# Security analysis of $\varepsilon$-almost dual universal ${ }_{2}$ hash functions: smoothing of min entropy vs. smoothing of Rényi entropy of order 2 

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#### Abstract

Recently, $\varepsilon$-almost dual universal ${ }_{2}$ hash functions has been proposed as a new and wider class of hash functions. Using this class of hash functions, several efficient hash functions were proposed. This paper evaluates the security performance when we apply this kind of hash functions. We evaluate the security in several kinds of setting based on the $L_{1}$ distinguishability criterion and the modified mutual information criterion. The obtained evaluation is based on smoothing of Rényi entropy of order 2 and/or min entropy. We clarify the difference between these two methods.


## Index Terms

$\varepsilon$-almost dual universal ${ }_{2}$ hash function, secret key generation, exponential decreasing rate, single-shot setting, equivocation rate

## I. Introduction

## A. Tight exponential evaluation of $L_{1}$ distinguishability under $\varepsilon$-almost dual universality ${ }_{2}$

Secure key generation is an important problem in information theoretic security. When a part of keys are leaked to a third party, we cannot use the key. In this case, we need to apply a hash function to the keys. Bennett et al. [4] and Håstad et al. [15] proposed to use universal ${ }_{2}$ hash functions for privacy amplification and derived two universal hashing lemma, which provides an upper bound for leaked information based on Rényi entropy of order 2. Two universal hashing lemma can guarantee the security only when the length of the generated keys is less than Rényi entropy of order 2 . In order to resolve this drawback, Renner [16] attached the smoothing to min entropy, which is a lower bound of conditional Rényi entropy of order 2 . The smoothing is the method to replace the true distribution by a good distribution that approximates the true distribution. This method works well when the security is evaluated variational distance between the real distribution and the ideal distribution, which is often called the $L_{1}$ distinguishability criterion.

Now, we consider the case when a random variable $A$ leaked to the third party $E$ is given as $n$-fold independent and identical distribution [7], [6]. Under this setting, the optimal asymptotic secure key generation rate is the conditional entropy [7], [6]. The smoothing to min entropy shows that universal ${ }_{2}$ hash functions asymptotically achieves the conditional entropy date. When the key generation rate is smaller than the conditional entropy date, the $L_{1}$ distinguishability criterion goes to zero exponentially. The previous paper [12] derived an exponentially decreasing rate under the universality ${ }_{2}$. Its tightness was also shown in [36]. Note that the importance of exponentially decreasing rate has been explained in the previous papers [12], [56].

Recently, Tsurumaru et al. [14] proposed to use $\varepsilon$-almost dual universal ${ }_{2}$ hash functions, which is a generalization of liner universal ${ }_{2}$ hash functions, and obtained a different version of two universal hashing lemma for this class of hash functions. Further, the recent paper [33] proposed several practical hash functions under the condition of the $\varepsilon$-almost dual universality ${ }_{2}$. The hash functions [33] have a smaller calculation amount and a smaller number of random variables than the concatenation of Toeplitz matrix and the identity matrix, which is a typical example of universal ${ }_{2}$ hash functions. Therefore, it is better to evaluate the security under the $\varepsilon$-almost dual universality ${ }_{2}$ rather than under the universality ${ }_{2}$. However, the above results in [12], [36] were given under the universality ${ }_{2}$. In this paper, we show that the above optimal exponential rate can be attained by $\varepsilon$-almost dual universal ${ }_{2}$ hash functions. Indeed, although the previous paper [56] obtained a similar result in the quantum setting, the exponent in [56] is strictly worse than the optimal exponent even in the commutative case.

## B. Evaluation of modified mutual information

When the key generation rate is larger than the conditional entropy date, it is helpful to evaluate how much information is leaked to the third party. In this case, the $L_{1}$ distinguishability does not go to zero and does not reflect the amount of leaked information properly. The mutual information seems to work more properly. Indeed, many papers [50], [52], [7], [6], [53], [54], [24], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [25] employ the mutual information as the security criterion. In the case of secure random number generation, we need to consider the uniformity as well as the independence.

[^0]For this purpose, Csiszár and Narayan [51] modified the mutual information. Then, we call the criterion the modified mutual information [65], [56]. In the above situation, the amount of leaked information is expected to increase linearly. To reflect this requirement, it is natural to surpass the chain rule for the criterion. In this paper, we show that only the modified mutual information satisfies several natural conditions for our security criteria including the chain rule. Since these natural conditions for our security criteria uniquely determine the security criterion, only the modified mutual information suits the situation when the key generation rate is larger than the conditional entropy date. Although the previous paper [56] gave a similar characterization in a quantum setting, the previous characterization [56] could not determine the security criterion uniquely.

When the key generation rate is smaller than the conditional entropy date, the modified mutual information does not go to zero and increases in proportion to the number $n$. The linear coefficient reflects the amount of leaked information, and is called the equivocation rate. The previous paper [65] showed that the optimal equivocation rate can be attained by universal ${ }_{2}$ hash functions. However, it was not shown whether the optimal equivocation rate can be attained by $\varepsilon$-almost dual universal ${ }_{2}$ hash functions. In this paper, we show that the above optimal equivocation rate can be attained by $\varepsilon$-almost dual universal ${ }_{2}$ hash functions.

Further, due to the Pinsker inequality, the modified mutual information goes to zero when the $L_{1}$ distinguishability criterion goes to zero. However, the exponential decreasing rate of the $L_{1}$ distinguishability criterion cannot determine the exponential decreasing rate of the modified mutual information because the Pinsker inequality is not so tight. The previous paper [13] also derived an lower bound of the exponentially decreasing rate of the modified mutual information when we apply universal ${ }_{2}$ hash functions. In this paper, we show that the same lower bound can be attained even when we apply $\varepsilon$-almost dual universal ${ }_{2}$ hash functions.

## C. Smoothing of min entropy vs. smoothing of Rényi entropy of order 2

To discuss the asymptotic performance, the paper [16] applies the smoothing of the min entropy. The previous paper [12] applied the smoothing of Renyi entropy of order 2 when the no leaked information. Since Renyi entropy of order 2 gives a better evaluation than the min entropy, the smoothing of the min entropy cannot surpass that of Renyi entropy of order 2 . The previous paper [12] also showed that the smoothing of the min entropy cannot realize the optimal exponential decreasing rate of the $L_{1}$ distinguishability criterion without any information leakage to the third party. However, the previous paper [12] did not discuss whether the smoothing of the min entropy can realize the optimal exponential decreasing rate of the $L_{1}$ distinguishability criterion when a partial information is leaked to the third party. It is needed to clarify whether the smoothing of the min entropy can realize the optimal exponential decreasing rate of the $L_{1}$ distinguishability criterion in this situation because this general situation is more important from the practical viewpoint and many people still believe the importance of the smoothing of min entropy.

On the other hand, recently, many researchers are interested in second order analysis [18], [20], [22], [21], [19]. Since the papers [20], [22] for second order analysis employ the method of information spectrum, which has been established by Han and Vérdu in their seminal papers [57], [58], [59], [60], [26] and the book [23], many people are interested in how powerful the method of information spectrum is. As is explained in Section V the smoothing of the min entropy is essentially the same as the method of information spectrum 1 . Hence, it is important to clarify the limit of the smoothing of the min entropy.

In this paper, we show that the smoothing of the min entropy cannot realize the optimal exponential decreasing rate of the $L_{1}$ distinguishability criterion even when a partial information leaked to the third party. Then, we arise another question when the smoothing of the min entropy can realize the optimal asymptotic performance. To answer this question, we show that the smoothing of the min entropy can attain the optimal second order key generation rate when the required the $L_{1}$ distinguishability criterion is fixed although the same result with the fidelity distance was obtained in the previous paper [17]. We also show that the smoothing of the min entropy can attain the optimal equivocation rate. Here, we should explain that the smoothing of the min entropy is almost same as the method of information spectrum, which is a powerful and general tool for information theory. Information spectrum has been established by Han and Vérdu in their seminal papers [57], [58], [59], [60], [26] and the book [23]. This method can derive asymptotically tight bounds of the optimal performances of various information processings.

These obtained results are summarized as Table $\square$

## D. Significance from information theoretical viewpoint

Before describing the organization of this paper, we need to think the current situation of the study of information theoretic security. Although the information theoretic security has information theoretic formulation, it has been mainly studied by the community of cryptography not by information theory community. Further, many important papers [16], [27], [63], [65], [56], [17], [66], [10], [14] in this direction were written with the quantum terminology. Since the information theoretic security even with the non-quantum setting has a sufficient significance from the practical viewpoint and its formulation has a sufficient

[^1]TABLE I
SUMMARY OF OBTAINED RESULTS.

| setting | single-shot /asymptotic | $L_{1}$ | MMI |
| :---: | :---: | :---: | :---: |
| exponent (Rényi 2) | single-shot | (53) in Theorem 15 | (55) in Theorem 15$]$ <br> 78$)$ in Theorem 23 |
|  | asymptotic | 86) in Theorem 26 | (87) in Theorem 26 |
| exponent (min) | single-shot | 66) in Theorem 18 | (67) in Theorem 18 |
|  | asymptotic | (95) in Theorem 28 | 96) in Theorem 28 |
| second order (min) | single-shot | (66) in Theorem 18 | ${ }_{-}^{-}$ |
|  | asymptotic | 86) in Theorem 25 | - |
| equivocation (min) | single-shot | $\underbrace{-}_{-}$ | (101) in Theorem 30 |
|  | asymptotic | - | 107) in Theorem 32 |

$L_{1}$ is the $L_{1}$ distinguishability criterion. MMI is the modified mutual information criterion. (min) means the result derived by the smoothing of min entropy. (Rényi 2) means the results derived by the smoothing of Rényi entropy of order 2.
similarity to information theory, it should be studied from information theory more actively. Indeed, this paper deals with a non-quantum topic. So, non-quantum researchers should be contained in the reader of this paper. However, the above mentioned situation obstructs the non-quantum researchers to access the papers in the information theoretic security even with the non-quantum setting. To resolve this situation, this paper needs to contain surveys of results originally obtained in quantum information, which should be written in the non-quantum terminology.

## E. Organization

The remaining part of this paper is organized as follows. Now, we give the outline of the preliminary parts. In Section II, we prepare the information quantities for evaluating the security and derive several useful inequalities for the quantum case. We also give a clear definition for security criteria. The contents in Section $\Pi$ except for Lemma 7 and Theorem 8 are known. However, since they are given in quantum terminology, these contents are not familiar for people in information theory. For readers in information theory, their proofs are given in Appendixes.

In Section III, we introduce several class of hash functions (universal ${ }_{2}$ hash functions and $\varepsilon$-almost dual universal ${ }_{2}$ hash functions). We clarify the relation between $\varepsilon$-almost dual universal ${ }_{2}$ hash functions and $\delta$-biased ensemble. We also derive an $\varepsilon$-almost dual universal ${ }_{2}$ version of two universal hashing lemma based on Lemma for $\delta$-biased ensemble given by Dodis et al [9]. The latter preliminary parts are more technical and used for proofs of the main results. Although the contents are given the previous paper [14] with terminologies in quantum information, since they are necessary for the latter discussion, they are presented in this paper with non-quantum terminologies.

In Section IV] under the $\varepsilon$-almost dual universal ${ }_{2}$ condition, we evaluate the $L_{1}$ distinguishability criterion and the modified mutual information based on the smoothing of min entropy and Rényi entropy of order 2. These parts give the definitions for concepts and quantities describing the main results. These parts are almost included in the papers [14], [56]. So, the larger part of Sections IIII and IV are surveys with non-quantum terminology.

Next, we outline the main results. In Section $\bar{V}$, using the tail probability of a proper event, we evaluate upper bounds given by the smoothing of min entropy in Section IV with the single-shot setting. This tail probability plays a central role in information spectrum. The bounds obtained in this section have smaller complexity for calculation than those given in Section IV] In Section VI, using the information quantities given in Section II we evaluate upper bounds given in Section IV. The bounds obtained in this section have smaller complexity for calculation than those given in Sections $\mathbf{V}$ and IV] In Section VII, we derive an exponential decreasing rate for both criteria when we simply apply hash functions. In Section VIII we also discuss the case when the key generation rate is greater than the conditional entropy rate.

## II. Preparation

## A. Rényi relative entropy

In order to discuss the security problem, we prepare several information quantities for sub-distributions $P_{A} Q_{A}$ on a space $\mathcal{A}$. That is, these are assumed to satisfy the conditions $P_{A}(a) \geq 0$ and $\sum_{a} P_{A}(a) \leq 1$. Rényi introduced Rényi relative entropy

$$
\begin{equation*}
D_{1+s}\left(P_{A} \| Q_{A}\right):=\frac{1}{s} \log \sum_{a \in \mathcal{A}} P_{A}(a)^{1+s} Q_{A}(a)^{-s} \tag{1}
\end{equation*}
$$

as a generalization of relative entropy

$$
\begin{equation*}
D\left(P_{A} \| Q_{A}\right):=\sum_{a \in \mathcal{A}} P_{A}(a) \log \frac{P_{A}(a)}{Q_{A}(a)} \tag{2}
\end{equation*}
$$

When we apply a stochastic matrix $\Lambda$ on $\mathcal{A}$, the information processing inequalities

$$
\begin{equation*}
D\left(\Lambda\left(P_{A}\right) \| \Lambda\left(Q_{A}\right)\right) \leq D\left(P_{A} \| Q_{A}\right), \quad D_{1+s}\left(\Lambda\left(P_{A}\right) \| \Lambda\left(Q_{A}\right)\right) \leq D_{1+s}\left(P_{A} \| Q_{A}\right) \tag{3}
\end{equation*}
$$

hold for $s \in(0,1]$. Since the map $s \mapsto s D_{1+s}\left(P_{A} \| Q_{A}\right)$ is convex, we have the following lemma.
Lemma 1: $D_{1+s}\left(P_{A} \| Q_{A}\right)$ is monotonically increasing for $s$ in $(-\infty, 0) \cup(0, \infty)$.
When $P_{A}$ and $Q_{A}$ are normalized distributions, we have $\left.s D_{1+s}\left(P_{A} \| Q_{A}\right)\right|_{s=0}=0$. Hence, the concavity of $s \mapsto s D_{1+s}\left(P_{A} \| Q_{A}\right)$ implies $\lim _{s \rightarrow 0} D_{1+s}\left(P_{A} \| Q_{A}\right)=D\left(P_{A} \| Q_{A}\right)$. Then, Lemma yields the following lemma.

Lemma 2: When $P_{A}$ and $Q_{A}$ are normalized distributions,

$$
\begin{equation*}
D_{1-s}\left(P_{A} \| Q_{A}\right) \leq D\left(P_{A} \| Q_{A}\right) \leq D_{1+s}\left(P_{A} \| Q_{A}\right) \tag{4}
\end{equation*}
$$

for $s>0$.

## B. Conditional Rényi entropy

1) Case of joint sub-distribution: Next, we prepare the conditional Rényi entropy for a joint sub-distribution $P_{A, E}$ on subsets $\mathcal{A}$ and $\mathcal{E}$. In the following discussion, the sub-distribution $P_{A}$ and $P_{A, E}$ is not necessarily normalized, and is assumed to satisfy the condition $\sum_{a} P_{A}(a) \leq 1$ or $\sum_{a, e} P_{A, E}(a, e) \leq 1$. For the sub-distributions $P_{A}$ and $P_{A, E}$, we define the normalized distributions $P_{A, \text { normal }}$ and $P_{A, E, \text { normal }}$ by $P_{A, \text { normal }}(a):=P_{A}(a) / \sum_{a} P_{A}(a)$ and $P_{A, E, \text { normal }}(a, e):=P_{A, E}(a, e) / \sum_{a, e} P_{A, E}(a, e)$. For a sub-distribution $P_{A, E}$, we define the marginal sub-distribution $P_{A}$ on $\mathcal{A}$ by $P_{A}(a):=\sum_{e \in \mathcal{E}} P_{A, E}(a, e)$. Then, we define the conditional sub-distribution $P_{A \mid E}$ on $\mathcal{A}$ by $P_{A \mid E}(a \mid e):=P_{A, E}(a, e) / P_{E, \text { normal }}(e)$. The conditional entropy is given as

$$
H\left(A|E| P_{A, E}\right):=H\left(A, E \mid P_{A, E}\right)-H\left(E \mid P_{E, \text { normal }}\right) .
$$

When we replace $P_{E, \text { normal }}$ by another normalized distribution $Q_{E}$ on $\mathcal{E}$, we can generalize the above quantities.

$$
\begin{align*}
H\left(A|E| P_{A, E} \| Q_{E}\right) & :=\log |\mathcal{A}|-D\left(P_{A, E} \| P_{\text {mix }, \mathcal{A}} \times Q_{E}\right) \\
& =-\sum_{a, e} P_{A, E}(a, e) \log \frac{P_{A, E}(a, e)}{Q_{E}(e)} \\
& =H\left(A|E| P_{A, E}\right)+D\left(P_{E} \| Q_{E}\right) \\
& \geq H\left(A|E| P_{A, E}\right), \tag{5}
\end{align*}
$$

where $P_{\text {mix }, \mathcal{A}}$ is the uniform distribution on the set that the random variable $A$ takes values in. By using the Rényi relative entropy, the conditional Rényi entropies and the conditional min entropy are given in the way relative to $Q_{E}$ as

$$
\begin{align*}
H_{1+s}\left(A|E| P_{A, E} \| Q_{E}\right): & =\log |\mathcal{A}|-D_{1+s}\left(P_{A, E} \| P_{\mathrm{mix}, \mathcal{A}} \times Q_{E}\right) \\
& =\frac{-1}{s} \log \sum_{a, e} P_{A, E}(a, e)^{1+s} Q_{E}(e)^{-s}, \\
H_{\min }\left(A|E| P_{A, E} \| Q_{E}\right): & =-\log \max _{(a, e): Q_{E}(e)>0} \frac{P_{A, E}(a, e)}{Q_{E}(e)} . \tag{6}
\end{align*}
$$

Applying Lemma we obtain the following lemma.
Lemma 3: The quantity $H_{1+s}\left(A|E| P_{A, E} \| Q_{E}\right)$ is monotonically decreasing for $s$ in $(-\infty, 0) \cup(0, \infty)$.
Since $\sum_{e} P_{E, \text { normal }}(e) \sum_{a} P_{A \mid E}(a \mid e) P_{A, E}(a, e)^{s} Q_{E}(e)^{-s} \leq \max _{a, e: P_{E}(e)>0} P_{A, E}(a, e)^{s} Q_{E}(e)^{-s}$ for $s>0$, we have

$$
\begin{equation*}
H_{1+s}\left(A|E| P_{A, E} \| Q_{E}\right) \geq H_{\min }\left(A|E| P_{A, E} \| Q_{E}\right) . \tag{7}
\end{equation*}
$$

Taking the limit, we obtain the equality

$$
\begin{equation*}
\lim _{s \rightarrow+\infty} H_{1+s}\left(A|E| P_{A, E} \| Q_{E}\right)=H_{\min }\left(A|E| P_{A, E} \| Q_{E}\right) . \tag{8}
\end{equation*}
$$

Due to (3), when we apply an operation $\Lambda$ on $\mathcal{E}$, it does not act on the system $\mathcal{A}$. Then,

$$
\begin{align*}
H\left(A|E| \Lambda\left(P_{A, E}\right) \| \Lambda\left(Q_{E}\right)\right) & \geq H\left(A|E| P_{A, E} \| Q_{E}\right)  \tag{9}\\
H_{1+s}\left(A|E| \Lambda\left(P_{A, E}\right) \| \Lambda\left(Q_{E}\right)\right) & \geq H_{1+s}\left(A|E| P_{A, E} \| Q_{E}\right) . \tag{10}
\end{align*}
$$

In particular, the inequalities

$$
H\left(A|E| \Lambda\left(P_{A, E}\right)\right) \geq H\left(A|E| P_{A, E}\right)
$$

hold. Conversely, when we apply the function $f$ to the random number $a \in \mathcal{A}$, we have

$$
\begin{equation*}
H\left(f(A)|E| P_{A, E}\right) \leq H\left(A|E| P_{A, E}\right) . \tag{11}
\end{equation*}
$$

Now, we introduce two kinds of conditional Rényi entropies by specifying $Q_{E}$. The first type is defined by substituting $P_{E, \text { normal }}$ into $Q_{E}$ as follows

$$
\begin{aligned}
H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right) & :=H_{1+s}\left(A|E| P_{A, E} \| P_{E, \text { normal }}\right) \\
& =\frac{-1}{s} \log \sum_{e} P_{E, \text { normal }}(e) \sum_{a} P_{A \mid E}(a \mid e)^{1+s} \\
H_{\min }^{\downarrow}\left(A|E| P_{A, E}\right) & :=H_{\min }\left(A|E| P_{A, E} \| P_{E, \text { normal }}\right) \\
& =-\log \underset{(a, e): P_{E, \text { normal }}(e)>0}{\max } P_{A \mid E}(a \mid e)
\end{aligned}
$$

with $s \in \mathbb{R} \backslash\{0\}$. Then, as a special case of (10), we have

$$
\begin{equation*}
H_{1+s}^{\downarrow}\left(A|E| \Lambda\left(P_{A, E}\right)\right) \geq H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right) \tag{12}
\end{equation*}
$$

The second type is defined as

$$
\begin{equation*}
H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right):=\max _{Q_{E}} H_{1+s}\left(A|E| P_{A, E} \| Q_{E}\right) \tag{13}
\end{equation*}
$$

This quantity has another expression as follows.
Lemma 4: A joint sub-distribution $P_{A, E}$ satisfies the relation

$$
\begin{equation*}
H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)=-\frac{1+s}{s} \log \sum_{e}\left(\sum_{a} P_{A, E}(a, e)^{1+s}\right)^{\frac{1}{1+s}} \tag{14}
\end{equation*}
$$

for $s \in[-1, \infty) \backslash\{0\}$. The maximum in (13) can be realized when $Q_{E}(e)=\left(\sum_{a} P_{A, E}(a, e)^{1+s}\right)^{1 /(1+s)} / \sum_{e}\left(\sum_{a} P_{A, E}(a, e)^{1+s}\right)^{1 /(1+s)}$. For reader's convenience, the proof of Lemma 4 is given in Appendix A In information theory, we often employ Gallager-type [8] function [12]:

$$
\begin{aligned}
\phi\left(s|A| E \mid P_{A, E}\right) & :=\log \sum_{e}\left(\sum_{a} P_{A, E}(a, e)^{1 /(1-s)}\right)^{1-s} \\
& =\log \sum_{e} P_{E}(e)\left(\sum_{a} P_{A \mid E}(a \mid e)^{1 /(1-s)}\right)^{1-s}
\end{aligned}
$$

The quantity $H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ can be expressed as

$$
H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)=-\frac{1+s}{s} \phi\left(\left.\frac{s}{1+s}|A| E \right\rvert\, P_{A, E}\right)
$$

Although $H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ can be lowerly bounded by $H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)$ due to the definition, we have the opposite inequality as follows.

