# Secret Key Agreement: General Capacity and Second-Order Asymptotics

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#### **Abstract**

We revisit the problem of secret key agreement using interactive public communication for two parties and propose a new secret key agreement protocol. The protocol attains the secret key capacity for general observations and attains the second-order asymptotic term in the maximum length of a secret key for independent and identically distributed observations. In contrast to the previously suggested secret key agreement protocols, the proposed protocol uses interactive communication. In fact, the standard one-way communication protocol used prior to this work fails to attain the asymptotic results above. Our converse proofs rely on a recently established upper bound for secret key lengths. Both our lower and upper bounds are derived in a *single-shot* setup and the asymptotic results are obtained as corollaries.

## I. Introduction

Two parties observing random variables (RVs) X and Y seek to agree on a secret key. They can communicate interactively over an error-free, authenticated, albeit insecure, communication channel of unlimited capacity. The secret key must be concealed from an eavesdropper with access to the communication and an additional side information Z. What is the maximum length  $S(X, Y \mid Z)$  of a secret key that the parties can agree upon?

A study of this question was initiated by Maurer [19] and Ahlswede and Csiszár [1] for the case where the observations of the parties and the eavesdropper consist of n independent and identically distributed

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(IID) repetitions  $(X^n, Y^n, Z^n)$  of RVs (X, Y, Z). For the case when  $X \Leftrightarrow Y \Leftrightarrow Z$  form a Markov chain, it was shown in [19], [1] that the *secret key capacity* equals  $I(X \land Y|Z)$ , namely

$$S(X^n, Y^n \mid Z^n) = nI(X \land Y \mid Z) + o(n).$$

However, in several applications (see, for instance, [6]) the observed data is not IID or even if the observations are IID, the observation length n is limited and a more precise asymptotic analysis is needed.

In this paper, we address the secret key agreement problem for these two important practical situations. First, when the observations consist of general sources (cf. [11], [10])  $(X_n, Y_n, Z_n)$  such that  $X_n \oplus Y_n \oplus Z_n$  is a Markov chain, we show that

$$S(X_n, Y_n \mid Z_n) = n\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}) + o(n),$$

where  $\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z})$  is the *inf-conditional information* of  $\mathbf{X}$  and  $\mathbf{Y}$  given  $\mathbf{Z}$ . Next, for the IID case with  $X \to Y \to Z$ , we identify the second-order asymptotic term<sup>1</sup> in  $S(X^n, Y^n \mid Z^n)$ . Specifically, denoting by  $S_{\epsilon,\delta}(X,Y \mid Z)$  the maximum length of a secret key over which the parties agree with probability greater than  $1 - \epsilon$  and with secrecy parameter less than  $\delta$ , we show that

$$S_{\epsilon,\delta}\left(X^{n},Y^{n}\mid Z^{n}\right)=nI(X\wedge Y\mid Z)-\sqrt{nV}Q^{-1}(\epsilon+\delta)\pm\mathcal{O}(\log n),$$

where Q is the tail probability of the standard Gaussian distribution and

$$V := \mathbb{V}\mathrm{ar}\left[\log\frac{\mathrm{P}_{XY\mid Z}\left(X,Y\mid Z\right)}{\mathrm{P}_{X\mid Z}\left(X\mid Z\right)\mathrm{P}_{Y\mid Z}\left(Y\mid Z\right)}\right].$$

In particular, our bounds allow us to evaluate the *gap to secret key capacity* at a finite blocklength n. In Figure 1 we illustrate this gap between the maximum possible rate of a secret key at a fixed n and the secret key capacity for the case where Z is a random bit, Y is obtained by flipping Z with probability 0.25 and X given by flipping Y with probability 0.125; see Example 1 in Section VI for details.

Underlying these results is a general single-shot characterization of the secret key length which shows that, when X - Z,  $S_{\epsilon,\delta}(X,Y|Z)$  roughly equals the  $(\epsilon + \delta)$ -tail of the random variable

$$i(X \wedge Y|Z) = \log \frac{\mathrm{P}_{XY|Z}\left(X,Y|Z\right)}{\mathrm{P}_{X|Z}\left(X|Z\right)\mathrm{P}_{Y|Z}\left(Y|Z\right)}.$$

<sup>&</sup>lt;sup>1</sup>Following the pioneering work of Strassen [28], study of these second-order terms in coding theorems has been revived recently by Hayashi [13], [14] and Polyanskiy, Poor, and Verdú [23].

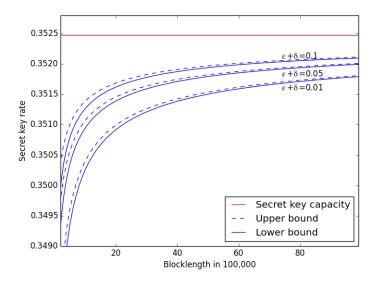


Fig. 1. Gap to secret key capacity at finite n for  $\epsilon + \delta = 0.01, 0.05, 0.1$ .

Our main technical contribution in proving this result is a new single-shot secret key agreement protocol which uses interactive communication and attains the desired optimal performance. It was observed in [19], [1] that a simple one-way communication protocol suffices to attain the secret key capacity, when the Markov relation X - Y - Z holds. Also, for a multiterminal setup with constant Z, [4] showed that a noninteractive communication protocol achieves the secret key capacity. Prior to this work, when the Markov relation X - Z holds, only such noninteractive communication protocols were used for generating secret keys<sup>2</sup>, even in single-shot setups (*cf.* [25]). In contrast, our proposed protocol uses interactive communication. We note in Remark 4 that none of the standard one-way communication protocols achieve the optimal asymptotic bounds, suggesting that perhaps interaction is necessary for generating a secret key of optimal length (see Section VII for further discussion and an illustrative example).

Typically, secret key agreement protocols consist of two steps: *information reconciliation* and *privacy amplification*. In the first step, the parties communicate to generate some shared random bits, termed *common randomness*. However, the communication used leaks some information about the generated common randomness. Therefore, a second privacy amplification step is employed to extract from the common randomness secure random bits that are almost independent of the communication used. For IID

<sup>&</sup>lt;sup>2</sup>Interaction is known to help in some cases where neither X - Y - Z nor Y - Z is satisfied [19], [35], [9].

observations, the information reconciliation step of the standard one-way secret key agreement protocol entails the two parties agreeing on X using a one-way communication of rate H(X|Y). In the privacy amplification step, the rate H(X|Z) residual randomness of X, which is almost independent of Z, is used to extract a secret key of rate H(X|Z) - H(X|Y) which is independent jointly of Z and the communication used in the information reconciliation stage. Under the Markov condition  $X \leftrightarrow Y \leftrightarrow Z$ , the resulting secret key attains the secret key capacity  $I(X \land Y \mid Z)$ . However, in the single-shot regime, the behavior of RVs  $-\log P_{X\mid Z}(X\mid Z)$  and  $-\log P_{X\mid Y}(X\mid Y)$ , rather than their expected values, becomes relevant. The difficulty in extending the standard one-way communication protocol to the single-shot setup lies in the spread of the *information spectrums*<sup>3</sup> of  $P_{X\mid Y}$  and  $P_{X\mid Z}$ . Specifically, while we require the random variable  $-\log P_{X\mid Y}(X\mid Y)$  itself to show up as the length of communication in the information reconciliation step, a naive extension requires as much communication as a large probability tail of  $-\log P_{X\mid Y}(X\mid Y)$ . To remedy this, we *slice the spectrum* of  $P_{X\mid Y}$  into slices<sup>4</sup> of length  $\Delta$  each and adapt the protocol to the slice which contains (X,Y). However, since neither party knows the value of  $-\log P_{X\mid Y}(X\mid Y)$ , this adaptation requires interactive communication.

Motivating this work, and underlying our converse proof, is a recently established single-shot upper bound on secret key lengths for the multiparty secret key agreement problem [33] (see, also, [31]). The proof relies on relating secret key agreement to binary hypothesis testing. In spirit, this result can be regarded as a multiterminal variant of a similar single-shot converse for the channel coding problem which appeared first in [21], [15] and has been termed the meta-converse by Polyanskiy, Poor, and Verdú [23], [22] (see, also, [34] and [12, Section 4.6]).

The basic concepts of secret key agreement and a general result for converting a high reliability protocol to a high secrecy protocol are given in the next section. In Section III, we review the single-shot upper bound of [33] for the two party case. Our new secret key agreement protocol and its single-shot performance analysis is presented in Section IV. The single-shot results are applied to general sources in Section V and to IID sources in Section VI. The final section contains a discussion on the role of interaction in our secret key agreement protocols.

 $<sup>^3</sup>$ The range of the log-likelihood  $-\log P_X(x)$  (conditional log-likelihood  $-\log P_{X|Y}(x|y)$ ) is referred to as information spectrum of  $P_X$  (conditional information spectrum of  $P_{X|Y}$ ). This notion was introduced in the seminal work [11] and is appropriate for deriving single-shot coding theorems, without making assumptions on the underlying distribution. See [10] for a detailed account.

 $<sup>^4</sup>$  See Appendix C for a secret key agreement based on slicing the spectrum of  $\mathrm{P}_X$ .

#### II. SECRET KEYS

We consider the problem of secret key agreement using interactive public communication by two (trusted) parties observing RVs X and Y taking values in countable sets  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively. Upon making these observations, the parties communicate interactively over a public communication channel that is accessible by an eavesdropper. We assume that the communication channel is error-free and authenticated. Specifically, the communication is sent over r rounds of interaction<sup>5</sup>. In the jth round of communication,  $1 \le j \le r$ , each party sends a message which is a function of its observation, locally generated randomness denoted by f0 and f1, and the previously observed communication. The overall interactive communication is denoted by f1. In addition to f2, the eavesdropper observes a RV f2 taking values in a countable set f2. The joint distribution f3 known to all parties.

Using the interactive communication  $\mathbf{F}$  and their local observations, the parties agree on a secret key. A RV K constitutes a secret key if the two parties form estimates that agree with K with probability close to 1 and K is concealed, in effect, from an eavesdropper with access to  $(\mathbf{F}, Z)$ . Formally, we have the following definition.

**Definition 1.** A RV K with range K constitutes an  $(\epsilon, \delta)$ -secret key  $((\epsilon, \delta)$ -SK) if there exist functions  $K_x$  and  $K_y$  of  $(U_x, X, \mathbf{F})$  and  $(U_y, Y, \mathbf{F})$ , respectively, such that the following two conditions are satisfied

$$P(K_x = K_y = K) \ge 1 - \epsilon, \tag{1}$$

$$\|\mathbf{P}_{K\mathbf{F}Z} - \mathbf{P}_{\mathtt{unif}} \times \mathbf{P}_{\mathbf{F}Z}\|_{1} \le \delta, \tag{2}$$

where  $P_{\mathtt{unif}}$  is the uniform distribution on  $\mathcal{K}$  and

$$\|P - Q\|_1 = \frac{1}{2} \sum_{u} |P(u) - Q(u)|.$$

The first condition above represents the *reliability* of the secret key and the second condition guarantees *secrecy*.

**Definition 2.** Given  $\epsilon, \delta \in [0, 1)$ , the supremum over the lengths  $\log |\mathcal{K}|$  of an  $(\epsilon, \delta)$ -SK is denoted by  $S_{\epsilon, \delta}(X, Y \mid Z)$ .

**Remark 1.** The only interesting case is when  $\epsilon + \delta < 1$ , since otherwise  $S_{\epsilon,\delta}(X,Y\mid Z)$  is unbounded.

<sup>&</sup>lt;sup>5</sup> In the asymptotic regime considered in Sections V, the number of rounds r may depend on the block length n.

<sup>&</sup>lt;sup>6</sup>The RVs  $U_x$  and  $U_y$  are mutually independent and independent jointly of (X,Y).

Indeed, consider two trivial secret keys  $K_1$  and  $K_2$  with range  $\mathcal{K}$  generated as follows: For  $K_1$ , the first party generates  $K_x = K_1$  uniformly over  $\mathcal{K}$  and sends it to the second party. Thus,  $K_1$  constitutes a  $(0, 1 - 1/|\mathcal{K}|)$ -SK, and therefore, also a (0, 1)-SK. For  $K_2$ , the first party generates  $K_x = K_2$  uniformly over  $\mathcal{K}$  and the second party generates  $K_y$  uniformly over  $\mathcal{K}$ . Then,  $K_2$  constitutes a  $(1 - 1/|\mathcal{K}|, 0)$ -SK, and therefore, also a (1, 0)-SK. If  $\epsilon + \delta \geq 1$ , the RV K which equals  $K_1$  with probability  $(1 - \epsilon)$  and  $K_2$  with probability  $\epsilon$  constitutes  $(\epsilon, 1 - \epsilon)$ -SK of length  $\log |\mathcal{K}|$ , and therefore, also an  $(\epsilon, \delta)$ -SK of the same length. Since  $\mathcal{K}$  was arbitrary,  $S_{\epsilon,\delta}(X,Y\mid Z) = \infty$ .

