On the hardness of the decoding and the minimum distance problems for rank codes

Philippe Gaborit*

Gilles Zémor[†]

April 15, 2014

Abstract

In this paper we give a randomized reduction for the Rank Syndrome Decoding problem and Rank Minimum Distance problem for rank codes. Our results are based on an embedding from linear codes equipped with Hamming distance unto linear codes over an extension field equipped with the rank metric. We prove that if both previous problems for rank metric are in $ZPP = RP \cap coRP$, then we would have NP = ZPP. We also give complexity results for the respective approximation problems in rank metric.

1 Introduction

1.1 General presentation

The syndrome decoding problem for Hamming distance is a fundamental problem in complexity theory, which gave rise to many papers over more than 30 years, since the seminal paper of Berlekamp, McEliece and van Tilborg [5], who proved the NP-completeness of the problem. The problem of decoding codes is of first importance regarding applications, in particular for information theory and also for its connections with lattices.

Besides the notion of Hamming distance for error-correcting codes and the notion of Euclidean distance for lattices, the concept of rank metric was introduced in 1951 by Loo-Keng Hua [10] as "arithmetic distance" for matrices over a field \mathbb{F}_q . Given two $n \times n$ matrices A and B over a finite field \mathbb{F}_q , the rank distance between A and B is defined as $d_R(A, B) = Rank(A - B)$. In 1978, Delsarte defined in [8] the notion of rank distance on a

^{*}XLIM, Université de Limoges, 123, Av. Albert Thomas, 87000 Limoges, France. gaborit@unilim.fr

†Université de Bordeaux , Institut de Mathématiques de Bordeaux, 351 cours de la Libération, 33405
Talence. zemor@math.u-bordeaux1.fr

set of bilinear form (which can also be seen as the set of rectangular matrices) and proposed a construction of optimal matrix codes in bilinear form representation. A matrix code over \mathbb{F}_q for the rank metric is defined as the set of \mathbb{F}_q -linear combinations of a set \mathcal{M} of $m \times n$ matrices over \mathbb{F}_q . Such codes are linear over \mathbb{F}_q and the number k of independent matrices in \mathcal{M} , is bounded from above by nm. Then in 1985, Gabidulin introduced in [11] the notion of rank codes in vector representation (as opposed to matrix representation) over an extension field \mathbb{F}_Q of \mathbb{F}_q (for $Q=q^m$). A rank code C[n,k] of length n and dimension k over \mathbb{F}_Q in vector representation is defined as a subspace over \mathbb{F}_Q of dimension k of \mathbb{F}_Q^n . It is possible to associate to any vector x of \mathbb{F}_Q^n an $m \times n$ matrix X over \mathbb{F}_q in the following way: let $x = (x_1, \dots, x_n)$ in \mathbb{F}_Q^n and let \mathcal{B} be a basis of \mathbb{F}_Q over \mathbb{F}_q . One can write any x_i of the extension field \mathbb{F}_Q , in the \mathbb{F}_q -linear basis \mathcal{B} , as a column vector $(x_{i1}, \dots, x_{im})^t$ of \mathbb{F}_q^m , so that one can associate an $m \times n$ matrix X over \mathbb{F}_q to any x in \mathbb{F}_Q^n . The rank weight of x is then defined as $w_R(x) = rank(X)$ and the rank distance between x and y in \mathbb{F}_{Q}^{n} , is defined as $d_{R}(x,y) = rank(X-Y)$. Rank codes in vector representation can be seen as classical error-correcting codes over \mathbb{F}_Q but embedded with the rank metric rather than with the Hamming metric, and one can define standard notions like generator and parity check matrices. Naturally any rank code C[n,k] in vector representation is $\mathbb{F}_{\mathcal{O}}$ -linear and can be seen as a matrix code defined with $k \times m$ matrices over \mathbb{F}_q , but the converse is not true and any rank matrix code has not, in general, a vector representation. The vector representation is interesting because such codes are more compact to describe and to handle. In the following we will simply denote by rank code, a rank code in vector representation.

In 1985 [11], Gabidulin introduced an optimal class of rank codes (in vector representation): the so-called Gabidulin codes, which are evaluation codes, analogous to Reed-Solomon codes but in a rank metric context, where monomial of the form x^p are replaced by linearized monomial of the form x^{q^p} introduced by Ore in 1933 in [30].

By analogy with the Hamming distance it is possible to define the two following problems:

Rank Syndrome Decoding problem (RSD)

Instance: a $(n-k) \times n$ matrix H over \mathbb{F}_Q^n , a syndrome s in \mathbb{F}_Q^{n-k} and an integer w Question: does there exist $x \in \mathbb{F}_Q^n$ such that $H.x^t = s$ and $w_R(x) \leq w$?

and

Rank Minimum Distance Problem (RMD)

Instance: a rank code C[n, k], an integer w,

Question: does there exist $\mathbf{x} \in C$ such that $w_R(x) \leq w$?

Remark: the two previous problems fundamentally differ from the so-called MinRank problem, which is also related to the rank metric but in a more general case as it is explained in the next section.

The purpose of this paper is to study the computational complexity of the RSD problem and the RMD problem, our main result reads as follows:

Theorem 1 If the Rank Minimum Distance Problem for rank codes is in ZPP = RP \cap coRP, then we must have NP = ZPP. Similarly, if the Rank Syndrome Decoding Problem for rank codes is in ZPP, we must have NP = ZPP.

