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Communication and Randomness Lower Bounds for Secure Computation

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Abstract—In secure multiparty computation (MPC), mutually distrusting users collaborate to compute a function of their private data without revealing any additional information about their data to the other users. While it is known that information theoretically secure MPC is possible among n users having access to private randomness and are pairwise connected by secure, noiseless, and bidirectional links against the collusion of less than n/2 users (in the *honest-but-curious* model; the threshold is n/3 in the *malicious model*), relatively little is known about the communication and randomness complexity of secure computation, i.e., the amount of communication and randomness required to compute securely.

In this work, we employ information theoretic techniques to obtain lower bounds on communication and randomness complexity of secure MPC. We restrict ourselves to a concrete interactive setting involving three users under which all functions are securely computable against corruption of individual users in the honest-but-curious model. We derive lower bounds for both the perfect security case (i.e., zero-error and no leakage of information) and asymptotic security (where the probability of error and information leakage vanish as block-length goes to ∞).

Our techniques include the use of a data processing inequality for residual information (i.e., the gap between mutual information and Gács-Körner common information), a new information inequality for 3-user protocols, and the idea of distribution switching by which lower bounds computed under certain worstcase scenarios can be shown to apply for the general case.

Our lower bounds are shown to be tight for various functions of interest. In particular, we show concrete functions which have "communication-ideal" protocols, i.e., which achieve the minimum communication simultaneously on all links in the network. Also, we obtain the first explicit example of a function that incurs a higher communication cost than the input length, in the secure computation model of Feige, Kilian, and Naor [1], who had shown that such functions exist. We also show that our communication bounds imply tight lower bounds on the amount of randomness required by MPC protocols for many interesting functions.

I. INTRODUCTION

Secure multiparty computation (MPC) allows mutually distrusting users to collaborate in computational tasks. In particular, it allows such parties to compute a function of their private data without revealing any additional information about their data to the other users. This can be trivially achieved in the presence of a trusted central server as all the users can send their private data to the server, which performs the computation and sends back the function value to all the users. The goal of secure MPC - often referred to simply as MPC – is to emulate this trusted server by a protocol in which the users communicate among themselves, and learn nothing but what they would have learned by interacting with the trusted server. MPC has several important potential applications including privacy-preserving data mining, secure auction, secure machine learning, secure benchmarking (see, e.g., [2]).

MPC was pioneered by the seminal works of Shamir, Rivest, and Adleman [3], Rabin [4], Blum [5], Yao [6], [7], Goldreich, Micali, and Wigderson [8], and others. All of these early results were based on computational limitations of adversaries and some cryptographic assumptions, such as the existence of one-way functions, hardness of factoring large integers, etc. In a remarkable result, Ben-Or, Goldwasser, and Wigderson [9], and independently Chaum, Crépeau, and Damgård [10], showed that information theoretically MPC is possible among n users having access to private randomness and are pairwise connected by private, noiseless, and bidirectional links against the collusion of at most $\lfloor \frac{n-1}{2} \rfloor$ users in the *honest-but-curious* model and against the collusion of at most $\lfloor \frac{n-1}{3} \rfloor$ users in the malicious model. In the honest-but-curious model, users follow the protocol honestly, but they retain all the messages exchanged during the entire execution of the protocol, and in the end, the colluders pool their information (data, private randomness, and messages) together and try to find additional information about other users' data. This is also referred to as the semi-honest model or a model with passive corruption. In the malicious model, dishonest users may arbitrarily deviate from the protocol. These thresholds are known to be tight, in the sense that there are functions which cannot be securely computed if these thresholds are exceeded. There is another line of work on information theoretically secure computation which relies on stochastic resources, such as noisy channel or distributed sources; and there, secure computation is possible even if these thresholds are exceeded [11]. In this work we focus on the model of [9] where such resources are

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unavailable.

Communication is a critical resource in any distributed computation. Communication complexity of multi-party computation without any security requirements has been widely studied since [12] (see [13]), and more recently has seen the use of information-theoretic tools as well, in [14] and subsequent works. Independently, in the information theory literature, communication requirements of interactive function computation have been studied (e.g. [15], [16]). In secure distributed computation, in addition to communication, private randomness is also a crucial resource; it is known that secure computation of nontrivial functions is not possible deterministically [17]. However, relatively less is known about the *lower* bounds on the amount of communication and randomness required by secure computation protocols, with a few notable exceptions [18], [19], [20], [1], [21], [17], [22], [23], which provide lower bounds for very specific functions (mostly for modular-addition). For 2-user secure computation, Kushilevitz [18] combinatorially characterized the communication complexity of securely computable functions with security against passive corruption of a single user. For n-user secure protocols, Franklin and Yung [19] proved an $\Omega(n^2)$ lower bound on the number of messages exchanged for XOR function if the security is required against the corruption of $t = \Omega(n)$ users, which matches the upper bound (up to constants); if a stronger corruption model (fail-stop corruption) is assumed, then [19] showed matching *amortized* upper and lower bounds for modular-addition for $t = \Omega(n)$, which implies that parallelization does not help in this stronger model. Further, Chor and Kushilevitz [20] gave tight lower and upper bounds (exact constants) on the number of messages exchanged for modularaddition against corruption of $t \le n-2$ users. For a restricted class of 3-user secure protocols, Feige, Kilian, and Naor [1], along with positive results, obtained a modest communication lower bound (more than the input length) for random Boolean functions. For secure computation of XOR, Gál and Rosén [23] proved an $\Omega(\log n)$ lower bound on the amount of randomness required, which matches the upper bound $O(t^2 \log(n/t))$ of Kushilevitz and Mansour [21] for any fixed t. Kushilevitz and Rosén [17] studied the tradeoff between randomness and number of rounds required in *n*-user secure computation of Boolean functions. Using information theoretic tools, Blundo et al. [22] proved matching (exact constants) lower and upper bounds on the randomness complexity for modular-addition against the corruption of any t = n - 2 users.

Obtaining strong lower bounds for communication and randomness in information-theoretically secure MPC protocols is considered difficult, as it has implications to other longstanding open problems in theoretical computer science. In particular, Ishai and Kushilevitz [24] showed that establishing strong MPC communication lower bounds (even with restrictions on the number of rounds) would imply breakthrough lower bound results for well-studied problems like private information retrieval and locally decodable codes. Further, due to the protocols of [9], [10], lower bounds for MPC communication (with a constant number of players) that are super-linear in the input size would imply super-linear lower bounds for circuit complexity – a notoriously hard problem. The protocol of Damgård and Ishai [25] extended this to a non-constant number of players. Kushilevitz et al. [26] showed that this relation to circuit size holds even for MPC protocols that use only a constant number of random bits, if security is required only against semi-honest corruption of a single player. One of the goals of this work is to develop tools to tackle the difficult problem of establishing lower bounds for communication and randomness in MPC, even if they do not have immediate implications to circuit complexity, private information retrieval, or locally decodable codes.

In this work, we also consider a relaxed notion of security for MPC – namely *asymptotic security*. In the standard cryptographic definitions, security is required for every input. Also, often the security is required to be "perfect" in that the computation is always correct and there is no information leaked about the inputs beyond the output.¹

In contrast, in asymptotic security, users are given many independent copies of the inputs, and we allow the error in security (probability of error in the outcome and the leakage of information) to be "vanishing" as a function of the number of copies (i.e., eventually, the error becomes less than any given positive constant). While asymptotically correct interactive function computation without any privacy requirement has been investigated [15], [27], [28], as far as we know, there is very little work on asymptotically secure computation [29]. Lee and Abbe [29] considered the communication requirements for asymptotically secure computation under a restrictive model of protocols in which no private randomness is available to the users. In this work we provide communication and randomness lower bounds for asymptotically secure computation of any function with arbitrary input distribution. We also establish a gap between the communication requirements (and also randomness requirements) under asymptotic security and under perfect security by studying the modular addition function.

It is instructive to compare the problem of communication complexity lower bounds for secure multi-party computation with that when there is no security requirement involved. This latter problem has been extensively studied — over the last three and a half decades, starting with [12] — resulting in a rich collection of results and techniques. Unfortunately, many of the techniques in the communication complexity setting are not relevant in the setting of secure computation:² for instance, for communication complexity, Yao's minimax theorem allows one to consider only deterministic protocols with public randomness, but in the secure computation setting, one must allow private randomness, and hence it is not sufficient to consider only deterministic protocols. This rules out several powerful combinatorial approaches from the communication complexity

¹While in this paper we mostly focus on perfect security, the more general notion of *statistical security* – which allows the error in security to be "negligible" as a function of a security parameter (i.e., given any polynomial in the security parameter, eventually the error becomes less than its reciprocal) – is similar in that it also requires security for every input and there is no distribution over inputs.

²Of course, communication complexity lower bounds continue to hold for secure computation as well, but these bounds as such are (apparently) very loose. There is a trivial upper bound for communication complexity, which is at most the size of all inputs and outputs. This is often insufficient for secure computation [1]; also see Section III-D.1.

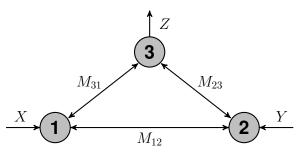


Fig. 1 A 3-user secure computation problem. Alice (user-1) has input X and Bob (user-2) has Y. We require that (i) Charlie (user-3) obtains an output Z, where Z is a (possibly randomized) function of the other two users' inputs, (ii) Alice and Bob learn no additional information about each other's inputs and the output, and (iii) Charlie learns nothing more about X, Y than what is revealed by Z. All users can talk to each other, over multiple rounds over bidirectional pairwise private links.

literature. But over the last decade or so (see for example, [30] and references therein), several information theoretic tools have been developed, which in many cases subsume more complicated combinatorial approaches. Information-theoretic techniques have also been successfully used in deriving bounds in various cryptographic problems like key agreement (e.g. [31]), secure two-party computation (e.g. [32]), and secret-sharing and its variants (e.g. [33] and [34]). Following this lead, the approach we take in this work is to develop an information-theoretic approach to obtain communication and randomness lower bounds for secure computation.

A. Results and Techniques

In this work we restrict our study to a concrete setting involving three users (with security against passive corruption of any single user), where two users, Alice and Bob have inputs, X and Y, and only the third user Charlie produces an output Z as a (possibly randomized) function of the inputs; see Figure 1. This is arguably the simplest setting of [9] where all functions can be securely evaluated, against passive corruption of a single user. Indeed, the functions considered in this model are the same as in the model of [1]; however, we allow fully interactive communication between all three users (as in [9]), whereas in the model of [1], there is no other communication except a single message each from Alice and Bob to Charlie, and a random string shared between Alice and Bob. Since we allow more general protocols, it is harder to establish lower bounds in our model. We obtain lower bounds on the entropy of the transcript between each pair of users which will imply lower bounds on the expected number of bits exchanged by these users. We also obtain lower bounds on the amount of randomness needed for secure computation.

At a high-level, the main ingredients in deriving our lower bounds for perfectly secure computation are the following:

• Firstly, we observe (in Lemma 3) that, since Alice and Bob do not obtain any outputs and therefore must not learn any additional information about each other's inputs, they are both forced to reveal their inputs fully (up to equivalent inputs) to the rest of the system, and further, Charlie's output depends on the inputs only through all the communication he has with the rest of the system. Combined with the privacy requirements, one can immediately obtain naïve lower bounds on the entropies of the transcripts: specifically, writing X, Y, Z as X_1, X_2, X_3 , we have $H(M_{ij}) \ge H(X_i, X_j | X_k)$, where $\{i, j, k\} = \{1, 2, 3\}$.³

We strengthen the naïve lower bounds by relying on a "secure data-processing inequality" (Lemma 1 due to Wolf and Wullschleger [35] and generalized in [36]) for *residual information* — i.e., the gap between mutual-information and (Gács-Körner) common information — which lets us relate the residual information of real-world views of a pair of users to the residual information of their ideal-world views.

- We can improve the lower bounds by exploiting the fact that, in a protocol, the transcripts have to be generated by the users interactively, rather than be created by an omniscient "dealer." (We formalize the latter notion of secure transcripts generated by a dealer as *Correlated Multi-Secret Sharing Schemes.*) A technical contribution of this work is a *new information inequality for 3-user protocols* (Lemma 6), which serves as a tool to separate the transcript generated by a secure protocol from one generated by a dealer.
- Our final tool, that is used to significantly improve the above lower bounds, is called *distribution switching*. The key idea is that the security requirement forces the distribution of the transcript on certain links to be independent of certain inputs. Hence we can optimize our bound using an appropriate distribution of inputs. In fact, we can take the different terms in our bound and *optimize each of them separately using different distributions over the inputs*. The resulting bounds are often stronger than what can be obtained by considering a single input distribution for the entire expression. Further, this shows that even if the protocol is allowed to depend on the input distribution, our bounds (which depend only on the function being evaluated) hold for every input distribution that has full support over the input domain.

For asymptotically secure computation, we show that for the same secure computation problems, the protocols for asymptotic security can provably be more communication efficient than the protocols for perfect security. Hence, the lower bounds derived for perfect security do not hold for asymptotic security. In deriving the lower bounds for asymptotic security, we use some of the basic ideas (cut-set, secure data-processing inequality, information inequality) from the

³ We point out a simple example for which one can obtain a tight bound from this naïve bound: addition (in any group) requires one group element to be communicated between every pair of players, even with amortization over several independent instances. Previous lower bounds for secure evaluation of addition [19], [20], while considering an arbitrary number of users, either restricted themselves to bounding the *number of messages* required, or relied on non-standard security requirements. In comparison, for the 3-user case, for semi-honest security, results of [19], [20] only imply that all three links should be used. [19] did give a lower bound on the number of bits communicated as well, but this was shown only under a non-standard security requirement called *unstoppability*.

bounds for perfectly secure computation, but the lower bound proofs here are slightly more involved. For instance, in any perfectly secure protocol, the information about a user's input must flow out through the links she/he is part of (Lemma 3); but this is not true for asymptotically secure computation – in fact, in some examples we show that our optimal protocol for asymptotically secure computation does not require users to reveal their inputs. The analogous (Lemma 7) turns out to be more involved.

While we restrict our attention to a 3-user setting, to the best of our knowledge, our lower bounds (for perfectly and asymptotically secure computations) are the first *generic* lower bounds which apply to any function. To illustrate their use, we apply them to several interesting example functions. In particular, we show the following:

- For several functions we prove that there are secure protocols which achieve *optimal communication complexity simultaneously on each link*. We call such a protocol a *communication-ideal* protocol.
- We show an explicit deterministic function $f: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^{n-1}$, which has a communication-ideal protocol in which Charlie's total communication cost is (and must be at least) 3n - 1bits. In contrast, [1] showed that there exist functions $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\},$ for which Charlie must receive at least 3n - 4 bits, where the protocol is required to be in their non-interactive model. (Note that our bound is incomparable to that of [1] since we require the output of our function to be longer; on the other hand, our bound uses an explicit function and continues to hold even if we allow unrestricted interaction.)
- Our lower bounds for communication complexity also yield lower bounds on the amount of randomness needed in secure computation protocols. We analyze secure protocols for several functions and prove that these protocols are *randomness-optimal*, i.e., they use the least amount of randomness.
- We also use our lower bounds to establish a separation between secret sharing and secure computation: we show that there exists a function (in fact, the AND function) which has a secret sharing scheme with a share strictly smaller than the number of bits in the transcript on the corresponding link in any secure computation protocol for that function. While such a separation is natural to expect, we note that proving it requires exploiting the properties of an interactive protocol.
- For asymptotically secure computation: we analyze asymptotically secure protocols for some functions and show that, under independent input distribution, these protocols are communication-ideal as well as randomness-optimal.

B. Outline of the Paper

We discuss the problem setup and some preliminaries in Section II. In Section III, we prove our lower bound results for perfectly secure computation; this is also the setting for classical positive results like that of [9]. Secure protocols for some functions of interest are given, and our bounds are analyzed for those functions. In Section IV, we prove our lower bound results for asymptotically secure computation and apply them to a few functions. In Section V, we conclude and give some open problems.

The lower bounds derived in this paper are for the honestbut-curious model against passive corruption of a single user. Typically these bounds continue to hold for active corruption as well – for many functionalities, every protocol secure against active corruption is a protocol secure against passive corruption.

II. PRELIMINARIES

Notation. We write p_X to denote the distribution of a discrete random variable X; $p_X(x)$ denotes $\Pr[X = x]$. When clear from the context, the subscript of p_X will be omitted. The conditional distribution denoted by $p_{Z|U}$ specifies $\Pr[Z = z|U = u]$, for each value z that Z can take and each value u that U can take. A randomized function of two variables, is specified by a probability distribution $p_{Z|XY}$, where X, Ydenote the two input variables, and Z denotes the output variable. For a sequence of random variables X_1, X_2, \ldots , we denote by X^n the vector (X_1, \ldots, X_n) . We abbreviate independent and identically distributed by i.i.d.

For random variables T, U, V, we write the *Markov chain* T - U - V to indicate that T and V are conditionally independent conditioned on U. All logarithms are to the base 2. The binary entropy function is denoted by $H_2(p) = -p \log p - (1-p) \log (1-p)$, $p \in (0,1)$.

Problem Definition. We consider three user computation functionalities, in which Alice and Bob (users 1 and 2) receive as inputs blocks of random variables $X^n \in \mathcal{X}^n$ and $Y^n \in \mathcal{Y}^n$, respectively, where $(X_i, Y_i) \sim p_{XY}$, i.i.d., and Charlie (user 3) wants to produce an output $Z^n \in \mathbb{Z}^n$, where Z_i 's are distributed according to a specified distribution $p_{Z|XY}$. In particular, we can consider a *deterministic function* evaluation functionality where $p_{Z|XY}(z|x,y) = 1_{z=f(x,y)}$ for some function $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$. The set \mathcal{X}, \mathcal{Y} , and \mathcal{Z} are always finite. We assume that every pair of users is connected by a noiseless, bidirectional link, which is secure from the other user, i.e., the other user cannot read or tamper with any message sent on that link. All the users have access to private randomness, which is independent between the users and also independent of their inputs. We study the secure computation problem in two settings: perfect security and asymptotically perfect security. Below, we consider protocols which can depend not only on $p_{Z|XY}$, but also p_{XY} (a protocol is formally defined later in this section); note that since we are interested in establishing lower bounds, this strengthens our results.

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- 1) **Perfectly secure computation:** A perfectly secure computation protocol $\Pi(p_{XY}, p_{Z|XY})$ satisfies the following conditions:
 - Correctness: Charlie's output Z^n should be distributed according to $p(z^n|x^n, y^n) = \prod_{i=1}^n p_{Z|XY}(z_i|x_i, y_i)$, where x^n and y^n are the inputs to Alice and Bob, respectively.
 - *Privacy:* Corresponding to privacy against Alice, Bob, and Charlie, respectively, we have the following three conditions:

$$I(M_{12}, M_{31}; Y^n, Z^n | X^n) = 0,$$

$$I(M_{12}, M_{23}; X^n, Z^n | Y^n) = 0,$$

$$I(M_{23}, M_{31}; X^n, Y^n | Z^n) = 0,$$

where M_{ij} is the collection of all the messages exchanged between users *i* and *j* on the *ij* link in either direction during the entire execution of the protocol (a formal definition is given later, along with the definition of a protocol).

Intuitively, the privacy conditions guarantee that even if one user, say Alice, is curious and retains her view (i.e., her input and all the messages exchanged during the entire execution of the protocol), this view reveals nothing more to her about the input and output of the other users (namely, Y^n, Z^n), than what her own input/output (namely, X^n) reveals. In other words, a curious user may as well simulate a view for herself based on just its input and output rather than retain the actual view it obtained from the protocol execution. A more formal definition of perfectly secure protocols is given in Definition 2 in Section III.

- Asymptotically secure computation: For asymptotically secure computation, for simplicity, in this paper we restrict ourselves to deterministic functions *f* : X × Y → Z. A sequence of asymptotically secure protocols Π_n(*f*, *p*_{XY}) satisfies the following conditions:
 - Correctness: Charlie's output Zⁿ should be close to the true output Zⁿ, where Z_i = f(X_i, Y_i), i = 1, 2, ..., n, in the sense that Pr{Zⁿ ≠ Zⁿ} → 0 as n → ∞, where probability is taken over the randomness of the input distribution and the protocol.
 - *Privacy:* Corresponding to privacy against Alice, Bob, and Charlie, respectively, we have the following three conditions as $n \to \infty$:

$$\begin{split} &I(M_{12}, M_{31}; Y^n, Z^n | X^n) \to 0, \\ &I(M_{12}, M_{23}; X^n, Z^n | Y^n) \to 0, \\ &I(M_{23}, M_{31}; X^n, Y^n | Z^n) \to 0. \end{split}$$

Intuitively, the privacy conditions guarantee that, from the protocol, any one user does not obtain non-negligible additional information about other users' inputs and output (if any). A more formal definition of asymptotically secure protocols is given in Definition 3 in Section IV.

A Normal Form for $(p_{XY}, p_{Z|XY})$. For a pair $(p_{XY}, p_{Z|XY})$, define the relations $x \cong x', y \cong y'$, and $z \cong z'$

as follows.

- 1) For any $x, x' \in \mathcal{X}$, let $\mathcal{S}_{x,x'} = \{y \in \mathcal{Y} : p_{XY}(x,y) > 0, p_{XY}(x',y) > 0\}$. We say that $x \cong x'$, if $\forall y \in \mathcal{S}_{x,x'}$ and $z \in \mathcal{Z}$, we have $p_{Z|XY}(z|x,y) = p_{Z|XY}(z|x',y)$.
- 2) For any $y, y' \in \mathcal{Y}$, let $\mathcal{S}_{y,y'} = \{x \in \mathcal{X} : p_{XY}(x,y) > 0, p_{XY}(x,y') > 0\}$. We say that $y \cong y'$, if $\forall x \in \mathcal{S}_{y,y'}$ and $z \in \mathcal{Z}$, we have $p_{Z|XY}(z|x,y) = p_{Z|XY}(z|x,y')$.
- 3) Let $S = \{(x, y) : p_{XY}(x, y) > 0\}$. For any $z, z' \in Z$, we say that $z \cong z'$, if $\exists c \ge 0$ such that $\forall (x, y) \in S$, we have $p_{Z|XY}(z|x, y) = c \cdot p_{Z|XY}(z'|x, y)$.

A pair $(p_{XY}, p_{Z|XY})$ is said to be in *normal form* if $x \cong x' \Rightarrow x = x', y \cong y' \Rightarrow y = y'$, and $z \cong z' \Rightarrow z = z'$.

In the paper we mostly deal with p_{XY} having full support. If p_{XY} has full support, then the above definition reduces to the following definition of normal form.

A Normal Form for Functionality $p_{Z|XY}$ (for p_{XY} with full support). For a randomized functionality $p_{Z|XY}$, we define the relation $x \equiv x'$ for $x, x' \in \mathcal{X}$ to hold if $\forall y \in \mathcal{Y}, z \in \mathcal{Z}$, p(z|x, y) = p(z|x', y); similarly we define $y \equiv y'$. For $z, z' \in \mathcal{Z}$, we define $z \equiv z'$ if there exists a constant c such that $\forall x \in \mathcal{X}, y \in \mathcal{Y}, p(z|x, y) = c \cdot p(z'|x, y)$. We say that $p_{Z|XY}$ is in *normal form* if $x \equiv x' \Rightarrow x = x', y \equiv y' \Rightarrow y = y'$, and $z \equiv z' \Rightarrow z = z'$.

Note that if $p_{Z|XY}$ is a deterministic mapping $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, then $x \equiv x'$ for $x, x' \in \mathcal{X}$ implies that $\forall y \in \mathcal{Y}$, f(x, y) = f(x', y); similarly $y \equiv y'$ is defined. We say that f is in normal form if $x \equiv x' \Rightarrow x = x'$ and $y \equiv y' \Rightarrow y = y'$.

