On Non-Interactive Simulation of Joint Distributions

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Abstract

We consider the following non-interactive simulation problem: Alice and Bob observe sequences X^n and Y^n respectively where $\{(X_i, Y_i)\}_{i=1}^n$ are drawn i.i.d. from P(x, y), and they output U and V respectively which is required to have a joint law that is close in total variation to a specified Q(u, v). It is known that the maximal correlation of U and V must necessarily be no bigger than that of X and Y if this is to be possible. Our main contribution is to bring hypercontractivity to bear as a tool on this problem. In particular, we show that if P(x, y) is the doubly symmetric binary source, then hypercontractivity provides stronger impossibility results than maximal correlation. Finally, we extend these tools to provide impossibility results for the k-agent version of this problem.

I. INTRODUCTION

The problem of simulating random variables by two agents with suitable resource constraints has had a rich history leading to different formulations of this problem in the literature. The general setup for the problem is as follows: Two or more agents wish to simulate a specified joint distribution under resource constraints in the form of limited communication, limited common randomness provided to all of them, or limited correlation between their observations. One then wishes to find the minimum resources required to achieve the desired goal.

The simulation problem has natural applications in numerous areas — from game-theoretic co-ordination in a network against an adversary to control of a dynamical system over a distributed network. These problems are expected to be important in many future technologies with remote-controlled applications, such as Amazon's drone-based delivery system [1] and robotic environmental cleanup, vegetation management, land clearing, and bio-mass harvesting [2]. In these technologies, individual robotic components would need to take randomized actions under limited or no communication with other components or the central system. Study of the simulation problem can provide fundamental limits on the capabilities of such robotic components and guide efficient usage of the available resources.

The earliest studied two-agent simulation problems were considered by Gács and Körner [3], and Wyner [4]. One may interpret their results, which we will describe shortly, in the framework of a generalization of both their problem setups as shown in Fig. 1. Let the random variables X, Y, U, V shown take values in finite sets.

In this formulation, two agents each having access to its own infinite stream of private randomness, observe n i.i.d. copies of samples generated according to a specified law P(x, y) as shown, and are required to output nR samples drawn from a distribution that is close (in total variation) to the the distribution constructed by taking i.i.d. copies of a specified law Q(u, v). Let the simulation capacity R^* be defined as the supremum of all rates for which given any $\epsilon > 0$, it is possible for some n to carry out this task to within total variation distance ϵ .

- When Q(u, v) is described by $U = V \sim \text{Ber}(1/2)$, and P(x, y) is a general distribution, this problem considers fundamental limits for *extracting* common randomness from the distribution of (X, Y). Gács and Körner showed in [3] that we have the *simulation capacity* $R^* = K(X;Y)$, which has come to be known as the Gács-Körner common information of X and Y. This quantity K(X;Y) can be described as $\sup H(\Theta)$ where $\Theta = f(X) = g(Y)$. In other words, the simulation capacity is non-zero only when the distribution of (X,Y) is *decomposable*, i.e. \mathcal{X} may be partitioned as $\mathcal{X}_1 \cup \mathcal{X}_2$ and \mathcal{Y} may be partitioned as $\mathcal{Y}_1 \cup \mathcal{Y}_2$ so that $\Pr(X \in \mathcal{X}_1, Y \in \mathcal{Y}_2) = \Pr(X \in \mathcal{X}_2, Y \in \mathcal{Y}_1) = 0$ and $\Pr(X \in \mathcal{X}_1, Y \in \mathcal{Y}_1)$, $\Pr(X \in \mathcal{X}_2, Y \in \mathcal{Y}_2) > 0$. Further, they showed that in general, $K(X;Y) \leq I(X;Y)$.
- When P(x, y) is described by $X = Y \sim Ber(1/2)$, and Q(u, v) is a general distribution, this problem considers fundamental limits for common randomness needed for *generating* the random variable pair (U, V). Wyner

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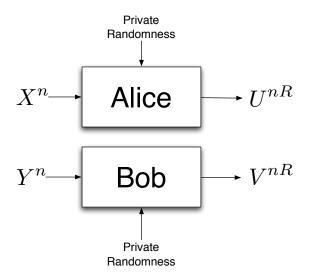


Fig. 1. A generalization of the problem setups considered by Gács-Körner [3] and Wyner [4]

showed in [4] that the amount of common information needed for generation per sample is $(R^*)^{-1} = C(U; V)$, which has come to be known as the Wyner common information of U and V. This quantity C(U; V) can be described as $\sup I(\Theta; U, V)$ over all Θ satisfying $U - \Theta - V$ with cardinality bound on the variable Θ given by $|\Theta| \leq |\mathcal{U}| \cdot |\mathcal{V}|$. Further, Wyner showed that $C(U; V) \geq I(U; V)$ in general. To be precise, Wyner considered a problem setting that required (U, V) to be simulated with vanishing normalized relative entropy, i.e. if $Q'(u^{nR}, v^{nR})$ is the law of the simulated samples, and Q(u, v) was the target distribution, then simulation is considered possible in Wyner's formulation if

$$\frac{1}{nR} D\left(Q'(u^{nR}, v^{nR}) || \Pi_{i=1}^{nR} Q(u_i, v_i)\right) \to 0.$$
(1)

It has been recognized that the simulation capacity remains the same under the vanishing total variation constraint [5, Lemma 5], [6, Lemma IV.1]. A recent work [7] considers a variant of Wyner's problem with *exact* generation of random variables as opposed to generation with a vanishing total variation distance.

The problem of characterizing R^* is open for general distributions P(x, y) and Q(u, v), and so is the problem of characterizing when $R^* > 0$.

In another stream of related work, the problem of simulation has been considered under rate-limited interaction between the agents. This began with the work of Cuff [8] who studied communication requirements for simulating a channel with rate-limited communication and rate-limited common randomness. [9] studied communication requirements for establishing dependence among nodes in a network setting. The former setup (of Cuff [8]) was generalized by Gohari and Anantharam in [10] (see Fig. 2). Two agents wish to simulate i.i.d. samples of a specified joint distribution P(x, y, u, v). Nature supplies i.i.d. copies of (X, Y) with the right marginal distribution as shown and the agents can use a certain rate of common randomness, certain rate-limited communication, and infinite streams of individual private randomness to accomplish the desired task. We want to understand the fundamental trade-offs between these rates to make this task possible. This problem was completely solved by Yassaee, Gohari, and Aref in [11]. However, this work does not address the problem of computing the simulation capacity R^* for the setup in Fig. 1, since the problem formulation there is different in two respects: In Fig. 2, the task is to output n samples while in Fig. 1, the task is to output nR samples. Furthermore, even if R were say chosen to be 1, in Fig. 2, the joint distribution of the quadruple (X^n, Y^n, U^n, V^n) is required to be close to i.i.d. copies of a specified joint distribution. However, in Fig. 1, the requirement is only on the marginal distribution of the output samples (U^n, V^n) and the quadruple (X^n, Y^n, U^n, V^n) need not even be close to an i.i.d. distribution.

In this paper, we consider the former non-interactive simulation setup à la Gács-Körner and Wyner (Fig. 1). Since the problem of characterizing whether $R^* > 0$ for general distributions P(x, y) and Q(u, v), is also non-trivial, we propose a relaxed problem where two agents observe an arbitrary finite number of samples drawn i.i.d. from P(x, y) as shown in Fig. 3 and are required to output one random variable each with the requirement that the output distribution be close in total variation to a specified Q(u, v). Clearly, if it is impossible to generate even a

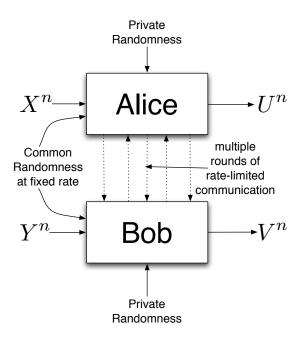


Fig. 2. Generalization of Cuff's formulation [8] by Gohari and Anantharam [10]

single sample, we must have $R^* = 0$. We therefore focus on impossibility results for this problem which will be relevant to the formulation in Fig. 1. It is not clear if the converse is true, i.e. it is unclear whether the feasibility of generating one sample asymptotically implies that we may generate samples at a rate R > 0.

Note that the notion of *simulation* we consider is distinct from the notion of *exact generation* wherein a certain distribution is required to be generated exactly. If we have a strategic setting, such as a distributed game, in which a player, represented by a number of distributed agents, is playing against an adversary, the agents would often need to generate a joint distribution exactly [12], to avoid providing unforeseen strategic advantages to the adversary.

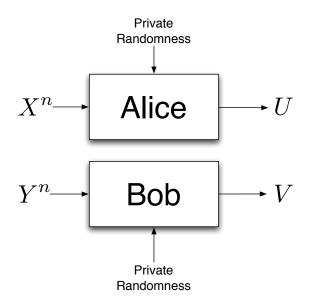


Fig. 3. The non-interactive simulation problem considered in this paper

When $(U, V) \sim Q(u, v)$ is described by $U = V \sim Ber(1/2)$ while P(x, y) is a general distribution, the problem has recently come to be called *non-interactive correlation distillation* [13], [14]. We therefore, call our formulation

the problem of *non-interactive simulation of joint distributions*. In a remarkable strengthening of the Gács-Körner result [3], Witsenhausen showed in [15] that unless the Gács-Körner common information K(X;Y) is positive (i.e. the joint distribution of (X,Y) is decomposable), non-interactive correlation distillation is impossible to achieve. The chief tool used in Witsenhausen's proof is the *maximal correlation* of two random variables, a quantity which will be of prime importance in the present paper as well.

The second tool that we will be using is *hypercontractivity*, which has found numerous applications in mathematics, physics, and theoretical computer science. The origins of hypercontractivity lie in the early works of Bonami [16], [17], of Nelson [18] in quantum field theory, of Gross [19] who first developed the connection to logarithmic Sobolev inequalities, and of Beckner [20]. The meaning of hypercontractivity was broadened by Borell [21] to what is sometimes called reverse hypercontractivity today [22]. Hypercontractivity has found powerful applications in a lot of fields, for example the study of influence of variables on Boolean functions [23], [24], [25] and in voting system theory [26]. Ahlswede and Gács [27] identified the use of hypercontractivity in studying the spreading of sets in high dimensional product spaces. In recent works, [28] showed an equivalence between hypercontractivity and strong data processing inequalities for Rényi divergences, [29] used hypercontractivity to show non-vanishing lower bounds on hypothesis testing, [30] studied hypercontractivity for a noise operator that computed spherical averages in Hamming space, [31] showed a connection between hypercontractivity and strong data processing inequalities for mutual information, and [32] used hypercontractivity to study the mutual information between Boolean functions. As we shall see, hypercontractivity has properties that make it naturally well-suited for studying the non-interactive simulation problem.

Let us formally set up the non-interactive simulation problem described earlier.

Definition 1. Let $\mathcal{X}, \mathcal{Y}, \mathcal{U}, \mathcal{V}$ denote finite sets. Given a *source distribution* P(x, y) over $\mathcal{X} \times \mathcal{Y}$ and a *target distribution* Q(u, v) over $\mathcal{U} \times \mathcal{V}$, we say that *non-interactive simulation* of Q(u, v) using P(x, y) is possible, if for any $\epsilon > 0$, there exists a positive integer n, a finite set \mathcal{R} , and functions $f : \mathcal{X}^n \times \mathcal{R} \mapsto \mathcal{U}, g : \mathcal{Y}^n \times \mathcal{R} \mapsto \mathcal{V}$ such that

$$d_{\mathrm{TV}}\left((f(X^n, M_X), g(Y^n, M_Y)); (U, V)\right) \le \epsilon$$

where $\{(X_i, Y_i)\}_{i=1}^n$ is a sequence of i.i.d. samples drawn from P(x, y), M_X, M_Y are uniformly distributed in \mathcal{R} and are mutually independent of each other and the samples from the source, (U, V) is drawn from Q(u, v) and $d_{\text{TV}}(\cdot; \cdot)$ is the total variation distance (defined as half the L_1 distance between the distributions).

For a fixed P(x, y), the set of distributions Q(u, v) on a fixed set $\mathcal{U} \times \mathcal{V}$ for which non-interactive simulation is possible is precisely the closure of the set of marginal distributions of (U, V) satisfying $U - X^k - Y^k - V$ for some k. However, this set of distributions appears to be very hard to characterize explicitly. In this paper, we focus on outer bounds on this set, or in other words impossibility results for non-interactive simulation.

Note that since we are interested only in determining the possibility of simulation and not in the simulation capacity, the problem does not have any less generality if we disallow the agents from using any private randomness, since agents can obtain as much private randomness as desired by using extended observations that are non-overlapping in time, i.e. the agents observe $n_1+n_2+n_3$ symbols, they use $(X_1, \ldots, X_{n_1}), (Y_1, \ldots, Y_{n_1})$ respectively as their correlated observations, Alice uses $X_{n_1+1}, \ldots, X_{n_2}$ as her private randomness, and Bob uses $Y_{n_2+1}, \ldots, Y_{n_3}$ as his private randomness. We make the choice to assume the availability of private randomness as part of the model.

We will consider two examples to motivate the focus of this study.

A. Example 1

Let X be a uniform Bernoulli random variable, $X \sim \text{Ber}(\frac{1}{2})$. Let Y be a noisy copy of X, i.e. Y = X + Nwhere $N \sim \text{Ber}(\alpha)$ for $0 < \alpha < \frac{1}{2}$, is independent of X. Here, the addition is modulo 2. We say that (X, Y)has the *doubly symmetric binary source* distribution with parameter α , denoted $\text{DSBS}(\alpha)$ following the notation of Wyner [4]. We consider $(U, V) \sim \text{DSBS}(\beta)$ for $0 \le \beta < \frac{1}{2}$. We may ask whether non-interactive simulation of $Q(u, v) = \text{DSBS}(\beta)$ using $P(x, y) = \text{DSBS}(\alpha)$ is possible. Witsenhausen answered this question in the negative when $\beta < \alpha$ in [15], thus significantly strengthening the result of Gács and Körner [3]. Witsenhausen established this by proving the tensorization of the maximal correlation of an arbitrary pair of random variables (both tensorization and maximal correlation are defined and discussed in Section II-A). This can be used to conclude that if noninteractive simulation is possible, then the maximal correlation of the target distribution can be no more than that of the source distribution. The parameter n has disappeared in this comparison thanks to the tensorization property. The maximal correlation of a pair of binary random variables distributed as $\text{DSBS}(\alpha)$ equals $|1 - 2\alpha|$. Thus, for instance, if the non-interactive simulation of $\text{DSBS}(\beta)$ using $\text{DSBS}(\alpha)$ is possible, with $0 \le \alpha, \beta \le \frac{1}{2}$, then we must have $\alpha \le \beta$. Furthermore, it is easy to see that if $\alpha \le \beta$, then non-interactive simulation is indeed possible: Alice outputs the first bit of her observation while Bob outputs a suitable noisy copy of his first bit. Thus, for $0 \le \alpha, \beta \le \frac{1}{2}$, non-interactive simulation of $DSBS(\beta)$ using $DSBS(\alpha)$ is possible if and only if $\alpha \le \beta$.

