

Linear Degrees of Freedom of MIMO Broadcast Channels with Reconfigurable Antennas in the Absence of CSIT

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Abstract

The K -user multiple-input and multiple-output (MIMO) broadcast channel (BC) with no channel state information at the transmitter (CSIT) is considered, where each receiver is assumed to be equipped with reconfigurable antennas capable of choosing a subset of receiving modes from several preset modes. Under general antenna configurations, the sum linear degrees of freedom (LDoF) of the K -user MIMO BC with reconfigurable antennas is completely characterized, which corresponds to the maximum sum DoF achievable by linear coding strategies. The LDoF region is further characterized for a class of antenna configurations. Similar analysis is extended to the K -user MIMO interference channels with reconfigurable antennas and the sum LDoF is characterized for a class of antenna configurations.

Index Terms

Blind interference alignment, broadcast channels, degrees of freedom (DoF), multiple-input and multiple-output (MIMO), reconfigurable antennas.

I. INTRODUCTION

Recently, there have been considerable researches on characterizing the *degrees of freedom* (DoF) of wireless networks. As current wireless networks become very complicated, exact capacity characterization is so difficult that many researchers have actively studied approximate capacity characterizations in the shape of DoF. The DoF is the prelog factor of capacity, providing an intuitive metric for the number of interference-free communication channels that wireless networks can attain at the high signal-to-noise ratio (SNR) regime. Hence, it is regarded as a primary performance metric for multiantenna and/or multiuser communication systems. Cadambe and Jafar recently made a remarkable progress on understanding DoF of multiuser wireless networks showing that the sum DoF of the K -user interference channel (IC) is given by $K/2$ [1]. An innovative methodology called *interference alignment* (IA) has been proposed to obtain $K/2$ DoF, which aligns multiple interfering signals into the same signal space at each receiver. The concept of such signal space alignment has been successfully adapted to various network environments, e.g., see [2]–[8] and the references therein. More recently, different strategies of IA were further developed in terms of ergodic IA [9]–[12] and real IA [13], [14].

Note that most of the previous researches including the aforementioned IA techniques have focused on DoF of wireless networks under the assumption that each transmitter perfectly knows global channel state information (CSI). However, for many practical communication systems, acquiring the exact CSI value at transmitters is very challenging due to channel feedback delay, system overhead, and so on. Motivated by these practical restrictions, implementing IA under a more relaxed CSI condition has been actively studied in the literature. Maddah-Ali and Tse made a breakthrough in [15] demonstrating that completely outdated CSI is still useful to improve DoF of the K -user multiple-input and single-output (MISO) broadcast channel (BC). Preceded by [15], there have been a series of researches for studying IA techniques exploiting outdated or delayed CSI at transmitters [16]–[20]. In [16]–[18], similar DoF gains were shown in MIMO BC under delayed CSIT and, in particular, the DoF region of the two-user MIMO BC with delayed CSIT was completely characterized in [18]. In the context of IC, it has been first shown in [21] that IA can achieve more than one DoF in the three-user SISO IC under delayed CSIT, which is then extended to the K -user case in [19], [20].

Although there is still a practical demand for further relaxing CSI requirements at the transmitter side, it has been proved in [22] that the DoF of the K -user MISO BC collapses to one for isotropic fading if the transmitter cannot acquire any information about CSI. In terms of isotropic fading and no CSIT, similar DoF degradation was further shown in MIMO BC and IC [23]–[26]. On the other hand, IA without CSIT, called *blind IA*, has been recently proposed in [27] for a class of

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heterogeneous block fading models¹ achieving larger DoF than that achievable for the isotropic fading model. In addition, it was shown that blind IA obtains similar DoF gain for a class of homogeneous block fading models² [28]–[30].

In [31], [32], Gou, Wang, and Jafar have first proposed a blind IA technique exploiting *reconfigurable antennas*. As shown in Fig. 1, reconfigurable antennas are capable of dynamically adjusting their radiation patterns in a controlled and reversible manner through various technologies such as solid state switches or microelectromechanical switches (MEMS), which can be conceptually modeled as antenna selection that each RF-chain of reconfigurable antennas chooses one of receiving mode among several preset modes at each time instant, see also [32, Section I] for the concept of reconfigurable antennas. Based on a remarkable observation that even for time-invariant channels, reconfigurable antenna can artificially create channel matrices correlated across time in some specific structure, the authors in [32] show that the optimal sum DoF of the K -user $M \times 1$ MISO BC is given by $\frac{MK}{M+K-1}$ when each user is equipped with a reconfigurable antenna whose RF-chain can choose one receiving mode from M preset modes. Subsequently, in [33], the achievability result in [32] is generalized to the K -user $M \times N$ MIMO BC where each user is equipped with a set of reconfigurable antennas whose RF-chains are able to choose N receiving modes from M preset modes, showing that the sum DoF of $\frac{MNK}{M+NK-N}$ is achievable. The idea of blind IA using reconfigurable antennas is further extended to ICs consisting of receivers with reconfigurable antennas [34]–[38].

In this paper, we consider the K -user MIMO BC assuming a general reconfigurable antenna environment. In particular, the transmitter is equipped with M antennas and user k , $k = 1, \dots, K$, is equipped with a set of reconfigurable antennas whose RF-chains can choose L_k receiving modes from N_k preset modes ($N_k \geq L_k$), which includes the conventional non-reconfigurable antenna model ($N_k = L_k$ for this case). We focus on the *linear DoF (LDoF) with no CSIT*, i.e., the maximum DoF achievable by linear coding strategies with no CSIT, see also [39]–[41] for the definition of LDoF. For general antenna configurations, we completely characterize the sum LDoF of the K -user MIMO BC with reconfigurable antennas in the absence of CSIT. We further characterize the LDoF region for a specific class of antenna configurations. Therefore, the main contributions of this paper are two-folds: 1) we generalize the previous achievability results in [32], [33] assuming a certain class of antenna configurations to general antenna configurations, 2) we show the converse of our achievable DoF in the LDoF sense, which implies that the achievability result in [33] is also optimal in the LDoF sense. Our analysis is further applied to a class of K -user MIMO IC with reconfigurable antennas and the sum LDoF is characterized for a class of antenna configurations, which generalizes the achievable sum DoF result in [36].

The rest of this paper is organized as follows. In Section II, we introduce the K -user MIMO BC with reconfigurable antennas. In Section III, we first define the LDoF and state the main result of this paper, the sum LDoF and LDoF region of the K -user MIMO BC with reconfigurable antennas. We present the converse and achievability of the main results in Section IV and V, respectively and finally conclude in Section VI.

II. SYSTEM MODEL

A. Notation

For integer values a and b , $a \setminus b$ and $a|b$ denote the quotient and the remainder respectively when dividing a by b . For a set \mathcal{A} , $|\mathcal{A}|$ is the cardinality of \mathcal{A} . For a vector space \mathcal{V} , $\dim(\mathcal{V})$ is the dimension of \mathcal{V} . For a matrix \mathbf{A} , \mathbf{A}^T , $|\mathbf{A}|$, $\text{rank}(\mathbf{A})$, and $\mathcal{R}(\mathbf{A})$ are the transpose, determinant, rank, and column space of \mathbf{A} respectively. For matrices \mathbf{A} and \mathbf{B} , $\mathbf{A} \otimes \mathbf{B}$ is the Kronecker product of \mathbf{A} and \mathbf{B} . For a set of matrices $\{\mathbf{A}_i\}_{i=1, \dots, n}$, $\text{diag}(\mathbf{A}_1, \dots, \mathbf{A}_n)$ denotes the block-diagonal matrix consisting of $\{\mathbf{A}_i\}$. Also \mathbf{I}_a , $\mathbf{1}_{a \times b}$, and $\mathbf{0}_{a \times b}$ denote the $a \times a$ identity matrix, the $a \times b$ all-one matrix, and the $a \times b$ all-zero matrix respectively and let $\mathbf{0}_a = \mathbf{0}_{a \times a}$.

B. K -user MIMO BC with Reconfigurable Antennas

Consider the K -user MIMO BC depicted in Fig. 2 in which the transmitter is equipped with M antennas and user $k \in \mathcal{K} = \{1, \dots, K\}$ is equipped with a set of reconfigurable antennas whose RF-chains are able to choose L_k receiving modes from N_k preset modes at every time instant, where $N_k \geq L_k$. Note that, if $N_k = L_k$, then user k is equivalent to be equipped with L_k conventional (non-reconfigurable) antennas.

The received signal vector of user k at time t is given by

$$\mathbf{y}_k(t) = \mathbf{\Gamma}_k(t)\mathbf{H}_k(t)\mathbf{x}(t) + \mathbf{z}_k(t) \quad (1)$$

where $\mathbf{H}_k(t) \in \mathbb{C}^{N_k \times M}$ is the channel matrix from the transmitter to N_k preset modes of user k at time t , $\mathbf{x}(t) \in \mathbb{C}^M$ is the transmit signal vector at time t , $\mathbf{z}_k(t) \in \mathbb{C}^{L_k}$ is the additive noise vector of user k at time t , and $\mathbf{\Gamma}_k(t) \in \{0, 1\}^{L_k \times N_k}$ is the selection matrix of user k at time t . In particular, each row vector of $\mathbf{\Gamma}_k(t)$ consists of zero values except for a single element of one value and is different from each other. That is, $\mathbf{\Gamma}_k(t)$ extracts L_k elements out of the N_k elements in $\mathbf{H}_k(t)\mathbf{x}(t)$ and if user k is equipped with conventional antennas, i.e., $L_k = N_k$, then $\mathbf{\Gamma}_k(t) = \mathbf{I}_{N_k}$ so that $\mathbf{\Gamma}_k(t)$ can be omitted in (1). The

¹Certain users experience smaller coherence time/bandwidth than others (See [27] for more details).

²All users experience independent block fading with the same coherence time, but different offsets (See [28] for more details).

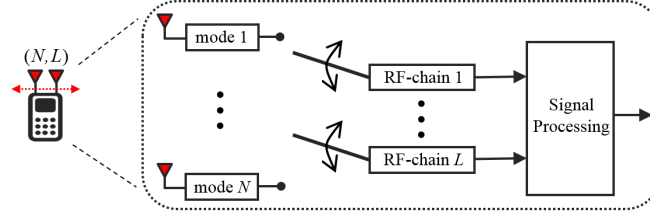


Fig. 1. Conceptual illustration of reconfigurable antennas, where each of L RF-chains chooses one receiving mode from N preset modes.

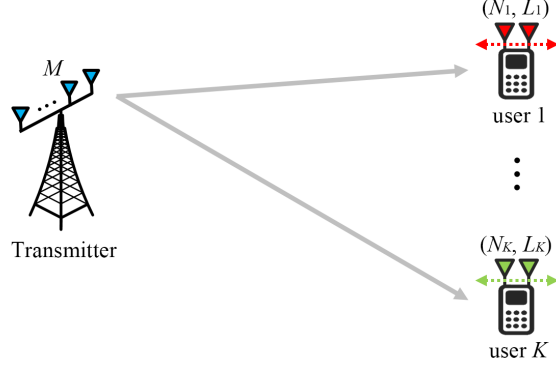


Fig. 2. K -user MIMO BC with reconfigurable antennas.

transmitter should satisfy the average power constraint P , i.e., $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \|\mathbf{x}(t)\|^2 \leq P$, where $\|\cdot\|$ denote the norm of a vector. The elements of $\mathbf{z}_k(t)$ are independent and identically distributed (i.i.d.) drawn from $\mathcal{CN}(0, 1)$.

We assume that channel coefficients are i.i.d. drawn from a continuous distribution and remain constant across time, i.e., $\mathbf{H}_k(t) = \mathbf{H}_k$ for all $t \in \mathbb{N}$. Global channel state information (CSI) is assumed to be available only at the users, but not at the transmitter. i.e., no CSIT. Furthermore, we assume that each user selects its receiving modes in a predetermined pattern independent of channel realization, which are revealed to the transmitter. That is, $\Gamma_k(t)$ is not a function of $\{\mathbf{H}_j\}_{j \in \mathcal{K}}$ for all $k \in \mathcal{K}$ and $t \in \mathbb{N}$.

For notational convenience, from (1), we define the n time-extended input–output relation as

$$\mathbf{y}_k^n = \Gamma_k^n \mathbf{H}_k^n \mathbf{x}^n + \mathbf{z}_k^n \quad (2)$$

where

$$\begin{aligned} \Gamma_k^n &= \text{diag}(\Gamma_k(1), \dots, \Gamma_k(n)), \\ \mathbf{H}_k^n &= \mathbf{I}_n \otimes \mathbf{H}_k, \\ \mathbf{y}_k^n &= [\mathbf{y}_k^T(1) \cdots \mathbf{y}_k^T(n)]^T, \\ \mathbf{x}_k^n &= [\mathbf{x}_k^T(1) \cdots \mathbf{x}_k^T(n)]^T, \\ \mathbf{z}_k^n &= [\mathbf{z}_k^T(1) \cdots \mathbf{z}_k^T(n)]^T. \end{aligned}$$

III. LINEAR DEGREES OF FREEDOM AND MAIN RESULTS

A. Linear Degrees of Freedom

In this paper, we confine the transmitter to use linear precoding techniques, in which DoF represents the dimension of the linear subspace of transmitted signals [40]. Consider a linear precoding scheme with block length n , in which the transmitter sends the information symbols of user k , denoted by $\mathbf{s}_k \in \mathbb{C}^{m_k(n)}$, through the n time-extended beamforming matrix $\mathbf{V}_k^n \in \mathbb{C}^{nM \times m_k(n)}$. Hence, the n time-extended transmit signal vector is given by

$$\mathbf{x}^n = \sum_{j=1}^K \mathbf{V}_j^n \mathbf{s}_j$$

and, from (2), the n time-extended received signal vector of user k is given by

$$\mathbf{y}_k^n = \sum_{j=1}^K \Gamma_k^n \mathbf{H}_k^n \mathbf{V}_j^n \mathbf{s}_j + \mathbf{z}_k^n.$$

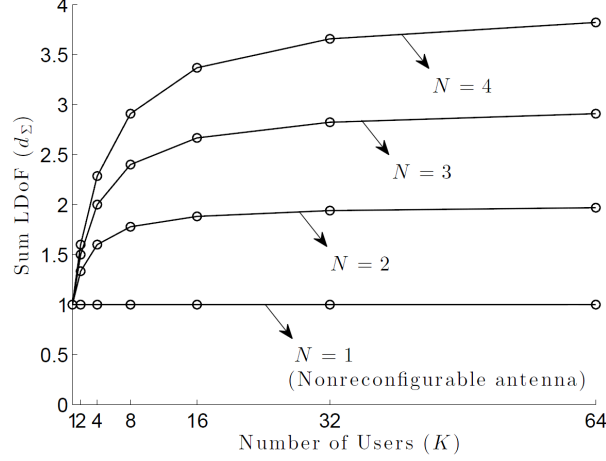


Fig. 3. Sum LDoF d_Σ with respect to K when $M = 4$ and $L = 1$.

Based on such a linear precoding scheme, we define the linear degrees of freedom as the follow, see also [40] for more details.

Definition 1: The linear degrees of freedom (LDoF) of K -tuple (d_1, \dots, d_K) is said to be achievable if there exist a set of beamforming matrices \mathbf{V}_j^n and selection matrices $\mathbf{\Gamma}_j^n$ for $j = 1, \dots, K$ almost surely satisfying

$$\dim \left(\text{Proj}_{\mathcal{I}_j^n} \mathcal{R}(\mathbf{\Gamma}_j^n \mathbf{H}_j^n \mathbf{V}_j^n) \right) = m_j(n),$$

$$d_j = \lim_{n \rightarrow \infty} \frac{m_j(n)}{n}$$

where $\mathcal{I}_j = \mathcal{R}(\mathbf{\Gamma}_j^n \mathbf{H}_j^n [\mathbf{V}_1^n \cdots \mathbf{V}_{j-1}^n \mathbf{V}_{j+1}^n \cdots \mathbf{V}_K^n])$ and $\text{Proj}_{\mathcal{A}^\perp} \mathcal{B}$ denotes the vector space induced by projecting the vector space \mathcal{B} onto the orthogonal complement of the vector space \mathcal{A} .