Remark 1 exhibits a high reliability (0,1)-SK and a high secrecy (1,0)-SK for the trivial case  $\epsilon+\delta\geq 1$ . The two constructions together sufficed to characterize  $S_{\epsilon,\delta}(X,Y\mid Z)$ . Following a similar approach for the regime  $\epsilon+\delta<1$ , we can construct a high reliability  $(\epsilon+\delta,0)$ -SK and a high secrecy  $(0,\epsilon+\delta)$ -SK and randomize over those two secret keys with probabilities  $\epsilon/(\epsilon+\delta)$  and  $\delta/(\epsilon+\delta)$  to obtain a hybrid,  $(\epsilon,\delta)$ -SK. However, the results below show that we do not need to construct both high secrecy and high reliability secret keys for the secrecy definition in (2) and a high reliability construction alone will suffice. We first show that any  $(\epsilon,\delta)$ -SK can be converted into a high secrecy,  $(\epsilon+\delta,0)$ -SK.

**Proposition 1** (Conversion to High Secrecy Protocol). Given an  $(\epsilon, \delta)$ -SK, there exists an  $(\epsilon + \delta, 0)$ -SK of the same length.

*Proof.* Let K be an  $(\epsilon, \delta)$ -SK using interactive communication F, with local estimates  $K_x$  and  $K_y$ . We construct a new  $(\epsilon + \delta, 0)$ -SK K' using the *maximal coupling lemma*, which asserts the following (cf. [29]): Given two distributions P and Q on a set  $\mathcal{X}$ , there exists a joint distribution  $P_{XX'}$  on  $\mathcal{X} \times \mathcal{X}$  such that the marginals are  $P_X = P$  and  $P_{X'} = Q$ , and under  $P_{XX'}$ 

$$P(X \neq X') = \|P - Q\|_1. \tag{3}$$

The distribution  $P_{XX'}$  is called the *maximal coupling* of P and Q.

For each fixed realization of  $(\mathbf{F}, Z)$ , let  $P_{KK'|\mathbf{F},Z}$  be the maximal coupling of  $P_{K|\mathbf{F}Z}$  and  $P_{\mathbf{unif}}$ . Then  $P_{K'\mathbf{F}Z} = P_{\mathbf{unif}} \times P_{\mathbf{F}Z}$ , and since K is an  $(\epsilon, \delta)$ -SK, we get by the maximal coupling property (3) that

$$\mathbf{P}\left(K \neq K'\right) = \left\|\mathbf{P}_{K\mathbf{F}Z} - \mathbf{P}_{\mathtt{unif}} \times \mathbf{P}_{\mathbf{F}Z}\right\|_{1} \leq \delta,$$

and define

$$P_{K'KK_xK_y\mathbf{F}XYZU_xU_y} := P_{K'|K\mathbf{F}Z}P_{KK_xK_y\mathbf{F}XYZU_xU_y}. \tag{4}$$

Since  $P(K = K_x = K_y) \ge 1 - \epsilon$ , under  $P_{K'KK_xK_yFXYZ}$  we have

$$P(K_x = K_y = K') \ge 1 - \epsilon - \delta.$$

Thus, K' constitutes an  $(\epsilon + \delta, 0)$ -SK.

Proposition 1 plays an important role in our secret key agreement protocol and allows us to convert a high reliability  $(\eta, \alpha)$ -SK with small  $\eta$  into an  $(\epsilon, \delta)$ -SK for any arbitrary  $\epsilon$  and  $\delta$  satisfying (roughly)  $\epsilon + \delta > \alpha$ . Formally, we have the following.

**Proposition 2 (Hybrid Protocol).** Given a protocol for generating  $(\eta, \alpha)$ -SK, there exists a protocol for generating an  $(\epsilon, \delta)$ -SK of the same length for every  $0 < \epsilon, \delta < 1$  such that

$$\epsilon \ge \eta$$
 
$$\epsilon + \delta \ge \alpha + \eta.$$

*Proof.* Given an  $(\eta, \alpha)$ -SK  $K_1$ , by Proposition 1 there exists an  $(\alpha + \eta, 0)$ -SK  $K_2$ . Let  $\theta = \delta/(\epsilon - \eta + \delta)$ . Consider a secret key K obtained by a hybrid use of the protocols for generating  $K_1$  and  $K_2$ , with the protocol for  $K_1$  executed with probability  $\theta$  and that for  $K_2$  with probability  $1 - \theta$ . Note from the proof of Proposition 1 that it is the same secret key agreement protocol  $(K_x, K_y, \mathbf{F})$  that generates both  $K_1$  and  $K_2$ . Thus, the claim follows for the time-shared secret key K since

$$P(K = K_x = K_y) = \theta P(K_1 = K_x = K_y) + (1 - \theta) P(K_2 = K_x = K_y)$$

$$\geq 1 - \frac{\delta \eta + (\epsilon - \eta)(\alpha + \eta)}{\epsilon - \eta + \delta}$$

$$\geq 1 - \frac{\delta \eta + (\epsilon - \eta)(\epsilon + \delta)}{\epsilon - \eta + \delta}$$

$$= 1 - \epsilon,$$
(5)

and

$$\|\mathbf{P}_{K\mathbf{F}Z} - \mathbf{P}_{\mathbf{unif}} \mathbf{P}_{\mathbf{F}Z}\|_{1} \leq \theta \|\mathbf{P}_{K_{1}\mathbf{F}Z} - \mathbf{P}_{\mathbf{unif}} \mathbf{P}_{\mathbf{F}Z}\|_{1} + (1 - \theta) \|\mathbf{P}_{K_{2}\mathbf{F}Z} - \mathbf{P}_{\mathbf{unif}} \mathbf{P}_{\mathbf{F}Z}\|_{1}$$

$$\leq \frac{\delta \alpha + (\epsilon - \eta) \cdot 0}{\epsilon - \eta + \delta}$$

$$\leq \frac{\delta (\epsilon + \delta - \eta)}{\epsilon - \eta + \delta}$$

$$= \delta,$$
(6)

where we have used the assumption  $\epsilon + \delta \ge \alpha + \eta$  in (5) and (6).

**Remark 2.** Note that the actual secret key K in Definition 1 is not available to any party and has only a formal role in the secret key agreement protocol. Interestingly, the proof above says that the estimates  $(K_x, K_y)$  of a high reliability  $(\eta, \alpha)$ -SK with  $\eta \approx 0$  constitute an  $(\epsilon, \delta)$ -SK as well for every  $\epsilon + \delta \gtrsim \alpha$ , albeit for a different hidden RVs K.

Thus, it suffices to exhibit a high reliability  $(\eta, \alpha)$ -SK with small  $\eta$  and desired  $\alpha$ . Such a protocol is given in Section IV and underlies all our achievability results.

A more demanding secrecy requirement. A more demanding secrecy requirement enforces one of the estimates  $K_x$  or  $K_y$  itself to be secure, *i.e.*,

$$\|\mathbf{P}_{K_x \mathbf{F} Z} - \mathbf{P}_{\mathtt{unif}} \mathbf{P}_{\mathbf{F} Z}\|_1 \le \delta \text{ or } \|\mathbf{P}_{K_y \mathbf{F} Z} - \mathbf{P}_{\mathtt{unif}} \mathbf{P}_{\mathbf{F} Z}\|_1 \le \delta.$$
 (7)

The validity of Proposition 1 for this secrecy requirement remains open. However, in the important special case when Z is a function of either X or Y, which includes the case of constant Z, Proposition 1 holds even under the more demanding secrecy requirement (7). Indeed, let  $(K_x, K_y)$  be an  $(\epsilon, \delta)$ -SK with  $K_x$  satisfying the more demanding secrecy requirement above. Proceeding as in the proof of Proposition 1 with  $K_x$  in the role of K, we obtain a RV K' such that  $P(K' \neq K_y) \leq \epsilon + \delta$  and  $P_{K'FZ} = P_{unif} \times P_{FZ}$ . Let the joint distribution  $P_{K'K_xK_yFXYZU_xU_y}$  be as in (4) with  $K_x$  replacing K. To claim that K' constitutes an  $(\epsilon + \delta, 0)$ -SK under (7), it suffices to show that one of the parties can simulate K'. To that end, the party observing K can first run the original secret key agreement protocol to get  $K_x$  and K and K' using the distribution  $P_{K'|K_xFZ}$ , which completes the proof of Proposition 1 under the more demanding secrecy requirement.

Note that the argument above relies on using local randomness to simulate K'. For the original secrecy requirement (2), Proposition 1 holds even when we restrict to deterministic protocols with no local randomness allowed. It turns out that this is not the case for the more demanding secrecy requirement (7), as the following simple counterexample shows: Let X be a binary RV taking 1 with probability  $p < \frac{1}{2}$ , and let Y = Z = constant. Then,  $K_x = X$  and constitutes a 1-bit (p, 1/2 - p)-SK under (7). If Proposition 1 holds, the parties should be able to generate a (1/2, 0)-SK. However, a (1/2, 0)-SK consists of an unbiased bit, which cannot be generated without additional randomness. Therefore, for secrecy requirement (7), Proposition 1 does not hold if we restrict to deterministic protocols.

To conclude, for the special case when Z is a function of X, it suffices to construct only a high

reliability protocol, provided that local randomness is available. In fact, the high reliability protocol proposed in Section IV satisfies the more demanding secrecy requirement (7) and, if Z is a function of either X or Y, all the results of this paper hold even under (7).

III. Upper bound on 
$$S_{\epsilon,\delta}(X,Y \mid Z)$$

We recall the *conditional independence testing* upper bound on  $S_{\epsilon,\delta}(X,Y\mid Z)$ , which was established recently in [33], [32]. In fact, the general upper bound in [33], [32] is a single-shot upper bound on the secret key length for a multiparty secret key agreement problem. We recall a specialization of the general result to the case at hand. In order to state our result, we need the following concept from binary hypothesis testing.

Consider a binary hypothesis testing problem with null hypothesis P and alternative hypothesis Q, where P and Q are distributions on the same alphabet  $\mathcal{V}$ . Upon observing a value  $v \in \mathcal{V}$ , the observer needs to decide if the value was generated by the distribution P or the distribution Q. To this end, the observer applies a stochastic test T, which is a conditional distribution on  $\{0,1\}$  given an observation  $v \in \mathcal{V}$ . When  $v \in \mathcal{V}$  is observed, the test T chooses the null hypothesis with probability T(0|v) and the alternative hypothesis with probability T(1|v) = 1 - T(0|v). For  $0 \le \epsilon < 1$ , denote by  $\beta_{\epsilon}(P, Q)$  the infimum of the probability of error of type II given that the probability of error of type I is less than  $\epsilon$ , i.e.,

$$\beta_{\epsilon}(P, Q) := \inf_{T: P[T] \ge 1 - \epsilon} Q[T], \tag{8}$$

where

$$P[T] = \sum_{v} P(v)T(0|v),$$

$$Q[T] = \sum_{v} Q(v)T(0|v).$$

The definition of a secret key used in [33], [32] is different from Definition 1. However, the two definitions are closely related, and the upper bound of [33], [32] can be extended to our case as well. We review the alternative definition in Appendix A and relate it to Definition 1 to derive the following upper bound, which will be instrumental in our converse proofs.

**Theorem 3** (Conditional independence testing bound). Given  $0 \le \epsilon + \delta < 1$ ,  $0 < \eta < 1 - \epsilon - \delta$ , the

following bound holds:

$$S_{\epsilon,\delta}(X,Y\mid Z) \le -\log \beta_{\epsilon+\delta+\eta}(P_{XYZ},Q_{X\mid Z}Q_{Y\mid Z}Q_{Z}) + 2\log(1/\eta),$$

for all joint distributions Q on  $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$  that render X and Y conditionally independent given Z.

#### IV. THE SECRET KEY AGREEMENT PROTOCOL

In this section we present our secret key agreement protocol, which will be used in all the achievability results of this paper. A typical secret key agreement protocol for two parties entails sharing the observations of one of the parties, referred to as *information reconciliation*, and then extracting a secret key out of the shared observations, referred to as *privacy amplification* (*cf.* [19], [1], [24]). In another interpretation, the parties communicate first to establish a *common randomness* [2] and then extract a secret key from the common randomness<sup>7</sup>. Our protocol below, too, has these two components but the rate of the communication for information reconciliation and the rate of the randomness extracted by privacy amplification have to be chosen carefully.

Heuristically, in the information reconciliation stage, the first party randomly bins X and sends it to the second party. If we do not use interaction, by the Slepian-Wolf theorem [27] (see [20],[10, Lemma 7.2.1], [18] for a single-shot version) the length of communication needed is roughly equals a large probability upper bound for  $h(X|Y) := -\log P_{X|Y}(X|Y)$ . However, in order to derive a lower bound that matches our upper bound, we expect to send X to Y using approximately h(X|Y) bits of communication, which can differ from the tail bound above by as much as the length of the spectrum of  $P_{X|Y}$ . To overcome this gap, we utilize spectrum slicing, a technique introduced in [10], to construct an adaptive scheme that can handle the *spread of information spectrum*. Specifically, we divide the spectrum of  $P_{X|Y}$  into slices of small lengths. The protocol proceeds interactively to adapt to the current slice index, allowing us to replace the spectrum length in the argument above with the length of a single slice.