B. Example 2

Let P(x, y) be given by $(X, Y) \sim \text{DSBS}(\alpha)$ with $0 < \alpha < \frac{1}{2}$. Consider binary random variables (U, V) distributed as Q(u, v) given by: $Q(0, 0) = 0, Q(0, 1) = Q(1, 0) = Q(1, 1) = \frac{1}{3}$. We ask if non-interactive simulation of Q(u, v) using $\text{DSBS}(\alpha)$ is possible. The maximal correlation of a $\text{DSBS}(\alpha)$ source distribution is $|1 - 2\alpha|$ while that of Q(u, v) is $\frac{1}{2}$. Since non-interactive simulation is impossible unless the maximal correlation of the source exceeds that of the target, we have non-interactive simulation impossible if $|1 - 2\alpha| \leq \frac{1}{2}$, i.e. $\frac{1}{4} < \alpha < \frac{1}{2}$. But what about the case when $0 < \alpha \leq \frac{1}{4}$? Can we come up with a suitable scheme to simulate Q(u, v)? The answer turns out to be no for each $0 < \alpha \leq \frac{1}{4}$ and can be proved using the following inequality which holds for $\{(X_i, Y_i)\}_{i=1}^n$ being i.i.d. $\text{DSBS}(\alpha)$, and for arbitrary sets $S, T \subseteq \{0, 1\}^n$:

$$\Pr\left(X^n \in S, Y^n \in T\right) \ge \Pr\left(X^n \in S\right)^{\frac{1}{2\alpha}} \Pr\left(Y^n \in T\right)^{\frac{1}{2\alpha}}.$$
(2)

The above inequality follows from a so-called reverse hypercontractive inequality [13, Thm. 3.4]. We will revisit this inequality in Section II-C. If non-interactive simulation of Q(u, v) using DSBS(α) were possible, we should be able to find sets S, T such that $\Pr(X^n \in S) \approx \frac{1}{3}, \Pr(Y^n \in T) \approx \frac{1}{3}$ and $\Pr(X^n \in S, Y^n \in T) \approx 0$. Inequality (2) rules out this possibility (assuming private randomness is not available, which we had argued is without loss of generality). Thus, hypercontractivity or reverse hypercontractivity can provide impossibility results when the maximal correlation approach cannot. Is it true that one is always stronger than the other? One of the main results in our paper is that hypercontractivity allows for stronger impossibility results than the maximal correlation when $P(x, y) = DSBS(\alpha)$. More generally, we give necessary and sufficient conditions on P(x, y) for this subsumption. This arises from an inequality obtained by Ahlswede and Gács [27] in the hypercontractive case which we extend to the reverse hypercontractive case.

The rest of the paper is organized as follows. Section II discusses preliminaries on maximal correlation and hypercontractivity. We present our main results in Section III. As mentioned earlier, one of our main results is a necessary and sufficient condition on the source distribution P(x, y) which allows one to definitively conclude that hypercontractivity will provide stronger impossibility results than maximal correlation. As our second main result, we give a characterization of a limiting hypercontractivity parameter (that we call s^*) as a strong data processing constant for KL divergences. This characterization was first proven by Ahlswede-Gacs [27]. However, our proof has the advantage of being more intuitive - arising naturally from a Taylor series expansion - while at the same time extending immediately to reverse hypercontractivity. This hypercontractivity parameter has recently been shown to also be the tightest constant in strong data processing inequalities for mutual information [31]. Section IV discusses the extension of the non-interactive simulation problem for $k \ge 3$ agents. We provide a couple of interesting threeuser non-interactive simulation examples where every two agents can simulate the corresponding pairwise marginal of the desired joint distribution but the triple cannot simulate the triple joint distribution.

II. MAIN TOOLS: MAXIMAL CORRELATION AND HYPERCONTRACTIVITY

In this paper, all sets are finite and all probability distributions are discrete and have finite support. We denote the marginals of P(x, y) and Q(u, v) by $P_X(x)$, $P_Y(y)$ and $Q_U(u)$, $Q_V(v)$ respectively. We will use $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ to denote non-negative reals and strictly positive reals respectively. In the following subsections, we will review the definition and properties of maximal correlation and hypercontractivity.

A. Maximal Correlation

For jointly distributed random variables (X, Y), define their maximal correlation $\rho_m(X; Y) := \sup \mathbb{E}f(X)g(Y)$ where the supremum is taken over $f : \mathcal{X} \mapsto \mathbb{R}, g : \mathcal{Y} \mapsto \mathbb{R}$ such that $\mathbb{E}f(X) = \mathbb{E}g(Y) = 0$ and $\mathbb{E}f(X)^2, \mathbb{E}g(Y)^2 \le 1$.

Example 1. If $(X, Y) \sim \text{DSBS}(\alpha)$, then the only functions f, g satisfying the conditions $\mathbb{E}f(X) = \mathbb{E}g(Y) = 0$ and $\mathbb{E}f(X)^2$, $\mathbb{E}g(Y)^2 \leq 1$ are $f(x) = a(1_{x=0} - 1_{x=1})$ and $g(y) = b(1_{y=0} - 1_{y=1})$ with $|a|, |b| \leq 1$. The optimum is then achieved with a = b = 1 if $\alpha < \frac{1}{2}$ and with a = b = -1 if $\alpha \geq \frac{1}{2}$. Thus,

$$\rho_m(X;Y) = |1 - 2\alpha|. \tag{3}$$

The following properties of the maximal correlation of two discrete random variables with finite support can be shown easily [33].

1) $0 \le \rho_m(X;Y) \le 1.$

- 2) $\rho_m(X;Y) = 0$ if and only if X is independent of Y.
- 3) $\rho_m(X;Y) = 1$ if and only if the Gács-Körner common information K(X;Y) > 0, i.e. if and only if (X,Y) is *decomposable*.

The three key properties of maximal correlation that are useful for the non-interactive simulation problem are as follows:

- (data processing inequality) For any functions $\phi, \psi, \rho_m(X;Y) \ge \rho_m(\phi(X), \psi(Y))$.
- (tensorization) If (X_1, Y_1) , (X_2, Y_2) are independent, then $\rho_m(X_1, X_2; Y_1, Y_2) = \max\{\rho_m(X_1; Y_1), \rho_m(X_2; Y_2)\}$ [15, Thm. 1].
- (lower semi-continuity) (Recall that if \mathcal{U} is a metric space, u is a point in \mathcal{U} and $f: \mathcal{U} \mapsto \mathbb{R}$ is a real-valued function, then we say f is lower semi-continuous at u if $u_n \to u$ implies $\liminf_n f(u_n) \ge f(u)$.) If the space of probability distributions on $\mathcal{X} \times \mathcal{Y}$ is endowed with the total variation distance metric, then $\rho_m(X;Y)$ is a lower semi-continuous function of the joint distribution P(x, y). [An example will be provided to show that ρ_m is not a continuous function of the joint distribution.]

To keep the paper self-contained, proofs of these properties are sketched in Appendix A. Now, using the above three properties, maximal correlation can be used to prove impossibility results for the non-interactive simulation problem.

Observation 1. Non-interactive simulation of $(U, V) \sim Q(u, v)$ using $(X, Y) \sim P(x, y)$ is possible only if $\rho_m(X; Y) \ge \rho_m(U; V)$.

Proof. Suppose non-interactive simulation of $(U, V) \sim Q(u, v)$ using $(X, Y) \sim P(x, y)$ is possible. This means, there exists a sequence of integers $(k_n : n \ge 1)$, a sequence of finite alphabets \mathcal{R}_n , and a sequence of functions $f_n : \mathcal{X}^{k_n} \times \mathcal{R}_n \mapsto \mathcal{U}, g_n : \mathcal{Y}^{k_n} \times \mathcal{R}_n \mapsto \mathcal{V}$, such that if $\{X_i, Y_i\}_{i=1}^{k_n}$ are drawn i.i.d. P(x, y) and M_X, M_Y are uniformly distributed in \mathcal{R}_n , with $\{X_i, Y_i\}_{i=1}^{k_n}, M_X, M_Y$ mutually independent, and $U_n = f_n(\mathcal{X}^{k_n}, M_X), V_n = g_n(\mathcal{Y}^{k_n}, M_Y)$, then $d_{\text{TV}}((U_n, V_n); (U, V)) \to 0$ as $n \to \infty$. We therefore, have

$$\rho_m(U_n; V_n) \le \rho_m(X^{k_n}, M_X; Y^{k_n}, M_Y) \text{ (Data Processing Inequality)}$$
(4)

$$= \max\{\rho_m(X_1; Y_1), \rho_m(X_2; Y_2), \dots, \rho_m(X_{k_n}, Y_{k_n}), \rho_m(M_X; M_Y)\}$$
(Tensorization) (5)

 $= \max\{\rho_m(X_1; Y_1), 0\}$ (6)

$$=\rho_m(X;Y) \tag{7}$$

By lower semi-continuity of ρ_m , $d_{\text{TV}}((U_n, V_n); (U, V)) \to 0$ implies

$$\rho_m(U;V) \le \liminf_{n \to \infty} \rho_m(U_n;V_n) \le \rho_m(X;Y).$$

B. Hypercontractivity

Definition 2. For any real-valued random variable W with finite support, and any real number p, define

$$||W||_{p} := \begin{cases} (\mathbb{E}|W|^{p})^{1/p}, & p \neq 0; \\ \exp(\mathbb{E}\log|W|) & p = 0, \end{cases}$$
(8)

with the understanding that for $p \le 0$, $||W||_p = 0$ if $\Pr(|W| = 0) > 0$.

 $||W||_p$ is continuous and non-decreasing in p. If W is not almost surely a constant, then $||W||_p$ is strictly increasing for $p \ge 0$. If in addition, $\Pr(|W| = 0) = 0$, then $||W||_p$ is strictly increasing for all p.

Definition 3. For any real $p \neq 0, 1$, define its *Hölder conjugate* p' by $\frac{1}{p} + \frac{1}{p'} = 1$. For p = 0, define p' = 0.

Suppose X, Y are real-valued random variables with finite support. We write $X \ge 0$ if $\Pr(X \ge 0) = 1$. The following are well-known [34]:

- (Minkowski's inequality) For $p \ge 1$, $||X + Y||_p \le ||X||_p + ||Y||_p$.
- (Reverse Minkowski's inequality) For $p \leq 1$ and $X, Y \geq 0$, $||X + Y||_p \geq ||X||_p + ||Y||_p$.

- (Hölder's inequality) For p > 1, $\mathbb{E}[XY] \le ||X||_{p'}||Y||_p$.
- (Reverse Hölder's inequality) For p < 1 and $X, Y \ge 0$, $\mathbb{E}[XY] \ge ||X||_{p'}||Y||_p$.

Definition 4. For a pair of random variables $(X, Y) \sim P(x, y)$ on $\mathcal{X} \times \mathcal{Y}$, we say (X, Y) is (p, q)-hypercontractive if

• $1 \le q \le p$, and

$$||\mathbb{E}[g(Y)|X]||_{p} \le ||g(Y)||_{q} \quad \forall g: \mathcal{Y} \mapsto \mathbb{R};$$

$$(9)$$

(If h(Y) = |g(Y)|, then $-\mathbb{E}[h(Y)|X] \le \mathbb{E}[g(Y)|X] \le \mathbb{E}[h(Y)|X]$ pointwise, thus we may equivalently restrict g to map to $\mathbb{R}_{\ge 0}$. If W_n supported on at most k values (for some fixed k) converges to W in distribution, then $||W_n||_p \to ||W||_p$ for any p, so we may further equivalently restrict g to map to $\mathbb{R}_{>0}$.)

• $1 \ge q \ge p$, and

$$||\mathbb{E}[g(Y)|X]||_p \ge ||g(Y)||_q \quad \forall g : \mathcal{Y} \mapsto \mathbb{R}_{\ge 0}.$$
(10)

(If W_n supported on at most k values (for some fixed k) converges to W in distribution, then $||W_n||_p \rightarrow ||W||_p$ for any p, so we may equivalently restrict g to map to $\mathbb{R}_{>0}$.)

Note that in the conventional definitions in (9) and (10), we have functions taking values in \mathbb{R} and $\mathbb{R}_{\geq 0}$ respectively. As explained above, for (9), we may restrict to functions taking values in $\mathbb{R}_{\geq 0}$. However, in (10), the functions must take non-negative values. This is conventional and necessary in various "reverse" inequalities such as the reverse Minkowski and reverse Hölder inequalities.

Define the hypercontractivity ribbon $\mathcal{R}(X;Y)$ as the set of pairs (p,q) for which (X,Y) is (p,q)-hypercontractive.

It is easy to check that the inequalities (9), (10) always hold for p = q. The conditional expectation operator is thus always contractive when $p \ge 1$, and reverse contractive for positive-valued functions when $p \le 1$. For random variables (X, Y) with a specific distribution P(x, y), the operator may be hypercontractive (i.e. more than contractive) in this precise sense. $\mathcal{R}(X;Y)$ is a region in \mathbb{R}^2 pinching to a point at (1,1) resembling a ribbon, explaining our choice of the name (see Fig. 4). Inequality (10) is also referred to as *reverse hypercontractivity* in the literature [22].

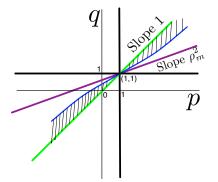


Fig. 4. The hypercontractivity ribbon $\mathcal{R}(X;Y)$ is the shaded region. Also shown a straight line of slope $\rho_m^2 := \rho_m^2(X;Y)$ through (1,1) (from Thm. 1).