It is easy to see that if p_{XY} has full support then one can transform any randomized function $p_{Z|XY}$ to one in normal form $p_{Z^*|X^*Y^*}$ with possibly smaller alphabets, so that any secure computation protocol for the former can be transformed to one for the latter with the same communication costs, and vice versa. To define X^* , \mathcal{X} is modified by replacing all xin an equivalence class of \equiv with a single representative; Y^* and Z^* are defined similarly. The modification to the protocol, in either direction, is for each user to locally map X to X^* etc., or vice versa; notice that the Z^* to Z map is potentially randomized.

Protocols. Given inputs X^n and Y^n to Alice and Bob, respectively, all the users engage in a protocol Π_n where they send messages to each other over several rounds, and at the end Charlie produces the output. We will omit the subscript n where it is clear from the context. A protocol consists of "next message functions" (Π^1, Π^2, Π^3) and an output function $\Pi^{3,\text{out}}$. The next message function $\Pi^i, i = 1, 2, 3$ specifies a distribution over $\mathbb{N} \times \{0,1\}^* \times \{0,1\}^*$ (which corresponds to the number of round and the messages on the two links to which user-i is associated) conditioned on the input of user-i (if any) and all the messages on these two links so far. The output function $\Pi^{3,out}$ defines the output of user-3 as a probabilistic function of all the messages it has seen so far. Specifically, it is a distribution over \mathbb{Z}^n conditioned on all the messages on the links of Charlie. We allow protocols to depend on the distribution of inputs to the users which would allow one to tune a protocol to be efficient for a given input distribution. We require that a valid protocol must terminate with probability 1, i.e., on each link, the (potentially

random) number of rounds after which the link remains unused must be finite with probability 1. We denote by M_{ijt} , the message sent from user i to user j during the round t and by M_{ij}^{t-1} , all the messages sent by user *i* to user *j* up to round t - 1. Let $M_{ij}^{t-1} = (M_{ij}^{t-1}, M_{ji}^{t-1})$ denote all the message exchanged between user i and user j up to round t-1. We denote by M_{ij} , the final transcript on link ij, which is the collection of all the messages exchanged between users i and j during the entire execution of the protocol. The message $M_{ij,t}$ (which may be an empty string) is a function of user *i*'s input (if any), M_{ij}^{t-1} , and its private randomness. Furthermore, we restrict the message M_{iit} to be a codeword of a (potentially random) prefix-free binary code $C_{ii,t}$, which itself can be determined (with probability 1) by the messages M_{ij}^{t-1} exchanged between users *i* and *j*. Although this restricts the kind of protocols we allow, (e.g., our definition does not allow $C_{12,t}$ to be determined by messages exchanged between users 1 and 2 via user-3 and not over the 12 link), this encompasses a fairly general class of protocols. Having the prefix-free requirement and also that the code be determined by previously exchanged messages allow the participating nodes to know when each message and the exchange over the link connecting them has come to an end without the need for an explicit end-of-message symbol. While we allow this generality, the protocols we provide have a deterministic number of rounds with deterministic message lengths. The generality is in order to prove impossibility results (communication and randomness lower bounds) with wide applicability.

We define $M_1 = (M_{12}, M_{31})$ as the transcripts that user 1 can see; M_2 and M_3 are defined similarly. We define the view of the i^{th} user, V_i to consist of M_i and that user's input and output (if any). Observe that a protocol along with an input distribution fully defines the joint distribution over all the inputs, outputs, and the joint transcripts on all the links.

Expected Number of Bits Exchanged and Entropy. As mentioned earlier, we require that the message sent at every round is a codeword in a prefix-free binary code which can be dynamically determined based on the previous messages exchanged over the link. This allows us to lower-bound the expected number of bits communicated in each link by the entropy of the transcript in that link.

Let $L_{i\overline{j},t} \in \{0,1,2,\ldots\}$ be the (potentially random) length of the message $M_{i\overline{j},t}$. Similarly, let $L_{ij,t}, L_{ij}^t$, and L_{ij} be the lengths of $M_{ij,t}, M_{ij}^t$, and M_{ij} , respectively. For a protocol Π_n , we define the rate quadruple $(R_{12}, R_{23}, R_{31}, \rho)$ as $R_{ij} := \frac{1}{n} \mathbb{E}[L_{ij}], i, j = 1, 2, 3, i \neq j$, and $\rho := \frac{1}{n} H(M_{12}, M_{23}, M_{31}, Z^n | X^n, Y^n)$.

We are interested in lower bounds for $\mathbb{E}[L_{ij}]$. We have

$$\begin{split} H(M_{ij}) &= \sum_{t=1}^{\infty} H(M_{i\overline{j},t}, M_{\overline{j}i,t} | M_{ij}^{t-1}) \\ &\leq \sum_{t=1}^{\infty} H(M_{i\overline{j},t} | M_{ij}^{t-1}) + H(M_{\overline{j}i,t} | M_{ij}^{t-1}) \\ &\stackrel{\text{(a)}}{=} \sum_{t=1}^{\infty} H(M_{i\overline{j},t} | M_{ij}^{t-1}, \mathcal{C}_{i\overline{j},t}) + H(M_{\overline{j}i,t} | M_{ij}^{t-1}, \mathcal{C}_{\overline{j}i,t}) \end{split}$$

$$\leq \sum_{t=1}^{\infty} H(M_{i\overline{j},t} | \mathcal{C}_{i\overline{j},t}) + H(M_{j\overline{i},t} | \mathcal{C}_{j\overline{i},t})$$

$$\leq \sum_{t=1}^{(b)} \mathbb{E}[L_{i\overline{j},t}] + \mathbb{E}[L_{j\overline{i},t}]$$

$$= \mathbb{E}[L_{ij}],$$

where (a) follows from the fact that the prefix-free codes $C_{i\vec{j},t}, C_{j\vec{i},t}$, of which $M_{i\vec{j},t}, M_{j\vec{i},t}$ are codewords, respectively, are functions of M_{ij}^{t-1} . (b) follows from the fact that the expected length L of a prefix-free binary code for a random variable U is lower-bounded by its entropy H(U) [37, Theorem 5.3.1].

Conditional Graph Entropy. Given a graph G = (V, E), where V is a finite collection of nodes and E is a collection of pairs of vertices from V. A subset $U \subseteq V$ of G is called an *independent set* of G if no two vertices of U have an edge (an edge is a pair of distinct vertices) between them in G. Let $\Gamma(G)$ denote the collection of all independent sets of G.

Witsenhausen [38] defined the *characteristic graph* $G_X = (V, E)$ for a pair (p_{XY}, f) , where $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is a deterministic function, as follows: its vertex set is the support set of X, and $E = \{\{x, x'\} : \exists y \in \mathcal{Y} \text{ such that } p_{XY}(x, y) \cdot p_{XY}(x', y) > 0 \text{ and } f(x, y) \neq f(x', y)\}$. G_Y can be defined similarly.

Definition 1 (Conditional Graph Entropy [15]). For a given pair (p_{XY}, f) , the conditional graph entropy of G_X is defined as follows:

$$H_{G_X}(X|Y) := \min_{\substack{p_{W|X:} \\ W-X-Y \\ X \in W}} I(W;X|Y), \qquad (1)$$

where the alphabet of W is $\Gamma(G_X)$ – the set of all independent sets of the characteristic graph G_X defined above. By the data-processing inequality, the minimization in (1) can be restricted to W ranging over maximal independent sets. Note that $0 \le H_{G_X}(X|Y) \le H(X|Y)$ hold in general; and if G_X is a complete graph then $H_{G_X}(X|Y) = H(X|Y)$.

Common Information and Residual Information. Gács and Körner [39] introduced the notion of common information to measure a certain aspect of correlation between two random variables. The Gács-Körner common information of a pair of correlated random variables (U, V) can be defined as $H(U \sqcap$ V), where $U \sqcap V$ is a random variable with maximum entropy among all random variables Q that are determined both by Uand by V (i.e., there are functions q and h such that Q =q(U) = h(V)). It is not hard to see that $U \sqcap V$ is equal to the random variable corresponding to the set of connected components of the *characteristic bipartite graph* of p_{UV} – for a distribution p_{UV} , a bipartite graph on vertex set $\mathcal{U} \cup \mathcal{V}$ is said to be the characteristic bipartite graph of p_{UV} , if $u \in \mathcal{U}$ and $v \in \mathcal{V}$ are connected whenever $p_{UV}(u, v) > 0$. Note that if p_{UV} is such that the characteristic bipartite graph is connected, then $U \sqcap V$ is constant and $H(U \sqcap V) = 0$. In [36], the gap between mutual information and common information was termed residual information: RI(U; V) := I(U; V) - $H(U \sqcap V).$

In [35], Wolf and Wullschleger identified (among other things) the following secure *data processing inequality* for residual information.

Lemma 1 (Secure data processing inequality [35]). If T, U, V, W are jointly distributed random variables such that the following two Markov chains hold: (i) U - T - W and (ii) T - W - V, then

$$RI(T;W) \le RI((U,T);(V,W))$$

The Markov chain conditions can be viewed as follows: let (U,T) and (V,W) be the views of any pair of users, where, for the user holding (U,T), U can be thought of as all the messages exchanged during the protocol and T can be thought of as its data (input and output) which is the idealworld view. (U,T) is the real-world view. Now the Markov chain U - T - W corresponds to the privacy requirement that U (the rest of this user's view) can be simulated based on its data T, independent of the other user's data W; and similarly for the second user. The lemma states that under this privacy condition, the residual information between the real-world views must be at least as large as that between the ideal-world views (i.e., the data).

In [36], the following alternate definition of residual information was given, which will be useful in lower-bounding conditional mutual information terms.

$$RI(U;V) := \min_{\substack{p_{Q|UV:}\\I(Q;V|U)=0\\I(Q;U|V)=0}} I(U;V|Q).$$
(2)

The random variable Q which achieves the minimum is, in fact, $U \sqcap V$. Note that the residual information is always non-negative.

Lemma 2 (*RI* tensorizes [35], [36]). For $(U^n; V^n)$, where (U_i, V_i) are *i.i.d.*, $RI(U^n; V^n) = nRI(U; V)$.

III. OUTER BOUNDS ON THE RATE-REGION FOR PERFECTLY SECURE COMPUTATION

This section is divided into four parts: in Section III-A we derive preliminary lower bounds for secure computation; in Section III-B we give some techniques which significantly improve the preliminary bounds and lead to our main theorems; in Section III-C we derive lower bounds on the amount of randomness required in secure computation protocols; and in Section III-D we consider some interesting examples – secure protocols are given, and our lower bound results are analyzed for these example functions.

We consider a 3-user secure computation problem specified by $(p_{XY}, p_{Z|XY})$, see Figure 2. Input X^n to Alice (user-1) and Y^n to Bob (user-2) are distributed according to $p_{X,Y}$, i.i.d. Charlie (user-3) wants to compute an output Z^n , which should be distributed according to $p(z^n|x^n, y^n) = \prod_{i=1}^n p_{Z|XY}(z_i|x_i, y_i)$. We say that a protocol $\prod_n(p_{XY}, p_{Z|XY})$ for this setup is *perfectly secure* if the output satisfies this and the protocol is perfectly secure against any single user as defined by (3)-(5) below. Recall that for a protocol \prod_n , we define the rate quadruple $(R_{12}, R_{23}, R_{31}, \rho)$

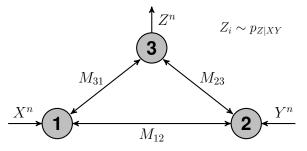


Fig. 2 A setup for 3-user secure computation; privacy is required against single users (i.e., no collusion). Here $(X, Y) \sim p_{XY}$ and $Z_i \sim p_{Z|XY}$ for all *i*.

as $R_{ij} := \frac{1}{n} \mathbb{E}[L_{ij}], \ i, j = 1, 2, 3, \ i \neq j$, and $\rho := \frac{1}{n} H(M_{12}, M_{23}, M_{31}, Z^n | X^n, Y^n)$.

Definition 2. For a secure computation problem $(p_{XY}, p_{Z|XY})$, the rate $(R_{12}, R_{23}, R_{31}, \rho)$ is achievable with perfect security for block-length n, if there is a protocol $\prod_n(p_{XY}, p_{Z|XY})$ with rate $(R_{12}, R_{23}, R_{31}, \rho)$ such that conditioned on inputs X^n, Y^n of Alice and Bob, Charlie's output Z^n is distributed according to $p(z^n|x^n, y^n) = \prod_{i=1}^n p_{Z|XY}(z_i|x_i, y_i)$, and the following holds:

$$I(M_{12}, M_{31}; Y^n, Z^n | X^n) = 0, (3)$$

$$I(M_{12}, M_{23}; X^n, Z^n | Y^n) = 0, (4)$$

$$I(M_{23}, M_{31}; X^n, Y^n | Z^n) = 0.$$
 (5)

Rate-region $\mathcal{R}^{n,PS}$ is the closure of the set of all rate quadruples achievable with perfect security for block-length n. We say that $(R_{12}, R_{23}, R_{31}, \rho)$ is achievable with perfect security if it is achievable with perfect security for some block-length n. And \mathcal{R}^{PS} is the closure of the set of all rate quadruples achievable with perfect security.

Here (3) ensures that Alice learns no additional information about (Y^n, Z^n) ; similarly for Bob; and (5) ensures that Charlie learns no additional information about (X^n, Y^n) than revealed by Z^n .

Remark 1. For perfectly secure computation, all our bounds are direct sum bounds, i.e., our outer bound on $\mathcal{R}^{1,\text{PS}}$ will also be an outer bound for \mathcal{R}^{PS} . So, for simplicity, we prove all our bounds for n = 1 and then show that it holds for \mathcal{R}^{PS} .

To make the presentation clear, in Section III-A and Section III-B we derive lower bounds only on the rates R_{12}, R_{23}, R_{31} , and lower bound on the randomness ρ is derived in Section III-C. We prove bounds on the entropies $H(M_{ij})$, which, as argued in Section II, is a lower bound on the expected length of the transcript M_{ij} .

A. Preliminary Lower Bounds

We first state the following basic lemma for any protocol for perfectly secure computation. Similar results have appeared in the literature earlier (for instance, special cases of Lemma 3 appear in [40], [41]).

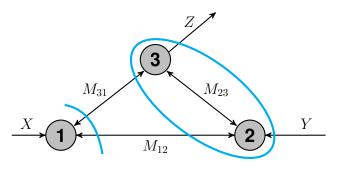


Fig. 3 A cut separating Alice from Bob & Charlie. Protocol Π induces a 2user secure computation protocol between Alice and *combined Bob-Charlie* with privacy requirement only against Alice.

Lemma 3. In any secure protocol $\Pi_1(p_{XY}, p_{Z|XY})$, where $(p_{XY}, p_{Z|XY})$ is in normal form, the following must hold:

$$H(X|M_{12}, M_{31}) = 0, (6)$$

$$H(Y|M_{12}, M_{23}) = 0, (7)$$

$$H(Z|M_{23}, M_{31}) = 0. (8)$$

We prove this lemma in Appendix A. Lemma 3 states the simple fact that, for $(p_{XY}, p_{Z|XY})$ in normal form,⁴ the cut separating Alice from Bob and Charlie must reveal Alice's input X (see Figure 3). The intuition is that, since Alice is not allowed to learn any new information about Y, correctness condition forces Alice to reveal X. Note that this conclusion crucially depends on the privacy requirement against Alice. For example, consider $X = (X_0, X_1), X_0, X_1 \in$ $\{0,1\}, Y \in \{0,1\}$, and X_0, X_1, Y are i.i.d. Bern(1/2). Let $f((X_0, X_1), Y) = X_Y$. Without the privacy condition, Bob may send Y to Alice who can compute $Z = X_Y$ and send this to Charlie. H(X) = 2, but here the cut (M_{12}, M_{31}) reveals only 1 bit of information about X. Similarly, the cut separating Bob from the rest of the users must reveal his input, and the cut separating Charlie must reveal his output. This relies on the fact that Alice and Bob obtain no output, and Charlie has no input in our model. We obtain a preliminary lower bound below by using the above lemma and the secure dataprocessing inequality for residual information (Lemma 1).

Theorem 1. For a secure computation problem $(p_{XY}, p_{Z|XY})$, where $(p_{XY}, p_{Z|XY})$ is in normal form, if $(R_{12}, R_{23}, R_{31}, \rho) \in \mathcal{R}^{PS}$, then,

$$R_{31} \ge \max\{RI(X;Y), RI(Y;Z)\} + H(X,Z|Y), \qquad (9)$$

$$R_{23} \ge \max\{RI(X;Y), RI(X;Z)\} + H(Y,Z|X), \quad (10)$$

$$R_{12} \ge \max\{RI(X;Z), RI(Y;Z)\} + H(X,Y|Z).$$
(11)

Proof: We shall prove (9) for $\mathcal{R}^{1,PS}$. The fact that it also holds for \mathcal{R}^{PS} follows from Lemma 2. The other two

inequalities can be shown similarly.

$$H(M_{31}) \ge \max\{H(M_{31}|M_{12}), H(M_{31}|M_{23})\}$$

= $\max\{I(M_{31}; M_{23}|M_{12}), I(M_{31}; M_{12}|M_{23})\}$
+ $H(M_{31}|M_{12}, M_{23})$ (12)

We can bound the last term of (12) as follows (to already get a naïve bound):

$$\begin{split} H(M_{31}|M_{12}, M_{23}) &\stackrel{\text{(a)}}{=} H(M_{31}, X, Z|M_{12}, M_{23}, Y) \\ &\geq H(X, Z|M_{12}, M_{23}, Y) \stackrel{\text{(b)}}{=} H(X, Z|Y), \end{split}$$

where (a) follows from Lemma 3 and (b) follows from (4), i.e., privacy against Bob. Next, we lower-bound the first term inside the max of (12) by RI(X;Y) as follows.

$$I(M_{31}; M_{23}|M_{12}) = I(M_{12}, M_{31}; M_{12}, M_{23}|M_{12})$$

$$\stackrel{(c)}{=} I(M_{12}, M_{31}, X; M_{12}, M_{23}, Y|M_{12})$$

$$\geq RI(M_{12}, M_{31}, X; M_{12}, M_{23}, Y),$$

where (c) follows from Lemma 3, and the last inequality follows from (2) by taking $Q = M_{12}$. Now, by privacy against Alice we have $(M_{12}, M_{31}) - X - Y$, and by privacy against Bob we have $(M_{12}, M_{23}) - Y - X$. Applying Lemma 1 with the above Markov chains, we get

$$RI(M_{12}, M_{31}, X; M_{12}, M_{23}, Y) \ge RI(X; Y).$$

Similarly, we can lower-bound the second term inside max of (12) by RI(Y; Z), completing the proof.

A consequence of Lemma 3 is that the transcripts in a secure computation protocol form shares in a "correlated multi-secret sharing scheme" (CMSS) for the same distribution $p_{XYZ} = p_{XY}p_{Z|XY}$; see Appendix B for details.⁵ Hence, lower bounds on the entropies of the shares in a CMSS imply lower bounds on the entropies of the messages in a secure computation protocol. In Appendix B we also derive stronger bounds on the sizes of these shares in CMSS.

To strengthen the preliminary bounds in Theorem 1, we will restrict our attention in the rest of the paper to p_{XY} having full support, which allows us to assume, without loss of generality, that the function $p_{Z|XY}$ is in normal form; see Section II for details.

B. Improved Lower Bounds

To improve the bounds in Theorem 1, we (i) give a technique, which we call *distribution switching*, and (ii) prove an information inequality for 3-user interactive protocols, using which we improve the above bounds and obtain our main theorems. We first prove the following lemma which gives an upper bound on the mutual information between the transcript on any link and the data (inputs and output) in terms of the Gács-Körner common information (which is equal to the difference between mutual information and residual

⁴ Note that when the input distribution p_{XY} has full support, this assumption is without loss of generality (see Section II). When p_{XY} has full support, information theoretic tools have proved to be successful in deriving optimal bounds for zero-error computation. But deriving bounds for zero-error computation with arbitrary input distribution is more amenable to combinatorial arguments [42]. Hence, for perfectly secure computation, in this paper we do not deal with arbitrary $(p_{XY}, p_{Z|XY})$; we confine our attention to either p_{XY} which have full support, or more generally, to $(p_{XY}, p_{Z|XY})$ which satisfy some technical conditions.

⁵We remark that our notion of multiple secret sharing schemes is different from that of [34], which (implicitly) required that secrets with different access structures be independent of each other. In our case, Z is typically strongly correlated with X, Y, often via a deterministic function.

information) between the data of the two users associated on that link.

Lemma 4. For any secure protocol $\Pi_1(p_{XY}, p_{Z|XY})$, where p_{XY} need not have full support and $p_{Z|XY}$ need not be in the normal form, the following hold:

$$I(M_{12}; X, Y, Z) \le I(X; Y) - RI(X; Y),$$
 (13)

$$I(M_{31}; X, Y, Z) \le I(X; Z) - RI(X; Z),$$
 (14)

$$I(M_{23}; X, Y, Z) \le I(Y; Z) - RI(Y; Z).$$
 (15)

Proof: We first show the bound in (13). Since $I(M_{12}; X, Y, Z) = I(M_{12}; X) + I(M_{12}; Y, Z|X)$, where second term is equal to zero by privacy against Alice (3), it is enough to show $I(M_{12}; X) \leq I(X; Y) - RI(X; Y)$.

$$I(M_{12}; X) = I(M_{12}, Y; X) - I(Y; X|M_{12})$$

= $I(Y; X) + I(M_{12}; X|Y) - I(Y; X|M_{12})$
= $I(X; Y) - I(X; Y|M_{12})$ (16)

 $\leq I(X;Y) - RI(X;Y). \tag{17}$

We get (16) by substituting $I(M_{12}; X|Y) = 0$, which follows from privacy against Bob (4). The inequality (17) is obtained by substituting $I(X; Y|M_{12}) \ge RI(X; Y)$, which can be proved by taking $Q = M_{12}$ in the definition of residual information (2) (where the Markov chain conditions $M_{12} - X - Y$ and $M_{12} - Y - X$ follow from privacy against Alice and privacy against Bob, respectively).

Similarly, we can show (14) using privacy against Alice and privacy against Charlie, and (15) using privacy against Bob and privacy against Charlie.

As mentioned in Section II, for any jointly distributed random variables U, V, if the characteristic bipartite graph of p_{UV} is connected, then I(U;V) = RI(U;V). Hence, as a simple consequence of the above lemma we obtain the following Lemma, which states that privacy requirements imply the independence of the transcript M_{12} generated by a secure protocol computing $p_{Z|XY}$ and the inputs. Moreover, if the function $p_{Z|XY}$ satisfies some additional constraints, the other two transcripts also become independent of the inputs.

Lemma 5. Consider a function $p_{Z|XY}$ not necessarily in normal form.

- 1) Suppose p_{XY} is such that the characteristic bipartite graph of p_{XY} is connected. Then any secure protocol $\Pi_1(p_{XY}, p_{Z|XY})$ satisfies $I(M_{12}; X, Y, Z) = 0$.
- 2) Suppose $(p_{XY}, p_{Z|XY})$ is such that the characteristic bipartite graph of the induced distribution p_{XZ} is connected. Then any secure protocol $\Pi_1(p_{XY}, p_{Z|XY})$ satisfies $I(M_{31}; X, Y, Z) = 0$.

The characteristic bipartite graph of p_{XZ} is connected if p_{XY} has full support and $p_{Z|XY}$ satisfies the following condition:

Condition 1. There is no non-trivial partition $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ (i.e., $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$ and neither \mathcal{X}_1 nor \mathcal{X}_2 is empty), such that if $\mathcal{Z}_k = \{z \in \mathcal{Z} : x \in \mathcal{X}_k, y \in \mathcal{Y}, p(z|x, y) > 0\}, k = 1, 2$, their intersection $\mathcal{Z}_1 \cap \mathcal{Z}_2$ is empty.