## A. Formal description of the protocol

We consider the essential spectrum of  $P_{X|Y}$ , *i.e.*, the set of values taken by h(X|Y) between  $\lambda_{\min}$  and  $\lambda_{\max}$ , where  $\lambda_{\min}$  and  $\lambda_{\max}$  are chosen such that h(X|Y) lies in  $(\lambda_{\min}, \lambda_{\max})$  with large probability. We divide the essential spectrum of  $P_{X|Y}$  into L slices, L of them of width  $\Delta$ . Specifically, for  $1 \leq j \leq L$ ,

<sup>&</sup>lt;sup>7</sup>For an interpretation of secrecy agreement in terms of common randomness decomposition, see [4], [5], [30].

the jth slice of the spectrum of  $P_{X|Y}$  is defined as follows

$$\mathcal{T}_{j} = \{(x, y) : \lambda_{j} \le -\log P_{X|Y}(x|y) < \lambda_{j} + \Delta\},\$$

where  $\lambda_j = \lambda_{\min} + (j-1)\Delta$ . Note that the slice index is not available to any one party. The proposed protocol proceeds assuming the lowest index j=1 and uses interactive communication to adapt to the actual slice index.

For information reconciliation, we simply send a random binning of X. However, the bin size is increased successively, where the incremental bin sizes  $M_1, \ldots, M_L$  are given by

$$\log M_j = \begin{cases} \lambda_1 + \Delta + \gamma, & j = 1\\ \Delta, & 1 < j \le L, \end{cases}$$

For privacy amplification, we will rely on the *leftover hash lemma* [17], [25]. Let  $\mathcal{F}$  be a 2-universal family of mappings  $f: \mathcal{X} \to \mathcal{K}$ , i.e., for each  $x' \neq x$ , the family  $\mathcal{F}$  satisfies

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \mathbb{1}(f(x) = f(x')) \le \frac{1}{|\mathcal{K}|}.$$
(9)

The following lemma is a slight modification of the known forms of the leftover hash lemma (*cf.* [24]); we give a proof in Appendix B for completeness.

**Lemma 4** (**Leftover Hash**). Consider RVs X, Z, and V taking values in X, Z and V, respectively, where X and Z are countable and V is finite. Let S be a random seed such that  $f_S$  is uniformly distributed over a 2-universal family as above. Then, for  $K = f_S(X)$  and for any  $Q_Z$  satisfying  $\operatorname{supp}(P_Z) \subset \operatorname{supp}(Q_Z)$ , we have

$$\|\mathbf{P}_{KVZS} - \mathbf{P}_{\mathtt{unif}} \mathbf{P}_{VZ} \mathbf{P}_S\|_1 \leq \frac{1}{2} \sqrt{|\mathcal{K}||\mathcal{V}| 2^{-H_{\min}(\mathbf{P}_{XZ}|\mathbf{Q}_Z)}},$$

where  $P_{\mathtt{unif}}$  is the uniform distribution on  $\mathcal K$  and

$$H_{\min}\left(\mathbf{P}_{XZ}\mid\mathbf{Q}_{Z}\right)=-\log\sup_{x,z:\mathbf{Q}_{Z}\left(z\right)>0}\frac{\mathbf{P}_{XZ}\left(x,z\right)}{\mathbf{Q}_{Z}\left(z\right)}.$$

The main benefit of the spectrum slicing approach above is that roughly  $h(X|Y) + L + \Delta$  bits are sent for each realization (X,Y). At the same time, we can estimate h(X|Y) up to a precision of  $\Delta$  when the protocol stops – a key property in our secrecy analysis.

The complete protocol is described in Protocol 1. We remark that the random seed based secret key

generation used in Protocol 1 is only for the ease of security analysis. A slight modification of our protocol can work with deterministic extractors.

# **Protocol 1:** Secret key agreement protocol

**Input**: Observations X and Y

**Output**: Secret key estimates  $K_x$  and  $K_y$ 

## Information reconciliation

Initiate the protocol with l = 1

## while $l \leq L$ and ACK not received do

First party sends the random bin index of X into  $M_l$  bins,  $B_l = F_{1l}(X)$ , to the second party if Second party find a unique x such that  $(x,Y) \in \mathcal{T}_l$  and  $F_{1j}(x) = B_j$ ,  $\forall 1 \leq j \leq l$  then Second party sets  $\hat{X} = x$  and sends back an ACK  $F_{2i} = 1$  to the first party else

Second party sends back a NACK  $F_{2i} = 0$ 

Parties update  $l \rightarrow l + 1$ 

## if No ACK received then

Protocol declares an error and aborts

#### else

# Privacy amplification

First party generates the random seed S and sends it to the second party using public communication

First party generates the secret key  $K_x = K = f_S(X)$ 

The second party generates the estimate  $K_y$  of K as  $K_y = f_S(\hat{X})$ 

**Remark 3.** Note that since each ACK-NACK signal will require 1-bit of communication to implement, the number of bits physically sent in the protocol is roughly  $h(X|Y) + \Delta + L$ . However, the log of the number of values taken by the transcript is much less, roughly  $h(X|Y) + \Delta + \log L$ , since the ACK-NACK sequence will be a stopped sequence consisting of NACKs followed by a single ACK. By Lemma 4, it is this latter quantity  $h(X|Y) + \Delta + \log L$  that captures the amount of information leaked to the eavesdropper by public communication, which will be used in our security analysis of the protocol.

## B. Performance analysis of the protocol

We now derive performance guarantees for the secret key agreement protocol of the previous section. In view of Proposition 2 and Remark 2, the protocol above will constitute an  $(\epsilon, \delta)$ -SK protocol for arbitrary  $\epsilon, \delta \in (0, 1)$  if it yields an  $(\eta, \epsilon + \delta)$ -SK with  $\eta \approx 0$ . The result below shows that Protocol 1

indeed yields such a high reliability secret key.

Denote by  $i_{XY}(x,y)$  the information density

$$i_{XY}(x,y) := \log \frac{P_{XY}(x,y)}{P_X(x) P_Y(y)}.$$

**Theorem 5.** For  $\lambda_{\min}$ ,  $\lambda_{\max}$ ,  $\Delta > 0$  with  $\lambda_{\max} \geq \lambda_{\min}$ , let

$$L = \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{\Delta}.$$

Then, for every  $\gamma > 0$  and  $\lambda \geq 0$ , there exists an  $(\epsilon, \delta)$ -SK K taking values in K with

$$\begin{split} \epsilon & \leq \mathrm{P}_{XY}\left(\mathcal{T}_{0}\right) + L2^{-\gamma}, \\ \delta & \leq \mathrm{P}\left(i_{XY}\left(X,Y\right) - i_{XZ}\left(X,Z\right) \leq \lambda + \Delta\right) \\ & + \frac{1}{2}\sqrt{|\mathcal{K}|2^{-(\lambda - \gamma - 3\log L)}} + \frac{1}{L} + \mathrm{P}_{XY}\left(\mathcal{T}_{0}\right) + L2^{-\gamma}, \end{split}$$

where

$$\mathcal{T}_0 := \left\{ (x, y) : -\log P_{X|Y}(x|y) \ge \lambda_{\max} \text{ or } -\log P_{X|Y}(x|y) < \lambda_{\min} \right\}$$

*Proof.* We begin by analyzing the reliability of Protocol 1. Let  $\rho(X,Y)$  denote the number of rounds after which the protocol stops when the observations are (X,Y). An error occurs if  $(X,Y) \in \mathcal{T}_0$  or if there exists a  $\hat{x} \neq X$  such that  $(\hat{x},Y) \in \mathcal{T}_l$  and  $F_{1j}(X) = F_{1j}(\hat{x})$  for all j such that  $1 \leq j \leq l$ , for some  $l \leq \rho(X,Y)$ . Note that for each  $1 \leq j \leq L$ ,

$$|\{x:(x,y)\in\mathcal{T}_j\}|\leq \exp(\lambda_j+\Delta)\quad\forall\,y\in\mathcal{Y}.$$

Therefore, using a slight modification of the usual probability of error analysis for random binning, the probability of error for Protocol 1 is bounded above as

$$P_{e} \leq P_{XY}(\mathcal{T}_{0}) + \sum_{x,y} P_{XY}(x,y) \sum_{l=1}^{\rho(x,y)} \sum_{\hat{x} \neq x} P(F_{1j}(x) = F_{1j}(\hat{x}), \forall 1 \leq j \leq l) \mathbb{1}((\hat{x},y) \in \mathcal{T}_{l})$$

$$\leq P_{XY}(\mathcal{T}_{0}) + \sum_{x,y} P_{XY}(x,y) \sum_{l=1}^{\rho(x,y)} \sum_{\hat{x} \neq x} \frac{1}{M_{1}...M_{l}} \mathbb{1}((\hat{x},y) \in \mathcal{T}_{l})$$

$$\leq P_{XY}(\mathcal{T}_{0}) + \sum_{x,y} P_{XY}(x,y) \sum_{l=1}^{\rho(x,y)} 2^{-\lambda_{l}-\Delta-\gamma} |\{\hat{x}: (\hat{x},y) \in \mathcal{T}_{l}\}|$$

$$\leq P_{XY}(\mathcal{T}_{0}) + L 2^{-\gamma}, \tag{10}$$

where we have used the fact that  $\log M_1...M_l = \lambda_1 + l\Delta + \gamma = \lambda_l + \Delta + \gamma$ .

We now establish the secrecy of the protocol. Our proof entails establishing secrecy of the protocol conditioned on each realization J=j of an appropriately defined RV, which roughly corresponds to the slice index for (X,Y). Specifically, denote by  $\mathcal{E}_1$  the set of (x,y) for which an error occurs in information reconciliation and by  $\mathcal{E}_2$  the set

$$\mathcal{E}_2 := \left\{ (x, y, z) : i_{XY}(x, y) - i_{XZ}(x, z) \le \lambda + \Delta \right\},\,$$

which is the same as

$$\left\{ \left( x,y,z \right) : \log \frac{1}{\mathrm{P}_{X|Z}\left( x|z \right)} - \log \frac{1}{\mathrm{P}_{X|Y}\left( x|y \right)} \le \lambda + \Delta \right\}$$

Let RV J taking values in the set  $\{0, 1, ..., L\}$  be defined as follows:

$$J = \begin{cases} &0, \text{ if } (X,Y) \in \mathcal{T}_0 \cup \mathcal{E}_1 \text{ or } (X,Y,Z) \in \mathcal{E}_2, \\ &j \text{ if } (X,Y) \in \mathcal{T}_j \cap \mathcal{E}_1^c \text{ and } (X,Y,Z) \in \mathcal{E}_2^c, 1 \leq j \leq L. \end{cases}$$

While we have used random coding in the information reconciliation stage, it is only for the ease of proof and the encoder can be easily derandomized. For the remainder of the proof, we assume a deterministic encoder; in particular, J is a function of (X, Y, Z).

We divide the indices  $0 \le j \le L$  into good indices  $\mathcal{I}_g$  and the bad indices  $\mathcal{I}_b = \mathcal{I}_g^c$ , where

$$\mathcal{I}_{g}=\left\{ j:j>0\text{ and }\mathrm{P}_{J}\left( j\right) \geq\frac{1}{L^{2}}
ight\} .$$

Denoting by  $\mathbf{F}^l$  the communication up to l round of the protocol, i.e.,  $\mathbf{F}^l := \{(F_{1j}, F_{2j}), 1 \leq j \leq l\}$  and by  $\mathbf{F} = \mathbf{F}^{\rho(X,Y)}$  the overall communication, we have

$$\|P_{KFZS} - P_{unif}P_{FZS}\|_{1}$$

$$\leq \|P_{KFZSJ} - P_{unif}P_{FZSJ}\|_{1}$$

$$\leq P(J \in \mathcal{I}_{b}) + \sum_{j \in \mathcal{I}_{g}} P_{J}(j) \|P_{KFZS|J=j} - P_{unif}P_{FZS|J=j}\|_{1}$$

$$\leq P_{XY}(\mathcal{T}_{0} \cup \mathcal{E}_{1}) + P_{XYZ}(\mathcal{E}_{2}) + \frac{1}{L} + \sum_{j \in \mathcal{I}_{g}} P_{J}(j) \|P_{KFZS|J=j} - P_{unif}P_{FZS|J=j}\|_{1}.$$
(11)

<sup>&</sup>lt;sup>8</sup>Since the final error probability is given by the expected probability of error, where the expectation is over the source distribution and the additional shared randomness used in communication, there exists a realization of the shared randomness for which the same expected error perforance with respect to the source distribution is attained.