Lemma 5: For $s \in[-1,1] \backslash\{0\}$, a joint sub-distribution $P_{A, E}$ satisfies the relation

$$
\begin{equation*}
H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right) \geq H_{\frac{1}{1-s}}^{\uparrow}\left(A|E| P_{A, E}\right) \tag{15}
\end{equation*}
$$

The equality holds only when $P_{A \mid E=e}$ is uniform distribution for all $e \in \mathcal{E}$.
Although Lemma 5 can be regarded as a special case of (47) or (48) of $66{ }^{2}$, we give its proof in Appendix B for reader's convenience because the proof in [66] given in quantum terminology.
2) Case of joint normalized distribution: When $P_{A, E}$ is a joint normalized distribution, the additional useful properties hold as follows. In this case, since $\lim _{s \rightarrow 0} s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)=0$, we have

$$
\begin{equation*}
\lim _{s \rightarrow 0} H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)=H\left(A|E| P_{A, E}\right) \tag{16}
\end{equation*}
$$

Hence, we define $H_{1}^{\downarrow}\left(A|E| P_{A, E}\right)$ and $H_{1}^{\uparrow}\left(A|E| P_{A, E}\right)$ to be $H\left(A|E| P_{A, E}\right)$. Further, applying Lemma 2 we obtain the following lemma.

Lemma 6: When $P_{A, E}$ and $Q_{E}$ are normalized distributions,

$$
\begin{equation*}
H_{1-s}\left(A|E| P_{A, E} \| Q_{E}\right) \geq H\left(A|E| P_{A, E} \| Q_{E}\right) \geq H_{1+s}\left(A|E| P_{A, E} \| Q_{E}\right) \tag{18}
\end{equation*}
$$

for $s>0$.
Similar properties hold for $H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ as follows.

[^2]Lemma 7:

$$
\begin{equation*}
\lim _{s \rightarrow 0} H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)=H\left(A|E| P_{A, E}\right) \tag{19}
\end{equation*}
$$

The map $s \mapsto s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ is concave and then the map $s \mapsto H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ is monotonically decreasing for $s \in(-1, \infty)$. In particular, when $P_{A \mid E=e}$ is not a uniform distribution for an element $e \in \mathcal{E}$, the map $s \mapsto s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ is strictly concave and then the map $s \mapsto H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ is strictly monotonically decreasing for $s \in(-1, \infty)$. Lemma 7 will be shown in Appendix C.

Hence, we define $H_{1}^{\uparrow}\left(A|E| P_{A, E}\right)$ to be $H\left(A|E| P_{A, E}\right)$. Then, the relations (19) and (13) hold even with $s=0$.
Remark 1: Iwamoto and Shikata [62] discussed conditional Rényi entropies in the different notations. They denote $H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)$ by $R_{1+s}^{\mathrm{H}}(A \mid E)$ and $H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ by $R_{1+s}^{\mathrm{A}}(A \mid E)$. They also compare these with other conditional Rényi entropies. MullerLennert et al [63] denoted $H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ by $H_{1+s}^{\downarrow}\left(P_{A, E} \mid E\right)$ in the quantum setting. Iwamoto and Shikata [62] pointed out that these quantities do not satisfy the chain rule. Instead, Muller-Lennert et al [63, Proposition 7] showed the inequality $H_{1+s}^{\uparrow}\left(A\left|E, E^{\prime}\right| P_{A, E, E^{\prime}}\right) \geq H_{1+s}^{\uparrow}\left(A, E^{\prime}|E| P_{A, E, E^{\prime}}\right)-\log \left|\mathcal{E}^{\prime}\right|$ for $s \in(-1, \infty)$. Also, the paper [64, Corollary 77] shows the inequality $H_{1+s(1-s)}\left(A|E| P_{A, E, E^{\prime}}\right) \geq H_{1+s}^{\downarrow}\left(A, E \mid P_{A, E, E^{\prime}}\right)-\log |\mathcal{E}|$ for $s \in[0,1)$.

## C. Criteria for secret random numbers

1) Case of joint sub-distribution: Next, we introduce criteria for the amount of the information leaked from the secret random number $A$ to $E$ for joint sub-distribution $P_{A, E}$. Using the $\ell_{1}$ norm, we can evaluate the secrecy for the state $P_{A, E}$ as follows:

$$
\begin{equation*}
d_{1}\left(A|E| P_{A, E}\right):=\left\|P_{A, E}-P_{A} \times P_{E}\right\|_{1} \tag{20}
\end{equation*}
$$

Taking into account the randomness, Renner [16] employed the $L_{1}$ distinguishability criteria for security of the secret random number $A$ :

$$
\begin{equation*}
d_{1}^{\prime}\left(A|E| P_{A, E}\right):=\left\|P_{A, E}-P_{\mathrm{mix}, \mathcal{A}} \times P_{E}\right\|_{1} \tag{21}
\end{equation*}
$$

which can be regarded as the difference between the true sub-distribution $P_{A, E}$ and the ideal sub-distribution $P_{\text {mix }, \mathcal{A}} \times P_{E}$. It is known that the quantity is universally composable [28].

Renner [16] defined the conditional $L_{2}$-distance from uniform of $P_{A, E}$ relative to a distribution $Q_{E}$ on $\mathcal{E}$ :

$$
\begin{aligned}
& d_{2}\left(A|E| P_{A, E} \| Q_{E}\right) \\
:= & \sum_{a, e}\left(P_{A, E}(a, e)-P_{\operatorname{mix}, \mathcal{A}}(a) P_{E}(e)\right)^{2} Q_{E}(e)^{-1} \\
= & \sum_{a, e} P_{A, E}(a, e)^{2} Q_{E}(e)^{-1}-\frac{1}{|\mathcal{A}|} \sum_{e} P_{E}(e)^{2} Q_{E}(e)^{-1} \\
= & e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)}-\frac{1}{|\mathcal{A}|} e^{D_{2}\left(P_{A} \| Q_{E}\right)}
\end{aligned}
$$

Using this value and a normalized distribution $Q_{E}$, we can evaluate $d_{1}^{\prime}\left(A|E| P_{A, E}\right)$ as follows [16, Lemma 5.2.3]:

$$
\begin{equation*}
d_{1}^{\prime}\left(A|E| P_{A, E}\right) \leq \sqrt{|\mathcal{A}|} \sqrt{d_{2}\left(A|E| P_{A, E} \| Q_{E}\right)} \tag{22}
\end{equation*}
$$

2) Case of joint normalized distribution: In the remaining part of this subsection, we assume that $P_{A, E}$ is a normalized distribution. The correlation between $A$ and $E$ can be evaluated by the mutual information

$$
\begin{equation*}
I\left(A: E \mid P_{A, E}\right):=D\left(P_{A, E} \| P_{A} \times P_{E}\right) \tag{23}
\end{equation*}
$$

By using the uniform distribution $P_{\text {mix }, \mathcal{A}}$ on $\mathcal{A}$, Csiszár and Narayan [51] modified the mutual information to

$$
\begin{equation*}
I^{\prime}\left(A|E| P_{A, E}\right):=D\left(P_{A, E} \| P_{\mathrm{mix}, \mathcal{A}} \times P_{E}\right) \tag{24}
\end{equation*}
$$

which is called the modified mutual information [56], [65] and satisfies

$$
\begin{equation*}
I^{\prime}\left(A|E| P_{A, E}\right)=I\left(A: E \mid P_{A, E}\right)+D\left(P_{A} \| P_{\mathrm{mix}, \mathcal{A}}\right) \tag{25}
\end{equation*}
$$

and

$$
\begin{equation*}
H\left(A|E| P_{A, E}\right)=-I^{\prime}\left(A|E| P_{A, E}\right)+\log |\mathcal{A}| \tag{26}
\end{equation*}
$$

Indeed, the quantity $I\left(A: E \mid P_{A, E}\right)$ represents the amount of information leaked by $E$, and the remaining quantity $D\left(P_{A} \| P_{\text {mix }, \mathcal{A}}\right)$ describes the difference of the random number $A$ from the uniform random number. So, if the quantity $I^{\prime}\left(A|E| P_{A, E}\right)$ is small, we can conclude that the random number $A$ has less correlation with $E$ and is close to the uniform random number.

Indeed, it is natural to adopt a quantity expressing the difference between the true distribution and the ideal distribution $P_{\text {mix }, \mathcal{A}} \times P_{E}$ as a security criterion. However, there are several quantities expressing the difference between two distributions. Both $d_{1}^{\prime}(A|E| P)$ and $I^{\prime}(A|E| P)$ are characterized in this way. Here, we show that the modified mutual criterion $I^{\prime}(A|E| P)$ can be derived in a more natural way in the following sense.

It is natural assume the following condition for the security criterion $C(A ; E \mid P)$ as well as the permutation invariance on $\mathcal{A}$ and $\mathcal{E}$.

C1 Chain rule $C(A, B|E| P)=C(B|E| P)+C(A|B, E| P)$.
C2 Linearity When the supports of two marginal distributions $P_{E, 1}$ and $P_{E, 2}$ are disjoint as subsets of $\mathcal{E}, C\left(A|E| \lambda P_{1}+\right.$ $\left.(1-\lambda) P_{2}\right)=\lambda C\left(A|E| P_{1}\right)+(1-\lambda) C\left(A|E| P_{2}\right)$.
C3 Range $\log |\mathcal{A}| \geq C(A|E| P) \geq 0$.
C4 Ideal case $C\left(A|E| P_{\text {mix, } \mathcal{A}} \otimes P_{E}\right)=0$.
C5 Normalization $C\left(A|E||a\rangle\langle a| \otimes P_{E}\right)=\log |\mathcal{A}|$.
Unfortunately, the $L_{1}$ distinguishability does not satisfies $\mathbf{C 1}$ Chain rule. However, we have the following theorem.
Theorem 8: $C(A|E| P)$ satisfies all of the above properties if and only if $C(A|E| P)$ coincides with the modified mutual information criterion $I^{\prime}(A|E| P)=\log |\mathcal{A}|-H(A|E| P)$.
For a proof, see Appendix D. Hence, it is natural to adopt the modified mutual information criterion $I^{\prime}(A|E| P)$ as a security criterion. In particular, if one emphasizes $\mathbf{C} 1$ Chain rule rather than the universal composability, it is better to employ the modified mutual information criterion $I^{\prime}(A|E| P)$.

In particular, if the quantity $I^{\prime}\left(A|E| P_{A, E}\right)$ goes to zero, $d_{1}^{\prime}\left(A|E| P_{A, E}\right)$ also goes to zero as follows. Using Pinsker inequality, we obtain

$$
\begin{align*}
d_{1}\left(A|E| P_{A, E}\right)^{2} & \leq 2 I\left(A|E| P_{A, E}\right)  \tag{27}\\
d_{1}^{\prime}\left(A|E| P_{A, E}\right)^{2} & \leq 2 I^{\prime}\left(A|E| P_{A, E}\right) \tag{28}
\end{align*}
$$

Conversely, we can evaluate $I\left(A: E \mid P_{A, E}\right)$ and $I^{\prime}\left(A|E| P_{A, E}\right)$ by using $d_{1}\left(A|E| P_{A, E}\right)$ and $d_{1}^{\prime}\left(A|E| P_{A, E}\right)$ in the following way. Applying the Fannes inequality, we obtain

$$
\begin{align*}
0 & \leq I\left(A: E \mid P_{A, E}\right)=H\left(A \mid P_{A}\right)+H\left(E \mid P_{E}\right)-H\left(A, E \mid P_{A, E}\right) \\
& =H\left(A, E \mid P_{A} \times P_{E}\right)-H\left(A, E \mid P_{A, E}\right) \\
& =\sum_{a} P_{A}(a) H\left(E \mid P_{E}\right)-H\left(E \mid P_{E \mid A=a}\right) \\
& \leq \sum_{a} P_{A}(a) \eta\left(\left\|P_{E \mid A=a}-P_{E}\right\|_{1}, \log |\mathcal{E}|\right) \\
& =\eta\left(\left\|P_{E, A}-P_{A} \times P_{E}\right\|_{1}, \log |\mathcal{E}|\right) \\
& =\eta\left(d_{1}\left(A|E| P_{A, E}\right), \log |\mathcal{E}|\right) \tag{29}
\end{align*}
$$

where $\eta(x, y):=-x \log x+x y$. Similarly, we obtain

$$
\begin{align*}
& 0 \leq I^{\prime}\left(A|E| P_{A, E}\right) \\
= & H\left(A \mid P_{\text {mix }, \mathcal{A}}\right)+H\left(E \mid P_{E}\right)-H\left(A, E \mid P_{A, E}\right) \\
= & H\left(A, E \mid P_{\text {mix }, \mathcal{A}} \times P_{E}\right)-H\left(A, E \mid P_{A, E}\right) \\
= & \sum_{e} P_{E}(e)\left(H\left(A \mid P_{\text {mix }, \mathcal{A}}\right)-H\left(A \mid P_{A \mid E=e}\right)\right) \\
\leq & \sum_{e} P_{E}(e)\left(\left\|P_{\text {mix }, \mathcal{A}}-H\left(A \mid P_{A \mid E=e}\right)\right\|_{1}, \log |\mathcal{A}|\right) \\
\leq & \eta\left(\left\|P_{\text {mix }, \mathcal{A}} \times P_{E}-P_{A, E}\right\|_{1}, \log |\mathcal{A}|\right) \\
= & \eta\left(d_{1}^{\prime}\left(A|E| P_{A, E}\right), \log |\mathcal{A}|\right) \tag{30}
\end{align*}
$$

## III. Random Hash functions

## A. General random hash functions

In this section, we focus on a random function $f_{\mathbf{X}}$ from $\mathcal{A}$ to $\mathcal{B}$, where $\mathbf{X}$ is a random variable identifying the function $f_{\mathbf{X}}$. In this case, the total information of Eve's system is written as $(E, \mathbf{X})$. Then, by using $P_{f_{\mathbf{X}}(A), E, \mathbf{X}}(b, e, x):=$
$\sum_{a \in f_{\mathbf{X}}^{-1}(b)} P_{A, E}(a, e) P_{\mathbf{X}}(x)$, the $L_{1}$ distinguishability criterion is written as

$$
\begin{align*}
& d_{1}^{\prime}\left(f_{\mathbf{X}}(A)|E, \mathbf{X}| P_{f_{\mathbf{X}}(A), E, \mathbf{X}}\right) \\
= & \left\|P_{f_{\mathbf{X}}(A), E, \mathbf{X}}-P_{\mathrm{mix}, \mathcal{B}} \times P_{E, \mathbf{X}}\right\|_{1} \\
= & \sum_{x} P_{\mathbf{X}}(x)\left\|P_{f_{\mathbf{X}=x}(A), E}-P_{\mathrm{mix}, \mathcal{B}} \times P_{E}\right\|_{1} \\
= & \mathrm{E}_{\mathbf{X}}\left\|P_{f_{\mathbf{X}}(A), E}-P_{\mathrm{mix}, \mathcal{B}} \times P_{E}\right\|_{1} \tag{31}
\end{align*}
$$

Also, the modified mutual information is written as

$$
\begin{align*}
& I^{\prime}\left(f_{\mathbf{X}}(A)|E, \mathbf{X}| P_{f_{\mathbf{X}}(A), E, \mathbf{X}}\right) \\
= & D\left(P_{f_{\mathbf{X}}(A), E, \mathbf{X}} \| P_{\text {mix }, \mathcal{B}} \times P_{E, \mathbf{X}}\right) \\
= & \sum_{x} P_{\mathbf{X}}(x) D\left(P_{f_{\mathbf{X}=x}(A), E, \mathbf{X}} \| P_{\mathrm{mix}, \mathcal{B}} \times P_{E}\right) \\
= & \mathrm{E}_{\mathbf{X}} D\left(P_{f_{\mathbf{X}}(A), E, \mathbf{X}} \| P_{\text {mix }, \mathcal{B}} \times P_{E}\right) \tag{32}
\end{align*}
$$

We say that a random function $f_{\mathbf{X}}$ is $\varepsilon$-almost universal [1], [2], [14], if, for any pair of different inputs $a_{1}, a_{2}$, the collision probability of their outputs is upper bounded as

$$
\begin{equation*}
\operatorname{Pr}\left[f_{\mathbf{X}}\left(a_{1}\right)=f_{\mathbf{X}}\left(a_{2}\right)\right] \leq \frac{\varepsilon}{|\mathcal{B}|} \tag{33}
\end{equation*}
$$

The parameter $\varepsilon$ appearing in (33) is shown to be confined in the region

$$
\begin{equation*}
\varepsilon \geq \frac{|\mathcal{A}|-|\mathcal{B}|}{|\mathcal{A}|-1} \tag{34}
\end{equation*}
$$

and in particular, a random function $f_{\mathbf{X}}$ with $\varepsilon=1$ is simply called a universal ${ }_{2}$ function.
Two important examples of universal ${ }_{2}$ hash function are the Toeplitz matrices (see, e.g., [3]), and multiplications over a finite field (see, e.g., [1], [4]). A modified form of the Toeplitz matrices is also shown to be universal ${ }_{2}$, which is given by a concatenation $(X, I)$ of the Toeplitz matrix $X$ and the identity matrix $I[13]$. The (modified) Toeplitz matrices are particularly useful in practice, because there exists an efficient multiplication algorithm using the fast Fourier transform algorithm with complexity $O(n \log n)$ (see, e.g., [5]).

The following proposition holds for any universal ${ }_{2}$ function.
Proposition 9 (Renner [16] Lemma 5.4.3]): Given any joint sub-distribution $P_{A, E}$ on $\mathcal{A} \times \mathcal{E}$ and any normalized distribution $Q_{E}$ on $\mathcal{E}$, any universal ${ }_{2}$ hash function $f_{\mathbf{X}}$ from $\mathcal{A}$ to $\mathcal{M}:=\{1, \ldots, \mathrm{M}\}$ satisfies

$$
\begin{equation*}
\mathrm{E}_{\mathbf{X}} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E} \| Q_{E}\right) \leq e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)} \tag{35}
\end{equation*}
$$

More precisely, the inequality

$$
\begin{align*}
& \mathrm{E}_{\mathbf{X}} e^{-H_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E} \| Q_{E}\right)} \\
\leq & \left(1-\frac{1}{\mathrm{M}}\right) e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)}+\frac{1}{\mathrm{M}} e^{D_{2}\left(P_{E} \| Q_{E}\right)} \tag{36}
\end{align*}
$$

holds.

## B. Ensemble of linear hash functions

Tsurumaru and Hayashi[14] focus on linear functions over the finite field $\mathbb{F}_{2}$. Now, we treat the case of linear functions over a finite field $\mathbb{F}_{q}$, where $q$ is a power of a prime number $p$. We assume that sets $\mathcal{A}, \mathcal{B}$ are $\mathbb{F}_{q}^{n}, \mathbb{F}_{q}^{m}$ respectively with $n \geq m$, and $f$ are linear functions over $\mathbb{F}_{q}$. Note that, in this case, there is a kernel $C$ corresponding to a given linear function $f$, which is a vector space of the dimension $n-m$ or more. Conversely, when given a vector subspace $C \subset \mathbb{F}_{q}^{n}$ of the dimension $n-m$ or more, we can always construct a linear function

$$
\begin{equation*}
f_{C}: \mathbb{F}_{q}^{n} \rightarrow \mathbb{F}_{q}^{n} / C \cong \mathbb{F}_{q}^{l}, \quad l \leq m \tag{37}
\end{equation*}
$$

That is, we can always identify a linear hash function $f_{C}$ and a code $C$.
When $C_{\mathbf{X}}=\operatorname{Ker} f_{\mathbf{X}}$, the definition of $\varepsilon$-universal ${ }_{2}$ function (33) takes the form

$$
\begin{equation*}
\forall x \in \mathbb{F}_{q}^{n} \backslash\{0\}, \quad \operatorname{Pr}\left[f_{\mathbf{X}}(x)=0\right] \leq q^{-m} \varepsilon \tag{38}
\end{equation*}
$$

which is equivalent with

$$
\begin{equation*}
\forall x \in \mathbb{F}_{q}^{n} \backslash\{0\}, \quad \operatorname{Pr}\left[x \in C_{\mathbf{X}}\right] \leq q^{-m} \varepsilon \tag{39}
\end{equation*}
$$

This shows that the kernel $C_{\mathbf{X}}$ contains sufficient information for determining if a random function $f_{\mathbf{X}}$ is $\varepsilon$-almost universal ${ }_{2}$ or not.

For a given random code $C_{\mathbf{X}}$, we define its minimum (respectively, maximum) dimension as $t_{\mathrm{min}}:=\min _{\mathbf{X}} \operatorname{dim} C_{\mathbf{X}}$ (respectively, $t_{\text {max }}:=\max _{r \in I} \operatorname{dim} C_{\mathbf{X}}$ ). Then, we say that a linear random code $C_{\mathbf{X}}$ of minimum (or maximum) dimension $t$ is an $\varepsilon$-almost universal ${ }_{2}$ code if the following condition is satisfied

$$
\begin{equation*}
\forall x \in \mathbb{F}_{q}^{n} \backslash\{0\}, \quad \operatorname{Pr}\left[x \in C_{\mathbf{X}}\right] \leq q^{t-n} \varepsilon . \tag{40}
\end{equation*}
$$

In particular, if $\varepsilon=1$, we call $C_{\mathbf{X}}$ a universal ${ }_{2}$ code.