The LDoF region \mathcal{D} is the closure of the set of all achievable LDoF tuples satisfying Definition 1 and the sum LDoF is then given by

$$d_\Sigma = \max_{(d_1, \dots, d_K) \in \mathcal{D}} \left\{ \sum_{k=1}^K d_k \right\}.$$

B. Main Results

For convenience of representation, the following parameters are defined.

$$\begin{aligned} L_{\max} &= \max_{k \in \mathcal{K}} \{L_k\}, \\ T_k &= \min(M, N_k) \text{ for } k \in \mathcal{K}, \\ \Lambda &= \{k \in \mathcal{K} : T_k > L_{\max}\}, \\ \eta &= \frac{\sum_{i \in \Lambda} \frac{T_i L_i}{T_i - L_i}}{1 + \sum_{i \in \Lambda} \frac{L_i}{T_i - L_i}}. \end{aligned} \quad (3)$$

In the following, we completely characterize the sum LDoF of the K -user MIMO BC with reconfigurable antennas.

Theorem 1: For the K -user MIMO BC with reconfigurable antennas defined in Section II, the sum LDoF is given by

$$d_\Sigma = \min(M, \max(L_{\max}, \eta)). \quad (4)$$

Proof: We refer to Section IV-A for the converse proof and Section V-A for the achievability proof. \blacksquare

Remark 1: From Theorem 1, N_k greater than M cannot further increase d_Σ . Therefore, the number of preset modes N_k for maximizing d_Σ is enough to set $N_k = M$ for $k \in \mathcal{K}$. Note that this remark is valid only in MIMO BC with reconfigurable antennas and it is shown in [37] that the number of preset modes greater than that of transmit antennas can increase sum DoF in MIMO IC with reconfigurable antennas.

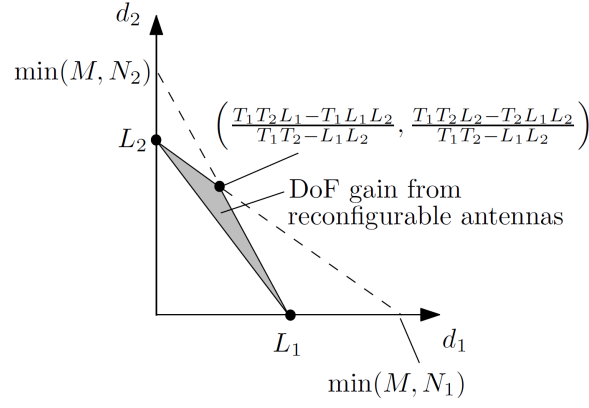


Fig. 4. LDoF region \mathcal{D} for the 2-user MIMO BC with reconfigurable antennas, where $M, N_1, N_2 > \max(L_1, L_2)$.

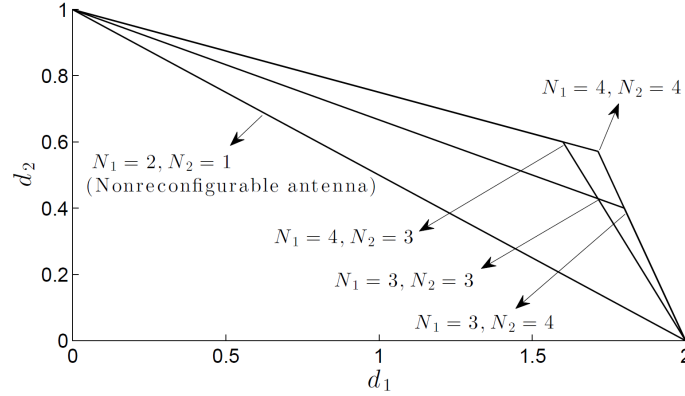


Fig. 5. LDoF region \mathcal{D} when $K = 2$, $M = 4$, $L_1 = 2$, and $L_2 = 1$.

Example 1: Consider the symmetric K -user MIMO BC with reconfigurable antennas in Section II in which $N_k = N$ and $L_k = L$ for all $k \in \mathcal{K}$. For this case,

$$d_\Sigma = \min \left(M, \max \left(L, \frac{KL \min(M, N)}{KL + \min(M, N) - L} \right) \right) \quad (5)$$

from Theorem 1. To figure out the impact of reconfigurable antennas, let us focus on the limiting case where K tends to infinity. Then

$$\lim_{K \rightarrow \infty} d_\Sigma = \min(M, \max(L, \min(M, N))) = \min(M, N) \quad (6)$$

regardless of L . Note that $d_\Sigma = \min(M, L)$ for the symmetric K -user MIMO BC without reconfigurable antennas, which corresponds to the case where $N = L$. Therefore, reconfigurable antennas can significantly improve the sum LDoF as both M and N increase. Figure 3 plots d_Σ with respect to K when $M = 4$ and $L = 1$. As the number of preset modes N increases, the DoF gain from reconfigurable antennas increases compared to the conventional (nonreconfigurable) antenna model, i.e., $N = L$.

We further derive the LDoF region \mathcal{D} for a class of antenna configurations in the following theorem.

Theorem 2: Consider the K -user MIMO BC with reconfigurable antennas defined in Section II. If $M > L_{\max}$ and $N_k > L_{\max}$ for all $k \in \mathcal{K}$, then the LDoF region \mathcal{D} consists of all K -tuples (d_1, \dots, d_K) satisfying

$$\frac{d_k}{L_k} + \sum_{j=1, j \neq k}^K \frac{d_j}{T_j} \leq 1 \quad (7)$$

for all $k \in \mathcal{K}$.

Proof: We refer to Section IV-B for the converse proof and Section V-B for the achievability proof. \blacksquare

Example 2: Consider the 2-user MIMO BC with reconfigurable antennas in Section II in which $M, N_1, N_2 > \max(L_1, L_2)$. From Theorem 2, the LDoF region \mathcal{D} is then given as in Fig. 4. For the conventional (nonreconfigurable) antenna model, where $N_1 = L_1$ and $N_2 = L_2$, \mathcal{D} is given by the time-sharing region between $(L_1, 0)$ and $(0, L_2)$. Hence \mathcal{D} enlarges as N_1

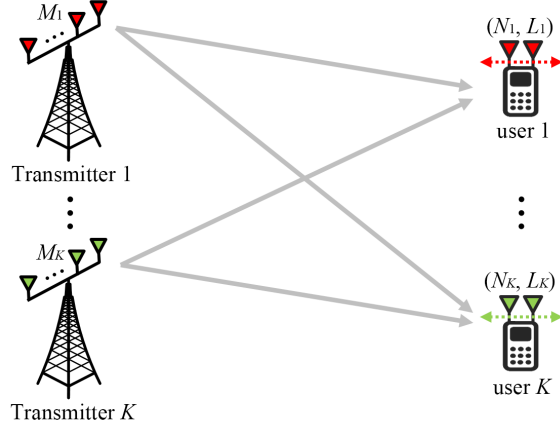


Fig. 6. K -user MIMO IC with reconfigurable antennas.

and N_2 increase, which demonstrate the benefit of reconfigurable antennas. Figure 5 plots \mathcal{D} when $K = 2$, $M = 4$, $L_1 = 2$, and $L_2 = 1$.

From Theorem 1, the sum LDoF is derived for a class of the K -user MIMO IC with reconfigurable antennas in the following. We omit the formal definition of LDoF for the K -user MIMO IC with reconfigurable antennas, which can be straightforwardly defined in the same manner as in Definition 1.

Corollary 1: Consider the K -user MIMO IC with reconfigurable antennas depicted in Fig. 6 in which transmitter $k \in \mathcal{K}$ is equipped with M_k antennas and user k is equipped with a set of reconfigurable antennas whose RF-chains are able to choose L_k receiving mode from N_k preset modes, where $N_k \geq L_k$. If $M_k \geq N_k$ for all $k \in \mathcal{K}$, then the sum LDoF is given by

$$d_{\Sigma, \text{IC}} = \max(L_{\max}, \eta_{\text{IC}}) \quad (8)$$

where η_{IC} is defined as η with $\Lambda = \{k \in \mathcal{K} : N_k > L_{\max}\}$ and $T_k = N_k$ for all $k \in \Lambda$

Proof: Obviously, the achievable LDoF of the K -user MIMO IC with reconfigurable antennas defined in Corollary 1 is upper bounded by d_{Σ} of the K -user MIMO BC with reconfigurable antennas where the transmitter is equipped with $\sum_{k=1}^K M_k$ antennas and user $k \in \mathcal{K}$ is equipped with a set of reconfigurable antennas whose RF-chains are able to choose L_k receiving modes from N_k preset modes. Hence, from Theorem 1, the LDoF of the considered K -user MIMO IC is upper bounded by (8), which completes the converse proof of Corollary 1. We refer to Section V-C for the achievability proof. ■

Example 3: Consider the symmetric MIMO IC with reconfigurable antennas in Fig. 6 in which $N_k = N$ and $L_k = L$ for all $k \in \mathcal{K}$, where $N \geq L$. If $M \geq N$, then from Corollary 1,

$$d_{\Sigma, \text{IC}} = \max\left(L, \frac{KLN}{KL + N - L}\right), \quad (9)$$

which attains $\lim_{K \rightarrow \infty} d_{\Sigma, \text{IC}} = N$. Note that the symmetric K -user MIMO IC without reconfigurable antennas is given by $d_{\Sigma, \text{IC}} = L$, which corresponds to the case where $M \geq N = L$. Therefore, similar to the symmetric MIMO BC case, reconfigurable antenna can significantly improve the sum LDoF as both M and N increase with $M \geq N$.

The following two remarks summarize the contributions of Theorem 1 and Corollary 1, compared with the previous results in [33], [36].

Remark 2: Consider the K -user MIMO BC with reconfigurable antennas defined in Section II. If $M = N_k$ and $L_k = L$ for all $k \in \mathcal{K}$ where $M > L$, then

$$d_{\Sigma} = \frac{MLK}{M + LK - L}$$

from Theorem 1, which coincides with the previous achievability result in [33]. Hence, Theorem 1 not only generalizes the result in [33] but it also shows the converse in the LDoF sense for general M , $\{N_k\}_{k \in \mathcal{K}}$, and $\{L_k\}_{k \in \mathcal{K}}$.

Remark 3: Consider the K -user MIMO IC with reconfigurable antennas defined in Corollary 1. If $M_k = N_k > 1$ and $L_k = 1$ for all $k \in \mathcal{K}$, then

$$d_{\Sigma, \text{IC}} = \frac{\sum_{k=1}^K \frac{N_k}{N_k - 1}}{1 + \sum_{k=1}^K \frac{1}{N_k - 1}}$$

from Corollary 1, which coincides with the previous achievability result in [36]. Hence, Corollary 1 not only generalizes the result in [36] but it also shows the converse in the LDoF sense for a broader class of antenna configurations.

IV. CONVERSE

In this section, we prove the converse of Theorem 1, 2.

A. Converse of Theorem 1

First divide the entire parameter space into three cases as follows:

- Case 1: $M \leq L_{\max}$.
- Case 2: $M > L_{\max}$ and $N_k \leq L_{\max}$ for all $k \in \mathcal{K}$.
- Case 3: $M > L_{\max}$ and $N_k > L_{\max}$ for some $k \in \mathcal{K}$.

Then the right hand side of (4) is given by

$$\min(M, \max(L_{\max}, \eta)) = \begin{cases} M & \text{for Case 1,} \\ L_{\max} & \text{for Case 2,} \\ \max(L_{\max}, \eta) & \text{for Case 3.} \end{cases} \quad (10)$$

For Case 1, an achievable sum LDoF is trivially upper bounded by the number of transmit antennas. Consequently, we have

$$d_{\Sigma} \leq M \quad \text{for Case 1.} \quad (11)$$

For Case 2, consider the *extended K-user MIMO BC* by substituting $N_k = L_{\max}$ and $L_k = L_{\max}$ for all $k \in \mathcal{K}$ from the original K -user MIMO BC with reconfigurable antennas. That is, for the extended K -user MIMO BC, all users are equipped with L_{\max} conventional antennas. Obviously, the sum DoF of the extended K -user MIMO BC provides an upper bound on d_{Σ} . From the fact that the received signals of all the user are statistically equivalent in the extended K -user MIMO BC so that any receiver can decode all messages from the transmitter, d_{Σ} is further bounded by the sum DoF of point-to-point MIMO BC where transmitter and receiver are equipped with M and L_{\max} conventional antennas respectively, given by $\min(M, L_{\max}) = L_{\max}$. Therefore, we have

$$d_{\Sigma} \leq L_{\max} \quad \text{for Case 2.} \quad (12)$$

Hence, for the rest of this subsection, we prove that

$$d_{\Sigma} \leq \max(L_{\max}, \eta)$$

by assuming that $M > L_{\max}$ and $N_k > L_{\max}$ for some $k \in \mathcal{K}$, which is Case 3. Suppose that user k satisfies the condition $T_k > L_{\max}$ (equivalently $k \in \Lambda$). Then, consider the *extended K-user MIMO BC with reconfigurable antennas at user k* by substituting $N_i = L_{\max}$ and $L_i = L_{\max}$ for all $i \in \mathcal{K} \setminus \Lambda$ and $L_i = N_i$ for all $i \in \Lambda \setminus \{k\}$ from the original K -user MIMO BC with reconfigurable antennas. Hence, users in $\mathcal{K} \setminus \Lambda$ have L_{\max} conventional antennas and user $i \in \Lambda \setminus \{k\}$ has N_i conventional antennas. Only user k is equipped with reconfigurable antennas in this extended model. Again, the sum DoF of this model provides an upper bound on d_{Σ} . Then, the received signal vector of user i is given by

$$\mathbf{y}_i(t) = \begin{cases} \mathbf{\Gamma}_k(t) \mathbf{H}_k \mathbf{x}(t) + \mathbf{z}_k(t) & \text{if } i = k, \\ \mathbf{G}_i \mathbf{x}(t) + \mathbf{z}_i(t) & \text{otherwise} \end{cases} \quad (13)$$

where $\mathbf{G}_i \in \mathbb{C}^{\max(N_i, L_{\max}) \times M}$ for $i \in \mathcal{K} \setminus \{k\}$ satisfies the channel assumption in Section II.

For convenience, we rearrange the users in ascending order of L_i and denote the new index of user i as $\sigma(i)$. We assume that $\sigma(k) = 1$ without loss of generality. From now on, we denote the index i as the rearranged user index. Fig. 7 illustrates the extended model based on the rearranged user index. Hence, the n time-extended received signal vector of user i with linear precoding is given by

$$\mathbf{y}_i^n = \begin{cases} \sum_{j=1}^K \mathbf{\Gamma}_i^n \mathbf{H}_i^n \mathbf{V}_j^n \mathbf{s}_j + \mathbf{z}_i^n & \text{if } i = 1, \\ \sum_{j=1}^K \mathbf{G}_i^n \mathbf{V}_j^n \mathbf{s}_j + \mathbf{z}_i^n & \text{otherwise} \end{cases} \quad (14)$$

where $\mathbf{G}_i^n = \mathbf{I}_n \otimes \mathbf{G}_i$ for $i \in \mathcal{K} \setminus \{1\}$. Also, we define an increasing sequence Δ_i for $i \in \mathcal{K}$ as

$$\Delta_i = \begin{cases} L_k & \text{if } \sigma^{-1}(i) = k, \\ L_{\max} & \text{if } \sigma^{-1}(i) \in \mathcal{K} \setminus \Lambda, \\ T_{\sigma^{-1}(i)} & \text{otherwise.} \end{cases}$$

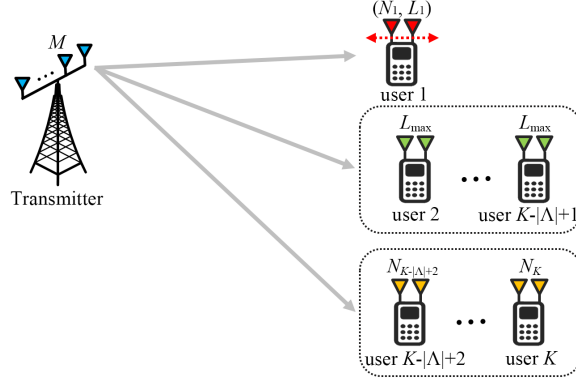


Fig. 7. Extended K -user MIMO BC with reconfigurable antennas based on the rearranged user index, where we assume that if $|\Lambda| = 1$, then the set of user $K - |\Lambda| + 2$ through user K is empty and if $|\Lambda| = K$, then the set of user 2 through $K - |\Lambda| + 1$ is empty.