3) Suppose $(p_{XY}, p_{Z|XY})$ is such that the characteristic bipartite graph of the induced distribution p_{YZ} is connected. Then any secure protocol $\Pi_1(p_{XY}, p_{Z|XY})$ satisfies $I(M_{23}; X, Y, Z) = 0$.

The characteristic bipartite graph of p_{YZ} is connected if p_{XY} has full support and $p_{Z|XY}$ satisfies the following condition:

Condition 2. There is no non-trivial partition $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2$ such that if $\mathcal{Z}_k = \{z \in \mathcal{Z} : x \in \mathcal{X}, y \in \mathcal{Y}_k, p(z|x, y) > 0\}, k = 1, 2$, their intersection $\mathcal{Z}_1 \cap \mathcal{Z}_2$ is empty.

In the context of 1) above, we point out that p_{XY} having a connected characteristic bipartite graph is a weaker condition than p_{XY} having full support.

1) Distribution Switching: We will argue that even if the protocol is allowed to depend on the input distribution (as we do here), privacy requirements will require that the lower bounds derived for when the distributions of the inputs are changed continue to hold for the original setting. The main idea can be summarized as follows: Any secure protocol $\Pi(p_{XY}, p_{Z|XY})$, where distribution p_{XY} has full support, continues to be a secure protocol even if we switch the input distribution to a different one $p_{X'Y'}$. This follows, as we show below, directly from examining the (correctness and privacy) conditions required for a protocol to be secure.

- Correctness: Note that we only change the input distribution, but the function being computed remains unchanged, i.e., $p_{Z'|X'Y'}(z|x,y) = p_{Z|XY}(z|x,y)$, for every $(x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$. The correctness condition requires that with the new input distribution, Charlie's output Z' should be distributed according to $p_{Z'|X'=x,Y'=y}$, where x and y are inputs of Alice and Bob respectively, which, as we argued before is equal to $p_{Z|X=x,Y=y}$.
- Privacy: We have to show that privacy conditions against Alice, Bob, and Charlie remain intact if we change the original input distribution p_{XY} with a different one $p_{X'Y'}$. In our secure protocol model, once Alice and Bob are given inputs X = xand Y = y, respectively, the protocol produces $(m_{12}, m_{23}, m_{31}, z)$ according to the conditional $p_{M_{12}M_{23}M_{31}Z|XY}(m_{12},m_{23},m_{31},z|x,y).$ distribution Note that this conditional distribution does not depend on p_{XY} . So, if we change the input distribution p_{XY} to $p_{X'Y'}$, the conditional distribution $p_{M'_{12}M'_{23}M'_{31}Z'|X'Y'}(m_{12},m_{23},m_{31},z|x,y)$ does not change. More precisely, the following holds for every distribution $p_{X'Y'}$ and $(x,y) \in \mathcal{X} \times \mathcal{Y}$ such that $p_{X'Y'}(x,y) > 0.$

$$p_{M'_{12}M'_{23}M'_{31}Z'|X'Y'}(m_{12}, m_{23}, m_{31}, z|x, y) = p_{M_{12}M_{23}M_{31}Z|XY}(m_{12}, m_{23}, m_{31}, z|x, y).$$
(18)

We only show the result for Alice, that is, we will prove that $I(M_1; (Y, Z)|X) = 0 \implies I(M'_1; (Y', Z')|X') = 0$, where $M'_1 = (M'_{12}, M'_{31})$ is generated by the original secure protocol when the inputs to Alice and Bob are

$$\begin{split} p_{M_{1}'Z'|X'Y'}(m_{1},z|x,y) &= p_{M_{1}Z|XY}(m_{1},z|x,y) \\ \Rightarrow p_{Z'|X'Y'}(z|x,y)p_{M_{1}'|X'Y'Z'}(m_{1}|x,y,z) \\ &= p_{Z|XY}(z|x,y)p_{M_{1}|XYZ}(m_{1}|x,y,z) \\ \overset{(a)}{\Rightarrow} p_{M_{1}'|X'Y'Z'}(m_{1}|x,y,z) &= p_{M_{1}|XYZ}(m_{1}|x,y,z) \\ \overset{(b)}{\Rightarrow} p_{M_{1}'|X'Y'Z'}(m_{1}|x,y,z) &= p_{M_{1}|X}(m_{1}|x), \end{split}$$

where (a) holds because $p_{Z'|X'Y'}(z|x,y) = p_{Z|XY}(z|x,y)$, and these are non-zero because the protocol produces z when Alice and Bob are given their respective inputs x and y; (b) follows from the privacy against Alice, i.e., $I(M_1; (Y,Z)|X) = 0$. Note that the last equality holds true for any distribution $p_{X'Y'}$, and it implies that $M'_1 - X' - (Y', Z')$, i.e., $I(M'_1; (Y', Z')|X') = 0$.

Privacy against Bob and Charlie are similarly proved.

2) An Information Inequality for Protocols: We exploit the fact that, in a protocol, transcripts are generated by the users interactively rather than by an omniscient dealer. Towards this, we derive an information inequality relating the transcripts on different links in general 3-user protocols, in which users do not share any common or correlated randomness or correlated inputs at the beginning of the protocol. Note that our model for protocols does indeed satisfy these conditions when the inputs are independent of each other.

Lemma 6. In any 3-user protocol (not necessarily secure), if the inputs to the users are independent of each other, then, for $\{\alpha, \beta, \gamma\} = \{1, 2, 3\},\$

$$I(M_{\gamma\alpha}; M_{\beta\gamma}) \ge I(M_{\gamma\alpha}; M_{\beta\gamma}|M_{\alpha\beta}).$$

This inequality provides us with a means to exploit the protocol structure behind transcripts and helps us to lowerbound $I(M_{\gamma\alpha}; M_{\beta\gamma})$ in terms of the distribution p_{XYZ} . This is achieved in a few easy steps: Lemma 6, combined with (2), lets us derive that $I(M_{\gamma\alpha}; M_{\beta\gamma}) \ge RI(M_{\gamma\alpha}, M_{\alpha\beta}; M_{\beta\gamma}, M_{\alpha\beta})$. Further combined with Lemma 3, the right hand side can be equated with the residual information of the views of the users α and β . Finally, by the secure data processing inequality (Lemma 1), this can be lower-bounded in terms of the residual information of (one of) the inputs and the output (if $\{\alpha, \beta\} \neq \{1, 2\}$) or both the inputs (if $\{\alpha, \beta\} = \{1, 2\}$).

Proof of Lemma 6: For any choice of distinct α, β, γ in $\{1, 2, 3\}$, the inequality of Lemma 6 is equivalent to the following inequality:

$$H(M_{12}) + H(M_{23}) + H(M_{31}) - H(M_{23}, M_{31}) - H(M_{31}, M_{12}) - H(M_{12}, M_{23}) + H(M_{12}, M_{23}, M_{31}) \ge 0.$$
(19)

To prove (19) we apply induction on the number of rounds of the protocol.

Base case: At the beginning of the protocol, all the transcripts M_{12}, M_{23}, M_{31} are empty. So the inequality (19) is trivially true.

Inductive step: Assume that the inequality (19) is true at the

end of round t, and we prove it for t + 1. Notice that the new message in round t + 1 can be one of the six different messages – from user-i to user-j, and vice versa; but since (19) is symmetric in all the transcripts, it is enough to prove the inequality when the new message sent in round t + 1 is, say, from user-1 to user-2.

For simplicity, let us denote the transcript on 1-2 link at the end of round t by M_{12} itself, the new message sent by user-1 to user-2 in round t+1 by ΔM_{12} , and the transcript on 1-2 link at the end of round t+1 by \widetilde{M}_{12} . Notation for the remaining transcripts are defined similarly. Hence, $\widetilde{M}_{12} = (M_{12}, \Delta M_{12})$, $\widetilde{M}_{23} = M_{23}$, $\widetilde{M}_{31} = M_{31}$. Since the users do not share any common or correlated randomness, the new message ΔM_{12} is conditionally independent of the transcript M_{23} between the other two users, conditioned on transcripts (M_{12}, M_{31}) on both the links to which user-1 is associated with. So we have the following Markov chain:

$$\Delta M_{\vec{12}} - (M_{12}, M_{31}) - M_{23} \tag{20}$$

Now we can show that the inequality in (19) holds at the end of round t + 1 as follows:

$$\begin{split} H(\widetilde{M}_{12}) &+ H(\widetilde{M}_{23}) + H(\widetilde{M}_{31}) - H(\widetilde{M}_{23}, \widetilde{M}_{31}) \\ &- H(\widetilde{M}_{31}, \widetilde{M}_{12}) - H(\widetilde{M}_{12}, \widetilde{M}_{23}) + H(\widetilde{M}_{12}, \widetilde{M}_{23}, \widetilde{M}_{31}) \\ &= H(M_{12}, \Delta M_{1\vec{2}}) + H(M_{23}) + H(M_{31}) - H(M_{23}, M_{31}) \\ &- H(M_{31}, M_{12}, \Delta M_{1\vec{2}}) - H(M_{12}, \Delta M_{1\vec{2}}, M_{23}) \\ &+ H(M_{12}, \Delta M_{1\vec{2}}, M_{23}, M_{31}) \\ &\geq H(\Delta M_{1\vec{2}}|M_{12}) - H(\Delta M_{1\vec{2}}|M_{12}, M_{31}) \\ &- H(\Delta M_{1\vec{2}}|M_{12}, M_{23}) + H(\Delta M_{1\vec{2}}|M_{12}, M_{23}, M_{31}) \ (21) \\ &= I(\Delta M_{1\vec{2}}; M_{23}|M_{12}) - I(\Delta M_{1\vec{2}}; M_{23}|M_{12}, M_{31}) \\ &\geq 0 \end{split}$$

In (21) we use the induction hypothesis, and last inequality follows from (20) and the fact that the conditional mutual information is always non-negative.

3) Main Lower Bounds:

Theorem 2. For a secure computation problem $(p_{XY}, p_{Z|XY})$, where p_{XY} has full support and $p_{Z|XY}$ is in normal form, if $(R_{12}, R_{23}, R_{31}, \rho) \in \mathcal{R}^{PS}$, then,

$$R_{31} \ge \max \left\{ \begin{array}{l} \max_{\substack{p_{X}p_{Y'} \\ p_{X}p_{Y'} \\ max} RI(X;Y'') + H(X,Z''|Y''), \\ max RI(Y';Z') + H(X,Z'|Y') \\ p_{XY'} \end{array} \right\},$$

$$R_{23} \ge \max \left\{ \begin{array}{l} \max_{\substack{p_{X'P_Y} \\ p_{X'P_Y} \\ max} RI(X';Z') \\ max RI(X';Y) + H(Y,Z''|X''), \\ max RI(X';Z') + H(Y,Z'|X') \\ p_{X'Y} \\ max RI(X';Z') + H(Y,Z'|X') \\ p_{X'Y} \end{array} \right\},$$

$$(23)$$

$$R_{12} \ge \max \left\{ \begin{array}{l} \max_{\substack{p_{X'}p_{Y'} \\ + \max_{p_{X'Y''}} RI(X';Z'') + H(X',Y''|Z''), \\ \max_{\substack{p_{X'P''} \\ p_{X'PY'} \\ + \max_{p_{X'Y'}} RI(X';Z') + H(X'',Y'|Z'') \\ & + \max_{p_{X'Y'}} RI(Y';Z'') + H(X'',Y'|Z'') \end{array} \right\},$$
(24)

where the maxima are over distributions having full support. The terms in the right hand side of (22) are evaluated using the distribution p_X of the data X of Alice. The terms in (22), for instance, are evaluated using

$$\begin{split} 1^{st} \ \textit{bound:} \ p_{X,Y',Z'}(x,y,z) &= p_X(x) p_{Y'}(y) p_{Z|X,Y}(z|x,y), \\ p_{X,Y'',Z''}(x,y,z) &= p_{XY''}(x,y) p_{Z|X,Y}(z|x,y), \\ 2^{nd} \ \textit{bound:} \ p_{X,Y',Z'}(x,y,z) &= p_{XY'}(x,y) p_{Z|X,Y}(z|x,y). \end{split}$$

Similarly, the terms in (23) are evaluated using the distribution p_Y of the data Y of Bob. The lower bound in (24) does not depend on the distributions p_X and p_Y of the data. The terms in the first bound of (24), for instance, are evaluated using

$$1^{st} \text{ bound: } p_{X',Y',Z'}(x,y,z) = p_{X'}(x)p_{Y'}(y)p_{Z|X,Y}(z|x,y),$$
$$p_{X',Y'',Z''}(x,y,z) = p_{X'Y''}(x,y)p_{Z|X,Y}(z|x,y).$$

Proof: We again prove this only for n = 1. To see the general case of, say, n = m we simply invoke the n = 1 result with inputs X^m, Y^m , and $p_{Z^m|X^m,Y^m}$ defined as the *m*-wise product of $p_{Z|X,Y}$. Making memoryless choices for the primed random variables in the bound, the result follows from Lemma 2.

Suppose we have a secure protocol for computing $p_{Z|XY}$ in the normal form under p_{XY} which has full support. Consider $H(M_{31})$,

$$H(M_{31}) = I(M_{31}; M_{12}) + I(M_{31}; M_{23}|M_{12}) + H(M_{31}|M_{12}, M_{23}).$$

By privacy against Alice, conditioned on X, M_{31} is independent of Y. So, by distribution switching, keeping the marginal distribution of X same, we may switch the distribution p_{XY} to, say, $p_{XY''}$, which also has full support, and the resulting M_{31} has the same distribution as under p_{XY} , i.e.,

$$H(M_{31}) = \max_{p_{XY''}} I(M_{31}; M_{12}) + I(M_{31}; M_{23}|M_{12}) + H(M_{31}|M_{12}, M_{23}).$$

Under this switched distribution, let us consider the first term $I(M_{31}; M_{12})$. Let us notice that, by privacy against Alice, (M_{31}, M_{12}) must again be independent of Y''. Hence, even if we switch the distribution of Y to, say, $p_{Y'}$ (keeping the marginal of X same), the joint distribution of (M_{31}, M_{12}) must remain unchanged. Hence, we have that $I(M_{31}; M_{12})$ under the distribution $p_{XY''}$ is the same as that under $p_{XY'}$. Therefore,

$$H(M_{31}) = \max_{\substack{p_{XY'} \\ p_{XY'}}} I(M_{31}; M_{12}) + \max_{\substack{p_{XY''} \\ p_{XY''}}} I(M_{31}; M_{23} | M_{12}) + H(M_{31} | M_{12}, M_{23}).$$

If we take the distribution under the first max to be the product distribution, i.e., $p_{XY'} = p_X p_{Y'}$, then we can apply Lemma 6 and get the following:

$$H(M_{31}) \ge \max_{p_X p_{Y'}} I(M_{31}; M_{12} | M_{23}) + \max_{p_{XY''}} I(M_{31}; M_{23} | M_{12}) + H(M_{31} | M_{12}, M_{23}).$$

We can bound each of the three terms separately as follows: (i) Using the definition of residual information, Lemma 3, and Lemma 1, we can show that $I(M_{31}; M_{12}|M_{23}) \ge RI(Y; Z)$ and $I(M_{31}; M_{23}|M_{12}) \ge RI(X; Y)$; and (ii) using Lemma 3 and privacy against Bob, we get $H(M_{31}|M_{12}, M_{23}) \ge$ H(X, Z|Y). This gives the first bound on $H(M_{31})$ (first row of (22)).

$$H(M_{31}) \ge \max_{p_X p_{Y'}} RI(Y'; Z') + \max_{p_{XY''}} RI(X; Y'') + H(X, Z''|Y'').$$

For the second bound on $H(M_{31})$ (second row of (22)), we first expand $H(M_{31})$ in a different way as follows:

$$H(M_{31}) = I(M_{31}; M_{23}) + I(M_{31}; M_{12}|M_{23}) + H(M_{31}|M_{12}, M_{23}).$$

Now drop the first term $I(M_{31}, M_{23})$ and proceed as above.

The bounds on $H(M_{23})$ follow in an identical fashion. To see the bounds on $H(M_{12})$, let us recall that M_{12} is independent of X, Y (by Lemma 5), and hence we may switch the distributions of both X and Y. Furthermore, let us note that we may write $H(M_{12})$ in two different ways.

$$H(M_{12}) = I(M_{12}; M_{31}) + I(M_{12}; M_{23}|M_{31}) + H(M_{12}|M_{23}, M_{31})$$
(25)
$$H(M_{12}) = I(M_{12}; M_{23}) + I(M_{12}; M_{31}|M_{23}) + H(M_{12}|M_{23}, M_{31}).$$
(26)

Using (25) and proceeding as we did for $H(M_{31})$ leads to the top row of the right hand side of (24), and (26) leads to the bottom row.

When the function satisfies certain additional constraints, we can strengthen the lower bounds on the $H(M_{23})$ and $H(M_{31})$ as shown below.

Theorem 3. For a secure computation problem $(p_{XY}, p_{Z|XY})$, where p_{XY} has full support and $p_{Z|XY}$ is in normal form, if $(R_{12}, R_{23}, R_{31}, \rho) \in \mathcal{R}^{PS}$, then,

1) Suppose the function $p_{Z|XY}$ satisfies Condition 1 of Lemma 5, that is, there is no non-trivial partition $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ (i.e., $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$ and neither \mathcal{X}_1 nor \mathcal{X}_2 is empty), such that if $\mathcal{Z}_k = \{z \in \mathcal{Z} : x \in \mathcal{X}_k, y \in \mathcal{Y}, p(z|x, y) > 0\}, k = 1, 2$, their intersection $\mathcal{Z}_1 \cap \mathcal{Z}_2$ is empty. Then we have the following strengthening of (22).

$$R_{31} \ge \max \begin{cases} \max_{p_{X'}p_{Y'}} RI(Y';Z') \\ + \max_{p_{X'Y''}} RI(X';Y'') + H(X',Z''|Y''), \\ \max_{p_{X'Y'}} RI(Y';Z') + H(X',Z'|Y') \end{cases}$$
(27)

where the maxima are over distributions having full support, and the terms in (27), for instance, are evaluated using

$$p_{X',Y',Z'}(x,y,z) = p_{X'}(x)p_{Y'}(y)p_{Z|X,Y}(z|x,y),$$

$$p_{X',Y'',Z''}(x,y,z) = p_{X'Y''}(x,y)p_{Z|X,Y}(z|x,y),$$

$$p_{X',Y',Z'}(x,y,z) = p_{X'Y'}(x,y)p_{Z|X,Y}(z|x,y).$$

Suppose the function p_{Z|XY} satisfies Condition 2 of Lemma 5, that is, there is no non-trivial partition Y = Y₁ ∪ Y₂ such that if Z_k = {z ∈ Z : x ∈ X, y ∈ Y_k, p(z|x, y) > 0}, k = 1, 2, their intersection Z₁∩Z₂ is empty. Then we have the following strengthening of (23).

$$R_{23} \ge \max \begin{cases} \max_{\substack{p_{X'}p_{Y'} \\ + \max_{p_{X'Y'}} RI(X'';Y') + H(Y',Z''|X''), \\ \max_{p_{X'Y'}} RI(X';Z') + H(Y',Z'|X') \\ \end{cases}$$
(28)

where the maxima are over distributions having full support, and the terms in (28), for instance, are evaluated using

$$\begin{split} p_{X',Y',Z'}(x,y,z) &= p_{X'}(x)p_{Y'}(y)p_{Z|X,Y}(z|x,y), \\ p_{X'',Y',Z''}(x,y,z) &= p_{X''Y'}(x,y)p_{Z|X,Y}(z|x,y), \\ p_{X',Y',Z'}(x,y,z) &= p_{X'Y'}(x,y)p_{Z|X,Y}(z|x,y). \end{split}$$

Proof: We again prove this only for the case of n = 1. The general case follows for the same reasons as in the proof of Theorem 2. By Lemma 5, M_{31} is independent of p_{XY} under condition 1. Hence, when condition 1 is satisfied, we may switch the distribution p_{XY} to $p_{X'Y'}$. Note that this is unlike what we did in the proof of the lower bound on $H(M_{31})$ in Theorem 2 where we switched p_{XY} to $p_{XY'}$ keeping the marginal distribution of X same. Now proceeding similarly as there leads us to (27). Similarly, under condition 2, M_{23} is independent of X, Y which leads to (28).

Note that in Theorem 2 and Theorem 3, any choice of $p_{X'}, p_{Y'}, p_{X'Y'}, p_{XY'}, p_{XY'}, p_{XY''}, p_{XY''}, p_{XY''}, p_{XY'Y''}$ (with full support) yields a lower bound. For a given function, while all choices do yield valid lower bounds, one is often able to obtain the *best* of these lower bounds analytically (as in Theorem 7, where it is seen to be the best as it matches an upper bound) or numerically (as in Theorem 11).

To summarize, for any secure computation problem $(p_{XY}, p_{Z|XY})$, where $(p_{XY}, p_{Z|XY})$ is in normal form, Theorem 1 gives lower bounds on entropies of the transcripts on all three links. If p_{XY} has full support and $p_{Z|XY}$ is in normal form, then Theorem 2 gives improved lower bounds on entropies of the transcripts on all three links. In addition, if $p_{Z|XY}$ satisfies condition 1 of Lemma 5, then (27) gives further improvements on the lower bound for $H(M_{31})$; if $p_{Z|XY}$ satisfies condition 2 of Lemma 5, then (28) further improves the lower bound for $H(M_{23})$. The fact that these improvements can be strict will be seen through examples in Section III-D; see the paragraph following Theorem 11 therein.

C. Lower Bounds on Randomness

In this section we provide lower bounds on the amount of randomness required in secure computation protocols. Although our focus in this paper is to prove communication lower bounds, it turns out that we may apply the above lower bounds on communication to derive bounds on the amount of randomness required. We show in Section III-D that they give tight bounds on randomness required for the specific functions we analyze.

Theorem 4. For a secure computation problem $(p_{XY}, p_{Z|XY})$, where $(p_{XY}, p_{Z|XY})$ is in normal form, if $(R_{12}, R_{23}, R_{31}, \rho) \in \mathbb{R}^{PS}$, then

$$\begin{split} \rho &\geq \max\{RI(X;Y) + RI(X;Z), \\ &RI(X;Y) + RI(Y;Z), \\ &RI(X;Z) + RI(Y;Z)\} \\ &+ H(Y,Z|X) + H(X,Z|Y) + H(X,Y|Z) - H(X,Y). \end{split}$$

Proof: Again, we prove this for n = 1, and the general result follows from Lemma 2. Fix a protocol $\Pi(p_{XY}, p_{Z|XY})$. We bound the randomness required by this protocol as follows:

$$\begin{split} \rho &= H(M_{12}, M_{23}, M_{31}, Z|X, Y), \quad \text{(by definition)} \\ &= H(M_{12}, M_{23}, M_{31}|X, Y, Z) + H(Z|X, Y) \\ &\geq \max\{H(M_{12}, M_{31}|X, Y, Z), H(M_{12}, M_{23}|X, Y, Z), \\ &\quad + H(M_{23}, M_{31}|X, Y, Z)\} + H(Z|X, Y) \quad (29) \\ &= \max\{H(M_{12}, M_{31}|X), H(M_{12}, M_{23}|Y), \\ &\quad + H(M_{23}, M_{31}|Z)\} + H(Z|X, Y) \quad (30) \\ &\geq \max\{H(M_{12}, M_{31}) - H(X), H(M_{12}, M_{23}) - H(Y) \\ &\quad + H(M_{23}, M_{31}) - H(Z)\} + H(Z|X, Y), \quad (31) \end{split}$$

where, in (30) we used privacy conditions (3)-(5), and in (31) we used $H(M_{12}, M_{31}|X) = H(M_{12}, M_{31}, X) - H(X) \ge$ $H(M_{12}, M_{31}) - H(X)$. We can bound $H(M_{12}, M_{31})$, $H(M_{12}, M_{23})$, and $H(M_{23}, M_{31})$ using the bounds we have already developed. First we bound $H(M_{12}, M_{31})$ as follows:

$$H(M_{12}, M_{31}) = H(M_{12}) + H(M_{31}|M_{12}).$$
(32)

Notice that one of the bounds on $H(M_{31})$ in Theorem 1 was obtained by bounding $H(M_{31}|M_{12})$ as follows:

$$H(M_{31}|M_{12}) \ge RI(X;Y) + H(X,Z|Y).$$
 (33)

Substituting the bound on $H(M_{12})$ from Theorem 1 and bound on $H(M_{31}|M_{12})$ from (33) into (32) we get:

$$H(M_{12}, M_{31}) \ge \max\{RI(X; Z), RI(Y; Z)\} + H(X, Y|Z) + RI(X; Y) + H(X, Z|Y).$$

Substituting this into the first term inside the \max in (31) we get:

$$\begin{split} \rho &\geq \max\{RI(X;Z), RI(Y;Z)\} + H(X,Y|Z) + RI(X;Y) \\ &+ H(X,Z|Y) - H(X) + H(Z|X,Y) \\ &= \max\{RI(X;Y) + RI(X;Z), RI(X;Y) + RI(Y;Z)\} \\ &+ H(Y,Z|X) + H(X,Z|Y) + H(X,Y|Z) - H(X,Y). \end{split}$$

If we expand $H(M_{12}, M_{31})$ in another way as $H(M_{31}) + H(M_{12}|M_{31})$ and proceed similarly as above, we get the following:

$$\rho \ge \max\{RI(X;Y) + RI(X;Z), RI(X;Z) + RI(Y;Z)\} + H(Y,Z|X) + H(X,Z|Y) + H(X,Y|Z) - H(X,Y).$$

Combining the above two bounds gives the desired result. Notice that bounding $H(M_{12}, M_{23})$ and $H(M_{23}, M_{31})$ result in the same bound.