To bound each term  $\|\mathbf{P}_{K\mathbf{F}ZS|J=j} - \mathbf{P}_{\mathbf{unif}}\mathbf{P}_{\mathbf{F}ZS|J=j}\|_1$ ,  $j \in \mathcal{I}_g$ , first note that under each event  $J = j \in \mathcal{I}_g$  information reconciliation succeeds and  $\mathbf{F} = \mathbf{F}^j$ . Furthermore, the number of possible transcripts sent in the information reconciliation stage up to jth round, i.e., the cardinality  $\|\mathbf{F}^j\|$  of the range of RV  $\mathbf{F}^j$ , satisfies (cf. Remark 3)

$$\log \|\mathbf{F}^j\| \le \lambda_j + \Delta + \gamma + \log j.$$

Let  $P_j$  be the probability distribution of X, Y, Z given J = j, i.e.,

$$P_{j}\left(x,y,z\right):=\frac{P_{XYZ}\left(x,y,z\right)\mathbb{1}\left(J(x,y,z)=j\right)}{P_{J}\left(j\right)},\quad x\in\mathcal{X},y\in\mathcal{Y},z\in\mathcal{Z},0\leq j\leq L.$$

With  $P_{j,XZ}$  denoting the marginal on  $\mathcal{X} \times \mathcal{Z}$  induced by  $P_j$ , for all  $j \in \mathcal{I}_g$ , we have

$$\log \frac{P_{j,XZ}(x,z)}{P_{Z}(z)} = \log \frac{\sum_{y} P_{XYZ}(x,y,z) \mathbb{1} (J(x,y,z) = j)}{P_{J}(j) P_{Z}(z)}$$

$$\leq \log \frac{\sum_{y} 2^{-\lambda - \Delta} P_{X|Y}(x|y) P_{Y|XZ}(y|x,z) \mathbb{1} (J(x,y,z) = j)}{P_{J}(j)}$$

$$\leq \log \frac{\sum_{y} 2^{-\lambda_{i} - \lambda - \Delta} P_{Y|XZ}(y|x,z) \mathbb{1} (J(x,y,z) = j)}{P_{J}(j)}$$

$$\leq \log \frac{2^{-\lambda_{i} - \lambda - \Delta}}{P_{J}(j)}$$

$$\leq -\lambda_{i} - \lambda - \Delta + 2 \log L,$$

where the first inequality holds since J(x,y,z)>0 implies  $(x,y,z)\in\mathcal{E}_2^c$ , the second inequality holds since J(x,y,z)=j implies  $(x,y)\in\mathcal{T}_j$ , and the last inequality holds since  $j\in\mathcal{I}_g$  implies  $P_J(j)>\frac{1}{L^2}$ . Thus, we obtain the following bound on  $H_{\min}(P_{j,XZ}|P_Z)$ :

$$H_{\min}(P_{j,XZ}|P_Z) \ge \lambda_j + \lambda + \Delta - 2\log L.$$

Therefore, noting that S is independent of  $(X, Z, \mathbf{F}, J)$  and using Lemma 4, we get

$$\begin{split} \left\| \mathbf{P}_{K\mathbf{F}ZS|J=j} - \mathbf{P}_{\mathtt{unif}} \mathbf{P}_{\mathbf{F}ZS|J=j} \right\|_1 &= \left\| \mathbf{P}_{K\mathbf{F}^j ZS|J=j} - \mathbf{P}_{\mathtt{unif}} \mathbf{P}_{\mathbf{F}^j ZS|J=j} \right\|_1 \\ &\leq \frac{1}{2} \sqrt{|\mathcal{K}| \|\mathbf{F}^j \| 2^{-H_{\min}(\mathbf{P}_{j,XZ}|\mathbf{P}_Z)}} \\ &\leq \frac{1}{2} \sqrt{|\mathcal{K}| 2^{-(\lambda-\gamma-3\log L)}}, \quad j \in \mathcal{I}_g. \end{split}$$

which gives the claimed secrecy by using the definition of  $\mathcal{E}_2$  and bounding  $P_{XY}(\mathcal{T}_0 \cup \mathcal{E}_1)$  using the union bound as in (10).

Thus, when the secret key length  $\log |\mathcal{K}| \approx \lambda$ , the reliability parameter  $\epsilon$  for Protocol 1 can be made very small by appropriately choosing parameters  $\lambda_{\min}$  and  $\lambda_{\max}$ , and the secrecy parameter  $\delta$  can be made roughly as small as the tail-probability  $P\left(i_{XY}\left(X,Y\right)-i_{XZ}\left(X,Z\right)\leq\lambda\right)$ . In fact, Proposition 2 allows us to shift this constraint on  $\delta$  to a constraint on  $\epsilon+\delta$  and Protocol 1 yields an  $(\epsilon,\delta)$ -SK of length roughly equal to  $\lambda$  as long as  $\epsilon+\delta$  is greater than  $P\left(i_{XY}\left(X,Y\right)-i_{XZ}\left(X,Z\right)\leq\lambda\right)$ . Formally, we have the following simple corollary of Theorem 5.

**Corollary 6.** For  $\lambda_{\max}, \lambda_{\min}, \Delta > 0$  with  $\lambda_{\max} \geq \lambda_{\min}$ , let

$$L = \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{\Delta},$$

and let

$$\mathcal{T}_{0} := \left\{ (x, y) : -\log \mathcal{P}_{X|Y} \left( x | y \right) \ge \lambda_{\max} \ or \ -\log \mathcal{P}_{X|Y} \left( x | y \right) < \lambda_{\min} \right\}.$$

Then, for every  $\lambda \geq 0$  and every  $\epsilon$  and  $\delta$  satisfying

$$\epsilon \ge P_{XY}(\mathcal{T}_0) + \frac{1}{4} \left( L^4 |\mathcal{K}| 2^{-\lambda} \right)^{\frac{1}{3}},\tag{12}$$

$$\epsilon + \delta \ge P\left(i_{XY}\left(X,Y\right) - i_{XZ}\left(X,Z\right) \le \lambda + \Delta\right) + \frac{1}{L} + 2P_{XY}\left(\mathcal{T}_{0}\right) + \frac{3}{2}\left(L^{4}|\mathcal{K}|2^{-\lambda}\right)^{\frac{1}{3}},\tag{13}$$

there exists an  $(\epsilon, \delta)$ -SK K taking values in K.

Proof: Let

$$\eta(\gamma) := \mathrm{P}_{XY}\left(\mathcal{T}_{0}\right) + L2^{-\gamma},$$

$$\alpha(\gamma) := \mathrm{P}\left(i_{XY}\left(X,Y\right) - i_{XZ}\left(X,Z\right) \leq \lambda + \Delta\right) + \frac{1}{2}\sqrt{|\mathcal{K}|2^{-(\lambda - \gamma - 3\log L)}} + \frac{1}{L} + \mathrm{P}_{XY}\left(\mathcal{T}_{0}\right) + L2^{-\gamma}.$$

We first optimize

$$\eta(\gamma) + \alpha(\gamma)$$

$$= P\left(i_{XY}\left(X,Y\right) - i_{XZ}\left(X,Z\right) \le \lambda + \Delta\right) + \frac{1}{2}\sqrt{|\mathcal{K}|2^{-(\lambda - \gamma - 3\log L)}} + \frac{1}{L} + 2P_{XY}\left(\mathcal{T}_{0}\right) + 2L2^{-\gamma}$$

over  $\gamma$ . By setting  $a=L2^{-\gamma}$  and by noting that the function  $f(a)=2a+\frac{a^{-1/2}L^2}{2}\sqrt{|\mathcal{K}|2^{-\lambda}}$  has minimum

value  $\frac{3}{2} \left( L^4 |\mathcal{K}| 2^{-\lambda} \right)^{\frac{1}{3}}$  with  $a = \frac{1}{4} \left( L^4 |\mathcal{K}| 2^{-\lambda} \right)^{\frac{1}{3}}$ , the minimum of  $\eta(\gamma) + \alpha(\gamma)$  is achieved when

$$\eta = \eta^* := P_{XY}(\mathcal{T}_0) + \frac{1}{4} \left( L^4 |\mathcal{K}| 2^{-\lambda} \right)^{\frac{1}{3}}, 
\alpha = \alpha^* := P\left( i_{XY}(X, Y) - i_{XZ}(X, Z) \le \lambda + \Delta \right) + \frac{1}{L} + P_{XY}(\mathcal{T}_0) + \frac{5}{4} \left( L^4 |\mathcal{K}| 2^{-\lambda} \right)^{\frac{1}{3}}.$$

The corollary follows by applying Proposition 2 with  $\eta = \eta^*$  and  $\alpha^*$  to the resulting  $(\eta^*, \alpha^*)$ -SK.

#### V. SECRET KEY CAPACITY FOR GENERAL SOURCES

In this section we will establish the secret key capacity for a sequence of general sources  $(X_n, Y_n, Z_n)$  with joint distribution  $P_{X_n Y_n Z_n}$ . The secret key capacity for general sources is defined as follows [19], [1], [4].

**Definition 3.** The secret key capacity C is defined as

$$C := \sup_{\epsilon_n, \delta_n} \liminf_{n \to \infty} \frac{1}{n} S_{\epsilon_n, \delta_n} (X_n, Y_n \mid Z_n),$$

where the sup is over all  $\epsilon_n, \delta_n \geq 0$  such that

$$\lim_{n\to\infty} \epsilon_n + \delta_n = 0.$$

To state our result, we need the following concepts from the *information spectrum* method; see [10] for a detailed account. For RVs  $(X_n, Y_n, Z_n)_{n=1}^{\infty}$ , the *inf-conditional entropy rate*  $\underline{H}(\mathbf{X} \mid \mathbf{Y})$  and the *sup-conditional entropy rate*  $\overline{H}(\mathbf{X} \mid \mathbf{Y})$  are defined as follows:

$$\begin{split} &\underline{H}(\mathbf{X}\mid\mathbf{Y}) = \sup\left\{\alpha\mid\lim_{n\to\infty}\mathrm{P}\left(-\frac{1}{n}\log\mathrm{P}_{X_n\mid Y_n}\left(X_n\mid Y_n\right) < \alpha\right) = 0\right\},\\ &\overline{H}(\mathbf{X}\mid\mathbf{Y}) = \inf\left\{\alpha\mid\lim_{n\to\infty}\mathrm{P}\left(-\frac{1}{n}\log\mathrm{P}_{X_n\mid Y_n}\left(X_n\mid Y_n\right) > \alpha\right) = 0\right\}. \end{split}$$

Similarly, the *inf-conditional information rate*  $\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z})$  is defined as

$$\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}) = \sup \left\{ \alpha \mid \lim_{n \to \infty} P\left(\frac{1}{n} i(X_n, Y_n \mid Z_n) < \alpha\right) = 0 \right\},$$

where, with a slight abuse of notation,  $i(X_n, Y_n \mid Z_n)$  denotes the conditional information density

$$i(X_n, Y_n \mid Z_n) = \log \frac{P_{X_n Y_n \mid Z_n} (X_n, Y_n \mid Z_n)}{P_{X_n \mid Z_n} (X_n \mid Z_n) P_{Y_n \mid Z_n} (Y_n \mid Z_n)}.$$

<sup>&</sup>lt;sup>9</sup>The distributions  $P_{X_nY_nZ_n}$  need not satisfy the consistency conditions.

We also need the following result credited to Verdú.

# **Lemma 7.** [10, Theorem 4.1.1] For every $\epsilon_n$ such that

$$\lim_{n\to\infty} \epsilon_n = 0,$$

it holds that

$$\liminf_{n} -\frac{1}{n} \log \beta_{\epsilon_{n}} \left( P_{X_{n}Y_{n}Z_{n}}, P_{X_{n}\mid Z_{n}} P_{Y_{n}\mid Z_{n}} P_{Z_{n}} \right) \leq \underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}),$$

where  $\beta_{\epsilon_n}$  is defined in (8).

Our result below characterizes the secret key capacity C for general sources for the special case when  $X_n - Y_n - Z_n$  is a Markov chain.

