Estimation for models defined by conditions on their **L-moments**

Michel Broniatowski $^{(1)}$, Alexis Decurninge $^{(1,\ast)}$

Abstract

This paper extends the empirical minimum divergence approach for models which satisfy linear constraints with respect to the probability measure of the underlying variable (moment constraints) to the case where such constraints pertain to its quantile measure (called here semi parametric quantile models). The case when these constraints describe shape conditions as handled by the L-moments is considered and both the description of these models as well as the resulting non classical minimum divergence procedures are presented. These models describe neighborhoods of classical models used mainly for their tail behavior, for example neighborhoods of Pareto or Weibull distributions, with which they may share the same first L-moments. A parallel is drawn with similar problems held in optimal transportation problems. The properties of the resulting estimators are illustrated by simulated examples comparing Maximum Likelihood estimators on Pareto and Weibull models to the minimum Chi-square empirical divergence approach on semi parametric quantile models, and others.

⁽¹⁾LSTA, Université Pierre et Marie Curie, Paris, France

^(*)Corresponding author.

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1 Motivation and notation

For univariate distributions, L-moments are expressed as the expectation of a particular linear combination of order statistics. Let us consider r independent copies $X_1, ..., X_r$ of a random variable X with $\mathbb{E}(|X|)$ a finite number. The r-th L-moment is defined by

$$\lambda_r = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \mathbb{E}[X_{r-k:r}]$$
(1.1)

where $X_{1:r} \leq ... \leq X_{r:r}$ denotes the order statistics. The four first L-moment can be considered as a measure of location, dispersion, skewness and kurtosis. Indeed $\lambda_1 = \mathbb{E}(X)$, λ_2 is expressed as $\lambda_2 = (1/2) \mathbb{E}(|X - Y|)$ with Y an independent copy of X, λ_3 indicates the expected distance between the mean of the extreme terms and the median one in a sample of three i.i.d. replications of X, and λ_4 is an indicator of the expected distance between the two central terms of a sample of four replicates of X with respect to a multiple of the distance between the two central terms.

L-moments constitute a robust alternative to traditional moments as descriptors of a distribution since only the existence of $\mathbb{E}(|X|)$ is needed in order to insure their existence. Since their introduction in Hosking's paper in 1990 ([19]), methods based on L-moments have become popular especially in applications dealing with heavy-tailed distributions. As mentioned in [19] and [20]:"The main advantage of L-moments over conventional moments is that L-moments, being linear functions of the data, suffer less from the effect of sampling variability: L-moments are more robust than conventional moments to outliers in the data and enable more secure inferences to be made from small samples about an underlying probability distribution. Also as seen through (1.1) the L-moments are determined by the expectation of extreme order statistics, and vice versa". This motivates their success for the inference in models pertaining to the tail behavior of random phenomenons.

In this article, we will consider semi-parametric models conditioned by constraints on a finite number of Lmoments. Let us mention three examples of such models; the two first examples describe neighborhoods of the Weibull and the Pareto models, which are classical benchmarks for the description of tail properties, and the third one describes a family of distributions which express some loose symmetry property. **Example 1.1** We first consider the model which is the family of all the distributions of a r.v. X whose second, third and fourth L-moments verify :

$$\begin{cases} \lambda_2 = \sigma (1 - 2^{-1/\nu}) \Gamma (1 + 1/\nu) \\ \lambda_3 = \lambda_2 [3 - 2 \frac{1 - 3^{-1/\nu}}{1 - 2^{-1/\nu}})] \\ \lambda_4 = \lambda_2 [6 + \frac{5(1 - 4^{-1/\nu}) - 10(1 - 3^{-1/\nu})}{1 - 2^{-1/\nu}})] \end{cases}$$
(1.2)

for any $\sigma > 0, \nu > 0$. These distributions share their first L-moments of order 2, 3 and 4 with those of a Weibull distribution with scale and shape parameter σ and ν . When X is substituted by Y := X + a for some real number a then the distribution of Y is Weibull with a shifted support, hence with the same parameters σ and ν as X; the r.v. Y shares the same L-moments λ_r with those of X but for r = 1 and the model (1.2) describes a neighborhood of the continuum of all Weibull distributions on $[a, \infty)$ or on $(-\infty, a]$ when a belongs to \mathbb{R} . Hence this model aims at describing a shape constraint on the tail of the distribution of the data, independently of its location.

Example 1.2 Secondly, we consider the model which is the space of the distributions whose second, third and fourth *L*-moments verify :

$$\begin{cases} \lambda_2 = \frac{\sigma}{(1-\nu)(2-\nu)} \\ \lambda_3 = \lambda_2 \frac{1+\nu}{3-\nu} \\ \lambda_4 = \lambda_2 \frac{(1+\nu)(2+\nu)}{(3-\nu)(4-\nu)} \end{cases}$$
(1.3)

for any $\sigma > 0, \nu \in \mathbb{R}$. These distributions share their first L-moments with those of a generalized Pareto distribution with scale and shape parameter σ and ν . The same remark as in the above example holds; model (1.3) describes a neighborhood of the whole continuum of Pareto distributions on $[a, \infty)$ or on $(-\infty, a]$ when a belongs to \mathbb{R} .

Example 1.3 Let finally be given an appealing example based on order statistics, namely

$$\begin{cases} \mathbb{E}[X_{1:3}] = \theta - \nu \\ \mathbb{E}[X_{2:3}] = \theta \\ \mathbb{E}[X_{3:3}] = \theta + \nu \end{cases}$$

for any $\theta \in \mathbb{R}, \nu > 0$.

Before any further discussion on the scope of the present paper, a few notation seems useful. For a non decreasing function F with bounded variation on any interval of \mathbb{R} we denote \mathbf{F} the corresponding positive σ -finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For example when F is the distribution function of a probability measure, then this measure is denoted \mathbf{F} or dF. Denote in this case

$$F^{-1}(u) := \inf \{ x \in \mathbb{R} \text{ s.t. } F(x) \ge u \} \text{ for } u \in (0, 1) \}$$

the generalized inverse of F, a left continuous non decreasing function which is the quantile function of the probability measure \mathbf{F} . Denote accordingly \mathbf{F}^{-1} or dF^{-1} , indifferently, the quantile measure with distribution function F^{-1} . If x_1, \ldots, x_n are *n* realizations of a random variable X with absolutely continuous probability measure \mathbf{F} then the gaps in the empirical distribution function

$$F_n(x) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{(-\infty,x]} (x_i)$$

are of size 1/n and are located on the X_i 's; the empirical quantile function satisfies

$$F_n^{-1}(u) = x_{i:n}$$
 when $\frac{i-1}{n} < u \le \frac{i}{n}$

and its gaps are given by

$$F_n^{-1}\left((i/n)^+\right) - F_n^{-1}\left((i/n)\right) = \mathbf{F}_n^{-1}\left(i/n\right) = x_{i+1:n} - x_{i:n}$$

where $x_{1:n} \leq ... \leq x_{n:n}$ denotes the ordered sample; those gaps will be denoted $\mathbf{F}_n^{-1}(i/n)$ or $dF_n^{-1}(i/n)$ indifferently; the empirical quantile measure has as its support the uniformely sparsed points $\{1/n, 2/n, ..., 1\}$ and attributes masses equal to sampled spacings at those points; it follows that the empirical quantile measure is a positive finite measure with finite support. The quantile measure associated with the distribution function F^{-1} is also a positive σ -finite measure, defined on (0, 1). The above construction defined the quantile measure from the probability measure, but the reciprocal construction will be used, starting from a quantile measure, defining its distribution function, turning to its inverse to define a distribution of a probability measure, and then to the probability measure itself.

We now turn back to our topics.

Models defined as in the above examples extend the classical parametric ones, and are defined through some constraints on the form of the distributions. They can be paralleled with models defined through moments conditions defined as follows.

Let θ in Θ , an open subset of \mathbb{R}^d and let $g : (x, \theta) \in \mathbb{R} \times \Theta \to \mathbb{R}^l$ be a *l*-valued function, each component of which is parametrized by $\theta \in \Theta \subset \mathbb{R}^d$. Define

$$M_{\theta} := \left\{ \mathbf{F} \text{ s.t. } \int_{\mathbb{R}} g(x, \theta) \mathbf{F}(dx) = 0 \right\}$$

and the semi parametric model defined by moment conditions is the collection of probability measures in

$$\mathcal{M} := \bigcup_{\theta \in \Theta} M_{\theta}. \tag{1.4}$$

These semiparametric models are defined by l conditions pertaining to l moments of the distributions and are widely used in applied statistics. When the dimension d of the parameter space exceeds l, no plug-in method can achieve any inference on θ ; however, various techniques have been proposed in this case; see for example Hansen [17], who defined the Generalized Method of Moments (GMM) and Owen, who defined the so-called empirical likelihood approach [26]. Later, Newey and Smith [25] or Broniatowski and Keziou [7] proposed a refinement of the GMM approach minimizing a divergence criterion over the model. A major feature of models defined by (1.4) lies in their linearity with respect to the cumulative distribution function (cdf) which brings a dual formulation of the minimization problem. Duality results easily lead to the consistency and the asymptotic normality of the estimators of θ ; see [7][25].

Similarly as for models defined by (1.4), we can introduce semiparametric linear quantile (SPLQ) models through

$$\bigcup_{\theta \in \Theta} L_{\theta} := \bigcup_{\theta \in \Theta} \left\{ \mathbf{F} \text{ s.t. } \int_{0}^{1} F^{-1}(u) k(u, \theta) du = f(\theta) \right\}$$
(1.5)

where $\Theta \subset \mathbb{R}^d$, $k : (u, \theta) \in [0; 1] \times \Theta \to \mathbb{R}^l$ and $f : \Theta \to \mathbb{R}^l$. In the above display, in accordance with the above notation, F^{-1} denotes the generalized inverse function of F, the distribution function of the measure **F**. Examples 1.1,1.2 and 1.3 can be written through (1.5); see Section 3.2. We will consider the case when k is a function of u only; this class contains many examples, typically models defined by a finite number of constraints on functions of the moments of the order statistics.

It is natural to propose similar estimation procedures for SPLQ models based on a minimization of a divergence. Models (1.5) do not enjoy linearity with respect to the cdf but with respect to the quantile function. Thus, as developed for models defined by (1.4), we propose to minimize a divergence criterion built on quantiles.

We will reformulate this criterion into a minimization of the energy of a deformation of the empirical distribution. A duality result and the subsequent consistency and asymptotic normality for the corresponding family of estimators are presented in Sections 5 and 7.

Section 6 draws a parallel with with an optimal transportation approach.

In the following, the transpose of a vector A will be denoted A^T and if F and G are two cdf's, $F \ll G$ means that F is absolutely continuous with respect to G. The Lebesgue measure on \mathbb{R} is denoted $d\lambda$ or dx, according to the common use in the context.

2 L-moments

2.1 Definition and characterizations

Let us consider data consisting in $\underline{X} = (x_1, ..., x_r)$, which are r realizations of real-valued independent and identically distributed (iid) copies $X_1, ..., X_r$ of a random variable (r.v.) X with distribution function F. The r-th L-moment λ_r is defined by

$$\lambda_r = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \mathbb{E}[X_{r-k:r}]$$
(2.1)

where $X_{1:r} \leq X_{2:r} \leq ... \leq X_{r:r}$ denotes the order statistics of $X_1, ..., X_r$. From the above definition all L-moments λ_r but λ_1 are shift invariant, hence independent upon λ_1 . If F is continuous, the expectation of the *j*-th order statistics $X_{j:r}$ is (see David p.33[12])

$$\mathbb{E}[X_{j:r}] = \frac{r!}{(j-1)!(r-j)!} \int_{\mathbb{R}} x F(x)^{j-1} (1-F(x))^{r-j} \mathbf{F}(dx).$$
(2.2)

The first four L-moments are

$$\lambda_{1} = \mathbb{E}[X]$$

$$\lambda_{2} = \frac{1}{2}\mathbb{E}[X_{2:2} - X_{1:2}]$$

$$\lambda_{3} = \frac{1}{3}\mathbb{E}[X_{3:3} - 2X_{2:3} + X_{1:3}]$$

$$\lambda_{4} = \frac{1}{4}\mathbb{E}[X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}].$$

Remark 2.1 The second L-moment is equal to the half of the absolute mean difference

TT3[TZ]

$$\lambda_2 = \frac{1}{2}\mathbb{E}[|X - Y|]$$

where X and Y are independently sampled from the same distribution F. The ratio $\frac{\lambda_2}{\lambda_1}$ is known as the Gini coefficient.

The expectations of the extreme order statistics characterize a distribution: if $\mathbb{E}(|X|)$ is finite, either of the sets $\{\mathbb{E}(X_{1:n}), n = 1, ..\}$ or $\{\mathbb{E}(X_{n:n}), n = 1, ..\}$ characterize the distribution of X; see [9] and [21]. Since the moments of order statistics are defined by the family of L-moments, those also characterize the distribution of X.

The *r*-th L-moment ratio is defined for $r \ge 2$ by

$$\tau_r = \frac{\lambda_r}{\lambda_2}.$$

The interpretation of $\lambda_1, \lambda_2, \tau_3, \tau_4$ as measures of location, scale, skewness and kurtosis respectively and the existence of all L-moments whenever $\int |x| \mathbf{F}(dx) < \infty$ makes them good alternatives to moments.

Remark 2.2 We can define from the quantile function $F^{-1}: [0;1] \to \mathbb{R}$ an associated measure on $\mathcal{B}([0;1])$

$$\mathbf{F}^{-1}(B) = \int_0^1 \mathbb{1}_{x \in B} dF^{-1}(x) \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

The above integral is a Riemann-Stieltjes integral. It defines a σ -finite measure since F^{-1} has bounded variations on every interval of the form [a, b] with $0 < a \le b < 1$. For any \mathbf{F}^{-1} -measurable function $a : \mathbb{R} \to \mathbb{R}$, it holds

$$\int_0^1 a(x)dF^{-1}(x) = \int_0^1 a(x)\mathbf{F}^{-1}(dx)$$

Writing the L-moments of a distribution F as an inner product of the corresponding quantile function with a specific complete orthogonal system of polynomials in $L^2(0, 1)$ is a cornerstone in the derivation of statistical inference in SPLQ models. The shifted Legendre polynomials define such a system of functions.