1.2 Previous work

Surprisingly the theoretical computational complexities of the RSD and RMD problems for the rank metric are not known, whereas the problem (and its variations) has been intensively studied for Hamming distance or for lattices. In particular besides the NP-completeness of the syndrome decoding problem for Hamming distance proven in [32], the minimum distance problem for Hamming distance has been proven NP-complete by Vardy in [32], as are also variations on the problem [6].

As explained earlier in this introduction it is possible to consider the decoding and minimum distance problems in the rank metric, but for matrix codes. These problems can be seen as generalizations of the RSD and RMD problems. For instance for the case of the decoding problem for rank matrix codes, we are given a set $\mathcal{M} = \{M_1, \cdots, M_k\}$ of $n \times n$ matrices over \mathbb{F}_q , a matrix M over \mathbb{F}_q , and an integer w. The question is to decide whether there exists an \mathbb{F}_q -matrix M_0 of rank $\leq w$ such that $M-M_0$ can be expressed as an \mathbb{F}_q -linear combination of matrices of \mathcal{M} (i.e. is in the \mathbb{F}_q -linear matrix code generated by the matrices of \mathcal{M}). Note that we have linearity over the small field \mathbb{F}_q for the code, but not necessarily over the extension field \mathbb{F}_Q . The latter decoding problem and its minimum distance variant have appeared, in slightly generalized forms, somewhat confusingly both under the name of "MinRank" in the literature. Courtois makes the observation in [7] that both the above problems for rank codes in matrix representation are NP-complete, by remarking that a Hamming metric code in \mathbb{F}_q^n can be "lifted" into a rank metric code in matrix representation simply by transforming any vector x of \mathbb{F}_a^n into a diagonal matrix with x written on the diagonal. By this process a Hamming code of dimension k with minimum distance d is lifted unto a rank-metric code in matrix representation with \mathcal{M} a set with k matrix element, with rank minimum distance d. This transformation yield the NP-completeness of the previous decoding problem for matrix codes from the NP-completeness of Syndrome Decoding problem for classical Hamming codes. The NP-completeness of the associated Minimum Rank Distance problem follows similarly from the NP-completeness of minimum distance problem for Hamming metric.

However, in the case of matrix representations the structure of the linear matrices over \mathbb{F}_q is simpler than the structure for rank [n,k] codes in vector representation which are linear over the extension field \mathbb{F}_Q and not only on the base field \mathbb{F}_q . The "MinRank" problem appears as a weakly-structured variation of the RSD and RMD problems. The above remark by Courtois works well for \mathbb{F}_q -linear matrix codes but clearly does not apply for \mathbb{F}_Q -linear [n,k] rank codes.

1.3 Applications of the rank metric

Over the years the notion of rank metric has become a very central tool for new applications of coding theory and has also very interesting applications to cryptography.

Applications to coding theory. Concerning coding theory, from the end of '90s new application contexts appeared for coding theory: space-time coding [31] in 1997 and network coding in 2001 [28].

Space time coding was introduced by Tarokh, Jafarkhani and Calderbank in 1998 in [31]. One strives to improve the reliability of data transmission in wireless communication systems using multiple transmission antennas. This redundancy results in a higher chance of being able to use one or more of the received copies to correctly decode the received signal. In fact, space—time coding combines all the copies of the received signal in an optimal way to extract as much information from each of them as possible. A full rank criterion was proposed for choosing rank matrices with full rank difference, which enables one to decode errors in this context.

For network coding introduced in 2001 in [28], the idea is optimize information sent in given time slots, when the information is sent from a single source to a single destination through a network with nodes which send random linear combination of received information. Koetter and Kschichang introduced in 2007 [27] the notion of subspace metric (which is a small variation on the rank metric [13]), and the so-called Koetter-Kschichang codes which are an adaptation of the Gabidulin codes in a subspace metric context.

More generally a lot of work has also been done for decoding Gabidulin or Koetter-Kschichang codes, though admittedly somewhat less than for Reed-Solomon codes, their Hamming distance counterparts: in particular list-decoding algorithms are known only for subclasses of Gabidulin codes and not yet for the general family of codes [19, 18, 23].

Applications to cryptography

Rank-based cryptography belongs to the larger class of post-quantum cryptosystems, which is an alternative class of cryptosystems which are *a priori* resistant to a putative quantum computer. The first cryptosystem was proposed in 1991 by Gabidulin, Paramonov and Tretjakov (the GPT cryptosystem [12] which adapts the McEliece cryptosystem to the rank metric and Gabidulin codes).

The particular interest of rank metric based problems compared to lattices or (Hamming) codes based problems is that the practical complexity of the best known attacks for rank-based problems [15] grows very quickly compared to their Hamming counterpart [3]. Indeed such attacks have a quadratic term (related to parameters of the rank code) in their exponential coefficient, while for Hamming distance problems (and somehow also for heuristic LLL attacks for lattice-based cryptographic), the best practical attacks have only an exponential term whose exponent is linear in the code parameters. This translates into rank codes having a decoding complexity that behaves as $exp(\Omega(N^{2/3}))$ rather than $exp(\Omega(N^{1/2}))$ for Hamming codes, where N is the input size, i.e. the number of q-ary

symbols needed to describe the code.