1) Interpretation of hypercontractivity as Hölder-contractivity: It is well-known [22] that an equivalent definition of $\mathcal{R}(X;Y)$ can be given by observing how much the corresponding Hölder's and reverse Hölder's inequalities may be tightened:

- $(1,1) \in \mathcal{R}(X;Y);$
- For $1 \le q \le p, 1 < p$ we have $(p,q) \in \mathbb{R}(X;Y)$ iff

$$\mathbb{E}f(X)g(Y) \le ||f(X)||_{p'}||g(Y)||_q \quad \forall f: \mathcal{X} \mapsto \mathbb{R}, g: \mathcal{Y} \to \mathbb{R};$$
(11)

• For $1 \ge q \ge p, 1 > p$ we have $(p,q) \in \mathbb{R}(X;Y)$ iff

$$\mathbb{E}f(X)g(Y) \ge ||f(X)||_{p'}||g(Y)||_q \quad \forall f: \mathcal{X} \mapsto \mathbb{R}_{>0}, g: \mathcal{Y} \mapsto \mathbb{R}_{>0}; \tag{12}$$

We will refer to inequalities (11), (12) as Hölder-contractive inequalities since they tighten Hölder's inequality (using the knowledge that X and Y are not 'too correlated' in a suitable sense).

To see the equivalence for $1 \ge q \ge p, 1 > p$ observe that if (10) holds for any strictly positive-valued function g, then for any fixed strictly positive-valued function f, we have

$$\mathbb{E}f(X)g(Y) = \mathbb{E}\left[f(X)\mathbb{E}[g(Y)|X]\right]$$
(13)

$$\geq ||f(X)||_{p'} ||\mathbb{E}[g(Y)|X]||_{p} \quad \text{(Reverse Hölder's inequality and } \mathbb{E}[g(Y)|X] > 0) \tag{14}$$

$$\geq ||f(X)||_{p'} ||g(Y)||_{q}. \tag{15}$$

Conversely, suppose (12) holds for any strictly positive-valued functions f, g. First assume $p \neq 0$. By fixing g and choosing $f(X) = \mathbb{E}[g(Y)|X]^{p-1}$, we get

$$\mathbb{E}\left[\mathbb{E}[g(Y)|X]^p\right] = \mathbb{E}\left[\mathbb{E}[g(Y)|X]^{p-1}g(Y)\right]$$
(16)

$$\geq ||\mathbb{E}[g(Y)|X]^{p-1}||_{p'}||g(Y)||_q \tag{17}$$

$$= \left(\mathbb{E}\left[\mathbb{E}[g(Y)|X]^{p}\right]\right)^{1-\frac{1}{p}} ||g(Y)||_{q}.$$
(18)

Since $\mathbb{E}[g(Y)|X] > 0$, we obtain $||\mathbb{E}[g(Y)|X]||_p \ge ||g(Y)||_q$.

Now, consider the case p = 0. If (12) holds for any strictly positive-valued functions f, g with p = p' = 0, then by monotonicity of $|| \cdot ||_r$ in r, we also have

$$\mathbb{E}f(X)g(Y) \ge ||f(X)||_{-\epsilon}||g(Y)||_q \quad \forall f: \mathcal{X} \mapsto \mathbb{R}_{>0}, g: \mathcal{Y} \mapsto \mathbb{R}_{>0};$$
(19)

By our previous argument, this gives $||\mathbb{E}[g(Y)|X]||_{\frac{\epsilon}{1+\epsilon}} \ge ||g(Y)||_q$. Since this holds for each $\epsilon > 0$, we get from continuity of $||\cdot||_p$ in p that $||\mathbb{E}[g(Y)|X]||_0 \ge ||g(Y)||_q$.

The equivalence for the case $1 \le q \le p, 1 < p$ is similar. We only need to note that for (X, Y) to be (p, q)-hypercontractive with $1 \le q \le p$, it suffices to have $||\mathbb{E}[g(Y)|X]||_p \le ||g(Y)||_q$ hold only for all strictly positive functions g > 0. The rest of the proof is identical.

2) Duality between $\mathcal{R}(X;Y)$ and $\mathcal{R}(Y;X)$: The equivalent description of $\mathcal{R}(X;Y)$ in (11), (12) immediately gives the following duality between $\mathcal{R}(X;Y)$ and $\mathcal{R}(Y;X)$:

$$(p,q) \in \mathcal{R}(X;Y) \Leftrightarrow (q',p') \in \mathcal{R}(Y;X), \quad p,q \neq 1.$$
 (20)

 $\mathcal{R}(X;Y)$ is completely specified by its non-trivial boundary $q_p^*(X;Y)$ defined for $p \neq 1$ as

$$q_p^*(X;Y) := \begin{cases} \inf\{q \ge 1 : ||\mathbb{E}[g(Y)|X]||_p \le ||g(Y)||_q \quad \forall g : \mathcal{Y} \mapsto \mathbb{R}\} & p > 1; \\ \sup\{q \le 1 : ||\mathbb{E}[g(Y)|X]||_p \ge ||g(Y)||_q \quad \forall g : \mathcal{Y} \mapsto \mathbb{R}_{>0}\} & p < 1. \end{cases}$$
(21)

We will find it useful to define the 'slope at p' by $s_p(X;Y) := \frac{q_p^*(X;Y)-1}{p-1}$ for $p \neq 1$. The following properties may be easily shown.

- 1) $0 \le s_p(X;Y) \le 1.$
- 2) $s_p(X;Y) = 0$ if and only if X is independent of Y. [This is a consequence of Thm. 1 and the corresponding property for $\rho_m(X;Y)$.]

One can show that for any $p \neq 1$, $s_p(X;Y)$ satisfies the same three key properties that maximal correlation satisfies (proofs of these properties are sketched in Appendix B).

- (data processing inequality) For any functions $\phi, \psi, s_p(X; Y) \ge s_p(\phi(X), \psi(Y))$.
- (tensorization) If $(X_1, Y_1), (X_2, Y_2)$ are independent, then $s_p(X_1, X_2; Y_1, Y_2) = \max\{s_p(X_1; Y_1), s_p(X_2; Y_2)\}$ [15].
- (lower semi-continuity) If the space of probability distributions on $\mathcal{X} \times \mathcal{Y}$ is endowed with the total variation distance metric, then $s_p(X;Y)$ is a lower semi-continuous function of the joint distribution P(x,y). [An example will be provided to show that s_p is not a continuous function of the joint distribution.]

Thus, we can use hypercontractivity to obtain impossibility results for the non-interactive simulation problem.

Observation 2. Non-interactive simulation of $(U,V) \sim Q(u,v)$ using $(X,Y) \sim P(x,y)$ is possible only if $s_p(X;Y) \geq s_p(U;V)$ for each $p \neq 1$, in other words, only if $\mathcal{R}(X;Y) \subseteq \mathcal{R}(U;V)$.

Example 2. A classical result states that for $(X, Y) \sim DSBS(\alpha)$,

$$\frac{q_p^*(X;Y) - 1}{p - 1} = s_p(X;Y) = (1 - 2\alpha)^2, \quad p \neq 1.$$
(22)

This was proved by Bonami [17] and Beckner [20, Lemma 1, Appendix Sec. 2] for p > 1 and by Borell [21, Thm 3.2] for p < 1.

C. Proving impossibility results for non-interactive simulation using the hypercontractivity ribbon $\mathcal{R}(X;Y)$

In this subsection, we state explicitly a simple observation that is well-known. Suppose non-interactive simulation of $(U,V) \sim Q(u,v)$ using $(X,Y) \sim P(x,y)$ is possible. This means, there exists a sequence of integers $(k_n : n \ge 1)$, a sequence of finite alphabets \mathcal{R}_n , and a sequence of functions $f_n : \mathcal{X}^{k_n} \times \mathcal{R}_n \mapsto \mathcal{U}$, $g_n : \mathcal{Y}^{k_n} \times \mathcal{R}_n \mapsto \mathcal{V}$, such that if $\{X_i, Y_i\}_{i=1}^{k_n}$ are drawn i.i.d. P(x,y) and M_X, M_Y are uniformly distributed in \mathcal{R}_n , with $\{X_i, Y_i\}_{i=1}^{k_n}, M_X, M_Y$ mutually independent, and $U_n = f_n(X^{k_n}, M_X), V_n = g_n(Y^{k_n}, M_Y)$, then $d_{\text{TV}}((U_n, V_n); (U, V)) \to 0$ as $n \to \infty$. Let $(U_n, V_n) \sim Q_n(u, v)$.

A traditional approach to prove impossibility results for non-interactive simulation is as follows. Fix n. Suppose (X, Y) is (p, q)-hypercontractive with $1 \le q \le p$. Then, by tensorization $((X^{k_n}, M_X), (Y^{k_n}, M_Y))$ is (p, q)-hypercontractive.

Consider the functions ϕ_n, ψ_n defined as:

$$\phi_n(x^{k_n}, m_x) = \sum_{u \in \mathcal{U}} \lambda_u \mathbb{1}_{[f_n(x^{k_n}, m_x) = u]},\tag{23}$$

$$\psi_n(y^{k_n}, m_y) = \sum_{v \in \mathcal{V}} \mu_v \mathbb{1}_{[g_n(y^{k_n}, m_y) = v]}.$$
(24)

By using (11), we get

$$\mathbb{E}\phi_n(X^{k_n}, M_X)\psi(Y^{k_n}, M_Y) \le ||\phi(X^{k_n}, M_X)||_{p'} ||\psi(Y^{k_n}, M_Y)||_q,$$
(25)

which is

$$\sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \lambda_u \mu_v Q_n(u, v) \le \left(\sum_{u \in \mathcal{U}} \lambda_u^{p'} Q_n(u) \right)^{1/p'} \cdot \left(\sum_{v \in \mathcal{V}} \mu_v^q Q_n(v) \right)^{1/q}.$$
(26)

By letting $n \to \infty$, we get

$$\sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \lambda_u \mu_v Q(u, v) \le \left(\sum_{u \in \mathcal{U}} \lambda_u^{p'} Q(u) \right)^{1/p'} \cdot \left(\sum_{v \in \mathcal{V}} \mu_v^q Q(v) \right)^{1/q}.$$
(27)

For any fixed λ_u, μ_v , we find that non-interactive simulation of $(U, V) \sim Q(u, v)$ from $(X, Y) \sim P(x, y)$ is possible only if Q satisfies the inequality (27).

Similarly, if (X, Y) is (p, q)-hypercontractive with $1 \ge q \ge p$ then, for any fixed $\lambda_u, \mu_v > 0$, non-interactive simulation of $(U, V) \sim Q(u, v)$ from $(X, Y) \sim P(x, y)$ is possible only if Q satisfies the following inequality:

$$\sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \lambda_u \mu_v Q(u, v) \ge \left(\sum_{u \in \mathcal{U}} \lambda_u^{p'} Q(u) \right)^{1/p'} \cdot \left(\sum_{v \in \mathcal{V}} \mu_v^q Q(v) \right)^{1/q}.$$
(28)

Indeed, (2) is a version of (28). Let $(X, Y) \sim \text{DSBS}(\alpha)$. Then, (X, Y) is $\left(-\frac{2\alpha}{1-2\alpha}, 2\alpha\right)$ -hypercontractive from (22). Choosing $\lambda_0 = \mu_0 = 1, \lambda_1 = \mu_1 = \epsilon$ with $\epsilon \to 0$, we obtain (2) where $\mathcal{U} = \mathcal{V} = \{0, 1\}$.

The inclusion $\mathcal{R}(X;Y) \subseteq \mathcal{R}(U;V)$ implies the collection of inequalities (27) for any choice of real $\{\lambda_u\}_{u \in \mathcal{U}}, \{\mu_v\}_{v \in \mathcal{V}}$ and the collection of inequalities (28) for any choice of positive valued $\{\lambda_u\}_{u \in \mathcal{U}}, \{\mu_v\}_{v \in \mathcal{V}}$. One can also easily show that the reverse implication from the collection of inequalities (27), (28) to $\mathcal{R}(X;Y) \subseteq \mathcal{R}(U;V)$ holds (using the equivalent interpretation of hypercontractivity as Hölder-contractivity).

Thus, $\mathcal{R}(X;Y) \subseteq \mathcal{R}(U;V)$ is powerful enough to subsume the application of all possible instantiations of λ_u, μ_v in the corresponding Hölder-contractive inequalities.

The reader should note the importance of the above observation in the context of thinking abstractly about the hypercontractivity ribbon and its usefulness when invoking an automated computer search for proving an impossibility of non-interactive simulation result. If non-interactive simulation of (U, V) using (X, Y) is possible, then any Hölder-contractive inequality satisfied by (X, Y) will also be satisfied by (U, V). Therefore, if any such inequality satisfied by all functions of X and Y is violated by some pair of functions of U and V, then we can conlude non-simulability, i.e. that simulation of (U, V) using (X, Y) is impossible. However, violation of any such Hölder-contractive inequality implies failure of the inclusion $\mathcal{R}(X;Y) \subseteq \mathcal{R}(U;V)$, so one can get the same conclusion from the result that failure of the inclusion $\mathcal{R}(X;Y) \subseteq \mathcal{R}(U;V)$ implies non-simulability. Further, it is easier to show failure of inclusion of the hypercontractivity ribbons than it is to show violation of any specific such Hölder-contractive inequality, simply because violation of any Hölder-contractive inequality implies failure of inclusion of the hypercontractivity ribbons but failure of inclusion of the hypercontractivity ribbons just implies that *some* Hölder-contractive inequality is violated. Thus, if one wishes to show non-simulability using a computer search, it suffices to compute the non-trivial boundaries of the two hypercontractivity ribbons $q_p^*(X;Y)$ and $q_p^*(U;V)$ (and the corresponding $s_p(X;Y)$ and $s_p(U;V)$) and find that $s_p(X;Y) < s_p(U;V)$ for some $p \neq 1$ without ever having to prove for some specific Hölder-contractive inequality that it is the one being violated.

To the best of our knowledge, there is no algorithm better than a brute force search following suitable discretization to compute the hypercontractivity ribbons. However, the observation above simplifies the approach of proving an impossibility result using instantiations of λ_u and μ_v .

III. MAIN RESULTS

In this section, we state and prove our main results.

A. Connection between maximal correlation and the hypercontractivity ribbon

Our first result is a geometric connection between maximal correlation and the hypercontractivity ribbon.

Theorem 1. If (X, Y) is (p, q)-hypercontractive and $p \neq 1$, then

$$\rho_m^2(X;Y) \le \frac{q-1}{p-1}.$$
(29)

Remark 1. For the case p > 1, Thm. 1 is obtained in [27]. In the current form of the statement of Thm. 1, the maximal correlation is afforded a geometric meaning, namely its square is the slope of a straight line bound constraining the hypercontractivity ribbon (see Fig 4). For $(X, Y) \sim \text{DSBS}(\alpha)$, we have from (3) and (22) that the hypercontractivity ribbon $\mathcal{R}(X;Y)$ is precisely the wedge obtained by the straight lines p = q, and the straight line corresponding to the maximal correlation bound $\frac{q-1}{p-1} = \rho_m^2(X;Y)$.