Definition 2.1 The shifted Legendre polynomial of order r is

$$L_{r}(t) = \sum_{k=0}^{r} (-1)^{k} {\binom{r}{k}}^{2} t^{r-k} (1-t)^{k} = \sum_{k=0}^{r} (-1)^{r-k} {\binom{r}{k}} {\binom{r+k}{k}} t^{k}.$$
 (2.3)

For $r \geq 1$ define K_r as the integrated shifted Legendre polynomials

$$K_r(t) = \int_0^t L_{r-1}(u) du = -t(1-t) \frac{J_{r-2}^{(1,1)}(2t-1)}{r-1}$$
(2.4)

with $J_{r-2}^{(1,1)}$ the corresponding Jacobi polynomial (see [18])

$$J_{r-2}^{(1,1)}(2t-1) = \frac{\Gamma(r)}{(r-2)!\Gamma(r+1)} \sum_{k=0}^{r-2} \binom{k}{r-2} \frac{\Gamma(r+1+k)}{\Gamma(2+k)} (t-1)^k.$$

The following result holds.

Proposition 2.1 Let F be any cdf and assume that $\int |x| dF(x)$ is finite. Then for any $r \ge 1$, it holds

$$\lambda_r = \int_0^1 F^{-1}(t) L_{r-1}(t) dt = \int_0^1 F^{-1}(t) dK_r(t)$$
(2.5)

where the last integral is the Stieltjes integral of F^{-1} with respect to the function $t \mapsto K_r(t)$.

Proof. The proof is based on the following fundamental Lemma, whose proof is deferred to the Appendix.

Lemma 2.1 Let U be a uniform random variable on [0;1] and X be a random variable with F. Then $F^{-1}(U) =_d X$.

Let $U_1, ..., U_r$ be r independent random variable uniformly distributed on [0; 1] and denote by $U_{1:r} \le ... \le U_{r:r}$ the ordered statistics. Then

$$(X_{1:r},...,X_{r:r}) \stackrel{d}{=} (F^{-1}(U_{1:r}),...,F^{-1}(U_{r:r}));$$

hence for $1 \leq j \leq r$

$$\mathbb{E}[X_{j:r}] = \mathbb{E}[F^{-1}(U_{j:r})] = \frac{r!}{(j-1)!(r-j)!} \int_0^1 F^{-1}(t)t^{j-1}(1-t)^{r-j}dt,$$

which ends the proof of Proposition 2.1. \blacksquare

Before going any further, we present an useful Lemma, the proof of which is also deferred to the Appendix.

Lemma 2.2 Let a be a real-valued function such that $\int_{\mathbb{R}} a(x) dF(x) < \infty$. Then

$$\int_{\mathbb{R}} a(x) d\mathbf{F}(x) = \int_{0}^{1} a(F^{-1}(t)) dt.$$
(2.6)

Similarly if $t \to b(t)$ is a real-valued function such that $\int_0^1 b(t) \mathbf{F}^{-1}(dt) < \infty$. Then

$$\int_{0}^{1} b(t) \mathbf{F}^{-1}(dt) = \int_{0}^{1} b(F(x)) dx.$$
(2.7)



Figure 1: Weights $w_i^{(r)}$ for the uniform law with a support containing 10 points

Remark 2.3 As a consequence of Lemma 2.2 and equation (2.5), it holds

$$\lambda_r = \int_0^1 x dK_r(F(x)).$$

Remark 2.4 If we consider a multinomial distribution with support $x_1 \le x_2 \le ... \le x_n$ and associated weights $\pi_1, ..., \pi_n$ ($\sum_{i=1}^n \pi_i = 1$), we get

$$\lambda_r = \sum_{i=1}^n w_i^{(r)} x_i = \sum_{i=1}^n \left[K_r \left(\sum_{a=1}^i \pi_a \right) - K_r \left(\sum_{a=1}^{i-1} \pi_a \right) \right] x_i = \int_0^1 L_{r-1}(t) Q_\pi(t) dt$$

with

$$Q_{\pi}(t) = \begin{cases} x_1 & \text{if } 0 \le t \le \pi_1 \\ x_i & \text{if } \sum_{a=1}^{i-1} \pi_a < t \le \sum_{a=1}^{i} \pi_a \end{cases}$$

This example illustrates Remark 2.3.

Figure 1 provides the first weight $w_i^{(r)}$ when the x_i 's are equally sparsed on [0,1] with equal weights $\pi_1 = ... = \pi_n = 1/n$.

The following characterization for the L-moments with order larger or equal to 2 is used in Section 3.2.

Proposition 2.2 If $r \ge 2$ and $\int_{\mathbb{R}} |x| dF(x) < +\infty$, then

$$\lambda_r = \int_0^1 F^{-1}(t) dK_r(t) = -\int_0^1 K_r(t) \mathbf{F}^{-1}(dt).$$
(2.8)

Proof. This result follows as an application of Fubini-Tonelli Theorem. Indeed

$$\lambda_r = \int_0^1 F^{-1}(t) dK_r(t)$$

= $\int_0^1 \int_0^t \mathbf{F}^{-1}(du) dK_r(t)$
= $\int_0^1 \int_0^1 \mathbb{1}_{0 \le u \le t} \mathbf{F}^{-1}(du) dK_r(t)$

This last equality holds since $(u, t) \mapsto \mathbb{1}_{0 \le u \le t}$ is measurable with respect to the measure $\mathbf{F}^{-1} \times dK_r$ since $\mathbb{E}[X] < \infty$. Applying Fubini-Tonelli Theorem, it holds

$$\lambda_r = \int_0^1 \int_0^1 \mathbb{1}_{0 \le u \le t} dK_r(t) \mathbf{F}^{-1}(du)$$

= $\int_0^1 \int_0^1 [K_r(1) - K_r(u)] \mathbf{F}^{-1}(du)$
= $-\int_0^1 K_r(u) \mathbf{F}^{-1}(du)$

since $K_r(1) = 0$ for r > 1.

Remark 2.5 That (2.8) does not hold for r = 1 follows from the fact that if G = F(. + a) for some $a \in \mathbb{R}$, then $\mathbf{G}^{-1} = \mathbf{F}^{-1}$. Hence, SPLQ models are shift-invariant. This can also be seen setting r = 1 in the right-hand side of (2.8); in this case, the integral is infinite (but if $supp(\mathbf{F})$ is bounded) whereas λ_1 is supposed to be finite.

2.2 Estimation of L-moments

Let $x_1, ..., x_n$ be iid realizations of a random variable X with distribution F and L-moments λ_r . Define F_n the empirical cdf of the sample and l_r the corresponding plug-in estimator of λ_r ,

$$l_r = \int_0^1 F_n^{-1}(t) L_{r-1}(t) dt.$$
(2.9)

This estimator of λ_r is biased as quoted in [19] and [32]. l_r is usually termed as a V-statistic. As noted upon in [19] and [32], the unbiased estimators of L-moments are the following U-statistics

$$l_r^{(u)} = \frac{1}{\binom{n}{r}} \sum_{1 \le i_1 < \dots < i_r \le n} \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} x_{i_{r-k}:n}.$$

Remark 2.6 An alternative definition for l_r as in (2.9) can be stated as follows. Conditionally on the realizations $x = (x_1, ..., x_n)$, define the uniform distribution on x. Then l_r is the discrete L-moment of order r of this conditional distribution. It can therefore be defined through

$$l_r = \frac{1}{\binom{r+n-1}{n-1}} \sum_{1 \le i_1 \le \dots \le i_r \le n} \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} x_{i_{r-k}:n}.$$

4

Let us now extend Definition 2.1 of the L-moments as follows. Let $(i_1, ..., i_r)$ be drawn without replacement from $\{1, ..., r\}$. We then define $x_{(i_1)} \leq ... \leq x_{(i_r)}$ the corresponding ordered observations and

$$\lambda_r^{(u)} = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \mathbb{E}[x_{(i_{r-k})}]$$

where the expectation is taken under the extraction process. Then $\lambda_r^{(u)}$ and $l_r^{(u)}$ coincide. Although $l_r^{(u)}$ is unbiased, for sake of simplicity only l_r which is asymptotically unbiased, will be used in the sequel.

These two estimators l_r and $l_r^{(u)}$ of the L-moment λ_r have the same asymptotic properties.

Proposition 2.3 Let us suppose that F has finite variance. Then, for any $m \ge 1$

$$\sqrt{n} \left[\begin{pmatrix} l_1 \\ \vdots \\ l_m \end{pmatrix} - \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \right] \to_d \mathcal{N}_m(0, \Lambda)$$

where \mathcal{N}_m denotes the multivariate normal distribution and the elements of Λ are given by

$$\Lambda_{rs} = \int \int_{x < y} \left[L_{r-1}(F(x)) L_{s-1}(F(y)) + L_{r-1}(F(y)) L_{s-1}(F(x)) \right] F(x) (1 - F(y)) dx dy$$

Furthermore, the same property holds for $l_1, ..., l_r$ substituted by $l_1^{(u)}, ..., l_m^{(u)}$.

Proof. This is a plain consequence of Theorem 6 in [29]. See also [19] for an evaluation of the bias of l_r .

3 Models defined by moment and L-moment equations

3.1 Models defined by moment conditions

Let us consider n iid random variables $X_1,...,X_n$ drawn from the same distribution function F. Semi-parametric models are often defined through equations :

$$\int_{\mathbb{R}} g(x,\theta) \mathbf{F}(dx) = \mathbb{E}[g(X,\theta)] = 0$$

where $g : \mathbb{R} \times \Theta \to \mathbb{R}^l$ and $\Theta \subset \mathbb{R}^d$ is a space of parameters, as quoted in Section 1.

Example 3.1 We can sometimes face distributions with constraints pertaining to the two first moments. For example, Godambe and Thompson [14] considered the distributions verifying $\mathbb{E}[X] = \theta$ and $\mathbb{E}[X^2] = h(\theta)$ with a known function h. Then, with our notations l = 2 and $g(x, \theta) = (x - \theta, x^2 - h(\theta))$

Example 3.2 Consider the distributions F such that for some θ it holds $F(y) = 1 - F(-y) = \theta$ [7]. This corresponds to a moment condition model with l = 2 and $g(x, \theta) = (\mathbb{1}_{]-\infty;y]}(x) - \theta, \mathbb{1}_{[y;+\infty[}(x) - \theta)$. The condition on the model is the existence of some θ such that the left and right quantiles of order θ are -y and +y for some given y.

3.2 Models defined by L-moments conditions

In the present paper we consider models defined by l constraints on their first L-moments, namely satisfying

$$-\mathbb{E}\left[\frac{1}{r}\sum_{k=0}^{r-1}(-1)^{k}\binom{r-1}{k}X_{k:r}\right] = f_{r}(\theta) \quad 1 \le r \le l$$
(3.1)

where Θ is some open set in \mathbb{R}^d and $f_r: \Theta \to \mathbb{R}$ are some given functions defined on Θ , $1 \le r \le l$. Those models are SPLQ, with $(u, \theta) \mapsto k(u, \theta)$ independent on θ , defined by

$$k(u,\theta) = -L(u) := -\begin{pmatrix} L_1(u) \\ \vdots \\ L_l(u) \end{pmatrix}$$
(3.2)

where the shifted Legendre polynomials L_r are as in Definition 2.1.

The SPLQ model (1.5) may be written as

$$\mathcal{L} := \bigcup_{\theta \in \Theta} L_{\theta} = \bigcup_{\theta \in \Theta} \left\{ \mathbf{F} \text{ s.t. } \int_{0}^{1} L(u) F^{-1}(u) du = -f(\theta) \right\}.$$
(3.3)

Due to Proposition 2.2 we may write equation (3.1) for $r \ge 2$ as follows, making use of the integrated shifted Legendre polynomials K_r in lieu of L_r .

$$-\mathbb{E}\left[\frac{1}{r}\sum_{k=0}^{r-1}(-1)^{k}\binom{r-1}{k}X_{k:n}\right] = \int_{0}^{1}K_{r}(u)\mathbf{F}^{-1}(du) = f_{r}(\theta).$$
(3.4)

Example 3.3 Turning back to Example 1.1, we define k and f by

$$k(u,\theta) = -\begin{pmatrix} L_2(u)\\ L_3(u)\\ L_4(u) \end{pmatrix}$$

and

$$f(\theta) = \begin{pmatrix} f_2(\theta) \\ f_3(\theta) \\ f_4(\theta) \end{pmatrix} = \begin{pmatrix} \sigma(1 - 2^{-1/\nu})\Gamma(1 + 1/\nu) \\ f_2(\theta)[3 - 2\frac{1 - 3^{-1/\nu}}{1 - 2^{-1/\nu}})] \\ f_2(\theta)[6 + \frac{5(1 - 4^{-1/\nu}) - 10(1 - 3^{-1/\nu})}{1 - 2^{-1/\nu}})] \end{pmatrix}$$

where $\theta = (\sigma, \nu) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+$ and $u \in [0; 1]$; hence (1.5) holds.

Example 3.4 Similarly, in case we consider Example 1.2, we define k and f by

$$k(u,\theta) = -\begin{pmatrix} L_2(u)\\ L_3(u)\\ L_4(u) \end{pmatrix}$$

and

$$f(\theta) = \begin{pmatrix} f_2(\theta) \\ f_3(\theta) \\ f_4(\theta) \end{pmatrix} = \begin{pmatrix} \frac{\sigma}{(1+\nu)(2+\nu)} \\ f_2(\theta)\frac{1-\nu}{3+\nu} \\ f_2(\theta)\frac{(1-\nu)(2-\nu)}{(3+\nu)(4+\nu)} \end{pmatrix}$$

where $\theta = (\sigma, \nu) \in \mathbb{R}^*_+ \times \mathbb{R}$ and $u \in [0, 1]$, which also validates (1.5).