In practice it means that it is possible to obtain secure practical parameters for random instances in rank metric of only a few thousand bits related to a hard problem, when at least a hundred thousand bits are needed for Hamming distance or for lattices. Such random instances for rank metric are used for instance, for zero-knowledge authentication in [17], and weakly structured instances are used for the recent LRPC cryptosystem [14] (similar to the NTRU cryptosystem [26] for lattices and the recent MDPC cryptosystem for codes) or for signature [16]. Of course with (Hamming) codes and lattices it is possible to decrease the size of parameters to a few thousand bits with additional structure [4, 22], but then the reduction properties to hard problems are lost because they are reduced to decoding problems for special classes of codes whose complexity is not known.

Overall because of the practical complexity of best known attacks, rank-based cryptography has very good potential for cryptography, furthermore, our present results show that cryptographic protocols whose security can be reduced to the decoding problem for rank codes will have both reduction to a proven hard problem and the potential for small keys. Finally, we remark that since the codes actually used for rank-metric applications, cryptographic or otherwise, tend to be rank-codes in the sense of this paper, i.e. with linearity over the large field, the decoding and minimum distance problems for these codes are more relevant than the same problems for the looser matrix code class, whose NP-completeness has been referred to a number of times in the past.

Organization of the paper: the paper is organized as follows, in Section 2, we give an overview of our results and describe our embedding technique, in Section 3 we give a probabilistic analysis of our reduction setting, Section 4 describes our main results, and finally Section 5 considers further results on approximation problems for the rank metric.

2 Overview

It is clear that Courtois's diagonal embedding of the Hamming space into the rank metric space works well for rank codes in matrix form linear over \mathbb{F}_q but does not work for rank codes with linearity over the extension field \mathbb{F}_Q . We shall therefore introduce a different embedding strategy defined as follows:

Definition 2 Let $m \ge n$ and $Q = q^m$. Let $\alpha = (\alpha_1, \dots \alpha_n)$ be an n-tuple of elements of \mathbb{F}_Q . Define the embedding of \mathbb{F}_q^n into \mathbb{F}_Q^n

$$\psi_{\alpha}: \quad \mathbb{F}_q^n \quad \to \quad \mathbb{F}_Q^n$$

$$x = (x_1, \dots, x_n) \quad \mapsto \quad \mathbf{x} = (x_1 \alpha_1, \dots x_n \alpha_n)$$

and for any \mathbb{F}_q -linear code C in \mathbb{F}_q^n , define $\mathfrak{C} = \mathfrak{C}(C, \boldsymbol{\alpha})$ as the \mathbb{F}_Q -linear code generated by $\psi_{\boldsymbol{\alpha}}(C)$, i.e. the set of \mathbb{F}_Q -linear combinations of elements of $\psi_{\boldsymbol{\alpha}}(C)$.

Remark: The condition $m \ge n$ ensures that, by adjoining m - n zeros to vectors of \mathbb{F}_Q^n , they may be seen as $m \times m$ matrices so that the code \mathbb{C} may be seen as a rank code.

It should be clear that for any α , the rank weight of $\psi_{\alpha}(x)$ is at most the Hamming weight of x: therefore the Minimum Rank Distance of $\mathbb C$ never exceeds the Hamming minimum distance of the original code C. It may however be less. For example, if $\alpha = 1 = (1, 1, \ldots, 1)$ then the rank weight of $\psi_{\alpha}(x)$ is always 1 for every $x \neq 0$. The minimum rank weight of $\mathbb C(C, \alpha)$ may also be less than the minimum Hamming distance of C for more sophisticated reasons. In particular, if $\alpha_1, \ldots \alpha_n$ are $\mathbb F_q$ -linearly independent, we have that the rank weight of $\psi_{\alpha}(x)$ is always equal to the Hamming weight of x, but the minimum rank weight of x may still be less than the minimum Hamming distance of x. Consider for instance the binary code x of words of even weight of length 3, we have that

$$\mathbf{x} = \alpha_2 \psi_{\alpha}(101) + \alpha_1 \psi_{\alpha}(011) = (\alpha_1 \alpha_2, \alpha_1 \alpha_2, \alpha_3(\alpha_1 + \alpha_2)).$$

Now if α_3 happens to have been chosen equal to $\alpha_1\alpha_2(\alpha_1 + \alpha_2)^{-1}$, we will have that $\operatorname{rank}(x) = 1 < d_{\operatorname{Hmin}}(C)$ even though α may very well be of rank 3.

If, given any Hamming code C, we could efficiently find an n-tuple α that would guarantee that $C(C, \alpha)$ has minimum rank distance equal to $d_{\text{Hmin}}(C)$, we would have a polynomial reduction that would derive the NP-completeness of the Minimum Rank Distance problem for rank codes from the NP-completeness of the classical minimum Hamming distance problem. We have not been able to see how to do this in any deterministic way. However, we shall show that when α is chosen randomly, for m = O(n), then we probability tending to 1 we have $d_{\text{Rmin}}(\mathcal{C}(C, \alpha)) = d_{\text{Hmin}}(C)$. This makes the Rank Minimum Distance hard for NP under unfaithful random reductions (UR reductions, in the terminology of [20]). As a consequence we have that if the Rank Minimum Distance Problem were in coRP we would have NP \subset coRP. With a further transformation we shall obtain that if the Rank Minimum Distance Problem were in RP then we would have also NP \subset RP: our results will therefore show that if the Rank Minimum Distance Problem were in ZPP = RP \cap coRP, then we would have NP=ZPP.

