-1} + \mathbf{B}_L^*)\mathbf{V}_L$$
$$= 0, \tag{76}$$

where the last step comes from $\mathbf{B}_L^*\mathbf{V}_L = 0$ in Proposition 4 in Theorem 2.

On the other hand, by Proposition 3 in Theorem 2, $\mathbf{V}_L^T \mathbf{\Delta}_L \mathbf{V}_L \preceq 0$, we see that

$$\mathrm{tr}\Big\{ \mathbf{V}_L\mathbf{V}_L^T \mathbf{\Delta}_L \mathbf{V}_L\mathbf{V}_L^T \Big(J(\mathbf{X}_{L,\gamma}|M_L) - (\mathbf{K}^{-1} + \mathbf{B}_L^*) \Big) \Big\}$$
$$\le \mathrm{tr}\Big\{ \mathbf{V}_L^T \mathbf{\Delta}_L \mathbf{V}_L \cdot \mathbf{0} \Big\} = 0.$$

∎

Lemma 11: For the second term I_7 and the third term I_3 in (74),

$$I_7 + I_3 \le 0. \tag{77}$$

Proof: By the definition of $\mathbf{\Delta}_i$:

$$\mathbf{\Delta}_i \triangleq \frac{\mu_i}{2}(\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*)^{-1} - \mathbf{\Psi}_i,$$

we can write $I_7 + I_3$ in the following form:

$$I_7 + I_3$$
$$= -\sum_{i=1}^{L} \mathrm{tr}\Big\{ 2\mathbf{U}_i \mathbf{U}_i^T \mathbf{\Psi}_i \mathbf{U}_i \mathbf{U}_i^T$$
$$\cdot \Big((\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*) - \mathbf{\Sigma}_i^{-1} \mathrm{cov}(\mathbf{Y}_{i,\gamma}|\mathbf{X}, M_i, Q)\mathbf{\Sigma}_i^{-1} \Big) \Big\} \tag{78}$$

Considering that

$$\mathrm{cov}(\mathbf{Y}_{i,\gamma}|\mathbf{X}, M_i, Q)$$
$$\overset{(a)}{=} (1-\gamma)\mathrm{cov}(\mathbf{Y}_i|\mathbf{X}, M_i, Q) + \gamma(\mathbf{\Sigma}_i - \mathbf{\Sigma}_i\mathbf{B}_i^*\mathbf{\Sigma}_i)$$
$$\preceq (1-\gamma)\mathrm{cov}(\mathbf{Y}_i|\mathbf{X}) + \gamma(\mathbf{\Sigma}_i - \mathbf{\Sigma}_i\mathbf{B}_i^*\mathbf{\Sigma}_i)$$
$$= \mathbf{\Sigma}_i - \gamma\mathbf{\Sigma}_i\mathbf{B}_i^*\mathbf{\Sigma}_i, \tag{79}$$

in which (a) is from the definition of random vector $\{\mathbf{Y}_{i,\gamma}\}$ in Section IV, we have

$$I_7 + I_3$$
$$\le -\mathrm{tr}\Big\{ 2\mathbf{U}_i\mathbf{U}_i^T \mathbf{\Psi}_i \mathbf{U}_i\mathbf{U}_i^T \Big((\mathbf{\Sigma}_i^{-1} - \mathbf{B}_i^*) - (\mathbf{\Sigma}_i^{-1} - \gamma\mathbf{B}_i^*) \Big) \Big\}$$
$$= \mathrm{tr}\Big\{ 2\mathbf{U}_i^T \mathbf{\Psi}_i \mathbf{U}_i\mathbf{U}_i^T(1-\gamma)\mathbf{B}_i^*\mathbf{U}_i \Big\}$$
$$\overset{(a)}{=} 2(1-\gamma)\mathrm{tr}\Big\{ \mathbf{U}_i^T \mathbf{\Psi}_i \mathbf{U}_i\mathbf{U}_i^T \mathbf{B}_i^*\mathbf{U}_i + \mathbf{V}_i^T \mathbf{\Psi}_i \mathbf{V}_i\mathbf{V}_i^T \mathbf{B}_i^*\mathbf{V}_i \Big\}$$
$$= 2(1-\gamma)\mathrm{tr}\Big\{ \mathbf{\Psi}_i \mathbf{U}_i\mathbf{U}_i^T \mathbf{B}_i^* \Big\}$$
$$= 2(1-\gamma)\mathrm{tr}\Big\{ \mathbf{U}_i\mathbf{U}_i^T \mathbf{B}_i^*\mathbf{\Psi}_i \Big\}$$
$$\overset{(b)}{=} 0 \tag{80}$$

where (a) is from $\mathbf{B}_i\mathbf{V}_i = 0$ of Proposition 3 in Theorem 2, (b) is from complementary slackness conditions in KKT conditions (14): $\mathbf{B}_i^*\mathbf{\Psi}_i = 0$.