Proof of Theorem 1. The proof uses a perturbative argument. Let $(X, Y) \sim P(x, y)$. The claim is obvious when either X or Y is a constant almost surely. So, assume this is not the case and fix functions $\phi : \mathcal{X} \mapsto \mathbb{R}, \psi : \mathcal{Y} \mapsto \mathbb{R}$ such that

$$\mathbb{E}\phi(X) = \mathbb{E}\psi(Y) = 0, \ \mathbb{E}\phi(X)^2 = \mathbb{E}\psi(Y)^2 = 1.$$
(30)

Fix r > 0. Define $f : \mathcal{X} \mapsto \mathbb{R}_{>0}, g : \mathcal{Y} \mapsto \mathbb{R}_{>0}$ by $f(x) = 1 + \frac{\sigma}{r}\phi(x), g(y) = 1 + \sigma r\psi(y)$. Note that for sufficiently small σ , the functions f, g do take only positive values. Fix $(p,q) \in \mathbb{R}_{X;Y}$ with p < 1. We also assume $p \neq 0$ using the standard limit argument to deal with the case p = 0. Using (12) with the functions f, g we just defined, we have

$$\mathbb{E}[(1+\frac{\sigma}{r}\phi(X))(1+\sigma r\psi(Y))] \ge \left(\mathbb{E}[(1+\frac{\sigma}{r}\phi(X))^{p'}]\right)^{1/p'} \cdot \left(\mathbb{E}[(1+\sigma r\psi(Y))^{q}]\right)^{1/q}.$$
(31)

For Z satisfying $\mathbb{E}Z = 0, \mathbb{E}Z^2 = 1$,

$$\left(\mathbb{E}[(1+aZ)^{l}]\right)^{1/l} = \left(1+l\cdot a\mathbb{E}Z + \frac{l(l-1)}{2}\cdot a^{2}\mathbb{E}Z^{2} + O(a^{3})\right)^{1/l}$$
$$= 1 + \frac{l-1}{2}a^{2} + O(a^{3}).$$

Using this in (31), we get

$$1 + \sigma^2 \mathbb{E}[\phi(X)\psi(Y)] \ge \left(1 + \frac{p'-1}{2r^2}\sigma^2 + O(\sigma^3)\right) \left(1 + \frac{(q-1)r^2}{2}\sigma^2 + O(\sigma^3)\right) \ .$$

Comparing the coefficient of σ^2 on both sides, we get

$$\mathbb{E}\phi(X)\psi(Y)\geq \frac{p'-1}{2r^2}+\frac{(q-1)r^2}{2}$$

Noting that p' - 1, q - 1 < 0 and taking the supremum over all r > 0, we get

$$\mathbb{E}\phi(X)\psi(Y) \ge -\sqrt{\frac{q-1}{p-1}} \quad \text{or} \quad -\mathbb{E}\phi(X)\psi(Y) \le \sqrt{\frac{q-1}{p-1}}.$$
(32)

Taking the supremum over all $-\phi$ and ψ satisfying (30), we get

$$\rho_m(X;Y) \le \sqrt{\frac{q-1}{p-1}}$$

We can similarly prove the inequality in the case when p > 1. This completes the proof.

The main implication of Thm. 1 for the problem of non-interactive simulation is the following corollary, which gives a necessary and sufficient condition on the source distribution P(x, y) for which Observation 2 will prove impossibility results that are at least as strong as Observation 1. This condition is satisfied for example, when P(x, y) is a DSBS(ϵ) distribution.

Corollary 1. Fix a distribution $(X, Y) \sim P(x, y)$. Then the following are equivalent: (a) For all $(U, V) \sim Q(u, v)$, $\mathcal{R}(X; Y) \subseteq \mathcal{R}(U; V) \implies \rho_m(X; Y) \ge \rho_m(U; V)$. (b)

$$\rho_m(X;Y) = \inf_{(p,q)\in\mathcal{R}(X;Y), p\neq 1} \sqrt{\frac{q-1}{p-1}}.$$
(33)

 $\begin{array}{ll} \textit{Proof of Corollary 1. (b) \implies} (a): \textit{Assume (b) holds for } P(x,y). \textit{ If } \mathcal{R}(X;Y) \subseteq \mathcal{R}(U;V), \textit{ then } \inf_{(p,q) \in \mathcal{R}(X;Y), p \neq 1} \sqrt{\frac{q-1}{p-1}} \geq \\ \inf_{(p,q) \in \mathcal{R}(U;V), p \neq 1} \sqrt{\frac{q-1}{p-1}}. \textit{ Now, by hypothesis, } \inf_{(p,q) \in \mathcal{R}(X;Y), p \neq 1} \sqrt{\frac{q-1}{p-1}} = \rho_m(X;Y) \textit{ and from Thm. 1, we have} \\ \inf_{(p,q) \in \mathcal{R}(U;V), p \neq 1} \sqrt{\frac{q-1}{p-1}} \geq \rho_m(U;V). \\ \sim (b) \implies \sim (a): \textit{ Suppose that for } (X,Y) \sim P(x,y), \textit{ we have for some } \delta \neq 0, \end{array}$

$$\rho_m(X;Y) = \inf_{(p,q)\in\mathcal{R}(X;Y), p\neq 1} \sqrt{\frac{q-1}{p-1}} - \delta.$$

By Theorem 1, $\delta > 0$. From (22), we know that if $(U, V) \sim \text{DSBS}(\epsilon)$, then for any $p \neq 1$,

$$\frac{q_p^*(U;V) - 1}{p - 1} = (1 - 2\epsilon)^2 = \rho_m(U;V)^2.$$

Choosing ϵ so that $\rho_m(U;V) = 1 - 2\epsilon = \inf_{(p,q) \in \mathcal{R}(X;Y), p \neq 1} \sqrt{\frac{q-1}{p-1}}$, we have $\rho_m(X;Y) < \rho_m(U;V)$ and $\mathcal{R}(X;Y) \subset \mathcal{R}(U;V).$

B. Limiting chordal slope of the hypercontractivity ribbon

Our second result proves the existence of $\lim_{p\to 1} s_p(X;Y)$ and provides a characterization of the limit in terms of a strong data processing constant for relative entropies that was studied first in [27].

Definition 5. Let $D(\mu(z)||\nu(z)) = \sum_{z} \mu(z) \log \frac{\mu(z)}{\nu(z)}$ denote the relative entropy of μ with respect to ν . Consider finite sets \mathcal{X} and \mathcal{Y} , and let P(x, y) be a joint distribution over the product set $\mathcal{X} \times \mathcal{Y}$. Let $R_X(x)$ be an arbitrary probability distribution on \mathcal{X} . Let $R_Y(y)$ be the probability distribution on \mathcal{Y} whose probability mass at y is $\sum_{x \in \mathcal{X}} \frac{P(x,y)}{P_X(x)} R_X(x)$. If $(X,Y) \sim P_X(x,y)$, then define the strong data processing constant for relative entropies corresponding to (X, Y) as

$$s^*(X;Y) := \sup \frac{D(R_Y(y)||P_Y(y))}{D(R_X(x)||P_X(x))}$$

where the supremum is taken over all $R_X(x)$ satisfying $R_X(x) \neq P_X(x)$ and $R_X(x) < P_X(x)$.

Remark 2. In a recent work [31], it is shown that s^* is also the tightest constant for data processing inequalities involving mutual information in Markov chains:

$$s^*(X;Y) = \sup_{U:U-X-Y} \frac{I(U;Y)}{I(U;X)}$$
.

Our result can be stated as follows.

Theorem 2.

$$\lim_{p \to 1} s_p(X;Y) = \lim_{p \to 1} \frac{q_p^*(X;Y) - 1}{p - 1} = s^*(Y;X).$$
(34)

The proof of Thm. 2 follows from a natural Taylor series calculation, and can be found in Appendix C. The following corollary shows that $\lim_{p\to\infty} s_p(X;Y) = \lim_{p\to-\infty} s_p(X;Y) = s^*(X;Y)$. The former was established in [27] while the latter result is new. We believe that using Theorems 1 and 2, we acquire a more intuitive proof of the result $\lim_{p\to\infty} s_p(X;Y) = s^*(X;Y)$ that was obtained in [27], while also showing the reverse hypercontractive case: $\lim_{p\to-\infty} s_p(X;Y) = s^*(X;Y)$

Corollary 2.

$$\lim_{p \to \infty} \frac{q_p^*(X;Y) - 1}{p - 1} = \lim_{p \to -\infty} \frac{q_p^*(X;Y) - 1}{p - 1} = s^*(X;Y).$$
(35)

The proof of Corollary 2 is in Appendix C. Corollary 3, which follows immediately from Corollary 1, Thm. 2 and Corollary 2 provides a sufficient condition for (33) to hold.

Corollary 3. If
$$\rho_m(X;Y) = \min\{\sqrt{s^*(X;Y)}, \sqrt{s^*(Y;X)}\}$$
, then for any $(U,V) \sim Q(u,v)$, we have
 $\mathcal{R}(X;Y) \subseteq \mathcal{R}(U;V) \implies \rho_m(X;Y) \ge \rho_m(U;V).$

Note that from (3), (22) and Thm. 2, DSBS sources always satisfy the condition in Corollary 3. One can also show that the condition holds for source distributions corresponding to the input-output pair resulting from a uniformly distributed input into a binary input symmetric output channel. The above ideas suggest that for a recent conjecture regarding Boolean functions [35], hypercontractivity is going to be a more useful tool than maximal correlation. Indeed, evidence for this can be found in [32], where usage of s^* helps in an automated proof of an inequality that cannot be proved using maximal correlation.

Example 3. Suppose we choose P(x, y) to be $DSBS(\epsilon)$, and Q(u, v) specified by Q(U = 1) = s, Q(V = 1|U = 0) = c, Q(V = 0|U = 1) = d. For certain values of s, c, d, non-interactive simulation is possible and for others, it is impossible. For fixed values of s, this is shown graphically in Fig. 5.

IV. Non-interactive simulation with $k \ge 3$ agents

The non-interactive simulation problem we have considered can be naturally extended to k-agents.

Definition 6. Let $\mathcal{X}_i, \mathcal{U}_i$ denote finite sets for i = 1, 2, ..., k. Given a source distribution $P(x_1, x_2, ..., x_k)$ over $\prod_{i=1}^k \mathcal{X}_i$ and a *target distribution* $Q(u_1, u_2, ..., u_k)$ over $\prod_{i=1}^k \mathcal{U}_i$, we say that *non-interactive simulation* of $Q(u_1, u_2, ..., u_k)$ using $P(x_1, x_2, ..., x_k)$ is possible if for any $\epsilon > 0$, there exists a positive integer n, a finite set \mathcal{R} and functions $f_i : \mathcal{X}_i^n \times \mathcal{R} \mapsto \mathcal{U}_i$ for i = 1, 2, ..., k such that

$$d_{\text{TV}}\left((f_1(X_1^n, M_1), f_2(X_2^n, M_2), \dots, f_k(X_k^n, M_k)); (U_1, U_2, \dots, U_k)\right) \le \epsilon$$

where $\{(X_{1,j}, X_{2,j}, \ldots, X_{k,j})\}_{j=1}^n$ is a sequence of i.i.d. samples drawn from $P(x_1, x_2, \ldots, x_k), M_1, M_2, \ldots, M_k$ are uniformly distributed in \mathcal{R} , mutually independent of each other and of the samples drawn from the source, (U_1, U_2, \ldots, U_k) is drawn from $Q(u_1, u_2, \ldots, u_k)$, and $d_{\text{TV}}(\cdot; \cdot)$ is the total variation distance.

In this section, we make simple observations about how hypercontractivity and maximal correlation may be used to prove impossibility results for this non-interactive simulation problem with k agents. For any set $A \subseteq \{1, 2, ..., k\}$, let us use the notation $X_A := (X_i : i \in A), U_A := (U_i : i \in A)$.

Recall that for the case of two random variables (X, Y) and $1 \le q < p$, we have $(p, q) \in \mathcal{R}(X; Y)$ if either of the two following equivalent conditions hold:

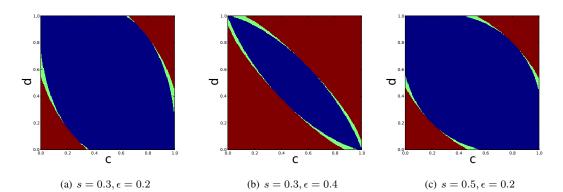


Fig. 5. Suppose the source distribution P(x, y) is DSBS(ϵ), and the target distribution is Q(u, v) specified by Q(U = 1) = s, Q(V = 1) = s. 1|U=0) = c, Q(V=0|U=1) = d. The plots above show restrictions on the space of distributions (s, c, d) that can be simulated. The X co-ordinate represents c and the Y co-ordinate represents d. In each plot, we fix $s_i \in$ as specifed and p = 1.5. The blue region indicates $\rho_m^2 \leq (1 - 2\epsilon)^2 < s_p$ and finally, the red region indicates $(1 - 2\epsilon)^2 < \rho_m^2 \leq s_p$. Thus, with p = 1.5, the red region is ruled out as impossible by ρ_m and s_p , the green region is ruled out by s_p , and the blue region is ruled out by neither ρ_m nor by s_p . Note that this does not mean all points in the blue region can be simulated by suitable choice of functions, only that our tools (using this particular choice of p) fail to prove impossibility for those points. Note that along the c = d line, (U, V) is a DSBS source as well, so both maximal correlation and hypercontractivity (for any p) give an impossibility result if and only if $c < \epsilon$ or $c > 1 - \epsilon$ in accordance with Sec. I-A.