If p_{XY} has full support (which allows us to assume, without loss of generality, that $p_{Z|XY}$ is in normal form), we can strengthen Theorem 4 using the ideas from Section III-B. For instance, now we can bound $H(M_{12})$ from Theorem 2. Notice that by (4), i.e., privacy against Alice, (M_{12}, M_{31}) is conditionally independent of Y conditioned on X. This means that the distribution of (M_{12}, M_{31}) will not change if we change the input distribution to $p_{XY'}$ keeping the marginal of Alice's input the same. Applying this observation to (33) we get the following:

$$H(M_{31}|M_{12}) \ge \max_{p_{Y'|X}} RI(X;Y') + H(X,Z'|Y'), \quad (34)$$

where the terms on the right hand side are evaluated using the following distribution:

$$p_{XY'Z'}(x, y, z) = p_X(x) \cdot p_{Y'|X}(y|x) \cdot p_{Z|XY}(z|x, y).$$

Similarly we can prove the following bounds, where right hand side expressions are evaluated using appropriate distributions:

$$H(M_{23}|M_{12}) \ge \max_{p_{X'|Y}} RI(X';Y) + H(Y,Z'|X'), \quad (35)$$

$$H(M_{12}|M_{31}) \ge \max_{p_{Y'|X}} RI(X;Z') + H(X,Y'|Z'), \quad (36)$$

$$H(M_{12}|M_{23}) \ge \max_{p_{X'|Y}} RI(Y;Z') + H(X',Y|Z'), \quad (37)$$

$$H(M_{23}|M_{31}) \ge \max_{p_{X'Y'}} RI(X';Z) + H(Y',Z|X'), \quad (38)$$

$$H(M_{31}|M_{23}) \ge \max_{p_{X'Y'}} RI(Y';Z) + H(X',Z|Y'), \quad (39)$$

where the $p_{X'Y'}$ in the maximization in (38) and (39) are over distributions such that they result in the same marginal distribution of Z, i.e., $p_{X'Y'}$ should be such that the following holds for every value z in the alphabet \mathcal{Z} :

$$\sum_{x,y} p_{X'Y'}(x,y) p_{Z|XY}(z|x,y) = \sum_{x,y} p_{XY}(x,y) p_{Z|XY}(z|x,y)$$

The above observations, along with (31), lead to the following theorem.

Theorem 5. For a secure computation problem $(p_{XY}, p_{Z|XY})$, where p_{XY} has full support and $p_{Z|XY}$ is in normal form, $(R_{12}, R_{23}, R_{31}, \rho) \in \mathcal{R}^{PS}$ only if

$$\begin{split} \rho &\geq H(M_{12}) + H(M_{31}|M_{12}) - H(X) + H(Z|XY), \\ \rho &\geq H(M_{12}) + H(M_{23}|M_{12}) - H(Y) + H(Z|XY), \\ \rho &\geq H(M_{31}) + H(M_{12}|M_{31}) - H(X) + H(Z|XY), \\ \rho &\geq H(M_{31}) + H(M_{23}|M_{31}) - H(Z) + H(Z|XY), \\ \rho &\geq H(M_{23}) + H(M_{12}|M_{23}) - H(Y) + H(Z|XY), \\ \rho &\geq H(M_{23}) + H(M_{31}|M_{23}) - H(Z) + H(Z|XY), \end{split}$$

where lower bounds on the entropy terms may be taken from *Theorem 2*, and lower bounds on the conditional entropy terms may be taken from (34)-(39).

Note that a consequence of the first bound in Theorem 5, combined with (34) (with Y' taken to be independent of X, so that $H(X, Z'|Y') \ge H(X)$), is that

$$\rho \ge H(M_{12}). \tag{40}$$

When the function satisfies certain additional constraints, we can further strengthen Theorem 5 using the improved bounds on $H(M_{31})$ and $H(M_{23})$ from Theorem 3. The resulting bounds on randomness are stated in the following theorem.

Theorem 6. For a secure computation problem $(p_{XY}, p_{Z|XY})$, where p_{XY} has full support and $p_{Z|XY}$ is in normal form, if $(R_{12}, R_{23}, R_{31}, \rho) \in \mathcal{R}^{PS}$ then following must hold.

1) If $p_{Z|XY}$ satisfies condition 1 of Lemma 5, then

$$\rho \ge H(M_{31}) + H(M_{12}|M_{31}) - H(X) + H(Z|XY),$$

$$\rho \ge H(M_{31}) + H(M_{23}|M_{31}) - H(Z) + H(Z|XY),$$

where lower bound on $H(M_{31})$ may be taken from *Theorem 3*, and lower bounds on the conditional entropy terms may be taken from (36) and (38).

2) If $p_{Z|XY}$ satisfies condition 2 of Lemma 5, then

$$\rho \ge H(M_{23}) + H(M_{12}|M_{23}) - H(Y) + H(Z|XY),$$

$$\rho \ge H(M_{23}) + H(M_{31}|M_{23}) - H(Z) + H(Z|XY),$$

where lower bound on $H(M_{23})$ may be taken from *Theorem 3*, and lower bounds on the conditional entropy terms may be taken from (37) and (39).

To summarize, for any secure computation problem $(p_{XY}, p_{Z|XY})$, where $(p_{XY}, p_{Z|XY})$ is in normal form, Theorem 4 gives a lower bound on the randomness. If p_{XY} has full support and $p_{Z|XY}$ is in normal form, then Theorem 5 gives an improved lower bound on randomness. In addition, if $p_{Z|XY}$ satisfies condition 1 or condition 2 of Lemma 5, then lower bound of Theorem 6 may, in some cases, be better than that of Theorem 5.

Remark 2. As argued in Section II, for communication requirements, if p_{XY} has full support, we can assume, without loss of generality, that $p_{Z|XY}$ is in normal form. In case $p_{Z|XY}$ is not in normal form, we redefine the problem to $(p_{X^*Y^*}, p_{Z^*|X^*Y^*})$, where $p_{Z^*|X^*Y^*}$ is in normal form, such that any secure protocol for the former can be transformed to a secure protocol for the latter with the same communication costs, and vice versa. This implies that the communication lower bounds developed for the modified problem $(p_{X^*Y^*}, p_{Z^*|X^*Y^*})$ are equally good for the original problem $(p_{XY}, p_{Z|XY})$. But this may not hold for randomness complexity, i.e., we do not know how to transform a secure protocol for $(p_{X^*Y^*}, p_{Z^*|X^*Y^*})$ to a secure protocol for $(p_{XY}, p_{Z|XY})$ with the same randomness requirement. Hence, although the lower bounds for randomness developed for the problem $(p_{X*Y*}, p_{Z*|X*Y*})$ also serve as valid lower bounds for the original problem $(p_{XY}, p_{Z|XY})$, improvements may be possible by considering the original problem.

D. Application to Specific Functions

In this section we consider a few important examples where we will apply our generic lower bounds from above and also give secure protocols. In some cases we will obtain the optimal rate regions. While some of these results are natural to conjecture, they are not easy to prove (see, for instance, Footnote 3). In all the examples except one⁶, our protocols are for a block-length of 1; hence, where they are optimal, $\mathcal{R}^{1,PS} = \mathcal{R}^{PS}$. Before presenting our examples, we make the following two definitions:

Communication-Ideal Protocol. We say that a protocol $\Pi(p_{XY}, p_{Z|XY})$ for securely computing a randomized function $p_{Z|XY}$ for a distribution p_{XY} is *communication-ideal*, if for each $ij \in \{12, 23, 31\}$,

$$H(M_{ij}^{\Pi}) = \inf_{\Pi'(p_{XY}, p_{Z|XY})} H(M_{ij}^{\Pi'}),$$

where the infimum is over all secure protocols for $p_{Z|XY}$ with the same distribution p_{XY} . That is, a communication-ideal protocol achieves the least entropy possible for every link, simultaneously. We remark that it is not clear, *a priori*, how to determine if a given function $p_{Z|XY}$ has a communicationideal protocol for a given distribution p_{XY} .

Randomness-Optimal Protocol. We say that a protocol $\Pi(p_{XY}, p_{Z|XY})$ for securely computing a randomized function $p_{Z|XY}$ for a distribution p_{XY} is *randomness-optimal*, if

$$\rho^{\Pi}(p_{XY}, p_{Z|XY}) = \inf_{\Pi'(p_{XY}, p_{Z|XY})} \rho^{\Pi'}(p_{XY}, p_{Z|XY}),$$

where the infimum is over all secure protocols for $p_{Z|XY}$ with the same distribution p_{XY} . That is, a protocol is randomness optimal if the randomness (measured in bits) used by the protocol is the least among all protocols. As defined in the beginning of Section III, the amount of randomness required by a protocol Π is $\rho^{\Pi}(p_{XY}, p_{Z|XY}) = H(M_{12}, M_{23}, M_{31}, Z|X, Y)$.

1. Secure Computation of Remote Oblivious Transfer REMOTE $\binom{m}{1}$ -OTⁿ₂: The REMOTE $\binom{m}{1}$ -OTⁿ₂ function, is defined as follows: Alice's input $X = (X_0, X_1, \ldots, X_{m-1})$ is made up of m bit-strings each of length n, and Bob has an input $Y \in \{0, 1, \ldots, m-1\}$. Charlie wants to compute $Z = f(X, Y) = X_Y$. This can be seen as a 3 user variant of oblivious transfer [43], [44]. Figure 4 gives the simple protocol for this function from [1] (rephrased as a protocol in our model). It requires nm bits to be exchanged over the Alice-Charlie (31) link, $n + \log m$ bits over the Bob-Charlie (23) link, and $nm + \log m$ bits over the Alice-Bob (12) link. The total number of random bits used in the protocol is $nm + \log m$. We show that this protocol achieves the optimal rate region, i.e., it is a communication-ideal as well as randomness-optimal protocol.

Theorem 7. Any secure protocol for computing REMOTE $\binom{m}{1}$ -OTⁿ₂ for inputs X and Y, where p_{XY} has full support, must

Algorithm 1: Secure Computation of REMOTE $\binom{m}{1}$ -OTⁿ₂

- **Require:** Alice has m input bit strings $X_0, X_1, \ldots, X_{m-1}$ each of length n & Bob has an input $Y \in \{0, 1, \ldots, m-1\}$.
- **Ensure:** Charlie securely computes the REMOTE $\binom{m}{1}$ -OT₂ⁿ: $Z = X_Y$.
- Alice samples nm + log m independent, uniformly distributed bits from her private randomness. Denote the first m blocks each of length n of this random string by K₀, K₁,..., K_{m-1} and the last log m bits by π. Alice sends it to Bob as M_{12,1} = (K₀, K₁,..., K_{m-1}, π).
- sends it to Bob as $M_{\vec{12},1} = (K_0, K_1, \dots, K_{m-1}, \pi)$. 2: Alice computes $M^{(i)} = X_{\pi+i \pmod{m}} \oplus K_{\pi+i \pmod{m}}$, $i \in \{0, 1, \dots, m-1\}$ and sends to Charlie $M_{\vec{13},2} = (M^{(0)}, M^{(1)}, \dots, M^{(m-1)})$. Bob computes $C = Y - \pi \pmod{m}, K = K_Y$ and sends to Charlie $M_{\vec{23},2} = (C, K)$.
- 3: Charlie outputs $Z = M^{(C)} \oplus K$.

satisfy

$$R_{31} \ge nm, \quad R_{23} \ge n + \log m, \text{ and}$$

 $R_{12}, \rho \ge nm + \log m.$

Hence the protocol in Figure 4 is optimal.

Proof: REMOTE $\binom{m}{1}$ -OTⁿ₂ satisfies Condition 1 and Condition 2 of Lemma 5, which implies RI(Y; Z) = I(Y; Z) and RI(Z; X) = I(Z; X). For $H(M_{31})$ and $H(M_{23})$ the bottom rows in (27) and (28) simplify to the following:

$$H(M_{31}) \ge \max_{p_{X'Y'}} I(Y'; Z') + H(X'|Y'),$$

$$H(M_{23}) \ge \max_{p_{X'Y'}} I(X'; Z') + H(Y'|X').$$

Taking X' and Y' to be independent and uniform in their respective domains, we get $H(M_{31}) \ge nm$. To derive a lower bound on $H(M_{23})$, take X', Y' to be independent with Y' ~ unif $\{0, 1, \ldots, m-1\}$ and X' distributed as below:

$$p_{X'_0,X'_1,\dots,X'_{m-1}}(x_0,x_1,\dots,x_{m-1}) = \begin{cases} \frac{1}{2^n} - \epsilon, & x_0 = x_1 = \dots = x_{m-1}, \\ \frac{\epsilon}{(2^{n(m-1)} - 1)}, & \text{otherwise}, \end{cases}$$

where $\epsilon > 0$ can be made arbitrarily small to make I(Z'; X')as close to n as desired. This gives a bound of $H(M_{23}) \ge n + \log m$. For $H(M_{12})$, the bottom row of (24) simplifies to

$$H(M_{12}) \ge \max_{p_{X'}p_{Y'}} I(X';Z') + \max_{p_{X''Y'}} I(Y';Z'') + H(X'',Y'|Z'').$$

⁶The exceptional case arises as a special case of CONTROLLED ERASURE where we need to exploit blocks of input for source compression.

Fig. 4 A protocol to securely compute REMOTE $\binom{m}{1}$ -OTⁿ₂, which is a special case of the general protocol given in [1]. The protocol requires nm bits to be exchanged over the Alice-Charlie (13) link, $n + \log m$ bits over the Bob-Charlie (23) link, and $nm + \log m$ bits over the Alice-Bob (12) link. We show optimality of our protocol by showing that any protocol must exchange an expected nm bits over the Alice-Charlie (31) link, $n + \log m$ bits over the Bob-Charlie (23) link, and $nm + \log m$ bits over the Alice-Bob (12) link.

Taking X'' and Y' to be independent and uniform (in the second maximum), and X' to be distributed as below

$$p_{X'_0,X'_1,\dots,X'_{m-1}}(x_0,x_1,\dots,x_{m-1}) = \begin{cases} \frac{1}{2^n} - \epsilon, & x_0 = x_1 = \dots = x_{m-1}, \\ \frac{\epsilon}{2^n}(2^{n(m-1)} - 1), & \text{otherwise}, \end{cases}$$

where $\epsilon > 0$ can be made arbitrarily small to make I(X'; Z') as close to n as desired. This gives a bound of $H(M_{12}) \ge nm + \log m$.

Finally, from (40) and the above bound on $H(M_{12})$, we get $\rho \ge H(M_{12}) \ge nm + \log m$. This implies that the above protocol is randomness-optimal.

The above optimality result for REMOTE OT function has the following implications:

- (i) Optimality of the FKN Protocol. Feige et al. [1] provided a generic (non-interactive) secure computation protocol for all 3-user functions in our model. This protocol uses a straight-forward (but "inefficient") reduction from an arbitrary function to the remote OT function, and then gives a simple protocol for the remote OT function. While the resulting protocol turns out to be suboptimal for most functions, Theorem 7 shows that the protocol that [1] used for REMOTE OT itself is optimal.
- (ii) Separating Secure and Insecure Computation. A basic question of secure computation is whether it needs more bits to be communicated to the user who wants to learn the output than the input-size itself (which suffices for insecure computation). While natural to expect, it is not easy to prove this. [1], in their restricted model⁷, showed a non-explicit result, that for securely computing *most* Boolean functions on the domain $\{0, 1\}^n \times \{0, 1\}^n$, Charlie is required to receive at least 3n 4 bits, which is significantly more than the 2n bits sufficient for insecure computation.

REMOTE $\binom{2}{1}$ -OTⁿ₂ from above already gives us an explicit example of a function where this is true: the total input size is 2n+1, but the communication is at least $H(M_{31})+$ $H(M_{23}) \geq 3n + 1$. To present an easy comparison to the lower bound of [1], we can consider a symmetrized variant of REMOTE $\binom{2}{1}$ -OT $^{n}_{2}$, in which two instances of REMOTE $\binom{2}{1}$ -OTⁿ₂ are combined, one in each direction. More specifically, $X = (A_0, A_1, a)$ where A_0, A_1 are of length (n-1)/2 (for an odd n) and a is a single bit; similarly $Y = (B_0, B_1, b)$; the output of the function is defined as an (n-1) bit string $f(X,Y) = (A_b, B_a)$. Considering (say) the uniform input distribution over X, Y, the bounds for REMOTE $\binom{2}{1}$ -OTⁿ₂ add up to give us $H(M_{31}) \ge 3(n-1)/2+1$ and $H(M_{23}) \ge 3(n-1)/2+1$, so that the communication with Charlie is lower-bounded by $H(M_{31}) + H(M_{23}) \ge 3n - 1$.

This compares favourably with the bound of [1] in many ways: our lower bound holds even in a model that allows interaction; in particular, this makes the gap between insecure computation (n - 1) bits in our case, 2n bits for [1]) and secure computation (about 3n bits for both) somewhat larger. More importantly, our lower bound is explicit (and tight for the specific function we use), whereas that of [1] is existential. However, our bound does not subsume that of [1], who considered *Boolean* functions. Our results do not yield a bound significantly larger than the input size, when the output is a single bit. It appears that this regime is more amenable to combinatorial arguments, as pursued in [1], rather than information theoretic arguments. We leave it as a fascinating open problem to obtain tight bounds in this regime, possibly by combining combinatorial and informationtheoretic approaches.

2. Secure Computation of GROUP-ADD: Let \mathbb{G} be a (possibly non-abelian) group with binary operation +. The function GROUP-ADD is defined as follows: Alice has an input $X \in \mathbb{G}$, Bob has an input $Y \in \mathbb{G}$ and Charlie should get Z = f(X, Y) = X + Y.

In Figure 5, we recapitulate a well-known simple protocol for securely computing the above function. The protocol requires a $|\mathbb{G}|$ -ary symbol to be exchanged per computation over each link. We show below that this protocol is a communication-ideal as well as randomness-optimal for any input distribution with full support. As mentioned in Footnote 3, this is easy to see for the uniform distribution, and using distribution switching, we can see that the same holds as long as the input distribution has full support.

Algorithm 2: Secure Computation of GROUP-ADD Require: Alice & Bob have input $X, Y \in \mathbb{G}$, respectively. Ensure: Charlie securely computes Z = X + Y.

- 1: Charlie samples a uniformly distributed element K from \mathbb{G} using his private randomness; sends it to Bob as $M_{\vec{32}} = K$.
- 2: Bob sends $M_{\vec{21}} = Y + M_{\vec{32}}$ to Alice.
- 3: Alice sends $\tilde{M}_{\vec{13}} = X + \tilde{M}_{\vec{21}}$ to Charlie.
- 4: Charlie outputs $Z = M_{\vec{13}} K$.

Fig. 5 An optimal protocol for secure computation in any group \mathbb{G} . The protocol requires a $|\mathbb{G}|$ -ary symbol to be exchanged over each link.

Theorem 8. Any secure protocol for computing in a Group \mathbb{G} , where p_{XY} has full support over $\mathbb{G} \times \mathbb{G}$, must satisfy

$$R_{12}, R_{23}, R_{31}, \rho \ge \log |\mathbb{G}|.$$

Hence the protocol in Figure 5 is optimal.

Proof: It is easy to see that the above function satisfies Condition 1 and Condition 2 of Lemma 5. We will only need the last terms (corresponding to the naïve bounds H(X', Y''|Z'') etc., but with distribution switching) of (24), (27), and (28) for $H(M_{12})$, $H(M_{31})$, and $H(M_{23})$, respectively. Since we are computing a deterministic function, and Y can be determined from (X, Z), the last terms in each of

 $^{^{7}}$ Recall that the model of [1] can be thought of as our protocol model with the following restrictions: (i) Alice and Bob share a common random variable independent of their inputs, but are otherwise unable communicate with each other. (ii) Alice and Bob may send only one message each to Charlie who may not send any messages to the other users.

$$H(M_{12}) \ge \max_{p_{X'Y''}} H(X'|Z''),$$

$$H(M_{31}) \ge \max_{p_{X'Y''}} H(X'|Y''),$$

$$H(M_{23}) \ge \max_{p_{X'Y'}} H(Y'|X'').$$

The optimum bounds for M_{12} , M_{31} , and M_{23} are obtained by evaluating all the expressions above with product distributions, where each random variable is uniformly distributed over \mathbb{G} ; this gives $H(M_{12}), H(M_{31}), H(M_{23}) \ge \log |\mathbb{G}|$.

Finally, from (40) and the above bound on $H(M_{12})$, we get $\rho \ge H(M_{12}) \ge \log |\mathbb{G}|$, which implies that the above protocol is randomness-optimal.

3. Secure Computation of CONTROLLED ERASURE: The controlled erasure function is defined as follows: Alice and Bob have bits X and Y, respectively. Alice's input X acts as the "control", which decides whether Charlie receives an erasure (Δ) or Bob's input Y.

$$\begin{array}{c|c} & y \\ \hline x & 0 & 1 \\ \hline 0 & \Delta & \Delta \\ 1 & 0 & 1 \end{array}$$

Notice that Charlie always finds out Alice's control bit, but does not learn Bob's bit when it is erased. This function does not satisfy Condition 1 of Lemma 5.

Figure 6 gives a protocol for securely computing this function on each location of strings of length n. Bob sends his input string to Charlie under the cover of a one-time pad and reveals the key used to Alice. Alice sends his input to Charlie compressed using a Huffman code (replaced by Lempel-Ziv if we want the protocol to be distribution independent). He also sends to Charlie those key bits he received from Bob that corresponds to the locations where there is no erasure (i.e., where his input bit is 1). When $X \sim \text{Bernoulli}(p)$ and $Y \sim$ Bernoulli(q), i.i.d., where $p, q \in (0, 1)$, the expected message length for Alice-Charlie link is $\mathbb{E}[L_{31}] < nH_2(p) + 1 + np$, the messages lengths on the other two links are deterministically n each, $L_{12} = L_{23} = n$. Here we prove the optimality of this protocol for $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Bernoulli}(q)$, where $p, q \in (0, 1)$. We also prove that this protocol is randomnessoptimal.