**Theorem 8.** For a sequence of sources  $\{X_n, Y_n, Z_n\}_{n=1}^{\infty}$  such that  $X_n \oplus Y_n \oplus Z_n$  form a Markov chain for all n, the secret key capacity C is given by  $S_n \oplus S_n \oplus S_n$ 

$$C = \underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}).$$

*Proof.* Applying Theorem 3 with  $\eta = \eta_n = n^{-1}$ , along with Lemma 7, gives

$$C \leq I(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}).$$

For the other direction, we construct a sequence of  $(\epsilon_n, \delta_n)$ -SKs  $K = K_n$  with  $\epsilon_n, \delta_n \to 0$  and rate approximately  $\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z})$ . Indeed, in Theorem 5 choose

$$\lambda_{\max} = n \left( \overline{H}(\mathbf{X} \mid \mathbf{Y}) + \Delta \right),$$

$$\lambda_{\min} = n \left( \underline{H}(\mathbf{X} \mid \mathbf{Y}) - \Delta \right),$$

$$\gamma = \gamma_n = n\Delta/2,$$

$$\lambda = \lambda_n = n \left( I\left(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}\right) - \Delta \right);$$

thus,

$$L = L_n = \frac{n\left(\overline{H}(\mathbf{X} \mid \mathbf{Y}) - \underline{H}(\mathbf{X} \mid \mathbf{Y}) + 2\Delta\right)}{\Delta}.$$

Since  $i_{XY}(X,Y) - i_{XZ}(X,Z) = i(X,Y|Z)$  if  $X \multimap Y \multimap Z$  form a Markov chain, there exists an

<sup>&</sup>lt;sup>10</sup>We assume that  $\overline{H}(\mathbf{X} \mid \mathbf{Y}) < \infty$ .

 $(\epsilon_n, \delta_n)$ -SK  $K_n$  of rate given by

$$\frac{1}{n}\log|\mathcal{K}| = \frac{1}{n}(\lambda_n - 3\log L_n) - \Delta$$
$$= \underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}) - 2\Delta - o(n),$$

such that  $\epsilon_n, \delta_n \to 0$  as  $n \to \infty$ . Rates arbitrarily close to  $\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z})$  are achieved by this scheme as  $\Delta > 0$  is arbitrary.

In the achievability part of the proof above, we actually show that, in general, our protocol generates a secret key of rate

$$\sup \left\{ \alpha \mid \lim_{n \to \infty} P\left(\frac{1}{n}[i(X_n, Y_n) - i(X_n, Z_n)] < \alpha \right) = 0 \right\},\,$$

which matches the converse bound of  $\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z})$  in the special case when  $X_n - Y_n - Z_n$  holds.

#### VI. SECOND-ORDER ASYMPTOTICS OF SECRET KEY RATES

The results of the previous section show that for with  $\epsilon_n, \delta_n \to 0$ , the largest length  $S_{\epsilon_n, \delta_n}(X_n, Y_n \mid Z_n)$  of an  $(\epsilon_n, \delta_n)$ -SK K is

$$\sup_{\epsilon_n, \delta_n} S_{\epsilon_n, \delta_n}(X_n, Y_n \mid Z_n) = n\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}) + o(n), \tag{14}$$

if  $X_n Y_n Z_n$  form a Markov chain. For the case when  $(X_n, Y_n, Z_n) = (X^n, Y^n, Z^n)$  is the *n*-IID repetition of (X, Y, Z) where X Z, we have

$$\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}) = I(X \wedge Y \mid Z).$$

Furthermore, (14) holds even without  $\epsilon_n, \delta_n \to 0$ . In fact, a finer asymptotic analysis is possible and the second-order asymptotic term in the maximum length of an  $(\epsilon, \delta)$ -SK can be established; this is the subject-matter of the current section.

Let

$$V := \mathbb{V}\mathrm{ar}\left[i(X, Y \mid Z)\right],$$

and let

$$Q(a) := \int_{a}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right] dt$$

be the tail probability of the standard Gaussian distribution. Under the assumptions

$$V_{X|Y} := \mathbb{V}\operatorname{ar}\left[-\log P_{X|Y}\left(X|Y\right)\right] < \infty, \tag{15}$$

$$T := \mathbb{E}\left[\left|i(X, Y \mid Y) - I(X \land Y \mid Z)\right|^{3}\right] < \infty, \tag{16}$$

the result below establishes the second-order asymptotic term in  $S_{\epsilon,\delta}(X^n,Y^n\mid Z^n)$ .

**Theorem 9.** For every  $\epsilon, \delta > 0$  such that  $\epsilon + \delta < 1$  and IID RVs  $(X^n, Y^n, Z^n)$  such that  $X - \Phi Y - \Phi Z$  is a Markov chain, we have

$$S_{\epsilon,\delta}(X^n, Y^n \mid Z^n) = nI(X \wedge Y \mid Z) - \sqrt{nV}Q^{-1}(\epsilon + \delta) \pm \mathcal{O}(\log n),$$

*Proof.* For the converse part, we proceed along the lines of [23, Lemma 58]. Recall the following simple bound for  $\beta_{\epsilon}(P, Q)$  (cf. [10, Lemma 4.1.2]):

$$-\log \beta_{\epsilon}(P, Q) \le \lambda - \log \left( P\left( \left\{ x : \log \frac{P(x)}{Q(x)} \le \lambda \right\} \right) - \epsilon \right).$$

Thus, applying Theorem 3 with  $P_{XYZ} = P_{X^nY^nZ^n}$ ,  $Q_{XYZ} = P_{X^n|Z^n}P_{Y^nZ^n}$ , and  $\eta = \eta_n = n^{-1/2}$ , and choosing

$$\lambda = nI(X \wedge Y \mid Z) - \sqrt{nV}Q^{-1} \left(\epsilon + \delta + \theta_n\right),\,$$

where

$$\theta_n = \frac{2}{\sqrt{n}} + \frac{T^3}{2V^{3/2}\sqrt{n}},$$

we get by the Berry-Esséen theorem (cf. [7], [26]) that

$$P(i(X^n, Y^n) - i(X^n, Z^n) \le \lambda) \ge \epsilon + \delta + \frac{2}{\sqrt{n}},$$

which implies

$$S_{\epsilon,\delta}(X^n, Y^n \mid Z^n) \le \lambda - \log\left(P\left(i(X^n, Y^n) - i(X^n, Z^n) \le \lambda\right) - \epsilon - \delta - n^{-1/2}\right) + \log n$$

$$\le nI(X \land Y \mid Z) - \sqrt{nV}Q^{-1}\left(\epsilon + \delta + \theta_n\right) + \frac{3}{2}\log n. \tag{17}$$

Thus, we have the desired converse by using Taylor approximation of  $Q(\cdot)$  to remove  $\theta_n$ .

For the direct part, we use Corollary 6 by setting

$$\lambda_{\max} = n(H(X|Y) + \Delta/2),$$
 
$$\lambda_{\min} = n(H(X|Y) - \Delta/2),$$
 
$$\lambda = nI(X \wedge Y \mid Z) - \sqrt{nV}Q^{-1}(\epsilon + \delta - \theta'_n) - \Delta,$$

and

$$\log |\mathcal{K}| = nI(X \wedge Y \mid Z) - \sqrt{nV}Q^{-1}(\epsilon + \delta - \theta_n') - \frac{11}{2}\log n - \Delta, \tag{18}$$

where  $0 < \Delta < 2H(X \mid Y)$ , and

$$\theta'_n = \frac{8V_{X|Y}}{n\Delta^2} + \frac{T^3}{2V^{3/2}\sqrt{n}} + \frac{1}{n} + \frac{3}{2\sqrt{n}}.$$

Note that L=n. Upon bounding the term  $P_{XY}(\mathcal{T}_0)$  in (12) and (13) by  $\frac{4V_{X|Y}}{n\Delta^2}$  using Chebyshev's inequality, the condition (12) is satisfied for sufficiently large n. Furthermore, upon bounding the first term of (13) by the Berry-Esséen theorem, the condition (13) is also satisfied. Thus, it follows that there exists an  $(\epsilon, \delta)$ -SK K taking values on K. The direct part follows by using Taylor approximation of  $Q(\cdot)$  to remove  $\theta'_n$ .

**Remark 4.** Note that a standard noninteractive secret key agreement protocol based on information reconciliation and privacy amplification (cf. [25]) only gives the following suboptimal achievability bound on the second-order asymptotic term:

$$S_{\epsilon,\delta}\left(X^{n},Y^{n}\mid Z^{n}\right) \geq nI(X\wedge Y\mid Z) - \sqrt{nV_{X\mid Y}}Q^{-1}(\epsilon) - \sqrt{nV_{X\mid Z}}Q^{-1}(\delta) + o(\sqrt{n}),$$

where  $V_{X|Y}$  and  $V_{X|Z}$  are the variances of the conditional log-likelihoods of X given Y and Z respectively (cf. (15)).

We close this section with a numerical example that illustrates the utility of our bounds in characterizing the gap to secret key capacity at a fixed n.

**Example 1** (Gap to secret key capacity). For  $\alpha_0, \alpha_1 \in (0, 1/2)$ , let  $B_0$  and  $B_1$  be independent random bits taking value 1 with probability  $\alpha_0$  and  $\alpha_1$ , respectively. Consider binary X, Y, Z where Z is a uniform random bit independent jointly of  $B_0$  and  $B_1, Y = Z \oplus B_0$ , and  $X = Y \oplus B_1$ . We consider the rate  $S_{\epsilon,\delta}(X^n,Y^n \mid Z^n)/n$  of an  $(\epsilon,\delta)$ -SK that can be generated using n IID copies of X and Y when

the eavesdropper observes  $\mathbb{Z}^n$ . The following quantities, needed to evaluate (17) and (18), can be easily evaluated:

$$I(X \wedge Y \mid Z) = h(\alpha_0 * \alpha_1) - h(\alpha_1), \quad V = \mu_2, \quad T = \mu_3,$$
 
$$V_{X|Y} = \alpha_1 (\log \alpha_1 - h(\alpha_1))^2 + (1 - \alpha_1)(\log(1 - \alpha_1) - h(\alpha_1))^2,$$

where  $\mu_r$  is the rth central moment of  $i(X,Y\mid Z)$  and is given by

$$\mu_{r} = \alpha_{0}\alpha_{1} \left| \log \frac{\alpha_{1}}{1 - \alpha_{0} * \alpha_{1}} - I(X \wedge Y \mid Z) \right|^{r} + (1 - \alpha_{0})(1 - \alpha_{1}) \left| \log \frac{1 - \alpha_{1}}{1 - \alpha_{0} * \alpha_{1}} - I(X \wedge Y \mid Z) \right|^{r} + (1 - \alpha_{0})\alpha_{1} \left| \log \frac{\alpha_{1}}{\alpha_{0} * \alpha_{1}} - I(X \wedge Y \mid Z) \right|^{r} + \alpha_{0}(1 - \alpha_{1}) \left| \log \frac{1 - \alpha_{1}}{\alpha_{0} * \alpha_{1}} - I(X \wedge Y \mid Z) \right|^{r},$$

 $h(x) = -x \log x - (1-x) \log (1-x)$  is the binary entropy function and  $\alpha_0 * \alpha_1 = \alpha_0 (1-\alpha_1) + (1-\alpha_0) \alpha_1$ . In Figure 1 (given in Section I), we plot the upper bound on  $S_{\epsilon,\delta}(X^n,Y^n\mid Z^n)/n$  resulting from (17) and the lower bound resulting from (17) with  $\Delta=1$  for  $\alpha_0=0.25$  and  $\alpha_1=0.125$ .

#### VII. DISCUSSION: IS INTERACTION NECESSARY?

In contrast to the protocols in [19], [1], [4], [25], our proposed Protocol 1 for secret key agreement is interactive. In fact, the protocol requires as many rounds of interaction as the number of slices L, which can be pretty large in general. For instance, to obtain the second-order asymptotic term in the previous section, we chose L = n. In Appendix C, we present an alternative protocol which requires only 1-bit of feedback and, in the special case when Z is constant, achieves the asymptotic results of Sections V and VI. But is interaction necessary for attaining our asymptotic results? Below we present an example where none of the known (noninteractive) secret key agreement protocols achieves the general capacity of Theorem 8, suggesting that perhaps interaction is necessary.

For 
$$i=1,2$$
, let  $(X_i^n,Y_i^n,Z_i^n)$  be IID with  $\mathcal{X}=\mathcal{Y}=\mathcal{Z}=\{0,1\}$  such that

$$P_{X_i^n Y_i^n Z_i^n}(x^n, y^n, z^n) = \frac{1}{2^n} W_i^n(y^n | x^n) V^n(z^n | y^n),$$

where  $W_i$  and V, respectively, are binary symmetric channels with crossover probabilities  $p_i$  and q. Let  $(X_n, Y_n, Z_n)$  be the mixed source given by

$$P_{X_n Y_n Z_n}(x^n, y^n, z^n) = \frac{1}{2} P_{X_1^n Y_1^n Z_1^n}(x^n, y^n, z^n) + \frac{1}{2} P_{X_2^n Y_2^n Z_2^n}(x^n, y^n, z^n)$$
$$= \frac{1}{2^n} \left[ \frac{1}{2} W_1^n(y^n | x^n) + \frac{1}{2} W_2^n(y^n | x^n) \right] V^n(z^n | y^n).$$

Note that  $X_n - Y_n - Z_n$  forms a Markov chain. Suppose that  $0 < p_1 < p_2 < \frac{1}{2}$ . Then, we have

$$\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}) = \min[H(X_1|Z_1) - H(X_1|Y_1), H(X_2|Z_2) - H(X_2|Y_2)] 
= \min[h(p_1 * q) - h(p_1), h(p_2 * q) - h(p_2)] 
= h(p_2 * q) - h(p_2),$$

where  $h(\cdot)$  is the binary entropy function and \* is binary convolution. Using a standard noninteractive secret key agreement protocol based on information reconciliation and privacy amplification (cf. [25]), we can achieve only

$$\underline{H}(\mathbf{X} \mid \mathbf{Z}) - \overline{H}(\mathbf{X} \mid \mathbf{Y}) 
= \min[H(X_1|Z_1), H(X_2|Z_2)] - \max[H(X_1|Y_1), H(X_2|Y_2)] 
= H(X_1|Z_1) - H(X_2|Y_2) 
= h(p_1 * q) - h(p_2),$$

which is less than the general secret key capacity of Theorem 8. Proving a precise limitation result for noninteractive protocols is a direction for future research.