3 Probabilistic analysis of our embedding

3.1 Notation and definitions

We refer to [24] and [21] for general results on codes and rank codes. Let \mathbb{F}_q be a field with q elements and let \mathbb{F}_Q , with $Q = q^m$, be an extension of \mathbb{F}_q of degree m. In the following we consider two type of codes, codes with Hamming distance considered as $C[n, k, d_H]$ linear codes over the base field \mathbb{F}_q , for n and k the length and dimension of the code and d_H , its minimum Hamming distance. we also consider rank codes with rank distance written as $C[n, k, d_R]$ linear codes over the field \mathbb{F}_Q of length n, dimension k an minimum rank distance d_R , embedded with the rank metric.

We recall the Griesmer bound for linear codes over \mathbb{F}_q that ill be useful for our proofs:

Proposition 3 (Griesmer bound) Let C be a [n, k, d] over \mathbb{F}_q then

$$n \ge \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil$$

3.2 Probabilistic analysis of $\mathcal{C}(C, \alpha)$

Lemma 4 Let C^{\perp} be the dual code of C over \mathbb{F}_q . Let $\boldsymbol{\beta} = \boldsymbol{\alpha}^{-1} = (\alpha_1^{-1}, \dots, \alpha_n^{-1})$. Then $\mathfrak{C}(C^{\perp}, \boldsymbol{\beta})$ is the dual code of $\mathfrak{C}(C, \boldsymbol{\alpha})$ over \mathbb{F}_Q and $\dim_{\mathbb{F}_Q} \mathfrak{C}(C, \boldsymbol{\alpha}) = \dim_{\mathbb{F}_q} C$.

Proof. It should be clear that $\mathcal{C}(C, \boldsymbol{\alpha})$ and $\mathcal{C}(C^{\perp}, \boldsymbol{\beta})$ are orthogonal to each other. Choosing systematic generator matrices for C and C^{\perp} shows that $\dim_{\mathbb{F}_Q} \mathcal{C}(C, \boldsymbol{\alpha}) = \dim_{\mathbb{F}_q} C$ and $\dim_{\mathbb{F}_Q} \mathcal{C}(C^{\perp}, \boldsymbol{\beta}) = \dim_{\mathbb{F}_q} C^{\perp}$. \square

Lemma 5 Let C be an \mathbb{F}_q -linear code of \mathbb{F}_q^n and let $W \subset \{1, 2, ..., n\}$ be a set of coordinates such that no non-zero codeword of C has its Hamming support included in W. Then, for any $j \in W$ and for any α , there is a codeword \mathbf{x} of $\mathcal{C}(C, \alpha)^{\perp}$ such that $\sup(\mathbf{x}) \cap W = \{j\}$.

Proof. Since the code C, punctured so as to leave only the coordinates in W, has only the zero codeword, we have that for any $j \in W$ there is x in C^{\perp} such that $\operatorname{supp}(x) \cap W = \{j\}$. The conclusion follows by Lemma 4. \square

Corollary 6 Let C be an \mathbb{F}_q -linear code of \mathbb{F}_q^n with minimum Hamming distance d. Then, for any α , the Hamming minimum distance of the embedded code $\mathcal{C}(C, \alpha)$ is equal to d.

Proof. That it is at most d is clear by the definition of $\mathcal{C}(C, \boldsymbol{\alpha})$. To see that it is at least d follows from Lemma 5. \square

Lemma 7 Let C be an \mathbb{F}_q -linear code of \mathbb{F}_q^n with minimum Hamming distance d. Let $w < \frac{q+1}{q}d$. Then, for any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, the only codewords of $\mathfrak{C}(C, \boldsymbol{\alpha})$ of Hamming weight w are of the form $\lambda \psi_{\boldsymbol{\alpha}}(x)$, $\lambda \in \mathbb{F}_Q$, for some codeword of C. In particular, if $\alpha_1, \ldots, \alpha_n$ are linearly independent, then any codeword of $\mathfrak{C}(C, \boldsymbol{\alpha})$ of Hamming weight w is also of rank weight w.

Proof. Let $W \subset \{1, 2, ..., n\}$ be a set of w coordinates. Let $C_{|S|}$ be the corresponding shortened code of C, i.e. the set of codewords of C of support included in W. By Lemma 4, we have that the dual code of $\mathcal{C}(C, \boldsymbol{\alpha})_{|S|}$ has dimension $w - \dim C_{|S|}$ and therefore $\dim \mathcal{C}(C, \boldsymbol{\alpha})_{|S|} = \dim C_{|S|} = \dim \mathcal{C}(C_{|S|}, \boldsymbol{\alpha})$. By the Griesmer bound, the dimension of $C_{|S|}$ is at most 1. Therefore the only codewords of $\mathcal{C}(C_{|S|}, \boldsymbol{\alpha})$ are of the form $\lambda \psi_{\boldsymbol{\alpha}}(x)$. \square

Theorem 8 Subject to the condition m > 2qn, when α is chosen randomly and uniformly in \mathbb{F}_Q^n , then for any linear code $C \in \mathbb{F}_q^n$, the probability that the rank minimum distance of $\mathcal{C}(C, \alpha)$ differs from the Hamming minimum distance of C is bounded from above by a quantity that vanishes exponentially fast in n.

Proof. Let C be fixed and let d be its Hamming minimum distance. It suffices to prove that for any Hamming weight $w \leq n$, the probability P_w that there exists a codeword of $\mathcal{C}(C, \alpha)$ of Hamming weight w and of rank weight d vanishes exponentially fast.