Note that Δ_i is the rank of $\mathbf{\Gamma}_1(t)\mathbf{H}_1$ for $i = 1$ and the rank of \mathbf{G}_i for $i = 2, \dots, K$, almost surely.

In the following, we introduce three key lemmas used for proving the converse of Theorem 1. The first lemma provides an equivalent condition for decodability of messages [40].

Lemma 1 (Lashgari–Avestimehr–Suh): For two matrices \mathbf{A}, \mathbf{B} with the same row size,

$$\dim(\text{Proj}_{\mathcal{R}(\mathbf{A})^c} \mathcal{R}(\mathbf{B})) = \text{rank}([\mathbf{A} \ \mathbf{B}]) - \text{rank}(\mathbf{A}).$$

Proof: We refer to [40, Lemma 1] for the proof. ■

The second lemma states that mode switching does not decrease the dimension of the interference space of user 1 almost surely.

Lemma 2: Consider the extended K -user MIMO BC with reconfigurable antennas at user 1 depicted in Fig. 7. Let $\mathbf{G}_1 \in \mathbb{C}^{L_1 \times M}$ denote the matrix consisting of the first through the L_1 th row vectors of \mathbf{H}_1 and $\mathbf{G}_1^n = \mathbf{I}_n \otimes \mathbf{G}_1$. For any mode switching pattern $\mathbf{\Gamma}_1^n$, the following relation holds almost surely:

$$\text{rank}(\mathbf{\Gamma}_1^n \mathbf{H}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n]) \geq \text{rank}(\mathbf{G}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n]). \quad (15)$$

Proof: We refer to Appendix I-A for the proof. ■

Although the definition of \mathbf{G}_1^n in Lemma 2 is not consistent with those of \mathbf{G}_i^n for $i = 2, \dots, K$ in (14), we adopt this notation for easy presentation of the converse proof. The third lemma shows the relation of the dimensions of the interference space between user i and user $i - 1$.

Lemma 3: Consider the extended K -user MIMO BC with reconfigurable antennas at user 1 depicted in Fig. 7. The following relations hold almost surely:

$$\frac{1}{\Delta_{i-1}} \text{rank}(\mathbf{G}_{i-1}^n [\mathbf{V}_i^n \cdots \mathbf{V}_K^n]) \geq \frac{1}{\Delta_i} \text{rank}(\mathbf{G}_i^n [\mathbf{V}_{i+1}^n \cdots \mathbf{V}_K^n]) + \frac{1}{\Delta_i} \dim(\text{Proj}_{\mathcal{I}_i^c} \mathcal{R}(\mathbf{G}_i^n \mathbf{V}_i^n)), \quad (16)$$

for $i = 2, \dots, K - 1$ and

$$\frac{1}{\Delta_{K-1}} \text{rank}(\mathbf{G}_{K-1}^n \mathbf{V}_K^n) \geq \frac{1}{\Delta_K} \dim(\text{Proj}_{\mathcal{I}_K^c} \mathcal{R}(\mathbf{G}_K^n \mathbf{V}_K^n)). \quad (17)$$

Proof: We refer to Appendix I-B for the proof. ■

We are now ready to prove the converse of Theorem 1. From the definition of $m_1(n)$, we have

$$\begin{aligned} m_1(n) &= \dim(\text{Proj}_{\mathcal{I}_1^c} \mathcal{R}(\mathbf{\Gamma}_1^n \mathbf{H}_1^n \mathbf{V}_1^n)) \\ &= \text{rank}(\mathbf{\Gamma}_1^n \mathbf{H}_1^n [\mathbf{V}_1^n \cdots \mathbf{V}_K^n]) - \text{rank}(\mathbf{\Gamma}_1^n \mathbf{H}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n]) \end{aligned} \quad (18a)$$

$$\begin{aligned} &\leq n\Delta_1 - \text{rank}(\mathbf{\Gamma}_1^n \mathbf{H}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n]) \\ &\stackrel{a.s.}{\leq} n\Delta_1 - \text{rank}(\mathbf{G}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n]) \end{aligned} \quad (18b)$$

$$\stackrel{a.s.}{\leq} n\Delta_1 - \sum_{i=2}^K \frac{\Delta_1}{\Delta_i} \dim(\text{Proj}_{\mathcal{I}_i^c} \mathcal{R}(\mathbf{G}_i^n \mathbf{V}_i^n)) \quad (18c)$$

$$= n\Delta_1 - \sum_{i=2}^K \frac{\Delta_1}{\Delta_i} m_i(n) \quad (18d)$$

where (18a), (18b), (18c), and (18d) follow from Lemma 1, 2, 3, and Definition 1 respectively. Then, by dividing both sides by n and letting n to infinity, we have

$$\sum_{i=1}^K \frac{1}{\Delta_i} d_i \leq 1. \quad (19)$$

Rearranging (19) with respect to the original index, i.e., $\sigma^{-1}(i)$ provides

$$\frac{1}{L_k} d_k + \sum_{i \in \Lambda \setminus \{k\}} \frac{1}{T_i} d_i + \sum_{i \notin \Lambda} \frac{1}{L_{\max}} d_i \leq 1. \quad (20)$$

Since (20) holds for all $k \in \Lambda$, we have total $|\Lambda|$ inequalities composing the outer region of \mathcal{D} . Then, we obtain an upper bound on d_Σ by solving the linear programming in the following lemma.

Lemma 4: Consider the following optimization problem assuming that $M > L_{\max}$ and $N_k > L_{\max}$ for some $k \in \mathcal{K}$:

$$\begin{aligned} & \text{maximize } \sum_{i=1}^K d_i \\ & \text{subject to } \frac{1}{L_k} d_k + \sum_{i \in \Lambda \setminus \{k\}} \frac{1}{T_i} d_i + \sum_{i \notin \Lambda} \frac{1}{L_{\max}} d_i \leq 1, \quad \forall k \in \Lambda, \\ & \quad d_i \geq 0, \quad \forall i \in \mathcal{K}. \end{aligned}$$

Then

$$\sum_{i=1}^K d_i \leq \max(\eta, L_{\max}).$$

Proof: We refer to Appendix I-C for the proof. ■

Therefore, combining (10), (11), (12) and the result in Lemma 4 provides

$$d_\Sigma \leq \min(M, \max(L_{\max}, \eta)),$$

which completes the converse proof of Theorem 1.

B. Converse of Theorem 2

Notice that the condition $M > L_{\max}$ and $N_k > L_{\max}$ in Theorem 2 is a special class of Case 3 defined in Section IV-A satisfying that $\Lambda = \mathcal{K}$. Therefore, in the same manner in Section IV-A, we have (20) with $\Lambda = \mathcal{K}$. Therefore,

$$\frac{d_k}{L_k} + \sum_{i=1, i \neq k}^K \frac{d_i}{T_i} \leq 1, \quad \forall k \in \mathcal{K},$$

which completes the converse proof of Theorem 2.

V. ACHIEVABILITY

In this section, we prove achievability of Theorems 1, 2, and Corollary 1. The proposed blind IA scheme generalizes those in [31]–[33], but it cannot be straightforwardly obtained from [31]–[33] due to general antenna configurations of M , $\{N_k\}_{k \in \mathcal{K}}$, and $\{L_k\}_{k \in \mathcal{K}}$ considered in this paper. For better understanding, we also provide an example for the proposed blind IA scheme based on the two-user case in Appendix II.

A. Achievability of Theorem 1

First divide the entire parameter space into four cases as follows:

- Case 1: $M \leq L_{\max}$.
- Case 2: $M > L_{\max}$ and $N_k \leq L_{\max}$ for all $k \in \mathcal{K}$.
- Case 3-1: $M > L_{\max}$, $N_k > L_{\max}$ for some $k \in \mathcal{K}$, and $\eta \leq L_{\max}$,
- Case 3-2: $M > L_{\max}$, $N_k > L_{\max}$ for some $k \in \mathcal{K}$, and $\eta > L_{\max}$,

where Cases 1 and 2 are identical to those in Section IV-A and Case 3-1 and Case 3-2 are two partitions of Case 3 in Section IV-A. The right side of (4) is then given by

$$\min(M, \max(L_{\max}, \eta)) = \begin{cases} M & \text{for Case 1,} \\ L_{\max} & \text{for Case 2 or Case 3-1,} \\ \eta & \text{for Case 3-2.} \end{cases} \quad (21)$$

For Cases 1, 2, and 3-1, the sum DoF is trivially achievable by only supporting the user having the maximum number of RF-chains. Hence,

$$d_{\Sigma} = \begin{cases} M & \text{for Case 1,} \\ L_{\max} & \text{for Case 2 or Case 3-1} \end{cases} \quad (22)$$

is achievable. For the rest of this subsection, we prove that

$$d_{\Sigma} = \eta = \frac{\sum_{k \in \Lambda} \frac{T_k L_k}{T_k - L_k}}{1 + \sum_{k \in \Lambda} \frac{L_k}{T_k - L_k}} \quad (23)$$

is achievable by assuming that $M > L_{\max}$, $N_i > L_{\max}$ for some $i \in \mathcal{K}$, and $\eta > L_{\max}$, which is Case 3-2. For this case, only the users in Λ are supported, i.e., $d_i = 0$ for all $i \notin \Lambda$. Suppose that $\Lambda = \{1, 2, \dots, |\Lambda|\}$ without loss of generality. For easy representation, let us define S_i , U_i , and W_i for $i \in \Lambda$ and define U and W as

$$\begin{aligned} S_i &= \begin{cases} \frac{T_i}{L_i} - 1 & \text{if } T_i | L_i = 0, \\ T_i \setminus L_i & \text{otherwise,} \end{cases} \\ U_i &= S_i \prod_{p \in \Lambda \setminus \{i\}} (T_p - L_p), \\ W_i &= \prod_{p \in \Lambda \setminus \{i\}} S_p, \\ U &= \prod_{p \in \Lambda} (T_p - L_p), \\ W &= \prod_{p \in \Lambda} S_p. \end{aligned} \quad (24)$$

1) *Transmit beamforming design*: To construct transmit beamforming, we adopt a bottom-up approach as in the following steps.

Step 1 (Alignment block): As the first step, we construct alignment blocks, which will be used for building alignment units in the next step. The basic concept of alignment block in this paper is similar to those in [32], [33]. Define $\mathbf{I}_{M, T_i} = [\mathbf{I}_{T_i} \ \mathbf{0}_{T_i, M-T_i}]^T$ and the j th information vectors of user i as $\mathbf{s}_j^{[i]} \in \mathbb{C}^{L_i T_i}$, which consists of $L_i T_i$ independent information symbols, where $j = 1, \dots, U_i W_i$. Then the j th alignment block of user i , denoted by $\mathbf{v}_j^{[i]} \in \mathbb{C}^{M T_i}$, is defined as

$$\mathbf{v}_j^{[i]} = \left[(\mathbf{v}_{j,1}^{[i]})^T (\mathbf{v}_{j,2}^{[i]})^T \dots (\mathbf{v}_{j,S_i+1}^{[i]})^T \right]^T$$

where for $k = 1, \dots, S_i + 1$,

$$\mathbf{v}_{j,k}^{[i]} = \begin{cases} (\Phi \otimes \mathbf{I}_{M, T_i}) \mathbf{s}_j^{[i]} \in \mathbb{C}^{(T_i | L_i) M} & \text{if } T_i | L_i \neq 0 \text{ and } k = 1, \\ (\mathbf{I}_{L_i} \otimes \mathbf{I}_{M, T_i}) \mathbf{s}_j^{[i]} \in \mathbb{C}^{L_i M} & \text{otherwise.} \end{cases} \quad (25)$$

Here $\Phi \in \mathbb{C}^{T_i | L_i \times L_i}$ is a random matrix whose entries are i.i.d. drawn from a continuous distribution. From (25), the following relation holds:

$$\mathbf{v}_{j,k}^{[i]} = \begin{cases} (\Phi \otimes \mathbf{I}_M) \mathbf{v}_{j,S_i+1}^{[i]} & \text{if } T_i | L_i \neq 0 \text{ and } k = 1, \\ \mathbf{v}_{j,S_i+1}^{[i]} & \text{otherwise.} \end{cases} \quad (26)$$

Step 2 (Alignment unit): Next, we build an alignment unit using U_i alignment blocks. Specifically, $\mathbf{v}_{1+(j-1)U_i}^{[i]}$ through $\mathbf{v}_{jU_i}^{[i]}$ are used for building the j th alignment unit of user i , denoted by $\mathbf{u}_j^{[i]} \in \mathbb{C}^{M T_i U_i}$ where $j = 1, \dots, W_i$, which is given as

$$\mathbf{u}_j^{[i]} = \left[(\mathbf{u}_{j,1}^{[i]})^T (\mathbf{u}_{j,2}^{[i]})^T \dots (\mathbf{u}_{j,S_i+1}^{[i]})^T \right]^T \quad (27)$$

where

$$\mathbf{u}_{j,k}^{[i]} = \begin{bmatrix} \mathbf{v}_{1+(j-1)U_i,(1-k)|S_i+1}^{[i]} \\ \mathbf{v}_{2+(j-1)U_i,(2-k)|S_i+1}^{[i]} \\ \vdots \\ \mathbf{v}_{jU_i,(U_i-k)|S_i+1}^{[i]} \end{bmatrix} \in \mathbb{C}^{MU}$$

for $k = 1, \dots, S_i$ and

$$\mathbf{u}_{j,S_i+1}^{[i]} = \begin{bmatrix} \mathbf{v}_{1+(j-1)U_i,S_i+1}^{[i]} \\ \mathbf{v}_{2+(j-1)U_i,S_i+1}^{[i]} \\ \vdots \\ \mathbf{v}_{jU_i,S_i+1}^{[i]} \end{bmatrix} \in \mathbb{C}^{L_i MU_i}.$$

From (26), the following relations hold for $k = 1, \dots, S_i$:

$$\mathbf{u}_{j,k}^{[i]} = \begin{cases} \mathbf{u}_{j,S_i+1}^{[i]} & \text{if } T_i | L_i = 0, \\ (\mathbf{I}_{U_i/S_i} \otimes \mathbf{Q}_k^{[i]} \otimes \mathbf{I}_M) \mathbf{u}_{j,S_i+1}^{[i]} & \text{otherwise} \end{cases} \quad (28)$$

where $\mathbf{Q}_k^{[i]} \in \mathbb{C}^{(T_i-L_i) \times L_i S_i}$ is the block-diagonal matrix consisting of S_i blocks whose blocks are all \mathbf{I}_{L_i} except that the k th block is Φ . For convenience, let us call $\mathbf{u}_{j,k}^{[i]}$ as the k th sub-unit of $\mathbf{u}_j^{[i]}$.