# **APPENDIX**

## A. Proof of Theorem 3

The definition of a secret key used in [33], [32] is different from the one in Definition 1, and it conveniently combines the secrecy and the reliability requirements into a single expression. Instead of considering a separate RV K, the alternative definition directly works with the estimates  $K_x$  and  $K_y$ . Specifically, let  $K_x$  and  $K_y$  be functions of  $(U_x, X, \mathbf{F})$  and  $(U_y, Y, \mathbf{F})$ , respectively, where  $\mathbf{F}$  is an interactive communication. Then, RVs  $K_x$  and  $K_y$  with a common range K constitute an  $\epsilon$ -secret key  $(\epsilon$ -SK) if

$$\left\| P_{K_x K_y \mathbf{F} Z} - P_{\mathbf{unif}}^{(2)} \times P_{\mathbf{F} Z} \right\|_{1} \le \epsilon, \tag{19}$$

where, for a pmf P on  $\mathcal{X}$ ,  $P^{(m)}$  denotes its extension to  $\mathcal{X}^m$  given by

$$P^{(m)}(x_1,...,x_m) = P(x) \mathbb{1}(x_1 = ... = x_m), \quad (x_1,...,x_m) \in \mathcal{X}^m.$$

Note that the alternative definition captures reliability condition  $P(K_x = K_y) \ge 1 - \epsilon$  by requiring that the joint distribution  $P_{K_xK_y}$  is close to a uniform distribution on the diagonal of  $\mathcal{K} \times \mathcal{K}$ . The upper bound in [33], [32] holds under this alternative definition of a secret key. However, the next lemma says that this alternative definition is closely related to our Definition 1.

**Lemma 10.** Given  $\epsilon, \delta \in [0, 1)$  and an  $(\epsilon, \delta)$ -SK K, the local estimates  $K_x$  and  $K_y$  satisfy (19) with  $\epsilon + \delta$ , i.e.,

$$\left\| P_{K_x K_y \mathbf{F} Z} - P_{\mathtt{unif}}^{(2)} \times P_{\mathbf{F} Z} \right\|_1 \le \epsilon + \delta.$$

Conversely, if  $K_x$  and  $K_y$  satisfy (19), either  $K_x$  or  $K_y$  constitutes an  $(\epsilon, \epsilon)$ -SK.

*Proof.* We prove the direct part first. For an  $(\epsilon, \delta)$ -SK K,

$$\begin{split} & \left\| \mathbf{P}_{K_x K_y \mathbf{F}Z} - \mathbf{P}_{\mathtt{unif}}^{(2)} \times \mathbf{P}_{\mathbf{F}Z} \right\|_1 \\ & \leq \left\| \mathbf{P}_{K K_x K_y \mathbf{F}Z} - \mathbf{P}_{\mathtt{unif}}^{(3)} \times \mathbf{P}_{\mathbf{F}Z} \right\|_1 \\ & \leq \left\| \mathbf{P}_{K K_x K_y \mathbf{F}Z} - \mathbf{P}_{K | \mathbf{F}Z}^{(3)} \times \mathbf{P}_{\mathbf{F}Z} \right\|_1 + \left\| \mathbf{P}_{K | \mathbf{F}Z}^{(3)} \times \mathbf{P}_{\mathbf{F}Z} - \mathbf{P}_{\mathtt{unif}}^{(3)} \times \mathbf{P}_{\mathbf{F}Z} \right\|_1. \end{split}$$

Since

$$\left\|\mathbf{P} - \mathbf{Q}\right\|_1 = \mathbf{Q}(\left\{x : \mathbf{Q}(x) \geq \mathbf{P}(x)\right\}) - \mathbf{P}(\left\{x : \mathbf{Q}(x) \geq \mathbf{P}(x)\right\})$$

and

$$\{(k, k_x, k_y, f, z) : P_{K|FZ}^{(3)}(k, k_x, k_y|f, z) \ge P_{KK_xK_y|FZ}(k, k_x, k_y|f, z)\}$$

$$= \{(k, k_x, k_y, \mathbf{f}, z) : k = k_x = k_y\},$$

the first term on the right-side above satisfies

$$\left\| P_{KK_xK_yFZ} - P_{K|FZ}^{(3)} \times P_{FZ} \right\|_1 = 1 - P(K = K_x = K_y) \le \epsilon.$$
 (20)

Furthermore, the second term satisfies

$$\begin{split} & \left\| \mathbf{P}_{K|\mathbf{F}Z}^{(3)} \times \mathbf{P}_{\mathbf{F}Z} - \mathbf{P}_{\mathbf{unif}}^{(3)} \times \mathbf{P}_{\mathbf{F}Z} \right\|_{1} \\ &= \sum_{k,k_{x},k_{y},f,z} \mathbf{P}_{\mathbf{F}Z}\left(f,z\right) \left| \mathbf{P}_{K|\mathbf{F}Z}\left(k|f,z\right) \mathbb{1}(k=k_{x}=k_{y}) - \mathbb{1}(k=k_{x}=k_{y}) \frac{1}{|\mathcal{K}|} \right| \\ &= \left\| \mathbf{P}_{K|\mathbf{F}Z} \times \mathbf{P}_{\mathbf{F}Z} - \mathbf{P}_{\mathbf{unif}} \times \mathbf{P}_{\mathbf{F}Z} \right\|_{1} \\ &\leq \delta, \end{split}$$

where the last inequality is by the  $\delta$ -secrecy condition K. Combining the bounds on the two terms above, the direct part follows.

For the converse,  $\epsilon$ -secrecy of  $K_x$  (or  $K_y$ ) holds since by the monotonicity of the variational distance

$$\left\|\mathbf{P}_{K_x\mathbf{F}Z} - \mathbf{P}_{\mathtt{unif}} \times \mathbf{P}_{\mathbf{F}Z}\right\|_1 \le \left\|\mathbf{P}_{K_xK_y\mathbf{F}Z} - \mathbf{P}_{\mathtt{unif}}^{(2)} \times \mathbf{P}_{\mathbf{F}Z}\right\|_1 \le \epsilon.$$

The  $\epsilon$ -reliability condition, too, follows from the triangle inequality upon observing that

$$\begin{aligned} \left\| \mathbf{P}_{K_x K_y} - \mathbf{P}_{\mathtt{unif}}^{(2)} \right\|_1 &= \sum_{k_x, k_y} \left| \mathbf{P}_{K_x K_y} \left( k_x, k_y \right) - \mathbb{1} (k_x = k_y) \frac{1}{|\mathcal{K}|} \right| \\ &\geq \sum_{k_x \neq k_y} \mathbf{P}_{K_x K_y} \left( k_x, k_y \right) \\ &= \mathbf{P} \left( K_x \neq K_y \right). \end{aligned}$$

To prove Theorem 3, we first relate the length of a secret key satisfying (19) to the exponent of the probability of error of type II in a binary hypothesis testing problem where an observer of  $(K_x, K_y, \mathbf{F}, Z)$  seeks to find out if the underlying distribution was  $P_{XYZ}$  of  $Q_{XYZ} = Q_{X|Z}Q_{Y|Z}Q_{Z}$ . This result is stated next.

**Lemma 11.** For an  $\epsilon$ -SK  $(K_x, K_y)$  satisfying (19) generated by an interactive communication  $\mathbf{F}$ , let  $W_{K_xK_y\mathbf{F}|XYZ}$  denote the resulting conditional distribution on  $(K_x, K_y, \mathbf{F})$  given (X, Y, Z). Then, for every  $0 < \eta < 1 - \epsilon$  and every  $Q_{XYZ} = Q_{X|Z}Q_{Y|Z}Q_{Z}$ , we have

$$\log |\mathcal{K}| \le -\log \beta_{\epsilon+\eta}(P_{K_x K_y \mathbf{F} Z}, Q_{K_x K_y \mathbf{F} Z}) + 2\log(1/\eta), \tag{21}$$

where  $Q_{K_xK_y\mathbf{F}Z}$  is the marginal of  $(K_x, K_y, \mathbf{F}, Z)$  of the joint distribution

$$Q_{K_x K_y \mathbf{F} X Y Z} = Q_{XYZ} W_{K_x K_y \mathbf{F} | XYZ}.$$

To prove Lemma 11, we need the following basic property of interactive communication (cf. [31]).

**Lemma 12** (Interactive communication property). Given  $Q_{XYZ} = Q_{X|Z}Q_{Y|Z}Q_{Z}$  and an interactive communication  $\mathbf{F}$ , the following holds:

$$Q_{XY|\mathbf{F}Z} = Q_{X|\mathbf{F}Z} \times Q_{Y|\mathbf{F}Z},$$

i.e., conditionally independent observations remain so when conditioned additionally on an interactive communication.

*Proof of Lemma 11:* We establish (21) by constructing a test for the hypothesis testing problem with null hypothesis  $P = P_{K_x K_y FZ}$  and alternative hypothesis  $Q = Q_{K_x K_y FZ}$ . Specifically, we use a deterministic test with the following acceptance region (for the null hypothesis)<sup>11</sup>:

$$\mathcal{A} := \left\{ (k_x, k_y, f, z) : \log \frac{\mathrm{P_{unif}^{(2)}}(k_x, k_y)}{\mathrm{Q}_{K_x K_y \mid \mathbf{F}Z}\left(k_x, k_y \mid f, z\right)} \ge \lambda \right\},\,$$

where

$$\lambda = \log |\mathcal{K}| - 2\log(1/\eta).$$

For this test, the probability of type II is bounded above as

$$Q_{K_{x}K_{y}\mathbf{F}Z}(\mathcal{A}) = \sum_{f,z} Q_{\mathbf{F}Z}(f,z) \sum_{\substack{k_{x},k_{y}:\\(k_{x},k_{y},f,z)\in\mathcal{A}}} Q_{K_{x}K_{y}|\mathbf{F}Z}(k_{x},k_{y}|f,z)$$

$$\leq 2^{-\lambda} \sum_{f,z} Q_{\mathbf{F}Z}(f,z) \sum_{k_{x},k_{y}} P_{\mathbf{unif}}^{(2)}(k_{x},k_{y})$$

$$= \frac{1}{|\mathcal{K}|\eta^{2}}.$$
(22)

<sup>11</sup>The values  $(k_x, k_y, f, z)$  with  $Q_{K_x K_y \mid \mathbf{F}Z}(k_x, k_y \mid f, z) = 0$  are included in  $\mathcal{A}$ .