3.3 Extension to models defined by order statistics conditions

The order statistics given by equation (2.2) can be written as

$$\mathbb{E}[X_{j:r}] = \int_0^1 P_{j:r}(u) F^{-1}(u) du$$

where the polynomials $P_{j:r}$ are given by

$$P_{j:r}(u) = \frac{r!}{(j-1)!(r-j)!} u^{j-1} (1-u)^{r-j}.$$

Any linear combination of moments of order statistics can be written as

$$-\sum_{i=1}^{r} a_j \mathbb{E}[X_{j:r}] = \int_0^1 P_a(u) F^{-1}(u) du$$

with coefficients a_i 's belonging to \mathbb{R} and

$$P_a(u) = -\sum_{i=1}^r a_j P_{j:r}(u)$$

These models are SPLQ (see 1.5) with

$$\mathcal{L} := \bigcup_{\theta} L_{\theta} = \bigcup_{\theta} \left\{ F \text{ s.t. } \int_{0}^{1} P(u) F^{-1}(u) du = -f(\theta) \right\}$$
(3.5)

where $P: u \in [0, 1] \mapsto P(u) \in \mathbb{R}^l$ is an array of l polynomials.

Example 3.5 *Turning back to Example 1.3, we define k and f by*

$$k(u,\theta) = \begin{pmatrix} P_{1:3}(u) \\ P_{2:3}(u) \\ P_{3:3}(u) \end{pmatrix}$$

and

$$f(\theta) = \begin{pmatrix} \theta - \nu \\ \theta \\ \theta + \nu \end{pmatrix}$$

where $\theta \in \mathbb{R}, \nu > 0$ and $u \in [0; 1]$.

4 Minimum of φ -divergence estimators

Estimation, confidence regions and tests based on moment conditions models have evolved over thirty years. Hansen and Owen respectively proposed the generalized method of moments (GMM)[16] and the empirical likelihood (EL) estimators [26]. Newey and Smith [25] introduced the generalized empirical likelihood (GEL) family of estimators encompassing the previous estimators. They also proposed the dual versions of the GEL estimators, the minimum discrepancy estimators (MD). These estimators are the solution of the minimization of a divergence with constraints corresponding to the model; see also Broniatowski and Keziou [7] for an approach through duality and properties of the inference under misspecification. In the quantiles framework, Gourieroux proposed an adaptation of GMM estimators in [15] for a parametric model seen through its quantile function $F^{-1}(t, \theta)$. In the following, we will consider inference based on divergences in order to present estimators for models defined by L-moments conditions.

4.1 φ -divergences

Let $\varphi : \mathbb{R} \to [0, +\infty]$ be a strictly convex function with $\varphi(1) = 0$ such that dom $(\varphi) = \{x \in \mathbb{R} | \varphi(x) < \infty\} := (a_{\varphi}, b_{\varphi})$ with $a_{\varphi} < 1 < b_{\varphi}$. If F and G are two σ -finite measures of $(\mathbb{R}, B(\mathbb{R}))$ such that G is absolutely continuous with respect to F, we define the divergence between F and G by :

$$D_{\varphi}(G,F) = \int_{\mathbb{R}} \varphi\left(\frac{dG}{dF}(x)\right) dF(x)$$
(4.1)

where $\frac{dG}{dF}$ is the Radon-Nikodym derivative. It is clear that when F = G, $D_{\varphi}(F,G) = 0$. Furthermore, as φ is supposed to be strictly convex,

$$D_{\varphi}(G, F) = 0$$
 if and only if $F = G$.

These divergences were independently introduced by Csiszar [10] or Ali and Silvey [1] in the context of probability measures. Definition 4.1 holds for any σ -finite measures even if our notation refers to probability measures. Indeed in the sequel we will consider divergences between quantile measure which are σ -finite but may be not finite. See Liese [23] who also considered divergences between σ -finite measures.

Example 4.1 The class of power divergences parametrized by $\gamma \ge 0$ is defined through the functions

$$x \mapsto \varphi_{\gamma}(x) = \frac{x^{\gamma} - \gamma x + \gamma - 1}{\gamma(\gamma - 1)}.$$

The domain of φ_{γ} depends on γ . The Kullback-Leibler divergence is associated to $x > 0 \mapsto \varphi_1(x) = x \log(x) - x + 1$, the modified Kullback-Leibler (KL_m) divergence to $x > 0 \mapsto \varphi_0(x) = -\log(x) + x - 1$, the χ^2 -divergence to $x \in \mathbb{R} \mapsto \varphi_2(x) = 1/2(x-1)^2$, etc.

4.2 M-estimates with L-moments constraints

4.2.1 Minimum of φ -divergences for probability measures

A plain approach to inference on θ consists in mimicking the empirical minimum divergence one, substituting the linear constraints with respect to the distribution by the corresponding linear constraints with respect to the quantile measure, and minimizing the divergence between all probability measures satisfying the constraint and the empirical measure \mathbf{F}_n pertaining to the data set. More formally this yields to the following program.

Denote by M the set of all probability measures defined on \mathbb{R} . For a given p.m. **F** in M we consider the submodel which consists in all p.m's **G** in M, absolutely continuous with respect to F, and which satisfy the constraints on their first L-moments for a given $\theta \in \Theta$. Identifying a measure **G** with its distribution function G we define

$$L_{\theta}^{(0)}(\mathbf{F}) = \left\{ \mathbf{G} \in M \text{ s.t. } \mathbf{G} \ll \mathbf{F}, \int_{0}^{1} L(t)G^{-1}(t)dt = -f(\theta) \right\}$$

Probability measures **G** satisfying the constraints and bearing their mass on the sample points belong to $L_{\theta}^{(0)}(\mathbf{F}_n)$. For any parameter $\theta \in \Theta$, the distance between **F** and the submodel $L_{\theta}^{(0)}(\mathbf{F})$ is defined by

$$D_{\varphi}(L_{\theta}^{(0)}(\mathbf{F}), \mathbf{F}) = \inf_{\mathbf{G} \in L_{\theta}^{(0)}(\mathbf{F})} D_{\varphi}(\mathbf{G}, \mathbf{F}),$$

and its plug-in estimator is

$$D_{\varphi}(L^{(0)}_{\theta}(\mathbf{F}_n),\mathbf{F}_n) = \inf_{\mathbf{G}\in L^{(0)}_{\theta}(\mathbf{F}_n)} D_{\varphi}(\mathbf{G},\mathbf{F}_n).$$

which measures the distance between the empirical measure \mathbf{F}_n and the class of all the probability measures supported by the sample and which satisfy the L-moment conditions for a given θ .

A natural estimator for θ may be defined by

$$\hat{\theta}_{n}^{(0)} = \arg \inf_{\theta \in \Theta} D_{\varphi}(L_{\theta}^{(0)}(\mathbf{F}_{n}), \mathbf{F}_{n}) = \arg \inf_{\theta \in \Theta} \inf_{\mathbf{G} \in L_{\theta}^{(0)}(F_{n})} \frac{1}{n} \sum_{i=1}^{n} \varphi(n\mathbf{G}(x_{i})).$$
(4.2)

Unfortunately, existence of this estimator may not hold. Indeed, we cannot assess that $L_{\theta}^{(0)}(\mathbf{F}_n)$ is not empty : its elements are multinomial distributions $\sum_{i=1}^{n} w_i \delta_{x_i}$ whose weights are solutions of a family of l-1 polynomial algebraic equation of degree l (with n unknowns $w_1, ..., w_n$)

$$\sum_{i=1}^{n} K_r \left(\sum_{a=1}^{i} w_a \right) (x_{i+1:n} - x_{i:n}) = -f_r(\theta); \ 1 < r \le l.$$

To our knowledge, general conditions of existence for the solutions of such problems do not exist even if we consider signed weights w_i .

Bertail in [2] proposes a linearization of the constraint in (4.2). We here prefer to switch to a different approach. If we consider the L-moment equation (3.4), we see that the quantile function plays a similar role as the distribution function in the classical moment equations. We will then change the functional to be minimized in order to be able to use duality for the optimization step.

4.2.2 Minimum of φ -divergences for quantile measures

We have seen that the characterization of the L-moments given by the equation (3.4) uses the quantile measure \mathbf{F}^{-1} , which is defined by the generalized inverse function of F. If \mathbf{F}^{-1} is absolutely continuous, we can define the quantile-density $q(u) = (F^{-1})'(u)$. This density was called "sparsity" function by Tukey [30] as it represents the sparsity of the distribution at the cumulating weight $u \in [0, 1]$. This is clear when we look at the empirical version of this measure which is composed by nothing but the increments of the sample. Some other approach, handling properties of the inverse function of $(F^{-1})'$, have been proposed by Parzen [27]. He claims that the inference procedures based on $(F^{-1})'$ possesses inherent robustness properties.

Define

$$K(u) = \begin{pmatrix} K_2(u) \\ \vdots \\ K_l(u) \end{pmatrix}$$

and

$$f(u) := f^{(2:l)}(u) = \begin{pmatrix} f_2(u) \\ \vdots \\ f_l(u) \end{pmatrix}.$$

For any θ in Θ the submodel which consists of all p.m's G with mass on the sample points is substituted by the set of all quantile measures denoted \mathbf{G}^{-1} which have masses on subsets of $\{1/n, 2/n, .., 1\}$ and whose distribution functions coincide with the generalized inverse functions of elements in $L_{\theta}^{(0)}(\mathbf{F}_n)$.

As in the case of divergence minimization for models constrained by moment conditions, we will relax the positivity for the masses of the quantile measures (see [7]). Let then N be the class of all σ -finite signed measures on \mathbb{R} . Let $L(u) := (L_2(u), ..., L_l(u))^T$ for all u in (0, 1). Introducing signed measures makes sense when the domain of the function φ is not restricted to \mathbb{R}^+ , as occurs for the chi-square divergence φ_2 . Making use of equation (3.4) define

$$L_{\theta}(\mathbf{F_n}^{-1}) := \left\{ \mathbf{G}^{-1} \in N \text{ s.t. } \mathbf{G}^{-1} \ll \mathbf{F_n^{-1}} \text{ and } \int_0^1 L(u)G^{-1}(u)du = -f(\theta) \right\}$$
$$= \left\{ \mathbf{G}^{-1} \in N \text{ s.t. } \mathbf{G}^{-1} \ll \mathbf{F_n^{-1}} \text{ and } \int_0^1 K(u)\mathbf{G}^{-1}(du) = f(\theta) \right\}$$

the family of all measures \mathbf{G}^{-1} with support included in $\{1/n, 2/n, .., 1\}$ which satisfy the l-1 constraints pertaining to the L-moments; see (3.3). Note that when \mathbf{F} bears an atom then for large enough n then \mathbf{G}^{-1} in $L_{\theta}(\mathbf{F_n}^{-1})$ has a support strictly included in $\{1/n, 2/n, .., 1\}$.

Since the measure \mathbf{G}^{-1} is not necessarily positive, its distribution function G^{-1} is not necessarily a generalized inverse of a function G; we will however inherit of the notation G^{-1} from the case when \mathbf{G}^{-1} is a positive measure to denote its distribution function. If \mathbf{G}^{-1} is positive, the mass of \mathbf{G}^{-1} at point i/n is a spacing $y_{i+1:n} - y_{i:n}$ where $y_{i:n}$ is the i - th order statistics of the sample $y_1, ..., y_n$ generating the empirical distribution function G.

A natural proposal for an estimation procedure in the SPLQ model is then to consider the minimum of a φ -divergence between quantile measures through

$$\hat{\theta}_n = \arg\inf_{\theta \in \Theta} \inf_{\mathbf{G}^{-1} \in L_{\theta}(\mathbf{F_n}^{-1})} \int_0^1 \varphi\left(\frac{d\mathbf{G}^{-1}}{d\mathbf{F}_n^{-1}}(u)\right) \mathbf{F}_n^{-1}(du)$$
(4.3)

$$= \arg \inf_{\theta \in \Theta} \inf_{\substack{(y_1, \dots, y_n) \in \mathbb{R}^n \text{ s.t.} \\ \sum_{i=1}^{n-1} K(i/n)(y_{i+1} - y_i) = f(\theta)}} \sum_{i=1}^{n-1} \varphi \left(\frac{y_{i+1} - y_i}{x_{i+1:n} - x_{i:n}} \right) (x_{i+1:n} - x_{i:n}).$$
(4.4)

Remark 4.1 The estimation defined by (4.3) produces estimators $\hat{\theta}_n$ which do not depend on the location of the sample, since a change the sample $(x_i \mapsto x_i + a)_{i=1...n}$ produces, independently on the value of a, the same measure

 \mathbf{F}_n^{-1} whose mass on point i/n is the gap $x_{i+1:n} - x_{i:n}$. The minimum discrepancy estimators defined by (4.4) are invariant with respect to the location of the underlying distribution of the data. Due to this fact, we consider the model defined by L-moments conditions only through equations of the form (3.4).

Both the constraint and the divergence criterion are expressed in function of G^{-1} and the constraint is linear with respect to this measure. This allows to use classical duality results in order to efficiently compute the estimator $\hat{\theta}_n$. Before that, we reformulate this criterion as a minimization of an "energy" of transformation of the sample.

5 Dual representations of the divergence under L-moment constraints

The minimization of φ -divergences under linear equality constraint is performed using Fenchel-Legendre duality. It transforms the constrained problems into an unconstrained one in the space of Lagrangian parameters. Let ψ denote the Fenchel-Legendre transform of φ , namely, for any $t \in \mathbb{R}$

$$\psi(t) := \sup_{x \in \mathbb{R}} \left\{ tx - \varphi(x) \right\}.$$

Let us recall that $dom(\varphi) = (a_{\varphi}, b_{\varphi})$. We can now present a general duality result for the two optimization problems that transform a constrained problem (possibly in an infinite dimensional space) into an unconstrained one in \mathbb{R}^{l} .