Step 3 (Transmit signal vector for user i): We then construct the transmit signal vector for user i using $\mathbf{u}_1^{[i]}$ through $\mathbf{u}_{W_i}^{[i]}$. The transmit signal for user i , denoted by $\mathbf{x}_i \in \mathbb{C}^{MUW+M \sum_{i \in \Lambda} L_i U_i W_i}$, is defined as

$$\mathbf{x}_i = [\mathbf{x}_{i,1}^T \quad \mathbf{0}_{C_{1,i-1} \times 1}^T \quad \mathbf{x}_{i,2}^T \quad \mathbf{0}_{C_{i+1,|\Lambda|} \times 1}^T]^T \quad (29)$$

where $C_{l,m} = \sum_{p=l}^m L_p MU_p W_p$ and $\mathbf{x}_{i,1}$ consists of $\{\mathbf{u}_{j,k}^{[i]}\}_{k=1, \dots, S_i}^{j=1, \dots, W_i}$, total W sub-units, and $\mathbf{x}_{i,2}$ consists of $\{\mathbf{u}_{j,k}^{[i]}\}_{k=S_i+1}^{j=1, \dots, W_i}$, total W_i sub-units, defined as in the followings.

$$\mathbf{x}_{i,1} = \begin{bmatrix} \mathbf{u}_{f^{[i]}(1)}^{[i]} \\ \mathbf{u}_{f^{[i]}(2)}^{[i]} \\ \vdots \\ \mathbf{u}_{f^{[i]}(W)}^{[i]} \end{bmatrix} \in \mathbb{C}^{MUW}, \quad \mathbf{x}_{i,2} = \begin{bmatrix} \mathbf{u}_{1,S_i+1}^{[i]} \\ \mathbf{u}_{2,S_i+1}^{[i]} \\ \vdots \\ \mathbf{u}_{W_i,S_i+1}^{[i]} \end{bmatrix} \in \mathbb{C}^{L_i MU_i W_i} \quad (30)$$

Here, $f^{[i]}$ for $i \in \Lambda$ is a function on $\{l \in \mathbb{N} : 1 \leq l \leq W\}$ to $\{(j,k) \in \mathbb{N}^2 : 1 \leq j \leq W_i, 1 \leq k \leq S_i\}$, defined by $f^{[i]}(l) = (f_1^{[i]}(l), f_2^{[i]}(l))$ such that

$$\begin{aligned} f_1^{[i]}(l) &= ((l-1) \setminus \prod_{p=1}^i S_p) \prod_{p=1}^{i-1} S_p + 1 + (l-1) \prod_{p=1}^{i-1} S_p, \\ f_2^{[i]}(l) &= ((l-1) \prod_{p=1}^i S_p) \setminus \prod_{p=1}^{i-1} S_p + 1. \end{aligned} \quad (31)$$

The following lemma shows that every element of $\{\mathbf{u}_{j,k}^{[i]}\}_{k=1, \dots, S_i}^{j=1, \dots, W_i}$ appears once in $\mathbf{x}_{i,1}$.

Lemma 5: Let $\mathcal{A} = \{l \in \mathbb{N} : 1 \leq l \leq W\}$, $\mathcal{B} = \{(j,k) \in \mathbb{N}^2 : 1 \leq j \leq W_i, 1 \leq k \leq S_i\}$. Let $f^{[i]}$ for $i \in \Lambda$ be a function on \mathcal{A} to \mathcal{B} defined in (31) and let $g^{[i]}$ for $i \in \Lambda$ be a function on \mathcal{B} to \mathcal{A} defined by

$$g^{[i]}(j,k) = 1 + ((j-1) \setminus \prod_{p=1}^{i-1} S_p) \prod_{p=1}^i S_p + (j-1) \prod_{p=1}^{i-1} S_p + (k-1) \prod_{p=1}^{i-1} S_p. \quad (32)$$

Then $g^{[i]}$ is the inverse function of $f^{[i]}$.

Proof: We refer to Appendix I-D for the proof. ■

Step 4 (Transmit signal vector): Finally, transmit signal vector \mathbf{x}^n is the sum of the transmit signal vector for each user as

$$\mathbf{x}^n = \sum_{i \in \Lambda} \mathbf{x}_i \quad (33)$$

where

$$n = UW + \sum_{i \in \Lambda} L_i U_i W_i \quad (34)$$

because $\mathbf{x}_i \in \mathbb{C}^{MUW+M \sum_{i \in \Lambda} L_i U_i W_i}$. That is, at time $t = 1, \dots, UW + \sum_{i \in \Lambda} L_i U_i W_i$, the transmitter sends from the $((t-1)M+1)$ th to the (tM) th elements of $\sum_{i \in \Lambda} \mathbf{x}_i$ through M antennas.

2) *Mode switching patterns at receivers:* Based on the proposed transmit beamforming stated above, we design the mode switching patterns at receivers, which is fixed regardless of channel realizations. From (29) and (33), we have

$$\mathbf{x}^n = \underbrace{\left[\left(\sum_{i \in \Lambda} \mathbf{x}_{i,1} \right)^T \right]}_{\text{block 1}} \underbrace{\left[\mathbf{x}_{1,2}^T \mathbf{x}_{2,2}^T \cdots \mathbf{x}_{|\Lambda|,2}^T \right]}_{\text{block 2}}^T. \quad (35)$$

Let us denote $\sum_{i \in \Lambda} \mathbf{x}_{i,1}$ and $[\mathbf{x}_{1,2}^T \mathbf{x}_{2,2}^T \cdots \mathbf{x}_{|\Lambda|,2}^T]^T$ in \mathbf{x}^n as block 1 and block 2 respectively. Subsequently, the received signal vector of user i is divided as

$$\mathbf{y}_i^n = \left[\mathbf{y}_{i,0}^T \mathbf{y}_{i,1}^T \cdots \mathbf{y}_{i,|\Lambda|}^T \right]^T$$

where $\mathbf{y}_{i,0}$ and $\mathbf{y}_{i,j}$ for $j = 1, \dots, |\Lambda|$ are the received signal vectors induced by $\sum_{i \in \Lambda} \mathbf{x}_{i,1}$ and $\mathbf{x}_{j,2}$ respectively. Now, we design each user's mode switching pattern during blocks 1 and 2 in the following. For convenience, we simply call a selection pattern (of user i at time t) to denote a specific selection matrix $\mathbf{\Gamma}_i(t)$. We omit rigorous description of selection patterns, nonetheless one can infer them from associated channel matrices induced by selection matrices, i.e., $\mathbf{\Gamma}_i(t)\mathbf{H}_i$.

Mode switching pattern during block 1: From (30), block 1 is divided as

$$\sum_{i \in \Lambda} \mathbf{x}_{i,1} = \begin{bmatrix} \sum_{i \in \Lambda} \mathbf{u}_{f^{[i]}(1)}^{[i]} \\ \vdots \\ \sum_{i \in \Lambda} \mathbf{u}_{f^{[i]}(W)}^{[i]} \end{bmatrix}. \quad (36)$$

Note that the time interval for transmitting block 1 is

$$1 \leq t \leq UW. \quad (37)$$

During block 1, user i exploits a set of S_i selection patterns repeatedly over the entire time interval in (37). The channel matrix associated with the j th selection pattern, denoted by $\mathbf{H}_{i,j} \in \mathbb{C}^{L_i \times M}$, is given by

$$\mathbf{H}_{i,j} = \begin{bmatrix} \mathbf{h}_{i,1+(j-1)L_i} \\ \mathbf{h}_{i,2+(j-1)L_i} \\ \vdots \\ \mathbf{h}_{i,jL_i} \end{bmatrix} \quad j = 1, \dots, S_i$$

where $\mathbf{h}_{k,l} \in \mathbb{C}^{1 \times M}$ is the l th row vector of \mathbf{H}_k for $k \in \Lambda$ and $l = 1, \dots, N_k$. For this case, at each time instant, each user chooses the selection pattern of which index is same as that of the currently transmitted sub-unit of his transmit signal vector. One can see from (36) that the sub-unit of user i transmitted at time $t = 1, \dots, UW$ is given by $\mathbf{u}_{f^{[i]}(l(t))}^{[i]}$ where $l(t) = 1 + (t-1) \setminus U$. Then, at time $t = 1, \dots, UW$, the user i receives the transmit signal vector using the $f_2^{[i]}(l(t))$ th selection pattern, associated with $\mathbf{H}_{i,f_2^{[i]}(l(t))}$. As a result, the received signal vector of user i during $t = 1, \dots, UW$ is given by

$$\mathbf{y}_{i,0} = \begin{bmatrix} (\mathbf{I}_U \otimes \mathbf{H}_{i,f_2^{[i]}(1)}) \sum_{j \in \Lambda} \mathbf{u}_{f^{[j]}(1)}^{[j]} \\ (\mathbf{I}_U \otimes \mathbf{H}_{i,f_2^{[i]}(2)}) \sum_{j \in \Lambda} \mathbf{u}_{f^{[j]}(2)}^{[j]} \\ \vdots \\ (\mathbf{I}_U \otimes \mathbf{H}_{i,f_2^{[i]}(W)}) \sum_{j \in \Lambda} \mathbf{u}_{f^{[j]}(W)}^{[j]} \end{bmatrix}. \quad (38)$$

Mode switching pattern during block 2: We divide block 2 into desired signal and interference signal parts of user i , in which desired signal part is $\mathbf{x}_{i,2}$ and interference signal part is the rest of block 2 except $\mathbf{x}_{i,2}$. Note that the time interval for transmitting $\mathbf{x}_{i,2}$ is

$$a_i + 1 \leq t \leq a_{i+1} \quad (39)$$

where $a_i = UW + \sum_{p=1}^{i-1} L_p U_p W_p$. Let us define $\mathbf{H}_{i,S_i+1} \in \mathbb{C}^{(T_i-L_i S_i) \times M}$ and $\mathbf{H}_{i,j,k} \in \mathbb{C}^{(L_i-T_i|L_i) \times M}$ as

$$\mathbf{H}_{i,S_i+1} = \begin{bmatrix} \mathbf{h}_{i,L_i S_i+1} \\ \mathbf{h}_{i,L_i S_i+2} \\ \vdots \\ \mathbf{h}_{i,T_i} \end{bmatrix} \quad (40)$$

and

$$\mathbf{H}_{i,j,k} = \begin{bmatrix} \mathbf{h}_{i,(j-1)L_i+(k-1)|L_i+1} \\ \mathbf{h}_{i,(j-1)L_i+k|L_i+1} \\ \vdots \\ \mathbf{h}_{i,(j-1)L_i+(k-2+L_i-(T_i|L_i))|L_i+1} \end{bmatrix} \quad (41)$$

for $j = 1, \dots, S_i$ and $k = 1, \dots, L_i$.

First consider the desired signal part of user i during block 2. For the transmission of $\mathbf{x}_{i,2}$, the mode switching pattern of user i differs according to the value of $T_i|L_i$. If $T_i|L_i = 0$, then user i exploits a single selection pattern repeatedly over the entire time interval in (39). The channel matrix associated with the selection pattern is given by (40), where $\mathbf{H}_{i,S_i+1} \in \mathbb{C}^{L_i \times M}$ in this case. If $T_i|L_i \neq 0$, then user i exploits a set of $L_i S_i$ selection patterns repeatedly over the entire time interval in (39), i.e., the number of $L_i U_i W_i / (L_i S_i) = \prod_{p \in \Lambda \setminus \{i\}} S_p (T_p - L_p)$ repetitions. The channel matrix associated with the j th selection pattern, denoted by $\mathbf{H}_{i,S_i+1,j} \in \mathbb{C}^{L_i \times M}$ for $j = 1, \dots, L_i S_i$ which can be constructed from (40) and (41), is given by

$$\mathbf{H}_{i,S_i+1,j} = \begin{bmatrix} \mathbf{H}_{i,S_i+1} \\ \mathbf{H}_{i,(j-1) \setminus L_i+1, (j-1)|L_i+1} \end{bmatrix}. \quad (42)$$

Then, at time $t = a_i + 1, \dots, a_{i+1}$, user i receives the transmit signal vector using the selection pattern corresponding to $\mathbf{H}_{i,S_i+1,l_i(t)}$ where $l_i(t) = 1 + (t - a_i - 1)|(L_i S_i)$. As a result, the received signal vector of user i during $t = a_i + 1, \dots, a_{i+1}$ is given by

$$\mathbf{y}_{i,i} = \begin{bmatrix} (\mathbf{I}_{U_i/S_i} \otimes \mathbf{H}'_{i,S_i+1}) \mathbf{u}_{1,S_i+1}^{[i]} \\ (\mathbf{I}_{U_i/S_i} \otimes \mathbf{H}'_{i,S_i+1}) \mathbf{u}_{2,S_i+1}^{[i]} \\ \vdots \\ (\mathbf{I}_{U_i/S_i} \otimes \mathbf{H}'_{i,S_i+1}) \mathbf{u}_{W_i,S_i+1}^{[i]} \end{bmatrix} \quad (43)$$

where

$$\mathbf{H}'_{i,S_i+1} = \begin{cases} \mathbf{I}_{L_i S_i} \otimes \mathbf{H}_{i,S_i+1} & \text{if } T_i|L_i = 0, \\ \text{diag}(\mathbf{H}_{i,S_i+1,1}, \mathbf{H}_{i,S_i+1,2}, \dots, \mathbf{H}_{i,S_i+1,L_i S_i}) & \text{otherwise.} \end{cases}$$

Now consider the interference signal part of user i during block 2. From (39), the time interval for transmitting $\mathbf{x}_{i',2}$, where $i' \in \Lambda \setminus \{i\}$, is given by

$$a_{i'} + 1 \leq t \leq a_{i'+1}. \quad (44)$$

For the transmission of $\mathbf{x}_{i',2}$, user i exploits the same set of S_i selection patterns used for block 1 again over the entire time interval in (44). For this case, at each time instant, each user chooses the selection pattern of which index is the same as that used to receive the first sub-unit of the alignment unit to which the currently transmitted sub-unit belongs. Specifically, from (30), the sub-unit of user i' transmitted at time $t = a_{i'} + 1, \dots, a_{i'+1}$ is given by $\mathbf{u}_{l(t),S_{i'}+1}^{[i']}$ where $l(t) = 1 + (t - 1 - a_{i'}) \setminus (L_{i'} U_{i'})$. Since $\mathbf{u}_{l(t),1}^{[i']} = \mathbf{u}_{f^{[i']}(g^{[i']}(l(t),1))}^{[i']}$ from Lemma 5, $\mathbf{u}_{l(t),1}^{[i']}$ is a summand of $\sum_{j \in \Lambda} \mathbf{u}_{f^{[j]}(g^{[j]}(l(t),1))}^{[j]}$, which means from (36) that $\mathbf{u}_{l(t),1}^{[i']}$ is transmitted simultaneously with $\mathbf{u}_{f^{[i]}(g^{[i]}(l(t),1))}^{[i]}$ in block 1. That is, user i exploits the $f_2^{[i]}(g^{[i]}(l(t),1))$ th selection pattern to receive $\mathbf{u}_{f^{[i]}(g^{[i]}(l(t),1))}^{[i]}$ so that, at time $t = a_{i'} + 1, \dots, a_{i'+1}$, user i receives the transmit signal vector using the $f_2^{[i]}(g^{[i]}(l(t),1))$ th selection pattern, associated with $\mathbf{H}_{i,f_2^{[i]}(g^{[i]}(l(t),1))}$. As a result, the received signal vector of user i induced by $\mathbf{x}_{i',2}$ is

$$\mathbf{y}_{i,i'} = \begin{bmatrix} (\mathbf{I}_{L_i U_i} \otimes \mathbf{H}_{i,f_2^{[i]}(g^{[i]}(1,1))}) \mathbf{u}_{1,S_{i'}+1}^{[i']} \\ (\mathbf{I}_{L_i U_i} \otimes \mathbf{H}_{i,f_2^{[i]}(g^{[i]}(2,1))}) \mathbf{u}_{2,S_{i'}+1}^{[i']} \\ \vdots \\ (\mathbf{I}_{L_i U_i} \otimes \mathbf{H}_{i,f_2^{[i]}(g^{[i]}(W_{i'},1))}) \mathbf{u}_{W_{i'},S_{i'}+1}^{[i']} \end{bmatrix}. \quad (45)$$

3) *Interference cancellation at receivers:* In the following, we show that user i can eliminate all interference signals contained in $\mathbf{y}_{i,0}$ in (38) using the received interference signal parts during block 2, i.e., $\mathbf{y}_{i,i'}$ in (45) for all $i' \in \Lambda \setminus \{i\}$. First, we introduce the following lemma, which plays a key role to verify such interference cancellation.