Theorem 9. Any secure protocol for computing CONTROLLED ERASURE with inputs X^n, Y^n , where $(X_i, Y_i) \sim p_{XY}$, i.i.d., has full support, and induced $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Bernoulli}(q)$ with $p, q \in (0, 1)$, must satisfy

$$R_{31} \ge n(H_2(p) + p), \quad R_{12}, R_{23}, \rho \ge n$$

Hence the protocol in Figure 6 is optimal.

Proof: It is easy to see that this function satisfies only Condition 2 of Lemma 5, which implies RI(Y; Z) = I(Y; Z); but Condition 1 of Lemma 5 is not satisfied – in fact RI(X; Z) = 0. Our best bounds for $H(M_{31})$ and $H(M_{23})$ are given by (22) and (28), respectively. The bottom row of

Algorithm 3: Secure Computation of CONTROLLED ERASURE

Require: Alice & Bob have input bits $X^n, Y^n \in \{0, 1\}^n$. **Ensure:** Charlie securely computes the CONTROLLED ERA-

SURE function

$$Z_i = f(X_i, Y_i), \qquad i = 1, \dots, n.$$

- Bob samples n i.i.d. uniformly distributed bits Kⁿ from his private randomness; sends it to Alice as M_{21,1} = Kⁿ. Bob sends to Charlie his input Yⁿ masked (bit-wise) with Kⁿ as M_{23,1} = Yⁿ ⊕ Kⁿ.
- 2: Alice sends his input X^n to Charlie compressed using a Huffman code (or Lempel-Ziv if we want the protocol to not depend on the input distribution of X^n); let $c(X^n)$ be the codeword. Alice also sends to Charlie the sequence of key bits K_i corresponding to the locations where his input X_i is 1.

$$M_{12,2} = c(X^n), (K_i)_{i:X_i=1}$$

3: Charlie outputs

$$Z_i = \begin{cases} \Delta & \text{if } X_i = 0, \\ (Y_i \oplus K_i) \oplus K_i & \text{if } X_i = 1. \end{cases}$$

Fig. 6 A protocol to compute CONTROLLED ERASURE function. For $X \sim$ Bernoulli(p) and $Y \sim$ Bernoulli(q), both i.i.d and $p, q \in (0, 1)$, the expected message lengths are $\mathbb{E}[L_{31}] < n(H_2(p) + p) + 1$, $L_{12} = n$, and $L_{23} = n$. We show that these are asymptotically optimal by showing the following lower bounds: $H(M_{31}) \ge n(H_2(p) + p)$, $H(M_{12}) \ge n$, and $H(M_{23}) \ge n$.

(22) simplifies to the following:

$$H(M_{31}) \ge \max_{p_{X^n Y'^n}} I(Y'^n; Z'^n) + H(X^n | Y'^n).$$

The optimum bound for $H(M_{31})$ is obtained by taking Y'^n , i.i.d., Bernoulli(1/2), independent of X^n ; this gives $H(M_{31}) \ge n(p+H_2(p))$. For $H(M_{23})$, the bottom row of (28) simplifies to the following:

$$H(M_{23}) \ge \max_{p_{X'^n} p_{Y'^n}} H({Y'^n} | {X'^n}).$$

Taking Y'^n , i.i.d., Bernoulli(1/2), independent of X'^n gives $H(M_{23}) \ge n$. For $H(M_{12})$, putting RI(X';Z') = 0 in the bottom row of (24) and simplifying further, we get the following:

$$H(M_{12}) \ge \max_{p_{X''^n Y'^n}} I(Y'^n; Z''^n) + H(X''^n, Y'^n | Z''^n).$$

Since X'' is a function of Z'' for CONTROLLED-ERASURE, the above expression simplifies to $H(M_{12}) \ge \sup_{p_{Y'n}} H(Y'^n)$, which gives $H(M_{12}) \ge n$ by taking Y'^n , i.i.d., Bernoulli(1/2).

Finally, from (40) and the above bound on $H(M_{12})$, we get $\rho \ge H(M_{12}) \ge n$, which implies that the above protocol is randomness-optimal.

4. Secure Computation of SUM: The SUM function is defined as follows: Alice and Bob have one bit input $X \in \{0,1\}$ and $Y \in \{0,1\}$, respectively, and Charlie wants to compute the arithmetic sum Z = f(X,Y) = X + Y. Figure 7 recapitulates a simple protocol for this function. This protocol requires a ternary symbol to be exchanged

per computation over each link. We show in below that our bounds give $H(M_{31}), H(M_{23}) \ge \log(3)$ and $H(M_{12}) \ge$ 1.5. Thus, while the protocol matches the lower bound on $H(M_{31})$ and $H(M_{23})$, there is a gap for $H(M_{12})$; while the protocol requires $H(M_{12}) = \log(3)$, the lower bound is only $H(M_{12}) \ge 1.5$. We also show that this protocol is randomness-optimal, which proves a recent conjecture of [29] for three users. For $U, V \in \{0, 1, 2\}$, we write U+V to denote the addition modulo-3.

Algorithm 4: Secure Computation of SUM

Require: Alice and Bob have input $X, Y \in \{0, 1\}$, respectively.

Ensure: Charlie securely computes SUM Z = X + Y.

- 1: Charlie samples a uniformly distributed element K from $\{0, 1, 2\}$ using his private randomness; sends it to Alice as $M_{3\vec{1}} = K$.
- 2: Alice sends $M_{\vec{12}} = M_{\vec{31}} + X$ to Bob.
- 3: Bob sends $M_{23} = M_{12} + Y$ to Charlie.
- 4: Charlie outputs $Z = \overline{M}_{23} K$.

Fig. 7 A protocol to compute SUM. The protocol requires a ternary symbol to be exchanged over all the three links. We show a lower bound of $\log(3)$ both on Alice-Charlie and Bob-Charlie links and a lower bound of 1.5 on Alice-Bob link.

Theorem 10. Any secure protocol for computing SUM, where p_{XY} has full support over $\{0,1\} \times \{0,1\}$, must satisfy

$$R_{31}, R_{23}, \rho \geq \log(3)$$
 and $R_{12} \geq 1.5$.

Proof: It is easy to see that SUM satisfies Condition 1 and Condition 2 of Lemma 5, which implies RI(Y; Z) = I(Y; Z)and RI(Z; X) = I(Z; X). Since X can be determined from (Y, Z) and Y can be determined from (X, Z), the bottom rows of the bounds in (27) and (28) for $H(M_{31})$ and $H(M_{23})$, respectively, simplify to the following:

$$H(M_{31}) \ge \max_{p_{X'Y'}} H(Z'),$$

 $H(M_{23}) \ge \max_{p_{X'Y'}} H(Z').$

For $H(M_{31})$, taking $p_{X'Y'}(0,0) = p_{X'Y'}(1,1) = 1/3$ and $p_{X'Y'}(0,1) = p_{X'Y'}(1,0) = 1/6$ gives $H(M_{31}), H(M_{23}) \ge \log(3)$. For $H(M_{12})$, the bound in top row in (24) simplifies to

$$H(M_{12}) \ge \max_{p_{X'}p_{Y'}} (H(Z') - H(X')) + \max_{p_{X'Y''}} H(X').$$

Taking $X', Y' \sim \text{Bern}(1/2)$ gives $H(M_{12}) \ge 1.5$.

For the SUM function the first bound in Theorem 6 simplifies to $\rho \ge H(M_{31})$. This together with the above bound on $H(M_{31})$ gives $\rho \ge \log(3)$, which implies randomness-optimality of the above protocol.

5. Secure Computation of AND: The AND function is defined as follows: Alice and Bob have one bit input $X \in \{0, 1\}$ and $Y \in \{0, 1\}$, respectively, and Charlie wants to compute $Z = f(X, Y) = X \land Y$. The best known protocol for AND first appeared in [1], and we recapitulate it here in Figure 8 (rephrased as a protocol in our model). This protocol requires

a ternary symbol to be exchanged over Alice-Charlie (13) and Bob-Charlie (23) links, and symbols from an alphabet of size 6 over the Alice-Bob (12) link. We show in below that our bounds give $H(M_{31}), H(M_{23}) \ge \log(3)$ and $H(M_{12}), \rho \ge$ 1.826. Thus, while the protocol matches the lower bound on $H(M_{31})$ and $H(M_{23})$, there is a gap for $H(M_{12})$ and ρ ; while the protocol requires $H(M_{12}) = \rho = \log(6) \approx 2.585$, the lower bound is only $H(M_{12}), \rho \ge 1.826$.

Algorithm 5: Secure Computation of AND

Require: Alice has an input bit X & Bob has a bit Y. **Ensure:** Charlie securely computes the AND $Z = X \wedge Y$.

- Alice samples a uniform random permutation (α, β, γ) of (0, 1, 2) from her private randomness; sends it to Bob M₁₂ = (α, β, γ) (using a symbol from an alphabet of size 6).
- 2: Alice sends α to Charlie if X = 1, and β if X = 0. Bob sends α to Charlie if Y = 1 and γ if Y = 1.

$$M_{31} = \begin{cases} \alpha & \text{if } X = 1, \\ \beta & \text{if } X = 0, \end{cases} \qquad M_{23} = \begin{cases} \alpha & \text{if } Y = 1, \\ \gamma & \text{if } Y = 0. \end{cases}$$

3: Charlie outputs Z = 1 if $M_{31} = M_{23}$, and 0 otherwise.

Theorem 11. Any secure protocol for computing AND for inputs X and Y, where p_{XY} has full support over $\{0,1\} \times \{0,1\}$, must satisfy

$$R_{31}, R_{23} \ge \log(3)$$
 and $R_{12}, \rho \ge 1.826$.

Proof: It is easy to see that AND satisfies Condition 1 and Condition 2 of Lemma 5, which implies RI(Y; Z) = I(Y; Z) and RI(Z; X) = I(Z; X). For $H(M_{31})$ and $H(M_{23})$ the bottom rows of (27) and (28) simplify to the following:

$$H(M_{31}) \ge \max_{p_{X'Y'}} I(Y'; Z') + H(X', Z'|Y'),$$

$$H(M_{23}) \ge \max_{p_{X'Y'}} I(X'; Z') + H(Y', Z'|X').$$

For $H(M_{31})$, take $p_{X'Y'}(0,0) = p_{X'Y'}(1,0) = p_{X'Y'}(1,1) = (1-\epsilon)/3$ and $p_{X'Y'}(0,1) = \epsilon$, where $\epsilon > 0$ can be made arbitrarily small to make $H(M_{31})$ as close to $\log(3)$ as we desire. For $H(M_{23})$, take $p_{X'Y'}(0,0) = p_{X'Y'}(0,1) = p_{X'Y'}(1,1) = (1-\epsilon)/3$ and $p_{X'Y'}(1,0) = \epsilon$, where $\epsilon > 0$ can be made arbitrarily small to make $H(M_{23})$ as close to $\log(3)$ as we desire.

For $H(M_{12})$, the top row of (24) simplifies to

$$H(M_{12}) \ge \max_{p_{X'}p_{Y'}} I(Y'; Z') + \max_{p_{X'Y''}} I(X'; Z'') + H(X', Y''|Z'').$$

The expression after the second maximum simplifies to $H(X') + p_{X'}(0)H(Y''|X' = 0)$, which is always upperbounded by $H(X') + p_{X'}(0)$ and can be made equal to this by taking $Y'' \sim$ Bernoulli(1/2) and independent of X'. Now taking $p_{X'}(1) = 0.456$ and $p_{Y'}(1) = 0.397$ gives $H(M_{12}) \ge 1.826$.

Finally, from (40) and the above bound on $H(M_{12})$, we get $\rho \ge H(M_{12}) \ge 1.826$, whereas the protocol requires $1 + \log 3 \approx 2.585$ random bits.

Here we explicitly show, through the above example AND, the progressive improvements on the communication lower bounds from applying Theorem 1 to Theorem 3. Let X, Y be i.i.d. binary random variables distributed uniformly in $\{0, 1\}$. For the secure computation of AND for this input distribution, Theorem 1 gives $(R_{31}, R_{23}, R_{12}) \ge (1.311, 1.311, 1.5)$, Theorem 2 gives $(R_{31}, R_{23}, R_{12}) \ge (1.5, 1.5, 1.826)$, and Theorem 3 gives $(R_{31}, R_{23}, R_{12}) \ge (\log(3), \log(3), 1.826)$. Notice that in Theorem 3 we only improve bounds on R_{31}, R_{23} over Theorem 2.

Separating Secure Computation and Secret Sharing: Another natural separation one expects is between the amount of communication needed when the views (or transcripts) are generated by a secure computation protocol, versus when they are generated by an omniscient "dealer" so that the security requirements are met. The latter setting corresponds to the share sizes in a CMSS scheme (see Appendix B). Again, while such a separation is expected, it is not very easy to establish this, especially with explicit examples. It requires us to establish a strong lower bound for the secure computation problem as well as provide a CMSS scheme that is better.

We establish the separation using the 3-user AND function. There is a CMSS scheme that achieves $log(3) \leq 1.6$ bits of entropy for all three shares M_{12}, M_{23} , and M_{31} (see Theorem 19 in Appendix B). However, Theorem 11 shows that in a secure computation protocol, $H(M_{12})$ should be strictly larger than this.

Note: We need the use of Lemma 6 (information inequality) only to improve the bound on $H(M_{12})$, in particular for SUM, REMOTE-OT, and AND. Bounds on the other two links in all the functions above do not need the use of information inequality.

IV. OUTER BOUNDS ON THE RATE-REGION FOR ASYMPTOTICALLY SECURE COMPUTATION

In this section we restrict ourselves to the secure computation of deterministic functions $f: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, i.e., where $p_{Z|XY}$ is a deterministic mapping of the inputs to the output. We consider block-wise computation (with block length n), where Alice has input $X^n = (X_1, X_2, \ldots, X_n)$, Bob has input $Y^n = (Y_1, Y_2, \ldots, Y_n)$, and Charlie wants to compute the output $Z^n = (Z_1, Z_2, \ldots, Z_n)$, where $Z_i = f(X_i, Y_i)$ and $(X_i, Y_i) \sim p_{XY}$, i.i.d.; see Figure 9. Protocol is allowed to be asymptotically secure, i.e., it can make an error in the function computation – Charlie may produce an output \hat{Z}^n such that $\Pr[\hat{Z}^n \neq Z^n] \to 0$ as $n \to \infty$, and it allows for small information leakage; see Definition 3 for a formal definition. Recall that for a protocol Π_n , we define the rate quadruple $(R_{12}, R_{23}, R_{31}, \rho)$ as $R_{ij} := \frac{1}{n} \mathbb{E}[L_{ij}], i, j = 1, 2, 3, i \neq j$, and $\rho := \frac{1}{n} H(M_{12}, M_{23}, M_{31}|X^n, Y^n)$.

Definition 3. For a secure computation problem (f, p_{XY}) , the rate $(R_{12}, R_{23}, R_{31}, \rho)$ is achievable with asymptotic security, if there exists a sequence of protocols Π_n with rate

Fig. 9 A setup for 3-user secure computation; privacy is required against single users (i.e., no collusion). Here $(X^n, Y^n) \sim p_{XY}$, i.i.d., and Z = f(X, Y) where f is the function being computed.

 $(R_{12}, R_{23}, R_{31}, \rho)$, such that for every $\epsilon > 0$, there is a large enough n, such that

 $\Pr[\hat{Z}^n \neq Z^n] \le \epsilon, \tag{41}$

$$I(M_{12}, M_{31}; Y^n | X^n) \le \epsilon,$$
 (42)

$$I(M_{12}, M_{23}; X^n | Y^n) \le \epsilon,$$
 (43)

$$I(M_{23}, M_{31}; X^n, Y^n | Z^n) \le \epsilon.$$
 (44)

The rate-region \mathcal{R}^{AS} is closure of the set of all achievable rate quadruples.

Here, (42)-(43) ensure that Alice and Bob learn negligible additional information about each other's inputs, and (44) ensures that Charlie learns negligible additional information about (X^n, Y^n) than revealed by Z^n .

We prove bounds on the entropies $H(M_{ij})$, which, as argued in Section II, is a lower bound on the expected length of the transcript M_{ij} . Let Π_n be a sequence of protocols which imply the achievability of rate $(R_{12}, R_{23}, R_{31}, \rho)$ as per Definition 3. Then, let $\epsilon > 0$ and n large enough be such that (41)-(44) are satisfied.

We need three key lemmas: Lemma 7, using cutset arguments, gives an upper bound on the amount of information present about inputs on different cuts; Lemma 10 gives a secure data-processing inequality for residual information; and Lemma 6, which gives an information inequality for interactive protocols.

Our main results provide outer bounds on the rate-region for secure computation of a given function f and an input distribution p_{XY} . The results are stated in Theorem 12 and Theorem 13. In Section IV-D, we present some example functions for which our outer bounds are tight.

A. Cutset Bounds

Our first lemma will imply that the cut separating Alice must reveal information about X^n at a rate of at least $H_{G_X}(X|Y)$ (see Figure 10). The rough intuition is that since Alice is not allowed to learn any (significant amount of) new information about Bob's input Y^n , this is essentially the function computation problem with one-sided communication of Orlitsky and Roche [15] for which the converse result there implies that Alice must send information about X^n at a rate of at least $H_{G_X}(X|Y)$.

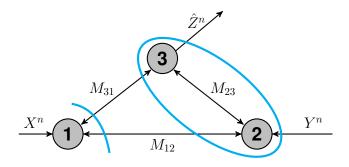


Fig. 10 A cut separating Alice from Bob & Charlie. Protocol Π_n induces a 2-user secure computation protocol between Alice and *combined Bob-Charlie* with privacy requirement only against Alice.

Lemma 7. The protocol Π_n satisfies the following:

$$H(X^n|M_{12}, M_{31}) \le n(H(X) - H_{G_X}(X|Y) + \epsilon_1),$$
 (45)

$$H(Y^{n}|M_{12}, M_{23}) \le n(H(Y) - H_{G_{Y}}(Y|X) + \epsilon_{2}), \quad (46)$$

$$H(Z^n | M_{23}, M_{31}) \le n\epsilon_3, \tag{47}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \rightarrow 0$ as $\epsilon \rightarrow 0$.

Remark 3. Note that this is *unlike* the case for perfectly secure computation. By Lemma 3, which is analogous to the above lemma for perfectly secure computation (if we restrict $p_{Z|XY}$ there to be a deterministic function f such that the pair (p_{XY}, f) is in normal form), all three conditional entropies on the left-hand-side above are equal to zero. However, for asymptotically secure computation (even if we restrict the pair (p_{XY}, f) to be in normal form), these conditional entropies may be far from zero – in fact, (45)-(46) above can hold with equality asymptotically (see Section IV-D).

$$Proof:$$

$$H(X^{n}|M_{12}, M_{31})$$

$$= I(X^{n}; Y^{n}|M_{12}, M_{31}) + H(X^{n}|M_{12}, M_{31}, Y^{n})$$

$$\leq I(M_{12}, M_{31}, X^{n}; Y^{n}) + H(X^{n}|M_{12}, M_{31}, Y^{n})$$

$$= I(X^{n}; Y^{n}) + \underbrace{I(M_{12}, M_{31}; Y^{n}|X^{n})}_{\leq \epsilon \text{ by } (42)} + H(X^{n}|M_{12}, M_{31}, Y^{n})$$

$$\leq nI(X; Y) + H(X^{n}|Y^{n}) - I(X^{n}; M_{12}M_{31}|Y^{n}) + \epsilon$$

$$\leq nH(X) - I(X^{n}; M_{12}M_{31}|Y^{n}) + \epsilon.$$
(48)

We apply cutset arguments, and use correctness (41) and privacy against Alice (42) to lower-bound the second term of (48). Consider the cut separating Alice from the other two users; Π_n induces a two-user secure computation protocol between Alice and *combined Bob-Charlie* (see Figure 10), with privacy requirement only against Alice. For $0 \le D \le 1$, we define

$$R_f^{WZ}(D) := \min_{\substack{p_U|_{XY}:\\U-X-Y\\ \exists g: \mathbb{E}[d_H(f(X,Y),g(U,Y))] \le D}} I(U;X|Y), \quad (49)$$

where $d_H : \mathcal{U} \times \mathcal{Y} \to \{0,1\}$ is the Hamming distortion function. $R_f^{WZ}(D)$ is the optimal rate of Csiszár-Körner's [45] extension (also see [46]) of Wyner-Ziv problem [47] specialized to this function computation (without any privacy) as used by Orlitsky and Roche [15]. **Lemma 8.** $I(X^n; M_{12}, M_{31}|Y^n) \ge n \left(R_f^{WZ}(0) - \delta_{\epsilon} \right)$, where $\delta_{\epsilon} \to 0$ as $\epsilon \to 0$.

The proof of the above lemma in Appendix C is along the lines of the converse of the Wyner-Ziv theorem in [48, Section 11.3] except for the following complication: in the Wyner-Ziv problem, communication is one-sided, but we allow messages in both directions over multiple rounds. However, as we show in Appendix C, privacy against Alice, $I(M_{12}, M_{31}; Y^n | X^n) \leq \epsilon$, which implies that very little new information about Y^n flows back to Alice, allows us to prove the lemma.

We can relate $R_f^{WZ}(D)$ with *conditional graph entropy* $H_{G_X}(X|Y)$ (defined in Definition 1) using the following result from [15].

Lemma 9 ([15], Theorem 2). For every p_{XY} and f

$$R_f^{WZ}(0) = H_{G_X}(X|Y).$$

From Lemma 8 and Lemma 9 we get the following:

$$I(X^{n}; M_{12}, M_{31}|Y^{n}) \ge n(H_{G_{X}}(X|Y) - \delta_{\epsilon}),$$
 (50)

where $\delta_{\epsilon} \to 0$ as $\epsilon \to 0$.

From (48) and (50) we get $H(X^n|M_{12}, M_{31}) \le n(H(X) - H_{G_X}(X|Y) + \epsilon_1)$, where $\epsilon_1 = \epsilon + \delta_{\epsilon}$, which proves (45). Similarly, by considering the cut separating Bob from Alice and Charlie, we can show the following (which proves (46)):

$$I(Y^{n}; M_{12}, M_{23} | X^{n}) \ge n(H_{G_{Y}}(Y | X) - \delta_{\epsilon}).$$
(51)

Applying Fano's inequality gives (47) as follows:

$$H(Z^{n}|M_{23}, M_{31}) \stackrel{\text{(b)}}{=} H(Z^{n}|M_{23}, M_{31}, \hat{Z}^{n})$$
$$\leq H(Z^{n}|\hat{Z}^{n}) \leq 1 + \epsilon(n \log |\mathcal{Z}| - 1) \leq n\epsilon_{3}.$$

where (b) follows from the Markov chain $\hat{Z}^n - (M_{23}, M_{31}) - (X^n, Y^n, M_{12})$, and $\epsilon_3 \to 0$ as $\epsilon \to 0$.

B. Asymptotically secure Data Processing Inequality

To prove Theorem 12 and Theorem 13, we need to prove an asymptotic version of the secure data processing inequality of Lemma 1.

Lemma 10 (Asymptotically secure data processing inequality). *Privacy conditions* (42)-(44) *imply the following:*

$$\frac{1}{n}RI(M_{12}, M_{31}, X^n; M_{23}, M_{31}, Z^n) \ge RI(X; Z) - \epsilon_{31},$$

$$\frac{1}{n}RI(M_{12}, M_{23}, Y^n; M_{23}, M_{31}, Z^n) \ge RI(Y; Z) - \epsilon_{23},$$

$$\frac{1}{n}RI(M_{12}, M_{31}, X^n; M_{12}, M_{23}, Y^n) \ge RI(X; Y) - \epsilon_{12},$$

where $\epsilon_{12}, \epsilon_{23}, \epsilon_{31} \rightarrow 0$ as $\epsilon \rightarrow 0$.