## C. Dual universality of a random code

Based on Tsurumaru and Hayashi[14], we define several variations of the universality of a error-correcting random code and the linear function as follows. First, we define the dual random code $C_{\mathbf{X}}^{\perp}$ of a given linear random code $C_{\mathbf{X}}$ as the dual code of $C_{\mathbf{X}}$. We also introduce the notion of dual universality as follows. We say that a random code $C_{\mathbf{X}}$ in $\mathbb{F}_{q}^{n}$ is $\varepsilon$-almost dual universal ${ }_{2}$ with minimum dimension $t$ (with maximum dimension $t$ ), if the dual random code $C_{\mathbf{X}}^{\perp}$ is $\varepsilon$-almost universal ${ }_{2}$ with maximum dimension $n-t$ (with minimum dimension $n-t$ ). Hence, we say that a linear random function $f_{\mathbf{X}}$ from $\mathbb{F}_{q}^{n}$ to $\mathbb{F}_{q}^{m}$ is $\varepsilon$-almost dual universal ${ }_{2}$, if the kernels $C_{\mathbf{X}}$ of $f_{\mathbf{X}}$ forms an $\varepsilon$-almost dual universal ${ }_{2}$ code with minimum dimension $n-m$. This condition is equivalent with the condition that the linear space spanned by the generating matrix of $f_{\mathbf{X}}$ forms an $\varepsilon$-almost universal ${ }_{2}$ random code with maximum dimension $m$. An explicit example of a dual universal ${ }_{2}$ function (with $\varepsilon=1$ ) can be given by the modified Toeplitz matrix mentioned earlier [11], i.e., a concatenation $(X, I)$ of the Toeplitz matrix $X$ and the identity matrix $I$. The modified Toeplitz matrix requires $n-1$ bits of random seeds $R$. This example is particularly useful in practice because it is both universal ${ }_{2}$ and dual universal ${ }_{2}$, and also because there exists an efficient algorithm with complexity $O(n \log n)$. When the random variable $R$ is not the uniform random number, the modified Toeplitz matrix is $q^{n-1} e^{-H_{\min }^{\downarrow}(R)}$-almost dual universal ${ }_{2}$, as shown in [33]. Therefore, we can evaluate the security of the modified Toeplitz matrix even with non-uniform random seeds. With these preliminaries, we present the following propositions in [14] with non-quantum terminologies and a general prime power $q$ :

Proposition 10 ( $\left[14\right.$ Corollary 2]): An $\varepsilon$-almost universal ${ }_{2}$ surjective liner random hash function $f_{\mathbf{X}}$ from $\mathbb{F}_{q}^{n}$ to $\mathbb{F}_{q}^{m}$ is $q\left(1-q^{m} \varepsilon\right)+(\varepsilon-1) q^{n-m}$-almost dual universal ${ }_{2}$ liner random hash function.
As a special case, we obtain the following.
Corollary 11: Any universal ${ }_{2}$ linear random function $f_{\mathbf{X}}$ over a finite filed $\mathbb{F}_{q}$ is a $q$-almost dual universal ${ }_{2}$ function.
Proposition 12 ( $\left[14\right.$ Lemma 3]): Given a joint sub-distribution $P_{A, E}$ on $\mathcal{A} \times \mathcal{E}$ and a normalized distribution $Q_{E}$ on $\mathcal{E}$. When $C_{\mathbf{X}}$ is an $\varepsilon$-almost dual universal ${ }_{2}$ code with minimum dimension $t$, the random hash function $f_{C_{\mathbf{X}}}$ satisfies

$$
\begin{equation*}
\mathrm{E}_{\mathbf{X}} d_{2}\left(f_{C_{\mathbf{X}}}(A)|E| P_{A, E} \| Q_{E}\right) \leq \varepsilon e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)} \tag{41}
\end{equation*}
$$

More precisely,

$$
\begin{align*}
& \mathrm{E}_{\mathbf{X}} e^{-H_{2}\left(f_{C_{\mathbf{X}}}(A)|E| P_{A, E} \| Q_{E}\right)} \\
\leq & \varepsilon e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)}+\frac{1}{q^{n-t}} e^{D_{2}\left(P_{E} \| Q_{E}\right)} . \tag{42}
\end{align*}
$$

In other words, an $\varepsilon$-almost dual universal ${ }_{2}$ function $f_{\mathrm{X}}$ from $\mathbb{F}_{2}^{n}$ to $\mathbb{F}_{2}^{n-t}$ satisfies (41) and (42).
Since Proposition 12 plays an central role instead of Proposition 9 in this paper and the proof in the previous paper [14] is given with quantum terminologies and the special case $q=2$, we give its proof in Appendix $\mathbb{E}$ without use of quantum terminologies for reader's convenience.

## IV. SECURITY Bounds with Rényi entropy of order 2 and min entropy

Firstly, we consider the secure key generation problem from a common random number $A \in \mathcal{A}$ which has been partially eavesdropped as an information by Eve. For this problem, it is assumed that Alice and Bob share a common random number $A \in \mathcal{A}$, and Eve has a random number $E$ correlated with the random number $A$, whose distribution is $P_{E}$. The task is to extract a common random number $f(A)$ from the random number $A \in \mathcal{A}$, which is almost independent of Eve's quantum state. Here, Alice and Bob are only allowed to apply the same function $f$ to the common random number $A \in \mathcal{A}$. Now, we focus on the random function $f_{\mathbf{X}}$ from $\mathcal{A}$ to $\mathcal{M}=\{1, \ldots, \mathrm{M}\}$, where $\mathbf{X}$ denotes a random variable describing the stochastic behavior of the function $f_{\mathbf{X}}$.

Renner [16, Lemma 5.2.3] essentially evaluated $\mathrm{E}_{\mathbf{X}} d_{1}^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right)$ by using $\mathrm{E}_{\mathbf{X}} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E} \| Q_{E}\right)$ as follows.
Lemma 13: When a state $Q_{E}$ is a normalized distribution on $\mathcal{E}$, any random hash function $f_{\mathbf{X}}$ from $\mathcal{A}$ to $\{1, \ldots, \mathrm{M}\}$ satisfies

$$
\begin{aligned}
& \mathrm{E}_{\mathbf{X}} d_{1}^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \\
\leq & \mathrm{M}^{\frac{1}{2}} \sqrt{\mathrm{E}_{\mathbf{X}} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E} \| Q_{E}\right)}
\end{aligned}
$$

Further, the inequalities used in proof of Renner[16, Corollary 5.6.1] imply that

$$
\begin{aligned}
& \mathrm{E}_{\mathbf{X}} d_{1}^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \\
\leq & 2\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}+\mathrm{E}_{\mathbf{X}} d_{1}^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime}\right) \\
\leq & 2\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}+\mathrm{M}^{\frac{1}{2}} \sqrt{\mathrm{E}_{\mathbf{X}} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| Q_{E}\right)} .
\end{aligned}
$$

Applying the same discussion to Shannon entropy, we can evaluate the average of the modified mutual information criterion by using $\mathrm{E}_{\mathbf{X}} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E} \| Q_{E}\right)$ as follows.

Lemma 14: Assume that $P_{A, E}$ is a normalized distribution on $\mathcal{A} \times \mathcal{E}$. Any random hash function $f_{\mathbf{X}}$ from $\mathcal{A}$ to $\mathcal{M}=$ $\{1, \ldots, M\}$ satisfies

$$
\begin{align*}
& \mathrm{E}_{\mathbf{X}} I^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \\
\leq & \log \left(1+\mathrm{ME}_{\mathbf{X}} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right)\right)  \tag{43}\\
\leq & \mathrm{ME}_{\mathbf{X}} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E} \| P_{E}\right) \tag{44}
\end{align*}
$$

Further, when a sub-distribution $P_{A, E}^{\prime}$ satisfies $P_{E}^{\prime}(e) \leq P_{E}(e)$ for any $e \in \mathcal{E}$ (we simplify this condition to $P^{\prime}{ }_{E} \leq P_{E}$ ), we obtain

$$
\begin{align*}
& \mathrm{E}_{\mathbf{X}} I^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \\
& \leq \eta\left(\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}, \log \mathrm{M}\right) \\
& \quad+\log \left(1+\mathrm{ME}_{\mathbf{X}} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| P_{E}\right)\right)  \tag{45}\\
& \leq \eta\left(\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}, \log \mathrm{M}\right) \\
&+\mathrm{ME}_{\mathbf{X}} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| P_{E}\right) \tag{46}
\end{align*}
$$

where $\eta(x, y):=x y-x \log x$.
Proof: The inequality $D_{2}\left(P^{\prime}{ }_{E} \| P_{E}\right) \leq 0$ holds due to the condition $P^{\prime}{ }_{E}(e) \leq P_{E}(e)$. Since

$$
\begin{align*}
& d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| P_{E}\right) \\
= & e^{-H_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| P_{E}\right)}-\frac{1}{\mathrm{M}} e^{D_{2}\left(P_{E}^{\prime} \| P_{E}\right)} \\
\geq & e^{-H_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| P_{E}\right)}-\frac{1}{\mathrm{M}} \tag{47}
\end{align*}
$$

we have

$$
e^{-H_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| P_{E}\right)} \leq d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| P_{E}\right)+\frac{1}{\mathrm{M}}
$$

Taking the logarithm, we obtain

$$
\begin{align*}
& -\log \mathrm{M}+\log \left(1+\mathrm{M} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| P_{E}\right)\right) \\
\geq & -H_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| P_{E}\right) \geq-H\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| P_{E}\right) \tag{48}
\end{align*}
$$

Substituting $P_{A, E}$ to $P_{A, E}^{\prime}$, we obtain $H\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| P_{E}\right)=H\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right)$ and

$$
\begin{aligned}
& \quad I^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right)=\log \mathrm{M}-H\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \\
& \leq \\
& \leq \log \left(1+\mathrm{M} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right)\right)
\end{aligned}
$$

Since the function $x \mapsto \log (1+x)$ is concave, we obtain

$$
\begin{aligned}
& \mathrm{E}_{\mathbf{X}} I^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \\
\leq & \log \left(1+\mathrm{ME}_{\mathbf{X}} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right)\right)
\end{aligned}
$$

which implies (43). The inequality $\log (1+x) \leq x$ and (43) yield (44).
Due to Fannes inequality, the normalized distribution $P_{A \mid E=e}(a):=\frac{P_{A, E}(a, e)}{P_{E}(e)}$ and the sub-distribution $P^{\prime} A \mid E=e(a):=$ $\frac{P_{A, E}^{\prime}(a, e)}{P_{E}(e)}$ satisfy

$$
\begin{align*}
& \left|H\left(f_{\mathbf{X}}(A) \mid P_{A \mid E=e}\right)-H\left(f_{\mathbf{X}}(A) \mid P_{A \mid E=e}^{\prime}\right)\right| \\
\leq & \eta\left(\left\|P_{A \mid E=e}-P_{A \mid E=e}^{\prime}\right\|_{1}, \log \mathrm{M}\right) . \tag{49}
\end{align*}
$$

Since $\sum_{e} P_{E}(e)\left\|P_{A \mid E=e}-P^{\prime}{ }_{A \mid E=e}\right\|_{1}=\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}$, taking the average under the distribution $P_{E}$, we obtain

$$
\begin{align*}
& \left|H\left(f_{\mathbf{X}}(A)|E| P_{A, E} \mid P_{E}\right)-H\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \mid P_{E}\right)\right| \\
= & \left|\sum_{e} P_{E}(e)\left(H\left(f_{\mathbf{X}}(A) \mid P_{A \mid E=e}\right)-H\left(f_{\mathbf{X}}(A) \mid P_{A \mid E=e}^{\prime}\right)\right)\right| \\
\leq & \sum_{e} P_{E}(e)\left|H\left(f_{\mathbf{X}}(A) \mid P_{A \mid E=e}\right)-H\left(f_{\mathbf{X}}(A) \mid P^{\prime}{ }_{A \mid E=e}\right)\right| \\
\leq & \sum_{e} P_{E}(e) \eta\left(\left\|P_{A \mid E=e}-P^{\prime}{ }_{A \mid E=e}\right\|_{1}, \log \mathrm{M}\right) \\
\leq & \eta\left(\sum_{e} P_{E}(e)\left\|P_{A \mid E=e}-P_{A \mid E=e}^{\prime}\right\|_{1}, \log \mathrm{M}\right) \\
= & \eta\left(\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}, \log \mathrm{M}\right) . \tag{50}
\end{align*}
$$

Therefore, using (50) and (48), we obtain

$$
\begin{aligned}
& I^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \\
= & \log \mathrm{M}-H\left(f_{\mathbf{X}}(A)|E| P_{A, E} \mid P_{E}\right) \\
\leq & \eta\left(\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}, \log \mathrm{M}\right) \\
& +\log \mathrm{M}-H\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \mid P_{E}\right) \\
\leq & \eta\left(\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}, \log \mathrm{M}\right) \\
& +\log \left(1+\mathrm{M} d_{2}\left(f_{\mathbf{X}}(A)|E| P_{A, E}^{\prime} \| P_{E}\right)\right)
\end{aligned}
$$

Taking the expectation of $\mathbf{X}$ and using the concavity of functions $x \mapsto \eta(x, \log \mathrm{M})$ and $x \mapsto \log (1+x)$, we obtain (45). The inequality $\log (1+x) \leq x$ yields (46). In this proof, the condition $P_{E}(e)^{\prime} \leq P_{E}(e)$ is crucial because Inequality 47) cannot be shown without this condition.

Now, we evaluate the security by combining Proposition 12 and Lemmas 13 and 14 For this purpose, we introduce the quantities:

$$
\begin{aligned}
\Delta_{d, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) & :=\min _{Q_{E}} \min _{P_{A, E}^{\prime}} 2\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}+\sqrt{\varepsilon} \mathrm{M}^{\frac{1}{2}} e^{-\frac{1}{2} H_{2}\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right)} \\
& =\min _{Q_{E}} \min _{\epsilon_{1}>0} 2 \epsilon_{1}+\sqrt{\varepsilon} \mathrm{M}^{\frac{1}{2}} e^{-\frac{1}{2} H_{2}^{\epsilon_{1}}\left(A|E| P_{A, E} \| Q_{E}\right)} \\
& =\min _{Q_{E}} \min _{R} 2 \min _{P_{A, E}^{\prime}: H_{2}\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right) \geq R}\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}+\sqrt{\varepsilon} \mathrm{M}^{\frac{1}{2}} e^{-\frac{1}{2} R}, \\
\Delta_{I, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) & :=\min _{P_{A, E}^{\prime}: P_{E}^{\prime} \leq P_{E}} \eta\left(\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}, \log \mathrm{M}\right)+\varepsilon \mathrm{M} e^{-H_{2}\left(A|E| P_{A, E}^{\prime} \| P_{E}\right)} \\
& =\min _{\epsilon_{1}>0} \eta\left(\epsilon_{1}, \log \mathrm{M}\right)+\varepsilon \mathrm{M} e^{-H_{2}^{\perp, \epsilon_{1}}\left(A|E| P_{A, E}\right)} \\
& =\min _{R} \eta\left({ }_{P_{A, E}^{\prime}: P_{E}^{\prime} \leq P_{E}, H_{2}\left(A|E| P_{A, E}^{\prime} \| P_{E}\right) \geq R}\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}, \log \mathrm{M}\right)+\varepsilon \mathrm{M} e^{-R},
\end{aligned}
$$

where

$$
\begin{align*}
H_{2}^{\downarrow, \epsilon_{1}}\left(A|E| P_{A, E} \| Q_{E}\right) & :=\max _{P_{A, E}^{\prime}:\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1} \leq \epsilon_{1}} H_{2}\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right)  \tag{51}\\
H_{2}^{\epsilon_{1}}\left(A|E| P_{A, E}\right) & :=\begin{array}{l}
P_{A, E}^{\prime}:\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1} \leq \epsilon_{1}, P_{E}^{\prime} \leq P_{E}
\end{array} H_{2}\left(A|E| P_{A, E}^{\prime} \| P_{E}\right) . \tag{52}
\end{align*}
$$

Note that $H_{2}^{\downarrow, \epsilon_{1}}\left(A|E| P_{A, E}\right)$ is different from $H_{2}^{\epsilon_{1}}\left(A|E| P_{A, E} \| P_{E}\right)$ because the definition of $H_{2}^{\downarrow, \epsilon_{1}}\left(A|E| P_{A, E}\right)$ has additional constraints for $P_{A, E}^{\prime}$. Then, we can evaluate the averages of both security criteria under the $\varepsilon$-almost dual universal ${ }_{2}$ condition.

Theorem 15: Assume that $Q_{E}$ is a normalized distribution on $\mathcal{E}, P_{A, E}$ is a sub-distribution on $\mathcal{A} \times \mathcal{E}$, and a linear random hash function $f_{\mathbf{X}}$ from $\mathcal{A}$ to $\mathcal{M}=\{1, \ldots, \mathrm{M}\}$ is $\varepsilon$-almost dual universal ${ }_{2}$. Then, the random hash function $f_{\mathbf{X}}$ satisfies

$$
\begin{align*}
& \mathrm{E}_{\mathbf{X}} d_{1}^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \\
& \leq \sqrt{\varepsilon} \mathrm{M}^{\frac{1}{2}} e^{-\frac{1}{2} H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)}, \\
& \mathrm{E}_{\mathbf{X}} d_{1}^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \\
& \leq \Delta_{d, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) . \tag{53}
\end{align*}
$$

When $P_{A, E}$ is a normalized joint distribution, it satisfies

$$
\begin{align*}
& \mathrm{E}_{\mathbf{X}} I^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \leq \log \left(1+\varepsilon \mathrm{M} e^{-H_{2}^{\downarrow}\left(A|E| P_{A, E}\right)}\right) \leq \varepsilon \mathrm{M} e^{-H_{2}^{\perp}\left(A|E| P_{A, E}\right)}  \tag{54}\\
& \mathrm{E}_{\mathbf{X}} I^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \leq \Delta_{I, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) \tag{55}
\end{align*}
$$

While the same evaluations for the $L_{1}$ distinguishability criterion under the universal ${ }_{2}$ condition has been shown in Renner 16, Corollary 5.6.1], those for the modified mutual information criterion have not been shown even under the universal ${ }_{2}$ condition. All of the above evaluations under the $\varepsilon$-almost dual universal ${ }_{2}$ condition have not been discussed in Renner.

Since the function $x \mapsto \eta(x, y)$ is concave, combing Inequality (30), we obtain the following corollary.
Corollary 16: When a linear random hash function $f_{\mathbf{X}}$ from $\mathcal{A}$ to $\mathcal{M}=\{1, \ldots, M\}$ is $\varepsilon$-almost dual universal ${ }_{2}$, any joint sub-distribution $P_{A, E}$ on $\mathcal{A}$ and $\mathcal{E}$ satisfies

$$
\begin{equation*}
\mathrm{E}_{\mathbf{X}} I^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \leq \eta\left(\Delta_{d, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right), \log |\mathcal{A}|\right) \tag{56}
\end{equation*}
$$

for $s \in(0,1 / 2]$.
Since the function $x \mapsto \sqrt{x}$ is concave, combing Inequality (28), we obtain the following corollary.
Corollary 17: When a linear random hash function $f_{\mathbf{X}}$ from $\mathcal{A}$ to $\mathcal{M}=\{1, \ldots, \mathrm{M}\}$ is $\varepsilon$-almost dual universal ${ }_{2}$, any joint normalized distribution $P_{A, E}$ on $\mathcal{A} \times \mathcal{E}$ satisfy

$$
\begin{equation*}
\mathrm{E}_{\mathbf{X}} d_{1}^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \leq \sqrt{2 \Delta_{I, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)} \tag{57}
\end{equation*}
$$

for $s \in(0,1 / 2]$.
Further, in the case of the universal ${ }_{2}$ condition, Renner[16, Corollary 5.6.1] proposed to replace $H_{2}\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right)$ by the min entropy $H_{\min }\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right)$ because $H_{2}\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right) \geq H_{\min }\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right)$. Based on $H_{\min }\left(A|E| P \| Q_{E}\right)$, Renner[16] introduced $\epsilon_{1}$-smooth min entropy as

$$
\begin{equation*}
H_{\min }^{\epsilon_{1}}\left(A|E| P_{A, E} \| Q_{E}\right):=\max _{\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1} \leq \epsilon_{1}} H_{\min }\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right) \tag{58}
\end{equation*}
$$

For the evaluation of $\mathrm{E}_{\mathbf{X}} I^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right)$, adding the condition $P^{\prime}{ }_{E} \leq P_{E}$, we define

$$
\begin{equation*}
H_{\min }^{\downarrow, \epsilon_{1}}\left(A|E| P_{A, E}\right):=\max _{\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1} \leq \epsilon_{1}, P^{\prime} E \leq P_{E}} H_{\min }\left(A|E| P_{A, E}^{\prime} \| P_{E}\right) \tag{59}
\end{equation*}
$$

As is shown in Lemma $19 H_{\min }^{\downarrow, \epsilon_{1}}\left(A|E| P_{A, E}\right)$ equals $H_{\min }^{\epsilon_{1}}\left(A|E| P_{A, E} \| P_{E}\right)$ while the former has an additional constraint. Defining the quantities

$$
\begin{align*}
\Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) & :=\min _{Q_{E}} \min _{P_{A, E}^{\prime}} 2\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}+\sqrt{\varepsilon} \mathrm{M}^{\frac{1}{2}} e^{-\frac{1}{2} H_{\min }\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right)}  \tag{60}\\
& =\min _{Q_{E}} \min _{\epsilon_{1}>0} 2 \epsilon_{1}+\sqrt{\varepsilon} \mathrm{M}^{\frac{1}{2}} e^{-\frac{1}{2} H_{\min }^{\epsilon_{1}}\left(A|E| P_{A, E} \| Q_{E}\right)}  \tag{61}\\
& =\min _{Q_{E}} \min _{R} 2{ }_{P_{A, E}^{\prime}: H_{\min }\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right) \geq R} \min \left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}+\sqrt{\varepsilon} \mathrm{M}^{\frac{1}{2}} e^{-\frac{1}{2} R},  \tag{62}\\
\Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) & :=\min _{Q_{E}} \min _{P_{A, E}^{\prime}: P_{E}^{\prime} \leq Q_{E},} \eta\left(\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}, \log \mathrm{M}\right)+\varepsilon \mathrm{M} e^{-H_{\min }\left(A|E| P_{A, E}^{\prime} \| P_{E}\right)}  \tag{63}\\
& =\min _{\epsilon_{1}>0} \eta\left(\epsilon_{1}, \log \mathrm{M}\right)+\varepsilon \mathrm{M} e^{-H_{\min }^{\downarrow, \epsilon_{1}}\left(A|E| P_{A, E}\right)}  \tag{64}\\
& =\min _{R} \eta\left(\min _{P_{A, E}^{\prime}: P_{E}^{\prime} \leq P_{E}, H_{\min }\left(A|E| P_{A, E}^{\prime} \| P_{E}\right) \geq R}\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}, \log \mathrm{M}\right)+\varepsilon \mathrm{M} e^{-R}, \tag{65}
\end{align*}
$$

we obtain the following theorem.
Theorem 18: Assume that $Q_{E}$ is a normalized distribution on $\mathcal{E}, P_{A, E}$ is a sub-distribution on $\mathcal{A} \times \mathcal{E}$, and a linear random hash function $f_{\mathbf{X}}$ from $\mathcal{A}$ to $\mathcal{M}=\{1, \ldots, \mathrm{M}\}$ is $\varepsilon$-almost dual universal ${ }_{2}$. Then, the random hash function $f_{\mathbf{X}}$ satisfies

$$
\begin{align*}
\mathrm{E}_{\mathbf{X}} d_{1}^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) & \leq \Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)  \tag{66}\\
\mathrm{E}_{\mathbf{X}} I^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) & \leq \Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) \tag{67}
\end{align*}
$$

That is, $\Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ and $\Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ are upper bounds for leaked information in the respective criteria when the smoothing of min entropy is applied.