Lemma 6: Let $\mathcal{A} = \{l \in \mathbb{N} : 1 \leq l \leq W\}$, $\mathcal{B} = \{(j, k) \in \mathbb{N}^2 : 1 \leq j \leq W_i, 1 \leq k \leq S_i\}$. Let $f^{[i]}$ for $i \in \Lambda$ be a function on \mathcal{A} to \mathcal{B} defined in (31) and let $g^{[i]}$ for $i \in \Lambda$ be a function on \mathcal{B} to \mathcal{A} defined in (32). For $i, i' \in \Lambda$ where $i \neq i'$ and $(j, k), (j, k') \in \mathcal{B}$, the following relation holds:

$$f_2^{[i]}(g^{[i']}(j, k)) = f_2^{[i]}(g^{[i']}(j, k'))$$

Proof: We refer to Appendix I-E for the proof. ■

Consider $(\mathbf{I}_U \otimes \mathbf{H}_{i, f_2^{[i]}(l)}) \mathbf{u}_{f^{[i']}(l)}^{[i']}$ for $i' \in \Lambda \setminus \{i\}$ and $1 \leq l \leq W$, which is an interference vector in $\mathbf{y}_{i,0}$. We have

$$\begin{aligned} (\mathbf{I}_U \otimes \mathbf{H}_{i, f_2^{[i]}(l)}) \mathbf{u}_{f^{[i']}(l)}^{[i']} &= (\mathbf{I}_U \otimes \mathbf{H}_{i, f_2^{[i]}(g^{[i']}(f^{[i']}(l), 1))}) \mathbf{u}_{f^{[i']}(l)}^{[i']} \\ &= (\mathbf{I}_U \otimes \mathbf{H}_{i, f_2^{[i]}(g^{[i']}(f_1^{[i']}(l), 1))}) \mathbf{u}_{f^{[i']}(l)}^{[i']} \end{aligned} \quad (46)$$

where the first and second equalities follow from Lemma 5 and Lemma 6 respectively. If $T_{i'}|L_{i'} = 0$, then, substituting (28) into (46), we have

$$(\mathbf{I}_U \otimes \mathbf{H}_{i, f_2^{[i]}(l)}) \mathbf{u}_{f^{[i']}(l)}^{[i']} = (\mathbf{I}_{L_i U_i} \otimes \mathbf{H}_{i, f_2^{[i]}(g^{[i']}(f_1^{[i']}(l), 1))}) \mathbf{u}_{f_1^{[i']}(l), S_{i'}+1}^{[i']} \quad (47)$$

where $U = L_i U_i$ for this case. If $T_{i'}|L_{i'} \neq 0$, then, substituting (28) into (46), we have

$$\begin{aligned} (\mathbf{I}_U \otimes \mathbf{H}_{i, f_2^{[i]}(l)}) \mathbf{u}_{f^{[i']}(l)}^{[i']} &= (\mathbf{I}_U \otimes \mathbf{H}_{i, f_2^{[i]}(g^{[i']}(f_1^{[i']}(l), 1))}) (\mathbf{I}_{U_i/S_i} \otimes \mathbf{Q}_{f_2^{[i']}(l)}^{[i']} \otimes \mathbf{I}_M) \mathbf{u}_{f_1^{[i']}(l), S_{i'}+1}^{[i']} \\ &= (\mathbf{I}_{U_i/S_i} \otimes \mathbf{Q}_{f_2^{[i']}(l)}^{[i']} \otimes \mathbf{I}_{L_i}) (\mathbf{I}_{L_i U_i} \otimes \mathbf{H}_{i, f_2^{[i]}(g^{[i']}(f_1^{[i']}(l), 1))}) \mathbf{u}_{f_1^{[i']}(l), S_{i'}+1}^{[i']}. \end{aligned} \quad (48)$$

Here (48) comes from the following relation:

$$\begin{aligned} &(\mathbf{I}_U \otimes \mathbf{H}_{i, f_2^{[i]}(g^{[i']}(f_1^{[i']}(l), 1))}) (\mathbf{I}_{U_i/S_i} \otimes \mathbf{Q}_{f_2^{[i']}(l)}^{[i']} \otimes \mathbf{I}_M) \\ &= (\mathbf{I}_{U_i/S_i} \otimes \mathbf{Q}_{f_2^{[i']}(l)}^{[i']} \otimes \mathbf{H}_{i, f_2^{[i]}(g^{[i']}(f_1^{[i']}(l), 1))}) \\ &= ((\mathbf{I}_{U_i/S_i} \otimes \mathbf{Q}_{f_2^{[i']}(l)}^{[i']}) \mathbf{I}_{L_i U_i}) \otimes (\mathbf{I}_{L_i} \mathbf{H}_{i, f_2^{[i]}(g^{[i']}(f_1^{[i']}(l), 1))}) \\ &= (\mathbf{I}_{U_i/S_i} \otimes \mathbf{Q}_{f_2^{[i']}(l)}^{[i']} \otimes \mathbf{I}_{L_i}) (\mathbf{I}_{L_i U_i} \otimes \mathbf{H}_{i, f_2^{[i]}(g^{[i']}(f_1^{[i']}(l), 1))}) \end{aligned} \quad (49)$$

where (49) follows from the mixed-product property that for matrices \mathbf{A} , \mathbf{B} , \mathbf{C} , and \mathbf{D} in which the matrix products \mathbf{AC} and \mathbf{BD} can be defined, $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$, see [42, Lemma 4.2.10].

From (45), user i is able to extract the following vector from $\mathbf{y}_{i,i'}$:

$$(\mathbf{I}_{L_i U_i} \otimes \mathbf{H}_{i, f_2^{[i]}(g^{[i']}(f_1^{[i']}(l), 1))}) \mathbf{u}_{f_1^{[i']}(l), S_{i'}+1}^{[i']} \quad (50)$$

Then, user i constructs $(\mathbf{I}_U \otimes \mathbf{H}_{i, f_2^{[i]}(l)}) \mathbf{u}_{f^{[i']}(l)}^{[i']}$ using (50) from the relations in (47) and (48) and subtracts it from $\mathbf{y}_{i,0}$. In the same manner, user i can remove all interference vectors in $\mathbf{y}_{i,0}$.

4) *Achievable LDoF:* Let us denote the remaining signal vector after cancelling all interference vectors in $\mathbf{y}_{i,0}$ as $\mathbf{y}'_{i,0}$. Combining $\mathbf{y}'_{i,0}$ with $\mathbf{y}_{i,i}$, user i has

$$\begin{bmatrix} \mathbf{y}'_{i,0} \\ \mathbf{y}_{i,i} \end{bmatrix} = \begin{bmatrix} (\mathbf{I}_U \otimes \mathbf{H}_{i, f_2^{[i]}(1)}) \mathbf{u}_{f^{[i]}(1)}^{[i]} \\ \vdots \\ (\mathbf{I}_U \otimes \mathbf{H}_{i, f_2^{[i]}(\prod_{p \in \Lambda} S_p)}) \mathbf{u}_{f^{[i]}(\prod_{p \in \Lambda} S_p)}^{[i]} \\ (\mathbf{I}_{U_i/S_i} \otimes \mathbf{H}'_{i, S_i+1}) \mathbf{u}_{1, S_i+1}^{[i]} \\ \vdots \\ (\mathbf{I}_{U_i/S_i} \otimes \mathbf{H}'_{i, S_i+1}) \mathbf{u}_{W_i, S_i+1}^{[i]} \end{bmatrix}. \quad (51)$$

By classifying (51) by alignment units, (52) is decomposed into W_i segments as follows:

$$\begin{bmatrix} (\mathbf{I}_U \otimes \mathbf{H}_{i,1}) \mathbf{u}_{j,1}^{[i]} \\ \vdots \\ (\mathbf{I}_U \otimes \mathbf{H}_{i, S_i}) \mathbf{u}_{j, S_i}^{[i]} \\ (\mathbf{I}_{U_i/S_i} \otimes \mathbf{H}'_{i, S_i+1}) \mathbf{u}_{j, S_i+1}^{[i]} \end{bmatrix} \quad j = 1, \dots, W_i. \quad (52)$$

By classifying (52) by alignment blocks, (51) is further decomposed into $U_i W_i$ segments as follows:

$$\begin{bmatrix} (\mathbf{I}_{N_i} \otimes \mathbf{H}_{i,(j-1)|S_i+1}) \mathbf{v}_{j,1}^{[i]} \\ (\mathbf{I}_{N_i} \otimes \mathbf{H}_{i,(j-2)|S_i+1}) \mathbf{v}_{j,2}^{[i]} \\ \vdots \\ (\mathbf{I}_{N_i} \otimes \mathbf{H}_{i,(j-S_i)|S_i+1}) \mathbf{v}_{j,S_i}^{[i]} \\ \mathbf{H}'_{i,S_i+1,j} \mathbf{v}_{j,S_i+1}^{[i]} \end{bmatrix} \quad j = 1, \dots, U_i W_i \quad (53)$$

where

$$\mathbf{H}'_{i,S_i+1,j} = \begin{cases} \mathbf{I}_{L_i} \otimes \mathbf{H}_{i,S_i+1} & \text{if } T_i|L_i = 0, \\ \text{diag}(\mathbf{H}_{i,S_i+1,((j-1)|S_i)L_i+1}, \mathbf{H}_{i,S_i+1,((j-1)|S_i)L_i+2}, \dots, \mathbf{H}_{i,S_i+1,((j-1)|S_i+1)L_i}) & \text{otherwise.} \end{cases}$$

If $T_i|L_i = 0$, then, substituting (25) into (53) and switching the rows, we have

$$(\mathbf{I}_{L_i} \otimes \mathbf{H}_i)(\mathbf{I}_{L_i} \otimes \mathbf{I}_{M,T_i}) \mathbf{s}_j^{[i]} = (\mathbf{I}_{L_i} \otimes [\mathbf{H}_i]_{T_i}) \mathbf{s}_j^{[i]} \quad (54)$$

where $[\mathbf{H}_i]_{T_i}$ is the leading principal minor of \mathbf{H}_i of order T_i . Since $(\mathbf{I}_{L_i} \otimes [\mathbf{H}_i]_{T_i})$ is non-singular almost surely, user i can obtain $\mathbf{s}_j^{[i]}$ from (54) almost surely.

If $T_i|L_i \neq 0$, then, substituting (25) into (53), we have

$$\underbrace{\begin{bmatrix} \Phi \otimes \mathbf{H}_{i,(j-1)|S_i+1} \\ \mathbf{I}_{N_i} \otimes \mathbf{H}_{i,(j-2)|S_i+1} \\ \vdots \\ \mathbf{I}_{N_i} \otimes \mathbf{H}_{i,(j-S_i)|S_i+1} \\ \mathbf{H}'_{i,S_i+1,j} \end{bmatrix}}_{\mathbf{B}_j^{[i]}} (\mathbf{I}_{L_i} \otimes \mathbf{I}_{M,T_i}) \mathbf{s}_j^{[i]}. \quad (55)$$

It can be easily verified that $\mathbf{B}_j^{[i]} = \mathbf{C}_j^{[i]} (\mathbf{I}_{L_i} \otimes [\mathbf{H}_i]_{T_i})$ where $\mathbf{C}_j^{[i]} \in \mathbb{C}^{L_i T_i \times L_i T_i}$ for $j = 1, \dots, U_i W_i$ is a non-singular matrix almost surely so that user i can obtain $\mathbf{s}_j^{[i]}$ from (55) almost surely. Consequently, user i is able to $\mathbf{s}_1^{[i]}$ through $\mathbf{s}_{U_i W_i}^{[i]}$ almost surely.

Since total $L_i T_i U_i W_i$ independent information symbols are delivered almost surely to user i during the period given in (34), the achievable LDoF of user i is given by

$$d_i = \frac{\frac{T_i L_i}{T_i - L_i} \prod_{p \in \Lambda} S_p (T_p - L_p)}{\prod_{p \in \Lambda} S_p (T_p - L_p) + \sum_{p \in \Lambda} \frac{L_p}{T_p - L_p} \prod_{q \in \Lambda} S_q (T_q - L_q)} \quad (56)$$

$$= \frac{\frac{T_i L_i}{T_i - L_i}}{1 + \sum_{p \in \Lambda} \frac{L_p}{T_p - L_p}}. \quad (57)$$

Therefore,

$$d_\Sigma = \sum_{i=1}^K d_i = \frac{\sum_{i \in \Lambda} \frac{T_i L_i}{T_i - L_i}}{1 + \sum_{i \in \Lambda} \frac{L_i}{T_i - L_i}} \quad (58)$$

is achievable for Case 3-2. In conclusion, from (21), (22), and (58), $d_\Sigma = \min(M, \max(L_{\max}, \eta))$ is achievable, which completes the achievability proof of Theorem 1.

B. Achievability of Theorem 2

For notational convenience, we define the inequality in (7) as I_{1k} and the inequality $d_k \geq 0$ as I_{2k} in the rest of this subsection. Since the LDoF region \mathcal{D} in Theorem 2 is a polyhedron, it suffices to show that all vertices of \mathcal{D} are achievable. Hence, our achievability proof begins with characterizing vertices of \mathcal{D} . The following lemma establishes a condition for a K -tuple in \mathbb{R}^K to be a vertex of \mathcal{D} .

Lemma 7: Consider the LDoF region \mathcal{D} in Theorem 2. If $\mathbf{d} \in \mathbb{R}^K$ is a vertex of \mathcal{D} , then only K inequalities among $\{I_{1k}, I_{2k}\}_{k \in \mathcal{K}}$ should be active ³ at \mathbf{d} while I_{1k} and I_{2k} for $k \in \mathcal{K}$ cannot be active simultaneously at \mathbf{d} .

³An inequality $f(\mathbf{x}) \leq 0$ is said to be active at \mathbf{x}^* if $f(\mathbf{x}^*) = 0$ [43, Definition 20.1].

Proof: Assume that I_{11} and I_{21} are active at $\mathbf{d} = (d_1, \dots, d_K) \in \mathcal{D}$. Combining I_{11} and I_{21} , it is followed by

$$\sum_{i=1}^K \frac{d_i}{T_i} = 1.$$

Hence, \mathbf{d} must not be a zero vector and we can find an index $k^* \in \mathcal{K}$ such that I_{2k^*} is not active at \mathbf{d} , i.e., $d_{k^*} > 0$. Then, we have

$$\frac{d_{k^*}}{L_{k^*}} + \sum_{i=1, i \neq k^*}^K \frac{d_i}{T_i} = 1 + \left(\frac{1}{L_{k^*}} - \frac{1}{T_{k^*}} \right) d_{k^*} > 1,$$

which means that \mathbf{d} does not satisfy I_{1k^*} so that $\mathbf{d} \notin \mathcal{D}$. Contradicting the assumption, I_{1k} and I_{2k} for $k \in \mathcal{K}$ cannot be active simultaneously at \mathbf{d} and, as a result, for $\mathbf{d} \in \mathcal{D}$, at most K inequalities among $\{I_{1k}, I_{2k}\}_{k \in \mathcal{K}}$ are active at \mathbf{d} . Furthermore, if \mathbf{d} is a vertex of \mathcal{D} , then at least K inequalities among $\{I_{1k}, I_{2k}\}_{k \in \mathcal{K}}$ should be active on \mathbf{d} because a vertex of a polyhedron is expressed as an intersection of at least K faces of the polyhedron. Therefore, only K inequalities among $\{I_{1k}, I_{2k}\}_{k \in \mathcal{K}}$ should be active at \mathbf{d} , which completes the proof of Lemma 7. ■

Consider a K -tuple $\mathbf{d} = (d_1, \dots, d_K) \in \mathbb{R}^K$ such that K inequalities among $\{I_{1k}, I_{2k}\}_{k \in \mathcal{K}}$ are active at \mathbf{d} while I_{1k} and I_{2k} for $k \in \mathcal{K}$ are not active simultaneously at \mathbf{d} and let $\Lambda_i = \{k \in \mathcal{K} : I_{ik} \text{ is active at } \mathbf{d}\}$ for $i = 1, 2$. Note that, from Lemma 7, $\{\Lambda_1, \Lambda_2\}$ is a partition of \mathcal{K} . Assume $\Lambda_1 = \{1, \dots, J\}$ and $\Lambda_2 = \{J+1, \dots, K\}$ without loss of generality. Composing the K inequalities active at \mathbf{d} , we have

$$\begin{aligned} \mathbf{A}_1 [d_1 \dots d_J]^T + \left[\frac{1}{T_{J+1}} \mathbf{1}_{J \times 1} \dots \frac{1}{T_K} \mathbf{1}_{J \times 1} \right] [d_{J+1} \dots d_K]^T &= \mathbf{1}_{J \times 1} \\ [d_{J+1} \dots d_K]^T &= \mathbf{0}_{(K-J) \times 1} \end{aligned} \quad (59)$$

where

$$\mathbf{A}_1 = \begin{bmatrix} \frac{1}{N_1} & \frac{1}{T_2} & \frac{1}{T_3} & \dots & \frac{1}{T_J} \\ \frac{1}{T_1} & \frac{1}{N_2} & \frac{1}{T_3} & \dots & \frac{1}{T_J} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{T_1} & \frac{1}{T_2} & \frac{1}{T_3} & \dots & \frac{1}{N_J} \end{bmatrix}.$$

Since \mathbf{A}_1 is non-singular from Lemma 10 in Appendix I-C, from (59), we have

$$\mathbf{d} = [(\mathbf{A}_1^{-1} \mathbf{1}_{J \times 1})^T \mathbf{0}_{(K-J) \times 1}^T]^T$$

From (69), $\mathbf{A}_1^{-1} \mathbf{1}_{J \times 1}$ can be calculated easily, which results that

$$d_i = \begin{cases} \frac{\frac{T_i N_i}{T_i - N_i}}{1 + \sum_{k \in \Lambda_1} \frac{N_k}{T_k - N_k}} & \text{if } i \in \Lambda_1, \\ 0 & \text{otherwise.} \end{cases} \quad (60)$$

Since (60) is achievable by supporting users in Λ_1 with the scheme proposed in Case 3-2 of Section V-A, all the vertices of \mathcal{D} are achievable, which completes the achievability proof of Theorem 2.