∎

Lemma 12: For the last term I_4 in (74), we have

$$I_4 \le 0. \tag{81}$$

Proof: Due to the spectral decomposition of $\mathbf{C}_1$:

$$\mathbf{C}_1 = \sum_{n=1}^{m} d_n e_n e_n^T,$$

we see that

$$
\begin{aligned}
&- I_4/2\lambda \\
&= \sum_{n=1}^{m} d_n^2 \operatorname{tr}\{\boldsymbol{e}_n^T J(\mathbf{X}_{1,\gamma}|M_1,\dots,M_L)\boldsymbol{e}_n - d_n^{-1}\} \\
&\geq \sum_{n=1}^{n_1} d_n^2 J(\boldsymbol{e}_n^T \mathbf{X}_{1,\gamma}|M_1,\dots,M_L) - \sum_{n=1}^{m} d_n,
\end{aligned} \tag{82}
$$

where the inequality in (82) is from [17, Corollary 1-b]: $J(\boldsymbol{\Lambda}\mathbf{N}) \preceq \boldsymbol{\Lambda}^T J(N)\boldsymbol{\Lambda}$ for any column orthogonal matrix $\boldsymbol{\Lambda}$.

Let

$$
c_n \triangleq \operatorname{cov}(\boldsymbol{e}_n^T \mathbf{X}|M_1, M_2, \dots, M_L).
$$

By the definition of $\{\mathbf{X}_{i,\gamma}\}$ and the Cramér–Rao lower bound in Lemma 1,

$$
\begin{aligned}
J(\boldsymbol{e}_n^T \mathbf{X}_{i,\gamma}|M_1,\dots,M_L)^{-1} &\leq \operatorname{cov}(\boldsymbol{e}_n^T \mathbf{X}_{i,\gamma}|M_1,\dots,M_L) \\
&= (1-\gamma)c_n + \gamma d_n
\end{aligned}
$$

To show that (82) is lower-bounded by 0 is equivalent to show:

$$
\sum_{n=1}^{m} d_n \frac{d_n}{(1-\gamma)c_n + \gamma d_n} \geq \sum_{n=1}^{m} d_n. \tag{83}
$$

According to Corollary 2, we have

$$
\operatorname{tr}(\mathbf{C}_1) = \sum_{n=1}^{m} d_n = d.
$$

Now consider the trace constraint

$$
\begin{aligned}
&\operatorname{tr}\{\operatorname{cov}(\mathbf{X}|M_1, M_2, \dots, M_L)\} \\
&= \operatorname{tr}\{\operatorname{cov}((\boldsymbol{e}_1^T, \boldsymbol{e}_2^T, \dots, \boldsymbol{e}_m^T)\mathbf{X}|M_1, M_2, \dots, M_L)\} \\
&= \sum_{n=1}^{m} \operatorname{cov}(\boldsymbol{e}_n^T \mathbf{X}|M_1, M_2, \dots, M_L) \\
&= \sum_{n=1}^{m} c_n \leq d.
\end{aligned}
$$

Since $f(x) = x^{-1}$ is convex, we have $\sum_{n=1}^{m} \alpha_n f(x_n) \geq f(\sum_{n=1}^{m} \alpha_n x_n)$, where $\sum_{n=1}^{m} \alpha_n = 1, \alpha_n \geq 0$.