## V. Relation with information spectrum

Information spectrum can derive asymptotically tight bounds of the optimal performances of various information processings by using only the asymptotic behavior of the tail probability, e.g., $P_{A, E}\left\{(a, e) \mid P_{A \mid E}(a \mid e) \geq e^{-R}\right\}$. Hence, it can be applied without any assumption for information sources. While information spectrum originally addresses the asymptotic setting, we bound the performances in the single-shot setting by using the tail probability. We call these upper and lower bounds single-shot information spectrum bounds.

In this section, we clarify the relation between the smoothing of min entropy and single-shot information spectrum bounds. In stead of the smooth min entropy $H_{\min }^{\downarrow, \epsilon_{1}}\left(A|E| P_{A, E}\right)$, we consider the bounds $\Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ and $\Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ as functions of $\min _{P_{A, E}^{\prime}: H_{\min }\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right) \geq R}\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}$ or $\min _{P_{A, E}^{\prime}: P_{E}^{\prime} \leq P_{E}, H_{\min }\left(A|E| P_{A, E}^{\prime} \| P_{E}\right) \geq R}\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}$. That is, we employ the formulas (62) and (65) rather than (61) and (64). Then, we give their relations with the tail probability, e.g., $P_{A, E}\left\{(a, e) \mid P_{A \mid E}(a \mid e) \geq e^{-R}\right\}$ as follows.

Lemma 19:

$$
\begin{align*}
& \min _{P_{A, E}^{\prime}: H_{\min }\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right) \geq R}\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1} \\
= & \min _{P_{A, E}^{\prime}: H_{\min }\left(A|E| P_{A, E}^{\prime} \| Q_{E}\right) \geq R, P_{A, E}^{\prime} \leq P_{A, E}}\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1} \\
= & P_{A, E}\left\{(a, e) \mid P_{A, E}(a, e)>e^{-R} Q_{E}(e)\right\}-e^{-R}|\mathcal{A}| P_{\operatorname{mix}, \mathcal{A}} \times Q_{E}\left\{(a, e) \mid P_{A, E}(a, e)>e^{-R} Q_{E}(e)\right\} . \tag{68}
\end{align*}
$$

and

$$
\begin{align*}
& \left(1-\frac{1}{c}\right) P_{A, E}\left\{(a, e) \mid P_{A, E}(a, e)>c e^{-R} Q_{E}(e)\right\} \\
\leq & P_{A, E}\left\{(a, e) \mid P_{A, E}(a, e)>e^{-R} Q_{E}(e)\right\}-e^{-R}|\mathcal{A}| P_{\operatorname{mix}, \mathcal{A}} \times Q_{E}\left\{(a, e) \mid P_{A, E}(a, e)>e^{-R} Q_{E}(e)\right\} \\
\leq & P_{A, E}\left\{(a, e) \mid P_{A, E}(a, e)>e^{-R} Q_{E}(e)\right\} \tag{69}
\end{align*}
$$

for $c>1$ and $R$.
Since the condition $P_{A, E}^{\prime} \leq P_{A, E}$ is more restrictive than $P_{A}^{\prime} \leq P_{A}$, we see that $H_{\min }^{\downarrow, \epsilon_{1}}\left(A|E| P_{A, E}\right)=H_{\min }^{\epsilon_{1}}\left(A|E| P_{A, E} \| P_{E}\right)$.
Proof: The optimal sub-distribution $P_{A, E}^{\prime}$ in the first line of (68) is given as

$$
P_{A, E}^{\prime}(a, e)= \begin{cases}e^{-R} Q_{E}(e) & \text { if } P_{A, E}(a, e)>e^{-R} Q_{E}(e)  \tag{70}\\ P_{A, E}(a, e) & \text { if } P_{A, E}(a, e) \leq e^{-R} Q_{E}(e)\end{cases}
$$

The sub-distribution is the optimal sub-distribution in the second line of 68). Substituting the above sub-distribution in to the first line, we obtain the third line of 68).

Next, we show (69). Since $c P_{A, E}\left\{(a, e) \mid P_{A, E}(a, e)>c e^{-R} Q_{E}(e)\right\} \geq e^{-R}|\mathcal{A}| P_{\text {mix }, \mathcal{A}} \times Q_{E}\left\{(a, e) \mid P_{A, E}(a, e)>c e^{-R} Q_{E}(e)\right\}$, we have

$$
\begin{align*}
& \left(1-\frac{1}{c}\right) P_{A, E}\left\{(a, e) \mid P_{A, E}(a, e)>c e^{-R} Q_{E}(e)\right\} \\
= & P_{A, E}\left\{(a, e) \mid P_{A, E}(a, e)>c e^{-R} Q_{E}(e)\right\}-c P_{A, E}\left\{(a, e) \mid P_{A, E}(a, e)>c e^{-R} Q_{E}(e)\right\} \\
\leq & P_{A, E}\left\{(a, e) \mid P_{A, E}(a, e)>c e^{-R} Q_{E}(e)\right\}-e^{-R}|\mathcal{A}| P_{\operatorname{mix}, \mathcal{A}} \times Q_{E}\left\{(a, e) \mid P_{A, E}(a, e)>c e^{-R} Q_{E}(e)\right\} \\
\leq & P_{A, E}\left\{(a, e) \mid P_{A, E}(a, e)>e^{-R} Q_{E}(e)\right\}-e^{-R}|\mathcal{A}| P_{\operatorname{mix}, \mathcal{A}} \times Q_{E}\left\{(a, e) \mid P_{A, E}(a, e)>e^{-R} Q_{E}(e)\right\}  \tag{71}\\
\leq & P_{A, E}\left\{(a, e) \mid P_{A, E}(a, e)>e^{-R} Q_{E}(e)\right\},
\end{align*}
$$

where the inequality (71) follows from the fact that the maximum $\max _{\Omega} P_{A, E}(\Omega)-e^{-R}|\mathcal{A}| P_{\text {mix }, \mathcal{A}} \times Q_{E}(\Omega)$ can be realized by the set $\left\{(a, e) \mid P_{A, E}(a, e)>e^{-R} Q_{E}(e)\right\}$.

Therefore, using the formulas (62) and (65), we obtain the following theorem.
Theorem 20: The upper bounds $\Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ and $\Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ of leaked information by the smoothing of min entropy can be evaluated as follows.

$$
\begin{align*}
& 2\left(1-\frac{1}{c}\right) \min _{Q_{E}} \min _{R^{\prime}} P_{A, E}\left\{(a, e) \left\lvert\, \frac{P_{A, E}(a, e)}{Q_{E}(e)}>c e^{-R^{\prime}}\right.\right\}+\sqrt{\varepsilon} \mathrm{M}^{\frac{1}{2}} e^{-\frac{1}{2} R^{\prime}}  \tag{72}\\
\leq & \Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) \leq \min _{Q_{E}} \min _{R^{\prime}} 2 P_{A, E}\left\{(a, e) \left\lvert\, \frac{P_{A, E}(a, e)}{Q_{E}(e)}>e^{-R^{\prime}}\right.\right\}+\sqrt{\varepsilon} \mathrm{M}^{\frac{1}{2}} e^{-\frac{1}{2} R^{\prime}}  \tag{73}\\
& \left(1-\frac{1}{c}\right) \min _{R^{\prime}} \eta\left(P_{A, E}\left\{(a, e) \in \mathcal{A} \times \mathcal{E} \mid P_{A \mid E}(a \mid e) \geq c e^{-R^{\prime}}\right\}, \log \mathrm{M}\right)+\varepsilon \mathrm{M} e^{-R^{\prime}}  \tag{74}\\
\leq & \Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) \leq \min _{R^{\prime}} \eta\left(P_{A, E}\left\{(a, e) \in \mathcal{A} \times \mathcal{E} \mid P_{A \mid E}(a \mid e)>e^{-R^{\prime}}\right\}, \log \mathrm{M}\right)+\varepsilon \mathrm{M} e^{-R^{\prime}} \tag{75}
\end{align*}
$$

for $c>1$.

Theorem 20 explains that the bounds $\Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ and $\Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ by the smoothing of min entropy have almost the same values as the single-shot information spectrum bounds. Using this characterization, we evaluate the bounds $\Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ and $\Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ in the latter sections. However, the bounds by the smoothing of Rényi entropy of order 2 can not be characterized in the same way. This fact seems to indicate the possibility of the smoothing of Rényi entropy of order 2 beyond the smoothing of min entropy.

## VI. Secret key generation: Single-Shot case

In order to obtain useful upper bounds, we need to calculate or evaluate the quantities $\Delta_{d, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)^{1 / 2}, \Delta_{I, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)^{1 / 2}$, $\Delta_{d, \max }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)^{1 / 2}$, and $\Delta_{I, \max }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)^{1 / 2}$. We say that their exact value is the smoothing bound. Using the smoothing bound of Rényi entropy of order 2, the paper [12] derived the following proposition.

Proposition 21: The inequality

$$
\begin{equation*}
\Delta_{d, 2}\left(\mathrm{M}, 1 \mid P_{A, E}\right) \leq 3 \mathrm{M}^{s} e^{-s H_{\frac{1}{1-s}}^{\uparrow}\left(A|E| P_{A, E}\right)} \tag{76}
\end{equation*}
$$

holds for $s \in(0,1 / 2]$.
Using the same smoothing bound, we obtain the following evaluation.
Lemma 22: The inequality

$$
\begin{equation*}
\Delta_{d, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) \leq(2+\sqrt{\varepsilon}) \mathrm{M}^{s} e^{-s H_{\frac{1}{1-s}}^{\uparrow}\left(A|E| P_{A, E}\right)} \tag{77}
\end{equation*}
$$

holds for $s \in(0,1 / 2]$.
Similar to Theorem 15, we obtain an upper bound for $\Delta_{I, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$.
Theorem 23: The inequality

$$
\begin{equation*}
\Delta_{I, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) \leq \eta\left(\mathrm{M}^{s} e^{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)}, \varepsilon+\log \mathrm{M}\right) \tag{78}
\end{equation*}
$$

holds for $s \in(0,1]$.
Proof: For any integer M , we choose the subset $\Omega_{\mathrm{M}}:=\left\{P_{A \mid E}(a \mid e)>\mathrm{M}^{-1}\right\}$, and define the sub-distribution $P_{A, E: \mathrm{M}}$ by

$$
P_{A, E: \mathrm{M}}(a, e):= \begin{cases}0 & \text { if }(a, e) \in \Omega_{\mathrm{M}} \\ P_{A, E}(a, e) & \text { otherwise }\end{cases}
$$

For $0 \leq s \leq 1$, we can evaluate $e^{-H_{2}\left(A|E| P_{A, E: M} \| P_{E}\right)}$ and $d_{1}\left(P_{A, E}, P_{A, E: M}\right)$ as

$$
\begin{align*}
& e^{-H_{2}\left(A|E| P_{A, E: \mathrm{M}} \| P_{E}\right)}=\sum_{(a, e) \in \Omega_{\mathrm{M}}^{c}} P_{A, E}(a, e)^{2}\left(P_{E}(e)\right)^{-1} \\
\leq & \sum_{(a, e) \in \Omega_{\mathrm{M}}^{c}} P_{A, E}(a, e)^{1+s}\left(P_{E}(e)\right)^{-s} \mathrm{M}^{-(1-s)} \\
\leq & \sum_{(a, e)} P_{A, E}(a, e)^{1+s}\left(P_{E}(e)\right)^{-s} \mathrm{M}^{-(1-s)} \\
= & e^{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)} \mathrm{M}^{-(1-s)}  \tag{79}\\
& \left\|P_{A, E}-P_{A, E: \mathrm{M}}\right\|_{1} \\
= & P_{A, E}\left(\Omega_{\mathrm{M}}\right)=\sum_{(a, e) \in \Omega_{\mathrm{M}}} P_{A, E}(a, e) \\
\leq & \sum_{(a, e) \in \Omega_{\mathrm{M}}}\left(P_{A, E}(a, e)\right)^{1+s} \mathrm{M}^{s}\left(P_{E}(e)\right)^{-s} \\
\leq & \sum_{(a, e)}\left(P_{A, E}(a, e)\right)^{1+s} \mathrm{M}^{s}\left(P_{E}(e)\right)^{-s} \\
= & \mathrm{M}^{s} e^{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right.} . \tag{80}
\end{align*}
$$

Substituting (79) and (80) into (55), we obtain (57) because

$$
\begin{aligned}
& \eta\left(\mathrm{M}^{s} e^{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)}, \varepsilon+\log \mathrm{M}\right) \\
= & \eta\left(\mathrm{M}^{s} e^{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)}, \log \mathrm{M}\right)+\varepsilon \mathrm{M}^{s} e^{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)} .
\end{aligned}
$$

In the above proof, we choose $P_{A, E}^{\prime}$ to be $P_{A, E: \mathrm{M}}(a, e)$, we call the smoothing with this particular choice the information-spectrum-smoothing bound because this type smoothing bound is used to derive the entropic information spectrum in [17]. Indeed, the paper [12] also employed the information-spectrum-smoothing bound to derive Proposition 21,

Further, $\Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ and $\Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ can be evaluated as follows.
Theorem 24: The upper bounds $\Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ and $\Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ of leaked information by the smoothing bound of min entropy can be evaluated as follows.

$$
\begin{align*}
& \Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) \leq(2+\sqrt{\varepsilon}) \min _{0 \leq s} e^{\frac{-s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)+s R}{1+2 s}}  \tag{81}\\
& \Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) \leq \eta\left(\min _{0 \leq s} e^{\frac{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)+s R}{1+s}}, \varepsilon+\log \mathrm{M}\right) \tag{82}
\end{align*}
$$

Theorem 24 gives upper bounds on $\Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ and $\Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$. The combination of Theorems 20 and 24 shows the performance of the smoothing bound of min entropy. Using these bounds, we can show the tight exponential decreasing rates of $\Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ and $\Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$.

Proof: Since

$$
\begin{align*}
& P_{A, E}\left\{(a, e) \left\lvert\, \frac{P_{A, E}(a, e)}{Q_{E}(e)}>e^{-R^{\prime}}\right.\right\} \\
= & \sum_{(a, e): \frac{P_{A, E}(a, e)}{Q_{E}(e)}>e^{-R^{\prime}}} P_{A, E}(a, e) \\
\leq & \sum_{(a, e): \frac{P_{A, E}(a, e)}{Q_{E}(e)}>e^{-R^{\prime}}} P_{A, E}(a, e)\left(\frac{P_{A, E}(a, e)}{Q_{E}(e)} e^{R^{\prime}}\right)^{s} \\
\leq & \sum_{(a, e)} P_{A, E}(a, e)\left(\frac{P_{A, E}(a, e)}{Q_{E}(e)} e^{R^{\prime}}\right)^{s} \\
= & e^{-s H_{1+s}\left(A|E| P_{A, E} \mid Q_{E}\right)+s R^{\prime}}, \tag{83}
\end{align*}
$$

choosing $R^{\prime}=\frac{\log \mathrm{M}+2 s H_{1+s}\left(A|E| P_{A, E} \mid Q_{E}\right)}{1+2 s}$, we have

$$
\begin{aligned}
& 2 P_{A, E}\left\{(a, e) \left\lvert\, \frac{P_{A, E}(a, e)}{Q_{E}(e)}>e^{-R^{\prime}}\right.\right\}+\sqrt{\varepsilon} \mathrm{M}^{\frac{1}{2}} e^{-\frac{1}{2} R^{\prime}} \\
\leq & 2 e^{-s H_{1+s}\left(A|E| P_{A, E} \mid Q_{E}\right)+s R^{\prime}}+\sqrt{\varepsilon} \mathrm{M}^{\frac{1}{2}} e^{-\frac{1}{2} R^{\prime}} \\
\leq & (2+\sqrt{\varepsilon}) e^{\frac{-(1+s) s H_{1+s}\left(A|E| P_{A, E} \mid Q_{E}\right)+s R}{1+2 s}}
\end{aligned}
$$

Since the above inequality holds for $s \geq 0$, Lemma 4 yields that

$$
\begin{aligned}
& \min _{Q_{E}} \min _{R^{\prime}} 2 P_{A, E}\left\{(a, e) \left\lvert\, \frac{P_{A, E}(a, e)}{Q_{E}(e)}>e^{-R^{\prime}}\right.\right\}+\sqrt{\varepsilon} \mathrm{M}^{\frac{1}{2}} e^{-\frac{1}{2} R^{\prime}} \\
\leq & \min _{0 \leq s} \min _{Q_{E}}(2+\sqrt{\varepsilon}) e^{\frac{-(1+s) s H_{1+s}\left(A|E| P_{A, E} \mid Q_{E}\right)+s R}{1+2 s}} \\
= & (2+\sqrt{\varepsilon}) \min _{0 \leq s} e^{\frac{-s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)+s R}{1+2 s}}
\end{aligned}
$$

Hence, combining (73), we obtain (81).
Choosing $R^{\prime}=\frac{\log \mathrm{M}+s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)}{1+s}$, we have

$$
\begin{aligned}
& \eta\left(P_{A, E}\left\{(a, e) \mid P_{A \mid E}(a \mid e)>e^{-R^{\prime}}\right\}, \log \mathrm{M}\right)+\varepsilon \mathrm{M} e^{-R^{\prime}} \\
\leq & \eta\left(e^{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)+s R^{\prime}}, \log \mathrm{M}\right)+\varepsilon \mathrm{M} e^{-R^{\prime}} \\
\leq & \eta\left(e^{\frac{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)+s R}{1+s}}, \log \mathrm{M}\right)+\varepsilon e^{\frac{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)+s R}{1+s}} \\
= & \eta\left(e^{\frac{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)+s R}{1+s}}, \varepsilon+\log \mathrm{M}\right) .
\end{aligned}
$$

Since the above inequality holds for $s \geq 0$, we have

$$
\begin{aligned}
& \min _{R^{\prime}} \eta\left(P_{A, E}\left\{(a, e) \mid P_{A \mid E}(a \mid e)>e^{-R^{\prime}}\right\}, \log \mathrm{M}\right)+\varepsilon \mathrm{M} e^{-R^{\prime}} \\
\leq & \min _{0 \leq s} \eta\left(e^{\frac{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)+s R}{1+s}}, \varepsilon+\log \mathrm{M}\right) \\
= & \eta\left(\min _{0 \leq s} e^{\frac{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)+s R}{1+s}}, \varepsilon+\log \mathrm{M}\right),
\end{aligned}
$$

Hence, combining (75), we obtain 82).
Remark 2: Here, we compare the calculation amount of obtained bounds in Sections IV V and VI In order to calculate the bounds $\Delta_{d, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right), \Delta_{I, 2}\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right), \Delta_{d, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$, and $\Delta_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right)$ based on the smoothing, we need calculate the smooth entropies, which contains several optimizations. Hence, the calculation of these bounds requires at least double optimization process. Then, they need higher calculation amounts. In particular, if the block size becomes larger, their calculation amounts increase heavily.

The bounds given in Section $\nabla$ are calculated from the tail probability. For example, the tail probability $P_{A, E}\left\{(a, e) \mid P_{A \mid E}(a \mid e)>\right.$ $\left.e^{-R^{\prime}}\right\}$ can be characterized as the tail probability with respect to the random variable $\log P_{A \mid E}(a \mid e)$ because $P_{A, E}\left\{(a, e) \mid P_{A \mid E}(a \mid e)>\right.$ $\left.e^{-R^{\prime}}\right\}=P_{A, E}\left\{(a, e) \mid \log P_{A \mid E}(a \mid e)>-R^{\prime}\right\}$. Hence, in the i.i.d. case, this probability can be calculated by using statistical packages. While the calculation amount increases with a rise in the block size, it is not as large as the above cases because statistical packages can be used.

The calculation amounts of the bounds given in Section VI are quite small. In particular, in the i.i.d. case, the calculation amounts do not depend on the block size. These bounds have great advantages with respect to their calculation amounts.