C. Achievability of Corollary 1

If $L_{\max} \geq \eta_{IC}$, then it is achievable by supporting a user with the maximum number of RF-chains. For the rest of this section we prove that $d_{\Sigma, IC} = \eta_{IC}$ assuming $L_{\max} < \eta_{IC}$. It can be shown that (8) is achievable by modifying the achievable scheme derived for Case 3-2 in Section V-A as follows. Transmitter $k \in \{k \in \mathcal{K} : N_k > L_{\max}\}$ constructs transmit signal vector for user k in accordance with Step 1 through Step 3 in Section V-A1 by setting $M = M_k$ and $T_k = N_k$ and sends it as in Step 4 in Section V-A1. Note that since the beamforming strategy in Section V-A1 does not require cooperation among transmitters, it can be directly applied to interference channels. Then each user receives and decodes the transmitted signal in accordance with the procedure in Section V-A2. Note that it can be easily shown that (8) is achievable, which completes the achievability proof of Corollary 1.

VI. CONCLUDING REMARKS

In this paper, the DoF of the K -user MIMO BC with reconfigurable antennas under no CSIT has been studied. We completely characterized the sum LDoF of the K -user MIMO BC with reconfigurable antennas under general antenna configurations and further characterized the LDoF region for a class of antenna configurations. Our results provide a comprehensive understanding of reconfigurable antennas on the LDoF of the K -user MIMO BC, which demonstrates that reconfigurable antennas are beneficial for a broad class of antenna configurations. In particular, the DoF gain from reconfigurable antennas enlarges as both the number of transmit antennas and the number of preset modes increase. Our analysis has been further extended to characterizing the sum LDoF of the K -user MIMO IC with reconfigurable antennas for a class of antenna configuration, which leads to similar argument for the K -user MIMO BC with reconfigurable antennas.

APPENDIX I

PROOF OF TECHNICAL LEMMAS

A. Proof of Lemma 2

Let us define $\mathbf{G}_1^c \in \mathbb{C}^{(N_1-L_1) \times M}$ as the submatrix consisting of the $(L_1 + 1)$ th through the N_1 th rows of \mathbf{H}_1 . We will prove Lemma 2 for given realization of \mathbf{G}_1 and $[\mathbf{V}_2^n \cdots \mathbf{V}_K^n]$. That is, ‘almost sure’ in the rest of the proof is due to the randomness of \mathbf{G}_1^c . Since Lemma 2 trivially holds if $\mathbf{I}_1^n \mathbf{H}_1^n = \mathbf{G}_1^n$ or $\mathbf{G}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n] = \mathbf{0}$, we assume $\mathbf{I}_1^n \mathbf{H}_1^n \neq \mathbf{G}_1^n$ and $\mathbf{G}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n] \neq \mathbf{0}$ from now on.

For convenience, denote $\text{rank}(\mathbf{G}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n]) = r \geq 1$. Define the set of column indices consisting of r linearly independent columns of $\mathbf{G}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n]$ as \mathcal{I} . Then construct $\mathbf{A}_1 \in \mathbb{C}^{nL_1 \times r}$ by choosing r column vectors of $\mathbf{G}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n]$ whose indexes are in \mathcal{I} and construct $\mathbf{A}_2 \in \mathbb{C}^{nL_1 \times r}$ by choosing r column vectors of $\mathbf{I}_1^n \mathbf{H}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n]$ whose indexes are in \mathcal{I} . Clearly, \mathbf{A}_1 is of full-rank. There exist $\binom{nL_1}{r}$ choices of constructing $r \times r$ submatrices from \mathbf{A}_1 (or \mathbf{A}_2) and the determinant of each of these submatrices can be expressed as a polynomial with respect to the entries of \mathbf{G}_1^c .

Suppose that all $r \times r$ submatrix of \mathbf{A}_2 of which determinant is a zero polynomial with respect to the entries of \mathbf{G}_1^c , i.e., a constant polynomial whose coefficients are all equal to zero. In this case, \mathbf{A}_2 is not of full-rank regardless of the entries of \mathbf{G}_1^c . Hence, any matrix constructed from \mathbf{A}_2 by substituting the entries of \mathbf{G}_1^c with arbitrary values is not of full-rank either. Let us now define \mathbf{A}_3 constructed from \mathbf{A}_2 by substituting \mathbf{G}_1^c with $\mathbf{P}\mathbf{G}_1^c$, where $\mathbf{P} \in \mathbb{C}^{(N_1-L_1) \times L_1}$, which is not of full-rank from the above argument. Then, we can represent \mathbf{A}_3 as $\mathbf{A}_3 = \mathbf{Q}\mathbf{A}_1$ for some matrix $\mathbf{Q} \in \mathbb{C}^{nL_1 \times nL_1}$. If all square submatrices of \mathbf{P} are non-singular, \mathbf{Q} becomes invertible so that \mathbf{A}_1 and \mathbf{A}_3 have the same rank. We can easily find such \mathbf{P} , for example, Vandermonde matrix or Cauchy matrix [44]. However, from the fact that \mathbf{A}_3 is not of full-rank, the result that \mathbf{A}_3 and \mathbf{A}_1 have the same rank contradicts the assumption that \mathbf{A}_1 is of full-rank. Consequently, there exists at least one $r \times r$ submatrix of \mathbf{A}_2 of which determinant is not a zero polynomial with respect to the entries of \mathbf{G}_1^c .

Then now consider some $r \times r$ submatrix of \mathbf{A}_2 of which determinant is not a zero polynomial with respect to the entries of \mathbf{G}_1^c . Since the entries of \mathbf{G}_1^c are i.i.d drawn from a continuous distribution, for given \mathbf{G}_1 and $[\mathbf{V}_2^n \cdots \mathbf{V}_K^n]$, the determinant of the considered submatrix of \mathbf{A}_2 is non-zero almost surely. Hence, \mathbf{A}_2 is of full-rank almost surely. Since \mathbf{A}_2 is a submatrix of $\mathbf{I}_1^n \mathbf{H}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n]$, the rank of $\mathbf{I}_1^n \mathbf{H}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n]$ is greater than or equal to that of $\mathbf{G}_1^n [\mathbf{V}_2^n \cdots \mathbf{V}_K^n]$ almost surely, which complete the proof of Lemma 2.

B. Proof of Lemma 3

In order to prove Lemma 3, we need the following lemmas. The first lemma comes from the submodularity property for rank of matrices [40], [45].

Lemma 8 (Lovász): For matrices \mathbf{A} , \mathbf{B} , and \mathbf{C} with the same number of rows,

$$\text{rank}[\mathbf{AC}] - \text{rank}[\mathbf{C}] \geq \text{rank}[\mathbf{ABC}] - \text{rank}[\mathbf{BC}].$$

Proof: We refer to [45] for the proof. ■

The second lemma is one of the key properties for multiantenna systems without CSIT, which means that there is no spatial preference in the received signal space without CSIT.

Lemma 9: Let $\mathbf{A}_{i,m}$ be the submatrix consisting of arbitrary m row vectors of \mathbf{G}_i and $\mathbf{B}_{j,m}$ be the submatrix consisting of arbitrary m row vectors of \mathbf{G}_j . Then, for all $i, j, k \in \mathcal{K}$, the following property holds almost surely:

$$\text{rank}(\mathbf{A}_{i,m}^n [\mathbf{V}_k^n \cdots \mathbf{V}_K^n]) = \text{rank}(\mathbf{B}_{j,m}^n [\mathbf{V}_k^n \cdots \mathbf{V}_K^n])$$

where $\mathbf{A}_{i,m}^n = \mathbf{I}_n \otimes \mathbf{A}_{i,m}$ and $\mathbf{B}_{j,m}^n = \mathbf{I}_n \otimes \mathbf{B}_{j,m}$.⁴

Proof: For $i = j$, it can be straightforwardly derived from the proof in Lemma 2. Hence, we assume $i \neq j$ in the rest of the proof. We first prove that

$$\text{rank}(\mathbf{A}_{i,m}^n [\mathbf{V}_k^n \cdots \mathbf{V}_K^n]) \stackrel{a.s.}{\leq} \text{rank}(\mathbf{B}_{j,m}^n [\mathbf{V}_k^n \cdots \mathbf{V}_K^n]) \quad (61)$$

⁴The maximum value of m depends on i and j , see the definition of $\{\mathbf{G}_l\}_{l \in \mathcal{K}}$.

for given realizations of $\mathbf{A}_{i,m}$ and $[\mathbf{V}_k^n \cdots \mathbf{V}_K^n]$. Note that, for given $\mathbf{A}_{i,m}$ and $[\mathbf{V}_k^n \cdots \mathbf{V}_K^n]$, $\mathbf{A}_{i,m}[\mathbf{V}_k^n \cdots \mathbf{V}_K^n]$ is deterministic but $\mathbf{B}_{j,m}[\mathbf{V}_k^n \cdots \mathbf{V}_K^n]$ is random induced by $\mathbf{B}_{j,m}$.

If $\mathbf{A}_{i,m}[\mathbf{V}_k^n \cdots \mathbf{V}_K^n] = \mathbf{0}$, then (61) trivially holds. Then now consider the case where $\mathbf{A}_{i,m}[\mathbf{V}_k^n \cdots \mathbf{V}_K^n] \neq \mathbf{0}$. For convenience, denote $\text{rank}(\mathbf{A}_{i,m}[\mathbf{V}_k^n \cdots \mathbf{V}_K^n]) = r \geq 1$. Define the set of column indices consisting of r linearly independent columns of $\mathbf{A}_{i,m}[\mathbf{V}_k^n \cdots \mathbf{V}_K^n]$ as \mathcal{I} . Then construct $\mathbf{C}_1 \in \mathbb{C}^{nm \times r}$ by choosing r column vectors of $\mathbf{A}_{i,m}[\mathbf{V}_k^n \cdots \mathbf{V}_K^n]$ whose indexes are in \mathcal{I} and $\mathbf{C}_2 \in \mathbb{C}^{nm \times r}$ by choosing r column vectors of $\mathbf{B}_{j,m}[\mathbf{V}_k^n \cdots \mathbf{V}_K^n]$ whose indexes are in \mathcal{I} . Clearly, \mathbf{C}_1 is of full-rank.

There exist $\binom{nm}{r}$ $r \times r$ choices of constructing $r \times r$ submatrices from \mathbf{C}_2 and the determinant of each of these submatrices can be expressed as a polynomial with respect to the entries of $\mathbf{B}_{j,m}$. Then from the same argument in the proof of Lemma 2, we can show that there exists at least one $r \times r$ submatrix of \mathbf{C}_2 of which determinant is not a zero polynomial with respect to the entries of $\mathbf{B}_{j,m}$. Now consider one of such $r \times r$ submatrices of \mathbf{C}_2 . Since the entries of $\mathbf{B}_{j,m}$ are i.i.d drawn from a continuous distribution, for given $\mathbf{A}_{i,m}$ and $[\mathbf{V}_k^n \cdots \mathbf{V}_K^n]$, its determinant is non-zero almost surely. Hence, \mathbf{C}_2 is of full-rank almost surely and, as a result, (61) holds. Similarly, we can also prove $\text{rank}(\mathbf{A}_{i,m}[\mathbf{V}_k^n \cdots \mathbf{V}_K^n]) \stackrel{a.s.}{\geq} \text{rank}(\mathbf{B}_{j,m}[\mathbf{V}_k^n \cdots \mathbf{V}_K^n])$. In conclusion, Lemma 9 holds. \blacksquare

We are now ready to prove Lemma 3. Let us define $\mathbf{Z}^{i,j} = (\mathbf{G}_i^n[\mathbf{V}_j^n \cdots \mathbf{V}_K^n])^T$ and $\mathbf{Z}_k^{i,j} = (\mathbf{g}_{i,k}^n[\mathbf{V}_j^n \cdots \mathbf{V}_K^n])^T$, where $\mathbf{g}_{i,k}$ is the k th row vector of \mathbf{G}_i and $\mathbf{g}_{i,k}^n = \mathbf{I}_n \otimes \mathbf{g}_{i,k}$. Then, for $i = 2, \dots, K$,

$$\text{rank}(\mathbf{Z}^{i-1,i}) \stackrel{a.s.}{=} \text{rank}[\mathbf{Z}_{\Delta_{i-1}}^{i-1,i} \cdots \mathbf{Z}_1^{i-1,i}] \quad (62a)$$

$$\begin{aligned} &= \text{rank}(\mathbf{Z}_1^{i-1,i}) + \sum_{k=2}^{\Delta_{i-1}} \left(\text{rank}[\mathbf{Z}_k^{i-1,i} \cdots \mathbf{Z}_1^{i-1,i}] - \text{rank}[\mathbf{Z}_{k-1}^{i-1,i} \cdots \mathbf{Z}_1^{i-1,i}] \right) \\ &\stackrel{a.s.}{=} \text{rank}(\mathbf{Z}_{\Delta_{i-1}}^{i-1,i}) + \sum_{k=2}^{\Delta_{i-1}} \left(\text{rank}[\mathbf{Z}_{\Delta_{i-1}}^{i-1,i} \mathbf{Z}_{k-1}^{i-1,i} \cdots \mathbf{Z}_1^{i-1,i}] - \text{rank}[\mathbf{Z}_{k-1}^{i-1,i} \cdots \mathbf{Z}_1^{i-1,i}] \right) \end{aligned} \quad (62b)$$

$$\geq \sum_{k=1}^{\Delta_{i-1}} \left(\text{rank}[\mathbf{Z}_{\Delta_{i-1}}^{i-1,i} \cdots \mathbf{Z}_1^{i,i}] - \text{rank}[\mathbf{Z}_{\Delta_{i-1}-1}^{i-1,i} \cdots \mathbf{Z}_1^{i-1,i}] \right) \quad (62c)$$

$$\stackrel{a.s.}{=} \Delta_{i-1} \left(\text{rank}[\mathbf{Z}_{\Delta_{i-1}}^{i,i} \cdots \mathbf{Z}_1^{i,i}] - \text{rank}[\mathbf{Z}_{\Delta_{i-1}-1}^{i,i} \cdots \mathbf{Z}_1^{i,i}] \right) \quad (62d)$$

$$\stackrel{a.s.}{=} \frac{\Delta_{i-1}}{(\Delta_i - \Delta_{i-1})} \sum_{k=1}^{\Delta_i - \Delta_{i-1}} \left(\text{rank}[\mathbf{Z}_{\Delta_{i-1}+k}^{i,i} \mathbf{Z}_{\Delta_{i-1}-1}^{i,i} \cdots \mathbf{Z}_1^{i,i}] - \text{rank}[\mathbf{Z}_{\Delta_{i-1}-1}^{i,i} \cdots \mathbf{Z}_1^{i,i}] \right) \quad (62e)$$

$$\geq \frac{\Delta_{i-1}}{(\Delta_i - \Delta_{i-1})} \sum_{k=1}^{\Delta_i - \Delta_{i-1}} \left(\text{rank}[\mathbf{Z}_{\Delta_{i-1}+k}^{i,i} \cdots \mathbf{Z}_1^{i,i}] - \text{rank}[\mathbf{Z}_{\Delta_{i-1}+k-1}^{i,i} \cdots \mathbf{Z}_1^{i,i}] \right) \quad (62f)$$

$$= \frac{\Delta_{i-1}}{(\Delta_i - \Delta_{i-1})} \left(\text{rank}[\mathbf{Z}_{\Delta_i}^{i,i} \cdots \mathbf{Z}_1^{i,i}] - \text{rank}[\mathbf{Z}_{\Delta_{i-1}}^{i,i} \cdots \mathbf{Z}_1^{i,i}] \right)$$

$$\stackrel{a.s.}{=} \frac{\Delta_{i-1}}{(\Delta_i - \Delta_{i-1})} \left(\text{rank}[\mathbf{Z}_{\Delta_i}^{i,i} \cdots \mathbf{Z}_1^{i,i}] - \text{rank}[\mathbf{Z}_{\Delta_{i-1}}^{i-1,i} \cdots \mathbf{Z}_1^{i-1,i}] \right) \quad (62g)$$

$$\stackrel{a.s.}{=} \frac{\Delta_{i-1}}{(\Delta_i - \Delta_{i-1})} \left(\text{rank}(\mathbf{Z}^{i,i}) - \text{rank}(\mathbf{Z}^{i-1,i}) \right). \quad (62h)$$