Let

$$
\alpha_n = \frac{d_n}{d}, \qquad x_n = \frac{(1-\gamma)c_n + \gamma d_n}{d_n}.
$$

It can be seen that

$$
\begin{aligned}
&\sum_{n=1}^{m} \frac{d_n}{d} \frac{d_n}{(1-\gamma)c_n + \gamma d_n} \geq \left(\sum_{n=1}^{m} \frac{d_n}{d} \frac{(1-\gamma)c_n + \gamma d_n}{d_n}\right)^{-1} \\
&= \frac{d}{(1-\gamma)\sum_{n=1}^{m} c_n + \gamma \sum_{n=1}^{m} d_n} \geq 1,
\end{aligned} \tag{84}
$$

which implies (83). Thus I_4 indeed upper-bounded by 0. ∎

This completes the proof of Theorem 4 as well as the extremal inequality in Theorem 3.

C. Rate Distortion Region

Now we proceed to prove Theorem 1, *i.e.* $R(d) \geq R^{BT}(d)$. To this end we need the Wagner-Anantharam single-letter outer bound [19] on $\mathcal{R}(d)$.

Theorem 5: [19, Theorem 1] The rate region $\mathcal{R}(d)$ is contained in the union of rate tuples $(R_1, R_2, \dots, R_L)$ such that

$$
\begin{aligned}
&\sum_{i=j}^{L} R_j \\
&\geq I(\mathbf{X}; M_1, \dots, M_i | M_{i+1}, \dots, M_L) + \sum_{i=j}^{L} I(\mathbf{Y}_j; M_j | \mathbf{X}, Q)
\end{aligned}
$$

where the union is over all joint distributions $p(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_L, m_1, \dots, m_L, q)$, which can be factorized as follows:

$$
\begin{aligned}
&p(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_L, m_1, \dots, m_L, q) \\
&= p(\mathbf{x})p(q) \prod_{i=1}^{L} p(\mathbf{y}_i|\mathbf{x})p(m_i|\mathbf{y}_i, q),
\end{aligned}
$$

and $\operatorname{tr}\{\operatorname{cov}(\mathbf{X}|M_1, \dots, M_L)\} \leq d$.

According to this single-letter outer bound, we have

$$
\begin{aligned}
&\sum_{i=1}^{L} \mu_i R_i \\
&\geq \sum_{i=1}^{L-1} (\mu_i - \mu_{i+1}) I(\mathbf{X}; M_1, \dots, M_i | M_{i+1}, \dots, M_L) \\
&\quad + \mu_L I(\mathbf{X}; M_1, \dots, M_L) + \sum_{i=1}^{L} I(\mathbf{Y}_i; M_i | \mathbf{X}, Q) \\
&= \sum_{i=1}^{L-1} (\mu_i - \mu_{i+1}) h(\mathbf{X}|M_{i+1}, \dots, M_L) \\
&\quad - \mu_1 h(\mathbf{X}|M_1, \dots, M_L) - \sum_{i=1}^{L} \mu_i h(\mathbf{Y}_i|\mathbf{X}, M_i, Q) \\
&\quad + \mu_L h(\mathbf{X}) + \sum_{i=1}^{L} h(\mathbf{Y}_i|\mathbf{X}).
\end{aligned} \tag{85}
$$

Notice that the term (85) equals the l.h.s of extremal inequality (49) in Theorem 3, so that we have

$$
\begin{aligned}
R(d) &= \inf_{(R_1,\dots,R_L)\in\mathcal{R}(d)} \sum_{i=1}^{L} \mu_i R_i \\
&\geq \inf_{\operatorname{tr}\{\operatorname{cov}(\mathbf{X}|M_1,\dots,M_L)\leq d\}} \sum_{i=1}^{L-1} (\mu_i - \mu_{i+1}) h(\mathbf{X}|M_{i+1}, \dots, M_L) \\
&\quad - \mu_1 h(\mathbf{X}|M_1, \dots, M_L) - \sum_{i=1}^{L} \mu_i h(\mathbf{Y}_i|\mathbf{X}, M_i, Q) \\
&\quad + \mu_L h(\mathbf{X}) + \sum_{i=1}^{L} h(\mathbf{Y}_i|\mathbf{X}) \\
&\geq \sum_{i=1}^{L-1} \frac{\mu_i - \mu_{i+1}}{2} \log |(2\pi e)(\mathbf{K}^{-1} + \sum_{j=i+1}^{L} \mathbf{B}_j^*)^{-1}|
\end{aligned}
$$

$$-\frac{\mu_1}{2}\log|(2\pi e)(\mathbf{K}^{-1}+\sum_{j=1}^{L}\mathbf{B}_j^*)^{-1}|$$