- w < d + d/q. If w < d, Then by Corollary 6, $P_w = 0$. Otherwise, by Lemma 7, P_w is bounded from above by the probability that $\alpha_1 \dots \alpha_n$ are linearly dependent, which is exponentially small in n.
- $w \geq d + d/q$. We bound from above P_w by the expected number of codewords of $\mathcal{C}(C, \boldsymbol{\alpha})$ of rank weight < d and Hamming weight w. Let \mathbf{x} be a vector of \mathbb{F}_Q^n of Hamming weight w and let W be the Hamming support of \mathbf{x} , so that w = |W|. Let J be a maximal subset of coordinates of W such that no nonzero codeword of C has its support included in I. By Lemma 5, we have that there are |J| parity-checks for the event $\mathbf{x} \in \mathcal{C}(C, \boldsymbol{\alpha})$ that are satisfied each with probability 1/Q and, truncated to W, are independent over \mathbb{F}_Q and therefore are satisfied independently in the sense of probability. Hence, the probability that \mathbf{x} is a codeword of $\mathcal{C}(C, \boldsymbol{\alpha})$ is at most $1/Q^{|J|}$. By the Griesmer bound, we have $|J| \geq d + d/q 1$. Bounding from above the number N_w of vectors of \mathbb{F}_Q^n of Hamming weight w and rank weight d-1 by:

$$N_w \le \binom{n}{w} Q^{d-1} (q^{d-1})^w \le 2^n q^{m(d-1) + w(d-1)}$$

we obtain

$$P_w \le N_w \frac{1}{q^{md+md/q-m}}$$
$$< 2^n q^{w(d-1)-md/q}$$

and the result follows from the hypothesis m/q > 2n.

4 The syndrome decoding problem

Let us recall the syndrome decoding problem:

Instance: an $r \times n$ matrix $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_n]$ over a field \mathbb{F} , a column vector

 $\mathbf{s} \in \mathbb{F}^r$, an integer w

Question: does there exist $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{F}^n$ of weight at most w such that

 $\sigma(\mathbf{x}) = \mathbf{H}^t \mathbf{x} = \sum_{i=1}^n x_i \mathbf{h}_i = \mathbf{s}$?

When $\mathbb{F} = \mathbb{F}_q$ and the weight refers to the Hamming weight, we have the classical or *Hamming* syndrome decoding problem: when $\mathbb{F} = \mathbb{F}_Q$ and the weight refers to the rank (or rank weight) we have the *rank* syndrome decoding problem. It is classical that the syndrome decoding problem is equivalent to the decoding (or closest vector) problem, because looking for the closest codeword to a given vector \mathbf{y} amounts to computing the syndrome $\mathbf{s} = \sigma(\mathbf{y})$ of \mathbf{y} and solving the syndrome decoding problem for \mathbf{s} (subtracting the solution to \mathbf{y} gives the closest codeword).

Since the Hamming syndrome decoding problem is known to be NP-complete, it is natural to try and devise a transformation from it to the rank syndrome decoding problem. For this purpose, let us introduce the following notation: for any $r \times n$ matrix $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_n]$ of elements of \mathbb{F}_q , and for any $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$, $\beta_i \in \mathbb{F}_Q$, denote by $\mathbf{H}(\boldsymbol{\beta})$ the matrix

$$\mathbf{H}(\boldsymbol{\beta}) = [\beta_1 \mathbf{h}_1, \beta_2, \mathbf{h}_2, \dots, \beta_n \mathbf{h}_n].$$

Our strategy is, given an instance $(\mathbf{H}, \mathbf{s}, w)$ of the Hamming syndrome decoding problem, to associate to it the transformed instance $(\mathbf{H}(\boldsymbol{\beta}), \mathbf{s}, w)$ of the rank decoding problem. It is clear that if x is a solution to the Hamming syndrome decoding problem, then $\mathbf{x} = \psi_{\boldsymbol{\alpha}}(x)$ is a solution to the associated rank syndrome decoding problem with $\boldsymbol{\alpha} = \boldsymbol{\beta}^{-1} = (\beta_1^{-1}, \dots, \beta_n^{-1})$, the rank weight of \mathbf{x} being not more than the Hamming weight of x. Again, we strive to show that when choosing $\boldsymbol{\beta}$ randomly and uniformly, the smallest rank weight of a solution to $(\mathbf{H}(\boldsymbol{\beta}), \mathbf{s}, w)$ is very probably equal to the smallest Hamming weight of a solution to $(\mathbf{H}, \mathbf{s}, w)$.

Lemma 9 Let \mathbf{H} be an $r \times n$ matrix and let \mathbf{s} be a column vector of \mathbb{F}_q^r . Let w_H be the minimum Hamming weight of a vector x of \mathbb{F}_q^n of syndrome $\sigma(x) = \mathbf{H}^t x = \mathbf{s}$. Let $\mathbf{x} \in \mathbb{F}_Q^n$ be such that $\sigma_{\boldsymbol{\beta}}(\mathbf{x}) = \mathbf{H}(\boldsymbol{\beta})^t \mathbf{x} = \mathbf{s}$. Then, if $J \subset \{1, \ldots, n\}$ is the Hamming support of \mathbf{x} , there exists a subset $W \subset J$ such that $|W| = w_H$ and the columns $(\mathbf{h}_j)_{j \in W}$ of \mathbf{H} indexed by W are \mathbb{F}_Q -linearly independent.