Here (62a) holds since $[\mathbf{Z}_{\Delta_{i-1}}^{i-1,i} \cdots \mathbf{Z}_1^{i-1,i}]$ is a submatrix of $\mathbf{Z}^{i-1,i}$ and $\mathbf{Z}^{i-1,i} = [\mathbf{Z}_{\Delta_{i-1}}^{i-1,i} \cdots \mathbf{Z}_1^{i-1,i}] \mathbf{F}$ almost surely for a matrix \mathbf{F} such that

$$\mathbf{F} = \begin{cases} \mathbf{I}_n \otimes \left[\mathbf{I}_{\Delta_{i-1}}^T \left(([\mathbf{G}_{i-1}]_{\Delta_{i-1}}^T)^{-1} ([\mathbf{G}_{i-1}]_{\Delta_{i-1}}^c)^T \right)^T \right]^T & \text{if } \mathbf{G}_{i-1} \text{ is a tall matrix,} \\ \mathbf{I}_{n\Delta_{i-1}} & \text{otherwise} \end{cases} \quad (63)$$

where $[\mathbf{G}_{i-1}]_{\Delta_{i-1}}$ is the leading principal minor of \mathbf{G}_{i-1} of order Δ_{i-1} , $[\mathbf{G}_{i-1}]_{\Delta_{i-1}}^c$ is the remainder part of \mathbf{G}_{i-1} except $[\mathbf{G}_{i-1}]_{\Delta_{i-1}}$. Lemma 9 is used for (62b), (62d), (62e), and (62g) and Lemma 8 is used for (62c) and (62f). Also, (62h) follows since $\text{rank}[\mathbf{Z}_{\Delta_i}^{i,i} \cdots \mathbf{Z}_1^{i,i}] \stackrel{a.s.}{=} \text{rank}(\mathbf{Z}^{i,i})$ and $\text{rank}[\mathbf{Z}_{\Delta_{i-1}}^{i-1,i} \cdots \mathbf{Z}_1^{i-1,i}] \stackrel{a.s.}{=} \text{rank}(\mathbf{Z}^{i-1,i})$. From the fact that $\text{rank } \mathbf{A} = \text{rank } \mathbf{A}^T$ for a matrix \mathbf{A} whose elements are complex numbers [46], (62) becomes

$$\frac{1}{\Delta_{i-1}} \text{rank}(\mathbf{G}_{i-1}^n[\mathbf{V}_i^n \cdots \mathbf{V}_K^n]) \stackrel{a.s.}{\geq} \frac{1}{\Delta_i} \text{rank}(\mathbf{G}_i^n[\mathbf{V}_i^n \cdots \mathbf{V}_K^n]).$$

Then, for $i = 2, \dots, K$, we have

$$\begin{aligned} \frac{1}{\Delta_{i-1}} \text{rank}(\mathbf{G}_{i-1}^n[\mathbf{V}_i^n \cdots \mathbf{V}_K^n]) &\stackrel{a.s.}{\geq} \frac{1}{\Delta_i} \text{rank}(\mathbf{G}_i^n[\mathbf{V}_i^n \cdots \mathbf{V}_K^n]) \\ &= \frac{1}{\Delta_i} \left(\text{rank}(\mathbf{G}_i^n[\mathbf{V}_{i+1}^n \cdots \mathbf{V}_K^n]) + \dim(\text{Proj}_{(\mathcal{I}'_i)^c} \mathcal{R}(\mathbf{G}_i^n \mathbf{V}_i^n)) \right) \end{aligned} \quad (64a)$$

$$\geq \frac{1}{\Delta_i} \left(\text{rank}(\mathbf{G}_i^n[\mathbf{V}_{i+1}^n \cdots \mathbf{V}_K^n]) + \dim(\text{Proj}_{\mathcal{I}_i^c} \mathcal{R}(\mathbf{G}_i^n \mathbf{V}_i^n)) \right) \quad (64b)$$

where $\mathcal{I}'_i = \mathcal{R}(\mathbf{G}_i^n[\mathbf{V}_{i+1}^n \cdots \mathbf{V}_K^n])$. Here (64a) follows from Lemma 1 and (64b) follows since $\mathcal{I}'_i \subseteq \mathcal{I}_i$, which is given by $\mathcal{I}_i = \mathcal{R}(\mathbf{G}_i^n[\mathbf{V}_1^n \cdots \mathbf{V}_{i-1}^n, \mathbf{V}_{i+1}^n, \dots, \mathbf{V}_K^n])$ from Definition 1. Therefore (16) holds. In the same manner, we can proof (17), which completes the proof of Lemma 3.

C. Proof of Lemma 4

Let us assume that $M > L_{\max}$ and $N_k > L_{\max}$ for some $k \in \mathcal{K}$, i.e., $\Lambda \neq \emptyset$. Let $\Lambda = \{1, \dots, |\Lambda|\}$ without loss of generality and $\mathbf{A} = [\mathbf{A}_1 \mathbf{A}_2]$ such that

$$\mathbf{A}_1 = \begin{bmatrix} \frac{1}{L_1} & \frac{1}{T_2} & \frac{1}{T_3} & \cdots & \frac{1}{T_{|\Lambda|}} \\ \frac{1}{T_1} & \frac{1}{L_2} & \frac{1}{T_3} & \cdots & \frac{1}{T_{|\Lambda|}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{T_1} & \frac{1}{T_2} & \frac{1}{T_3} & \cdots & \frac{1}{L_{|\Lambda|}} \end{bmatrix} \quad (65)$$

and $\mathbf{A}_2 = \frac{1}{L_{\max}} \mathbf{1}_{|\Lambda| \times (K-|\Lambda|)}$. Then the optimization problem in Lemma 4 is rewritten as

$$\begin{aligned} &\text{maximize } \sum_{i=1}^K d_i \\ &\text{subject to } \mathbf{A}\mathbf{d} + \mathbf{x} = \mathbf{1}_{|\Lambda| \times 1}, \\ &\quad \mathbf{d} \geq \mathbf{0}, \mathbf{x} \geq \mathbf{0} \end{aligned} \quad (66)$$

where $\mathbf{d} = [d_1 \cdots d_K]^T$, $\mathbf{x} = [x_1 \cdots x_{|\Lambda|}]^T$, see also [43].

The following lemma provides non-singularity of \mathbf{A}_1 .

Lemma 10: For the matrix \mathbf{A}_1 defined in (65), the determinant of \mathbf{A}_1 is given by

$$|\mathbf{A}_1| = \prod_{k \in \Lambda} \frac{T_k - L_k}{T_k L_k} \left(1 + \sum_{k \in \Lambda} \frac{L_j}{T_j - L_j} \right). \quad (67)$$

Consequently, since $T_i > L_i$ for $i \in \Lambda$, \mathbf{A}_1 is non-singular.

Proof: It can be easily verified by mathematical induction. ■

Notice the the optimal \mathbf{d} should satisfy (66). Then, subtracting \mathbf{x} from both sides of (66) and multiplying them by $\mathbf{1}_{1 \times |\Lambda|} \mathbf{A}_1^{-1}$, which is possible from Lemma 10, we have

$$\sum_{i=1}^{|\Lambda|} d_i + \frac{\mathbf{1}_{1 \times |\Lambda|} \mathbf{A}_1^{-1} \mathbf{1}_{|\Lambda| \times 1}}{L_{\max}} \sum_{i=|\Lambda|+1}^K d_i = \mathbf{1}_{1 \times |\Lambda|} \mathbf{A}_1^{-1} \mathbf{1}_{|\Lambda| \times 1} - \mathbf{1}_{1 \times |\Lambda|} \mathbf{A}_1^{-1} \mathbf{x}. \quad (68)$$

Note that

$$\begin{aligned} \mathbf{1}_{1 \times |\Lambda|} \mathbf{A}_1^{-1} \mathbf{1}_{|\Lambda| \times 1} &= \sum_{i \in \Lambda} [\mathbf{1}_{1 \times |\Lambda|} \mathbf{A}_1^{-1}]_i \\ &= \sum_{i \in \Lambda} \frac{|\mathbf{A}_1|_{T_i=L_i=1}}{|\mathbf{A}_1|} \\ &= \sum_{i \in \Lambda} \frac{\prod_{k \in \Lambda, k \neq i} \frac{T_k - L_k}{T_k L_k}}{\prod_{k \in \Lambda} \frac{T_k - L_k}{T_k L_k} \left(1 + \sum_{j \in \Lambda} \frac{L_j}{T_j - L_j} \right)} \\ &= \frac{\sum_{i \in \Lambda} \frac{T_i L_i}{T_i - L_i}}{1 + \sum_{i \in \Lambda} \frac{L_i}{T_i - L_i}} = \eta. \end{aligned} \quad (69)$$

Here the third and fourth equalities follow from Cramer's rule [47, Lemma 176] and Lemma 10 respectively. Substituting (69) into (68), we have

$$\sum_{i=1}^{|\Lambda|} d_i + \frac{\eta}{L_{\max}} \sum_{i=|\Lambda|+1}^K d_i = \eta - \mathbf{1}_{1 \times |\Lambda|} \mathbf{A}_1^{-1} \mathbf{x}. \quad (70)$$

Therefore,

$$\begin{aligned} \sum_{i=1}^K d_i &\leq \max(\eta, L_{\max}) \left(\frac{1}{\eta} \sum_{i=1}^{|\Lambda|} d_i + \frac{1}{L_{\max}} \sum_{i=|\Lambda|+1}^K d_i \right) \\ &= \max(\eta, L_{\max}) \frac{1}{\eta} (\eta - \mathbf{1}_{1 \times |\Lambda|} \mathbf{A}_1^{-1} \mathbf{x}) \end{aligned} \quad (71)$$

$$\leq \max(\eta, L_{\max}) \quad (72)$$

where (71) follows from (70) and (72) follows since $\mathbf{1}_{1 \times |\Lambda|} \mathbf{A}_1^{-1} \mathbf{x} \geq 0$. In conclusion, Lemma 4 holds.

D. Proof of Lemma 5

We have

$$\begin{aligned} g^{[i]}(f^{[i]}(l)) &= ((l-1) \setminus \prod_{p=1}^i S_p) \prod_{p=1}^i S_p + (l-1) \prod_{p=1}^{i-1} S_p + ((l-1) \prod_{p=1}^i S_p) \setminus \prod_{p=1}^{i-1} S_p \prod_{p=1}^{i-1} S_p + 1 \\ &= ((l-1) \setminus \prod_{p=1}^i S_p) \prod_{p=1}^i S_p + (l-1) \prod_{p=1}^i S_p + 1 \\ &= l. \end{aligned} \quad (73)$$

Here (73) follows that $(l-1) \prod_{p=1}^{i-1} S_p = ((l-1) \prod_{p=1}^i S_p) \prod_{p=1}^{i-1} S_p$. In conclusion, Lemma 5 holds.

E. Proof of Lemma 6

We now prove that, for $i, i' \in \mathcal{K}$ where $i \neq i'$,

$$f_2^{[i]}(g^{[i']}(j, k)) = \begin{cases} ((j-1) \prod_{p=1}^{i'-1} S_p) \setminus \prod_{p=1}^{i'-1} S_p + 1 & \text{if } i < i', \\ ((j-1) \setminus \prod_{p=1}^{i'-1} S_p) \prod_{p=i'+1}^i S_p \setminus \prod_{p=i'+1}^{i-1} S_p + 1 & \text{if } i > i'. \end{cases} \quad (74)$$

From the definition of $f^{[i]}$ and $g^{[i']}$, (74) holds trivially for $i < i'$. Hence, assume $i > i'$ in the rest of this section.

For easy representation of the proof, for $i > i'$, denote,

$$\begin{aligned} a_0 &= (j-1) \setminus \prod_{p=1}^{i'-1} S_p \\ a_1 &= a_0 \setminus \prod_{p=i'+1}^i S_p, \quad b_1 = a_0 \prod_{p=i'+1}^i S_p \\ a_2 &= b_1 \setminus \prod_{p=i'+1}^{i-1} S_p, \quad b_2 = b_1 \prod_{p=i'+1}^{i-1} S_p \end{aligned}$$

From the definition of $g^{[i']}$, the following relation holds for $j \in \mathcal{A}$, $k \in \mathcal{B}$, and $i > i'$:

$$\begin{aligned} g^{[i']}(j, k) - 1 &= a_0 \prod_{p=1}^{i'} S_p + c \\ &= a_1 \prod_{p=1}^i S_p + b_1 \prod_{p=1}^{i'} S_p + c \end{aligned} \quad (75)$$

where $c = (j-1)|\prod_{p=1}^{i'-1} S_p + (k-1)\prod_{p=1}^{i'-1} S_p$ and the second equality follows since $a_0 = a_1 \prod_{p=i'+1}^i S_p + b_1$. From the fact that $b_1 \leq \prod_{p=i'+1}^i S_p - 1$ and $c < \prod_{p=1}^i S_p$, one can see that $b_1 \prod_{p=1}^{i'} S_p + c < \prod_{p=1}^i S_p$, which results from (75) that

$$\begin{aligned} (g^{[i']}(j, k) - 1) \prod_{p=1}^i S_p &= b_1 \prod_{p=1}^{i'} S_p + c \\ &= a_2 \prod_{p=1}^{i-1} S_p + b_2 \prod_{p=1}^{i'} S_p + c \end{aligned} \quad (76)$$

where the second equality follows that $b_1 = a_2 \prod_{p=i'+1}^{i-1} S_p + b_2$. Since $b_2 \leq \prod_{p=i'+1}^{i-1} S_p - 1$ and $c < \prod_{p=1}^{i'} S_p$, one can see that $b_2 \prod_{p=1}^{i'} S_p + c < \prod_{p=1}^{i-1} S_p$, which results from (76) that

$$\begin{aligned} f_2^{[i]}(g^{[i']}(j, k)) &= ((g^{[i']}(j, k) - 1) \prod_{p=1}^i S_p) \setminus \prod_{p=1}^{i-1} S_p + 1 \\ &= a_2 + 1, \end{aligned}$$

which completes the proof of Lemma 6.

APPENDIX II BLIND IA FOR A TWO-USER EXAMPLE

For better understanding of the proposed blind IA stated in Section V-A, we provide a two-user example here. Consider the two-user MIMO BC with reconfigurable antennas defined in Section II where $M = N_1 = N_2 = 3$, $L_1 = 1$, and $L_2 = 2$. From (24), $T_1 = T_2 = 3$, $S_2 = W_1 = 1$, and $S_2 = U_1 = U_2 = U = W_2 = W = 2$.