## VII. Secret key generation: Asymptotic case

Next, we consider the case when the information source is given by the $n$-fold independent and identical distribution $P_{A, E}^{n}$ of $P_{A, E}$, i.e., $P_{A_{n}, E_{n}}=P_{A, E}^{n}$. In this case, Ahlswede and Csiszár [7] showed that the optimal generation rate

$$
G\left(P_{A E}\right):=\sup _{\left\{\left(f_{n}, \mathrm{M}_{n}\right)\right\}}\left\{\left.\lim _{n \rightarrow \infty} \frac{\log \mathrm{M}_{n}}{n} \right\rvert\, d_{1}^{\prime}\left(f_{n}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right) \rightarrow 0\right\}
$$

equals the conditional entropy $H(A \mid E)$, where $f_{n}$ is a function from $\mathcal{A}^{n}$ to $\left\{1, \ldots, \mathrm{M}_{n}\right\}$. That is, when the generation rate $R=\lim _{n \rightarrow \infty} \frac{\log \mathrm{M}_{n}}{n}$ is smaller than $H(A \mid E)$, the quantity $d_{1}^{\prime}\left(f_{n}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right)$ goes to zero. In order to treat the speed of this convergence, we focus on the supremum of the exponential rate of decrease (exponent) for $d_{1}^{\prime}\left(f_{n}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right)$ and $I^{\prime}\left(f_{n}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right)=I\left(f_{n}\left(A_{n}\right): E_{n} \mid P_{A, E}^{n}\right)+D\left(P_{f_{n}\left(A_{n}\right)} \| P_{\text {mix }, f_{n}\left(\mathcal{A}_{n}\right)}\right)$ for a given $R$.

Due to (30), when $d_{1}^{\prime}\left(f_{C_{n}}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right)$ goes to zero, $I^{\prime}\left(f_{C_{n}}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right)$ goes to zero. Conversely, due to (28), when $I^{\prime}\left(f_{C_{n}}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right)$ goes to zero, $d_{1}^{\prime}\left(f_{C_{n}}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right)$ goes to zero. So, even if we replace the security criterion by $I^{\prime}\left(f_{C_{n}}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right)$, the optimal generation rate does not change.

Now, we consider the case when the length of generated keys behaves as $n H(A|E| P)+\sqrt{n} R$. It is known in [29, Subsection II-D] that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \min _{f} d_{1}^{\prime}\left(f\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right)=2 \int_{-\infty}^{R / \sqrt{V(P)}} \frac{1}{\sqrt{2 \pi}} e^{-x^{2} / 2} d x \tag{84}
\end{equation*}
$$

Then, using Theorem 24 we obtain the following theorem.
Theorem 25: We choose a polynomial $P(n)$. When a random linear function $f_{\mathbf{X}^{n}}$ from $\mathcal{A}^{n}$ to $\left\{1, \ldots,\left\lfloor e^{n H(A|E| P)+\sqrt{n} R}\right\rfloor\right\}$ is $P(n)$-almost dual universal ${ }_{2}$, the relations

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathrm{E}_{\mathbf{X}_{n}} d_{1}^{\prime}\left(f_{\mathbf{X}^{n}}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right)=\lim _{n \rightarrow \infty} \min _{f} d_{1}^{\prime}\left(f\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right)=2 \int_{-\infty}^{R / \sqrt{V(P)}} \frac{1}{\sqrt{2 \pi}} e^{-x^{2} / 2} d x \tag{85}
\end{equation*}
$$

hold, where we take the minimum under the condition that $f$ is a function from $\mathcal{A}^{n}$ to $\left\{1, \ldots,\left\lfloor e^{n H(A|E| P)+\sqrt{n} R}\right\rfloor\right\}$ and $V(P):=\sum_{a, e} P_{A, E}(a, e)\left(\log P_{A \mid E}(a \mid e)-H(A|E| P)\right)^{2}$.

Lemma 25 implies that any $P(n)$-almost dual universal ${ }_{2}$ hash function realizes the optimality in the sense of the second order asymptotics when we employ the $L_{1}$ distinguishability criterion. This analysis is obtained from the smoothing bound of min entropy. That is, this analysis does not require the smoothing bound of Rényi entropy of order 2 . The second order analysis with the mutual information criterion is not so easy. This topic will be discussed in a future paper.

Proof: We applying (73) in Theorem 20 with $R^{\prime}=n H(A|E| P)+\sqrt{n} R+n^{1 / 4}$. Then, the central limit theorem guarantees that

$$
\begin{aligned}
& \mathrm{E}_{\mathbf{X}_{n}} d_{1}^{\prime}\left(f_{\mathbf{X}^{n}}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right) \leq \Delta_{d, \min }\left(e^{n H(A|E| P)+\sqrt{n} R+n^{1 / 4}}, P(n) \mid P_{A, E}^{n}\right) \\
\leq & 2 P_{A, E}^{n}\left\{(a, e) \mid P_{A \mid E}^{n}(a \mid e)>e^{-n H(A|E| P)-\sqrt{n} R-n^{1 / 4}}\right\}+\sqrt{P(n)} e^{-n^{1 / 4} / 2} \\
\rightarrow & 2 \int_{-\infty}^{R / \sqrt{V(P)}} \frac{1}{\sqrt{2 \pi}} e^{-x^{2} / 2} d x
\end{aligned}
$$

Since $\min _{f} d_{1}^{\prime}\left(f\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right) \leq d_{1}^{\prime}\left(f_{\mathbf{X}^{n}}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right)$, combining 84], we obtain 85].
Now, we proceed to the exponential decreasing rate when we choose the key generation rate $R$ is greater than $H(A|E| P)$. Since the discussion for the exponential decreasing rate is more complex, more delicate treatment is required. First, we should remark that the exponential decreasing rate depends on the choice of the security criterion. Then, we obtain the following theorem.

Theorem 26: We choose a polynomial $P(n)$. When a linear random function $f_{\mathbf{X}^{n}}$ from $\mathcal{A}^{n}$ to $\left\{1, \ldots,\left\lfloor e^{n R}\right\rfloor\right\}$ is $P(n)$-almost dual universal ${ }_{2}$, the relations

$$
\begin{align*}
& \liminf _{n \rightarrow \infty} \frac{-1}{n} \log \mathrm{E}_{\mathbf{X}_{n}} d_{1}^{\prime}\left(f_{\mathbf{X}^{n}}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right) \geq \liminf _{n \rightarrow \infty} \frac{-1}{n} \log \Delta_{d, 2}\left(e^{n R}, P(n) \mid P_{A, E}^{n}\right) \geq e_{d}\left(P_{A, E} \mid R\right)  \tag{86}\\
& \liminf _{n \rightarrow \infty} \frac{-1}{n} \log \mathrm{E}_{\mathbf{X}_{n}} I^{\prime}\left(f_{\mathbf{X}^{n}}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right) \geq \liminf _{n \rightarrow \infty} \frac{-1}{n} \log \Delta_{I, 2}\left(e^{n R}, P(n) \mid P_{A, E}^{n}\right) \geq e_{I}\left(P_{A, E} \mid R\right) \tag{87}
\end{align*}
$$

hold, where

$$
\begin{align*}
e_{d}\left(P_{A, E} \mid R\right) & :=\max _{0 \leq t \leq \frac{1}{2}} t\left(H_{\frac{1}{1-t}}^{\uparrow}\left(A|E| P_{A, E}\right)-R\right)  \tag{88}\\
e_{I}\left(P_{A, E} \mid R\right) & :=\max _{0 \leq s \leq 1} s\left(H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-R\right) . \tag{89}
\end{align*}
$$

Proof: (86) can be shown by Theorem 22 (87) can be shown by Theorem 23
As is shown in Appendix F-A, the following relation between two exponents $e_{I}\left(P_{A, E} \mid R\right)$ and $e_{d}\left(P_{A, E} \mid R\right)$ holds.
Lemma 27: we obtain

$$
\begin{align*}
\frac{1}{2} e_{I}\left(P_{A, E} \mid R\right) & \leq e_{d}\left(P_{A, E} \mid R\right)  \tag{90}\\
e_{I}\left(P_{A, E} \mid R\right) & \geq e_{d}\left(P_{A, E} \mid R\right) \tag{91}
\end{align*}
$$

First, we consider the tightness of Inequality (86). Corollary 17 yields the exponent $\frac{e_{I}\left(P_{A, E} \mid R\right)}{2}$ for the $L_{1}$ distinguishability criterion. Lemma 27 shows that the exponents by Theorem 22 is better than that by Corollary 17 Further, it is also shown in [36. Theorem 30] that there exists a sequence of universal ${ }_{2}$ functions $f_{\mathbf{X}^{n}}$ from $\mathcal{A}^{n}$ to $\left\{1, \ldots,\left\lfloor e^{n R}\right\rfloor\right\}$ such that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \frac{-1}{n} \log \mathrm{E}_{\mathbf{X}_{n}} d_{1}^{\prime}\left(f_{\mathbf{X}^{n}}\left(A_{n}\right)\left|E_{n}\right| P_{A, E}^{n}\right) \leq \bar{e}_{d}\left(P_{A, E} \mid R\right) \tag{92}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{e}_{d}\left(P_{A, E} \mid R\right):=\max _{0 \leq t} t\left(H_{\frac{1}{1-t}}^{\uparrow}\left(A|E| P_{A, E}\right)-R\right) \tag{93}
\end{equation*}
$$

When the maximum $\max _{0 \leq t} t\left(H_{\frac{1}{1-t}}^{\uparrow}\left(A|E| P_{A, E}\right)-R\right)$ is attained with $t \in\left(0, \frac{1}{2}\right]$, we have $e_{d}\left(P_{A, E} \mid R\right)=\bar{e}_{d}\left(P_{A, E} \mid R\right)$. Assume that $P(n) \geq 1$. Then, Since $\Delta_{d, 2}\left(e^{n R}, 1\right) \leq \Delta_{d, 2}\left(e^{n R}, P(n) \mid P_{A, E}^{n}\right) \leq \sqrt{P(n)} \Delta_{d, 2}\left(e^{n R}, 1 \mid P_{A, E}^{n}\right)$, combining (76), 86), and (92) we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{-1}{n} \log \Delta_{d, 2}\left(e^{n R}, P(n) \mid P_{A, E}^{n}\right)=\lim _{n \rightarrow \infty} \frac{-1}{n} \log \Delta_{d, 2}\left(e^{n R}, 1 \mid P_{A, E}^{n}\right)=e_{d}\left(P_{A, E} \mid R\right) \tag{94}
\end{equation*}
$$

That is, our evaluation 86) for $\Delta_{d, 2}\left(e^{n R}, P(n) \mid P_{A, E}^{n}\right)$ is sufficiently tight in the large deviation sense.
Next, we consider the tightness of Inequality 87). Corollary 16 yields the exponent $e_{d}\left(P_{A, E} \mid R\right)$ for the modified mutual information criterion. Lemma 27 shows that the exponent by Theorem 23 is better than that by Corollary 16 Further, the lower bound of the exponent $e_{d}\left(P_{A, E} \mid R\right)$ is the same as that given in the previous paper [13] under the universal ${ }_{2}$ condition. Since the bound given in [13] is the best lower bound of the exponent, our evaluation 87] for $\Delta_{I, 2}\left(e^{n R}, P(n) \mid P_{A, E}^{n}\right)$ is as good as the existing evaluation [13] in the large deviation sense.

From the above discussion, we find that the exponents directly obtained by the smoothing bound of Renyi entropy of order 2 are better than the exponents derived from the combination of Inequality $28 / \sqrt{30}$ and the exponent of the other criterion. This fact indicates that we need to choose the smoothing bound dependently of the security criterion.

Remark 3: Now, we consider the relation with the recent paper [27] discussing the quantum case as including the nonquantum case. When $\mathcal{A}=\mathbb{F}_{q}$, we focus on a $1+P(n) q^{-n+\lfloor n R\rfloor}$-almost universal ${ }_{2}$ surjective linear function $f_{\mathbf{X}^{n}}$ over the field
$\mathbb{F}_{q}$ from $\mathbb{F}_{q}^{n}$ to $\mathbb{F}_{q}^{\lfloor n R\rfloor}$. Thanks to Proposition 10, the surjective linear random function $f_{\mathbf{X}^{n}}$ over the field $\mathbb{F}_{q}$ is $q+P(n)$-almost dual universal ${ }_{2}$. Hence, we obtain (86), which can recover a part of the result by [27] with the case of linear functions in the non-quantum case. The paper [27] showed the security with an $\epsilon_{n}$-almost universal ${ }_{2}$ hash function when $\epsilon_{n}$ approaches to 1 . Since we assume the surjectivity, our method cannot recover the result by [27] with the linear hash function perfectly.

Now, we clarify how better our smoothing bound of Rényi entropy of order 2 is than the smoothing bound of min entropy. As is shown in Appendix $G$ we obtain the following theorem.

Theorem 28: The relations

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \frac{-1}{n} \log \Delta_{d, \min }\left(e^{n R}, \varepsilon \mid P_{A, E}^{n}\right) \\
= & \tilde{e}_{d}\left(P_{A, E} \mid R\right):=\max _{0 \leq s} \frac{s\left(H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-R\right)}{1+2 s}  \tag{95}\\
& \lim _{n \rightarrow \infty} \frac{-1}{n} \log \Delta_{I, \min }\left(e^{n R}, \varepsilon \mid P_{A, E}^{n}\right) \\
= & \tilde{e}_{I}\left(P_{A, E} \mid R\right):=\max _{0 \leq s} \frac{s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R}{1+s} \tag{96}
\end{align*}
$$

hold.
For the comparison of the exponents by the smoothing bound of min entropy and Rényi entropy of order 2, as is shown in Appendix F-B we have the following lemma by using Theorem 24

Lemma 29: The inequalities

$$
\begin{align*}
& e_{d}\left(P_{A, E} \mid R\right)>\tilde{e}_{d}\left(P_{A, E} \mid R\right)  \tag{97}\\
& e_{I}\left(P_{A, E} \mid R\right)>\tilde{e}_{I}\left(P_{A, E} \mid R\right) \tag{98}
\end{align*}
$$

hold when $P_{A \mid E=e}$ is not a uniform distribution for an element $e \in \mathcal{E}$. The equalities $e_{d}\left(P_{A, E} \mid R\right)=\tilde{e}_{d}\left(P_{A, E} \mid R\right)$ and $e_{I}\left(P_{A, E} \mid R\right)=\tilde{e}_{I}\left(P_{A, E} \mid R\right)$ hold when $P_{A \mid E=e}$ is a uniform distribution for any element $e \in \mathcal{E}$.

Theorem 28 and Lemma 29 show that the smoothing bound of min entropy cannot attain the exponents $e_{d}\left(P_{A, E} \mid R\right)$ and $e_{I}\left(P_{A, E} \mid R\right)$. That is, the bounds $\Delta_{d, 2}\left(e^{n R}, \varepsilon \mid P_{A, E}^{n}\right)$ and $\Delta_{I, 2}\left(e^{n R}, \varepsilon \mid P_{A, E}^{n}\right)$ by the smoothing bound of Rényi entropy of order 2 are strictly better than the bounds $\Delta_{d, \min }\left(e^{n R}, \varepsilon \mid P_{A, E}^{n}\right)$ and $\Delta_{I, \min }\left(e^{n R}, \varepsilon \mid P_{A, E}^{n}\right)$ by the smoothing bound of min entropy in the sense of large deviation. This fact indicates the importance of smoothing bound of Rényi entropy of order 2.

In summary, while the smoothing bound of min entropy yields the tight bound in the sense of the second order asymptotics, the smoothing bound of min entropy cannot yield the tight bound in the sense of the exponential decreasing rate.

Remark 4: Here, we give the relation with the results in the quantum case [56]. The paper [56] showed that

$$
\begin{align*}
& \liminf _{n \rightarrow \infty} \frac{-1}{n} \log \Delta_{d, 2}\left(e^{n R}, P(n) \mid P_{A, E}^{n}\right) \geq \max _{0 \leq t \leq \frac{1}{2}} \frac{t}{2(1-t)}\left(H_{\frac{1}{1-t}}^{\uparrow}\left(A|E| P_{A, E}\right)-R\right)  \tag{99}\\
& \liminf _{n \rightarrow \infty} \frac{-1}{n} \log \Delta_{I, 2}\left(e^{n R}, P(n) \mid P_{A, E}^{n}\right) \geq \max _{0 \leq s \leq 1} \frac{s}{2-s}\left(H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-R\right) \tag{100}
\end{align*}
$$

The RHSs of (99) and 100) are smaller than $e_{d}\left(P_{A, E} \mid R\right)$ and $e_{I}\left(P_{A, E} \mid R\right)$, respectively. Hence, our result is better in the non-quantum case.

## VIII. EQUivocation rate of secret key generation

When the key generation rate $R$ is larger than the conditional entropy $H\left(A|E| P_{A, E}\right)$, the leaked information does not go to zero. In this case, it is natural to consider the rate of the conditional entropy rate of generated keys or the rate of the modified mutual information [30]. The former rate is called the equivocation rate, and is known to be less than the conditional entropy $H\left(A|E| P_{A, E}\right)$ [30]. That is, the rate of the modified mutual information is larger than $R-H\left(A|E| P_{A, E}\right)$. Now, we show that the minimum rate of the modified mutual information $R-H\left(A|E| P_{A, E}\right)$ can be achieved by an $\varepsilon$-almost dual universal ${ }_{2}$ hash function. For this purpose, we employ (45) instead of (46). Then, we obtain a slightly stronger evaluation than Theorem 18

Theorem 30: Assume that $Q_{E}$ is a normalized distribution on $\mathcal{E}, P_{A, E}$ is a sub-distribution on $\mathcal{A} \times \mathcal{E}$, and a linear random hash function $f_{\mathbf{X}}$ from $\mathcal{A}$ to $\mathcal{M}=\{1, \ldots, \mathrm{M}\}$ is $\varepsilon$-almost dual universal ${ }_{2}$. Then, the random hash function $f_{\mathbf{X}}$ satisfies

$$
\begin{equation*}
\mathrm{E}_{\mathbf{X}} I^{\prime}\left(f_{\mathbf{X}}(A)|E| P_{A, E}\right) \leq \underline{\Delta}_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) \tag{101}
\end{equation*}
$$

where

$$
\begin{align*}
\underline{\Delta}_{I, \min }\left(\mathrm{M}, \varepsilon \mid P_{A, E}\right) & :=\min _{Q_{E}} \min _{P_{A, E}^{\prime}: P_{E}^{\prime} \leq Q_{E},} \eta\left(\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}, \log \mathrm{M}\right)+\log \left(1+\varepsilon M e^{-H_{\min }\left(A|E| P_{A, E}^{\prime} \| P_{E}\right)}\right)  \tag{102}\\
& =\min _{\epsilon_{1}>0} \eta\left(\epsilon_{1}, \log \mathrm{M}\right)+\log \left(1+\varepsilon \mathrm{M} e^{-H_{\min }^{\downarrow, \epsilon_{1}}\left(A|E| P_{A, E}\right)}\right)  \tag{103}\\
& =\min _{R^{\prime}} \eta\left(\min _{P_{A, E}^{\prime}: P_{E}^{\prime} \leq P_{E}, H_{\min }\left(A|E| P_{A, E}^{\prime} \| P_{E}\right) \geq R}\left\|P_{A, E}-P_{A, E}^{\prime}\right\|_{1}, \log \mathrm{M}\right)+\log \left(1+\varepsilon \mathrm{M} e^{-R^{\prime}}\right) . \tag{104}
\end{align*}
$$

Further, by using similar discussions as Sections $\mathbf{V}$ and VI the upper bound $\underline{\Delta}_{I, \min }\left(\mathrm{M}, \varepsilon\left|P_{A, E}\right| P_{A, E}\right)$ can be evaluated as follows.

Theorem 31:

$$
\begin{align*}
\underline{\Delta}_{I, \min }\left(\mathrm{M}, \varepsilon\left|P_{A, E}\right| P_{A, E}\right) & \leq \min _{R^{\prime}} \eta\left(P_{A, E}\left\{(a, e) \mid P_{A \mid E}(a \mid e)>e^{-R^{\prime}}\right\}, \log \mathrm{M}\right)+\log \left(1+\varepsilon \mathrm{M} e^{-R^{\prime}}\right)  \tag{105}\\
& \leq \min _{R^{\prime}} \eta\left(\min _{s \geq 0} e^{s\left(R^{\prime}-H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)\right.}, \log \mathrm{M}\right)+\log \left(1+\varepsilon \mathrm{M} e^{-R^{\prime}}\right) \tag{106}
\end{align*}
$$

Proof: Inequality (105) follows from Lemma 19 and 104). Inequality (106) follows from (83) with $Q_{E}=P_{E}$.
Now, we consider the asymptotic behavior of $\underline{\Delta}_{I, \min }\left(\left\lceil e^{n R}\right\rceil, \varepsilon \mid P_{A, E}^{n}\right)$.
Theorem 32: Any polynomial $P(n)$ satisfies

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n} \underline{\Delta}_{I, \min }\left(\left\lceil e^{n R}\right\rceil, P(n) \mid P_{A, E}^{n}\right)=R-H\left(A|E| P_{A, E}\right) \tag{107}
\end{equation*}
$$

for $R \geq H\left(A|E| P_{A, E}\right)$.
Theorem 32 shows that $\varepsilon$-almost dual universal ${ }_{2}$ hash functions realize the asymptotically optimal performance in the sense of equivocation rate. Further, Theorem 32 clarifies that the smoothing bound of min entropy yields the optimal evaluation in the sense of equivocation rate.