Proof. Let W be a maximal subset of the support of \mathbf{x} such $(\beta_j \mathbf{h}_j)_{j \in W}$ is \mathbb{F}_Q -linearly independent. Since $\sigma_{\beta}(\mathbf{x}) = \mathbf{s}$, we must also have that \mathbf{s} belongs to the \mathbb{F}_Q -linear span of $(\beta_j \mathbf{h}_j)_{j \in W}$. Now by Lemma 4 we have that \mathbb{F}_Q -linear independence of $(\beta_j \mathbf{h}_j)_{j \in I}$ (and therefore also simply of $(\mathbf{h}_j)_{j \in I}$) is equivalent to \mathbb{F}_q -linear independence of $(\mathbf{h}_j)_{j \in I}$ for any set I of coordinates. Since any set of columns of \mathbf{H} that generate \mathbb{F}_q -linearly \mathbf{s} must be of size at least w_H by definition of w_H we have $|W| \geq w_H$. \square

Theorem 10 Subject to the condition $m > n^2$, when $\boldsymbol{\beta}$ is chosen randomly and uniformly in \mathbb{F}_Q , then for any $r \times n$ matrix \mathbf{H} over \mathbb{F}_q and any column vector $\mathbf{s} \in \mathbb{F}_q^r$, denoting by w_H the minimum Hamming weight of a vector of \mathbb{F}_q^n of syndrome \mathbf{s} by \mathbf{H} and by w_R the minimum rank weight of a vector of \mathbb{F}_Q^n of syndrome \mathbf{s} by $\mathbf{H}(\boldsymbol{\beta})$, we have that the probability that $w_H \neq w_R$ is bounded from above by a quantity that vanishes exponentially fast in n.

Proof. Let \mathbf{H} , \mathbf{s} and w_H be fixed. Since we have remarked that $w_R \leq w_H$, It suffices to show for every integer $w \leq n$ that the probability P_w that there exists a codeword of \mathbb{F}_Q^n of syndrome \mathbf{s} by $\mathbf{H}(\boldsymbol{\beta})$ and of Hamming weight w and rank weight $< w_H$, is a quantity that vanishes exponentially fast with n.

By Lemma 9, if $w < w_H$ we have $P_w = 0$. Suppose therefore $w \ge w_H$. Let \mathbf{x} be a vector of \mathbb{F}_Q^n of Hamming weight w. Lemma 9 implies that there are at least w_H columns of \mathbf{H} indexed by nonzero coordinates of \mathbf{x} that are \mathbb{F}_Q -linearly independent. This implies that the span of \mathbb{F}_Q -linear combinations of these w_H columns has size Q^{w_H} , and therefore the probability that the syndrome by $\mathbf{H}(\boldsymbol{\beta})$ of \mathbf{x} equals \mathbf{s} is at most $1/Q^{w_H}$.

Bounding from above P_w by the expectation of the number of codewords of Hamming weight w and rank weight $\leq w_H - 1$, we have:

$$P_w \le \binom{n}{w} Q^{w_H - 1} (q^{w_H - 1})^w \frac{1}{Q^{w_H}}$$
$$\le 2^n q^{n(w_H - 1)} \frac{1}{Q}$$
$$\le q^{nw_H - m}$$

which proves the result since the case $w_H = n$ is easily dealt with separately. \square

Theorems 8 and 10 yield Theorem 1, our main result stated in the introduction:

Proof of Theorem 1

Proof. That NP \subset coRP follows directly from the NP-completeness of the Hamming Minimum Distance Problem and Theorem 8 in the first case and from the NP-completeness of the Hamming Syndrome Decoding Problem and Theorem 10 in the second case: the original Hamming problem is simply transformed by a probabilistic embedding into the corresponding Rank metric problem. To be precise, the hypothesis that the Rank Minimum Distance Problem is in coRP means that there is probabilistic polynomial time algorithm that always outputs "yes" on "yes" instances and often outputs "no" on "no" instances. Applied to a code $\mathcal{C}(C, \alpha)$ for a random α , we obtain an algorithm that, for the original Hamming Minimum Distance Problem always outputs "yes" (the minimum distance is not more than a given value) on "yes" instances and often "no" otherwise.

Next we deduce from the hypothesis that the Rank Minimum Distance Problem for rank codes is in RP that $NP \subset RP$. We need to construct a probabilistic algorithm that given

an integer w and a Hamming code with minimum distance $d_W > w$ always decides "no", and often decides "yes" when the minimum distance d_H is not more than w. To achieve this we find a witness for $d_H \leq w$. The hypothesis that the Rank Minimum Distance Problem is in RP means that there is a probabilistic polynomial time algorithm that always decides "no" when the rank minimum distance d_R is above w and very often decides "yes" when it $d_R \leq w$. Suppose that the Hamming code C is such that $d_H \leq w$. We transform C into a random $\mathcal{C}(C, \boldsymbol{\alpha})$ and ask the probabilistic machine for the Minimum Rank Distance whether $d_R \leq w$. If the answer is "no" we output a "no". If it is "yes", we remove the first column from a fixed parity-check matrix of C and start the procedure (create another random rank-metric code from the shortened version of C) again. If the answer is "no", we put back the removed column and start again by removing the second column, until we either run out of columns to remove in which case we output a final "no", or we obtain a "yes", in which case we continue removing columns, always of a larger index than the columns we have previously tried to remove. We stop removing columns if we reach a point when only w columns remain. At this point we check that the thus shortened Hamming code has dimension at least 1, in which case we "output" a "yes". In all other cases we output a "no".