$$-\sum_{i=1}^{L}\frac{\mu_i}{2}\log|(2\pi e)(\boldsymbol{\Sigma}_i-\boldsymbol{\Sigma}_i\mathbf{B}_i^*\boldsymbol{\Sigma}_i)|$$

$$+\frac{\mu_1}{2}\log|(2\pi e)\mathbf{K}|+\sum_{i=1}^{L}\frac{\mu_i}{2}\log|(2\pi e)\boldsymbol{\Sigma}_i|$$

$$=R^{BT}(d).$$

This completes the proof of Theorem 1 and establishes the tightness of Berger-Tung inner bound for the vector Gaussian CEO problem with trace distortion constraint.

VI. CONCLUSION

This paper provides a complete characterization of the rate region of the vector Gaussian CEO problem with the trace distortion constraint. Our proof is based on, among other things, a careful analysis of the KKT conditions for the optimization problem associated with the Berger-Tung scheme. In particular, we exploit the special structure of the KKT conditions to bound the rate region by considering the projection into different subspaces, and the inherent symmetry of the CEO problem enables us to perform the projection procedure recursively.

It should be stressed that the approach in this work does not apply directly to the setting considered in [8], [9] where a covariance constraint instead of a trace constraint is imposed. However, our work indicates that a more thorough analysis of the KKT conditions might lead to some progress towards that direction.

APPENDIX A
PROOF OF LEMMA 6

Note that

$$\gamma^2\,\text{cov}(\mathbf{X}|U)+(1-\gamma)^2\boldsymbol{\Sigma}$$
$$\overset{(a)}{\succeq}(\gamma(\gamma\,\text{cov}(\mathbf{X}|U))^{-1}+(1-\gamma)((1-\gamma)\boldsymbol{\Sigma})^{-1})^{-1}$$
$$=(\text{cov}(\mathbf{X}|U)^{-1}+\boldsymbol{\Sigma}^{-1})^{-1},$$

which (a) is because $\mathbf{A}^{-1}$ is matrix concave in $\mathbf{A}$. This together with the fact (see, e.g., [7, footnote 2])

$$(\text{cov}(\mathbf{X}|U)^{-1}+\boldsymbol{\Sigma}^{-1})^{-1}\succeq\text{cov}(\mathbf{X}|\mathbf{X}+\mathbf{N},U)$$

completes the proof of Lemma 6.

APPENDIX B
EXISTENSE OF KKT CONDITIONS FOR $R^{BT}(d)$

The proof is similar to those in [11, Appendix IV] and [13, Appendix B]. One can refer to [20, Sections 4-5] for the background materials. We first rewrite the optimization problem $R^{BT}(d)$ in a general form:

$$\min_{\boldsymbol{b}}\quad f(\boldsymbol{b})$$
$$\text{subject to}\quad g(\boldsymbol{b})\le 0,$$
$$\boldsymbol{b}\in\mathcal{B}\triangleq\mathcal{B}_1\cap\mathcal{B}_2\cap\ldots\cap\mathcal{B}_L.\qquad(86)$$

The vector $\boldsymbol{b}\in\mathbb{R}^{Lm^2\times 1}$ is constructed by concatenating the columns of $m\times m$ matricies $\mathbf{B}_1$ through $\mathbf{B}_L$; moreover,

$$f(\boldsymbol{b})\triangleq\sum_{i=1}^{L-1}\frac{\mu_i-\mu_{i+1}}{2}\log\frac{|\mathbf{K}^{-1}+\sum_{j=1}^{L}\mathbf{B}_j|}{|\mathbf{K}^{-1}+\sum_{j=i+1}^{L}\mathbf{B}_j|}$$
$$+\sum_{i=1}^{L}\frac{\mu_i}{2}\log\frac{|\boldsymbol{\Sigma}_i^{-1}|}{|\boldsymbol{\Sigma}_i^{-1}-\mathbf{B}_i|}+\frac{\mu_L}{2}\log\frac{|\mathbf{K}^{-1}+\sum_{j=1}^{L}\mathbf{B}_j|}{|\mathbf{K}^{-1}|},$$

$$g(\boldsymbol{b})\triangleq\text{tr}\{(\mathbf{K}^{-1}+\sum_{i=1}^{L}\mathbf{B}_i)^{-1}\}-d,$$

and

$$\mathcal{B}_i\triangleq\{\text{column concatenation of }(\mathbf{B}_1,\mathbf{B}_2,\ldots,\mathbf{B}_L):\mathbf{B}_i\succeq 0\},$$
$$i=1,2,\ldots,L.$$