Proof: It is known by [30] that any sequence of hash function from $\mathcal{A}$ to $\left\{1, \ldots,\left\lceil e^{n R}\right\rceil\right\}$ satisfies

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \frac{1}{n} \mathrm{E}_{\mathbf{X}, n} I^{\prime}\left(f_{\mathbf{X}, n}(A)|E| P_{A, E}\right) \geq R-H\left(A|E| P_{A, E}\right) \tag{108}
\end{equation*}
$$

Hence, it is enough to show that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \frac{1}{n} \underline{\Delta}_{I, \min }\left(\left\lceil e^{n R}\right\rceil, P(n) \mid P_{A, E}^{n}\right) \leq R-H\left(A|E| P_{A, E}\right) \tag{109}
\end{equation*}
$$

We choose $R^{\prime}<H\left(A|E| P_{A, E}\right)$. Relation (106) implies that

$$
\begin{equation*}
\frac{1}{n} \underline{\Delta}_{I, \min }\left(\left\lceil e^{n R}\right\rceil, P(n) \mid P_{A, E}^{n}\right) \leq \frac{1}{n} \eta\left(\min _{s \geq 0} e^{s n\left(R^{\prime}-H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)\right.}, n R\right)+\frac{1}{n} \log \left(1+P(n) e^{n\left(R-R^{\prime}\right)}\right) \tag{110}
\end{equation*}
$$

Since $R^{\prime}<H\left(A|E| P_{A, E}\right)$, the value $\min _{s \geq 0} e^{s n\left(R^{\prime}-H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)\right.}$ goes to zero exponentially. Hence, the term $\frac{1}{n} \eta\left(\min _{s \geq 0} e^{s n\left(R^{\prime}-H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)\right.}, n R\right)$ goes to zero. Since $\frac{1}{n} \log \left(1+P(n) e^{n\left(R-R^{\prime}\right)}\right) \leq R-R^{\prime}+\frac{1}{n} \log (1+P(n)) \rightarrow R-R^{\prime}$, we have

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \frac{1}{n} \underline{\Delta}_{I, \min }\left(\left\lceil e^{n R}\right\rceil, P(n) \mid P_{A, E}^{n}\right) \leq R-R^{\prime} \tag{111}
\end{equation*}
$$

Since $R^{\prime}$ is an arbitrary real number satisfying $R^{\prime}<H\left(A|E| P_{A, E}\right)$, we obtain 109).

## IX. Conclusion

We have derived upper bounds for the leaked information in the modified mutual information criterion and the $L_{1}$ distinguishability criterion when we apply an $\varepsilon$-almost dual universal ${ }_{2}$ hash function for privacy amplification. (Theorems 23 and 22 in Section VI). Then, we have derived lower bounds on their exponential decreasing rates in the i.i.d. setting. (Theorem 26 in Section VII).

We have rigorously compared the exponents by the smoothing bound of min-entropy and Rényi entropy of order 2. That is, we have clarified the upper bounds of leaked information via the smoothing of min-entropy in the both criteria. That is, we have compared $\Delta_{d, 2}\left(M, \varepsilon \mid P_{A, E}\right)$ and $\Delta_{d, \min }\left(M, \varepsilon \mid P_{A, E}\right)$ for Rényi entropy of order 2, and have done $\Delta_{I, 2}\left(M, \varepsilon \mid P_{A, E}\right)$ and $\Delta_{I, \min }\left(M, \varepsilon \mid P_{A, E}\right)$ for modified mutual information criterion. We have derived the exponents of the upper bounds (Theorem 28 in Section VI), and have shown that the exponents are strictly worse than the exponents by the smoothing bound of Renyi entropy of order 2 (Lemma 29 in Section VI). This fact shows the importance of the smoothing of Rényi entropy of order 2. The obtained exponents are summarized in Table II

Due to Pinsker inequality and Inequality (30), the exponential convergence of one criterion yields the exponential convergence of the other criterion. However, we have shown that better exponential decreasing rates can be obtained by separate derivations.

For example, the smoothing of Rényi entropy of order 2 yields the exponent $e_{d}\left(P_{A, E} \mid R\right)$ for the $L_{1}$ distinguishability criterion, which yields the exponent $e_{d}\left(P_{A, E} \mid R\right)$ for the modified mutual information criterion by using Pinsker inequality. Similarly, the smoothing of Rényi entropy of order 2 yields the exponent $e_{I}\left(P_{A, E} \mid R\right)$ for the modified mutual information criterion, which yields the exponent $\frac{e_{I}\left(P_{A, E} \mid R\right)}{2}$ for the $L_{1}$ distinguishability criterion by Inequality 30 . Since $e_{d}\left(P_{A, E} \mid R\right) \geq \frac{e_{I}\left(P_{A, E} \mid R\right)}{2}$ and $e_{I}\left(P_{A, E} \mid R\right) \geq e_{d}\left(P_{A, E} \mid R\right)$, the exponents directly derived by the smoothing of Rényi entropy of order 2 are better than the exponents derived from the combination of the exponent for the other criterion and the inequality.

TABLE II
SUMMARY OF OBTAINED LOWER BOUNDS ON EXPONENTS.

| Method | $L_{1}$ | MMI |
| :--- | :---: | :---: |
| smooth Rényi 2 | $e_{d}\left(P_{A, E} \mid R\right)$ | $e_{I}\left(P_{A, E} \mid R\right)$ |
| smooth min | $\tilde{e}_{d}\left(P_{A, E} \mid R\right)$ | $\tilde{e}_{I}\left(P_{A, E} \mid R\right)$ |

smooth Rényi 2 is the exponent for privacy amplification via the smoothing of Rényi entropy of order 2 . smooth min is the exponent for privacy amplification via the smoothing of min entropy. L2 is the $L_{1}$ distinguishability criterion. MMI is the modified mutual information criterion.

We have also shown that the application of $\varepsilon$-almost dual universal hash function attains the asymptotically optimal performance in the sense of the second order asymptotics as well as in that of the asymptotic equivocation rate. These facts have been shown by using the smoothing of min entropy. We can conclude that $\varepsilon$-almost dual universal hash functions are very a useful class of hash functions. Further, these discussions show that the smoothing of min entropy is sufficiently powerful except for the exponential decreasing rate. That is, the exponential decreasing rate requires more delicate evaluation than other settings.

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## Appendix A

## Proof of Lemma4

For two non-negative functions $X(e)$ and $Y(e)$, the reverse Hölder inequality [34]

$$
\sum_{e} X(e) Y(e) \geq\left(\sum_{e} X(e)^{1 /(1+s)}\right)^{1+s}\left(\sum_{e} Y(e)^{-1 / s}\right)^{-s}
$$

holds for $s \in(0, \infty]$. Substituting $\sum_{a} P_{A, E}(a, e)^{1+s}$ and $Q_{E}(e)^{-s}$ to $X(e)$ and $Y(e)$, we obtain

$$
\begin{aligned}
& e^{-s H_{1+s}\left(A|E| P_{A, E} \| Q_{E}\right)} \\
= & \sum_{e} \sum_{a} P_{A, E}(a, e)^{1+s} Q_{E}(e)^{-s} \\
\geq & \left(\sum_{e}\left(\sum_{a} P_{A, E}(a, e)^{1+s}\right)^{1 /(1+s)}\right)^{1+s}\left(\sum_{e} Q_{E}(e)^{-s \cdot-1 / s}\right)^{-s} \\
= & \left(\sum_{e}\left(\sum_{a} P_{A, E}(a, e)^{1+s}\right)^{1 /(1+s)}\right)^{1+s} \\
= & \left(\sum_{e}\left(\sum_{a} P_{A, E}(a, e)^{1+s}\right)^{\frac{1}{1+s}}\right)^{1+s}
\end{aligned}
$$

for $s \in(0, \infty]$. Since the equality holds when $Q_{E}(e)=\left(\sum_{a} P_{A, E}(a, e)^{1+s}\right)^{1 /(1+s)} / \sum_{e}\left(\sum_{a} P_{A, E}(a, e)^{1+s}\right)^{1 /(1+s)}$, we obtain

$$
e^{-s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)},=\min _{Q_{E}} e^{-s H_{1+s}\left(A|E| P_{A, E} \| Q_{E}\right)}=\left(\sum_{e}\left(\sum_{a} P_{A, E}(a, e)^{1+s}\right)^{\frac{1}{1+s}}\right)^{1+s}
$$

which implies (13) with $s \in(0, \infty]$.
For two non-negative functions $X(e)$ and $Y(e)$, the Hölder inequality

$$
\sum_{e} X(e) Y(e) \leq\left(\sum_{e} X(e)^{1 /(1+s)}\right)^{1+s}\left(\sum_{e} Y(e)^{-1 / s}\right)^{-s}
$$

holds for $s \in[-1,0)$. The same substitution yields

$$
e^{-s H_{1+s}\left(A|E| P_{A, E} \| Q_{E}\right)} \leq\left(\sum_{e}\left(\sum_{a} P_{A, E}(a, e)^{1+s}\right)^{\frac{1}{1+s}}\right)^{1+s}
$$

for $s \in[-1,0)$. Hence, similarly we obtain (13) with $s \in[-1,0)$.

## Appendix B

Proof of Lemma 5
For $s \in(0,1]$ and two functions $X(a)$ and $Y(a)$, the Hölder inequality

$$
\sum_{a} X(a) Y(a) \leq\left(\sum_{a}|X(a)|^{1 /(1-s)}\right)^{1-s}\left(\sum_{a}|Y(a)|^{1 / s}\right)^{s}
$$

holds. The equality holds only when $X(a)$ is a constant times of $Y(a)$. Substituting $P_{A, E}(a, e)$ and $\left(\frac{P_{A, E}(a, e)}{P_{E}(e)}\right)^{s}$ to $X(a)$ and $Y(a)$, we obtain

$$
\begin{aligned}
& e^{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)} \\
= & \sum_{e} \sum_{a} P_{A, E}(a, e)\left(\frac{P_{A, E}(a, e)}{P_{E}(e)}\right)^{s} \\
\leq & \sum_{e}\left(\sum_{a} P_{A, E}(a, e)^{1 /(1-s)}\right)^{1-s}\left(\sum_{a} \frac{P_{A, E}(a, e)}{P_{E, \text { normal }}(e)}\right)^{s} \\
= & \sum_{e}\left(\sum_{a} P_{A, E}(a, e)^{1 /(1-s)}\right)^{1-s} \\
= & e^{-s H_{\frac{1}{1-s}}^{\uparrow}}\left(A|E| P_{A, E}\right)
\end{aligned}
$$

for $s \in(0,1]$ because $\sum_{a} \frac{P_{A, E}(a, e)}{P_{E, \text { normal }}(e)}=\frac{P_{E}(e)}{P_{E, \text { normal }}(e)} \leq 1$. The equality condition holds only when $P_{A \mid E=e}$ is uniform distribution for all $e \in \mathcal{E}$.

For $s \in[-1,0)$ and two functions $X(a)$ and $Y(a)$, the reverse Hölder inequality [34]

$$
\sum_{a} X(a) Y(a) \geq\left(\sum_{a}|X(a)|^{1 /(1-s)}\right)^{1-s}\left(\sum_{a}|Y(a)|^{1 / s}\right)^{s}
$$

holds. The same substitution yields

$$
e^{-s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)} \geq e^{-s H_{\frac{1}{1-s}}^{\uparrow}\left(A|E| P_{A, E}\right)}
$$

for $s \in[-1,0)$ because $\left(\sum_{a} \frac{P_{A, E}(a, e)}{P_{E, \text { normal }}(e)}\right)^{s}=\left(\frac{P_{E}(e)}{P_{E, \text { normal }}(e)}\right)^{s} \geq 1$. The equality condition holds only when $P_{A \mid E=e}$ is uniform distribution for all $e \in \mathcal{E}$.

## Appendix C

## Proof of Lemma 7

First, we show (19). Taking the limit $s \rightarrow 0$, we obtain

$$
\begin{align*}
& H\left(A|E| P_{A, E}\right)=-\left.\frac{d \phi\left(s|A| E \mid P_{A, E}\right)}{d s}\right|_{s=0} \\
= & -\lim _{s \rightarrow 0} \frac{\phi\left(s|A| E \mid P_{A, E}\right)}{s}=\lim _{s \rightarrow 0} H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right) . \tag{112}
\end{align*}
$$

The remaining properties are shown by the following lemma.
Lemma 33:

$$
\begin{align*}
& -\frac{d}{d s} s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right) \\
= & \sum_{a, e} P_{A, E ; s}(a, e)\left(\log P_{A \mid E}(a \mid e)-\frac{1}{1+s} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right)+\phi\left(\left.\frac{s}{1+s}|A| E \right\rvert\, P_{A, E}\right),  \tag{113}\\
& -\frac{d^{2}}{d s^{2}} s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right) \\
= & (1+s) \sum_{a, e} P_{A, E ; s}(a, e)\left(\frac{1}{1+s} \log P_{A \mid E}(a \mid e)-\frac{1}{(1+s)^{2}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right)^{2} \\
& -(1+s)\left(\sum_{a, e} P_{A, E ; s}(a, e)\left(\frac{1}{1+s} \log P_{A \mid E}(a \mid e)-\frac{1}{(1+s)^{2}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right)\right)^{2} . \tag{114}
\end{align*}
$$

Hence, when we regard $H_{1}^{\uparrow}\left(A|E| P_{A, E}\right)$ as $H\left(A|E| P_{A, E}\right)$ and $P_{A \mid E=e}$ is not a uniform distribution for an element $e \in \mathcal{E}$, the function $s \mapsto-s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ is strictly convex in $(-1, \infty)$. That is, the map $s \mapsto s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ is strictly concave and then the map $s \mapsto H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ is strictly monotonically decreasing for $s \in(-1, \infty)$.

Proof: We define

$$
\varphi(s):=\sum_{e} P_{E}(e)\left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)^{\frac{1}{1+s}} .
$$

Then,

$$
\begin{aligned}
& \frac{d \varphi(s)}{d s} \\
= & \sum_{a, e} \frac{P_{A \mid E}(a \mid e)^{1+s} P_{E}(e)}{\left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)^{\frac{s}{1+s}}\left(\sum_{e} P_{E}(e)\right.}\left(\frac{1}{1+s} \log P_{A \mid E}(a \mid e)-\frac{1}{(1+s)^{2}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right) \\
= & \varphi(s) \sum_{a, e} P_{A, E ; s}(a, e)\left(\frac{1}{1+s} \log P_{A \mid E}(a \mid e)-\frac{1}{(1+s)^{2}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right) .
\end{aligned}
$$

Since

$$
\begin{aligned}
& -\frac{d}{d s} s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right) \\
= & \phi\left(\left.\frac{s}{1+s}|A| E \right\rvert\, P_{A, E}\right)+(1+s) \frac{d \varphi(s)}{d s} \varphi(s)^{-1},
\end{aligned}
$$

we obtain (113).
Next, we show (114). Since

$$
\begin{aligned}
& \frac{d^{2} \varphi(s)}{d s^{2}} \\
= & \sum_{a, e} \frac{P_{A \mid E}(a \mid e)^{1+s} P_{E}(e)}{\left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)^{\frac{s}{1+s}}\left(\sum_{e} P_{E}(e)\right.}\left(\frac{1}{1+s} \log P_{A \mid E}(a \mid e)-\frac{1}{(1+s)^{2}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right)^{2} \\
& +\sum_{a, e} \frac{P_{A \mid E}(a \mid e)^{1+s} P_{E}(e)}{\left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)^{\frac{s}{1+s}}\left(\sum_{e} P_{E}(e)\right.}\left(-\frac{2}{(1+s)^{2}} \log P_{A \mid E}(a \mid e)+\frac{2}{(1+s)^{3}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right) \\
= & \varphi(s) \sum_{a, e} P_{A, E ; s}(a, e)\left(\frac{1}{1+s} \log P_{A \mid E}(a \mid e)-\frac{1}{(1+s)^{2}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right)^{2} \\
& -\frac{2}{(1+s)} \frac{d \varphi(s)}{d s},
\end{aligned}
$$

we have

$$
\begin{aligned}
& \frac{d^{2}}{d s^{2}}(1+s) \phi\left(\left.\frac{s}{1+s}|A| E \right\rvert\, P_{A, E}\right) \\
= & (1+s) \frac{d^{2}}{d s^{2}} \phi\left(\left.\frac{s}{1+s}|A| E \right\rvert\, P_{A, E}\right)+2 \frac{d}{d s} \phi\left(\left.\frac{s}{1+s}|A| E \right\rvert\, P_{A, E}\right) \\
= & (1+s) \frac{\varphi(s) \frac{d^{2} \varphi(s)}{d s^{2}}-\frac{d \varphi(s)^{2}}{d s}}{\varphi(s)^{2}}+2 \frac{\frac{d \varphi(s)}{d s}}{\varphi(s)} \\
= & (1+s) \frac{\varphi(s) \frac{d^{2} \varphi(s)}{d s^{2}}-\frac{d \varphi(s)^{2}}{d s}}{\varphi(s)^{2}}+2 \frac{\frac{d \varphi(s)}{d s}}{\varphi(s)} \\
= & (1+s) \sum_{a, e} P_{A, E ; s}(a, e)\left(\frac{1}{1+s} \log P_{A \mid E}(a \mid e)-\frac{1}{(1+s)^{2}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right)^{2} \\
& -(1+s)\left(\sum_{a, e} P_{A, E ; s}(a, e)\left(\frac{1}{1+s} \log P_{A \mid E}(a \mid e)-\frac{1}{(1+s)^{2}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right)\right)^{2},
\end{aligned}
$$

which implies (114).

## APPENDIX D <br> Proof of Theorem 8

First, we show that the modified mutual information criterion $I^{\prime}(A|E| P)=\log |\mathcal{A}|-H(A|E| P)$ satisfies all of the above conditions. We can trivially check the conditions C4 Ideal case and C5 Normalization. We show other conditions. C1 Chain rule can be shown as follows.

$$
\begin{aligned}
& I^{\prime}(A, B|E| P)=\log |\mathcal{A}|+\log |\mathcal{B}|-H(A, B, E \mid P)+H(E \mid P) \\
= & \log |\mathcal{A}|+\log |\mathcal{B}|-H(B, E \mid P)+H(E \mid P)-H(A, B, E \mid P)+H(B, E \mid P) \\
= & \log |\mathcal{A}|+\log |\mathcal{B}|-H(B|E| P)-H(A|B, E| P)=I^{\prime}(A|B, E| P)+I^{\prime}(B|E| P) .
\end{aligned}
$$

When two marginal distributions $P_{E, 1}$ and $P_{E, 2}$ are distinghuishable on $\mathcal{E}$,

$$
\begin{aligned}
& I^{\prime}\left(A|E| \lambda P_{1}+(1-\lambda) P_{2}\right)=\log |\mathcal{A}|-H\left(A, E \mid \lambda P_{1}+(1-\lambda) P_{2}\right)+H\left(E \mid \lambda P_{1}+(1-\lambda) P_{2}\right) \\
= & \log |\mathcal{A}|-\lambda H\left(A, E \mid P_{1}\right)-(1-\lambda) H\left(A, E \mid P_{2}\right)-h(\lambda)+\lambda H\left(E \mid P_{1}\right)+(1-\lambda) H\left(E \mid P_{2}\right)+h(\lambda) \\
= & \log |\mathcal{A}|-\lambda H\left(A, E \mid P_{1}\right)-(1-\lambda) H\left(A, E \mid P_{2}\right)+\lambda H\left(E \mid P_{1}\right)+(1-\lambda) H\left(E \mid P_{2}\right) \\
= & \lambda I^{\prime}\left(A|E| P_{1}\right)+(1-\lambda) I^{\prime}\left(A|E| P_{2}\right),
\end{aligned}
$$

which implies C2 Linearity. $I^{\prime}(A|E| P)=D\left(P \| P_{\mathrm{mix}, \mathcal{A}} \otimes P_{E}\right) \geq 0$. Since $H(A, E \mid P) \geq 0, I^{\prime}(A|E| P)$ satisfies C3 Range. Thus, $I^{\prime}(A|E| P)$ satisfies all of the above properties.

Next, we show that an quantity satisfying all of the above properties is the modified mutual information criterion $I^{\prime}(A|E| P)=$ $\log |\mathcal{A}|-H(A|E| P)$. For this purpose, we focus on $\tilde{H}(A|E| P):=\log |\mathcal{A}|-C(A|E| P)$. Due to C1 Linearity, we have

$$
\tilde{H}(A|E| P)=\sum_{e} P_{E}(e) \tilde{H}\left(A|E| P_{A \mid E=e}\right)
$$

Further, we see that the quantity $\tilde{H}\left(A|E| P_{A \mid E=e}\right)$ satisfies Khinchin's axioms [55] for entropy because of the remaining properties. Hence, we find that $\tilde{H}\left(A|E| P_{A \mid E=e}\right)=H\left(P_{A \mid E=e}\right)$. Thus, $\tilde{H}(A|E| P)$ is equal to the conditional entropy $H(A|E| P)$. Hence, $C(A|E| P)=I^{\prime}(A|E| P)$.

## Appendix E <br> Proof of Proposition 12

Since the proof of Proposition 12 is related to $\delta$-biased ensemble, we make several preparations before starting the proof of Proposition 12] According to Dodis and Smith[9], we introduce $\delta$-biased ensemble of random variables $W_{\mathbf{X}}$ on a vector space over a general finite field $\mathbb{F}_{q}$, where $q$ is the power of the prime $p$. First, we fix a non-degenerate bilinear form (, ) from $\mathbb{F}_{q}^{2}$ to $\mathbb{F}_{p}$. Then, we define $(x \cdot y) \in \mathbb{F}_{p}$ for $x, y \in \mathbb{F}_{q}^{n}$ as $(x \cdot y):=\sum_{j=1}^{n} x_{j} \cdot y_{j}$. For a given $\delta>0$, an ensemble of random variables $\left\{W_{\mathbf{X}}\right\}$ on $\mathbb{F}_{q}^{n}$ is called $\delta$-biased when the inequality

$$
\begin{equation*}
\mathrm{E}_{\mathbf{X}}\left|\mathrm{E}_{W_{\mathbf{X}}} \omega_{p}^{\left(x \cdot W_{\mathbf{x}}\right)}\right|^{2} \leq \delta^{2} \tag{115}
\end{equation*}
$$

holds for any $x \neq 0 \in \mathbb{F}_{q}^{n}$, where $\omega_{p}:=e^{\frac{2 \pi i}{p}}$.
We denote the random variable subject to the uniform distribution on a code $C \in \mathbb{F}_{q}^{n}$, by $W_{C}$. Then,

$$
\mathrm{E}_{W_{C}} \omega_{p}^{\left(x \cdot W_{C}\right)}= \begin{cases}0 & \text { if } x \notin C^{\perp}  \tag{116}\\ 1 & \text { if } x \in C^{\perp}\end{cases}
$$

Using the above relation, as is suggested in [9, Case 2], we obtain the following lemma.
Lemma 34: When a random code $C_{\mathbf{X}}$ in $\mathbb{F}_{q}^{n}$ is $\varepsilon$-almost dual universal with minimum dimension $t$, the ensemble of random variables $W_{C_{\mathbf{x}}}$ in $\mathbb{F}_{q}^{n}$ is $\sqrt{\varepsilon q^{-t}}$-biased.