Let $C: \Omega \to \mathbb{R}^l$ and $a \in \mathbb{R}^l$. Denote

$$L_{C,a} = \left\{ g : \Omega \to \mathbb{R} \text{ s.t. } \int_{\Omega} g(t)C(t)\mu(dt) = a \right\}.$$

Proposition 5.1 Let μ be a σ -finite measure on $\Omega \subset \mathbb{R}$. Let $C : \Omega \to \mathbb{R}^l$ be an array of functions such that

$$\int_{\Omega} \|C(t)\|\mu(dt) < \infty.$$

If there exists some g in $L_{C,a}$ such that $a_{\varphi} < g < b_{\varphi} \mu$ -a.s. then the duality gap is zero i.e.

$$\inf_{g \in L_{C,a}} \int_{\Omega} \varphi(g) \, d\mu = \sup_{\xi \in \mathbb{R}^l} \langle \xi, a \rangle - \int_{\Omega} \psi(\langle \xi, C(x) \rangle) \mu(dx).$$
(5.1)

Moreover, if ψ is differentiable, if μ is positive and if there exists a solution ξ^* of the dual problem which is an interior point of

$$\left\{\xi \in \mathbb{R}^l \text{ s.t. } \int_{\Omega} \psi(\langle \xi, C(x) \rangle) \mu(dx) < \infty \right\},\$$

then ξ^* is the unique maximum in (5.1) and

$$\int \psi'(\langle \xi^*, C(x) \rangle) C(x) \mu(dx) = a$$

Furthermore the mapping $a \mapsto \xi^*(a)$ is continuous.

Proof. The proof is delayed to the Appendix.

Remark 5.1 When $\mathbf{G}^{-1} \ll \mathbf{F}^{-1}$, denoting $g^* = d\mathbf{G}^{-1}/d\mathbf{F}^{-1}$ and assuming $g^* \in L_{K,f(\theta)}$, and when $\mu = \mathbf{F}^{-1}$ it holds

$$\int \varphi(g^*) d\mu = D_{\varphi} \left(\mathbf{G}^{-1}, \mathbf{F}^{-1} \right).$$

Remark 5.2 Here, the classical assumption of finiteness of μ is replaced by

$$\int_{\Omega} \|C(x)\|\mu(dx) < \infty$$

which is needed for the application of the dominated convergence Theorem; also we refer to the illuminating paper by Csiszár and Matúš [24] for the description of the geometric tools used in the proof of Proposition 5.1.

We now apply the above Proposition 5.1 to the case when the array of functions C is equal to K, the measure μ is the quantile measure \mathbf{F}^{-1} pertaining to the distribution function F of a probability measure and when the class of functions $L_{C,a}$ is substituted by the class of functions $\mathbf{dG}^{-1}/\mathbf{dF}^{-1}$ when defined. Let $\theta \in \Theta$ and F be fixed. Let us recall that for any reference cdf F

$$L_{\theta}(\mathbf{F}^{-1}) := \left\{ \mathbf{G}^{-1} \ll \mathbf{F}^{-1} \text{ s.t. } \int_{\mathbb{R}} K(u) \mathbf{G}^{-1}(du) = f(\theta) \right\}.$$
(5.2)

Corollary 5.1 If there exists some \mathbf{G}^{-1} in $L_{\theta}(\mathbf{F}^{-1})$ such that $a_{\varphi} < d\mathbf{G}^{-1}/d\mathbf{F}^{-1} < b_{\varphi} \mathbf{F}^{-1}$ -a.s. then

$$\inf_{\mathbf{G}^{-1}\in L_{\theta}(\mathbf{F}^{-1})} \int_{0}^{1} \varphi\left(\frac{d\mathbf{G}^{-1}}{d\mathbf{F}^{-1}}\right) d\mathbf{F}^{-1} = \sup_{\xi\in\mathbb{R}^{l}} \langle\xi, f(\theta)\rangle - \int_{0}^{1} \psi(\langle\xi, K(u)\rangle) \mathbf{F}^{-1}(du).$$
(5.3)

Moreover, if ψ is differentiable and if there exists a solution ξ^* of the dual problem which is an interior point of

$$\left\{\xi \in \mathbb{R}^l \text{ s.t. } \int_{\mathbb{R}} \psi(\langle \xi, K(u) \rangle) \mathbf{F}^{-1}(du) < \infty \right\}$$

then ξ^* is the unique maximum in (5.3) and

$$\int \psi'^*(\langle \xi, K(u) \rangle) K(u) \mathbf{F}^{-1}(du) = f(\theta).$$

Remark 5.3 The above Corollary 5.1 is the cornerstone for the plug-in estimator of $D_{\omega}(\mathbf{G}, \mathbf{F})$.

Let us present an other application of the above Proposition 5.1 leading to the same dual problem. Denote by λ the Lebesgue measure on \mathbb{R} and $L'_{\theta}(F)$ be the set of all functions g defined by

$$L'_{\theta}(F) = \left\{ g : \mathbb{R} \to \mathbb{R} \text{ s.t. } \int_{\mathbb{R}} K(F(x))g(x)\lambda(dx) = f(\theta) \right\},$$

whenever non void.

Corollary 5.2 If there exists some g in $L'_{\theta}(F)$ such that $a_{\varphi} < g < b_{\varphi} \lambda$ -a.s. then

$$\inf_{g \in L'_{\theta}(F)} \int_{\mathbb{R}} \varphi(g) \, d\lambda = \sup_{\xi \in \mathbb{R}^l} \langle \xi, f(\theta) \rangle - \int_{\mathbb{R}} \psi(\langle \xi, K(F(x)) \rangle) dx.$$
(5.4)

Moreover, if ψ is differentiable and if there exists a solution ξ^* of the dual problem which is an interior point of

$$\left\{\xi \in \mathbb{R}^l \text{ s.t. } \int_{\mathbb{R}} \psi(\langle \xi, K(F(x)) \rangle) dx < \infty \right\},\$$

then ξ^* is the unique maximizer in (5.4). It satisfies

$$\int \psi'(\langle \xi^*, K(F(x)) \rangle) dx = f(\theta)$$
(5.5)

Proof. We will detail the proof of Corollary 5.2. Corollary 5.1 is proved similarly.

We apply the above Proposition 5.1 for $\Omega = \mathbb{R}$, $\mu = \lambda$, the array of functions C substituted by the array of functions $x \mapsto K(F(x))$ and $a = f(\theta)$.

Consequently, the class of functions g depends upon F, and $L_{C,a} = L'_{\theta}(F)$. We need then to show that

$$\int_{\mathbb{R}} \|K(F(x))\| dx < \infty.$$

Denote $K := (K_{i_1}, ..., K_{i_l})$ with $i_j \ge 2$ for all j. Recall that from equation (2.4)

$$K_{i_j}(t) = -t(1-t)\frac{J_{i_j-2}^{(1,1)}(2t-1)}{i_j-1}.$$

It is clear that there exists C > 0 such that $\left| \frac{J_{i_j-2}^{(1,1)}(2t-1)}{i_j-1} \right| < C$. Hence

$$\int_{\mathbb{R}} \|K(F(x))\| dx < lC \int_{\mathbb{R}} F(x)(1 - F(x)) dx < +\infty$$

since F is the cdf of a random variable with finite expectation. By applying Proposition 5.1, it then holds

$$\inf_{g \in L_{\theta}''(F)} \int_{\mathbb{R}} \varphi(g) \, d\lambda = \sup_{\xi \in \mathbb{R}^l} \langle \xi, f(\theta) \rangle - \int_{\mathbb{R}} \psi(\langle \xi, K(F(x)) \rangle) dx.$$

Remark 5.4 If we consider the class of functions

$$L''_{\theta}(F) = \left\{ T : \mathbb{R} \to \mathbb{R} \text{ s.t. } T \text{ derivable } \lambda - a.e. \text{ and } \int_{\mathbb{R}} K(F(x)) \frac{dT}{d\lambda}(x) \lambda(dx) = f(\theta) \right\},$$

containing the functions $T := x \mapsto \int_{-\infty}^{x} g(t) dt$ rather than the class of functions g, it holds that $T \in L''_{\theta}(F)$ if and only if $dT/d\lambda \in L'_{\theta}(F)$. Therefore,

$$\inf_{T \in L_{\theta}''(F)} \int_{\mathbb{R}} \varphi\left(\frac{dT}{d\lambda}\right) d\lambda = \inf_{g \in L_{\theta}'(F)} \int_{\mathbb{R}} \varphi\left(g\right) d\lambda$$

This seemingly formal definition of the function T makes sense since we can view T as a deformation function, as detailed in the following Section 6.

6 Reformulation of divergence projections and extensions

6.1 Minimum of an energy of deformation

6.1.1 The case of models defined by moments constraints

Let us suppose for a while that \mathbf{F} and \mathbf{G} are both absolutely continuous with respect to the Lebesgue measure defined on \mathbb{R} . Define the function $T = G \circ F^{-1}$. Then T is derivable a.e. and $T' = \frac{dT}{d\lambda}$. It holds

$$D_{\varphi}(\mathbf{G}, \mathbf{F}) = \int_{\mathbb{R}} \varphi\left(\frac{d\mathbf{G}}{d\mathbf{F}}(x)\right) \mathbf{F}(dx) = \int_{0}^{1} \varphi\left(T'(u)\right) du$$

even if G is not a positive measure, as far as the integrand in the central term of the above display is defined. The function T can be viewed as a measure of the deformation of F into G and

$$E_1(T) = \int \varphi\left(\frac{dT}{d\lambda}\right) d\lambda$$

as an energy of this deformation.

It can be seen that the absolute continuity assumption of both F and G with respect to the Lebesgue measure can be relaxed.

Proposition 6.1 Let F and G be two arbitrary cdf's and λ be the Lebesgue measure. Let us define

$$M_{\theta}(\mathbf{F}) = \left\{ \mathbf{G} \ll \mathbf{F} \text{ s.t. } \int_{\mathbb{R}} g(x,\theta) \mathbf{G}(dx) = 0 \right\}$$

and let $M'_{\theta}(\mathbf{F})$ denote the class of all functions T which are a.e derivable on [0; 1] defined through

$$M'_{\theta}(\mathbf{F}) = \left\{ T : [0;1] \to \mathbb{R} \text{ s.t. } \int_0^1 g(F^{-1}(u),\theta) \frac{dT}{d\lambda}(u)\lambda(du) = 0 \right\}.$$
(6.1)

Then if there exists $T \in M'_{\theta}(\mathbf{F})$ such that $a_{\varphi} < \frac{dT}{d\lambda} < b_{\varphi}$ and $\mathbf{G} \in M_{\theta}(\mathbf{F})$ such that $a_{\varphi} < \frac{d\mathbf{G}}{d\mathbf{F}} < b_{\varphi}$

$$\inf_{G \in M_{\theta}(\mathbf{F})} \int_{\mathbb{R}} \varphi\left(\frac{d\mathbf{G}}{d\mathbf{F}}(x)\right) \mathbf{F}(dx) = \inf_{T \in M_{\theta}'(\mathbf{F})} E_1(T)$$

Proof. This results from Proposition 5.1 applied twice.

First, if $C = g(., \theta)$, a = 0, $\mu = \mathbf{F}$ and $g = d\mathbf{G}/d\mathbf{F}$, it holds

$$\inf_{G \in M_{\theta}(\mathbf{F})} \int_{\mathbb{R}} \varphi\left(\frac{d\mathbf{G}}{d\mathbf{F}}(x)\right) \mathbf{F}(dx) = \sup_{\xi \in \mathbb{R}^{l}} - \int_{\mathbb{R}} \psi\left(\langle \xi, g(x,\theta) \rangle\right) \mathbf{F}(dx).$$

Secondly, if $C = g(F^{-1}(.), \theta)$, a = 0, $\mu = \lambda$ and $g = dT/d\lambda$, it holds

$$\inf_{T \in M'_{\theta}(\mathbf{F})} \int_{0}^{1} \varphi\left(\frac{dT}{d\lambda}\right) d\lambda = \sup_{\xi \in \mathbb{R}^{l}} - \int_{0}^{1} \psi\left(\langle \xi, g(F^{-1}(u), \theta) \rangle\right) \lambda(du).$$

Lemma 2.2 concludes the proof.

The estimators of minimum divergence used in [25] and [7] can be expressed in terms of T, introducing the empirical distribution of the sample in place of the true unknown distribution \mathbf{F}_{θ_0} . For each θ in Θ it holds

$$\inf_{\mathbf{G}\in M_{\theta}(\mathbf{F}_n)} \int_{\mathbb{R}} \varphi\left(\frac{d\mathbf{G}}{d\mathbf{F}_n}\right) \mathbf{F}_n(dx) = \inf_{T\in M_{\theta}'(\mathbf{F}_n)} E_1(T)$$

and

$$\theta_n := \arg \inf_{\theta \in \Theta} \inf_{T \in M'_{\theta}(\mathbf{F}_n)} E_1(T).$$

Remark 6.1 Note that if $T \in M'_{\theta}(\mathbf{F}_n)$, $T : [0;1] \to [0;1]$ is λ -a.e. derivable and verifies

$$\sum_{i=1}^{n-1} g(x_{i:n}, \theta) \left(T\left(\frac{i+1}{n}\right) - T\left(\frac{i}{n}\right) \right) = 0.$$

The plug-in estimator that realizes the minimum of the divergence between a given distribution and the submodel \mathcal{M}_{θ} results from the minimum of an energy of a deformation of the uniform grid on [0, 1] under constraints envolving the observed sample. Therefore the classical minimum divergence approach under moment conditions turns out to be a transformation of the uniform measure on the sample points, represented by the uniform grid on [0, 1] onto a projected measure on the same sample points, and the projected measure \mathbf{G}_n which solves the primal problem has support $x_1, ..., x_n$ and has a distribution function $G_n = T(F_n)$ where T solves

$$\inf_{T \in M'_{\theta}(\mathbf{F}_n)} E_1(T)$$

Turning now to the case of models defined by L-moments, we will now see that the approach of Section 4.2.2 consists in minimizing a deformation of the points of the distribution of interest instead of the weights.

6.1.2 The case of models defined by L-moment constraints

Similarly as for the case of models defined by moment constraints we now see that the solution of the minimum divergence problem (primal problem) holds without assuming \mathbf{F}^{-1} absolutely continuous with respect to the Lebesgue measure.