We see that the number of times we use randomness and access the rank minimum distance oracle is at most n. Furthermore, if it is true that $d_H \leq w$ for the original code, then with probability exponentially close to 1 for large n and fixed q we will obtain a "yes", and if it is not true that $d_H \leq w$ we will always obtain a "no". This is a random polynomial time algorithm that puts an NP-complete problem (Minimum Distance for Hamming linear codes) in RP, hence the result.

To reach the same conclusion from the hypothesis that the Rank Syndrome Decoding Problem for rank codes is in RP we use a very similar witness constructing technique for the Hamming syndrome decoding problem. \Box

Remark. The reduction is somewhat looser in the Decoding case where an extension field of quadratic degree in n is needed, than in the Minimum distance case where a degree linear in n was sufficient. This is somewhat surprising, since in the more well-known Hamming distance and Lattice cases, the Minimum Distance problem has been more difficult to reduce than the Decoding problem.

5 Further results on approximation problems for rank metric

The syndrome decoding problem and the minimum distance problem for Hamming distance are connected to other interesting problems. It is natural to consider generalizations of these problems from the Hamming distance to rank metric, especially with the use of our very versatile embedding. In the following as an example of application of our embedding

we consider the case of two particular well known approximation problems in Hamming distance: the Gap Minimum Distance Problem (GapMDP), for which we want to approximate the minimum distance of a code up to a constant and Gap Nearest Codeword (GapNCP) in which we want to approximate the decoding distance. Notice that equivalently the previous (GapNCP) problem can be stated in terms of Syndrome Decoding with a parity check matrix.

These approximation problems are stated in the following way:

Definition 11 ($GapMDP_{q,\gamma}$) For a prime power q and $\gamma \geq 1$, an instance of the $GapMinimum\ Distance\ problem\ GapMDP_{q,\gamma}$ is a linear code C over \mathbb{F}_q , given by its generator matrix, and an integer t such that:

- it is a YES instance if $d_H(C) \leq t$;
- it is a NO instance if $d_H(C) > \gamma t$

Definition 12 ($GapNCP_{q,\gamma}$) For a prime power q and $\gamma \geq 1$, an instance (C, v, t) of the Gap Minimum Distance problem $GapNCP_{q,\gamma}$ is a linear code C over \mathbb{F}_q , given by its generator matrix, $v \in \mathbb{F}_q^n$ and a positive integer t.

- it is a YES instance if $d_H(v, C) \leq t$;
- it is a NO instance if $d_H(v,C) > \gamma t$

Both these promise problems have been proven NP-complete for Hamming distance for $\gamma > 1$ respectively in [6] (see also [9]) and [1].

The generalization of these problems to the rank metric is straightforward: we may define Gap Rank Minimum Distance (GapRMPD) and Gap Rank Nearest Codeword Problem (GapRNCP):

Definition 13 ($GapRMDP_{q,\gamma}$) For a prime power q, an integer m and $\gamma \geq 1$, an instance of the $Gap\ Rank\ Minimum\ Distance\ problem\ GapMDP_{q,\gamma}$ is a linear rank code C over F_{q^m} , given by its generator matrix, and an integer t such that:

- it is a YES instance if $d_R(C) \le t$;
- it is a NO instance if $d_R(C) > \gamma t$

Definition 14 ($GapRNCP_{q,\gamma}$) For a prime power q, an integer m and $\gamma \geq 1$, an instance (C, v, t) of the $Gap\ Rank\ Minimum\ Distance\ problem\ GapRNCP_{q,\gamma}$ is a linear rank code C over F_{q^m} , given by its generator matrix, $v \in F_{q^m}^n$ and a positive integer t.

- it is a YES instance if $d_R(v, C) \leq t$;
- it is a NO instance if $d_R(v, C) > \gamma t$

We then deduce the following corollary:

Corollary 15 If the problems $GapRMDP_{q,\gamma}$ and $GapRNCP_{q,\gamma}$ are in coRP then NP=ZPP.

Proof. We use the same embedding technique as for Theorem 1. Since the Hamming distance is always greater or equal than the rank distance, we obtain a Unfaithful Random (UR) reduction between the respective approximation Hamming distance problems and rank distance problems and hence by the result of ([20],p.118), the result follows. \Box

6 Conclusion

In this paper we proved the hardness of the minimum distance and syndrome decoding problems for rank codes and rank distance under a randomized UR reduction. If we compare to other type of metrics like Hamming or Euclidean distance, we see that, for the decoding problem the reductions for codes equipped with Hamming distance and lattices with Euclidean distance are deterministic and for minimum distance, reductions are randomized for lattices and deterministic for codes (see [25] and references therein). A worthwhile challenge would be to obtain a deterministic reduction also for rank metric.

ACKNOWLEDGEMENT

The first author thanks O. Ruatta for helpful discussions.