On the other hand, the probability of error of type I is bounded above as

$$P_{K_x K_y \mathbf{F}Z} (\mathcal{A}^c) \le \left\| P_{K_x K_y \mathbf{F}Z} - P_{\mathbf{unif}}^{(2)} \times P_{\mathbf{F}Z} \right\|_1 + P_{\mathbf{unif}}^{(2)} \times P_{\mathbf{F}Z} (\mathcal{A}^c)$$

$$\le \epsilon + P_{\mathbf{unif}}^{(2)} \times P_{\mathbf{F}Z} (\mathcal{A}^c), \tag{23}$$

where the first inequality follows from the definition of variational distance, and the second is a consequence of the security criterion (19). The second term above can be expressed as follows:

$$P_{\text{unif}}^{(2)} \times P_{\mathbf{F}Z}(\mathcal{A}^{c}) = \sum_{f,z} P_{\mathbf{F}Z}(f,z) \frac{1}{|\mathcal{K}|} \sum_{k} \mathbb{1}\left((k,k,f,z) \in \mathcal{A}^{c}\right)$$

$$= \sum_{f,z} P_{\mathbf{F}Z}(f,z) \frac{1}{|\mathcal{K}|} \sum_{k} \mathbb{1}\left(Q_{K_{x}K_{y}|\mathbf{F}Z}(k,k,|f,z)|\mathcal{K}|^{2}\eta^{2} > 1\right). \tag{24}$$

The inner sum can be further upper bounded as

$$\sum_{k} \mathbb{1} \left( Q_{K_{x}K_{y}|\mathbf{F}Z}(k,k,|f,z) |\mathcal{K}|^{2} \eta^{2} > 1 \right) \leq \sum_{k} \left( Q_{K_{x}K_{y}|\mathbf{F}Z}(k,k,|f,z) |\mathcal{K}|^{2} \eta^{2} \right)^{\frac{1}{2}}$$

$$= |\mathcal{K}| \eta \sum_{k} Q_{K_{x}K_{y}|\mathbf{F}Z}(k,k,|f,z)^{\frac{1}{2}}$$

$$= |\mathcal{K}| \eta \sum_{k} Q_{K_{x}|\mathbf{F}Z}(k|f,z)^{\frac{1}{2}} Q_{K_{y}|\mathbf{F}Z}(k|f,z)^{\frac{1}{2}}, \qquad (25)$$

where the previous equality uses Lemma 12 and the fact that given  $\mathbf{F}$ ,  $K_x$  and  $K_y$  are functions of  $(X, U_x)$  and  $(Y, U_y)$ , respectively. Next, an application of the Cauchy-Schwartz inequality to the sum on the right-side of (25) yields

$$\sum_{k} Q_{K_{x}|\mathbf{F}Z} (k|f,z)^{\frac{1}{2}} Q_{K_{y}|\mathbf{F}Z} (k|f,z)^{\frac{1}{2}} \leq \left( \sum_{k_{x}} Q_{K_{x}|\mathbf{F}Z} (k_{x}|f,z) \right)^{\frac{1}{2}} \left( \sum_{k_{y}} Q_{K_{y}|\mathbf{F}Z} (k_{y}|f,z) \right)^{\frac{1}{2}}$$

$$= 1.$$
(26)

Upon combining (24)-(26), we obtain

$$P_{\mathtt{unif}}^{(2)} \times P_{\mathbf{F}Z}(\mathcal{A}^c) \leq \eta,$$

which along with (23) gives

$$P_{K_{\tau}K_{\eta}FZ}(\mathcal{A}^c) \le \epsilon + \eta. \tag{27}$$

It follows from (27) and (22) that

$$\beta_{\epsilon+\eta}(P_{K_xK_y\mathbf{F}Z}, Q_{K_xK_y\mathbf{F}Z}) \le \frac{1}{|\mathcal{K}|\eta^2},$$

which completes the proof.

Finally, we derive the upper bound for  $S_{\epsilon,\delta}(X,Y\mid Z)$  using the data processing property of  $\beta_{\epsilon}$ : let W be a stochastic mapping from  $\mathcal V$  to  $\mathcal V'$ , i.e., for each  $v\in \mathcal V$ ,  $W(\cdot|v)$  is a distribution on  $\mathcal V'$ . Then, since the map W followed by a test on  $\mathcal V'$  can be regarded as a stochastic test on  $\mathcal V$ ,

$$\beta_{\epsilon}(P, Q) \le \beta_{\epsilon}(P \circ W, Q \circ W),$$
 (28)

where  $(P \circ W)(v') = \sum_{v} P(v) W(v'|v)$ .

Proof of Theorem 3: Using the data processing inequality (28) with  $P = P_{XYZ}$ ,  $Q = Q_{XYZ}$ , and  $W = W_{K_xK_y\mathbf{F}|XYZ}$ , Lemma 11 implies that any  $(K_x, K_y)$  satisfying the secrecy criterion (19) must satisfy

$$\log |\mathcal{K}| \le -\log \beta_{\epsilon+\eta}(P_{XYZ}, Q_{XYZ}) + 2\log(1/\eta). \tag{29}$$

Furthermore, from Lemma 10,  $(\epsilon, \delta)$ -SK implies existence of local estimates  $K_x$  and  $K_y$  satisfying (19) with  $(\epsilon + \delta)$  in place of  $\epsilon$ . Thus, an  $(\epsilon, \delta)$ -SK with range K must satisfy (29) with  $\epsilon$  replaced by  $(\epsilon + \delta)$ , which completes the proof.

# B. Proof of Lemma 4

Let  $K_s=f_s(X)$  be the key for a fixed seed. By using the Cauchy-Schwarz inequality,

$$\begin{split} \|\mathbf{P}_{K_{s}VZ} - \mathbf{P}_{\mathtt{unif}} \mathbf{P}_{VZ} \|_{1} &= \frac{1}{2} \sum_{k,v,z} \left| \mathbf{P}_{K_{s}VZ} \left( k,v,z \right) - \frac{1}{|\mathcal{K}|} \mathbf{P}_{VZ} \left( v,z \right) \right| \\ &= \frac{1}{2} \sum_{k,v,z} \sqrt{\mathbf{Q}_{Z} \left( z \right)} \left| \frac{\mathbf{P}_{K_{s}VZ} \left( k,v,z \right) - \frac{1}{|\mathcal{K}|} \mathbf{P}_{VZ} \left( v,z \right)}{\sqrt{\mathbf{Q}_{Z} \left( z \right)}} \right| \\ &\leq \frac{1}{2} \sqrt{|\mathcal{K}||\mathcal{V}| \sum_{k,v,z} \frac{\left( \mathbf{P}_{K_{s}VZ} \left( k,v,z \right) - \frac{1}{|\mathcal{K}|} \mathbf{P}_{VZ} \left( v,z \right) \right)^{2}}{\mathbf{Q}_{Z} \left( z \right)}}. \end{split}$$

Thus, by the concavity of  $\sqrt{\cdot}$ ,

$$\left\|\mathbf{P}_{KVZS}-\mathbf{P}_{\mathtt{unif}}\mathbf{P}_{VZ}\mathbf{P}_{S}\right\|_{1} \leq \frac{1}{2}\sqrt{\left|\mathcal{K}||\mathcal{V}|\sum_{k,v,z,s}\mathbf{P}_{S}\left(s\right)\frac{\left(\mathbf{P}_{K_{s}VZ}\left(k,v,z\right)-\frac{1}{|\mathcal{K}|}\mathbf{P}_{VZ}\left(v,z\right)\right)^{2}}{\mathbf{Q}_{Z}\left(z\right)}}.$$

The numerator of the sum can be rewritten as

$$\begin{split} & \sum_{k,s} \mathrm{P}_{S}\left(s\right) \left( \mathrm{P}_{K_{s}VZ}\left(k,v,z\right) - \frac{1}{|\mathcal{K}|} \mathrm{P}_{VZ}\left(v,z\right) \right)^{2} \\ & = \sum_{s} \mathrm{P}_{S}\left(s\right) \sum_{k} \left[ \mathrm{P}_{K_{s}VZ}\left(k,v,z\right)^{2} - 2 \mathrm{P}_{K_{s}VZ}\left(k,v,z\right) \frac{1}{|\mathcal{K}|} \mathrm{P}_{VZ}\left(v,z\right) + \frac{1}{|\mathcal{K}|^{2}} \mathrm{P}_{VZ}\left(v,z\right)^{2} \right] \\ & = \sum_{s} \mathrm{P}_{S}\left(s\right) \left[ \sum_{k} \mathrm{P}_{K_{s}VZ}\left(k,v,z\right)^{2} - \frac{1}{|\mathcal{K}|} \mathrm{P}_{VZ}\left(v,z\right)^{2} \right] \\ & = \sum_{s} \mathrm{P}_{S}\left(s\right) \left[ \sum_{x,x'} \mathrm{P}_{XVZ}\left(x,v,z\right) \mathrm{P}_{XVZ}\left(x',v,z\right) \left\{ \mathbb{1} \left( f_{s}(x) = f_{s}(x') \right) - \frac{1}{|\mathcal{K}|} \right\} \right] \\ & = \sum_{x} \mathrm{P}_{XVZ}\left(x,v,z\right)^{2} \sum_{s} \mathrm{P}_{S}\left(s\right) \left\{ 1 - \frac{1}{|\mathcal{K}|} \right\} \\ & + \sum_{x \neq x'} \mathrm{P}_{XVZ}\left(x,z,v\right) \mathrm{P}_{XVZ}\left(x',v,z\right) \sum_{s} \mathrm{P}_{S}\left(s\right) \left\{ \mathbb{1} \left( f_{s}(x) = f_{s}(x') \right) - \frac{1}{|\mathcal{K}|} \right\} \\ & \leq \sum_{x} \mathrm{P}_{XVZ}\left(x,v,z\right)^{2}, \end{split}$$

where we used the property of two-universality (9) in the last inequality. Thus, we have

$$\begin{split} \|\mathbf{P}_{KVZS} - \mathbf{P}_{\mathtt{unif}} \mathbf{P}_{VZ} \mathbf{P}_{S}\|_{1} &\leq \frac{1}{2} \sqrt{|\mathcal{K}||\mathcal{V}| \sum_{x,v,z} \frac{\mathbf{P}_{XVZ}\left(x,v,z\right)^{2}}{\mathbf{Q}_{Z}\left(z\right)}} \\ &\leq \frac{1}{2} \sqrt{|\mathcal{K}||\mathcal{V}| \sum_{x,v,z} \frac{\mathbf{P}_{XVZ}\left(x,v,z\right) \mathbf{P}_{XZ}\left(x,z\right)}{\mathbf{Q}_{Z}\left(z\right)}} \\ &= \frac{1}{2} \sqrt{|\mathcal{K}||\mathcal{V}| \sum_{x,z} \frac{\mathbf{P}_{XZ}\left(x,z\right)^{2}}{\mathbf{Q}_{Z}\left(z\right)}} \\ &\leq \frac{1}{2} \sqrt{|\mathcal{K}||\mathcal{V}| 2^{-H_{\min}(\mathbf{P}_{XZ}|\mathbf{Q}_{Z})}}. \end{split}$$

Note that the last step in the proof above shows that it is, in fact, the conditional Rényi entropy of order 2 that determines the leakage (see [3] for a similar observation). However, the weaker bound proved above suffices for our case, as it does for many other cases (*cf.* [24]).

# C. A secret key agreement protocol requiring 1-bit feedback

In this section, we present a secret key agreement protocol which requires only 1-bit of feedback for generating an  $(\epsilon, \delta)$ -SK, in the special case when Z is a constant. The main component is a high secrecy protocol which achieves arbitrarily high secrecy and required reliability. In contrast to Protocol 1, which

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relied on slicing the spectrum of  $P_{X|Y}$ , the high secrecy protocol is based on slicing the spectrum of  $P_X$ . Since the party observing X can determine the corresponding slice index, feedback is not needed and one-way communication suffices. We then convert this high secrecy protocol into a high reliability protocol, using a 1-bit feedback. The required protocol for generating an  $(\epsilon, \delta)$ -SK is obtained by randomizing between the high secrecy and the high reliability protocols. This protocol appeared in a conference version containing some of the results of this paper [16], but was discovered independently by [8] in a slightly different setting. Note that this is a different approach from the one used in Section IV where a high reliability protocol was constructed and Proposition 1 was invoked to obtain a high secrecy protocol.

**Description of the high secrecy protocol.** We now describe our protocol formally. The information reconciliation step of our protocol relies on a single-shot version of the classical Slepian-Wolf theorem [27] in distributed source coding for two sources [20], [10, Lemma 7.2.1] (see, also, [18]). We need a slight modification of the standard version – the encoder is still a random binning but for decoding, instead of using a "typical-set" decoder for the underlying distribution, we use a mismatched typical-set decoder. We provide a proof for completeness.

**Lemma 13** (Slepian-Wolf Coding). Given two distributions  $P_{XY}$  and  $Q_{XY}$  on  $\mathcal{X} \times \mathcal{Y}$ , for every  $\gamma > 0$  there exists a code (e,d) of size M with encoder  $e: \mathcal{X} \to \{1,...,M\}$ , and a decoder  $d: \{1,...,M\} \times \mathcal{Y} \to \mathcal{X}$ , such that

$$P_{XY}(\{(x,y) : x \neq d(e(x),y)\}) \le P_{XY}(\{(x,y) : -\log Q_{X|Y}(x \mid y) \ge \log M - \gamma\}) + 2^{-\gamma}.$$

*Proof.* We use a random encoder given by random binning, *i.e.*, for each  $x \in \mathcal{X}$ , we independently randomly assign i = 1, ..., M. For the decoder, we use a typicality-like argument, but instead of using the standard typical set defined via  $P_{X|Y}$ , we use the mismatched typical-set

$$\mathcal{T}_{\mathbf{Q}_{X|Y}} := \left\{ (x, y) : -\log \mathbf{Q}_{X|Y}(x|y) < \log M - \gamma \right\}.$$

Then, upon receiving  $i \in \{1, ..., M\}$ , the decoder outputs  $\hat{x}$  if there exists a unique  $\hat{x}$  satisfying  $e(\hat{x}) = i$  and  $(\hat{x}, y) \in \mathcal{T}_{Q_{X|Y}}$ . An error occur if  $(X, Y) \notin \mathcal{T}_{Q_{X|Y}}$  or there exists  $\tilde{x} \neq X$  such that  $(\tilde{x}, Y) \in \mathcal{T}_{Q_{X|Y}}$  and  $e(\tilde{x}) = e(X)$ . The former error event occurs with probability

$$P_{XY}\left(\left\{(x,y):-\log Q_{X\mid Y}(x\mid y)\geq \log M-\gamma\right\}\right).$$

The probability of the second error event averaged over the random binning is bounded as

$$\mathbb{E}\left[\mathbf{P}\left(\exists \tilde{x} \neq X \text{ s.t. } e(\tilde{x}) = e(X), \ (\tilde{x}, Y) \in \mathcal{T}_{\mathbf{Q}_{X|Y}}\right)\right]$$

$$\leq \sum_{x,y} \mathbf{P}_{XY}\left(x,y\right) \mathbb{E}\left[\sum_{\tilde{x} \neq x} \mathbb{1}\left(e(\tilde{x}) = e(x)\right) \cdot \mathbb{1}\left((\tilde{x},y) \in \mathcal{T}_{\mathbf{Q}_{X|Y}}\right)\right]$$

$$= \sum_{x,y} \mathbf{P}_{XY}\left(x,y\right) \sum_{\tilde{x} \neq x} \frac{1}{M} \mathbb{1}\left((\tilde{x},y) \in \mathcal{T}_{\mathbf{Q}_{X|Y}}\right)$$

$$\leq \sum_{x,y} \mathbf{P}_{XY}\left(x,y\right) 2^{-\gamma}$$

$$= 2^{-\gamma},$$

where the expectation is over the random encoder e. The first inequality above is by the union bound, the first equality is a property of random binning, and the second inequality follows from

$$|\{x:(x,y)\in\mathcal{T}_{Q_{X|Y}}\}|\leq M2^{-\gamma}\quad\forall y\in\mathcal{Y}.$$

Thus, there exists a code (e, d) satisfying the desired bound.