- $||\mathbb{E}[g(Y)|X]||_p \leq ||g(Y)||_q \quad \forall g : \mathcal{Y} \mapsto \mathbb{R};$ $\mathbb{E}f(X)g(Y) \leq ||f(X)||_{p'}||g(Y)||_q \quad \forall f : \mathcal{X} \mapsto \mathbb{R}, \forall g : \mathcal{Y} \mapsto \mathbb{R}.$

Similarly, for $1 \ge q > p$, we have $(p,q) \in \mathcal{R}(X;Y)$ if either of the two following equivalent conditions hold:

- $||\mathbb{E}[g(Y)|X]||_p \ge ||g(Y)||_q \quad \forall g : \mathcal{Y} \mapsto \mathbb{R}_{>0};$ $\mathbb{E}f(X)g(Y) \ge ||f(X)||_{p'}||g(Y)||_q \quad \forall f : \mathcal{X} \mapsto \mathbb{R}_{>0}, \forall g : \mathcal{Y} \mapsto \mathbb{R}_{>0}.$

We can define a Hölder-contraction region $\mathcal{H}(X;Y)$ by observing how much Hölder's inequality and the reverse Hölder's inequality may be tightened. Define $(p_1, p_2) \in \mathcal{H}(X; Y)$ if

- $p_1, p_2 \ge 1$, and $\forall f : \mathcal{X} \mapsto \mathbb{R}, \forall g : \mathcal{Y} \mapsto \mathbb{R}$, we have $\mathbb{E}f(X)g(Y) \le ||f(X)||_{p_1}||g(Y)||_{p_2}$; $p_1, p_2 \le 1$, and $\forall f : \mathcal{X} \mapsto \mathbb{R}_{>0}, \forall g : \mathcal{Y} \mapsto \mathbb{R}_{>0}$, we have $\mathbb{E}f(X)g(Y) \ge ||f(X)||_{p_1}||g(Y)||_{p_2}$.

This prompts a natural extension to k-random variables using the k-random variable Hölder inequalities. The most general Hölder and reverse Hölder inequalities for k random variables are respectively given by:

$$\mathbb{E}\Pi_{i=1}^{k} W_{i} \le \Pi_{i=1}^{k} ||W_{i}||_{p_{i}}, \qquad p_{i} > 1, \sum_{i=1}^{k} \frac{1}{p_{i}} = 1;$$
(36)

$$\mathbb{E}\Pi_{i=1}^{k} W_{i} \ge \Pi_{i=1}^{k} ||W_{i}||_{p_{i}}, \qquad p_{i} < 1, p_{i} \neq 0, \text{ exactly one } p_{i} > 0, \sum_{i=1}^{k} \frac{1}{p_{i}} = 1, W_{i} \ge 0.$$
(37)

Proof of Hölder and reverse Hölder inequalities. By the weighted arithmetic mean-geometric mean inequality, we have for any real numbers $y_1, y_2, \ldots, y_k \ge 0$, and $p_1, p_2, \ldots, p_k > 1$ satisfying $\sum_{i=1}^k \frac{1}{p_i} = 1$,

$$\Pi_{i=1}^{k} y_i \le \sum_{i=1}^{k} \frac{y_i^{p_i}}{p_i}.$$
(38)

Setting $y_i = \frac{|W_i|}{||W_i||_{p_i}}$ and taking expectations gives the Hölder inequality.

Now, if $0 < p_1 < 1, p_2, p_3, \dots, p_k < 0$ satisfying $\sum_{i=1}^k \frac{1}{p_i} = 1$, we may set $q_1 = \frac{1}{p_1}, q_i = \frac{-p_i}{p_1}, i = 2, 3, \dots, k$, so that $q_i > 1$ and $\sum_{i=1}^k \frac{1}{q_i} = 1$. Using (38) with q_i 's, we get

$$\Pi_{i=1}^{k} y_{i} \le p_{1} y_{1}^{\frac{1}{p_{1}}} + \sum_{i=2}^{k} \frac{p_{1}}{-p_{i}} y_{i}^{-\frac{p_{i}}{p_{1}}}.$$
(39)

For any $x_1, x_2, \ldots, x_k > 0$, choose $y_1 = \prod_{i=1}^k x_i^{p_1}$ and $y_i = x_i^{-p_1}$ for $i = 2, 3, \ldots, k$, to get

$$\sum_{i=1}^{k} \frac{x_i^{p_i}}{p_i} \le \prod_{i=1}^{k} x_i.$$
(40)

Setting $x_i = \frac{|W_i|}{||W_i||_{p_i}}$ and taking expectations proves the reverse Hölder inequality for $W_i > 0$ almost surely, $i = 1, 2, \ldots, k$. If $W_i \ge 0$, we can set $W'_i = W_i + \epsilon$ and let $\epsilon \downarrow 0$ to complete the proof.

Remark 3. Both Hölder and reverse Hölder inequalities can also be proved by recursively invoking the inequalities for two variables. As a demonstration, fix any 0 < p, q < 1. For any non-negative real-valued W_1, W_2, W_3 ,

$$\begin{split} \mathbb{E}W_1 W_2 W_3 &\geq ||W_1 W_2||_p |W_3|_{\frac{-p}{1-p}} \\ &= \left(\mathbb{E}(W_1 W_2)^p\right)^{\frac{1}{p}} |W_3|_{\frac{-p}{1-p}} \\ &\geq \left(||W_1^p||_q ||W_2^p||_{\frac{-q}{1-q}}\right)^{\frac{1}{p}} |W_3|_{\frac{-p}{1-p}} \\ &= ||W_1||_{pq} ||W_2||_{\frac{-pq}{1-q}} ||W_3||_{\frac{-p}{1-p}}. \end{split}$$

It is easy to check that any reverse Hölder inequality may be obtained in this way by suitable choice of p, q. Remark 4. The reverse Hölder inequality will also hold if some of the p_i were equal to zero as long as the point (p_1, p_2, \ldots, p_k) is the limit of points satisfying $p_i \leq 1, p_i \neq 0$, exactly one $p_i > 0, \sum_{i=1}^k \frac{1}{p_i} = 1$. In particular, if we set for any integer $M > 1, p_1^{(M)} = \frac{1}{Mk}, p_2^{(M)} = p_3^{(M)} = \ldots = p_k^{(M)} = -\frac{k-1}{Mk-1}$, then $(p_1^{(M)}, p_2, {}^{(M)}, \ldots, p_k^{(M)})$ is a legitimate choice for the reverse Hölder's inequality. Taking the limit as $M \to \infty$, we get the inequality $\mathbb{E}\prod_{i=1}^k W_i \geq \prod_{i=1}^k ||W_i||_0$, which is also valid for all random variables $W_i \geq 0$ and is a reverse Hölder's inequality. Remark 5. The restriction in reverse Hölder inequality that exactly one $p_i > 0$ is necessary. If no such p_i exists, then the inequality is a consequence of $\mathbb{E}\prod_{i=1}^k W_i \geq \prod_{i=1}^k ||W_i||_0$ in the previous remark and the montonicity of norms. On the other hand, if more than one such p_i exists, say $p_1, p_2 > 0$, then we can choose any mutually exclusive events A, B such that $P(A \cap B) = 0, P(A) > 0, P(B) > 0$. Set $W_1 = 1_A, W_2 = 1_B, W_3 = W_4 = \ldots, W_k = 1$. The reverse Hölder inequality, if true, would then yield $P(A \cap B) \geq P(A)^{\frac{1}{p_1}} P(B)^{\frac{1}{p_2}}$ which is false.

Define $(p_1, p_2, ..., p_k) \in \mathcal{H}(X_1; X_2; ...; X_k)$ if

• $p_1, p_2, \ldots, p_k \ge 1$, and $\forall f_i : \mathcal{X}_i \mapsto \mathbb{R}, i = 1, 2, \ldots, k$ we have

 $\mathbb{E}\Pi_{i=1}^{k} f_{i}(X_{i}) \leq \Pi_{i=1}^{k} ||f_{i}(X_{i})||_{p_{i}};$

• $p_1, p_2, \ldots, p_k \leq 1$, and $\forall f_i : \mathcal{X}_i \mapsto \mathbb{R}_{>0}, i = 1, 2, \ldots, k$ we have

$$\mathbb{E}\Pi_{i=1}^{k} f_{i}(X_{i}) \geq \Pi_{i=1}^{k} ||f_{i}(X_{i})||_{p_{i}};$$

Remark 6. The restriction to the orthant $p_1, p_2, \ldots, p_k \ge 1$ for the forward Hölder contraction is without loss of generality: Assuming X_1 is a non-constant random variable and f_1 is chosen so that $f_1(X_1)$ is non-constant and f_2, f_3, \ldots, f_k are chosen to be constants, the inequality will hold only if $p_1 \ge 1$. Likewise, the restriction to the orthant $p_1, p_2, \ldots, p_k \le 1$, for the reverse Hölder contraction is without loss of generality.

It is easy to check that tensorization, data processing and appropriate semi-continuity properties continue to hold for $\mathcal{H}(X_1; X_2; \ldots; X_k)$ so we have the following observation.

Observation 3. Non-interactive simulation of $(U_1, U_2, \ldots, U_k) \sim Q(u_1, u_2, \ldots, u_k)$ using $(X_1, X_2, \ldots, X_k) \sim P(x_1, x_2, \ldots, x_k)$ is possible only if, for all non-empty subsets $S_1, S_2, \ldots, S_m \subseteq \{1, 2, \ldots, k\}, \mathcal{H}(X_{S_1}; X_{S_2}; \ldots; X_{S_m}) \subseteq \mathcal{H}(U_{S_1}; U_{S_2}; \ldots; U_{S_m}).$

Similarly, using maximal correlation, we can make the following observation:

Observation 4. Non-interactive simulation of $(U_1, U_2, \ldots, U_k) \sim Q(u_1, u_2, \ldots, u_k)$ using $(X_1, X_2, \ldots, X_k) \sim P(x_1, x_2, \ldots, x_k)$ is possible only if for all non-empty subsets $S_1, S_2 \subseteq \{1, 2, \ldots, k\}$, we have $\rho_m(X_{S_1}; X_{S_2}) \geq \rho_m(U_{S_1}; U_{S_2})$.

Example 4. We define the following distributions of *DSBS triples* as shown in Fig. 6. For chosen $0 \le \epsilon_X, \epsilon_Y, \epsilon_Z < \frac{1}{2}$, we define $(X, Y, Z) \sim \text{DSBS-triple}(\epsilon_X, \epsilon_Y, \epsilon_Z)$ as the unique triple joint distribution satisfying $(Y, Z) \sim \frac{1}{2}$.

 $DSBS(\epsilon_X), (X, Y) \sim DSBS(\epsilon_Z), (X, Z) \sim DSBS(\epsilon_Y)$ (note that there are two such distributions if $\epsilon_X = \epsilon_Y = \epsilon_Z = \frac{1}{2}$). Such a distribution exists as long as the triangle inequalities $\epsilon_X + \epsilon_Y \ge \epsilon_Z, \epsilon_X + \epsilon_Z \ge \epsilon_Y, \epsilon_Z + \epsilon_Y \ge \epsilon_X$ are satisfied and the joint distribution of (X, Y, Z) is given by:

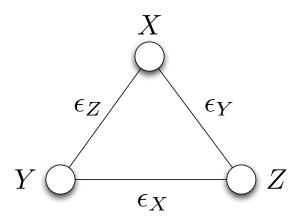


Fig. 6. $(X, Y, Z) \sim \text{DSBS-triple}(\epsilon_X, \epsilon_Y, \epsilon_Z)$

$$P_{X,Y,Z}(0,0,0) = P_{X,Y,Z}(1,1,1) = \frac{2 - \epsilon_X - \epsilon_Y - \epsilon_Z}{4}$$
(41)

$$P_{X,Y,Z}(0,0,1) = P_{X,Y,Z}(1,1,0) = \frac{\epsilon_X + \epsilon_Y - \epsilon_Z}{4}$$
(42)

$$P_{X,Y,Z}(0,1,0) = P_{X,Y,Z}(1,0,1) = \frac{\epsilon_X - \epsilon_Y + \epsilon_Z}{4}$$
(43)

$$P_{X,Y,Z}(0,1,1) = P_{X,Y,Z}(1,0,0) = \frac{-\epsilon_X + \epsilon_Y + \epsilon_Z}{4}.$$
(44)

If either A or B is binary-valued, then one can simply write [15]

$$\rho_m^2(A;B) = -1 + \sum_{a,b} \frac{p_{A,B}(a,b)^2}{p_A(a)p_B(b)}.$$
(45)

Using this simple formula, we find that the various maximal correlation terms for $(X, Y, Z) \sim \text{DSBS-triple}(\epsilon_X, \epsilon_Y, \epsilon_Z)$ are given by:

$$\rho_m(X;Y) = 1 - 2\epsilon_Z,\tag{46}$$

$$\rho_m(X;Y,Z) = \sqrt{\frac{(\epsilon_Y - \epsilon_Z)^2}{\epsilon_X} + \frac{(1 - \epsilon_Y - \epsilon_Z)^2}{1 - \epsilon_X}}.$$
(47)

Now, consider the following three-agent non-interactive simulation problem. Agents Alice, Bob, and Charlie observe X^n, Y^n, Z^n respectively and output (as a function of their observations and their private randomness) $\tilde{U}, \tilde{V}, \tilde{W}$ respectively, which is required to be close in total variation to the target distribution (U, V, W) as shown in Fig. 7.

Suppose that for some $\epsilon < \frac{1}{2}$, the source and target distributions are specified by $(X, Y, Z) \sim \text{DSBS-triple}(\epsilon, \epsilon, \epsilon)$ and $(U, V, W) \sim \text{DSBS-triple}(\epsilon, 2\epsilon(1 - \epsilon), \epsilon)$ as shown in Fig. 8. In Section I-A, we pointed out that for a twoagent problem, non-interactive simulation of a DSBS target distribution with parameter $\beta < \frac{1}{2}$ using a DSBS source distribution with parameter $\alpha < \frac{1}{2}$ is possible if and only if the target distribution is more noisy, i.e. $\alpha \leq \beta$. Thus, for this example, each pair of agents can perform the marginal pair simulation desired of them. However, the three agents cannot simulate the desired triple joint distribution.

Using the formula (47), we get

$$\rho_m(X,Z;Y) = \frac{1-2\epsilon}{\sqrt{1-\epsilon}},\tag{48}$$

$$\rho_m(U,W;V) = \frac{1-2\epsilon}{\sqrt{1-2\epsilon+2\epsilon^2}}.$$
(49)

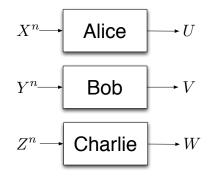


Fig. 7. Three-user non-interactive simulation problem

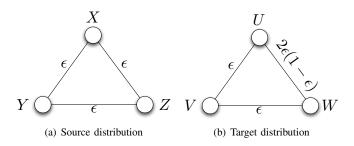


Fig. 8. Three random variable simulation example: Every pair of agents can achieve the desired simulation but the triple cannot.