References

- [1] Sanjeev Arora, László Babai, Jacques Stern, Z. Sweedyk: The Hardness of Approximate Optima in Lattices, Codes, and Systems of Linear Equations. J. Comput. Syst. Sci. 54(2): 317-331 (1997)
- [2] Razvan Barbulescu and Pierrick Gaudry and Antoine Joux and Emmanuel Thomé, "A quasi-polynomial algorithm for discrete logarithm in finite fields of small characteristic", eprint iacr 2013/400
- [3] Anja Becker, Antoine Joux, Alexander May, Alexander Meurer: Decoding Random Binary Linear Codes in 2 n/20: How 1+1=0 Improves Information Set Decoding. EUROCRYPT 2012: 520-536
- [4] Thierry P. Berger, Pierre-Louis Cayrel, Philippe Gaborit, Ayoub Otmani: Reducing Key Length of the McEliece Cryptosystem. AFRICACRYPT 2009: 77-97

- [5] Elwyn R. Berlekamp, Robert J. McEliece, Henk C. A. van Tilborg: On the inherent intractability of certain coding problems (Corresp.). IEEE Transactions on Information Theory 24(3): 384-386 (1978)
- [6] Qi Cheng, Daqing Wan: A deterministic reduction for the gap minimum distance problem: [extended abstract]. STOC 2009: 33-38
- [7] N. Courtois, "Efficient zero-knowledge authentication based on a linear algebra problem MinRank", Asiacrypt 2001, LNCS 2248, pp. 402–421.
- [8] P. Delsarte, "Bilinear forms overe a finite fied with applications to coding theory", Journal of Comb. Theory A, vol. 25, pp. 226-241, 1978.
- [9] Ilya Dumer, Daniele Micciancio, Madhu Sudan: Hardness of Approximating the Minimum Distance of a Linear Code. FOCS 1999: 475-485
- [10] H. Loo-Kengn "A theorem on matrices over a field and its applications", Chinese mathematical society, Vol. 1, No. 2, pp. 109-163,1951.
- [11] Ernst M. Gabidulin, Theory of Codes with Maximum Rank Distance, Probl. Peredachi Inf, (21), pp. 3-16 (1985).
- [12] Ernst M. Gabidulin, A. V. Paramonov, O. V. Tretjakov: Ideals over a Non-Commutative Ring and thier Applications in Cryptology. EUROCRYPT 1991: 482-489
- [13] E. M. Gabidulin, Nina I. Pilipchuk: Error and erasure correcting algorithms for rank codes. Des. Codes Cryptography 49(1-3): 105-122 (2008)
- [14] Philippe Gaborit, Gaétan Murat, Olivier Ruatta and Gilles Zémor: Low Rank Parity Check Codes and their application in cryptography, in the preproceedings of Workshop on Coding and Cryptography (WCC) 2013, Borgen, Norway, p. 167-179.
- [15] Philippe Gaborit, Olivier Ruatta and Julien Schrek: On the complexity of the rank syndrome decoding problem, eprint (submitted to IEEE trans. Information Theory)
- [16] Philippe Gaborit, Olivier Ruatta, Julien Schrek and Gilles Zémor: RankSign: an efficient signature algorithm based on the rank metric, eprint iacr, submitted
- [17] Philippe Gaborit, Julien Schrek, Gilles Zémor: Full Cryptanalysis of the Chen Identification Protocol. PQCrypto 2011: 35-50
- [18] Venkatesan Guruswami, Srivatsan Narayanan, Carol Wang: List decoding subspace codes from insertions and deletions. ITCS 2012: 183-189
- [19] Venkatesan Guruswami, Chaoping Xing: List decoding reed-solomon, algebraic-geometric, and gabidulin subcodes up to the singleton bound. STOC 2013: 843-852

- [20] D. S. Johnson, "A Catalog of Complexity Classes", Chapter 2 in J. van Leeuwen (ed.) Handbook of theoretical computer science Vol. A (Algorithms and Complexity), Elsevier Science, 1990.
- [21] P. Loidreau, Properties of codes in rank metric, http://arxiv.org/abs/cs/0610057
- [22] Vadim Lyubashevsky, Chris Peikert, Oded Regev: On Ideal Lattices and Learning with Errors over Rings. J. ACM 60(6): 43 (2013)
- [23] Hessam Mahdavifar, Alexander Vardy: Algebraic List-Decoding of Subspace Codes. IEEE Transactions on Information Theory 59(12): 7814-7828 (2013)
- [24] J. McWilliams and N.J.A Sloane, "The theory of error-correcting", Eslevier eds., 1977
- [25] Daniele Micciancio: Inapproximability of the Shortest Vector Problem: Toward a Deterministic Reduction. Theory of Computing 8(1): 487-512 (2012)
- [26] Jeffrey Hoffstein, Jill Pipher, Joseph H. Silverman: NTRU: A Ring-Based Public Key Cryptosystem. ANTS 1998: 267-288
- [27] Ralf Koetter, Frank R. Kschischang: Coding for Errors and Erasures in Random Network Coding. IEEE Transactions on Information Theory 54(8): 3579-3591 (2008) (short version ISIT 2007)
- [28] Shuo-Yen Robert Li, Raymond W. Yeung, Ning Cai: Linear network coding. IEEE Transactions on Information Theory 49(2): 371-381 (2003)
- [29] Rafael Misoczki and Jean-Pierre Tillich and Nicolas Sendrier and Paulo S. L. M. Barreto, MDPC-McEliece: New McEliece Variants from Moderate Density Parity-Check Codes Cryptology, ISIT 2013
- [30] O. Ore, On a special class of polynomials, Trans. American Math. Soc. (1933)
- [31] Vahid Tarokh, Nambi Seshadri, A. Robert Calderbank: Space-Time Codes for High Data Rate Wireless Communications: Performance criterion and Code Construction. IEEE Transactions on Information Theory 44(2): 744-765 (1998) (short version ISIT 1997).
- [32] Alexander Vardy: Algorithmic Complexity in Coding Theory and the Minimum Distance Problem. STOC 1997: 92-109