We prove this Lemma in Appendix C. Our proof only uses "weak" privacy constraints, i.e., the privacy conditions are upper bounded by $n\epsilon$, as opposed to the "strong" ones in (42)-(44) which are upper bounded by ϵ .

C. Main Lower Bounds

Theorem 12. For a secure computation problem (f, p_{XY}) , if $(R_{12}, R_{23}, R_{31}, \rho) \in \mathcal{R}^{AS}$, then

$$\begin{split} R_{12} &\geq H_{G_X}(X|Y) + H_{G_Y}(Y|X) - H(Z) \\ &\quad + \max\{RI(X;Z), RI(Y;Z)\}, \\ R_{23} &\geq H_{G_Y}(Y|X) + RI(X;Z), \\ R_{31} &\geq H_{G_X}(X|Y) + RI(Y;Z), \\ \rho &\geq H_{G_X}(X|Y) + H_{G_Y}(Y|X) - H(Z) \\ &\quad + RI(X;Z) + RI(Y;Z), \\ \rho &\geq H_{G_X}(X|Y) + H_{G_Y}(Y|X) - I(X;Y) \\ &\quad + RI(X;Y) + \max\{RI(X;Z), RI(Y;Z)\} - H(Z). \end{split}$$

Proof:

$$H(M_{12}) \ge H(M_{12}|M_{31})$$

$$= H(M_{12}|M_{31}, M_{23}) + I(M_{12}; M_{23}|M_{31})$$

$$= H(M_{12}, X^n|M_{31}, M_{23}) - H(X^n|M_{12}, M_{31}, M_{23})$$

$$+ I(M_{12}, X^n; M_{23}|M_{31}) - I(X^n; M_{23}|M_{12}, M_{31})$$

$$= H(M_{12}, X^n|M_{31}, M_{23}) - H(X^n|M_{12}, M_{31})$$

$$+ I(M_{12}, X^n; M_{23}|M_{31})$$

$$\geq H(M_{12}, X^n|M_{31}, M_{23}) + I(M_{12}, X^n; M_{23}|M_{31})$$

$$- n(H(X) - H_{G_X}(X|Y) + \epsilon_1),$$
(52)

where (53) follows from (45). The first two terms of (53) can be bounded easily as follows:

$$H(M_{12}, X^{n}|M_{31}, M_{23}) = H(M_{12}, X^{n}, Y^{n}|M_{31}, M_{23}) - H(Y^{n}|M_{12}, M_{23}, M_{31}, X^{n}) \geq H(X^{n}, Y^{n}|M_{31}, M_{23}, Z^{n}) - H(Y^{n}|M_{12}, M_{23}, X^{n}) \geq n(H(X, Y|Z) - \epsilon/n) - n(H(Y|X) - H_{G_{Y}}(Y|X) + \delta_{\epsilon}). = n(H(X) - H(Z) + H_{G_{Y}}(Y|X) - \epsilon/n - \delta_{\epsilon})$$
(54)

In the last inequality we use privacy against Charlie (44) and $H(Y^n|M_{12}, M_{23}, X^n) \leq n(H(Y|X) - H_{G_Y}(Y|X) + \delta_{\epsilon})$ from (51).

$$I(M_{12}, X^{n}; M_{23}|M_{31}) = I(M_{12}, X^{n}; M_{23}, Z^{n}|M_{31}) - \underbrace{I(M_{12}, X^{n}; Z^{n}|M_{23}, M_{31})}_{\leq H(Z^{n}|M_{23}, M_{31}) \leq n\epsilon_{3} \text{ by } (47)}_{\leq RI(M_{12}, M_{31}, X^{n}; M_{23}, M_{31}, Z^{n}) - n\epsilon_{3}}$$
(55)

$$\geq n(RI(X;Z) - \epsilon_{31} - \epsilon_3) \quad \text{(by Lemma 10)}, \tag{56}$$

where (55) follows from the definition of residual information in (2) by taking $U = (M_{12}, M_{31}, X^n)$, $V = (M_{23}, M_{31}, Z^n)$, and $Q = M_{31}$. Substituting from (54) and (56) into (53) and simplifying further, we get:

$$H(M_{12}) \ge n(H_{G_X}(X|Y) + H_{G_Y}(Y|X) - H(Z) + RI(X;Z) - \epsilon'_{12}),$$

where $\epsilon'_{12} = \epsilon/n + \epsilon_1 + \epsilon_3 + \epsilon_{31} + \delta_{\epsilon}$ and $\epsilon'_{12} \to 0$ as $\epsilon \to 0$. By symmetry and letting $\epsilon \downarrow 0$, we have

$$R_{12} \ge H_{G_X}(X|Y) + H_{G_Y}(Y|X) - H(Z) + \max\{RI(X;Z), RI(Y;Z)\}.$$

The remaining bounds on R_{23}, R_{31} and ρ are proved below.

$$\begin{split} &H(M_{23}) \geq H(M_{23}|M_{31}) \\ &= H(M_{23}|M_{12}, M_{31}, X^n) + I(M_{23}; M_{12}, X^n|M_{31}) \\ &= H(M_{23}, Y^n|M_{12}, M_{31}, X^n) - H(Y^n|M_{12}, M_{23}, M_{31}, X^n) \\ &+ I(M_{23}, Z^n; M_{12}, X^n|M_{31}) - I(Z^n; M_{12}, X^n|M_{23}, M_{31}) \\ &\geq \underbrace{H(Y^n|M_{12}, M_{31}, X^n)}_{\geq H(Y^n|X^n) - \epsilon \text{ by } (42)} - \underbrace{H(Y^n|M_{12}, M_{23}, X^n)}_{\leq n(H(Y|X) - H_{G_Y}(Y|X) + \delta_{\epsilon}) \text{ by } (51)} \\ &+ \underbrace{RI(M_{23}, M_{31}, Z^n; M_{12}, M_{31}, X^n)}_{\geq n(RI(Z;X) - \epsilon_{31}) \text{ by Lemma } 10} - \underbrace{H(Z^n|M_{23}, M_{31})}_{\leq n\epsilon_3 \text{ by } (47)} \\ &\geq n(H_{G_Y}(Y|X) + RI(X; Z) - \delta_{23}) \\ &\quad (\text{where } \delta_{23} = \epsilon/n + \epsilon_3 + \epsilon_{31} + \delta_{\epsilon}, \text{ and } \delta_{23} \to 0 \text{ as } \epsilon \to 0) \end{split}$$

By letting $\epsilon \downarrow 0$, we get

$$R_{23} \ge H_{G_Y}(Y|X) + RI(X;Z).$$

Similarly, we can prove the bound on $H(M_{31})$ by first expanding as $H(M_{31}) \ge H(M_{31}|M_{23})$ and then proceed as in $H(M_{23})$. For the rate of private randomness ρ required, we bound $H(M_{12}, M_{23}, M_{31}|X^n, Y^n)$ as follows:

$$n\rho_{n} = H(M_{12}, M_{23}, M_{31}|X^{n}, Y^{n})$$

$$\geq H(M_{12}, M_{31}|X^{n}, Y^{n})$$

$$= H(M_{12}, M_{31}|X^{n}) - \underbrace{I(M_{12}, M_{31}; Y^{n}|X^{n})}_{\leq \epsilon \text{ by } (42)}$$

$$\geq H(M_{12}, M_{31}, X^{n}) - H(X^{n}) - \epsilon.$$
(57)

We bound the first term of (57) as follows:

$$H(M_{12}, M_{31}, X^n) = H(M_{31}) + H(M_{12}|M_{31}) + H(X^n|M_{12}, M_{31}).$$
(58)

We can bound the second and third term of (58) together as follows:

$$H(M_{12}|M_{31}) + H(X^{n}|M_{12}, M_{31})$$

$$\geq H(M_{12}, X^{n}|M_{23}, M_{31}) + I(M_{12}, X^{n}; M_{23}|M_{31})$$

$$\geq n(H(X) - H(Z) + H_{G_{Y}}(Y|X) - \epsilon/n - \delta_{\epsilon})$$

$$+ n(RI(X; Z) - \epsilon_{31} - \epsilon_{3}), \qquad (59)$$

where the first inequality follows from (52); (59) follows from (54) and (56). We lower-bound the first term of (58) as follows:

$$H(M_{31}) \ge H(M_{31}|M_{23})$$

= $H(M_{31}|M_{12}, M_{23}, Y^n) + I(M_{31}; M_{12}, Y^n|M_{23})$
= $H(M_{31}, X^n|M_{12}, M_{23}, Y^n) - H(X^n|M_{12}, M_{23}, M_{31}, Y^n)$
+ $I(M_{31}, Z^n; M_{12}, Y^n|M_{23}) - I(Z^n; M_{12}, Y^n|M_{23}, M_{31})$

$$\stackrel{\text{(a)}}{\geq} \underbrace{H(X^{n}|M_{12}, M_{23}, Y^{n})}_{\geq H(X^{n}|Y^{n}) - \epsilon \text{ by } (43)} - \underbrace{H(X^{n}|M_{12}, M_{31}, Y^{n})}_{\leq n(H(X|Y) - H_{G_{X}}(X|Y) + \delta_{\epsilon})} \text{ by } (50) \\ + RI(M_{23}, M_{31}, Z^{n}; M_{12}, M_{23}, Y^{n}) - H(Z^{n}|M_{23}, M_{31}) \\ \geq n(H_{G_{X}}(X|Y) - \epsilon/n - \delta_{\epsilon}) + n(RI(Y; Z) - \epsilon_{23}) - n\epsilon_{3},$$

$$(60)$$

where, in (a) we use the definition of residual information (2) and simple Shannon information inequalities. In (60) we use

Lemma 10 and (47). From (57)-(60) and letting $\epsilon \downarrow 0$, we get the following:

$$\rho \ge H_{G_X}(X|Y) + H_{G_Y}(Y|X) + RI(X;Z) + RI(Y;Z) - H(Z).$$
(61)

There is another way to bound the first term of (57) as follows:

$$H(M_{12}, M_{31}, X^n) = H(M_{12}) + H(M_{31}|M_{12}) + H(X^n|M_{12}, M_{31}) \\ \ge H(M_{12}|M_{31}) + H(X^n|M_{12}, M_{31}) + H(M_{31}|M_{12}).$$
(62)

The first two terms of (62) can be bounded as in (59). We bound the last term of (62) as follows:

$$\begin{split} H(M_{31}|M_{12}) &= H(M_{31}|M_{12}, M_{23}) + I(M_{31}; M_{23}|M_{12}) \\ &= H(M_{31}|M_{12}, M_{23}, Y^n) + I(M_{31}; Y^n|M_{12}, M_{23}) \\ &\quad + I(M_{31}; M_{23}|M_{12}) \\ &= H(M_{31}, X^n|M_{12}, M_{23}, Y^n) - H(X^n|M_{12}, M_{23}, M_{31}, Y^n) \\ &\quad + I(M_{31}; M_{23}, Y^n|M_{12}) \\ &\geq \underbrace{H(X^n|M_{12}, M_{23}, Y^n)}_{\geq H(X^n|Y^n) - \epsilon \text{ by } (43)} + \underbrace{I(M_{31}, X^n; M_{23}, Y^n|M_{12})}_{\geq n(RI(X;Y) - \epsilon_{12}) \text{ by } (2) \text{ and Lemma 10} \\ - I(X^n; M_{23}, Y^n|M_{12}, M_{31}) - H(X^n|M_{12}, M_{23}, M_{31}, Y^n) \\ &\geq H(X^n|Y^n) - \epsilon + n(RI(X;Y) - \epsilon_{12}) - H(X^n|M_{12}, M_{31}) \\ &\geq n(H_{G_X}(X|Y) + RI(X;Y) - I(X;Y) - \epsilon/n - \epsilon_1 - \epsilon_{12}). \end{split}$$
(63)

Last inequality (63) follows from (45). From (57), (59), (62), (63), and by letting $\epsilon \downarrow 0$, we get the following:

$$\rho \ge H_{G_X}(X|Y) + H_{G_Y}(Y|X) + RI(X;Y) + RI(X;Z) - I(X;Y) - H(Z).$$

By symmetry, we have

$$\rho \ge H_{G_X}(X|Y) + H_{G_Y}(Y|X) + RI(X;Y) + \max\{RI(X;Z), RI(Y;Z)\} - I(X;Y) - H(Z).$$
(64)

(61) and (64) together prove the bounds on ρ in Theorem 12.

If the input distribution p_{XY} is a product distribution, i.e., $p_{XY} = p_X p_Y$, then we can improve Theorem 12. In the case of independent inputs we can assume, without loss of generality, that p_X and p_Y have full support; and as observed earlier, the input distribution having full support allows us to assume, without loss of generality, that the function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ is in normal form (see Section II for details), which implies that the characteristic graphs G_X, G_Y are complete graphs, and therefore, the conditional graph entropies are equal to conditional entropies, i.e., $H_{G_X}(X|Y) = H(X|Y), H_{G_Y}(Y|X) = H(Y|X)$. For independent inputs we have I(X;Y) = 0, and the bounds in Lemma 7 reduce to the following: $H(X^n|M_{12}, M_{31}) \leq n\epsilon_1$, $H(Y^n|M_{12}, M_{23}) \leq n\epsilon_2$, and $H(Z^n|M_{23}, M_{31}) \leq n\epsilon_3$. **Theorem 13.** Consider a secure computation problem $(f, p_X p_Y)$, where p_X and p_Y have full support and f is in normal form. If $(R_{12}, R_{23}, R_{31}, \rho) \in \mathcal{R}^{AS}$, then

$$R_{12} \ge H(X, Y|Z) + RI(X; Z) + RI(Y; Z),$$

$$R_{23} \ge H(Y|X) + RI(X; Z),$$

$$R_{31} \ge H(X|Y) + RI(Y; Z),$$

$$\rho \ge H(X, Y|Z) + RI(X; Z) + RI(Y; Z).$$

Proof: We crucially use the information inequality for interactive protocols (Lemma 6) to improve the bounds in Theorem 12. The improvement is only on R_{12} . The other bounds on R_{23} , R_{31} , ρ can directly be obtained by substituting the conditional graph entropies by conditional entropies in Theorem 12.

$$H(M_{12}) = H(M_{12}|M_{23}, M_{31}) + I(M_{12}; M_{23}, M_{31})$$

= $H(M_{12}|M_{23}, M_{31}) + I(M_{12}; M_{23}) + I(M_{12}; M_{31}|M_{23})$
 $\geq H(M_{12}|M_{23}, M_{31}) + I(M_{12}; M_{23}|M_{31})$
+ $I(M_{12}; M_{31}|M_{23})$ (by Lemma 6) (65)

We bound the first term of (65) from below as follows:

$$H(M_{12}|M_{23}, M_{31}) = H(M_{12}, X^n|M_{23}, M_{31}) - \underbrace{H(X^n|M_{12}, M_{23}, M_{31})}_{\leq H(X^n|M_{12}, M_{31}) \leq n\epsilon_1} \\ \geq H(M_{12}, X^n, Y^n|M_{23}, M_{31}) - \underbrace{H(Y^n|M_{12}, M_{23}, M_{31}, X^n)}_{\leq H(Y^n|M_{12}, M_{23}) \leq n\epsilon_2} \\ - n\epsilon_1$$

$$\geq H(X^{n}, Y^{n} | M_{23}, M_{31}, Z^{n}) - n(\epsilon_{1} + \epsilon_{2})$$

$$\geq n(H(X^{n}, Y^{n} | Z^{n}) - \epsilon/n - \epsilon_{1} - \epsilon_{2}) \quad \text{by (44)}$$
(66)

We bound the second term of (65) from below as follows:

$$I(M_{12}; M_{23}|M_{31}) = I(M_{12}, X^n; M_{23}|M_{31}) - \underbrace{I(X^n; M_{23}|M_{12}, M_{31})}_{\leq H(X^n|M_{12}, M_{31}) \leq n\epsilon_1}$$

$$\geq I(M_{12}, X^n; M_{23}, Z^n|M_{31}) - \underbrace{I(M_{12}, X^n; Z^n|M_{23}, M_{31})}_{\leq H(Z^n|M_{23}, M_{31}) \leq n\epsilon_3}$$

$$- n\epsilon_1$$

$$\stackrel{\text{(a)}}{\geq} RI(M_{12}, M_{31}, X^n; M_{23}, M_{31}, Z^n) - n(\epsilon_1 + \epsilon_3) \\ \geq n(RI(X; Z) - \epsilon_1 - \epsilon_3 - \epsilon_{31}), \quad \text{(by Lemma 10)}$$
(67)

 $\langle - \rangle$

where (a) follows from the definition of residual information (2), by taking $U = (M_{12}, M_{31}, X^n)$, $V = (M_{23}, M_{31}, Z^n)$, and $Q = M_{31}$. Similarly, we can bound the third term of (65) as follows:

$$I(M_{12}; M_{31}|M_{23}) \ge n(RI(Y; Z) - \epsilon_2 - \epsilon_3 - \epsilon_{23}).$$
 (68)

Substituting the values from (66)-(68) into (65), we get

$$H(M_{12}) \ge n(H(X, Y|Z) + RI(X; Z) + RI(Y; Z) - \gamma_{12}),$$

where $\gamma_{12} = \epsilon/n + 2(\epsilon_1 + \epsilon_2 + \epsilon_3) + \epsilon_{31} + \epsilon_{23}$, and $\gamma_{12} \to 0$ as $\epsilon \to 0$. By letting $\epsilon \downarrow 0$, we have

$$R_{12} \ge H(X, Y|Z) + RI(X; Z) + RI(Y; Z)$$

Remark 4. The observation in Remark 2 on the fact that working with normal form may not be without loss of generality as far as randomness requirement is concerned even if p_{XY} has full support holds here as well.

D. Application to Specific Functions

In this section we present some examples and show that secure protocols for some of these achieve the optimal rate-region.

1. Secure computation of ADDITION **in a finite field:** Let $(\mathbb{F}, +, \times)$ be a finite field and $(X, Y) \sim p_{XY}$, where p_{XY} is a joint distribution over \mathbb{F}^2 . The function ADDITION is defined as follows: Z = X + Y, where + is performed in \mathbb{F} . Our protocol uses the following fact about data compression of a discrete memoryless source.

Fact 1 [49]. Let U^n be a sequence of n i.i.d. random variables, each distributed over \mathbb{F} . For fix $\epsilon > 0$, let $R = H(U)/\log |\mathbb{F}| + \epsilon$, then there is a sequence of linear encoder and decoder pairs (A_n, D_n) , where $A_n \in \mathbb{F}^{nR \times n}$ and $D_n : \mathbb{F}^{nR} \to \mathbb{F}^n$, such that $\Pr[D_n(A_n U^n) \neq U^n] \to 0$ as $n \to \infty$.

The protocol in Figure 11 is a secure version of Körner-Marton scheme [50] in finite fields. All the arithmetic is in \mathbb{F} . The protocol requires $n\rho = |M_{12}| = |M_{23}| = |M_{31}| = n(H(Z) + \epsilon)$ (in bits). Below we show that if p_{XY} is a product distribution, i.e., $p_{XY} = p_X p_Y$, then this protocol achieves the optimal rate-region.

Algorithm 6: Secure Computation of ADDITION in a finite field

Require: Alice & Bob have input vectors $X^n, Y^n \in \mathbb{F}^n$. **Ensure:** Charlie securely computes \hat{Z}^n with $\Pr[\hat{Z}^n \neq Z^n] \rightarrow 0$, where $Z^n = (Z_1, Z_2, \dots, Z_n), Z_i = X_i + Y_i$.

- For fix ε, let R = H(Z)/log |F| + ε and A_n be a nR×n matrix in F whose existence is ensured by the fact 1. Alice and Bob share K ~ Unif(F^{nR}) over 1-2 link.
- 2: Alice sends $M_{\vec{13}} := A_n X^n + K$ (component-wise addition) to Charlie.
- 3: Bob sends $M_{\vec{23}} := A_n Y^n K$ to Charlie.
- 4: Charlie computes $M_{13} + M_{23}$, which is equal to $A_n Z^n$, and recovers Z^n with high probability.

Fig. 11 An optimal protocol for block-wise secure computation of ADDITION in any finite field \mathbb{F} . The protocol requires roughly nH(Z) bits to be exchanged on average over each link.

Theorem 14. For any secure protocol for computing ADDI-TION in a finite field \mathbb{F} for independent $X \sim p_X$, $Y \sim p_Y$ with full support, we have the following optimal bound on the rate-region:

$$R_{12}, R_{23}, R_{31}, \rho \ge H(Z).$$

Proof: For this function with $p_X p_Y$ having full support, we have RI(X;Z) = I(X;Z) and RI(Y;Z) = I(Y;Z). For a product distribution, i.e., $p_{XY} = p_X p_Y$, it can be verified easily that the all four bounds on $R_{12}, R_{23}, R_{31}, \rho$ in Theorem 13 reduce to H(Z) (in bits), thereby achieving the optimal rate-region. Note that the converse of the optimality of this protocol needs full force of Lemma 6.

Separating Perfectly and Asymptotically Secure Computation: Note that the result of Theorem 8 in Section III-D also holds when restricted to independent inputs taking values in finite fields. Comparing that with the result of above Theorem 14 establishes a gap in the rate regions of perfectly secure computation and asymptotically secure computation.

We give a tight characterization of the rate-region of this function only for independent input distributions, and we leave it as an interesting open problem to characterize the rate-region of this function for arbitrary p_{XY} . For arbitrary p_{XY} our bounds in Theorem 12 reduce to ρ , R_{23} , $R_{31} \ge H(Z)$ but $R_{12} \ge \max\{H(X|Y), H(Y|X))\}$. In general, the bound on R_{12} does not match what our protocol achieves. But for the special case of the joint distribution of Körner-Marton [50]: $p_{XY}(x,y) = \frac{p}{2} \mathbf{1}_{x \neq y} + \frac{1-p}{2} \mathbf{1}_{x = y}$, where $x, y \in \{0, 1\}$ and $0 \le p \le 1/2$, we have H(X|Y) = H(Y|X) = H(Z). This distribution is sometimes referred to as the doubly symmetric binary source (DSBS) with parameter p. Thus the secure computation of modular addition in a binary field with DSBS source requires $R_{12}, R_{23}, R_{31}, \rho \ge H(Z)$, which is also an example with dependent inputs that separates perfect secure computation from asymptotically secure computation.

2. Secure computation of CONTROLLED-ERASURE: We again study the controlled erasure function from Section III-D here in the asymptotic setting. In this function Alice and Bob have one bit input X and Y, respectively, where Alice's input X acts as the "control" which decides whether Charlie receives an erasure (Δ) or Bob's input Y.

Let $(X, Y) \sim p_{XY}$, where p_{XY} is a joint distribution over $\{0, 1\}^2$ with marginal distributions $X \sim \text{Bern}(p)$ and $Y \sim \text{Bern}(q)$, $p, q \in (0, 1)$. The protocol in Figure 12 requires $\mathbb{E}[L_{31}] < n(H_2(p) + p) + 1$, $|M_{12}| = n, \rho = n$, and

$$|M_{23}| = nH(Y \oplus K|G, X) = n(pH(Y|X = 1) + (1-p) \cdot 1)$$

= $n(H(Y|X) + (1-p)(1-H(Y|X = 0))).$

Theorem 15. For any secure protocol for computing CONTROLLED-ERASURE function with $(X, Y) \sim p_{XY}$, $X \sim$ Bern(p) and $Y \sim Bern(q)$, $p, q \in (0, 1)$, we have the following bound on the rate region:

$$R_{12}, R_{23}, \rho \ge H(Y|X),$$

 $R_{31} \ge H(X) + pH(Y|X = 1)$

If X and Y are independent and q = 1/2 (irrespective of the value of p), we have ρ , R_{12} , $R_{23} \ge n$ and $R_{31} \ge H_2(p) + p$, achieving the optimal rate region.