We are now in a position to describe our protocol, which is based on slicing the spectrum of  $P_X$ . We first slice the spectrum of  $P_X$  into L+1 parts. Specifically, for  $1 \le i \le L$ , let  $\lambda_i = \lambda_{\min} + (i-1)\Delta$  and define

$$\mathcal{X}_i := \{x : \lambda_i \le -\log P_X(x) < \lambda_i + \Delta\}. \tag{30}$$

We also define

$$\mathcal{X}_0 := \left\{ (x, y) : -\log P_X(x) \ge \lambda_{\max} \text{ or } -\log P_X(x) < \lambda_{\min} \right\}. \tag{31}$$

Denote by J the RV such that the event  $\{J=j\}$  corresponds to  $\mathcal{X}_j, \ 0 \leq j \leq L$ . We divide the indices  $0 \leq j \leq L$  into "good" indices  $\mathcal{I}_g$  and the "bad" indices  $\mathcal{I}_b = \mathcal{I}_g^c$ , where

$$\mathcal{I}_{g}=\left\{ j:j>0\text{ and }\mathrm{P}_{J}\left( j\right) \geq\frac{1}{L^{2}}\right\} .$$

Denote by  $P_j$  the conditional distribution of X, Y given J = j, i.e.,

$$P_{j}(x,y) = \frac{P_{XY}(x,y)}{P_{X}(\mathcal{X}_{j})} \mathbf{1}(x \in \mathcal{X}_{j}), \quad x \in \mathcal{X}, \ y \in \mathcal{Y}, \ 0 \le j \le L.$$

Note that J is a function of X and can be computed by the first party, i.e., the party observing X. In our

protocol below, the first party computes J and sends it to the second party as public communication. If  $J \in \mathcal{I}_b$ , the protocol declares a reconciliation error and aborts. Otherwise, the protocol generates a secret key conditioned on the event  $\mathcal{X}_J$ .

For  $1 \leq j \leq L$ , let  $(e_j, d_j)$  be the Slepian-Wolf code of Lemma 13 for  $P_{XY} = P_j$  and  $Q_{XY} = P_{XY}$ . Further, let  $\mathcal{F}$  be a 2-universal family of mappings  $f : \mathcal{X} \to \mathcal{K}$ , and let S be random seed such that  $f_S$  denotes a randomly chosen member of  $\mathcal{F}$ .

Our secret key agreement protocol is given in Protocol 2.

# Protocol 2: High secrecy protocol

**Input**: Observations X and Y

**Output**: Secret key estimates  $K_x$  and  $K_y$ 

Information reconciliation

First party (observing X) finds the index  $J \in \{0, 1, ..., L\}$  such that  $X \in \mathcal{X}_J$ 

if  $J \in \mathcal{I}_b$  then

The protocol declares an error and aborts

else

First party sends  $(J, e_J(X))$  to the second party

Second party computes  $\hat{X} = d_J(Y, e_J(X))$ 

Privacy amplification

First party generates the random seed S and sends it to the second party using public communication

First party generates the secret key  $K_x = K = f_S(X)$ 

The second party generates the estimate  $K_y$  of K as  $K_y = f_S(\hat{X})$ 

**Performance bounds for Protocol 2.** The next result shows that Protocol 2 attains arbitrary high secrecy and required reliability.

**Theorem 14.** For every  $\gamma > 0$  and  $0 \le \lambda \le \lambda_{\min}$ , Protocol 2 yields an  $(\epsilon, \delta)$ -SK K taking values in K with

$$\epsilon \leq P\left(i_{XY}\left(X,Y\right) \leq \lambda + \gamma + \Delta\right) + P_{XY}\left(\mathcal{X}_{0}\right) + 2^{-\gamma} + \frac{1}{L},$$
$$\delta \leq \frac{1}{2}\sqrt{|\mathcal{K}|2^{-(\lambda - 2\log L)}},$$

where, with  $\lambda_{\text{max}} = \lambda_{\text{min}} + L\Delta$ ,  $\mathcal{X}_0$  is given by (31).

*Proof.* To bound the error in information reconciliation, note that for all  $j \in \mathcal{I}_q$  by Lemma 13 with

$$P_{XY} = P_j$$
 and  $Q_{XY} = P_{XY}$ 

$$\begin{aligned} & \mathbf{P}_{j}\left(\left\{\left(x,y\right):x\neq d_{j}(e_{j}(x),y)\right\}\right)-2^{-\gamma} \\ & \leq \mathbf{P}_{j}\left(\left\{\left(x,y\right):-\log \mathbf{P}_{X\mid Y}(x\mid y)\geq \log M_{j}-\gamma\right\}\right) \\ & = \mathbf{P}_{j}\left(\left\{\left(x,y\right):-\log \mathbf{P}_{X}\left(x\right)-i_{XY}\left(x,y\right)\geq \log M_{j}-\gamma\right\}\right) \\ & \leq \mathbf{P}_{j}\left(\left\{\left(x,y\right):\lambda_{j}+\Delta-i_{XY}\left(x,y\right)\geq \log M_{j}-\gamma\right\}\right), \end{aligned}$$

where the previous inequality uses the definition of  $P_i$  and (30). On choosing

$$\log M_i = \lambda_i - \lambda,$$

we get

$$P_{i}(\{(x,y): x \neq d_{i}(e_{i}(x),y)\}) \leq P_{i}(\{(x,y): i_{XY}(x,y) \leq \lambda + \gamma + \Delta\}) + 2^{-\gamma}.$$

An error in information reconciliation occurs if either  $J \notin \mathcal{I}_g$  or if  $j \in \mathcal{I}_g$  and  $X \neq d_j(e_j(X), Y)$ . From the bound above

$$\begin{split} \epsilon & \leq \mathrm{P}_{XY}\left(J \notin \mathcal{I}_g\right) + 2^{-\gamma} + \sum_{j \in \mathcal{I}_g} \mathrm{P}_J\left(j\right) \mathrm{P}_j\left(\left\{\left(x,y\right) : i_{XY}\left(x,y\right) \leq \lambda + \gamma + \Delta\right\}\right) \\ & \leq \mathrm{P}_{XY}\left(J \notin \mathcal{I}_g\right) + 2^{-\gamma} + \mathrm{P}_{XY}\left(\left\{\left(x,y\right) : i_{XY}\left(x,y\right) \leq \lambda + \gamma + \Delta\right\}\right), \end{split}$$

which using

$$P_{J}(\mathcal{I}_{b}) = \sum_{j \in \mathcal{I}_{b}} P_{J}(j) \leq P_{J}(0) + \frac{1}{L}$$

gives

$$\epsilon \leq \mathrm{P}_{X}\left(\mathcal{X}_{0}\right) + 2^{-\gamma} + \frac{1}{L} + \mathrm{P}_{XY}\left(\left\{\left(x,y\right) : i_{XY}\left(x,y\right) \leq \lambda + \gamma + \Delta\right\}\right),$$

proving the reliability bound of the theorem.

We proceed to secrecy analysis. Note that the protocol only defines the secret key for the case  $J \in \mathcal{I}_g$ . For concreteness, let

$$K = \begin{cases} f_S(X), & J \in \mathcal{I}_g, \\ ext{unif}(\mathcal{K}), & ext{otherwise}, \end{cases}$$

K is perfectly secure when  $J \in \mathcal{I}_b$ . Denoting the communication  $(J, e_J(X))$  by F, we get

$$\|\mathbf{P}_{K\mathbf{F}S} - \mathbf{P}_{\mathbf{unif}} \mathbf{P}_{\mathbf{F}S}\|_{1}$$

$$= \sum_{j \notin \mathcal{I}_{g}} \mathbf{P}_{J}(j) \cdot 0 + \sum_{j \in \mathcal{I}_{g}} \mathbf{P}_{J}(j) \cdot \|\mathbf{P}_{Ke_{J}(X)S|J=j} - \mathbf{P}_{\mathbf{unif}} \mathbf{P}_{e_{J}(X)S|J=j}\|_{1}.$$
(32)

To bound  $\|P_{Ke_j(X_j)S|J=j} - P_{\mathtt{unif}}P_{e_J(X)S|J=j}\|_1$ , denote by  $P_{j,X}$  the marginal on  $\mathcal X$  induced by  $P_j$ . Note that for each  $j \in \mathcal I_g$ 

$$-\log P_{j,X}(x) = -\log \frac{P_X(x)}{P_X(\mathcal{X}_j)}$$
$$\geq \lambda_j - 2\log L,$$

where the last inequality uses the definition of  $\mathcal{X}_j$  and  $\mathcal{I}_g$ . It follows that

$$H_{\min}(P_{j,X}) \ge \lambda_j - 2\log L.$$

Therefore, upon noting that S is independent of (X, Y, J) even upon conditioning on J = j, for each  $j \in \mathcal{I}_g$  an application of Lemma 4 implies that

$$\begin{split} \left\| \mathbf{P}_{Ke_j(X)S|J=j} - \mathbf{P}_{\mathtt{unif}} \mathbf{P}_{e_J(X)S|J=j} \right\|_1 &\leq \frac{1}{2} \sqrt{|\mathcal{K}| M_j 2^{-H_{\min}(\mathbf{P}_{j,X})}} \\ &\leq \frac{1}{2} \sqrt{|\mathcal{K}| 2^{-(\lambda-2\log L)}}, \quad j \in \mathcal{I}_g, \end{split}$$

which together with (32) gives

$$\left\|\mathbf{P}_{K\mathbf{F}S} - \mathbf{P}_{\mathtt{unif}}\mathbf{P}_{\mathbf{F}S}\right\|_1 \leq \frac{1}{2}\sqrt{|\mathcal{K}|2^{-(\lambda-2\log L)}},$$

which in turn proves the secrecy bound claimed in the theorem.

From high secrecy protocol to a high reliability protocol. By Theorem 14, the secrecy parameter  $\delta$  of Protocol 2 can be made small by choosing  $\log |\mathcal{K}| \approx \lambda$ , but its reliability parameter  $\epsilon$  is limited by the tail-probability  $P(i_{XY}(X,Y) \leq \lambda)$ . Thus, in contrast to Protocol 1, Protocol 2 constitutes a high secrecy protocol. Note that while any high reliability protocol can be converted into a high secrecy protocol using Proposition 1, it is unclear if a high secrecy protocol can be converted to a high reliability protocol in general. However, high secrecy Protocol 2 can be converted into a high reliability protocol as follows: The second party upon decoding X computes the indicator of the error event  $\mathcal{E}_j := \{-\log P_{X|Y}(X|Y) \geq \log M_j - \gamma\}$  and sends it back to the first party. If  $\mathcal{E}_j$  doesn't occur, the secret key K is as in the protocol above. Otherwise, K is chosen to be a constant. For this modified secret key, the event  $\mathcal{E}_j$  is accounted

for in the secrecy parameter  $\delta$  and not in  $\epsilon$  as earlier. Thus, Theorem 14 holds for the modified secret key where the leading term  $P(i_{XY}(X,Y) \leq \lambda + \gamma + \Delta)$  is moved from the upper bound on  $\epsilon$  to that on  $\delta$ , and the resulting protocol has high reliability. Furthermore, the high reliability protocol uses just 1-bit of feedback from the second party to the first.

Finally, a protocol for generating an arbitrary  $(\epsilon, \delta)$ -SK can be obtained by a hybrid use of the high reliability and high secrecy protocols as in Proposition 2.

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