Since f and g are continuously differentiable, the Fritz-John necessary conditions [20, Definition 5.2.1] hold: there exist $\mu,\lambda\ge 0$ for the local minima $\boldsymbol{b}^*$ such that

$$-(\mu\nabla f(\boldsymbol{b}^*)+\lambda\nabla g(\boldsymbol{b}^*))\in T_{\mathcal{B}}(\boldsymbol{b}^*)^*,\qquad(87)$$

where $T_{\mathcal{B}}(\boldsymbol{b}^*)$ is the *tangent cone* of $\mathcal{B}$ at $\boldsymbol{b}^*$ and $T_{\mathcal{B}}(\boldsymbol{b}^*)^*$ is its *polar cone*.

As $\mathcal{B}_i,i=1,2,\ldots,L$ are nonempty convex sets such that $\text{ri}(\boldsymbol{b}_1^*)^*\cap\text{ri}(\boldsymbol{b}_2^*)^*\cap\cdots\cap\text{ri}(\boldsymbol{b}_L^*)^*$ is nonempty, it follows [20, Problem 4.23] and [20, Proposition 4.63] that

$$T_{\mathcal{B}}(\boldsymbol{b}^*)^*=T_{\mathcal{B}_1}(\boldsymbol{b}^*)^*+T_{\mathcal{B}_2}(\boldsymbol{b}^*)^*+\cdots+T_{\mathcal{B}_L}(\boldsymbol{b}^*)^*.$$

As in [13, Section B], it can be verified that

$$T_{\mathcal{B}_i}(\boldsymbol{b}^*)^*\cap\mathcal{A}\subseteq\{\text{column concatenation of }(\mathbf{O},\ldots,-\boldsymbol{\Psi}_i,$$
$$\ldots,\mathbf{O}):\boldsymbol{\Psi}_i\succeq 0,\text{tr}\{\boldsymbol{\Psi}_i\mathbf{B}_i^*\}=0\}\qquad(88)$$

in which $\mathcal{A}$ is the set of vectors constructed by concatenating the columns of L symmetric matrices.

Since l.h.s of equation (87) is also in $\mathcal{A}$, to complete the proof of the existence of KKT conditions, we need to show $\mu\ne 0$. As in [11, Appendix IV], we will verify the constraint qualifications (CQ5a in [20, Section 5.4]), i.e., there exists a vector

$$\boldsymbol{d}\in T_{\mathcal{B}}(\boldsymbol{b}^*)=T_{\mathcal{B}_1}(\boldsymbol{b}^*)\cap T_{\mathcal{B}_2}(\boldsymbol{b}^*)\cap\cdots\cap T_{\mathcal{B}_L}(\boldsymbol{b}^*),$$

such that $\nabla g(\boldsymbol{b}^*)^T\boldsymbol{d}<0$.

Given any $\alpha>1$, let's define a set of $m^2\times 1$ vectors

$$\boldsymbol{b}_i=vec(\mathbf{B}_i)\triangleq vec\left(\alpha\mathbf{B}_i^*+\frac{\alpha-1}{L}\mathbf{K}^{-1}\right),i=1,2,\ldots,L.$$
$$(89)$$

Here $vec(\cdot)$ is the vectorization operator. It can be seen that $\boldsymbol{b}_i\in\mathcal{B}_i$ since $\mathbf{B}_i\succeq 0$. We denote $\boldsymbol{d}_i=\boldsymbol{b}_i-(\boldsymbol{b}^*)_i$, where $(\boldsymbol{b}^*)_i$ denotes the ith L-components in $\boldsymbol{b}^*$. By [20, Definition 4.6.1] and [20, Proposition 4.6.2], we have $\boldsymbol{d}_i\in T_{\mathcal{B}_i}(\boldsymbol{b}^*)$. Now $\boldsymbol{d}$ can be constructed by

$$\boldsymbol{d}=vec(\boldsymbol{d}_1,\boldsymbol{d}_2,\ldots,\boldsymbol{d}_L).$$