Proposition 6.2 Let F and G be two arbitrary cdf's. Let $L''_{\theta}(\mathbf{F}^{-1})$ denote the class of all functions T which are a.e derivable on \mathbb{R} defined through

$$L_{\theta}''(\mathbf{F}^{-1}) = \left\{ T : \mathbb{R} \to \mathbb{R} \text{ s.t. } \int_{\mathbb{R}} K(F(x)) \frac{dT}{d\lambda}(x) \lambda(dx) = f(\theta) \right\}.$$

Then, with $L_{\theta}(\mathbf{F}^{-1})$ defined in (5.2), if there exists $T \in L''_{\theta}(\mathbf{F}^{-1})$ such that $a_{\varphi} < \frac{d\mathbf{T}}{d\lambda} < b_{\varphi}$ and $\mathbf{G}^{-1} \in L_{\theta}(\mathbf{F}^{-1})$ such that $a_{\varphi} < \frac{d\mathbf{G}^{-1}}{d\mathbf{F}^{-1}} < b_{\varphi}$

$$\inf_{\mathbf{G}^{-1}\in L_{\theta}(\mathbf{F}^{-1})} \int_{0}^{1} \varphi\left(\frac{d\mathbf{G}^{-1}}{d\mathbf{F}^{-1}}\left(u\right)\right) \mathbf{F}^{-1}(du) = \inf_{T\in L_{\theta}''(\mathbf{F}^{-1})} \int_{\mathbb{R}} \varphi\left(\frac{dT}{d\lambda}\right) d\lambda$$

Proof. This results from a combination of Corollaries 5.1 and 5.2. ■

In the following, we consider the estimator of θ

$$\hat{\theta}_n = \arg \inf_{\theta \in \Theta} \inf_{T \in L''_{\theta}(\mathbf{F}_n^{-1})} \int_{\mathbb{R}} \varphi\left(\frac{dT}{d\lambda}\right) d\lambda.$$
(6.2)

The estimator $\hat{\theta}_n$ defined in (6.2) coincides with (4.3) thanks to the above Proposition 6.2.

Remark 6.2 $\cup_{\theta} L_{\theta}(\mathbf{F}^{-1})$ and $\cup_{\theta} L''_{\theta}(\mathbf{F}^{-1})$ both represent the same model with L-moments constraints, seen through a reference measure \mathbf{F}^{-1} . This model is either expressed as the space of quantile measures absolutely continuous with respect to \mathbf{F}^{-1} satisfying the L-moment constraints or as the space of all deformations $\mathbf{F}^{-1} \to T \circ F^{-1}$ of the reference measure \mathbf{F}^{-1} such that the deformed measure satisfies the L-moment constraints. In the second point of view T is derivable λ -a.e. even if the reference measure is \mathbf{F}_n^{-1} .

Remark 6.3 For the set of deformations $L''_{\theta}(\mathbf{F}_n^{-1})$ (whenever non void), the duality for finite distributions is expressed through the following equality :

$$\inf_{T \in L_{\theta}^{"}(\mathbf{F}_{n}^{-1})} \int \varphi\left(\frac{dT}{d\lambda}\right) d\lambda = \sup_{\xi \in \mathbb{R}^{l}} \xi^{T} f(\theta) - \sum_{i=1}^{n-1} \psi\left(\xi^{T} K\left(\frac{i}{n}\right)\right) (x_{i+1:n} - x_{i:n}).$$

Remark that we incorporate the requirement that for any T in the model $L''_{\theta}(\mathbf{F}_n^{-1})$, $a_{\varphi} < \frac{dT_1}{d\lambda} < b_{\varphi} \lambda$ -a.s. holds.

Example 6.1 If we consider the χ^2 -divergence $\varphi(x) = \frac{(x-1)^2}{2}$, then $\psi(t) = \frac{1}{2}t^2 + t$ and the solution ξ_1^* of the equation (5.5) is

$$\xi_1^* = \Omega^{-1} \left(f(\theta) - \int K(F(x)) d\lambda \right)$$

with

$$\Omega = \int K(F(x))K(F(x))^T d\lambda.$$

If we set $\Omega_n = \int K(F_n(x))K(F_n(x))d\lambda$, the estimator shares similarities with the GMM estimator. Indeed

$$\hat{\theta}_n = \arg \inf_{\theta \in \Theta} \left(f(\theta) - \int K(F_n(x)) d\lambda \right) \Omega_n^{-1} \left(f(\theta) - \int K(F_n(x)) d\lambda \right).$$

This divergence should thus be favored for its fast implementation.

Remark 6.4 We did not consider the constraints of positivity classically assumed in moment estimating equations for the sake of simplicity of dual representations. We could suppose that the transformation T is an increasing mapping. It would be the case if, for example, the divergence chosen is the Kullback-Leibler one. Indeed, in this case, problem (6.2) is well defined since $\varphi(x) = +\infty$ for all $x \leq 0$.

6.2 Transportation functionals and multivariate generalization

The notion of a deformation which was introduced in the above section is close to the notion of a transportation. The reformulation presented in Proposition 6.2 calls for a natural extension in this respect. Let us recall the definition of a transportation in \mathbb{R} .

Definition 6.1 The pushforward measure of **F** through T is the measure denoted by $T # \mathbf{F}$ satisfying

$$T # \mathbf{F}(B) = \mathbf{F}(T^{-1}(B))$$
 for every Borel subset B of \mathbb{R} .

T is said to be a transportation map between **F** and **G** if $T#\mathbf{F} = \mathbf{G}$. If *X* and *Y* are associated with respective cdf *F* and *G* then $T(X) =_d Y$.

We write L_{θ} (equation (3.3)) as a space of σ -measures

$$L_{\theta} = \left\{ \mathbf{G} \in M \text{ s.t. } \int_{0}^{1} L(u)G^{-1}(u)du = -f(\theta) \right\}.$$

Let furthermore C_A denote the space of absolutely continuous functions defined on \mathbb{R} . It follows that an alternative to the estimator (6.2) may be defined by

$$\hat{\theta}_{n}^{(tr)} = \arg \inf_{\theta \in \Theta} \inf_{T \in \mathcal{C}_{A}: T \# \mathbf{F}_{n} \in L_{\theta}} \int_{\mathbb{R}} \varphi\left(\frac{dT}{d\lambda}\right) d\lambda$$
(6.3)

where

- \mathbf{F}_n is the empirical measure on the observed sample $x_1, ..., x_n$
- $E(T) := \int_{\mathbb{R}} \varphi\left(\frac{dT}{d\lambda}\right) d\lambda$ stands for the energy which transports \mathbf{F}_n onto some \mathbf{G} .

We can give a rewriting of this transport estimator similar to Equation (4.4).

Proposition 6.3 If there exists some absolutely continuous T_0 such that $T_0 \# \mathbf{F}_n \in L_{\theta}$ and $a_{\varphi} < \frac{dT_0}{d\lambda} < b_{\varphi}$, then

$$\hat{\theta}_n^{(tr)} = \arg \inf_{\theta \in \Theta} \inf_{\substack{y \in \mathbb{R}^n \\ \sum_{i=1}^{n-1} K(i/n)(y_{i+1:n} - y_{i:n}) = f(\theta)}} D_{\varphi}(x, y)$$
(6.4)

with

$$D_{\varphi}(x,y) = \sum_{i=1}^{n-1} \varphi\left(\frac{y_{i+1} - y_i}{x_{i+1:n} - x_{i:n}}\right) (x_{i+1:n} - x_{i:n})$$

 $\hat{\theta}_n^{(tr)} = \hat{\theta}_n$

If moreover, $\varphi(x) = +\infty$ *for any* $x \le 0$ *then*

Proof. The proof is postponed to the Appendix.

Remark 6.5 The fact that T is absolutely continuous is necessary. Indeed, stating

$$\hat{\theta}_n^{(tr0)} := \arg \inf_{\theta \in \Theta} \inf_{T:T \# \mathbf{F}_n \in L_{\theta}} \int_{\mathbb{R}} \varphi\left(\frac{dT}{d\lambda}\right) d\lambda$$

may not lead to a well defined estimator; consider any discrete uniform distribution in L_{θ} (i.e any distribution in the submodel $L_{\theta}(\mathbf{F}_n^{-1})$). Let us denote its support by $y := \{y_1, ..., y_n\}$, and define T_y a.e. derivable such that

$$T_y(x) = \begin{cases} y_i \text{ if } x = x_i \\ x \text{ otherwise } \end{cases},$$

(6.5)

then $T_u \# \mathbf{F}_n \in L_{\theta}$ and

$$\int_{\mathbb{R}} \varphi\left(\frac{dT_y}{d\lambda}\right) d\lambda = 0$$

since $\varphi(1) = 0$. So when $L_{\theta}(\mathbf{F}_n^{-1})$ is not reduced to a unique measure, this estimator is undefined : the solution of the infimum problem is not unique.

In transportation theory, it is customary to define a cost function instead of an energy function. Given a convex cost function $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, an alternative version to (6.2) is

$$\hat{\theta}_n = \arg \inf_{\theta \in \Theta} \inf_{T:T \# \mathbf{F}_n \in L_{\theta}} \int_{\mathbb{R}} c(x, T(x)) \mathbf{F}_n(dx).$$
(6.6)

Remark 6.6 Whereas the estimator given by Equation (6.2) minimizes an energy expressed in function of T' (the estimation process then penalizes big values of T'), the optimal transportation estimator depends on the function T itself and penalizes the distance between each x_i and $T(x_i)$ i.e. the "initial" state and the deformed state.

Example 6.2 The following estimator stems from the optimal transportation problem (6.6) in the context of models constrained by L-moments equations.

Consider the cost function $c(x, y) = (x - y)^2$. The transportation problem reduces to (see e.g. [31])

$$\inf_{T:T\#\mathbf{F}_{n}\in L_{\theta}}\int_{\mathbb{R}} (x-T(x))^{2} \mathbf{F}_{n}(dx) := \inf_{\mathbf{G}\in L_{\theta}} W_{2}(\mathbf{F}_{n},\mathbf{G})^{2} = \inf_{\mathbf{G}\in L_{\theta}} \int_{0}^{1} \left|F_{n}^{-1}(t) - G^{-1}(t)\right|^{2} dt,$$

 W_2 is called the Wasserstein distance. The estimator (6.6) will then be defined by

$$\hat{\theta}_n := \arg \inf_{\theta \in \Theta} \inf_{T:T \# \mathbf{F}_n \in L_\theta} \int_{\mathbb{R}} (x - T(x))^2 \mathbf{F}_n(dx)$$
$$= \arg \inf_{\theta \in \Theta} \min_{\substack{Y \in \mathbb{R}^n \\ \sum_{i=1}^{n-1} K(i/n)(y_{i+1:n} - y_{i:n}) = f(\theta)}} \frac{1}{n} \sum_{i=1}^n |x_{i:n} - y_{i:n}|^2$$

with l_r given by equation (2.9).

As transportation is well defined for measure in \mathbb{R}^d in contrast with quantile measures, this may appear as a way to generalize L-moments constrained models and associated estimators of the form (6.3); we could also consider estimators of the form (6.6), importing henceforth optimal transportation concepts in the field of multivariate quantile models; see [13].

6.3 Relation to elasticity theory

It may be of interest for the statistician to observe that, besides the probabilistic context of semiparametrics, the minimization of a φ divergence over a class of functions defined by L-moments (see (6.2)) is in the same vein as finding the deformation of a solid under a given force L and given boundary constraints. Let us consider a solid defining a domain $\Omega \subset \mathbb{R}^3$. This solid can be deformed under the action of volumetric or surface forces. This deformation can be described by a function $T : \Omega \to \mathbb{R}^3$. The deformed solid will be defined on the volume $T(\Omega)$. The gradient of deformation is then ∇T .

The general equations describing the equilibrium of the solid under volumetric forces L defined on Ω read (we omit boundary forces)

$$-\operatorname{div} S = L$$

where S is a tensor describing the configuration of the solid [4]. Hyper-elasticity is often assumed i.e. the solid is supposed to dissipate no energy during the deformation. In mathematical terms, this means the existence of a function φ such that

$$S(T) = \frac{\partial \varphi}{\partial T}(T)$$

From these above relations, the energy of deformation is expressed on the form [22][4]

$$\mathcal{E}(T) = \int_{\Omega} \varphi(\nabla T(x)) dx - \int_{\Omega} L(x) \cdot T(x) dx.$$

 φ is usually convex and represents physical properties of the solid. It is then customary in mechanical physics to assume the principle of least action and to study the T minimizing the variational problem

$$\inf_{T \text{ admissible}} \mathcal{E}(T).$$

The space of admissible T describes the constraints, such as boundary conditions. If we could write the volumetric force term (namely the right hand side of $\mathcal{E}(T)$) as fixed constraints, we remark similarities with the estimation given by equation 6.2

$$\hat{\theta}_n = \arg \inf_{\theta \in \Theta} \inf_{\int_{\Omega} L(x) \cdot T(x) dx = f(\theta)} \int_{\Omega} \varphi\left(\nabla T(x)\right) dx$$

Moreover, microscopic and macroscopic scales can be related through convergence results. Let us present the microscopic models of the same solid represented by N particles $x_1, ..., x_N$, corresponding for example to the intersection of Ω with a lattice of scale ϵ . If V denotes an interaction potential, the energy of the solid subjected to a deformation T would be

$$\mathcal{E}_N(T) = \frac{\epsilon^3}{2} \sum_{i=1}^N \sum_{j \neq i} V\left(\frac{T(x_i) - T(x_j)}{\epsilon}\right) - \sum_{i=1}^N L(x_i) \cdot T(x_i)$$

where for any 3×3 matrix M

$$\varphi(M) = \frac{1}{2} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} V(Mk).$$

Under some assumptions (see [3]), it can be proved that if $\epsilon \to 0$ (i.e. $N \to \infty$), then

$$\mathcal{E}_N(T) \to_{N \to \infty} \mathcal{E}(T).$$

This short account may give us some intuition about the present estimation

7 Asymptotic properties of the L-moment estimators

In this section, we study the convergence of the estimator given by the equation (6.2). The proof of the two asymptotic theorems are postponed to the Appendix.