For $0 < \epsilon < \frac{1}{2}$, we have $1 - 2\epsilon + 2\epsilon^2 < 1 - \epsilon$, which gives $\rho_m(X, Z; Y) < \rho_m(U, W; V)$. This shows that even if agents Alice and Charlie were to combine their observations and their random variable generation tasks to form one agent Alice-Charlie, then Alice-Charlie and Bob cannot achieve the desired non-interactive simulation. *Example* 5. Consider the following choices of source distribution P(x, y, z) and target distribution Q(u, v, w).

$$P(x, y, z) = \begin{cases} a_0 & \text{ if } (x, y, z) = (0, 0, 0), \\ a_2 & \text{ if } (x, y, z) = (0, 1, 1), (1, 0, 1), (1, 1, 0), \end{cases}$$

where

$$a_0 + 3a_2 = 1, (50)$$

i.e. (X, Y, Z) take values on the 4 sequences that satisfy $X \oplus Y \oplus Z = 0$ (addition modulo 2).

$$Q(u, v, w) = \begin{cases} b_0 & \text{if } (u, v, w) = (0, 0, 0), \\ b_1 & \text{if } (u, v, w) = (0, 0, 1), (0, 1, 0), (1, 0, 0), \\ b_2 & \text{if } (u, v, w) = (0, 1, 1), (1, 0, 1), (1, 1, 0), \\ b_3 & \text{if } (u, v, w) = (1, 1, 1). \end{cases}$$

We will choose these parameters so that for some $0 < \gamma < 1$, we have

$$b_0 + b_1 = a_0 + 2a_2\gamma + a_2\gamma^2, \tag{51}$$

$$b_1 + b_2 = a_2(1 - \gamma^2), \tag{52}$$

$$b_2 + b_3 = a_2(1 - \gamma)^2 . (53)$$

Consider the question of whether (U, V, W) can be simulated from (X, Y, Z). For simulation of pair (U, V) from (X, Y), note that if $A_1, A_2 \sim \text{Ber}(\gamma)$ i.i.d. and mutually independent of (X, Y), then

$$(X \oplus (A_1 \cdot 1_{X=1}), Y \oplus (A_2 \cdot 1_{Y=1})) \stackrel{a}{=} (U, V)$$

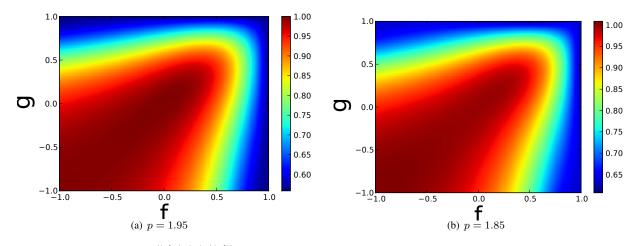


Fig. 9. Contour plots of the ratio $\frac{||\mathbb{E}[f(X)g(Y)|Z]||_{p'}}{||f(X)||_p||g(Y)||_p}$ where $f(x) = (1+f)\mathbf{1}_{x=1} + (1-f)\mathbf{1}_{x=0}$ and $g(y) = (1+g)\mathbf{1}_{y=1} + (1-g)\mathbf{1}_{y=0}$. The X-axis represents the variable $f \in [-1, 1]$ and the Y-axis represents the variable $g \in [-1, 1]$. We see numerically that for p = 1.95, the ratio is upper bounded by 1 everywhere, but for p = 1.85, the ratio is maximized at f = g = -1 where it takes the value 1.0088... This implies that $(1.95, 1.95, 1.95, 1.95) \in \mathcal{H}(X; Y; Z)$ but $(1.85, 1.85, 1.85) \notin \mathcal{H}(X; Y; Z)$.

because of conditions (51), (52), (53). By symmetry then, every pair of agents can achieve the desired simulation.

Now, if we imagine two agents observe (X, Y) and Z respectively and are required to simulate (U, V) and W respectively, then again this is possible since (X, Y) uniquely determines Z, so the agents now have access to shared randomness which can be used to generate any required joint distribution.

However, consider the specific choice:

$$a_0 = 0.825, \gamma = 0.2, b_0 = 0.8,$$

so that the other parameters are fixed from (50), (51), (52), (53) to be:

 $a_2 = 0.058333..., \quad b_1 = 0.05066666..., \quad b_2 = 0.005333..., \quad b_3 = 0.032.$

Here, we find computationally that

$$\kappa := \inf\{p \ge 1 : (p, p, p) \in \mathcal{H}(X; Y; Z)\} = 1.93...;$$
(54)

$$\zeta := \inf\{p \ge 1 : (p, p, p) \in \mathcal{H}(U; V; W)\} = 2.07....$$
(55)

We present numerical evidence supporting the above claims. Specifically, we will show that $1.85 < \kappa < 1.95$ and $\zeta > 2.05$.

Using Hölder's inequality, it is easy to verify that the following two statements are equivalent:

$$\mathbb{E}f(X)g(Y)h(Z) \le ||f(X)||_p ||g(Y)||_p ||h(Z)||_p, \ \forall f: \mathcal{X} \to \mathbb{R}, g: \mathcal{Y} \to \mathbb{R}, h: \mathcal{Z} \to \mathbb{R},$$
(56)

$$|\mathbb{E}[f(X)g(Y)|Z]||_{p'} \le ||f(X)||_p ||g(Y)||_p, \ \forall f: \mathcal{X} \to \mathbb{R}, g: \mathcal{Y} \to \mathbb{R},$$
(57)

and furthermore, equivalently, all functions above may have co-domain $\mathbb{R}_{\geq 0}$. We choose $f(x) = (1+f)1_{x=1} + (1-f)1_{x=0}$ and $g(y) = (1+g)1_{y=1} + (1-g)1_{y=0}$. It suffices to consider functions of this form since the inequalities above are homogeneous. Fig. 9 shows contour plots of the ratio $\frac{||\mathbb{E}[f(X)g(Y)|Z]||_{p'}}{||f(X)||_p||g(Y)||_p}$ where the X-axis represents the variable $f \in [-1, 1]$ and the Y-axis represents the variable $g \in [-1, 1]$. For p = 1.95, the ratio is upper-bounded by 1, whereas for p = 1.85, the ratio takes the value 1.0088... at f = g = -1. (Note that the color bar in Fig. 9 has a maximum value of 1.0 for p = 1.95 and a maximum value of a little greater than 1.0 for p = 1.85.) Thus, $(1.95, 1.95, 1.95) \in \mathcal{H}(X; Y; Z)$ but $(1.85, 1.85, 1.85) \notin \mathcal{H}(X; Y; Z)$ and so, $1.85 < \kappa < 1.95$.

Now, consider the function $\delta(\theta) = 9 \cdot 1_{\theta=1} + 1_{\theta=0}$. Then,

$$\mathbb{E}\delta(U)\delta(V)\delta(W) = 26.792\tag{58}$$

$$||\delta(U)||_{2.05}||\delta(V)||_{2.05}||\delta(W)||_{2.05} = (||\delta(U)||_{2.05})^3 = (2.9747...)^3 = 26.322... < 26.792.$$
(59)

This proves that $(2.05, 2.05, 2.05) \notin \mathcal{H}(U; V; W)$ and so, $\zeta > 2.05$.

Since $\kappa < 1.95$ and $\zeta > 2.05$, the inclusion $\mathcal{H}(X;Y;Z) \subseteq \mathcal{H}(U;V;W)$ is false and so, the simulation of (U,V,W) from (X,Y,Z) is impossible.

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APPENDIX

A. Proof of the claimed properties of ρ_m

In this subsection, we prove the claimed properties of maximal correlation.

- (data processing inequality) For any functions φ, ψ, ρ_m(X; Y) ≥ ρ_m(φ(X), ψ(Y)).
 Proof: This is straightforward from the definition of ρ_m.
- (tensorization) If (X₁, Y₁) and (X₂, Y₂) are independent, then ρ_m(X₁, X₂; Y₁, Y₂) = max{ρ_m(X₁; Y₁), ρ_m(X₂; Y₂)}. *Proof:* This property was shown by Witsenhausen [15]. The following exposition of Witsenhausen's proof is by Kumar [36]. If we define |X| × |Y| matrices P, Q by P_{x,y} = P(x, y) and Q_{x,y} = P(x,y)/√P(x)P(y), then the top two singular values of Q are σ₁(Q) = 1 and σ₂(Q) = ρ_m(X; Y) (for proof, see [36]). The tensorization property then follows from the fact that the singular values of the tensor product of two matrices A ⊗ B are given by σ_i(A)σ_i(B).
- (Lower semi-continuity) If the space of probability distributions on X × Y is endowed with the total variation distance metric, then ρ_m(X; Y) is a lower semi-continuous function of the joint distribution P(x, y).
 Proof: Suppose (X, Y), (X₁, Y₁), (X₂, Y₂),... are random variable pairs taking values in the finite set X × Y

satisfying $d_{\text{TV}}((X_n, Y_n); (X, Y)) \to 0$ as $n \to \infty$. We will show that $\rho := \liminf_{n \to \infty} \rho_m(X_n; Y_n) \ge \rho_m(X; Y)$. Let $\{j_n\}_{n=1}^{\infty}$ be a subsequence so that $\rho = \lim_{n \to \infty} \rho_m(X_{j_n}; Y_{j_n})$.

For any $\epsilon > 0$, there exists a $j(\epsilon)$ such that $\rho_m(X_{j_n}; Y_{j_n}) \leq \rho + \epsilon$ for all $j_n \geq j(\epsilon)$. Fix any functions $f : \mathcal{X} \mapsto \mathbb{R}, g : \mathcal{Y} \mapsto \mathbb{R}$ such that $\mathbb{E}f(X) = \mathbb{E}g(Y) = 0$ and $\mathbb{E}f(X)^2, \mathbb{E}g(Y)^2 \leq 1$. We will show $\mathbb{E}f(X)g(Y) \leq \rho$ which will complete the proof.

If $\mathbb{E}f(X)^2 = 0$ or $\mathbb{E}g(Y)^2 = 0$, there is nothing to prove. So, suppose $\mathbb{E}f(X)^2$, $\mathbb{E}g(Y)^2 > 0$. Since $\mathcal{X} \times \mathcal{Y}$ is a finite set, $d_{\mathrm{TV}}((X_{j_n}, Y_{j_n}); (X, Y)) \to 0$ implies that $\operatorname{Var}(f(X_{j_n})) \to \operatorname{Var}(f(X)) > 0$, $\operatorname{Var}(g(Y_{j_n})) \to \operatorname{Var}(g(Y)) > 0$. There exists j(f,g) such that $\operatorname{Var}(f(X_{j_n})) \ge \frac{\operatorname{Var}(f(X))}{2}$, $\operatorname{Var}(g(Y_{j_n})) \ge \frac{\operatorname{Var}(g(Y))}{2}$ for all $j \ge n(f,g)$.

Define for $j_n \ge \max\{j(\epsilon), j(f,g)\}$ the functions $f_{j_n} : \mathcal{X} \mapsto \mathbb{R}, g_{j_n} : \mathcal{Y} \mapsto \mathbb{R}$ given by

$$f_{j_n}(X) = \frac{f(X_{j_n}) - \mathbb{E}f(X_{j_n})}{\sqrt{\operatorname{Var}(f(X_{j_n}))}} ,$$
(60)

$$g_{j_n}(Y) = \frac{g(Y_{j_n}) - \mathbb{E}g(Y_{j_n})}{\sqrt{\operatorname{Var}(g(Y_{j_n}))}} , \qquad (61)$$

which is possible since for such j_n we have $\operatorname{Var}(f(X_{j_n})), \operatorname{Var}(g(Y_{j_n})) > 0$. Again, we will have $\mathbb{E}f_{j_n}(X)g_{j_n}(Y) \to \frac{\mathbb{E}f(X)g(Y)}{\sqrt{\mathbb{E}f(X)^2\mathbb{E}g(Y)^2}} \geq \mathbb{E}f(X)g(Y)$. But by definition, we have for $j_n \geq \max\{j(\epsilon), j(f,g)\}$ that $\mathbb{E}f_{j_n}(X)g_{j_n}(Y) \leq \rho_m(X_{j_n};Y_{j_n}) \leq \rho + \epsilon$. This gives $\mathbb{E}f(X)g(Y) \leq \rho + \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\mathbb{E}f(X)g(Y) \leq \rho$.

B. Proof of the claimed properties of s_p

In this subsection, we prove the claimed properties of s_p for $p \neq 1$.

• (data processing inequality) For any functions $\phi, \psi, s_p(X;Y) \ge s_p(\phi(X);\psi(Y))$. *Proof:* Let $W = \phi(X), Z = \psi(Y)$. Suppose for $1 \le q \le p$, we have $||\mathbb{E}[g(Y)|X]||_p \le ||g(Y)||_q$ for all functions $g: \mathcal{X} \mapsto \mathbb{R}$. For any function of Z, say $\theta(Z)$, we have

$$||\mathbb{E}[\theta(Z)|W]||_p = ||\mathbb{E}[\theta(\psi(Y))|\phi(X)]||_p \tag{62}$$

$$\stackrel{(a)}{=} ||\mathbb{E}[\mathbb{E}[\theta(\psi(Y))|X]|\phi(X)]||_p \tag{63}$$

$$\stackrel{(b)}{\leq} ||\mathbb{E}[\theta(\psi(Y))|X]||_{p} \tag{64}$$

$$< ||\theta(\psi(Y))||_a \tag{65}$$

$$= ||Q(7)||$$
(66)

$$= ||\theta(Z)||_q,\tag{60}$$

where (a) follows from successive conditioning and (b) follows from Jensen's inequality: $||\mathbb{E}[A|\phi(X)]||_p \leq ||A||_p$. Similarly, we can deal with the case $1 \geq q \geq p$. This completes the proof.

• (tensorization) If (X_1, Y_1) and (X_2, Y_2) are independent, then $s_p(X_1, X_2; Y_1, Y_2) = \max\{s_p(X_1; Y_1), s_p(X_2; Y_2)\}$.