Proof: $C_{\mathbf{X}}^{\perp}$ is $\varepsilon$-almost universal with maximum dimension $n-t$ in $\mathbb{F}_{q}^{n}$. Hence, for any $x \in \mathbb{F}_{q}^{n}$, the probability $\operatorname{Pr}\left\{x \in C_{\mathbf{X}}^{\perp}\right\}$ is less than $\varepsilon q^{-t}$. Thus, 116) guarantees that the ensemble of random variables $W_{C_{\mathbf{x}}}$ in $\mathbb{F}_{q}^{n}$ is $\sqrt{\varepsilon q^{-t}}$-biased.

In the following, we treat the case of $\mathcal{A}=\mathbb{F}_{q}^{n}$. Given a joint sub-distribution $P_{A, E}$ on $\mathcal{A} \times \mathcal{E}$ and a normalized distribution $P_{W}$ on $\mathcal{A}$, we define another joint sub-distribution $P_{A, E} * P_{W}(a, e):=\sum_{w} P_{W}(w) P_{A, E}(a-w, e)$. Using these concepts, Dodis and Smith[9] evaluated the average of $d_{2}\left(A|E| P_{A, E} * P_{W_{\mathbf{x}}} \| Q_{E}\right)$ as follows.

Proposition 35 ([9] Lemma 4]): For any joint sub-distribution $P_{A, E}$ on $\mathcal{A} \times \mathcal{E}$ and any normalized distribution $Q_{E}$ on $\mathcal{E}$, a $\delta$-biased ensemble of random variables $\left\{W_{\mathbf{X}}\right\}$ on $\mathcal{A}=\mathbb{F}_{q}^{n}$ satisfies

$$
\begin{equation*}
\mathrm{E}_{\mathbf{X}} d_{2}\left(A|E| P_{A, E} * P_{W_{\mathbf{x}}} \| Q_{E}\right) \leq \delta^{2} e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)} \tag{117}
\end{equation*}
$$

More precisely,

$$
\begin{align*}
& \mathrm{E}_{\mathbf{X}} d_{2}\left(A|E| P_{A, E} * P_{W_{\mathbf{X}}} \| Q_{E}\right) \\
\leq & \delta^{2} d_{2}\left(A|E| P_{A, E} \| Q_{E}\right) \tag{118}
\end{align*}
$$

The original proof by Dodis and Smith [9] discussed in the case with $q=2$. Fehr and Schaffner [10] extended this lemma to the quantum setting in the case with $q=2$. Their proof is based on Fourier analysis and easy to understand. The proof with a general prime power $q$ is given latter. by generalizing the idea by Fehr and Schaffner [10]. Dodis and Smith[9, Lemma 6] also considered the case with a general prime power $q$. They did not explicitly give Proposition 35 and the definition (115) with a general prime power $q$.

Proposition 12 essentially coincides with Proposition 35 However, the concept " $\delta$-biased" does not concern a linear random hash function while the concept " $\varepsilon$-almost dual universality ${ }_{2}$ " does it because the former is defined for the ensemble of random variables. That is, the latter is a generalization of a universal ${ }_{2}$ linear hash function while the former does not. Hence, Proposition 35 cannot directly provide the performance of a linear random hash function. In contrast, Proposition 12 gives how the privacy amplification by a linear hash function decreases the leaked information. Therefore, in the main part of this paper, using Proposition 12, we treat the exponential decreasing rate when we apply the privacy amplification by an $\varepsilon$-almost dual universal ${ }_{2}$ linear hash function.

Proof of Proposition 12, Due to Lemma 34 and 117, we obtain

$$
\begin{equation*}
\mathrm{E}_{\mathbf{X}} d_{2}\left(A|E| P_{A, E} * P_{W_{C_{\mathbf{X}}}} \| Q_{E}\right) \leq \varepsilon q^{-t} e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)} \tag{119}
\end{equation*}
$$

Denoting the quotient class with respect to the subspace $C$ with the representative $a \in \mathcal{A}$ by $[a]$, we obtain

$$
\begin{aligned}
P_{A, E} * P_{W_{C}}(a, e) & =\sum_{w \in C} q^{-t} P_{A, E}(a-w, e) \\
& =q^{-t} P_{A, E}([a], e)
\end{aligned}
$$

Now, we focus on the relation $\mathcal{A} \cong \mathcal{A} / C \times C \cong f_{C}(\mathcal{A}) \times C$. Then,

$$
P_{A, E} * P_{W_{C_{\mathbf{X}}}}(b, w, e)=q^{-t} P_{f_{C}(A), E}(b, e)
$$

Thus,

$$
\begin{align*}
& d_{2}\left(A|E| P_{A, E} * P_{W_{C}} \| Q_{E}\right) \\
= & q^{-t} d_{2}\left(f_{C}(A)|E| P_{f_{C}(A), E} \| Q_{E}\right) \\
= & q^{-t} d_{2}\left(f_{C}(A)|E| P_{A, E} \| Q_{E}\right) \tag{120}
\end{align*}
$$

Therefore, 119 implies

$$
\begin{aligned}
& \mathrm{E}_{\mathbf{X}} q^{-t} d_{2}\left(f_{C_{\mathbf{X}}}(A)|E| P_{A, E} \| Q_{E}\right) \\
\leq & \varepsilon q^{-t} e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)}
\end{aligned}
$$

which implies 41.
Similarly, Lemma 34, 118, and (120) imply that

$$
\begin{aligned}
& \mathrm{E}_{\mathbf{X}} q^{-t} d_{2}\left(f_{C \mathbf{X}}(A)|E| P_{A, E} \| Q_{E}\right) \\
\leq & \varepsilon q^{-t} e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)}
\end{aligned}
$$

Since $\mathrm{E}_{\mathbf{X}} d_{2}\left(f_{C_{\mathbf{X}}}(A)|E| P_{A, E} \| Q_{E}\right)=\mathrm{E}_{\mathbf{x}} e^{-H_{2}\left(f_{C_{\mathbf{X}}}(A)|E| P_{A, E} \| Q_{E}\right)}-\frac{1}{q^{n-t}} e^{D_{2}\left(P_{E} \| Q_{E}\right)}$, we have (42).
To start our proof of Proposition 35, we make preparation before our proof of Proposition 35, First, remember that $\mathcal{A}$ is a vector space $\mathbb{F}_{q}^{n}$ and $\mathcal{E}$ is a general discrete set. We define the $\ell^{2}$ norm over the space $L^{2}(\mathcal{A} \times \mathcal{E})$ as

$$
\begin{equation*}
\|f\|_{2}^{2}:=\sum_{a \in \mathcal{A}, e \in \mathcal{E}}|f(a, e)|^{2}, \quad \forall f \in L^{2}(\mathcal{A} \times \mathcal{E}) \tag{121}
\end{equation*}
$$

Then, we define the discrete Fourier transform $\mathcal{F}$ on $L^{2}(\mathcal{A} \times \mathcal{E})$ as

$$
\begin{equation*}
\mathcal{F}(f)\left(a^{\prime}, e\right):=q^{-\frac{n}{2}} \sum_{a \in \mathcal{A}} \omega_{p}^{\left(a^{\prime} \cdot a\right)} f(a, e), \quad \forall f \in L^{2}(\mathcal{A} \times \mathcal{E}), \forall a^{\prime} \in \mathcal{A}, \forall e \in \mathcal{E} \tag{122}
\end{equation*}
$$

which satisfies $\|\mathcal{F} f\|_{2}=\|f\|_{2}$. For $\forall f, g \in L^{2}(\mathcal{A} \times \mathcal{E})$, the convolution $f * g$ :

$$
\begin{equation*}
f * g(a, e):=\sum_{a^{\prime} \in \mathcal{A}} f\left(a-a^{\prime}, e\right) g\left(a^{\prime}, e\right) \tag{123}
\end{equation*}
$$

satisfies

$$
\begin{equation*}
\mathcal{F}(f * g)(a, e)=q^{\frac{n}{2}} \mathcal{F}(f)(a, e) \mathcal{F}(g)(a, e) \tag{124}
\end{equation*}
$$

We prepare the following lemma.

Lemma 36: When $f_{P_{A, E}, Q_{E}} \in L^{2}(\mathcal{A} \times \mathcal{E})$ is defined as

$$
\begin{equation*}
f_{P_{A, E}, Q_{E}}(a, e):=P_{A, E}(a, e) Q_{E}(e)^{-\frac{1}{2}} \tag{125}
\end{equation*}
$$

we have

$$
\begin{align*}
\left\|f_{P_{A, E}, Q_{E}}\right\|_{2}^{2} & =e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)}  \tag{126}\\
\sum_{e \in \mathcal{E}}\left|\mathcal{F}\left(f_{P_{A, E}, Q_{E}}\right)(0, e)\right|^{2} & =e^{D_{2}\left(P_{E} \| Q_{E}\right)}  \tag{127}\\
\sum_{a \neq 0 \in \mathcal{A} e \in \mathcal{E}}\left|\mathcal{F}\left(f_{P_{A, E}, Q_{E}}\right)(a, e)\right|^{2} & =d_{2}\left(A|E| P_{A, E} \| Q_{E}\right) \tag{128}
\end{align*}
$$

Proof: 126 and 127) are shown as follows.

$$
\begin{aligned}
\left\|f_{P_{A, E}, Q_{E}}\right\|_{2}^{2} & =\sum_{a, e}\left(P_{A, E}(a, e) Q_{E}(e)^{-\frac{1}{2}}\right)^{2}=e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)} \\
\sum_{e \in \mathcal{E}}\left|\mathcal{F}\left(f_{P_{A, E}, Q_{E}}\right)(0, e)\right|^{2} & =\sum_{e}\left(\sum_{a} P_{A, E}(a, e) Q_{E}(e)^{-\frac{1}{2}}\right)^{2}=\sum_{e}\left(P_{E}(e) Q_{E}(e)^{-\frac{1}{2}}\right)^{2}=e^{D_{2}\left(P_{E} \| Q_{E}\right)}
\end{aligned}
$$

(128) is shown as follows.

$$
\begin{aligned}
& \sum_{a \neq 0 \in \mathcal{A}, e \in \mathcal{E}}\left|\mathcal{F}\left(f_{P_{A, E}, Q_{E}}\right)(a, e)\right|^{2}=\left\|\mathcal{F}\left(f_{P_{A, E}, Q_{E}}\right)\right\|_{2}^{2}-\sum_{e \in \mathcal{E}}\left|\mathcal{F}\left(f_{P_{A, E}, Q_{E}}\right)(0, e)\right|^{2} \\
= & \left\|f_{P_{A, E}, Q_{E}}\right\|_{2}^{2}-\sum_{e \in \mathcal{E}}\left|\mathcal{F}\left(f_{P_{A, E}, Q_{E}}\right)(0, e)\right|^{2} \\
= & e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)}-e^{D_{2}\left(P_{E} \| Q_{E}\right)}=d_{2}\left(A|E| P_{A, E} \| Q_{E}\right) .
\end{aligned}
$$

Proof of Proposition 35. Now, we choose $g_{\mathbf{X}} \in L^{2}(\mathcal{A} \times \mathcal{E})$ as

$$
\begin{equation*}
g_{\mathbf{X}}(a, e):=P_{W_{\mathbf{X}}}(a) \tag{129}
\end{equation*}
$$

Then,

$$
\begin{equation*}
f_{P_{A, E}, Q_{E}} * g_{\mathbf{X}}=f_{P_{A, E} * P_{W_{\mathbf{X}}}, Q_{E}} \tag{130}
\end{equation*}
$$

The assumption yields that

$$
\begin{equation*}
\mathrm{E}_{\mathbf{X}}\left|\mathcal{F}\left(g_{\mathbf{X}}\right)(a, e)\right|^{2}=\mathrm{E}_{\mathbf{X}}\left|q^{-\frac{n}{2}} \sum_{a \in \mathcal{A}} \omega_{p}^{\left(a^{\prime} \cdot a\right)} P_{W_{\mathbf{X}}}(a)\right|^{2} \leq \delta^{2} q^{-n} \tag{131}
\end{equation*}
$$

for $a^{\prime} \neq 0 \in \mathcal{A}$. Hence,

$$
\begin{align*}
& \mathrm{E}_{\mathbf{X}} d_{2}\left(A|E| P_{A, E} * P_{W_{\mathbf{X}}} \| Q_{E}\right) \stackrel{(a)}{=} \mathrm{E}_{\mathbf{X}} \sum_{a \neq 0 \in \mathcal{A}, e \in \mathcal{E}}\left|\mathcal{F}\left(f_{P_{A, E} * P_{W_{\mathbf{X}}}, Q_{E}}\right)(a, e)\right|^{2} \\
& \stackrel{(b)}{=} \mathrm{E}_{\mathbf{X}} \sum_{a \neq 0 \in \mathcal{A}, e \in \mathcal{E}}\left|\mathcal{F}\left(f_{P_{A, E}, Q_{E}} * g_{\mathbf{X}}\right)(a, e)\right|^{2} \stackrel{(c)}{=} \mathrm{E}_{\mathbf{X}} \sum_{a \neq 0 \in \mathcal{A}, e \in \mathcal{E}}\left|q^{\frac{n}{2}} \mathcal{F}\left(f_{P_{A, E}, Q_{E}}\right)(a, e) \mathcal{F}\left(g_{\mathbf{X}}\right)(a, e)\right|^{2} \\
& \stackrel{(d)}{\leq} \delta^{2} \mathrm{E}_{\mathbf{X}} \sum_{a \neq 0, e}\left|\mathcal{F}\left(f_{P_{A, E}, Q_{E}}\right)(a, e)\right|^{2} \stackrel{(e)}{=} \delta^{2} d_{2}\left(A|E| P_{A, E} \| Q_{E}\right) \leq \delta^{2} e^{-H_{2}\left(A|E| P_{A, E} \| Q_{E}\right)} \tag{132}
\end{align*}
$$

which shows (117) and (118). Here, $(a),(b),(c),(d)$, and $(e)$ follow from (128), 130), 124), 131), and (128), respectively.

## Appendix F <br> PROOFS OF COMPARISONS OF EXPONENTS

## A. Proof of Lemma 27

Inequality (91) can be shown from (15). Lemma 4 yields that

$$
\begin{align*}
& \frac{1}{2} e_{I}\left(P_{A, E} \mid R\right) \\
= & \max _{0 \leq s \leq 1} \frac{s}{2} H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-\frac{s}{2} R \\
\leq & \max _{0 \leq s \leq 1} \frac{s}{2} H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-\frac{s}{2} R \\
= & \max _{0 \leq t \leq 1 / 2} \frac{t}{2(1-t)}\left(H_{\frac{1}{1-t}}^{\uparrow}\left(A|E| P_{A, E}\right)-R\right) \\
\leq & \max _{0 \leq t \leq 1 / 2} t\left(H_{\frac{1}{1-t}}\left(A|E| P_{A, E}\right)-R\right)  \tag{133}\\
= & e_{d}\left(P_{A, E} \mid R\right),
\end{align*}
$$

where $t=\frac{s}{1+s}$, i.e., $s=\frac{t}{1-t}$. Inequality (133) follows from the non-negativity of the RHS of (133) and the inequality $\frac{1}{2(1-t)} \leq 1$.

## B. Proof of Lemma 29

Lemma 7 implies that

$$
H_{\frac{1}{1-s}}^{\uparrow}\left(A|E| P_{A, E}\right)<H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)
$$

Choosing $t=\frac{s}{1+s}$, we have

$$
\begin{aligned}
& \max _{0 \leq s} \frac{s\left(H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-R\right)}{1+2 s} \\
= & \max _{0 \leq t \leq 1} \frac{t\left(H_{\frac{1}{1-t}}^{\uparrow}\left(A|E| P_{A, E}\right)-R\right)}{1+t} \\
< & \max _{0 \leq t \leq 1} \frac{t\left(H_{1+t}^{\uparrow}\left(A|E| P_{A, E}\right)-R\right)}{1+t},
\end{aligned}
$$

which implies 97). Similarly, since $H_{1+t}\left(A|E| P_{A, E}\right)$ is strictly monotonically increasing with respect to $t$,

$$
\begin{aligned}
& \max _{0 \leq s} \frac{s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R}{1+s} \\
= & \max _{0 \leq t \leq 1} t H_{\frac{1}{1-t}}\left(A|E| P_{A, E}\right)-t R \\
< & \max _{0 \leq t \leq 1} t H_{1+t}\left(A|E| P_{A, E}\right)-t R,
\end{aligned}
$$

which implies 98).
When $P_{A \mid E=e}$ is a uniform distribution for any element $e \in \mathcal{E}, H_{1+t}\left(A|E| P_{A, E}\right)$ and $H_{1+t}^{\uparrow}\left(A|E| P_{A, E}\right)$ do not depend on $t$.
Hence, we obtain $\max _{0 \leq s} \frac{s\left(H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-R\right)}{1+2 s}=\max _{0 \leq t \leq 1} \frac{t\left(H_{1+t}^{\uparrow}\left(A|E| P_{A, E}\right)-R\right)}{1+t}=\frac{H\left(A|E| P_{A, E}\right)-R}{2}$ and $\max _{0 \leq s} \frac{s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R}{1+s}=$ $\max _{0 \leq t \leq 1} t H_{1+t}\left(A|E| P_{A, E}\right)-t R=H\left(A|E| P_{A, E}\right)-R$, which imply the equalities $e_{d}\left(P_{A, E} \mid R\right)=\tilde{e}_{d}\left(P_{A, E} \mid R\right)$ and $e_{I}\left(P_{A, E} \mid R\right)=$ $\tilde{e}_{I}\left(P_{A, E} \mid R\right)$.

## Appendix G

## Smoothing bound of min entropy

## A. Proof of (96) of Theorem 28

First, $\Delta_{I, \min }\left(e^{n R}, \varepsilon \mid P_{A, E}^{n}\right)$ is the upper bound by the smoothing of min entropy in the modified mutual information criterion as is mentioned in 67]. Using the relation (82) in Theorem 24 we obtain

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \frac{-1}{n} \log \Delta_{I, \min }\left(e^{n R}, \varepsilon \mid P_{A, E}^{n}\right) \geq \max _{0 \leq s} \frac{s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R}{1+s} \tag{134}
\end{equation*}
$$

Now, we show the opposite inequality. Applying the Cramér Theorem [35], we obtain

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \frac{-1}{n} \log P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \mid P_{A \mid E}^{n}(a \mid e) \geq 2 e^{-n R^{\prime}}\right\} \\
= & \max _{0 \leq s} s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R^{\prime} . \tag{135}
\end{align*}
$$

Since $s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R^{\prime}$ is monotone decreasing with respect to $R^{\prime}$ and $R^{\prime}-R$ is monotone increasing with respect to $R^{\prime}$, we have

$$
\begin{equation*}
\max _{R^{\prime}} \min \left\{s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R^{\prime}, R^{\prime}-R\right\}=\frac{s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R}{1+s} \tag{136}
\end{equation*}
$$

because the solution of $s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R^{\prime}=R^{\prime}-R$ with respect to $R^{\prime}$ is $\frac{s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)+R}{1+s}$.
Using the lower bound (74) in Theorem 20 with $c=2$, 135), and (136), we have

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \frac{-1}{n} \log \min _{\varepsilon>0}\left(\eta(\varepsilon, n R)+e^{n R-H_{\min }^{\downarrow, \varepsilon}\left(A|E| P_{A, E}^{n}\right)}\right) \\
\leq & \lim _{n \rightarrow \infty} \frac{-1}{n} \log \min _{R^{\prime}} \eta\left(2 P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \mid P_{A \mid E}^{n}(a \mid e) \geq e^{-n 2 R^{\prime}}\right\}, \log e^{n R}\right)+e^{n R} e^{-n R^{\prime}} \\
= & \max _{R^{\prime}} \lim _{n \rightarrow \infty} \frac{-1}{n} \log \eta\left(2 P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \mid P_{A \mid E}^{n}(a \mid e) \geq e^{-n 2 R^{\prime}}\right\}, \log e^{n R}\right)+e^{n\left(R-R^{\prime}\right)} \\
= & \max _{R^{\prime}} \min \left\{\lim _{n \rightarrow \infty} \frac{-1}{n} \log \eta\left(2 P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \mid P_{A \mid E}^{n}(a \mid e) \geq e^{-n 2 R^{\prime}}\right\}, \log e^{n R}\right), R^{\prime}-R\right\} \\
= & \max _{R^{\prime}} \min \left\{\max _{0 \leq s} s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R^{\prime}, R^{\prime}-R\right\} \\
= & \max _{R^{\prime}} \max _{0 \leq s} \min \left\{s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R^{\prime}, R^{\prime}-R\right\} \\
= & \max _{0 \leq s} \max _{R^{\prime}} \min \left\{s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R^{\prime}, R^{\prime}-R\right\} \\
= & \max _{0 \leq s} \frac{s H_{1+s}^{\downarrow}\left(A|E| P_{A, E}\right)-s R}{1+s} . \tag{137}
\end{align*}
$$

Hence, we obtain (96).