Theorem 7.1 Let $x_1, ..., x_n$ be an observed sample drawn iid from a distribution F_0 with finite variance. Assume that

- there exists θ_0 such that $F_0 \in L_{\theta_0}$, θ_0 is the unique solution of the equation $f(\theta) = f(\theta_0)$
- f is continuous and $\Theta \subset \mathbb{R}^d$ is compact
- the matrix $\Omega_0 = \int K(F_0(x)) K(F_0(x))^T dx$ is non singular.

$$\theta_n \to \theta_0$$
 in probability as $n \to \infty$.

We may now turn to the limit distribution of the estimator. Let

- $J_0 = J_f(\theta_0)$ be the Jacobian of f with respect to θ in θ_0
- $M = (J_0^T \Omega^{-1} J_0)^{-1}$
- $H = M J_0^T \Omega^{-1}$
- $P = \Omega^{-1} \Omega^{-1} J_0 M J_0^T \Omega^{-1}$

Theorem 7.2 Let $x_1, ..., x_n$ be an observed sample drawn iid from a distribution F_0 with finite variance. We assume that the hypotheses of Theorem 7.1 holds. Moreover, we assume that

- $\theta_0 \in int(\Theta)$
- J_0 has full rank
- *f* is continuously differentiable in a neighborhood of θ_0

Then,

$$\sqrt{n} \begin{pmatrix} \hat{\theta}_n - \theta_0 \\ \hat{\xi}_n \end{pmatrix} \to_d \mathcal{N}_{d+l} \left(0, \begin{pmatrix} H\Sigma H^T & 0 \\ 0 & P\Sigma P^T \end{pmatrix} \right)$$

The estimator of the minimum of the divergence from **F** onto the model, namely $2n \left[\hat{\xi}_n^T f(\hat{\theta}_n) - \int \psi(\hat{\xi}_n^T K(F_n(x)) dx\right]$, does not converge to a χ^2 -distribution as in the case of moment condition models [25]. However, we can state an alternative result.

Corollary 7.1 Let us assume that the hypotheses of Theorem 7.2 hold. Let $S_n := n\hat{\xi}_n^T (P_n \Sigma_n P_n^T)^{-1} \hat{\xi}_n$ with P_n and Σ_n the respective empirical versions of P and Σ . If $P\Sigma P$ is non singular then

 $S_n \to_d \chi^2(l)$

where $\chi^2(l)$ denotes a chi-square distribution with l degrees of freedom.

Proof. From Theorem 7.2, we have that

$$n^{1/2}\hat{\xi}_n \to_d X = \mathcal{N}_l(0, P\Sigma P)$$

where X denotes such a multivariate Gaussian random vector. Furthermore

 $P_n \Sigma_n P_n \to_p P \Sigma P.$

Hence, for n large enough, $P_n \Sigma_n P_n$ is invertible and by Slutsky Theorem

$$n\hat{\xi}_n^T (P_n \Sigma_n P_n)^{-1} \hat{\xi}_n \to_p X^T X =_d \chi^2(l).$$

Since the weak convergence of S_n to a chi-square distribution is independent of the value of θ_0 , this result may be used in order to build confidence regions related to the semi-parametric model.

8 Numerical applications : Inference for Generalized Pareto family

8.1 Presentation

The Generalized Pareto Distributions (GPD) are known to be heavy-tailed distributions. They are classically parametrized by a location parameter m, which we assume to be 0, a scale parameter σ and a shape parameter ν . They can be defined through their density :

$$f_{\sigma,\nu}(x) = \begin{cases} \frac{1}{\sigma} \left(1 + \nu \frac{x}{\sigma} \right)^{-1-1/\nu} \mathbb{1}_{x>0} & \text{if } \nu > 0\\ \frac{1}{\sigma} \exp\left(\frac{x}{\sigma}\right) \mathbb{1}_{x>0} & \text{if } \nu = 0\\ \frac{1}{\sigma} \left(1 + \nu \frac{x}{\sigma} \right)^{-1-1/\nu} \mathbb{1}_{-\sigma/\nu>x>0} & \text{if } \nu < 0 \end{cases}$$

Let us remark that if $\nu \ge 1$, the GPD does not have a finite expectation. We perform different estimations of the scale and the shape parameter of a GPD from samples with size n = 100.

We will estimate the parameters in the model composed by the distributions of all r.v's X whose second, third and fourth L-moments verify

$$\begin{cases} \lambda_2 = \frac{\sigma}{(1-\nu)(2-\nu)} \\ \frac{\lambda_3}{\lambda_2} = \frac{1+\nu}{3-\nu} \\ \frac{\lambda_4}{\lambda_2} = \frac{(1+\nu)(2+\nu)}{(3-\nu)(4-\nu)} \end{cases}$$
(8.1)

for any $\sigma > 0, \nu \in \mathbb{R}$. These distributions share their first L-moments with those of a GPD with scale and shape parameter σ and ν (see [19]). This estimation will be compared with classical parametric estimators detailed hereafter.

8.2 Moments and L-moments calculus

The variance and the skewness of the GPD are given by

$$\begin{cases} var = \mathbb{E}[(X - \mathbb{E}[X])^2] = \frac{\sigma^2}{(1 - \nu)^2(1 - 2\nu)} \\ t_3 = \mathbb{E}[\left(\frac{X - \mathbb{E}[X]}{\mathbb{E}[(X - \mathbb{E}[X])^2]}\right)^3] = \frac{2(1 + \nu)\sqrt{1 - 2\nu}}{1 - 3\nu} \end{cases}$$

Let us remark that var and t_3 respectively exist since $\nu < 1/2$ and $\nu < 1/3$.

On the other hand, the first L-moments are given by equation 8.1. Assuming $\nu < 1$ entails existence of the L-moments.

8.3 Simulations

We perform N = 500 runs of the following estimators

- the estimation proposed in this article (equation (6.2)) for the χ^2 -divergence and the modified Kullback (KL_m) divergence with the constraints estimated on the L-moments of order 2, 3, 4
- the estimate defined through the L-moment method, based on the empirical second L-moment $\hat{\lambda}_2$ and the fourth L-moment ratio $\hat{\tau}_4 = \frac{\lambda_4}{\lambda_2}$

$$\hat{\nu} = \frac{7\hat{\tau}_4 + 3 - \sqrt{(\hat{\tau}_4^2 + 98\hat{\tau}_4 + 1)^2}}{2(\hat{\tau}_4 - 1)}$$
$$\hat{\sigma} = \hat{\lambda}_2(1 - \hat{\nu})(2 - \hat{\nu})$$

• the estimate defined through the moment method estimated from the empirical variance $v\hat{a}r$ and skewness \hat{t}_3

$$\hat{\nu} = \frac{2(1+\hat{t}_3)\sqrt{1-2\hat{t}_3}}{1-3\hat{t}_3}$$
$$\hat{\sigma} = \sqrt{v\hat{a}r(1-\hat{t}_3)^2(1-2\hat{t}_3)}$$

		n = 30			n = 100			
Estimation method	Parameter	Mean	Median	StD	Mean	Median	StD	
χ^2 -divergence	σ	4.68	4.41	2.52	3.80	3.75	0.90	
KL_m -divergence	σ	6.44	4.77	8.02	4.08	3.95	4.00	
L-moment method	σ	5.67	4.98	3.44	3.96	3.80	1.09	
Moment method	σ	17.17	10.45	62.95	17.15	11.64	19.52	
MLE	σ	3.33	3.17	1.14	3.08	3.07	0.57	
χ^2 -divergence	ν	0.38	0.39	0.24	0.55	0.55	0.16	
KL_m -divergence	ν	0.37	0.38	0.24	0.38	0.37	0.16	
L-moment method	ν	0.33	0.38	0.31	0.54	0.56	0.18	
Moment method	ν	0.08	0.12	0.12	0.21	0.22	0.06	
MLE	ν	0.61	0.63	0.33	0.68	0.69	0.17	

Table 1: Estimates of GPD scale and shape parameters for $\nu = 0.7$ and $\sigma = 3$ (the moment method has little sense since $\nu > 0.5$) for the first scenario without outliers

• the MLE defined in the GPD family

We present the following different features for any of the above estimators

- the mean of the N estimates based on the N runs
- the median of the N estimates based on the N runs
- the standard deviation of the N estimates
- the L_1 distance between the estimated generalized Pareto density and the true density, namely

$$\int_{x\geq 0} |f_{\hat{\sigma},\hat{\nu}}(x) - f_{\sigma,\nu}(x)| dx$$

which, by Scheffé Lemma, equals twice the maximum error committed substituting $f_{\sigma,\nu}$ by $f_{\hat{\sigma},\hat{\nu}}$

$$\int_{x\geq 0} |f_{\hat{\sigma},\hat{\nu}}(x) - f_{\sigma,\nu}(x)| dx = 2 \sup_{A\in\mathcal{B}(\mathbb{R})} \left| \int_A f_{\hat{\sigma},\hat{\nu}}(x) - \int_A f_{\sigma,\nu}(x)| dx \right|.$$

Finally, we present four different scenarios which illustrate robusness properties of any of the above estimators, as well as their behavior under misspecification:

- a first scenario without outliers : samples of size 30 or 100 are drawn from a GPD
- two more scenarios with 10% outliers : samples of size 27 or 90 are drawn from a GPD. The remaining points are drawn from a Dirac the value of which depends on the shape parameter
- a fourth scenario without outliers but with misspecification : samples of size 30 or 100 are drawn from a Weibull distribution.

Unsurprisingly, the MLE performs well under the model and the L-moment method has an overall better behavior than the classical moment method for the considered heavy-tailed distributions (see Table 1). Furthermore, we observe that the χ^2 - divergence is more robust than the modified Kullback as indeed expected.

The interesting result lies in their behavior with outliers and misspecification. Indeed, we can see that L-momentbased estimators perform well on the shape parameter whereas the MLE provides a good estimation of the scale

		n = 30			n = 100		
Estimation method	Parameter	Mean	Median	StD	Mean	Median	StD
χ^2 -divergence	σ	12.43	12.24	2.83	12.29	12.21	1.62
KL_m -divergence	σ	24.01	19.36	49.38	27.30	20.99	48.75
L-moment method	σ	22.27	20.83	5.69	21.68	21.03	3.09
Moment method	σ	80.97	76.27	20.89	80.93	76.84	31.09
MLE	σ	3.06	2.88	1.08	2.88	2.86	0.55
χ^2 -divergence	ν	0.55	0.55	0.05	0.54	0.54	0.04
KL_m -divergence	ν	0.50	0.52	0.24	0.54	0.49	0.27
L-moment method	ν	0.54	0.54	0.06	0.54	0.53	0.04
Moment method	ν	0.07	0.08	0.02	0.08	0.07	0.03
MLE	ν	1.48	1.44	0.22	1.50	1.49	0.11

Table 2: Estimates of GPD scale and shape parameters for $\nu = 0.7$ and $\sigma = 3$ for a sample with 10% outliers of value 300 (the moment method has little meaning since $\nu > 0.5$)

			n = 30		n = 100		
Estimation method	Parameter	Mean	Median	StD	Mean	Median	StD
χ^2 -divergence	σ	4.32	4.23	0.91	4.45	4.42	0.51
KL_m -divergence	σ	5.04	4.90	1.15	5.07	5.08	0.67
L-moment method	σ	5.18	5.04	1.44	5.11	5.04	0.75
Moment method	σ	8.64	8.44	0.92	8.54	8.48	0.50
MLE	σ	3.12	3.08	0.87	3.08	3.05	0.49
χ^2 -divergence	ν	0.27	0.28	0.08	0.27	0.27	0.05
KL_m -divergence	ν	0.25	0.25	0.09	0.24	0.24	0.05
L-moment method	ν	0.24	0.24	0.10	0.24	0.24	0.06
Moment method	ν	0.01	0.02	0.04	0.01	0.02	0.02
MLE	ν	0.56	0.54	0.17	0.55	0.55	0.09

Table 3: Estimates of GPD scale and shape parameters for $\nu = 0.1$ and $\sigma = 3$ for a sample with 10% outliers of value 30

	n = 30				n = 100			
Estimation method	Sc 1	Sc 2	Sc 3	Sc 4	Sc 1	Sc 2	Sc 3	Sc 4
χ^2 -divergence	2.53	7.20	3.16	2.63	1.55	7.32	3.28	1.80
L-moment method	3.10	10.07	4.09	4.31	1.70	9.93	4.07	3.51
Moment method	6.79	14.47	7.07	8.69	6.91	14.42	6.98	9.98
MLE	1.78	2.83	2.68	11.69	0.97	2.42	2.33	9.25

Table 4: L_1 -distances (to be multiplied by 10^{-4}) between GPD densities for different scenarios; Scenario (Sc) 1 corresponds to a simulated GPD with $\nu = 0.7$ and $\sigma = 3$; Scenario 2 corresponds to a simulated GPD with $\nu = 0.7$, $\sigma = 3$ and 10% outliers of value 300; Scenario 3 corresponds to a simulated GPD with $\nu = 0.1$, $\sigma = 3$ and 10% outliers of value 30; Scenario 4 corresponds to a simulated Weibull distribution with $\nu = 0.4$ and $\sigma = 3$





(a) Simulated GPD with $\nu = 0.7$ and $\sigma = 3$

(b) Simulated GPD with $\nu = 0.7, \sigma = 3$ and 10% outliers of value 300



(c) Simulated GPD with $\nu = 0.1$, $\sigma = 3$ and 10%(d) Simulated Weibull distribution with $\nu = 0.4$ and outliers of value 30 $\sigma = 3$

Figure 2: Estimated GPD densities with estimated parameters for simulated scenarios (with a logarithmic scale)

parameter but overestimates the shape parameter. In that sense, the L-moments method can be used for the robust estimation of the shape parameter of a GPD in case of contamination by outliers. However, even with outliers, the MLE performs well in term of L_1 -distance computed on the estimated densities. It is under misspecification that the performance of the MLE drops as measured by the L_1 criterion. This confirms the flexibility of models defined only through moment or L-moment equations that are less dependent on the GPD model.