Proof: For this function and p_{XY} having full support, we have RI(X; Z) = 0 and RI(Y; Z) = I(Y; Z), bounds in Theorem 12 reduce to the following:

$$R_{12}, R_{23}, \rho \ge H(Y|X),$$

 $R_{31} \ge H(X) + pH(Y|X = 1).$

Algorithm 7: Secure Computation of CONTROLLED ERASURE Require: Alice & Bob have input vectors $X^n, Y^n \in \{0, 1\}^n$

- with $(X, Y) \sim p_{XY}$; let $X \sim \text{Bern}(p)$ and $Y \sim \text{Bern}(q)$. **Ensure:** Charlie securely computes \hat{Z}^n with $\Pr[\hat{Z}^n \neq Z^n] \rightarrow$
- 0, where Z_i , i = 1, ..., n. is the CONTROLLED-ERASURE function of X_i, Y_i .
- 1: Alice and Bob share n random bits K^n over 1-2 link.
- 2: Alice sends $M_{13} := (C(X^n), (K_i)_{i \in \{j: X_j = 1\}})$ to Charlie, where $C(X^n)$ is the Huffman compression of X^n .
- 3: Let

$$G_i = \begin{cases} K_i & \text{if } X_i = 1 \\ \bot & \text{if } X_i = 0 \end{cases}$$

Charlie decodes $C(X^n)$ to get X^n and obtains $(i)_{i \in \{j:X_j=1\}}$, which, together with the second component of M_{13} gives $G^n = (G_1, G_2, \ldots, G_n)$. Since Charlie has (G^n, X^n) , by Slepian-Wolf theorem [48, Section 10.3], Bob only needs to send at rate $H(Y \oplus K|G, X)$ for Charlie to recover $Y^n \oplus K^n$ with high probability.

- 4: Now, having access to $Y^n \oplus K^n$ and G^n , Charlie can recover Z^n .
- Fig. 12 A protocol to securely compute CONTROLLED ERASURE function asymptotically. For $(X, Y) \sim p_{XY}$ with $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Bernoulli}(q)$, both i.i.d and $0 < p, q \le 1/2$.

It can be verified easily that for independent X, Y and q = 1/2 (irrespective of the value of p), the lower bounds match the protocol requirements, thereby achieving the optimal rate-region.

V. CONCLUSION

In this work we presented generic lower bounds on communication and randomness for perfectly and asymptotically secure 3-user computation, and showed that they yield tight bounds for some interesting examples. However, the general problem of obtaining tight lower bounds for communication and randomness complexity of secure computation remains open.

For perfectly secure computation, the standard upper bound on the total communication exchanged between all three users is linear in the size of the circuit computing the function [9], [10]. This implication to circuit lower bounds presents a "barrier" to obtaining super-linear bounds for explicit functions since circuit complexity lower bounds are notoriously difficult [51, Chapter 23]. We propose a possibly easier open problem: do there exist Boolean functions with super-linear communication complexity for secure computation? Note that lower bounds on circuit complexity do not directly translate to lower bounds on communication complexity of secure computation, as established by a sub-exponential upper bound of $2^{O(\sqrt{n})}$ for the latter [52]. Though it is plausible that for random Boolean functions, the actual communication cost is $2^{\Omega(n^{\epsilon})}$ for some $\epsilon > 0$, none of the current techniques appear capable of delivering such a result. Another interesting problem we leave open is to find an explicit example for a Boolean function in which the total communication to Charlie must be significantly larger than the total input size. Note that [1] gave an existential result (in their restricted model) and the explicit example in this work does not have Boolean output.

For asymptotically secure computation, the only generic feasibility results are the ones that were developed for standard (statistically or perfectly) secure computation, like the ones by Ben-Or, Goldwasser, and Wigderson [9] or Chaum, Crépeau, and Damgård [10]. In light of the result of Section IV-D, where we establish a gap between the communication and randomness requirements of perfectly secure computation and asymptotically secure computation, it is plausible that there are generic protocols for asymptotically secure computation with lower communication and randomness requirements than possible for standard secure computation.

We presented a new information inequality for 3-user interactive protocols, which was instrumental in obtaining our strongest bounds. This inequality requires users to have independent inputs in the beginning. It would be interesting to generalize this to settings where the inputs may be dependent. More generally, proving information inequalities for interactive protocols in larger networks is also of independent interest and might prove useful in establishing strong communication lower bounds in multiuser setting.

Two other directions we leave as important open directions are to develop communication and randomness lower bounds for secure multiparty computation involving more than 3 parties, and to obtain stronger lower bounds for security against active corruption than in the honest-but-curious setting (when computation is feasible in both models; indeed, it is well-known that general secure computation against active corruption is not possible when 1 out of 3 parties can be actively corrupted). There has been some prior work in the first direction as mentioned in Section I, these results have been mostly only for the modular addition function. While some of our techniques can be extended to more than 3 parties, we will need entirely new techniques for separating the communication requirements of the honest-but-curious and active corruption settings.

APPENDIX A Details Omitted from Section III

Proof of Lemma 3: Fix a protocol II. First we show $H(X|M_{12}, M_{13}) = 0$. We apply a cut-set argument. Consider the cut isolating Alice from Bob & Charlie. We need to show that for every m_{12}, m_{31} with $p(m_{12}, m_{31}) > 0$, there is a (necessarily unique) $x \in \mathcal{X}$ such that $p(x|m_{12}, m_{31}) = 1$. Suppose, to the contrary, that we have a secure protocol resulting in a p.m.f. $p(x, y, z, m_{12}, m_{31})$ such that there exists $x, x' \in \mathcal{X}, x \neq x'$, and m_{12}, m_{31} satisfying $p(m_{12}, m_{31}) > 0$, $p(x|m_{12}, m_{31}) > 0$, and $p(x'|m_{12}, m_{31}) > 0$. For these x, x', since $(p_{XY}, p_{Z|XY})$ is in the normal form, $\exists (y, z) \in \mathcal{Y} \times \mathcal{Z}$ such that $p_{XY}(x, y) > 0, p_{XY}(x', y) > 0$, and $p_{Z|X,Y}(z|x, y) \neq p_{Z|X,Y}(z|x', y)$.

- (i) The definition of a protocol implies that $p(x, y, z, m_{12}, m_{31})$ can be written as $p_{X,Y}(x, y)p(m_{12}, m_{31}|x, y)p(z|m_{12}, m_{31}, y)$.
- (ii) Privacy against Alice implies that $p(m_{12}, m_{31}|x, y, z) = p(m_{12}, m_{31}|x)$.

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- (iii) Using (ii) in (i), we get $p(x, y, z, m_{12}, m_{31}) = p_{X,Y}(x, y)p(m_{12}, m_{31}|x)p(z|m_{12}, m_{31}, y).$
- (iv) Correctness and (ii) imply that we can also write $p(x, y, z, m_{12}, m_{31}) = p_{X,Y}(x, y)p_{Z|X,Y}(z|x, y)p(m_{12}, m_{31}|x).$
- (v) Since $p_{X,Y}(x,y)p(m_{12},m_{31}|x) > 0$, from (iii) and (iv), we get $p(z|m_{12},m_{31},y) = p_{Z|X,Y}(z|x,y)$.

Applying the above arguments to $(x', y, z, m_{12}, m_{31})$ we get $p(z|m_{12}, m_{31}, y) = p_{Z|X,Y}(z|x', y)$, leading to the contradiction $p(z|m_{12}, m_{31}, y) \neq p(z|m_{12}, m_{31}, y)$, since by assumption $p_{Z|X,Y}(z|x, y) \neq p_{Z|X,Y}(z|x', y)$.

Similarly, by considering the cut separating Bob from Alice & Charlie we can show (7).

To show (8), i.e., $H(Z|M_{23}, M_{31}) = 0$, we need to show that for every m_{23}, m_{31} with $p(m_{23}, m_{31}) > 0$, there is a (necessarily unique) $z \in Z$ such that $p(z|m_{23}, m_{31}) = 1$. Suppose, to the contrary, that we have a secure protocol resulting in a p.m.f. $p(x, y, z, m_{23}, m_{31})$ such that there exists $z, z' \in Z, z \neq z'$ and m_{23}, m_{31} satisfying $p(m_{23}, m_{31}) > 0$, $p(z|m_{23}, m_{31}), p(z'|m_{23}, m_{31}) > 0$. By the assumption that $(p_{XY}, p_{Z|XY})$ is in normal form, there exists (x, y) s.t. $p_{XY}(x, y) > 0$ and $p_{Z|X,Y}(z|x, y) > 0$.

- (i) The definition of a protocol implies that $p(x, y, z, m_{23}, m_{31})$ can be written as $p_{X,Y}(x, y)p(m_{23}, m_{31}|x, y)p(z|m_{23}, m_{31})$.
- (ii) Privacy against Charlie implies that $p(x, y, z, m_{23}, m_{31})$ can be written as $p_{X,Y}(x, y)p(z|x, y)p(m_{23}, m_{31}|z)$.
- (iii) (i), (ii), and correctness give $p(m_{23}, m_{31}|x, y)p(z|m_{23}, m_{31}) = p_{Z|X,Y}(z|x, y)p(m_{23}, m_{31}|z).$

By assumption, $p(m_{23}, m_{31}) > 0$ and $p(z|m_{23}, m_{31}) > 0$, which imply that $p(m_{23}, m_{31}|z)$ 0. And since > $p_{Z|X,Y}(z|x,y)$ >0, we have from (iii) that $p(m_{23}, m_{31}|x, y) > 0$. Now consider (x, y, z'). By assumption, $p(m_{23}, m_{31}) > 0$ and $p(z'|m_{23}, m_{31}) > 0$, which imply $p(m_{23}, m_{31}|z') > 0$. Since $p(m_{23}, m_{31}|x, y) > 0$, running the same steps (i)-(iii) as above with $(x, y, z', m_{23}, m_{31})$, (iii) implies that $p_{Z|X,Y}(z'|x, y) > 0$. Define $\alpha \triangleq \frac{p(z|x,y)}{p(z'|x,y)}$. Since $(p_{XY}, p_{Z|XY})$ is in normal form, $\exists (x',y') \in (\mathcal{X},\mathcal{Y})$ s.t. $p_{XY}(x',y') > 0$ and $p_{Z|X,Y}(z|x',y') \neq \alpha \cdot p_{Z|X,Y}(z'|x',y')$. Since $\alpha \neq 0$, at least one of p(z|x', y') or p(z'|x', y') is non-zero. Assume that any one of these is non-zero, then applying the above arguments will give us that the other one should also be non-zero.

- (iv) Repeating the steps (i)-(iii) with $(x, y, z', m_{23}, m_{31})$ yields $p(m_{23}, m_{31}|x, y)p(z'|m_{23}, m_{31}) = p_{Z|X,Y}(z'|x, y)p(m_{23}, m_{31}|z').$
- (v) Dividing the expression in (iii) by the expression in (iv) gives $\frac{p(z|m_{23},m_{31})}{p(z'|m_{23},m_{31})} = \alpha \cdot \frac{p(m_{23},m_{31}|z)}{p(m_{23},m_{31}|z')}.$
- (vi) Repeating (i)-(v) for $(x', y', z, m_{23}, m_{31})$ and $(x', y', z', m_{23}, m_{31})$, we get $\frac{p(z|m_{23}, m_{31})}{p(z'|m_{23}, m_{31}|z')} \neq \alpha \cdot \frac{p(m_{23}, m_{31}|z)}{p(m_{23}, m_{31}|z')}$, which contradicts (v).

APPENDIX B

CONNECTIONS TO SECURE SAMPLING AND CORRELATED MULTI-SECRET SHARING

Secure Sampling. In *secure sampling* functionalities, none of the users receives any input, but all three users produce outputs. The functionality is specified by a joint distribution p_{XYZ} , and the protocol for sampling p_{XYZ} is specified by $\Pi(p_{XYZ})$. The correctness condition in this case is that the outputs of Alice, Bob, and Charlie are distributed according to p_{XYZ} . The security conditions remain the same as in the case of secure computation, that is, none of the users can infer anything about the other users' outputs other than what they can from their own outputs.

A Normal Form for p_{XYZ} . For a joint distribution p_{XYZ} , define the relation $x \sim x'$ for $x, x' \in \mathcal{X}$ to hold if $\exists c \geq 0$ such that $\forall y \in \mathcal{Y}, z \in \mathcal{Z}, p(x', y, z) = c \cdot p(x, y, z)$. Similarly, we define $y \sim y'$ for $y, y' \in \mathcal{Y}$ and $z \sim z'$ for $z, z' \in \mathcal{Z}$. We say that p_{XYZ} is in the normal form if $x \sim x' \Rightarrow x = x'$, $y \sim y' \Rightarrow y = y'$, and $z \sim z' \Rightarrow z = z'$.

It is easy to see that one can transform any distribution p_{XYZ} to one in normal form $p_{X'Y'Z'}$, with possibly smaller alphabets, so that any secure sampling protocol for the former can be transformed to one for the latter with the same communication costs, and vice versa. To define X', X is modified by removing all x such that p(x) = 0 and then replacing all x in an equivalence class of \sim with a single representative; Y' and Z' are defined similarly. The modification to the protocol, in either direction, is for each user to locally map X to X' etc., or vice versa. Hence it is enough to study the communication complexity of securely sampling distributions in the normal form.

Now, we show an analog of Lemma 3 for secure sampling protocols.

Lemma 11. Suppose p_{XYZ} is in normal form. Then, in any secure sampling protocol $\Pi(p_{XYZ})$, the cut isolating Alice from Bob and Charlie must determine Alice's output X, i.e., $H(X|M_{12}, M_{31}) = 0$. Similarly, $H(Y|M_{12}, M_{23}) = 0$ and $H(Z|M_{23}, M_{31}) = 0$.

Proof: We only prove $H(X|M_{12}, M_{31}) = 0$; the other ones, i.e., $H(Y|M_{12}, M_{23}) = 0$ and $H(Z|M_{23}, M_{31}) = 0$ can be proved similarly. We need to show that for every m_{12}, m_{31} with $p(m_{12}, m_{31}) > 0$, there is a (necessarily unique) $x \in \mathcal{X}$ such that $p(x|m_{12}, m_{31}) = 1$. Suppose, to the contrary, that we have a secure sampling protocol resulting in a p.m.f. $p(x, y, z, m_{12}, m_{31})$ such that there exists $x, x' \in \mathcal{X}, x \neq x'$ and m_{12}, m_{31} satisfying $p(m_{12}, m_{31}) > 0$, $p(x|m_{12}, m_{31}) > 0$, and $p(x'|m_{12}, m_{31}) > 0$. Since $p(m_{12}, m_{31}) > 0$ and $p(x|m_{12}, m_{31}) > 0$ imply $p_X(x) > 0$, there exists (y, z) s.t. $p_{XYZ}(x, y, z) > 0$.

- (i) The definition of a protocol implies that $p(x, y, z, m_{12}, m_{31})$ can be written as $p_{YZ}(y, z)p(m_{12}, m_{31}|y, z)p(x|m_{12}, m_{31}).$
- (ii) Privacy against Alice implies that $p(x, y, z, m_{12}, m_{31})$ can be written as $p_{XYZ}(x, y, z)p(m_{12}, m_{31}|x)$.
- (iii) (i) and (ii) gives $p_{YZ}(y, z)p(m_{12}, m_{31}|y, z)p(x|m_{12}, m_{31})$ = $p_{XYZ}(x, y, z)p(m_{12}, m_{31}|x)$.

By assumption, $p(m_{12}, m_{31}) > 0$ and $p(x|m_{12}, m_{31}) > 0$, which imply that $p(m_{12}, m_{31}|x) > 0$. And since $p_{XYZ}(x, y, z) > 0$, we have from (iii) that $p(m_{12}, m_{31}|y, z) > 0$. Now consider $(x', y, z, m_{12}, m_{31})$. By assumption, $p(m_{12}, m_{31}) > 0$ and $p(x'|m_{12}, m_{31}) > 0$, which imply $p(m_{12}, m_{31}|x') > 0$. Since $p(m_{12}, m_{31}|y, z) > 0$ from above, (iii) implies that $p_{XYZ}(x', y, z) > 0$. Define $\alpha \triangleq \frac{p(x, y, z)}{p(x', y, z)}$. Since p_{XYZ} is in normal form, $\exists (y', z') \in (\mathcal{Y}, \mathcal{Z})$ s.t. $p_{XYZ}(x, y', z') \neq \alpha \cdot p_{XYZ}(x', y', z')$. Since $\alpha \neq 0$, at least one of p(x, y', z') or p(x', y', z') is non-zero. Assume that any one of these is non-zero, then applying the above arguments will give us that the other one should also be non-zero.

- (iv) Repeating the steps (i)-(iii) with $(x', y, z, m_{12}, m_{31})$ yields $p_{YZ}(y, z)p(m_{12}, m_{31}|y, z)p(x'|m_{12}, m_{31}) = p_{XYZ}(x', y, z)p(m_{12}, m_{31}|x').$
- (v) Dividing the expression in (iii) by the expression in (iv) gives $\frac{p(x|m_{12},m_{31})}{p(x'|m_{12},m_{31})} = \alpha \cdot \frac{p(m_{12},m_{31}|x)}{p(m_{12},m_{31}|x')}$.
- (vi) Repeating (i)-(v) for $(x, y', z', m_{12}, m_{31})$ and $(x', y', z', m_{12}, m_{31})$, we get $\frac{p(x|m_{12}, m_{31})}{p(x'|m_{12}, m_{31})} \neq \alpha \cdot \frac{p(m_{12}, m_{31}|x)}{p(m_{12}, m_{31}|x')}$, which contradicts (v).

Theorem 16. Any secure sampling protocol $\Pi(p_{XYZ})$, where p_{XYZ} is in normal form, should satisfy the following lower bounds on the entropy of the transcripts on each link.

$$H(M_{23}) \ge RI(X;Z) + RI(X;Y) + H(Y,Z|X), H(M_{31}) \ge RI(Y;Z) + RI(X;Y) + H(X,Z|Y), H(M_{12}) \ge RI(X;Z) + RI(Y;Z) + H(X,Y|Z).$$

Proof: From Lemma 11 we have $H(X|M_{12}, M_{31}) = 0$, $H(Y|M_{12}, M_{23}) = 0$, and $H(Z|M_{23}, M_{31}) = 0$. Note that we can apply Lemma 6 for secure sampling of *dependent* X, Y, and Z, because, in the beginning users only have independent randomness, but no inputs. In the end, they output from a joint distribution p_{XYZ} , where X, Y and Z may be dependent, but this does not affect the requirements of Lemma 6 in any way. The proof for $H(M_{23})$ is given below; the other two bounds follows similarly.

$$\begin{split} H(M_{31}) &= I(M_{12}; M_{31}) + H(M_{31}|M_{12}) \\ &= I(M_{12}; M_{31}) + I(M_{31}; M_{23}|M_{12}) + H(M_{31}|M_{12}, M_{23}) \\ \stackrel{(a)}{\geq} I(M_{12}; M_{31}|M_{23}) + I(M_{31}; M_{23}|M_{12}) \\ &\quad + H(M_{31}|M_{12}, M_{23}) \\ \stackrel{(b)}{\geq} RI(Y; Z) + RI(X; Y) + H(X, Z|Y), \end{split}$$

where (a) used $I(M_{12}; M_{31}) \ge I(M_{12}; M_{31}|M_{23})$, which follows from Lemma 6; (b) used $I(M_{12}; M_{31}|M_{23}) \ge RI(Y; Z)$, $I(M_{31}; M_{23}|M_{12}) \ge RI(X; Y)$, and $H(M_{31}|M_{12}, M_{23}) \ge H(X, Z|Y)$, all of which we have shown in the proof of Theorem 1.

We remark that if the marginal distributions satisfy $p_{XY} = p_X p_Y$ (i.e., X and Y are independent), then a secure computation protocol for $p_{Z|XY}$ can be turned into a secure sampling protocol (with the same communication costs), by having Alice and Bob locally sample inputs X and Y according

Correlated Multi-Secret Sharing Schemes. We define a notion of secret-sharing, called Correlated Multi-Secret Sharing (CMSS) that is closely related to secure sampling/computation problem. We will show that lower bounds on the entropy of shares of such secret-sharing schemes will also be lower bounds on entropy of transcripts for the corresponding secure computation protocols. However, we shall show a separation between the efficiency of secret-sharing (where there is an omniscient dealer) and a protocol, using the stronger lower bounds we have established in Section III-B.

Definition 4. Given a graph G = (V, E), an adversary structure $\mathcal{A} \subseteq 2^V$, and a joint distribution $p_{(X_v)_{v \in V}}$ over random variables X_v indexed by $v \in V$, a correlated multiple secret sharing scheme for $(G, p_{(X_v)_{v \in V}})$ defines a distribution $p_{(M_e)_{e \in E}|(X_v)_{v \in V}}$ of shares M_e for each edge $e \in E$, such that the following hold. Below, for $S \subseteq E$, M_S stands for the collection of all M_e for $e \in S$; similarly X_T is defined for $T \subseteq V$; $E_v \subseteq E$ denotes the set of edges incident on a vertex V.

- Correctness: For all $v \in V$, $H(X_v|M_{E_v}) = 0$.
- Privacy: For every set $T \in \mathcal{A}$, let $E_T = \bigcup_{v \in T} E_v$; then, $I(X_{\overline{T}}; M_{E_T} | X_T) = 0.$

Below we give a specialised version of the above general definition which is suitable to our setting, where G is the clique over the vertex set $V = \{1, 2, 3\}$, and $\mathcal{A} = \{\{1\}, \{2\}, \{3\}\}$ (corresponding to 1-privacy).

We define Σ to be a correlated multi-secret sharing scheme for a joint distribution p_{XYZ} (with respect to our fixed adversary structures) if it probabilistically maps secrets (X, Y, Z)to shares M_{12}, M_{23}, M_{31} such that the following conditions hold:

- Correctness: $H(X|M_{12}, M_{31}) = H(Y|M_{12}, M_{23}) = H(Z|M_{23}, M_{31}) = 0.$
- Privacy:
 - $$\begin{split} &I(M_{12},M_{31};Y,Z|X)=0 \quad (\text{privacy against Alice}), \\ &I(M_{12},M_{23};X,Z|Y)=0 \quad (\text{privacy against Bob}), \\ &I(M_{23},M_{31};X,Y|Z)=0 \quad (\text{privacy against Charlie}). \end{split}$$

We point out that while the correctness condition relates only to the supports of X, Y, and Z individually, the privacy condition is crucially influenced by the joint distribution.

Theorem 17. Any CMSS scheme for any joint distribution p_{XYZ} satisfies

$$H(M_{12}) \ge \max\{RI(X;Z), RI(Y;Z)\} + H(X,Y|Z), H(M_{23}) \ge \max\{RI(X;Z), RI(X;Y)\} + H(Y,Z|X), H(M_{31}) \ge \max\{RI(Y;Z), RI(X;Y)\} + H(X,Z|Y).$$

Proof: We proceed along the lines of the proof of Theorem 1, except that here we do not need Lemma 3 to argue that $H(X|M_{12}, M_{31}) = H(Y|M_{12}, M_{23}) = H(Z|M_{23}, M_{31}) = 0$, instead, these follow from the correctness of CMSS.

If $p_{XYZ} = p_{XY}p_{Z|XY}$, where p_{XY} has full support and $p_{Z|XY}$ is in normal form, using Lemma 3, the bounds in Theorem 17 imply bounds in Theorem 1. If p_{XYZ} has full support, then we can further strengthen the bounds in Theorem 17 by applying distribution switching.