## B. Proof of (95) of Theorem 28

The quantity $\Delta_{d, \min }\left(e^{n R}, \varepsilon \mid P_{A, E}^{n}\right)$ is the upper bound by smoothing of min entropy in the $L_{1}$ distinguishability criterion as is mentioned in 66). Using the relation (81) in Theorem 24 we obtain

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \frac{-1}{n} \log \Delta_{d, \min }\left(e^{n R}, \varepsilon \mid P_{A, E}^{n}\right) \geq \max _{0 \leq s} \frac{s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-s R}{1+2 s} \tag{138}
\end{equation*}
$$

We show the opposite inequality in (95) by using the following lemma. The proof of Lemma 37 will be shown latter.
Lemma 37: The following inequality

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \frac{-1}{n} \log \min _{Q_{E, n}} P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \left\lvert\, \frac{P_{A, E}^{n}(a, e)}{Q_{E, n}(e)} \geq 2 e^{-n R^{\prime}}\right.\right\} \\
\leq & \max _{0 \leq s} s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-s R^{\prime} . \tag{139}
\end{align*}
$$

Using (139) in Lemma 37 and the lower bound (72) in Theorem 20 with $c=2$, we obtain

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \frac{-1}{n} \log \left(\min _{\epsilon_{2}>0} 2 \epsilon_{1}+e^{\frac{1}{2} n R} e^{-\frac{1}{2} H_{\min }^{\downarrow, \epsilon_{1}}\left(A|E| P_{A}^{n}\right)}\right) \\
\leq & \lim _{n \rightarrow \infty} \frac{-1}{n} \log \left(\min _{R^{\prime}} \min _{Q_{E, n}} P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \left\lvert\, \frac{P_{A, E}^{n}(a, e)}{Q_{E, n}(e)} \geq 2 e^{-n R^{\prime}}\right.\right\}+e^{\frac{1}{2} n\left(R-R^{\prime}\right)}\right) \\
= & \max _{R^{\prime}} \lim _{n \rightarrow \infty} \frac{-1}{n} \log \left(\min _{Q_{E, n}} P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \left\lvert\, \frac{P_{A, E}^{n}(a, e)}{Q_{E, n}(e)} \geq 2 e^{-n R^{\prime}}\right.\right\}+e^{\frac{1}{2} n\left(R-R^{\prime}\right)}\right) \\
= & \max _{R^{\prime}} \min \left\{\lim _{n \rightarrow \infty} \frac{-1}{n} \log \left(\min _{Q_{E, n}} P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \left\lvert\, \frac{P_{A, E}^{n}(a, e)}{Q_{E, n}(e)} \geq 2 e^{-n R^{\prime}}\right.\right\}\right), \frac{R^{\prime}-R}{2}\right\} \\
\leq & \max _{R^{\prime}} \min \left\{\max _{0 \leq s} s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-s R^{\prime}, \frac{R^{\prime}-R}{2}\right\} \\
= & \max _{R^{\prime}} \max _{0 \leq s} \min \left\{s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-s R^{\prime}, \frac{R^{\prime}-R}{2}\right\} \\
= & \max _{0 \leq s} \max _{R^{\prime}} \min \left\{s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-s R^{\prime}, \frac{R^{\prime}-R}{2}\right\} . \tag{140}
\end{align*}
$$

Further, $s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-s R^{\prime}$ is monotone increasing with respect to $R^{\prime}$ and $\frac{R-R^{\prime}}{2}$ is monotone decreasing with respect to $R^{\prime}$. Solving the equation $s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-s R^{\prime}=\frac{R^{\prime}-R}{2}$ with respect to $R^{\prime}$, we have $R^{\prime}=\frac{2 s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)+R}{1+2 s}$, which implies that

$$
\max _{R^{\prime}} \min \left\{s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-s R^{\prime}, \frac{R^{\prime}-R}{2}\right\}=\frac{s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-s R}{1+2 s}
$$

Thus,

$$
\begin{aligned}
& \max _{0 \leq s} \max _{R^{\prime}} \min \left\{s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-s R^{\prime}, \frac{R^{\prime}-R}{2}\right\} \\
= & \max _{0 \leq s} \frac{s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-s R}{1+2 s} .
\end{aligned}
$$

Hence, we obtain (95).

Proof of Lemma 37. We show Lemma 37 by using Lemmas 38 and 40 which will be given latter. For any distribution $Q_{E, n}$, we define the permutation invariant distribution $Q_{E, n, \text { inv }}$ by

$$
Q_{E, n, \operatorname{inv}}(e):=\sum_{g \in S_{n}} \frac{1}{n!} Q_{E, n}(g(e))
$$

where $S_{n}$ is the $n$-th permutation group and $g(e)$ is the element permuted from $e \in \mathcal{E}^{n}$ by $g \in S_{n}$. Then, we have

$$
\begin{aligned}
& P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \left\lvert\, \frac{P_{A, E}^{n}(a, e)}{Q_{E, n}(e)} \geq 2 e^{-n R^{\prime}}\right.\right\} \\
= & P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \mid P_{A, E}^{n}(a, e) \geq 2 e^{-n R^{\prime}} Q_{E, n}(e)\right\} \\
\geq & \frac{1}{2} P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \mid P_{A, E}^{n}(a, e) \geq 4 e^{-n R^{\prime}} Q_{E, n, \operatorname{inv}}(e)\right\} \\
= & \frac{1}{2} P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \left\lvert\, \frac{P_{A, E}^{n}(a, e)}{Q_{E, n, \text { inv }}(e)} \geq 4 e^{-n R^{\prime}}\right.\right\},
\end{aligned}
$$

where the inequality follows from Lemma 38 Here, we denote the set of types of $\mathcal{E}$ by $T_{n, \mathcal{E}}$. For any element $Q_{E} \in T_{n, \mathcal{E}}$, we denote the uniform distribution over the subset of elements whose type is $Q_{E}$ by $\hat{Q}_{E}$. Now, we define the distribution

$$
Q_{E, n, \text { inv }, \text { mix }}(e):=\frac{1}{\left|T_{n, \mathcal{E}}\right|} \sum_{Q_{E} \in T_{n, \mathcal{E}}} \hat{Q}_{E}(e)
$$

Since $Q_{E, n, \text { inv }}(e) \leq\left|T_{n, \mathcal{E}}\right| Q_{E, n, \text { inv,mix }}(e)$, we have

$$
\begin{aligned}
& \frac{1}{2} P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n} \mid P_{A, E}^{n}(a, e) \geq 4 e^{-n R^{\prime}} Q_{E, n, \mathrm{inv}}(e)\right\} \\
\geq & \frac{1}{2} P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n}\left|P_{A, E}^{n}(a, e) \geq 4\right| T_{n, \mathcal{E}} \mid e^{-n R^{\prime}} Q_{E, n, \text { inv }, \text { mix }}(e)\right\}
\end{aligned}
$$

For given sequence $(a, e) \in \mathcal{A} \times \mathcal{E}$, we denote the type of $(a, e)$ by $P_{A, E}^{\prime}$ and its marginal distribution over $\mathcal{E}$ of $P_{A, E}^{\prime}$ by $P_{E}^{\prime}$. Then, $P_{A, E}^{n}(a, e)=e^{-n\left(D\left(P_{A, E}^{\prime} \| P_{A, E}\right)+H\left(P_{A, E}^{\prime}\right)\right)}$ and $\left|T_{n, \mathcal{E}}\right| Q_{E, n, \text { inv, mix }}(e)=e^{-n H\left(P_{E}^{\prime}\right)}$. That is, the condition $P_{A, E}^{n}(a, e) \geq$ $4\left|T_{n, \mathcal{E}}\right| e^{-n R^{\prime}} Q_{E, n, \text { inv,mix }}(e)$ is equivalent to the condition $D\left(P_{A, E}^{\prime} \| P_{A, E}\right)+H\left(P_{A, E}^{\prime}\right) \leq \frac{\log 4}{n}+H\left(P_{E}^{\prime}\right)+R^{\prime}$. We denote the set of sequences whose types are $P_{A, E}^{\prime}$ by $T_{P_{A, E^{\prime}}}$. Hence,

$$
\begin{aligned}
& \frac{1}{2} P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n}\left|P_{A, E}^{n}(a, e) \geq 4\right| T_{n, \mathcal{E}} \mid e^{-n R^{\prime}} Q_{E, n, \mathrm{inv}, \operatorname{mix}}(e)\right\} \\
= & \sum_{P_{A, E}^{\prime} \in T_{n, \mathcal{A} \times \mathcal{E}}: D\left(P_{A, E}^{\prime} \| P_{A, E}\right)+H\left(P_{A, E}^{\prime}\right) \leq \frac{\log 4}{n}+H\left(P_{E}^{\prime}\right)+R^{\prime}} \frac{1}{2} P_{A, E}^{n}\left(T_{\left.P_{A, E^{\prime}}\right)}\right) \\
\geq & \max _{P_{A, E}^{\prime} \in T_{n, \mathcal{A} \times \mathcal{E}}: D\left(P_{A, E}^{\prime} \| P_{A, E}\right)+H\left(P_{A, E}^{\prime}\right) \leq \frac{\log 4}{n}+H\left(P_{E}^{\prime}\right)+R^{\prime}} \frac{1}{2} P_{A, E}^{n}\left(T_{P_{A, E}^{\prime}}\right) .
\end{aligned}
$$

Since $P_{A, E}^{n}\left(T_{P_{A, E}^{\prime}}\right) \cong e^{-n D\left(P_{A, E}^{\prime} \| P_{A, E}\right)}$, taking the limit, we have

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{-1}{n} \log \frac{1}{2} P_{A, E}^{n}\left\{(a, e) \in \mathcal{A}^{n} \times \mathcal{E}^{n}\left|P_{A, E}^{n}(a, e) \geq 4\right| T_{n, \mathcal{E}} \mid e^{-n R^{\prime}} Q_{E, n, \text { inv,mix }}(e)\right\} \\
\leq & \max _{P_{A, E}^{\prime}}\left\{D\left(P_{A, E}^{\prime} \| P_{A, E}\right) \mid D\left(P_{A, E}^{\prime} \| P_{A, E}\right)+H\left(P_{A, E}^{\prime}\right) \leq R^{\prime}+H\left(P_{E}^{\prime}\right)\right\} \\
= & \max _{P_{A, E}^{\prime}}\left\{D\left(P_{A, E}^{\prime} \| P_{A, E}\right) \mid D\left(P_{A, E}^{\prime} \| P_{A, E}\right)+H\left(A|E| P_{A, E}^{\prime}\right) \leq R^{\prime}\right\} .
\end{aligned}
$$

Hence, combining Lemma 40, we obtain (139).
Lemma 38: The relation

$$
\begin{equation*}
P_{A}^{n}\left\{a \in \mathcal{A}^{n} \mid c \geq f(a)\right\} \geq \frac{1}{2} P_{\operatorname{mix}, \mathcal{A}}\left\{a \in \mathcal{A}^{n} \left\lvert\, c \geq \frac{1}{n!} \sum_{g \in S_{n}} f(g(a))\right.\right\} \tag{141}
\end{equation*}
$$

holds for any function $f$.
Proof: Lemma 38 can be shown by applying Lemma 39 to all of distributions conditioned with type.
Lemma 39: The relation

$$
\begin{equation*}
P_{\text {mix }, \mathcal{A}}\{a \mid c \geq f(a)\} \geq \frac{1}{2} P_{\text {mix }, \mathcal{A}}\left\{a \left\lvert\, c \geq \frac{1}{|\mathcal{A}|} \sum_{a} f(a)\right.\right\} \tag{142}
\end{equation*}
$$

holds for any function $f$.
Proof: Markov inequality implies that

$$
P_{\mathrm{mix}, \mathcal{A}}\{a \mid c<f(a)\} \leq \frac{1}{c} \frac{1}{|\mathcal{A}|} \sum_{a} f(a)
$$

When $c \geq \frac{2}{|\mathcal{A}|} \sum_{a} f(a), 1-\frac{1}{c} \frac{1}{|\mathcal{A}|} \sum_{a} f(a)$ is greater than $\frac{1}{2}$. Hence,

$$
P_{\mathrm{mix}, \mathcal{A}}\{a \mid c \geq f(a)\}=1-P_{\mathrm{mix}, \mathcal{A}}\{a \mid c<f(a)\} \geq 1-\frac{1}{c} \frac{1}{|\mathcal{A}|} \sum_{a} f(a) \geq \frac{1}{2} P_{\mathrm{mix}, \mathcal{A}}\left\{a \left\lvert\, c \geq \frac{2}{|\mathcal{A}|} \sum_{a} f(a)\right.\right\}
$$

Lemma 40: The relation

$$
\begin{align*}
& \min _{P_{A, E}^{\prime}}\left\{D\left(P_{A, E}^{\prime} \| P_{A, E}\right) \mid D\left(P_{A, E}^{\prime} \| P_{A, E}\right)+H\left(A|E| P_{A, E}^{\prime}\right) \leq R^{\prime}\right\} \\
= & \max _{0 \leq s} s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)-s R^{\prime} . \tag{143}
\end{align*}
$$

holds.
Proof: We show Lemma 40 by using Lemma 33, which will be given latter. We employ a generalization of the method used in [61, Appendix D]. First, we define the distribution $P_{A, E ; s}$ as

$$
P_{A, E ; s}(a, e):=\frac{P_{A \mid E}(a \mid e)^{1+s} P_{E}(e)}{\left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)^{\frac{s}{1+s}}\left(\sum_{e} P_{E}(e)\left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)^{\frac{1}{1+s}}\right)}
$$

That is, we have

$$
\begin{aligned}
P_{A \mid E ; s}(a \mid e) & =\frac{P_{A \mid E}(a \mid e)^{1+s}}{\sum_{a} P_{A \mid E}(a \mid e)^{1+s}} \\
P_{E ; s}(e) & =\frac{P_{E}(e)\left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)^{\frac{1}{1+s}}}{\left(\sum_{e} P_{E}(e)\left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)^{\frac{1}{1+s}}\right)} .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& D\left(P_{A, E ; s} \| P_{A, E}\right) \\
= & \sum_{a, e} P_{A, E ; s}(a, e)\left(s \log P_{A \mid E}(a \mid e)-\frac{s}{1+s} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right) \frac{s}{1+s} H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right), \\
= & H\left(A|E| P_{A, E ; s}\right) \\
& \sum_{a, e} P_{A, E ; s}(a, e)\left(-(1+s) \log P_{A \mid E}(a \mid e)+\log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right) \\
= & \sum_{a, e} P_{A, E ; s}(a, e)\left(-\log P_{A \mid E}(a \mid e)+\frac{1}{1+s} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right) \frac{s}{1+s} H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right) .
\end{aligned}
$$

Given $s \geq 0$, we choose an arbitrary distribution $P_{A, E}^{\prime}$ such that

$$
D\left(P_{s}^{A, E} \| P_{A, E}\right)=D\left(P_{A, E}^{\prime} \| P_{A, E}\right)
$$

Since

$$
\begin{aligned}
D\left(P_{A, E}^{\prime} \| P_{A, E}\right)= & \sum_{a, e} P_{A, E}^{\prime}(a, e)\left(\log P_{A, E}^{\prime}(a, e)-\log P_{A, E}(a, e)\right) \\
D\left(P_{A, E}^{\prime} \| P_{s}^{A, E}\right)= & \sum_{a, e} P_{A, E}^{\prime}(a, e)\left(\log P_{A, E}^{\prime}(a, e)--(1+s) \log P_{A \mid E}(a \mid e)-\log P_{E}(e)\right. \\
& \left.+\frac{s}{1+s} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)-\frac{s}{1+s} H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)\right)
\end{aligned}
$$

we have

$$
\begin{aligned}
& D\left(P_{A, E}^{\prime} \| P_{A, E ; s}\right)=D\left(P_{A, E}^{\prime} \| P_{A, E ; s}\right)+D\left(P_{A, E ; s} \| P_{A, E}\right)-D\left(P_{A, E}^{\prime} \| P_{A, E}\right) \\
= & \sum_{a, e} P_{A, E ; s}(a, e)\left(s \log P_{A \mid E}(a \mid e)-\frac{s}{1+s} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right) \frac{s}{1+s} H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right) \\
& -\sum_{a, e} P_{A, E}^{\prime}(a, e)\left(s \log P_{A \mid E}(a \mid e)-\frac{s}{1+s} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right)-\frac{s}{1+s} H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right) \\
= & \sum_{a, e}\left(P_{A, E ; s}(a, e)-P_{A, E}^{\prime}(a, e)\right)\left(s \log P_{A \mid E}(a \mid e)-\frac{s}{1+s} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right) .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& H\left(A|E| P_{A, E ; s}\right)-H\left(A|E| P_{A, E}^{\prime}\right)+D\left(P_{E}^{\prime} \| P_{E ; s}\right) \\
= & H\left(A|E| P_{A, E ; s}\right)+D\left(P_{A, E ; s} \| P_{A, E}\right)-\left(H\left(A|E| P_{A, E}^{\prime}\right)-D\left(P_{A, E}^{\prime} \| P_{A, E}\right)\right)+D\left(P_{E}^{\prime} \| P_{E ; s}\right) \\
= & \sum_{a, e} P_{A, E ; s}(a, e)\left(-\log P_{A \mid E}(a \mid e)+\frac{1}{1+s} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right)+\frac{s}{1+s} H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right) \\
& -\sum_{a, e} P_{A, E}^{\prime}(a, e)\left(-\log P_{A \mid E}(a \mid e)+\frac{1}{1+s} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right)+\frac{s}{1+s} H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right) \\
= & \sum_{a, e}\left(P_{A, E ; s}(a, e)-P_{A, E}^{\prime}(a, e)\right)\left(-\log P_{A \mid E}(a \mid e)+\frac{1}{1+s} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s}\right)\right) \\
= & -s D\left(P_{A, E}^{\prime} \| P_{A, E ; s}\right) \leq 0 .
\end{aligned}
$$

Since $D\left(P_{E}^{\prime} \| P_{E ; s}\right) \geq 0$, we have $H\left(A|E| P_{A, E ; s}\right) \leq H\left(A|E| P_{A, E}^{\prime}\right)$, which implies

$$
H\left(A|E| P_{A, E ; s}\right)+D\left(P_{A, E ; s} \| P_{A, E}\right) \leq H\left(A|E| P_{A, E}^{\prime}\right)+D\left(P_{A, E}^{\prime} \| P_{A, E}\right)
$$

Since the map $s \mapsto D\left(P_{A, E ; s} \| P_{A, E}\right)$ is continuous, we have

$$
\begin{aligned}
& \min _{P_{A, E}^{\prime}}\left\{D\left(P_{A, E}^{\prime} \| P_{A, E}\right) \mid D\left(P_{A, E}^{\prime} \| P_{A, E}\right)+H\left(A|E| P_{A, E}^{\prime}\right) \leq R^{\prime}\right\} \\
= & \min _{s \geq 0}\left\{D\left(P_{A, E ; s} \| P_{A, E}\right) \mid D\left(P_{A, E ; s} \| P_{A, E}\right)+H\left(A|E| P_{A, E ; s}\right) \leq R^{\prime}\right\} .
\end{aligned}
$$

Now, we choose $s_{0} \geq 0$ such that

$$
\begin{aligned}
& D\left(P_{s_{0}}^{A, E} \| P_{A, E}\right)+H\left(A|E| P_{s_{0}}^{A, E}\right) \\
= & \sum_{a, e} P_{s_{0}}^{A, E}(a, e)\left(-\log P_{A \mid E}(a \mid e)+\frac{1}{1+s_{0}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s_{0}}\right)\right)+\frac{s_{0}}{1+s_{0}} H_{1+s_{0}}^{\uparrow}\left(A|E| P_{A, E}\right) \\
= & R^{\prime}
\end{aligned}
$$

which implies that

$$
\sum_{a, e} P_{s_{0}}^{A, E}(a, e)\left(-\log P_{A \mid E}(a \mid e)+\frac{1}{1+s_{0}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s_{0}}\right)\right)=R^{\prime}-\frac{s_{0}}{1+s_{0}} H_{1+s_{0}}^{\uparrow}\left(A|E| P_{A, E}\right)
$$

Then,

$$
\begin{aligned}
& \min _{s \geq 0}\left\{D\left(P_{A, E ; s} \| P_{A, E}\right) \mid D\left(P_{A, E ; s} \| P_{A, E}\right)+H\left(A|E| P_{A, E ; s}\right) \leq R^{\prime}\right\} \\
= & \sum_{a, e} P_{s_{0}}^{A, E}(a, e)\left(s_{0} \log P_{A \mid E}(a \mid e)-\frac{s_{0}}{1+s_{0}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s_{0}}\right)\right)+\frac{s_{0}}{1+s_{0}} H_{1+s_{0}}^{\uparrow}\left(A|E| P_{A, E}\right) \\
= & -s_{0} \sum_{a, e} P_{s_{0}}^{A, E}(a, e)\left(-\log P_{A \mid E}(a \mid e)+\frac{1}{1+s_{0}} \log \left(\sum_{a} P_{A \mid E}(a \mid e)^{1+s_{0}}\right)\right)+\frac{s_{0}}{1+s_{0}} H_{1+s_{0}}^{\uparrow}\left(A|E| P_{A, E}\right) \\
= & -s_{0}\left(R^{\prime}+\phi\left(\left.\frac{s_{0}}{1+s_{0}}|A| E \right\rvert\, P_{A, E}\right)\right)+\frac{s_{0}}{1+s_{0}} H_{1+s_{0}}^{\uparrow}\left(A|E| P_{A, E}\right) \\
= & -s_{0} R^{\prime}+s_{0} H_{1+s_{0}}^{\uparrow}\left(A|E| P_{A, E}\right) \\
= & \max _{s \geq 0}-s R^{\prime}+s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right),
\end{aligned}
$$

where the reason of the equation is the following. Due to Lemma 33, the function $s \mapsto-s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$ is convex, and $-R^{\prime}=-\frac{d}{d s} s H_{1+s}^{\uparrow}\left(A|E| P_{A, E}\right)$. Then, we obtain (143).

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[^1]:    ${ }^{1}$ This argument is true even in the classical case. In the quantum case, there are several variants for information spectrum. Hence, we cannot say that the smoothing of the min entropy is essentially the same as the method of information spectrum. Indeed, the previous paper [17] discussed this problem only with fidelity distance.

[^2]:    ${ }^{2}$ Historically, the earlier version of this paper showed Lemma 5 at the first time. Then, the paper [66] extended this inequality to the quantum setting.