Moreover, the L_1 -distance between the model and its estimation has an order between 10^{-3} and 10^{-4} . The error committed by the estimation under models defined through L-moments conditions is the most stable over the proposed scenarios. We can then affirm that we can estimate the probability of events if the true value of this probability is of order 10^{-3} (the error of estimation for the estimator based on L-moments method would approximately be of 30% depending on the size of the sample and the scenario).

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A Proofs

A.1 Proof of Lemma 2.1

Let $x \in \mathbb{R}$. We denote by F the cdf of X and by A_t the event

$$A_t = \{ x \in \mathbb{R} \text{ s.t. } F(x) \ge t \}$$

We then have $Q(t) = \inf A_t$. We wish to prove :

$$\{t \in [0;1] \text{ s.t. } Q(t) \le x\} = \{t \in [0;1] \text{ s.t. } t \le F(x)\}$$
(A.1)

We temporarily admit this assertion. Then

$$\mathbb{P}[Q(U) \le x] = \mathbb{P}[U \le F(x)] = F(x)$$

which ends the proof. It remains to prove (A.1).

First, the definition of Q yields

$$\left\{t \le F(x)\right\} \Rightarrow \left\{x \in A_t\right\} \Rightarrow \left\{Q(t) \le x\right\}.$$

Secondly, let t be such that $Q(t) \le x$. Then by monotonicity of F, $F(Q(t)) \le F(x)$. We then claim that

$$Q(t) \in A_t.$$

Indeed, let us suppose the contrary and consider a strictly decreasing sequence $x_n \in A_t$ such that

$$\lim_{n \to \infty} x_n = \inf A_t = Q(t).$$

By right continuity of F

$$\lim_{n \to \infty} F(x_n) = F(Q(t))$$

and, on the other hand, by definition of A_t ,

$$\lim_{n \to \infty} F(x_n) \ge t$$

i.e. $Q(t) \in A_t$ which contradicts the hypothesis. Then $Q(t) \in A_t$ i.e. $t \leq F(Q(t))$ thus $t \leq F(x)$. We have proved that

$$\left\{Q(t) \le x\right\} \Rightarrow \left\{t \le F(x)\right\}.$$

A.2 Proof of Lemma 2.2

Let us recall that the support of a measure μ defined on $X \subset \mathbb{R}$ is the largest closed set $C \subset X$ such that

$$U \in B(X)$$
 and $U \cap C \neq \emptyset \Rightarrow \mu(U \cap C) > 0$

where B(X) denotes the Borel sets in X. Let S be the support of \mathbf{F}^{-1} . Then $[0;1]\setminus S$ is an open set in [0;1] i.e. a countable union of intervals $\bigcup_{i\geq 1} t_{2i}, t_{2i+1}$ [and

$$\int_{0}^{1} a(F^{-1}(t))dt = \int_{S} a(F^{-1}(t))dt + \sum_{i\geq 1} \int_{]t_{2i};t_{2i+1}[} a(F^{-1}(t))dt$$
$$= \int_{F^{-1}(S)} a(x)dF(x) + \sum_{i\geq 1} a(F^{-1}(t_{2i}))(t_{2i+1} - t_{2i})$$
$$= \int_{F^{-1}(S)} a(x)dF(x) + \sum_{i\geq 1} \int_{\{F^{-1}(t_{2i})\}} a(x)dF(x)$$
$$= \int_{F^{-1}(S)\cup\left(\cup_{i\geq 1}\{F^{-1}(t_{2i})\}\right)} a(x)dF(x).$$

The second equality stems from the definition of the quantile as left-continuous function and from the fact that F^{-1} is strictly monotone on S.

As F^{-1} is constant on the open interval $[t_{2i}; t_{2i+1}[, \{F^{-1}(t_{2i})\} = F^{-1}([t_{2i}; t_{2i+1}[))$. Hence

$$F^{-1}(S) \cup \left(\bigcup_{i \ge 1} \{ F^{-1}(t_{2i}) \} \right) = F^{-1}([0;1])$$

= $\{ x \in \mathbb{R} \text{ s.t. there exists } t \text{ with } F^{-1}(t) = x \} = supp(F).$

We conclude the first part of the proof since

$$\int_{supp(F)} a(x)dF(x) = \int_{\mathbb{R}} a(x)dF(x).$$

The second part of the proof can be proved similarly since the above arguments are not particular to a specific measure.

A.3 **Proof of Proposition 5.1**

The proof is directly adapted from the proof of Theorem II.2 of Csiszár et al. [11]. Let us begin with the fundamental lemma inspired from Theorem 2.9 of Borwein and Lewis[5].

Lemma A.1 Let $C : \Omega \to \mathbb{R}^l$ be an array of bounded functions such that

$$\int_{\Omega} \|C(x)\| d\mu(x) < \infty.$$

We denote

$$L_{C,a} = \left\{ g \text{ s.t. } \int_{\Omega} g(t)C(t)d\mu(t) = a \right\}.$$

If there exists some g in $L_{C,a}$ such that $a_{\varphi} < g < b_{\varphi} \mu$ -a.s and $\int_{\Omega} ||g(t)C(t)|| d\mu(t) < \infty$, then there exists $a'_{\varphi} > a_{\varphi}$, $b'_{\varphi} < b_{\varphi}$ and $g_b \in L_{C,a}$ such that $a'_{\varphi} \leq g_b(x) \leq b'_{\varphi}$ for all $x \in \Omega$.

Proof. Let *L* denotes the subspace of \mathbb{R}^l composed by the vectors representable as $\int_{\Omega} gCd\mu$ for some $g: \Omega \to \mathbb{R}^l$. Let us denote by a_n a decreasing sequence $a_n \to a_{\varphi}$, by b_n a increasing one $b_n \to b_{\varphi}$ and let T_n be the set

$$T_n = \{x \in \Omega \text{ s.t. } a_n \le g(x) \le b_n\}$$
 .

We first claim that, for n large enough

$$L = L_n = \left\{ \int_{\Omega} hCd\mu \text{ with } h(x) = 0 \text{ if } x \notin T_n \text{ and } h \text{ bounded} \right\}.$$

Indeed, if not, we can build a sequence of vectors v_n such that $||v_n|| = 1$, $v_n \in L^{\perp}$ and $v_n \to v \in L$. Furthermore, $v_n \in L^{\perp}$ means

$$\langle v_n, \int_{\Omega} hCd\mu \rangle = \int_{\Omega} h\langle v_n, C \rangle d\mu = 0$$

then $\langle v_n, C \rangle = 0$ for all $x \in T_n \mu$ -a.s. Hence $\langle v, C \rangle = 0 \mu$ -a.s. and $v \in L^{\perp}$ which contradicts $v \in L$ with ||v|| = 1. Let us then fix some n_0 such that $L_{n_0} = L$. We denote by

$$L_n(\delta) = \left\{ \int_{\Omega} hCd\mu \text{ with } h(x) = 0 \text{ if } x \notin T_n \text{ and } |h(x)| < \delta \text{ for } x \in \Omega \right\}.$$

Then, the affine hull of $L_n(\delta)$ is the vector space L and $0 \in L_n(\delta)$. We can consider the function g_n

$$g_n(x) = \begin{cases} a_n & \text{if } g(x) < a_n \\ g(x) & \text{if } b_n \le g(x) \le a_n \\ b_n & \text{if } g(x) > b_n \end{cases}$$

Then $\|\int_{\Omega} (g_n - g)Cd\mu\| \to_{n\to\infty} 0$. Indeed we can apply the dominated convergence theorem since, for any $x \in \Omega$, $g_n(x) \to g$ and

$$\begin{aligned} \|(g_n(x) - g(x))C(x)\| &= \|\mathbb{1}_{g(x) < a_n}(a_n - g(x))C(x) + \mathbb{1}_{g(x) > b_n}(g(x) - b_n)C(x)\| \\ &\leq (\|a_0 - g(x)\| + \|b_0 - g(x)\|)\|C(x)\| \\ &\leq (\|a_0\| + \|b_0\|)\|C(x)\| + 2\|g(x)\|\|C(x)\| \end{aligned}$$

which is μ -measurable by hypothesis.

We conclude that $\int_{\Omega} (g_n - g)Cd\mu \in L_{n_0}(\delta)$ for n large enough because $0 \in L_{n_0}(\delta)$. Hence there exists h such that $\int_{\Omega} (g_n - g)Cd\mu = \int_{\Omega} hCd\mu$, |h(x)| = 0 for $x \notin T_{n_0}$ and $|h(x)| < \delta$ for x in T_{n_0} . Therefore for $x \in \Omega$, $\min(a_n, a_{n_0} - \delta) \leq g_n(x) + h(x) \leq \min(b_n, b_{n_0} + \delta)$ and $\int_{\Omega} (g_n + h)Cd\mu = \int_{\Omega} gCd\mu$. As δ

is arbitrarily small, h is the null function.

We can now prove the duality equality. Let note for $c \in \mathbb{R}^l I(c) = \inf_{\int gCd\mu = c} \int_{\Omega} \varphi(g) d\mu$ and

$$J(c) = \begin{cases} 0 & \text{if } c = a \\ +\infty & \text{otherwise} \end{cases}$$

Then

$$\inf_{g \in L_{C,a}} \int \varphi(g) d\mu = \inf_{c \in \mathbb{R}^l} I(c) + J(c).$$

Recall that the Fenchel duality theorem ([28] p327) states that if $ri(dom(I)) \cap ri(dom(J)) \neq \emptyset$ then

$$\inf_{c \in \mathbb{R}^l} I(c) + J(c) = \max_{\xi \in \mathbb{R}^l} -I^*(\xi) - J^*(-\xi)$$

We prove that $ri(dom(I)) \cap ri(dom(J)) \neq \emptyset$. Note that $ri(dom(J)) = \{a\}$. It suffices then to prove that a belongs to int(dom(I)) for the topology induced by L. By the above Lemma A.1 there exists g_b such that $a_{\varphi} < a'_{\varphi} \leq a'_{\varphi} \leq a'_{\varphi} \leq a'_{\varphi}$.

 $g_b(x) \leq b'_{\varphi} < b_{\varphi}$ for all $x \in \Omega$. Since $a + L_n(\delta)$ is a neighborhood of a included in dom(I) for δ sufficiently small, it holds that $a \in int(L_n(\delta)) \subset int(dom(I))$.

It remains now to compute the conjugates of I and J.

$$I^{*}(\xi) = \sup_{c \in \mathbb{R}^{l}} \langle \xi, c \rangle - \inf_{g, \int gCd\mu = c} \varphi(g)d\mu$$

$$= \sup_{c \in \mathbb{R}^{l}} \sup_{g, \int gCd\mu = c} \langle \xi, c \rangle - \varphi(g)d\mu$$

$$= \sup_{g} \langle \xi, \int gCd\mu \rangle - \varphi(g)d\mu$$

$$= \sup_{g} \int \langle \xi, C \rangle g - \varphi(g)d\mu$$

$$= \int \psi(\langle \xi, C \rangle)d\mu$$

This equality is referred to as the integral representation of I^* . The last equality can be rigorously justified (see for example [24]).

Furthermore, $J^*(-\xi) = -\langle \xi, a \rangle$ which closes the first part of the proof, namely

$$\inf_{g \in L_{C,a}} \int_{\Omega} \varphi(g) \, d\mu = \sup_{\xi \in \mathbb{R}^l} \langle \xi, a \rangle - \int_{\Omega} \psi(\langle \xi, C(x) \rangle) d\mu$$

As we assume ψ differentiable, then $\xi \mapsto \langle \xi, a \rangle - \int_{\Omega} \psi(\langle \xi, C(x) \rangle) d\mu$ is differentiable as well. It follows that any critical point is the solution of

$$\int_{\Omega} \psi'(\langle \xi, C(x) \rangle) C(x) d\mu = a.$$

Furthermore, as φ is strictly convex, ψ is strictly concave and for $\xi, \xi' \in \mathbb{R}^l$ and $t \in [0, 1]$ it holds

$$\begin{split} \langle (1-t)\xi + t\xi', a \rangle &- \int_{\Omega} \psi(\langle (1-t)\xi + t\xi', C(x) \rangle) d\mu \\ &= \langle (1-t)\xi + t\xi', a \rangle - \int_{\Omega} \psi((1-t)\langle \xi, C(x) \rangle + t\langle \xi', C(x) \rangle) d\mu \\ &< (1-t) \left[\langle \xi, a \rangle - \int_{\Omega} \psi(\xi, C(x) \rangle) d\mu \right] + t \left[\langle \xi', a \rangle - \int_{\Omega} \psi(\xi', C(x) \rangle) d\mu \right] \end{split}$$

i.e. the functional $\xi \to \langle \xi, a \rangle - \int_{\Omega} \psi(\xi, C(x)) d\mu$ is strictly convex which proves the uniqueness of ξ^* . The continuity of $a \mapsto \xi^*(a)$ comes from the implicit function theorem. If we note $D(\xi) = \int \psi'(\langle \xi, C(x) \rangle) C(x) d\mu$ then D is continuously differentiable with a Jacobian given by

$$J_D(\xi) = \int \psi''(\langle \xi, C(x) \rangle) C(x) C(x)^T d\mu$$

which is positive definite thanks to the strict convexity of ψ .