Theorem 18. Consider any CMSS scheme for a joint distribution p_{XYZ} , where p_{XYZ} has full support.

$$H(M_{12}) \ge \max \left\{ \begin{array}{c} \max_{p_{X'Y'Z'}} RI(X';Z') + H(X',Y'|Z'), \\ \max_{p_{X'Y'Z'}} RI(Y';Z') + H(X',Y'|Z') \end{array} \right\}$$

where $p_{X'Y'Z'}$ is any distribution for which the characteristic bipartite graph of $p_{X'Y'}$ is connected.

$$H(M_{23}) \ge \max \left\{ \begin{array}{c} \max_{p_{X'Y'Z'}} RI(X';Z') + H(Y',Z'|X'), \\ \max_{p_{X'Y'Z'}} RI(X';Y') + H(Y',Z'|X') \end{array} \right\}$$

where $p_{X'Y'Z'}$ is any distribution for which the characteristic bipartite graph of $p_{Y'Z'}$ is connected.

$$H(M_{31}) \ge \max \left\{ \begin{array}{c} \max_{p_{X'Y'Z'}} RI(Y';Z') + H(X',Z'|Y'), \\ \max_{p_{X'Y'Z'}} RI(X';Y') + H(X',Z'|Y') \end{array} \right\}$$

where $p_{X'Y'Z'}$ is any distribution for which the characteristic bipartite graph of $p_{X'Z'}$ is connected.

Proof: First we observe that we can apply distribution switching to CMSS schemes also, i.e., if we have a CMSS $\Sigma(p_{XYZ})$, where p_{XYZ} has full support, it will remain a CMSS if we change the distribution to a different one $p_{X'Y'Z'}$. This follows from the correctness and privacy conditions of a CMSS. Proceeding as in the proof of Lemma 5, we can show that for any CMSS $\Sigma(p_{XYZ})$, connectedness of the characteristic bipartite graph of p_{XY} implies $I(X, Y, Z; M_{12}) = 0$. The other two, i.e., connectedness of the characteristic bipartite graph of p_{XZ} implies $I(X, Y, Z; M_{31}) = 0$, and connectedness of the characteristic bipartite graph of p_{YZ} implies $I(X, Y, Z; M_{23}) = 0$, follow similarly. Now, we can apply the distribution switching to the bounds in Theorem 17.

It is easy to see that any secure sampling protocol $\Pi(p_{XYZ})$, where p_{XYZ} is in normal form, yields a CMSS scheme for the same joint distribution p_{XYZ} : An omniscient dealer can always produce the shares M_{12}, M_{23}, M_{31} which are precisely the transcripts produced by the secure sampling protocol. Now, correctness for this CMSS follows from Lemma 11, and privacy of CMSS scheme follows from the privacy of the secure sampling protocol. Thus the lower bounds on the transcripts produced by a CMSS scheme for a given p_{XYZ} in normal form, gives lower bounds on the corresponding links for any secure sampling protocol for this p_{XYZ} . Furthermore, if $p_{XYZ} = p_{XY}p_{Y|XY}$, where p_{XY} has full support and $p_{Z|XY}$ is in normal form, then lower bounds for CMSS schemes provide lower bounds for secure computation problems. As we discuss in page 17, these lower bounds are not tight in general for secure computation, i.e., there is a function (in fact the AND function) for which there is a CMSS scheme which requires less communication than what our lower bounds for secure computation for that function provide. Towards this, here we

give upper bounds on the share sizes of a 3-user CMSS for AND, which is defined as X and Y independent and uniformly distributed bits, and $Z = X \wedge Y$.

Theorem 19. For p_{XYZ} such that X and Y independent and uniformly distributed bits, and $Z = X \wedge Y$, there is a CMSS $\Sigma(p_{XYZ})$ which has $H(M_{12}) = H(M_{23}) = H(M_{31}) = \log(3)$.

Proof: Consider a CMSS scheme Σ defined as follows. Let (α, β, γ) be a random permutation of the set $\{0, 1, 2\}$. Let $M_{12} = \alpha$ and

$$M_{31} = \begin{cases} \alpha & \text{if } X = 1, \\ \beta & \text{if } X = 0, \end{cases} \qquad M_{23} = \begin{cases} \alpha & \text{if } Y = 1, \\ \gamma & \text{if } Y = 0. \end{cases}$$

It can be seen that this scheme satisfies the correctness and privacy requirements (in particular, (M_{12}, M_{31}) is uniformly random, conditioned on $M_{12} = M_{31}$ when X = 1 and conditioned on $M_{12} \neq M_{31}$ when X = 0). $H(M_{12}^{\Sigma}) = H(M_{23}^{\Sigma}) = H(M_{31}^{\Sigma}) = \log 3 < 1.585$.

Theorem 18 implies that this scheme is optimal.

APPENDIX C

DETAILS OMITTED FROM SECTION IV

Proof of Lemma 8: We define the following function:

$$R_f(\delta, D) := \min_{\substack{p_U \mid XY:\\I(U;Y|X) \le \delta\\ \exists g: \mathbb{E}[d_H(f(X,Y),g(U,Y))] \le D}} I(U;X|Y), \quad (69)$$

where d_H is the Hamming distortion function. Note that $R_f^{WZ}(D) = R_f(0, D)$. Now define the rate-region tradeoff corresponding to (69) as follows:

$$\mathcal{T}_{R_f}(X,Y) = \{ (R_1, R_2, D) : \exists p_{U|XY} \text{ and a function } g, \\ \text{for which } I(U;X|Y) \leq R_1, I(U;Y|X) \leq R_2, \\ \text{and } \mathbb{E}[d_H(f(X,Y), g(U,Y))] \leq D \}.$$
(70)

Lemma 12. $\mathcal{T}_{R_f}(X, Y)$, as defined in (70), is a closed and convex set. Hence, $R_f(\delta, D)$ is convex in (δ, D) .

Proof: Closedness: Let \mathcal{P}_{XY} denote the set of all conditional p.m.f.'s $p_{U|XY}$. Since \mathcal{X} and \mathcal{Y} are finite alphabets, it follows from the Fenchel-Eggleston's strengthening of Carathéodory's theorem [53, pg. 310], that we can restrict the alphabet size of U s.t. $|\mathcal{U}| \leq |\mathcal{X}| \cdot |\mathcal{Y}| + 2$. This implies that \mathcal{P}_{XY} is a compact set (since it is closed and bounded). For a fixed p_{XY} , consider the following function:

$$m(p_{U|XY}) = (I(U; X|Y), I(U; Y|X), \\ \min_{a} \mathbb{E}[d_H(f(X, Y), g(U, Y)]).$$
(71)

Note that $\mathcal{T}_{R_f}(X, Y)$ is the increasing hull of $\operatorname{image}(m)$ – image of the function m – where increasing hull of a set $S \subseteq \mathbb{R}^3$ is defined as $\{(a, b, c) \in \mathbb{R}^3 : \exists (a', b', c') \in S \text{ s.t. } a' \leq a, b' \leq b, \text{ and } c' \leq c\}$. Since the increasing hull of a closed set is always closed, it is enough to show that $\operatorname{image}(m)$ is closed. We show, below, that m is continuous in $p_{U|XY} \in \mathcal{P}_{XY}$; this proves closedness of the set $\operatorname{image}(m)$, since image of a compact set under a continuous function is always compact – and therefore closed. In order to show that $m(p_{U|XY})$ is continuous in $p_{U|XY}$, we need to show that I(U;X|Y), I(U;Y|X), and $\min_g \mathbb{E}[d_H(f(X,Y),g(U,Y)]$ are continuous in $p_{U|XY} \in \mathcal{P}_{XY}$. It is well known that conditional mutual information is a continuous function of the distribution. To show that $\min_g \mathbb{E}[d_H(f(X,Y),g(U,Y)]]$ is continuous in $p_{U|XY}$, it is sufficient to show that $\mathbb{E}[d_H(f(X,Y),g(U,Y)]]$ is continuous for every choice of g. This is because there are only finitely many functions $g: \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}$, min is a continuous function, and composition of two continuous functions is continuous.

Consider $p_{U|XY}, p_{U'|XY} \in \mathcal{P}_{XY}$ such that $\sum_{x,y,u} p_{XY}(x,y) |p_{U|XY}(u|x,y) - p_{U'|XY}(u|x,y)| \leq \gamma$. Since the value of Hamming distortion function d_H is at most 1, it can be easily seen that for every function g, $|\mathbb{E}[d_H(f(X,Y),g(U,Y)] - \mathbb{E}[d_H(f(X,Y),g(U',Y)]| \leq \gamma$. Hence $\mathbb{E}[d_H(f(X,Y),g(U,Y)]$ is continuous in $p_{U|XY}$ for every g.

This proves the closedness of $\mathcal{T}_{R_f}(X, Y)$, which justifies taking min, instead of inf, in the definition of $R_f(\delta, D)$ in (69).

$$I(U; X|Y) = I(\Phi, U_{\Phi}; X|Y) = I(U_{\Phi}; X|Y, \Phi)$$

= $\alpha I(U_1; X|Y) + (1 - \alpha)I(U_2; X|Y)$
 $\leq \alpha R_1^{(0)} + (1 - \alpha)R_1^{(1)}.$

Similarly we can show $I(U; Y|X) \leq \alpha R_2^{(0)} + (1 - \alpha) R_2^{(1)}$. For the third quantity:

$$\mathbb{E}[d_H(f(X,Y),g(U,Y)] = \alpha \mathbb{E}[d_H(f(X,Y),g_0(U_0,Y)] + (1-\alpha)\mathbb{E}[d_H(f(X,Y),g_1(U_1,Y)]] \le \alpha D^{(0)} + (1-\alpha)D^{(1)}.$$

Lemma 13. For a fixed pair (f, p_{XY}) , the function $R_f(\delta, D)$, defined in (69), is right continuous at $(\delta, D) = (0, 0)$.

Proof: This is proved using the property that the region $\mathcal{T}_{R_f}(X, Y)$ is closed. From the definition of $R_f(\delta, D)$ in (69), it is easy to see that it is a non-increasing function of (δ, D) . Now suppose, to the contrary, that $R_f(\delta, D)$ is not right continuous at $(\delta, D) = (0, 0)$. This implies that there exists a monotone decreasing sequence $(\delta_m, D_m) \downarrow 0$ (i.e., $\delta_m \geq \delta_{m+1}$ and $D_m \geq D_{m+1}, \forall m \in \mathbb{N}$, and $\delta_m \downarrow 0, D_m \downarrow 0$) and $\gamma > 0$ s.t. $R_f(\delta_m, D_m) \leq R_f(0, 0) - \gamma$ for all $m \in \mathbb{N}$. As observed earlier, $R_f(\delta_m, D_m)$ is a monotone non-decreasing sequence that is bounded above by $R_f(0, 0)$, which implies that it is convergent (since every monotone non-decreasing

sequence that is bounded above is convergent). Let $L = \lim_{m\to\infty} R_f(\delta_m, D_m)$ be the limit of this sequence. We have $L \leq R_f(0,0) - \gamma < R_f(0,0)$. This contradicts the fact that $R_f(0,0)$ is the minimum value r s.t. $(r,0,0) \in \mathcal{T}_{R_f}(X,Y)$, because $\mathcal{T}_{R_f}(X,Y)$ is closed (i.e., $\mathcal{T}_{R_f}(X,Y)$ contains all its limit points), implying that $(L,0,0) \in \mathcal{T}_{R_f}(X,Y)$.

Now we are ready to lower-bound $I(X^n; M_{12}, M_{31}|Y^n)$. For simplicity of notation, define $M := (M_{12}, M_{31})$. Note that \hat{Z}^n depends on M_{23} in the original problem of Figure 9. The transcript M_{23} and \hat{Z}^n can be sampled, conditioned on (Y^n, M) , by combined Bob-Charlie using additional private randomness Θ , which is independent of (X^n, Y^n, M) . So, we can assume that the *i*-th symbol \hat{Z}_i of the output is determined by a function $g_i(M, Y^n, \Theta)$. Let $U_i := (M, Y^{i-1}, Y^n_{i+1}, \Theta)$, then $\hat{Z}_i = g_i(U_i, Y_i)$.

$$\begin{split} I(X^{n}; M|Y^{n}) &= \sum_{i=1}^{n} H(X_{i}|Y_{i}) - H(X_{i}|Y^{n}, X^{i-1}, M) \quad ((X_{i}, Y_{i})'\text{s are i.i.d.}) \\ &\stackrel{(a)}{\geq} \sum_{i=1}^{n} H(X_{i}|Y_{i}) - H(X_{i}|Y^{n}, M, \Theta) \\ &= \sum_{i=1}^{n} I(U_{i}; X_{i}|Y_{i}) \quad (\text{where } U_{i} = (M, Y^{i-1}, Y_{i+1}^{n}, \Theta)) \\ &\stackrel{(72)}{\geq} \sum_{i=1}^{n} R_{f} \Big(I(U_{i}; Y_{i}|X_{i}), \mathbb{E}[d_{H}(f(X_{i}, Y_{i}), g_{i}(U_{i}, Y_{i}))] \Big) \\ &\stackrel{(c)}{\geq} nR_{f} \Big(\frac{1}{n} \sum_{i=1}^{n} I(U_{i}; Y_{i}|X_{i}), \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[d_{H}(f(X_{i}, Y_{i}), g_{i}(U_{i}, Y_{i}))] \Big) \\ &(73) \end{split}$$

(a) follows from independence of Θ and (M, X^n, Y^n) , and the fact that conditioning reduces entropy; (b) follows by the definition of R_f in (69); (c) follows from the convexity of $R_f(\delta, D)$, proved in Lemma 12. Now, we bound both the arguments of R_f in (73); we use privacy against Alice (42) for the first argument and correctness condition (41) for the second argument.

$$\begin{split} &\sum_{i=1}^{n} I(U_i;Y_i|X_i) = \sum_{i=1}^{n} I(M,Y^{i-1},Y_{i+1}^n,\Theta;Y_i|X_i) \\ &\stackrel{\text{(d)}}{=} \sum_{i=1}^{n} I(M,Y^{i-1},Y_{i+1}^n;Y_i|X_i) \\ &= \sum_{i=1}^{n} \underbrace{H(Y_i|X_i)}_{H(Y_i|X^n,Y^{i-1})} - H(Y_i|X_i,M,Y^{i-1},Y_{i+1}^n) \\ &\leq \sum_{i=1}^{n} H(Y_i|X^n,Y^{i-1}) - H(Y_i|X^n,M,Y^{i-1},Y_{i+1}^n) \\ &= \sum_{i=1}^{n} I(Y_i;M|X^n,Y^{i-1}) + \sum_{i=1}^{n} I(Y_i;Y_{i+1}^n|X^n,M,Y^{i-1}) \\ &\leq \underbrace{I(Y^n;M|X^n)}_{\leq \epsilon, \text{ by } (42)} + \sum_{i=1}^{n} I(Y_i;M,Y^{i-1},Y_{i+1}^n|X^n) \end{split}$$

$$\leq \sum_{i=1}^{n} \underbrace{I(Y_{i}; Y^{i-1}, Y_{i+1}^{n} | X^{n})}_{= 0} + I(Y_{i}; M | X^{n}, Y^{i-1}, Y_{i+1}^{n}) + \epsilon$$

$$= \sum_{i=1}^{n} \underbrace{I(Y^{n}; M | X^{n})}_{\leq \epsilon, \text{ by } (42)} - \underbrace{I(Y^{i-1}, Y_{i+1}^{n}; M | X^{n})}_{\geq 0} + \epsilon$$

$$\leq (n+1)\epsilon$$

$$\leq 2n\epsilon, \qquad (74)$$

where (d) follows from independence of Θ and (M, X^n, Y^n) . For the second argument of (73):

$$\sum_{i=1}^{n} \mathbb{E}[d_H(f(X_i, Y_i), g_i(U_i, Y_i))]$$

$$= \sum_{i=1}^{n} \mathbb{E}[d_H(f(X_i, Y_i), \hat{Z}_i)] \quad (\text{where } \hat{Z}_i = g_i(U_i, Y_i))$$

$$= \sum_{i=1}^{n} \Pr[\hat{Z}_i \neq f(X_i, Y_i)]$$

$$\leq \sum_{i=1}^{n} \Pr[\hat{Z}^n \neq Z^n] \leq \sum_{i=1}^{n} \epsilon = n\epsilon.$$
(75)

Now we can complete the proof by using (74)-(75) in (73):

$$\begin{split} I(X^{n}; M|Y^{n}) &\stackrel{(e)}{\geq} nR_{f}(2\epsilon, \epsilon) \\ &\stackrel{(f)}{\geq} n(R_{f}(0, 0) - \delta_{\epsilon}) \quad (\text{where } \delta_{\epsilon} \to 0 \text{ as } \epsilon \to 0) \\ &= n(R_{f}^{WZ}(0) - \delta_{\epsilon}), \end{split}$$

where (e) uses the fact that $(R_f(\delta, D)$ is non-increasing in (δ, D)), and (f) follows from Lemma 13.

Proof of Lemma 10: We prove only the first inequality of Lemma 10, and as stated there, we use only the weak privacy conditions – where (42)-(44) are upper-bounded by $n\epsilon$ – to prove this. The other two inequalities can be proved similarly. Let $M_1 := (M_{12}, M_{31})$ and $M_3 := (M_{23}, M_{31})$. For (X^n, Z^n) , we define the function $\frac{1}{n}RI_{n\epsilon}(X^n; Z^n)$ as follows:

$$\frac{1}{n}RI_{n\epsilon}(X^{n};Z^{n}) := \min_{\substack{p_{Q|X^{n}Z^{n}:}\\\frac{1}{n}I(Q;Z^{n}|X^{n}) \le \epsilon\\\frac{1}{n}I(Q;X^{n}|Z^{n}) \le \epsilon}} \frac{1}{n}I(X^{n};Z^{n}|Q).$$
(76)

For (X, Z), we define the function $RI_{\epsilon}(X; Z)$ as follows:

$$RI_{\epsilon}(X;Z) := \min_{\substack{p_{Q'|XZ}:\\I(Q';Z|X) \le \epsilon\\I(Q';X|Z) \le \epsilon}} I(X;Z|Q').$$
(77)

Note that $RI_0(X;Z) = RI(X;Z)$. We prove the result by proving the following three inequalities:

$$\frac{1}{n}RI(M_1, X^n; M_3, Z^n) \ge \frac{1}{n}RI_{n\epsilon}(X^n; Z^n)$$
(78)

$$\geq RI_{\epsilon}(X;Z) \tag{79}$$

$$\geq RI(X;Z) - \epsilon_{31}. \tag{80}$$

For (78), we proceed as follows:

• $I(Q; M_3, Z^n | M_1, X^n) = 0$, together with weak privacy against Alice $(I(M_1; Y^n, Z^n | X^n) \le n\epsilon)$ implies $I(Q; Z^n | X^n) \le n\epsilon$.

$$0 = I(Q; M_3, Z^n | M_1, X^n)$$

$$\geq I(Q; Z^{n}|M_{1}, X^{n})$$

$$= I(Q, M_{1}; Z^{n}|X^{n}) - \underbrace{I(M_{1}; Z^{n}|X^{n})}_{\leq n\epsilon}$$

$$\geq I(Q; Z^{n}|X^{n}) - n\epsilon$$

• Similarly, it can be shown that $I(Q; M_1, X^n | M_3, Z^n) = 0$ and weak privacy against Charlie $(I(M_3; X^n, Y^n | Z^n) < n\epsilon)$ imply $I(Q; X^n | Z^n) < n\epsilon$.

Now, $I((M_1, X^n); (M_3, Z^n)|Q) \ge I(X^n; Z^n|Q)$ (which is always true), together with the above two implications, implies (78). For (79) we proceed as follows:

$$\begin{split} \epsilon &\geq \frac{1}{n} I(Q; Z^{n} | X^{n}) \\ &\stackrel{\text{(a)}}{=} \frac{1}{n} \sum_{i=1}^{n} H(Z_{i} | X_{i}) - H(Z_{i} | X^{n}, Z^{i-1}, Q) \\ &\geq \frac{1}{n} \sum_{i=1}^{n} H(Z_{i} | X_{i}) - H(Z_{i} | X^{i-1}, X_{i}, Z^{i-1}, Q) \\ &= \frac{1}{n} \sum_{i=1}^{n} I(Q, X^{i-1}, Z^{i-1}; Z_{i} | X_{i}) \\ &\stackrel{\text{(b)}}{=} \sum_{i=1}^{n} p_{T}(i) I(Q, X^{i-1}, Z^{i-1}; Z_{i} | X_{i}, T = i) \\ &= I(Q, X^{T-1}, Z^{T-1}; Z_{T} | X_{T}, T) \\ &= I(Q, X^{T-1}, Z^{T-1}, T; Z_{T} | X_{T}) \\ &= I(Q_{T}, T; Z_{T} | X_{T}) \quad (\text{where } Q_{T} = (Q, X^{T-1}, Z^{T-1})) \end{split}$$

(a) follows because (X_i, Z_i) 's are i.i.d; the random variable Tin (b) is distributed with Unif $\{1, 2, ..., n\}$ and is independent of (Q, X^n, Z^n) . For the other constraint: $\frac{1}{n}I(Q; X^n|Z^n) \le \epsilon \implies I(Q_T, T; X^n|Z^n) \le \epsilon$, we can proceed similarly as above. For the objective function:

$$\frac{1}{n}I(X^n; Z^n | Q) \ge \sum_{i=1}^n \frac{1}{n}I(X_i; Z_i | Q, X^{i-1}, Z^{i-1})$$
$$= I(X_T; Z_T | Q_T, T).$$

So, we get the following:

$$\frac{1}{n}RI_{n\epsilon}(X^n;Z^n) \ge \min_{\substack{p_{Q|X^nZ^n}:\\I(Q_T,T;Z_T|X_T)\le\epsilon\\I(Q_T,T;X_T|Z_T)\le\epsilon}} I(X_T;Z_T|Q_T,T),$$
(81)

where, on the RHS, $T \sim \text{Unif}\{1, 2, \dots, n\}$ and is independent of (Q, X^n, Z^n) , and $Q_T = (Q, X^{T-1}, Z^{T-1})$. To get (79), we define $p_{Q'|XZ} := p_{Q_TT|X_TZ_T}$ to get

$$\min_{\substack{p_{Q|X^nZ^n}:\\I(Q_T,T;Z_T|X_T)\leq\epsilon\\I(Q_T,T;X_T|Z_T)\leq\epsilon}} I(X_T;Z_T|Q_T,T) \geq \min_{\substack{p_{Q'|XZ}:\\I(Q';Z|X)\leq\epsilon\\I(Q';X|Z)\leq\epsilon}} I(X;Z|Q'),$$

which proves (79).

For (80), we prove that for fixed joint distribution p_{XZ} , $RI_{\epsilon}(X; Z)$ is right continuous at $\epsilon = 0$. This is proved, below, using the property that the tension region $\mathfrak{T}(X; Z)$ is closed [36]. For simplicity of notation, we denote $RI_{\epsilon}(X; Z)$ by RI_{ϵ} . Note that $RI_0 = RI$.

From the definition of RI_{ϵ} in (77), it is easy to see that it is a non-increasing function of ϵ , that is to say, if $\epsilon < \epsilon'$, then $RI_{\epsilon} \ge RI_{\epsilon'}$. Now suppose, to the contrary, that RI_{ϵ} is not right continuous at $\epsilon = 0$. This implies that there exists a monotone decreasing sequence $\epsilon_m \downarrow 0$ and $\gamma > 0$ s.t. $RI_{\epsilon_m} \le RI_0 - \gamma$ for all $m \in \mathbb{N}$. Note that RI_{ϵ_m} is a monotone non-decreasing sequence that is bounded above by RI_0 , which implies that it is convergent (since every monotone non-decreasing sequence that is bounded above is convergent). Let $L = \lim_{m \to \infty} RI_{\epsilon_m}$ be the limit of this sequence. We have $L \le RI_0 - \gamma < RI_0$. This contradicts the fact that RI_0 is the minimum value r s.t. $(0, 0, r) \in \mathfrak{T}(X; Z)$, because $\mathfrak{T}(X; Z)$ is closed (i.e., $\mathfrak{T}(X; Z)$ contains all its limit points), implying that $(0, 0, L) \in \mathfrak{T}(X; Z)$.

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