In this way, the expression of $\nabla g(\boldsymbol{b}^*)^T\boldsymbol{d}$ can be written as

$$\sum_{i=1}^{L}\text{tr}\left\{(\mathbf{K}^{-1}+\sum_{i=1}^{L}\mathbf{B}_i^*)^{-2}(\mathbf{B}_i^*-\mathbf{B}_i)\right\}$$

11

$$
= \sum_{i=1}^{L} \operatorname{tr} \left\{ (\mathbf{K}^{-1} + \sum_{i=1}^{L} \mathbf{B}_i^*)^{-2} \left((1-\alpha)\mathbf{B}_i^* - \frac{\alpha-1}{L} \mathbf{K}^{-1} \right) \right\}
$$
$$
= (1-\alpha) \operatorname{tr} \left\{ (\mathbf{K}^{-1} + \sum_{i=1}^{L} \mathbf{B}_i^*)^{-1} \right\}
$$
$$
< 0,
$$

where the inequality is because $1 - \alpha < 0$ and $(\mathbf{K}^{-1} + \sum_{i=1}^{L} \mathbf{B}_i^*)^{-1} \succ 0$. This completes the proof of the existence of KKT conditions for the non-convex optimization problem $R^{BT}(d)$.

APPENDIX C
PROOF OF INEQUALITY (b) IN (73)

We shall show that

$$
J(\mathbf{X}|M_{i+1}, \ldots, M_L)
$$
$$
\preceq \mathbf{K}^{-1} + \sum_{j=i+1}^{L} \left(\mathbf{\Sigma}_j^{-1} - \mathbf{\Sigma}_j^{-1} \operatorname{cov}(\mathbf{Y}_j|\mathbf{X}, M_j, Q) \mathbf{\Sigma}_j^{-1} \right) \quad (90)
$$

Note that

$$
\mathbf{X} = \sum_{j=i+1}^{L} \mathbf{A}_j \mathbf{Y}_j + \mathbf{Z} \triangleq \bar{\mathbf{X}} + \mathbf{Z},
$$

where $\mathbf{Z}$ is a Gaussian random vector, independent of $(\mathbf{Y}_{i+1}, \ldots, \mathbf{Y}_L)$, with mean zero and covariance matrix $\mathbf{K}_{\mathbf{Z}} \triangleq (\mathbf{K}^{-1} + \sum_{j=i+1}^{L} \mathbf{\Sigma}_j^{-1})^{-1}$, and $\mathbf{A}_j \triangleq \mathbf{K}_{\mathbf{Z}} \mathbf{\Sigma}_j^{-1}$. Using the complementary relationship between Fisher information and MSE in Lemma 2, we have

$$
J(\mathbf{X}|M_{i+1}, \ldots, M_L)
$$
$$
\stackrel{(a)}{\preceq} J(\mathbf{X}|M_{i+1}, \ldots, M_L, Q)
$$
$$
= J(\bar{\mathbf{X}} + \mathbf{Z}|M_{i+1}, \ldots, M_L, Q)
$$
$$
= \mathbf{K}_{\mathbf{Z}}^{-1} - \mathbf{K}_{\mathbf{Z}}^{-1} \operatorname{cov}(\bar{\mathbf{X}}|\bar{\mathbf{X}} + \mathbf{Z}, M_{i+1}, \ldots, M_L, Q) \mathbf{K}_{\mathbf{Z}}^{-1}
$$
$$
\stackrel{(b)}{=} \mathbf{K}_{\mathbf{Z}}^{-1} - \sum_{j=i+1}^{L} \mathbf{\Sigma}_j^{-1} \operatorname{cov}(\mathbf{Y}_j|\mathbf{X}, M_j, Q) \mathbf{\Sigma}_j^{-1}
$$
$$
= \mathbf{K}^{-1} + \sum_{j=i+1}^{L} \left(\mathbf{\Sigma}_j^{-1} - \mathbf{\Sigma}_j^{-1} \operatorname{cov}(\mathbf{Y}_j|\mathbf{X}, M_j, Q) \mathbf{\Sigma}_j^{-1} \right),
$$
$$
(91)
$$

where (a) is from the data processing inequality in Lemma 4 and (b) is due to the fact that for any j, the Markov chain $(\mathbf{Y}_j, M_j) \leftrightarrow (\mathbf{X}, Q) \leftrightarrow (\mathbf{Y}_{\{j\}^c}, M_{\{j\}^c})$ holds.