A.4 Proof of Proposition 6.3

Note

$$D_{\varphi}(x,y) = \sum_{i=1}^{n-1} \varphi\left(\frac{y_{i+1} - y_i}{x_{i+1:n} - x_{i:n}}\right) (x_{i+1:n} - x_{i:n}).$$

Assuming that (6.4) holds then (6.5) follows from equation (4.4). Indeed, since φ is infinite for negative values, it holds

$$\inf_{\substack{y \in \mathbb{R}^n \\ \sum_{i=1}^{n-1} K(i/n)(y_{i+1:n} - y_{i:n}) = f(\theta) \\ = \inf_{\substack{(y_1 < \dots < y_n) \in \mathbb{R}^n \\ \sum_{i=1}^{n-1} K(i/n)(y_{i+1} - y_i) = f(\theta)}} D_{\varphi}(x, y).$$

We now turn to (6.4). The minimization problem can be decomposed into

$$= \inf_{\substack{T \in \mathcal{C}_A: T \# \mathbf{F}_n \in L_\theta \\ \sum_{i=1}^n K(i/n)(y_{i+1:n} - y_{i:n}) = f(\theta)}} \int_{\mathbb{R}} \varphi\left(\frac{dT}{d\lambda}\right) d\lambda$$

by denoting

$$I_{\varphi}(x,y) = \inf_{T \in \mathcal{C}_A: T(x_{i:n}) = y_i} \int_{\mathbb{R}} \varphi\left(\frac{dT}{d\lambda}\right) d\lambda$$

This minimization problem has an explicit solution. Indeed

$$I_{\varphi}(x,y) = \inf_{T \in \mathcal{C}_A: T(x_{i+1:n}) - T(x_{i:n}) = y_{i+1} - y_i} \int_{\mathbb{R}} \varphi\left(\frac{dT}{d\lambda}\right) d\lambda.$$

If $T \in \mathcal{C}_A$ satisfies $T(x_{i:n}) = y_i$ for $1 \le i \le n$ then as T is absolutely continuous, it holds for all $1 \le i \le n-1$

$$\int_{x_{i:n}}^{x_{i+1:n}} \frac{dT}{d\lambda} d\lambda = y_{i+1} - y_i.$$

Conversely, if $S : \mathbb{R} \to \mathbb{R}$ is such that for all i between 1 and n-1

$$\int_{x_{i:n}}^{x_{i+1:n}} S(x)\lambda(dx) = y_{i+1} - y_i$$

then $T: x \mapsto \int_0^x S(x)\lambda(dx) \in \mathcal{C}_A$ and $T(x_{i+1:n}) - T(x_{i:n}) = y_{i+1} - y_i$. We thus obtain

$$I_{\varphi}(x,y) = \inf_{\substack{S: \int_{\mathbb{R}} S(x) \mathbb{1}_{\{x_{i:n} \le x \le x_{i+1:n}\}} \lambda(dx) \\ = y_{i+1} - y_i, 1 \le i \le n-1}} \lambda(dx) \int_{\mathbb{R}} \varphi\left(S(x)\right) \lambda(dx)$$

From Proposition 5.1, it then holds, since $\psi(0) = 0$

$$\inf_{S: \int_{\mathbb{R}} S(x) \mathbb{1}_{\{x_{i:n} \le x \le x_{i+1:n}\}} \lambda(dx) = y_{i+1} - y_i} \int_{\mathbb{R}} \varphi(S(x)) \lambda(dx)$$

$$= \sup_{(\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}} \sum_{i=1}^{n-1} \xi_i (y_{i+1} - y_i) - \psi(\xi_i) (x_{i+1:n} - x_{i:n})$$

$$= \sum_{i=1}^{n-1} \sup_{\xi_i \in \mathbb{R}} \xi_i (y_{i+1} - y_i) - \psi(\xi_i) (x_{i+1:n} - x_{i:n})$$

$$= \sum_{i=1}^{n-1} (x_{i+1:n} - x_{i:n}) \sup_{\xi_i \in \mathbb{R}} \xi_i \frac{y_{i+1} - y_i}{x_{i+1:n} - x_{i:n}} - \psi(\xi_i)$$

$$= \sum_{i=1}^{n-1} (x_{i+1:n} - x_{i:n}) \varphi\left(\frac{y_{i+1} - y_i}{x_{i+1:n} - x_{i:n}}\right)$$

which concludes the proof.

A.5 **Proof of Theorem 7.1**

The arguments of this proof and of the following one are similar to the ones given by Newey and Smith in [25] for their Theorem 3.1; the essential argument is a Taylor expansion of the functionals in equation (5.4). Let begin with a lemma adapted from Theorem 6 due to Stigler [29] :

Lemma A.2 Let $x_1, ..., x_n$ be an observed sample drawn iid from a distribution F with finite variance. We note F_n the empirical distribution of the sample.

Let $A: [0;1] \to \mathbb{R}^l$ be a continuously derivable function such that A' is bounded F^{-1} -a.e. Then

$$n^{1/2}\left(\int x dA(F_n(x)) - \int x dA(F(x))\right) \to_d N(0, \Sigma_A)$$

with

$$\Sigma_A = \iint \left[F(\min(x,y)) - F(x)F(y) \right] A'(F(x))A'(F(y))^T dxdy.$$

In the following, we will note $\frac{dT}{d\lambda}(x)=T'(x)$ for all $x\in\mathbb{R}$.

First step : maximization step

Clearly, it holds

$$\inf_{T \in \cup_{\theta} L_{\theta}''(F_n)} \int_{\mathbb{R}} \varphi(T'(x)) dx \le \inf_{T \in L_{\theta_0}''(F_n)} \int_{\mathbb{R}} \varphi(T'(x)) dx.$$
(A.2)

By Taylor-Lagrange expansion, there exists some D > 0 such that for n large enough and for any t in $[1 - n^{-1/4}; 1 + n^{1/4}]$

$$\varphi(t) \le \frac{D}{2}(t-1)^2$$

holds.

We may then majorize the RHS in (A.2) by the solution of the quadratic case. Let

$$T'_{0,n}(x) := 1 + (f(\theta_0) - m_n)^T \Omega_n^{-1} K(F_n(x))$$

where $m_n := \int K(F_n(x)) dx$ and $\Omega_n := \int_{\mathbb{R}} K(F_n(x)) K(F_n(x))^T dx$. As $T'_{0,n} \in L''_{\theta}(F_n)$, it holds

$$\inf_{T \in L_{\theta_0}''(F_n)} \int_{\mathbb{R}} \varphi(T'(x)) dx \le \int_{\mathbb{R}} \varphi(T'_{0,n}(x)) dx$$

From Lemma A.2, we deduce that $\Omega_n \to \Omega$ in probability. As Ω is non singular, for *n* large enough, Ω_n is non singular and $T'_{0,n}$ is well defined.

As $||f(\theta_0) - m_n|| = O_P(n^{-1/2})$ from Lemma A.2 and $||\Omega_n^{-1}|| = O_P(1)$, for almost all $x \in \mathbb{R}$,

$$T'_{0,n}(x) = 1 + O_P(n^{-1/2})$$

and we can apply a Taylor-Lagrange maximization

$$\varphi(T'_{0,n}(x)) \le \frac{D}{2} (f(\theta_0) - m_n)^T \Omega_n^{-1} K(F_n(x)) K(F_n(x))^T \Omega_n^{-1} (f(\theta_0) - m_n).$$

By integration in the above display

$$\int_{\mathbb{R}} \varphi(T'_{0,n}(x)) dx \leq \frac{D}{2} (f(\theta_0) - m_n) \Omega_n^{-1} \left[\int_{\mathbb{R}} K(F_n(x)) K(F_n(x))^T dx \right] \Omega_n^{-1} (f(\theta_0) - m_n) \\ \leq \|f(\theta_0) - m_n\|^2 \|\Omega_n^{-1}\| = O_P(n^{-1}).$$

Second step : minimization step

Since Θ is compact, and φ is strictly convex, and $\theta \mapsto \inf_{T \in L'_{\theta}(F_n)} \int_{\mathbb{R}} \varphi(T'(x)) dx$ is continuous (see Proposition 5.1), it follows that $\hat{\theta}$ is well defined and the duality equality states

$$\inf_{T \in L_{\hat{\theta}_n}''(F_n)} \int_{\mathbb{R}} \varphi(T'(x)) dx = \sup_{\xi \in \mathbb{R}^l} \xi^T f(\hat{\theta}_n) - \int \psi(\xi^T K(F_n(x))) dx$$
$$\geq \xi_n^T f(\hat{\theta}_n) - \int \psi(\xi_n^T K(F_n(x))) dx$$

with

$$\xi_n = n^{-1/2} \frac{f(\theta_n) - m_n}{\|f(\hat{\theta}_n) - m_n\|}$$

Therefore

$$\xi_n^T K(F_n(x)) = O_P(n^{-1/2})$$
 for a.e $x \in \mathbb{R}$.

By Taylor-Lagrange expansion, there exists a constant C > 0 such that $|\psi(x) - x| < Cx^2$ in a neighborhood of 0. Thus, for n large enough

$$\int \psi(\xi_n^T K(F_n(x))) dx - \xi_n^T m_n < C \int \xi_n^T K(F_n(x)) K(F_n(x))^T \xi_n dx = C \xi_n^T \Omega_n \xi_n$$

and

$$\inf_{T \in L_{\hat{\theta}_n}''(F_n)} \int_{\mathbb{R}} \varphi(T'(x)) dx > \xi_n^T (f(\hat{\theta}_n) - m_n) - C \xi_n^T \Omega_n \xi_n$$

Conclusion

Combining the two inequalities, we have

$$n^{-1/2} \|f(\hat{\theta}_n) - m_n\| < C \|\Omega_n\| n^{-1} + \|f(\theta_0) - m_n\|^2 \|\Omega_n^{-1}\| = O_P(n^{-1})$$

i.e. $||f(\hat{\theta}_n) - m_n|| = O_P(n^{-1/2})$. By Lemma A.2, $||m_n - f(\theta_0)|| = O_P(n^{-1/2})$. Hence, $||f(\hat{\theta}_n) - f(\theta_0)|| = O_P(n^{-1/2})$. Since $f(\theta) = f(\theta_0)$ has a unique solution at θ_0 , $||f(\theta) - f(\theta_0)||$ is bounded away from zero outside some neighborhood of θ_0 . Therefore $\hat{\theta}_n$ is inside any neighborhood of θ_0 with probability approaching 1 i.e $\hat{\theta}_n \to \theta_0$ in probability.

A.6 Proof of Theorem 7.2

First we prove that

$$\hat{\xi}_n = \arg\max_{\xi} \xi^T f(\hat{\theta}_n) - \int \psi(\xi^T K(F_n(x)) dx = O_P(n^{-1/2}).$$

Consider

$$\xi_n = \arg \max_{\xi \in \mathbb{R}^l \text{ s.t. } \|\xi\| < n^{-1/4}} \xi^T f(\hat{\theta}_n) - \int \psi(\xi^T K(F_n(x)) dx,$$

where the maximum is taken on a ball of radius $n^{-1/4}$. The maximum is attained because of the concavity of the functional

$$U: \xi \mapsto \xi^T f(\hat{\theta}_n) - \int \psi(\xi^T K(F_n(x)) dx)$$

For all x in a neighborhood of 0, the inequality $y - \psi(y) < -Cy^2$ for some C > 0 holds. For n large enough, as $\|\xi_n\| < n^{-1/4}$ we can claim (as $\psi(0) = 0$)

$$0 \leq \xi_n^T f(\hat{\theta}_n) - \int \psi(\xi_n^T K(F_n(x))) dx$$

$$\leq \xi_n^T (f(\hat{\theta}_n) - m_n) - C\xi_n^T \Omega_n \xi_n$$

$$\leq \|\xi_n\| \cdot \|f(\hat{\theta}_n) - m_n\| - C\xi_n \Omega_n \xi_n$$

with $m_n := \int K(F_n(x)) dx$.

Furthermore, there exists D > 0 such that $\|\Omega_n\| \ge D > 0$ for n large enough and

$$CD \le C \frac{\xi_n^T}{\|\xi_n\|} \Omega_n \frac{\xi_n}{\|\xi_n\|} \le \frac{\|f(\hat{\theta}_n) - m_n\|}{\|\xi_n\|}.$$

It follows that $\xi_n = O_P(n^{-1/2})$ and that ξ_n is an interior point of $\{\xi \in \mathbb{R}^l \text{ s.t. } \|\xi\| < n^{-1/4}\}$; by concavity of the functional U, ξ_n is the unique maximizer, hence $\xi_n = \hat{\xi}_n$. We write the first order conditions of optimality of $(\hat{\theta}_n - \theta_0, \hat{\xi}_n)$:

$$\begin{cases} (f(\hat{\theta}_n) - f(\theta_0)) + (f(\theta_0) - m_n) - \int \left[\psi'(\hat{\xi}_n K(F_n(x)) - 1 \right] K(F_n(x)) dx = 0 \\ J_f(\hat{\theta}_n) \hat{\xi}_n = 0 \end{cases}$$

A mean value expansion (since $\theta_0 \in int(\Theta)$) gives the existence of $\bar{\xi}$ and $\bar{\theta}$ such that $\|\bar{\xi}\| < \|\hat{\xi}_n\|$ and $\|\bar{\theta} - \theta_0\| < \|\hat{\theta}_n - \theta_0\|$ such that

$$\begin{cases} J_f(\bar{\theta})(\theta - \theta_0) + (f(\theta_0) - m_n) - \left[\int \psi''(\bar{\xi}K(F_n(x))K(F_n(x))K(F_n(x))dx\right]\hat{\xi}_n = 0\\ J_f(\hat{\theta}_n)\hat{\xi}_n = 0 \end{cases}$$

It holds

$$A_n := \begin{pmatrix} J_f(\bar{\theta}) - \int \psi''(\bar{\xi}K(F_n(x))K(F_n(x))K(F_n(x))dx \\ 0 & J_f(\hat{\theta}_n) \end{pmatrix} \to_p A := \begin{pmatrix} J_0 - \Omega \\ 0 & J_0 \end{pmatrix}.$$

By the very definition of A_n ,

$$A_n\begin{pmatrix}\hat{\theta}_n - \theta_0\\\hat{\xi}_n\end{pmatrix} = \begin{pmatrix}m_n - f(\theta_0)\\0\end{pmatrix}$$

As Ω is non singular and J_0 has full rank, A is non singular and its inverse is given by

$$A^{-1} = \begin{pmatrix} H & M \\ P H - H^T \end{pmatrix}.$$

Hence by Lemma A.2

$$\sqrt{n} \begin{pmatrix} \hat{\theta}_n - \theta_0 \\ \hat{\xi}_n \end{pmatrix} = A_n^{-1} \begin{pmatrix} \sqrt{n}(m_n - f(\theta_0)) \\ 0 \end{pmatrix} \to_d A^{-1} \begin{pmatrix} \mathcal{N}_l(0, \Sigma) \\ 0 \end{pmatrix},$$

which ends the proof.