Proof: Suppose $(X_1, Y_1) \sim P_1(x_1, y_1)$ and $(X_2, Y_2) \sim P_2(x_2, y_2)$ are both (p, q)-hypercontractive, with $p < 1, p \neq 0$. We remark that for the case of p = 0, we take limits in the standard way. Then,

$$\mathbb{E}f(X_1)g(Y_1) \ge ||f(X_1)||_{p'}||g(Y_1)||_q \ \forall \ f : \mathcal{X}_1 \mapsto \mathbb{R}_{>0}, \ \forall \ g : \mathcal{Y}_1 \mapsto \mathbb{R}_{>0};$$
(67)

$$\mathbb{E}f(X_2)g(Y_2) \ge ||f(X_2)||_{p'}||g(Y_2)||_q \ \forall \ f : \mathcal{X}_2 \mapsto \mathbb{R}_{>0}, \ \forall \ g : \mathcal{Y}_2 \mapsto \mathbb{R}_{>0}.$$
(68)

Now, fix any positive-valued functions $f : \mathcal{X}_1 \times \mathcal{X}_2 \mapsto \mathbb{R}_{>0}, g : \mathcal{Y}_1 \times \mathcal{Y}_2 \mapsto \mathbb{R}_{>0}$.

$$\mathbb{E}f(X_1, X_2)g(Y_1, Y_2) = \sum_{x_1, y_1} P_1(x_1, y_1) \sum_{x_2, y_2} P_2(x_2, y_2)f(x_1, x_2)g(y_1, y_2)$$
(69)

$$\stackrel{(a)}{\geq} \sum_{x_1, y_1} P_1(x_1, y_1) \left(\sum_{x_2} P_{X_2}(x_2) f(x_1, x_2)^{p'} \right)^{\frac{1}{p'}} \left(\sum_{y_2} P_{Y_2}(y_2) g(y_1, y_2)^q \right)^{\frac{1}{q}}$$
(70)

$$\overset{(b)}{\geq} \left(\sum_{x_1, x_2} P_{X_1}(x_1) P_{X_2}(x_2) f(x_1, x_2)^{p'} \right)^{\frac{1}{p'}} \left(\sum_{y_1, y_2} P_{Y_1}(y_1) P_{Y_2}(y_2) g(y_1, y_2)^q \right)^{\frac{1}{q}}$$
(71)
= $||f(X_1, X_2)||_{p'} ||g(Y_1, Y_2)||_q,$ (72)

where (a) follows from (68) and (b) follows from (67). This means $((X_1, X_2), (Y_1, Y_2))$ is (p, q)-hypercontractive. It is easy to see that if one of (X_1, Y_1) or (X_2, Y_2) is not (p, q)-hypercontractive, then $((X_1, X_2), (Y_1, Y_2))$ is not (p, q)-hypercontractive. Thus,

$$q_p^*(X_1, X_2; Y_1, Y_2) = \min\{q_p^*(X_1; Y_1), q_p^*(X_2; Y_2)\},\$$

which gives

$$s_p(X_1, X_2; Y_1, Y_2) = \max\{s_p(X_1; Y_1), s_p(X_2; Y_2)\}.$$

For p > 1, the proof is similar; in this case, we find

$$q_p^*(X_1, X_2; Y_1, Y_2) = \max\{q_p^*(X_1; Y_1), q_p^*(X_2; Y_2)\}$$

and

$$s_p(X_1, X_2; Y_1, Y_2) = \max\{s_p(X_1; Y_1), s_p(X_2; Y_2)\}.$$

• (lower semi-continuity) If the space of probability distributions on $\mathcal{X} \times \mathcal{Y}$ is endowed with the total variation distance metric, then $s_p(X;Y)$ is a lower semi-continuous function of the joint distribution P(x,y).

Proof: Let us fix p < 1. An identical proof holds for the case of p > 1. Suppose $(X, Y), (X_1, Y_1), (X_2, Y_2), \ldots$ are random variable pairs taking values in the finite set $\mathcal{X} \times \mathcal{Y}$ satisfying $d_{\mathrm{TV}}((X_n, Y_n); (X, Y)) \to 0$ as $n \to \infty$. Let $s := \liminf_{n \to \infty} s_p(X_n; Y_n) \ge 0$. We will show that $s \ge s_p(X; Y)$. Let $\{j_n\}_{n=1}^{\infty}$ be a subsequence so that $s = \lim_{n \to \infty} s_p(X_{j_n}; Y_{j_n})$.

We may assume without loss of generality that s < 1. For any $\epsilon > 0$, there exists a $j(\epsilon)$ such that $s_p(X_{j_n}; Y_{j_n}) \leq s + \epsilon$ for all $j_n \geq j(\epsilon)$. We would like to show $s_p(X; Y) \leq s$, i.e., that for any functions $f : \mathcal{X} \mapsto \mathbb{R}_{>0}, g : \mathcal{Y} \mapsto \mathbb{R}_{>0}$, the following holds:

$$\mathbb{E}f(X)g(Y) \ge ||f(X)||_{p'}||g(Y)||_{1+s(p-1)}.$$
(73)

For any given functions $f : \mathcal{X} \mapsto \mathbb{R}_{>0}, g : \mathcal{Y} \mapsto \mathbb{R}_{>0}$, and any $j_n \ge j(\epsilon)$, we have from $s_p(X_{j_n}; Y_{j_n}) \le s + \epsilon$ that for $j_n \ge n(\epsilon)$,

$$\mathbb{E}f(X_{j_n})g(Y_{j_n}) \ge ||f(X_{j_n})||_{p'}||g(Y_{j_n})||_{1+(s+\epsilon)(p-1)}.$$
(74)

From the portmanteau lemma [37], we get

$$\mathbb{E}f(X)g(Y) \ge ||f(X)||_{p'}||g(Y)||_{1+(s+\epsilon)(p-1)}.$$
(75)

Since this is true for each $\epsilon > 0$, we get from continuity of $||.||_q$ in q that

$$\mathbb{E}f(X)g(Y) \ge ||f(X)||_{p'}||g(Y)||_{1+s(p-1)}.$$
(76)

Since this is true for any functions $f : \mathcal{X} \mapsto \mathbb{R}_{>0}, g : \mathcal{Y} \mapsto \mathbb{R}_{>0}$, we have $s_p(X;Y) \leq s$. *Remark* 7. Note that this implies that $q_p(X;Y) = 1 + s_p(X;Y)(p-1)$ is lower semi-continuous in the joint distribution for fixed p > 1 and upper semi-continuous in the joint distribution for fixed p < 1. Remark 8. Lower semi-continuity of ρ_m and s_p was enough for our purposes. Indeed, ρ_m and s_p are not continuous in the underlying joint distribution. As an example, let (X_n, Y_n) be binary-valued and have a joint probability distribution given by $\begin{bmatrix} \frac{1}{n} & 0\\ 0 & 1 - \frac{1}{n} \end{bmatrix}$. Then, $(X_n, Y_n) \stackrel{d}{\to} (X, Y)$ where (X, Y) has a joint probability distribution given by $\begin{bmatrix} 0 & 0\\ 0 & 1 \end{bmatrix}$. But $\rho_m(X_n; Y_n) = s_p(X_n; Y_n) = 1$ for each n and each $p \neq 1$, while $\rho_m(X; Y) = s_p(X; Y) = 0$.

However, it may be shown that if $(X, Y) \sim P(x, y)$ satisfies the assumption $P(x) > 0 \quad \forall x \in \mathcal{X}, P(y) > 0 \quad \forall y \in \mathcal{Y}$, then $(X_n, Y_n) \stackrel{d}{\to} (X, Y)$ implies $\lim_{n\to\infty} \rho_m(X_n; Y_n) = \rho_m(X; Y)$. To see this, use the characterization $\rho_m(X; Y) = \sigma_2(A_{X;Y})$, where the matrix $A_{X;Y}$ is specified by $[A_{X;Y}]_{x,y} = \frac{P(x,y)}{\sqrt{P(x)P(y)}}$ and $\sigma_2(\cdot)$ is the second largest singular value [15], [36]. Under the assumption, $A_{X_n;Y_n} \to A_{X;Y}$ and the second largest singular value is a continuous matrix functional.

C. Limiting properties of s_p : Proofs of Thm. 2 and Corollary 2

As in [22], we define for any non-negative random variable X, the function $\operatorname{Ent}(X) := \mathbb{E}[X \log X] - \mathbb{E}[X] \cdot \log \mathbb{E}[X]$, where by convention $0 \log 0 := 0$. By strict convexity of the function $x \mapsto x \log x$ and Jensen's inequality, we get that $\operatorname{Ent}(X) \ge 0$ and equality holds if and only if X is a constant almost surely. Also, we note that $\operatorname{Ent}(\cdot)$ is homogenous, that is, $\operatorname{Ent}(aX) = a \operatorname{Ent}(X)$ for any $a \ge 0$.

We begin by presenting first a simple lemma.

Lemma 1. For any random variable Z satisfying $0 \le Z \le K$ for some constant K > 0 and $\mathbb{E}Z = 1$ and $0 \le u \le 1$, we have

$$1 + u \operatorname{Ent}(Z) - u^2 L_1(K) \le ||Z||_{1+u} \le 1 + u \operatorname{Ent}(Z) + u^2 L_0(K),$$
(77)

where $L_0(K) = \frac{1}{2} \max\{K^u, 1\} \max_{0 \le z \le K} z(\log z)^2$ and $L_1(K) = (\max_{0 \le z \le K} |z \log z|) + \frac{1}{2} (\max_{0 \le z \le K} |z \log z|)^2$.

Proof of Lemma 1. For any constant $0 \le u \le 1$ and any $\theta \in \mathbb{R}$, a Taylor's series expansion yields

$$1 + u\theta \le e^{u\theta} \le 1 + u\theta + \frac{u^2}{2}\theta^2 \max\{e^{u\theta}, 1\}$$

Thus, for any $0 \le z \le K$ for some constant K > 0, and $0 \le u \le 1$, we have using $z^{1+u} = ze^{u \log z}$,

$$z + uz \log z \le z^{1+u} \le z + uz \log z + \frac{u^2}{2} z (\log z)^2 \max\{z^u, 1\}$$

For any random variable Z satisfying $0 \le Z \le K$ almost surely and any $0 \le u \le 1$,

$$\mathbb{E}Z + u\mathbb{E}[Z\log Z] \le \mathbb{E}[Z^{1+u}] \le \mathbb{E}Z + u\mathbb{E}[Z\log Z] + \frac{u^2}{2}\max\{K^u, 1\}\mathbb{E}[Z(\log Z)^2]$$
$$\le \mathbb{E}Z + u\mathbb{E}[Z\log Z] + u^2L_0(K) .$$
(78)

Now, again a Taylor's expansion yields that for $0 \le r \le 1$ and any $x \ge 0$, we have

$$1 + rx - \frac{x^2}{2}r(1-r) \le (1+x)^r \le 1 + rx .$$
⁽⁷⁹⁾

Suppose Z is any random variable that satisfies $0 \le Z \le K$ and $\mathbb{E}Z = 1$. Then $\mathbb{E}[Z \log Z] = \text{Ent}(Z) \ge 0$. For any $0 \le u \le 1$, we get using the lower bounds in both (78) and (79) with the choice $r = \frac{1}{1+u}$ and x = u Ent(Z),

$$1 + \frac{1}{1+u} u \operatorname{Ent}(Z) - \frac{u^2 \operatorname{Ent}(Z)^2}{2} \frac{1}{1+u} \frac{u}{1+u} \le \left(\mathbb{E}[Z^{1+u}] \right)^{\frac{1}{1+u}}.$$

Similarly, using the upper bounds in both (78) and (79) with the choice $r = \frac{1}{1+u}$ and $x = u \operatorname{Ent}(Z) + u^2 L_0(K)$, we get

$$\left(\mathbb{E}[Z^{1+u}]\right)^{\frac{1}{1+u}} \le 1 + \frac{1}{1+u}u\operatorname{Ent}(Z) + \frac{1}{1+u}u^2L_0(K)$$
.

Putting the above two inequalities together,

$$1 + \frac{1}{1+u}u\operatorname{Ent}(Z) - \frac{u^2\operatorname{Ent}(Z)^2}{2}\frac{1}{1+u}\frac{u}{1+u} \le \|Z\|_{1+u} \le 1 + \frac{1}{1+u}u\operatorname{Ent}(Z) + \frac{1}{1+u}u^2L_0(K).$$

Define $L_2(K) = \max_{0 \le z \le K} |z \log z|$ and observing that for $0 \le u \le 1$, we have $\frac{1}{2} \le \frac{1}{1+u} \le 1$, we obtain

$$1 + \frac{u\operatorname{Ent}(Z)}{1+u} - \frac{u^3}{2}L_2(K)^2 \le ||Z||_{1+u} \le 1 + \frac{u\operatorname{Ent}(Z)}{1+u} + u^2L_0(K).$$

Further using the fact that for $0 \le u \le 1$, we have $1 - u \le \frac{1}{1+u} \le 1$, we get

$$1 + u \operatorname{Ent}(Z) - u^2 L_2(K) - \frac{u^3}{2} L_2(K)^2 \le ||Z||_{1+u} \le 1 + u \operatorname{Ent}(Z) + u^2 L_0(K).$$

Finally, since $L_1(K) = L_2(K) + \frac{1}{2}L_2(K)^2$ and $u \leq 1$, we have

$$1 + u \operatorname{Ent}(Z) - u^2 L_1(K) \le ||Z||_{1+u} \le 1 + u \operatorname{Ent}(Z) + u^2 L_0(K).$$
(80)

Next, we present the proof of Thm. 2.

Proof of Theorem 2. The theorem is easily seen to be true when Y is a constant almost surely. We assume then that this is not the case and that $P_Y(y) > 0$ for all $y \in \mathcal{Y}$ and $P_X(x) > 0$ for all $x \in \mathcal{X}$ without loss of generality. Define $s := \sup \frac{\operatorname{Ent}(\mathbb{E}[g(Y)|X])}{\operatorname{Ent}(g(Y))}$, where the supremum is taken over functions $g : \mathcal{Y} \mapsto \mathbb{R}_{\geq 0}$ such that g(Y) is not a constant almost surely.

For any distribution $R_Y(y) \neq P_Y(y)$ consider the non-constant non-negative valued function g given by $g(y) := \frac{R_Y(y)}{P_Y(y)}$. This choice yields $\operatorname{Ent}(g(Y)) = D(R_Y(y)||P_Y(y))$ and $\operatorname{Ent}(\mathbb{E}[g(Y)|X]) = D(R_X(x)||P_X(x)))$, where $R_X(x) = \sum_y \frac{P_{X,Y}(x,y)}{P_Y(y)} R_Y(y)$. Along with homogeneity of $\operatorname{Ent}(\cdot)$, this means that $s = s^*(Y;X)$ and thus, from the data processing inequality $0 \le s \le 1$.