REFERENCES

[1] T. Berger, Z. Zhang, and H. Viswanathan, "The CEO problem," *IEEE Trans. Inf. Theory*, vol. 42, no. 3, pp. 887 –902, May 1996.

[2] Y. Oohama, "The rate-distortion function for the quadratic Gaussian CEO problem," *IEEE Trans. Inf. Theory*, vol. 43, no. 11, pp. 1912 –1923, Nov. 1997.

[3] ——, "Rate-distortion theory for Gaussian multiterminal source coding systems with several side information at the decoder," *IEEE Trans. Inf. Theory*, vol. 51, no. 7, pp. 2577 –2593, Jul. 2005.

[4] V. Prabhakaran, D. Tse, and K. Ramachandran, "Rate region of the quadratic Gaussian CEO problem," in *Proc. IEEE Int. Symp. Inf. Theory*, Jun. 2004, p. 119.

[5] S. Tavildar and P. Viswanath, "On the sum-rate of the vector Gaussian CEO problem," in *Proc. 39th Asilomar Conf. on Signal, Syst. and Comput.*, 2005, pp. 3–7.

[6] G. Zhang and W. Kleijn, "Bounding the rate region of the two-terminal vector Gaussian CEO problem," in *Proc. Data Comp. Conf.*, Mar. 2011, p. 488.

[7] J. Chen and J. Wang, "On the vector Gaussian CEO problem," in *Proc. IEEE Int. Symp. Inf. Theory*, Aug. 2011, pp. 2050 –2054.

[8] J. Wang and J. Chen, "On the vector Gaussian L-terminal CEO problem," in *Proc. IEEE Int. Symp. Inf. Theory*, July 2012, pp. 571 –575.

[9] E. Ekrem and S. Ulukus, "An outer bound for the vector Gaussian CEO problem," *IEEE Trans. Inf. Theory*, submitted for publication. [Online]. Available: arXiv:1202.0536.

[10] T. Liu and P. Viswanath, "An extremal inequality motivated by multiterminal information-theoretic problems," *IEEE Trans. Inf. Theory*, vol. 53, no. 5, pp. 1839 –1851, May 2007.

[11] H. Weingarten, Y. Steinberg, and S. Shamai, "The capacity region of the Gaussian multiple-input multiple-output broadcast channel," *IEEE Trans. Inf. Theory*, vol. 52, no. 9, pp. 3936 –3964, Sept. 2006.

[12] M. Rahman and A. Wagner, "Rate region of the Gaussian scalar-help-vector source-coding problem," *IEEE Trans. Inf. Theory*, vol. 58, no. 1, pp. 172 –188, Jan. 2012.

[13] ——, "Rate region of the vector Gaussian one-helper source-coding problem," *IEEE Trans. Inf. Theory*, submitted for publication. [Online]. Available: arXiv:1112.6367.

[14] Y. Xu and Q. Wang, "A perturbation proof of the vector Gaussian One-Help-One problem," in *Proc. IEEE Int. Symp. Inf. Theory*, Istanbul, Turkey, Jul. 2013.

[15] D. Palomar and S. Verdú, "Gradient of mutual information in linear vector Gaussian channels," *IEEE Trans. Inf. Theory*, vol. 52, no. 1, pp. 141–154, 2006.

[16] A. Dembo, T. Cover, and J. Thomas, "Information theoretic inequalities," *IEEE Trans. Inf. Theory*, vol. 37, no. 6, pp. 1501 –1518, Nov. 1991.

[17] R. Zamir, "A proof of the Fisher information inequality via a data processing argument," *IEEE Trans. Inf. Theory*, vol. 44, no. 3, pp. 1246 –1250, May 1998.

[18] O. Rioul, "Information theoretic proofs of entropy power inequalities," *IEEE Trans. Inf. Theory*, vol. 57, no. 1, pp. 33–55, 2011.

[19] A. Wagner and V. Anantharam, "An improved outer bound for multiterminal source coding," *IEEE Trans. Inf. Theory*, vol. 54, no. 5, pp. 1919–1937, 2008.

[20] D. P. Bertsekas, A. Nedić, and A. E. Ozdaglar, *Convex Analysis and Optimization*. Athena Scientific Belmont, 2003.