For non-negative g, we always have

$$||\mathbb{E}[g(Y)|X]||_1 = ||g(Y)||_1 \quad \forall g: \mathcal{Y} \mapsto \mathbb{R}_{\ge 0}.$$
(81)

Let \mathcal{G} be the set of all non-negative functions $g: \mathcal{Y} \mapsto \mathbb{R}_{\geq 0}$ that satisfy $||g(Y)||_1 = 1$. Note that for any $g \in \mathcal{G}$, both g(Y) and $\mathbb{E}[g(Y)|X]$ are bounded between 0 and $K := \frac{1}{\min_y P_Y(y)}$ almost surely. If $0 \leq m \leq 1$ is any parameter satisfying m < s, then $(1 + \tau, 1 + m\tau) \notin \mathcal{R}(X;Y)$ for all sufficiently small

If $0 \le m \le 1$ is any parameter satisfying m < s, then $(1 + \tau, 1 + m\tau) \notin \mathcal{R}(X;Y)$ for all sufficiently small $\tau > 0$. To see this, fix g_0 to be any function in \mathcal{G} that satisfies

$$\frac{\operatorname{Ent}(\mathbb{E}[g_0(Y)|X])}{\operatorname{Ent}(g_0(Y))} \ge m + \frac{\delta}{2},\tag{82}$$

where $\delta := s - m$. From Lemma 1, we have that for any $g \in \mathcal{G}$,

$$1 + m\tau \operatorname{Ent}(g(Y)) - m^2 \tau^2 L_1(K) \le ||g(Y)||_{1+m\tau} \le 1 + m\tau \operatorname{Ent}(g(Y)) + m^2 \tau^2 L_0(K),$$
(83)

$$1 + \tau \operatorname{Ent}(\mathbb{E}[g(Y)|X]) - \tau^2 L_1(K) \le ||\mathbb{E}[g(Y)|X]||_{1+\tau} \le 1 + \tau \operatorname{Ent}(\mathbb{E}[g(Y)|X]) + \tau^2 L_0(K).$$
(84)

Putting together (82), (83), (84), we get the existence of $\tau_0 > 0$ such that

$$||\mathbb{E}[g_0(Y)|X]||_{1+\tau} > ||g_0(Y)||_{1+m\tau} \quad \forall \tau : 0 < \tau \le \tau_0.$$
(85)

Thus, $s = s^*(Y; X) \ge \limsup_{p \to 1^+} s_p(X; Y) = \limsup_{p \to 1^+} \frac{q_p^*(X; Y) - 1}{p - 1}$. If for some $0 \le m \le 1$ we have m > s, then define for any $g \in \mathcal{G}$,

$$\tau(g) := \max\{\zeta : 0 \le \zeta \le 1, ||\mathbb{E}[g(Y)|X]||_{1+\eta} \le ||g(Y)||_{1+m\eta} \text{ for all } 0 \le \eta \le \zeta\}$$

From (81), we have $\tau(g) \ge 0$ for all $g \in \mathcal{G}$.

Let $g_1 \in \mathcal{G}$ denote the constant function 1. Then, $\tau(g_1) = 1$. Lemma 2 below shows that there is an open neighborhood U of g_1 in \mathcal{G} and a constant $\tau_0 > 0$ such that $\tau(g) \ge \tau_0 \ \forall g \in U$.

Over the compact set $\mathcal{G} \setminus U$, we define

$$\tau'(g) := \max\{\zeta : 0 \le \zeta \le 1, 1 + \eta \operatorname{Ent}(\mathbb{E}[g(Y)|X]) + \eta^2 L_0(K) \le 1 + m\eta \operatorname{Ent}(g(Y)) - m^2 \eta^2 L_1(K) \text{ for all } 0 \le \eta \le \zeta\}$$

Then, $\tau'(g) \leq \tau(g)$ from Lemma 1. And indeed,

$$\tau'(g) = \min\left\{\frac{m\operatorname{Ent}(g(Y)) - \operatorname{Ent}(\mathbb{E}[g(Y)|X])}{L_0(K) + m^2 L_1(K)}, 1\right\}.$$

Since $\tau'(g)$ is continuous in g over $\mathcal{G} \setminus U$, and furthermore strictly positive over that set (since m > s and because $\operatorname{Ent}(g(Y)) > 0$ for g non-constant), we have that τ' attains its infimum over the compact set $\mathcal{G} \setminus U$. Since $\tau'(g) \leq \tau(g)$, we also have that $\inf_{g \in \mathcal{G} \setminus U} \tau(g) > 0$.

Then, $\inf_{g \in \mathcal{G}} \tau(g) = \min \left\{ \tau_0, \inf_{g \in \mathcal{G} \setminus U} \tau(g) \right\} > 0$. Using homogeneity of the norm, this establishes that (1 + 1) $(\tau, 1 + m\tau) \in \mathcal{R}(X; Y)$ for all $0 \le \tau \le \tau_0$ for some $\tau_0 > 0$ and thus, that $s = s^*(Y; X) \le \liminf_{p \to 1^+} s_p(X; Y) = 0$ $\liminf_{p \to 1^+} \frac{q_p^*(X;Y) - 1}{p - 1}.$

Therefore, $s = s^*(Y; X) = \lim_{p \to 1^+} s_p(X; Y) = \lim_{p \to 1^+} \frac{q_p^*(X; Y) - 1}{p - 1}$. Similarly, we can show the reverse hypercontractive case namely, that $s = s^*(Y; X) = \lim_{p \to 1^-} s_p(X; Y) = \lim_{p \to 1^+} \frac{q_p^*(X; Y) - 1}{p - 1}$. $\lim_{p\to 1^-} \frac{q_p^*(X;Y)-1}{p-1}$. This completes the proof of the theorem.

Lemma 2. When $1 \ge m > s$, there exists an open neighborhood U of the constant function g_1 in \mathcal{G} and a constant $\tau_0 > 0$ such that $\tau(g) \ge \tau_0$ for all $g \in U$.

Proof of Lemma 2. Let \mathcal{F} denote the set of all functions $f: \mathcal{Y} \mapsto \mathbb{R}$ such that $\mathbb{E}[f(Y)] = 0$ and $\mathbb{E}[f(Y)^2] = 1$. For any $f \in \mathcal{F}$, and any $y \in \mathcal{Y}$, we have $|f(y)| \leq \frac{1}{\min_y \sqrt{P_Y(y)}}$.

For $0 < \epsilon_0 < \frac{1}{2} \min_y \sqrt{P_Y(y)}$, the set $U(\epsilon_0) := \{g_1 + \epsilon f : f \in \mathcal{F}, 0 \le \epsilon < \epsilon_0\}$ is an open neighborhood of the constant function g_1 in \mathcal{G} . Furthermore, $\frac{1}{2} \le g(y) \le \frac{3}{2}$ for all $y \in \mathcal{Y}$ and all $g \in U(\epsilon_0)$.

Let $m = (1 + \delta)s$ where s < 1 and $m \le 1$ and where $\delta > 0$.

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For $g \in \mathcal{G}$, denote $\chi_q(x) = \mathbb{E}[g(Y)|X = x]$ and note that $\frac{1}{2} \leq \chi_q(x) \leq \frac{3}{2}$ for all $x \in \mathcal{X}$. Now, for $0 \le \eta \le 1$,

$$||g(Y)||_{1+m\eta} = \left(\sum_{y} P_Y(y)g(y)e^{m\eta\log g(y)}\right)^{\frac{1}{1+m\eta}}$$
(86)

$$e^{\frac{m\eta}{1+m\eta}\operatorname{Ent}(g(Y))} \tag{87}$$

$$\geq e^{\frac{m\eta}{s(1+m\eta)}\operatorname{Ent}(\mathbb{E}[g(Y)|X])}$$
(88)

$$\geq e^{(1+\delta)\frac{\eta}{(1+\eta)}\operatorname{Ent}(\chi_g(X))}$$
(89)

$$\geq \left(1 + \eta(1+\delta)\operatorname{Ent}(\chi_g(X)) + \frac{\eta^2}{2}(1+\delta)^2\operatorname{Ent}(\chi_g(X))^2\right)^{\frac{1}{1+\eta}},\tag{90}$$

where (87) follows from convexity of the exponential function, (88) follows from the definition of s and (90)follows from $e^u \ge 1 + u + \frac{u^2}{2}$ for $u \ge 0$.

Likewise, we have

$$\|\mathbb{E}[g(Y)|X]\|_{1+\eta} = \left(\sum_{x} P_X(x)\chi_g(x)e^{\eta \log \chi_g(x)}\right)^{\frac{1}{1+\eta}}$$
(91)

$$\leq \left(\sum_{x} P_X(x)\chi_g(x) \left(1 + \eta \log \chi_g(x) + a\frac{\eta^2}{2} (\log \chi_g(x))^2\right)\right)^{\frac{1}{1+\eta}}$$
(92)

$$\leq \left(1 + \eta \operatorname{Ent}(\chi_g(X)) + a \frac{\eta^2}{2} \sum_x P_X(x) \chi_g(x) (\log \chi_g(x))^2 \right)^{\frac{1}{1+\eta}},$$
(93)

where a > 1 is a constant such that $e^u \le 1 + u + a\frac{u^2}{2}$ for $|u| \le \log 2$. Note that $\operatorname{Ent}(\chi_g(X)) = D(Q_X||P_X)$ where $Q_X(x) = P_X(x)\chi_g(x)$ for all $x \in \mathcal{X}$. By Pinsker's inequality,

$$\operatorname{Ent}(\chi_g(X)) \ge \frac{1}{2} \left(\sum_{x} |P_X(x)\chi_g(x) - P_X(x)| \right)^2.$$

Thus, for all $x \in \mathcal{X}$, we have

$$|\chi_g(x) - 1| \le \frac{1}{\min_x P_X(x)} \sqrt{2 \operatorname{Ent}(\chi_g(X))}.$$

If we define for $0 \le \alpha \le 1$, the function $\kappa(\alpha) := \max_{1-\alpha \le v \le 1+\alpha} v(\log v)^2$, where $\kappa(\alpha) \to 0$ as $\alpha \to 0$, then we have

$$\|\mathbb{E}[g(Y)|X]\|_{1+\eta} \le \left(1 + \eta \operatorname{Ent}(\chi_g(X)) + a\frac{\eta^2}{2}\kappa\left(\frac{\sqrt{2\operatorname{Ent}(\chi_g(X))}}{\min_x P_X(x)}\right)\right)^{\frac{1}{1+\eta}}.$$
(94)

Using (90) and (94), we find that for any $g \in U(\epsilon_0)$, we have $\tau(g) \ge \beta(\operatorname{Ent}(\chi_g(X)))$ where

$$\beta(\rho) := \begin{cases} 1 & \text{if } a\kappa \left(\frac{\sqrt{2\rho}}{\min_x P_X(x)}\right) - (1+\delta)^2 \rho^2 \le 0\\ \min\left\{\frac{2\delta\rho}{a\kappa \left(\frac{\sqrt{2\rho}}{\min_x P_X(x)}\right) - (1+\delta)^2 \rho^2}, 1\right\} & \text{else.} \end{cases}$$

Given any $\theta > 0$, there exists $0 < \epsilon_1 < \epsilon_0$ small enough so that $\operatorname{Ent}(\chi_g(X)) \leq \operatorname{Ent}(g(Y)) \leq \theta$ for all $g \in U(\epsilon_1)$. This means that for all $g \in U(\epsilon_1)$, we have $\tau(g) \geq \inf_{0 \leq \rho \leq \theta} \beta(\rho)$. Since $\kappa(\alpha) = \alpha^2 + O(\alpha^3)$ for small $\alpha > 0$, it follows that $\inf_{0 \leq \rho \leq \theta} \beta(\rho) > 0$ for sufficiently small θ . This completes the proof of the lemma.

Now, we present the proof of Corollary 2.

Proof of Corollary 2. If X and Y are independent, then it is clear that $\rho_m(X;Y) = s^*(X;Y) = 0$ and $q_p^*(X;Y) = 1$ for all $p \neq 1$. The claim is obvious in this case.

Suppose X and Y are not independent. Fix any ϵ satisfying $0 < \epsilon < s^*(Y; X)$. Note that by Theorems 1 and 2, we have $s^*(Y; X) = \lim_{p \to 1} s_p(X; Y) \ge \rho_m^2(X; Y) > 0$.

From Thm. 2, we have that there exists a $\delta > 0$ such that

$$0 < |p-1| \le \delta \implies s^*(Y;X) - \epsilon \le \frac{q_p^*(X;Y) - 1}{p-1} \le s^*(Y;X) + \epsilon.$$

$$(95)$$

Now, define

$$A(\epsilon) := \left\{ (p,q) : 0 < |p-1| \le \delta, s^*(Y;X) + \epsilon \le \frac{q-1}{p-1} \le 1 \right\},$$
(96)

$$B(\epsilon) := \left\{ (p,q) : 0 < |p-1| \le \delta, s^*(Y;X) - \epsilon \le \frac{q-1}{p-1} \le 1 \right\} \cup \{(1,1)\}$$
$$\cup \left\{ (p,q) : |p-1| \ge \delta, \rho_m^2(X;Y) \le \frac{q-1}{p-1} \le 1 \right\}.$$
(97)

From (95) and Thm. 1, it is clear that

$$A(\epsilon) \subseteq \mathcal{R}(X;Y) \subseteq B(\epsilon). \tag{98}$$

By using the duality $(p,q) \in \mathcal{R}(X;Y) \Leftrightarrow (q',p') \in \mathcal{R}(Y;X)$ for $p,q \neq 1$, we obtain

$$A_1(\epsilon) \subseteq \mathcal{R}(Y;X) \subseteq B_1(\epsilon),\tag{99}$$

where

$$A_1(\epsilon) := \left\{ (p,q) : |q-1| \ge \frac{1}{\delta}, s^*(Y;X) + \epsilon \le \frac{q-1}{p-1} \le 1 \right\},$$
(100)

$$B_{1}(\epsilon) := \left\{ (p,q) : |q-1| \ge \frac{1}{\delta}, s^{*}(Y;X) - \epsilon \le \frac{q-1}{p-1} \le 1 \right\} \cup \{(1,1)\}$$
$$\cup \left\{ (p,q) : 0 < |q-1| \le \frac{1}{\delta}, \rho_{m}^{2}(X;Y) \le \frac{q-1}{p-1} \le 1 \right\}.$$
(101)

This immediately gives

$$s^{*}(Y;X) - \epsilon \le \lim \inf_{p \to -\infty} \frac{q_{p}^{*}(Y;X) - 1}{p - 1} \le \lim \sup_{p \to -\infty} \frac{q_{p}^{*}(Y;X) - 1}{p - 1} \le s^{*}(Y;X) + \epsilon,$$
(102)

$$s^{*}(Y;X) - \epsilon \le \liminf_{p \to \infty} \frac{q_{p}^{*}(Y;X) - 1}{p - 1} \le \limsup_{p \to \infty} \frac{q_{p}^{*}(Y;X) - 1}{p - 1} \le s^{*}(Y;X) + \epsilon.$$
(103)

Since this is true for each sufficiently small $\epsilon > 0$, interchanging X and Y completes the proof.