1) *Transmit beamforming design:* In Step 1, user 1 needs two information vectors ($U_1 W_1 = 2$) of which size is three ($T_1 L_1 = 3$) and user 2 needs four information vectors ($U_2 W_2 = 4$) of which size is six ($T_2 L_2 = 6$). Let us denote the information vectors of user 1 as $\mathbf{s}_1^{[1]}$ and $\mathbf{s}_2^{[1]} \in \mathbb{C}^3$ and denote the information vectors of user 2 as $\mathbf{s}_1^{[2]}$, $\mathbf{s}_2^{[2]}$, $\mathbf{s}_3^{[2]}$, and $\mathbf{s}_4^{[2]} \in \mathbb{C}^6$. Then, from (25), the alignment block of user 1 $\mathbf{v}_j^{[1]}$ for $j = 1, 2$ is given by

$$\mathbf{v}_j^{[1]} = \begin{bmatrix} \frac{\mathbf{v}_{j,1}^{[1]}}{\mathbf{v}_{j,2}^{[1]}} \\ \frac{\mathbf{v}_{j,2}^{[1]}}{\mathbf{v}_{j,3}^{[1]}} \end{bmatrix} = \begin{bmatrix} \frac{(\mathbf{I}_1 \otimes \mathbf{I}_3) \mathbf{s}_j^{[1]}}{(\mathbf{I}_1 \otimes \mathbf{I}_3) \mathbf{s}_j^{[1]}} \\ \frac{(\mathbf{I}_1 \otimes \mathbf{I}_3) \mathbf{s}_j^{[1]}}{(\mathbf{I}_1 \otimes \mathbf{I}_3) \mathbf{s}_j^{[1]}} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{I}_3 \\ \mathbf{I}_3 \end{bmatrix} \mathbf{s}_j^{[1]} \in \mathbb{C}^9, \quad j = 1, 2$$

and from (25), the alignment block of user 2 $\mathbf{v}_j^{[2]}$ for $j = 1, 2, 3, 4$ are given by

$$\mathbf{v}_j^{[2]} = \begin{bmatrix} \frac{\mathbf{v}_{j,1}^{[1]}}{\mathbf{v}_{j,2}^{[1]}} \end{bmatrix} = \begin{bmatrix} \frac{(\Phi \otimes \mathbf{I}_3) \mathbf{s}_j^{[2]}}{(\mathbf{I}_2 \otimes \mathbf{I}_3) \mathbf{s}_j^{[2]}} \end{bmatrix} = \begin{bmatrix} \Phi \otimes \mathbf{I}_3 \\ \mathbf{I}_6 \end{bmatrix} \mathbf{s}_j^{[2]} \in \mathbb{C}^9, \quad j = 1, 2, 3, 4$$

where $\Phi = [\phi_{11} \ \phi_{12}] \in \mathbb{C}^{1 \times 2}$ is a random matrix of which entries are i.i.d. continuous random variables.

In Step 2, we construct one ($W_1 = 1$) alignment unit of user 1 and two ($W_2 = 2$) alignment units of user 2. Then, from (27), alignment unit of user 1 $\mathbf{u}_1^{[1]}$ is given by

$$\mathbf{u}_1^{[1]} = \begin{bmatrix} \frac{\mathbf{u}_{1,1}^{[1]}}{\mathbf{u}_{1,2}^{[1]}} \\ \frac{\mathbf{u}_{1,2}^{[1]}}{\mathbf{u}_{1,3}^{[1]}} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{v}_{1,1}^{[1]}}{\mathbf{v}_{2,2}^{[1]}} \\ \frac{\mathbf{v}_{1,2}^{[1]}}{\mathbf{v}_{2,1}^{[1]}} \\ \frac{\mathbf{v}_{1,3}^{[1]}}{\mathbf{v}_{2,3}^{[1]}} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_6 \\ \mathbf{I}_6 \\ \mathbf{I}_6 \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[1]} \\ \mathbf{s}_2^{[1]} \end{bmatrix} \in \mathbb{C}^{18}$$

and the alignment unit of user 2 $\mathbf{u}_j^{[2]}$ for $j = 1, 2$ is given by

$$\mathbf{u}_j^{[2]} = \begin{bmatrix} \frac{\mathbf{u}_{j,1}^{[2]}}{\mathbf{u}_{j,2}^{[2]}} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{v}_{2j-1,1}^{[2]}}{\mathbf{v}_{2j-1,2}^{[2]}} \\ \frac{\mathbf{v}_{2j-1,2}^{[2]}}{\mathbf{v}_{2j,2}^{[2]}} \end{bmatrix} = \begin{bmatrix} \Phi \otimes \mathbf{I}_3 & \mathbf{0}_{3 \times 6} \\ \mathbf{0}_{3 \times 6} & \Phi \otimes \mathbf{I}_3 \\ \mathbf{I}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & \mathbf{I}_6 \end{bmatrix} \begin{bmatrix} \mathbf{s}_{2j-1}^{[2]} \\ \mathbf{s}_{2j}^{[2]} \end{bmatrix} \in \mathbb{C}^{18}.$$

In Step 3, we construct the transmit signal vector for each user. For this case, $f^{[i]}$ for $i = 1, 2$ is given by

$$\begin{bmatrix} f^{[1]}(1) & f^{[1]}(2) \end{bmatrix} = [(1, 1) \ (1, 2)], \quad \begin{bmatrix} f^{[2]}(1) & f^{[2]}(2) \end{bmatrix} = [(1, 1) \ (2, 1)]$$

Then, from (30), we have

$$\begin{aligned} \mathbf{x}_{1,1} &= \left[(\mathbf{u}_{1,1}^{[1]})^T \ (\mathbf{u}_{1,2}^{[1]})^T \right]^T, \quad \mathbf{x}_{1,2} = \mathbf{u}_{1,3}^{[1]}, \\ \mathbf{x}_{2,1} &= \left[(\mathbf{u}_{1,1}^{[2]})^T \ (\mathbf{u}_{2,1}^{[2]})^T \right]^T, \quad \mathbf{x}_{2,2} = \left[(\mathbf{u}_{1,2}^{[2]})^T \ (\mathbf{u}_{2,2}^{[2]})^T \right]^T. \end{aligned}$$

Subsequently, from (29), the transmit signal vector for user 1 is given by

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{x}_{1,1} \\ \mathbf{x}_{1,2} \\ \mathbf{0}_{24 \times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_{1,1}^{[1]} \\ \mathbf{u}_{1,2}^{[1]} \\ \mathbf{u}_{1,3}^{[1]} \\ \mathbf{0}_{24 \times 6} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_6 \\ \mathbf{I}_6 \\ \mathbf{I}_6 \\ \mathbf{0}_{24 \times 6} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[1]} \\ \mathbf{s}_2^{[1]} \end{bmatrix} \in \mathbb{C}^{42}$$

and the transmit signal vector for user 2 is given by

$$\mathbf{x}_2 = \begin{bmatrix} \mathbf{x}_{2,1} \\ \mathbf{0}_{6 \times 1} \\ \mathbf{x}_{2,2} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_{1,1}^{[2]} \\ \mathbf{u}_{2,1}^{[2]} \\ \mathbf{0}_{6 \times 1} \\ \mathbf{u}_{1,2}^{[2]} \\ \mathbf{u}_{2,2}^{[2]} \end{bmatrix} = \begin{bmatrix} \Phi \otimes \mathbf{I}_3 & \mathbf{0}_{3 \times 6} & \mathbf{0}_{3 \times 6} & \mathbf{0}_{3 \times 6} \\ \mathbf{0}_{3 \times 6} & \Phi \otimes \mathbf{I}_3 & \mathbf{0}_{3 \times 6} & \mathbf{0}_{3 \times 6} \\ \mathbf{0}_{3 \times 6} & \mathbf{0}_{3 \times 6} & \Phi \otimes \mathbf{I}_3 & \mathbf{0}_{3 \times 6} \\ \mathbf{0}_{3 \times 6} & \mathbf{0}_{3 \times 6} & \mathbf{0}_{3 \times 6} & \Phi \otimes \mathbf{I}_3 \\ \hline \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 \\ \hline \mathbf{I}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & \mathbf{I}_6 & \mathbf{0}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{I}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{I}_6 \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[2]} \\ \mathbf{s}_2^{[2]} \\ \mathbf{s}_3^{[2]} \\ \mathbf{s}_4^{[2]} \end{bmatrix} \in \mathbb{C}^{42}.$$

In Step 4, the overall transmit signal vector is given by

$$\mathbf{x}^n = \mathbf{x}_1 + \mathbf{x}_2$$

where $n = 14$.

2) *Mode switching patterns at receivers:* From (37), the time interval for transmitting block 1 is given by $1 \leq t \leq 4$. For this case, user 1 has two selection patterns and user 2 has one selection pattern, in which the associated channel matrices are given by

$$\begin{aligned} \mathbf{H}_{1,j} &= \mathbf{h}_{1,j} \in \mathbb{C}^{1 \times 3}, \quad j = 1, 2, \\ \mathbf{H}_{2,1} &= [\mathbf{h}_{2,1}^T \ \mathbf{h}_{2,2}^T]^T \in \mathbb{C}^{2 \times 3} \end{aligned} \tag{77}$$

respectively. As explained in Section V-A, when block 1 is transmitted, each user chooses the selection pattern of which index is the same as that of the currently transmitted sub-unit of his transmit signal vector. Since the indexes of the transmitted sub-unit of users 1 and 2 are 1, 1, 2, 2 and 1, 1, 1, 1 respectively for $1 \leq t \leq 4$, from (38), the received signal vectors of users 1 and 2 during $1 \leq t \leq 4$ are given by

$$\mathbf{y}_{1,0} = \begin{bmatrix} \mathbf{H}_{1,1} & \mathbf{0}_{1 \times 3} \\ \mathbf{0}_{1 \times 3} & \mathbf{H}_{1,1} \\ \mathbf{H}_{1,2} & \mathbf{0}_{1 \times 3} \\ \mathbf{0}_{1 \times 3} & \mathbf{H}_{1,2} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[1]} \\ \mathbf{s}_2^{[1]} \end{bmatrix} + \begin{bmatrix} \Phi \otimes \mathbf{H}_{1,1} & \mathbf{0}_{1 \times 6} & \mathbf{0}_{1 \times 6} & \mathbf{0}_{1 \times 6} \\ \mathbf{0}_{1 \times 6} & \Phi \otimes \mathbf{H}_{1,1} & \mathbf{0}_{1 \times 6} & \mathbf{0}_{1 \times 6} \\ \mathbf{0}_{1 \times 6} & \mathbf{0}_{1 \times 6} & \Phi \otimes \mathbf{H}_{1,2} & \mathbf{0}_{1 \times 6} \\ \mathbf{0}_{1 \times 6} & \mathbf{0}_{1 \times 6} & \mathbf{0}_{1 \times 6} & \Phi \otimes \mathbf{H}_{1,2} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[2]} \\ \mathbf{s}_2^{[2]} \\ \mathbf{s}_3^{[2]} \\ \mathbf{s}_4^{[2]} \end{bmatrix}, \tag{78}$$

$$\mathbf{y}_{2,0} = \begin{bmatrix} \mathbf{H}_{2,1} & \mathbf{0}_{2 \times 3} \\ \mathbf{0}_{2 \times 3} & \mathbf{H}_{2,1} \\ \mathbf{H}_{2,1} & \mathbf{0}_{2 \times 3} \\ \mathbf{0}_{2 \times 3} & \mathbf{H}_{2,1} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[1]} \\ \mathbf{s}_2^{[1]} \end{bmatrix} + \begin{bmatrix} \Phi \otimes \mathbf{H}_{2,1} & \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} \\ \mathbf{0}_{2 \times 6} & \Phi \otimes \mathbf{H}_{2,1} & \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} \\ \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} & \Phi \otimes \mathbf{H}_{2,1} & \mathbf{0}_{2 \times 6} \\ \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} & \Phi \otimes \mathbf{H}_{2,1} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[2]} \\ \mathbf{s}_2^{[2]} \\ \mathbf{s}_3^{[2]} \\ \mathbf{s}_4^{[2]} \end{bmatrix} \tag{79}$$

respectively.

The time interval for transmitting block 2 is given by $5 \leq t \leq 14$. First, we consider the time interval for transmitting $\mathbf{x}_{1,2}$, given by $5 \leq t \leq 6$. Note that this part corresponds to the desired signal part of block 2 for user 1 and the interference signal part of block 2 for user 2. Since $T_1|L_1 = 0$, user 1 exploits one selection pattern repeatedly over the time interval $5 \leq t \leq 6$, in which the associated channel matrix is given by

$$\mathbf{H}_{1,3} = \mathbf{h}_{1,3} \in \mathbb{C}^{1 \times 3}.$$

On the other hands, user 2 exploits the same selection pattern used for receiving during block 1 over the time interval $5 \leq t \leq 6$, in which there is one selection pattern associated with $\mathbf{H}_{2,1}$ for this case. Then, user 2 receives the transmit signal for $5 \leq t \leq 6$ using the selection pattern associated with $\mathbf{H}_{2,1}$. As a result, from (43) and (45), the received signal vectors of users 1 and 2 during $5 \leq t \leq 6$ are given by

$$\mathbf{y}_{1,1} = \begin{bmatrix} \mathbf{H}_{1,3} & \mathbf{0}_{1 \times 3} \\ \mathbf{0}_{1 \times 3} & \mathbf{H}_{1,3} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[1]} \\ \mathbf{s}_2^{[1]} \end{bmatrix}, \mathbf{y}_{2,1} = \begin{bmatrix} \mathbf{H}_{2,1} & \mathbf{0}_{2 \times 3} \\ \mathbf{0}_{2 \times 3} & \mathbf{H}_{2,1} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[1]} \\ \mathbf{s}_2^{[1]} \end{bmatrix} \quad (80)$$

respectively. Next, we consider the time interval for transmitting $\mathbf{x}_{2,2}$, given by $7 \leq t \leq 14$. This part corresponds to the desired signal part of block 2 for user 2 and the interference signal part of block 2 for user 1. Since $T_2|L_2 \neq 0$, user 2 exploits two ($L_2 S_2 = 2$) selection patterns, which repeat four ($S_1(T_1 - L_1) = 4$) times periodically over the time interval $7 \leq t \leq 14$. The associated channel matrices are given by

$$\mathbf{H}_{1,2,1} = [\mathbf{h}_{2,3}^T \mathbf{h}_{2,1}^T]^T \in \mathbb{C}^{2 \times 3}, \mathbf{H}_{1,2,2} = [\mathbf{h}_{2,3}^T \mathbf{h}_{2,2}^T]^T \in \mathbb{C}^{2 \times 3}.$$

On the other hands, user 1 exploits the same selection pattern used for receiving block 1 over the time interval $7 \leq t \leq 14$, in which the associated channel matrices are given in (77). When the interference signal part of block 2 is transmitted, user 2 chooses the selection pattern of which index is the same as that used to receive the first sub-unit of the alignment unit to which the currently transmitted sub-unit belongs. One can see that the sub-units transmitted for $7 \leq t \leq 10$ and $11 \leq t \leq 14$ is $\mathbf{u}_{1,3}^{[2]}$ and $\mathbf{u}_{2,3}^{[2]}$ respectively and user 1 exploits the selection pattern associated with $\mathbf{H}_{1,1}$ to receive $\mathbf{u}_{1,1}^{[2]}$ and the selection pattern associated with $\mathbf{H}_{1,2}$ to receive $\mathbf{u}_{2,1}^{[2]}$ in block 1. Hence, user 1 exploits the selection pattern associated with $\mathbf{H}_{1,1}$ for $7 \leq t \leq 10$ and the selection pattern associated with $\mathbf{H}_{1,2}$ for $11 \leq t \leq 14$. As a result, from (43) and (45), the received signal vectors of user 1 and 2 during $7 \leq t \leq 14$ are given by

$$\mathbf{y}_{1,2} = \begin{bmatrix} \mathbf{I}_2 \otimes \mathbf{H}_{1,1} & \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} \\ \mathbf{0}_{2 \times 6} & \mathbf{I}_2 \otimes \mathbf{H}_{1,1} & \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} \\ \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} & \mathbf{I}_2 \otimes \mathbf{H}_{1,2} & \mathbf{0}_{2 \times 6} \\ \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} & \mathbf{I}_2 \otimes \mathbf{H}_{1,2} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[2]} \\ \mathbf{s}_2^{[2]} \\ \mathbf{s}_3^{[2]} \\ \mathbf{s}_4^{[2]} \end{bmatrix}, \quad (81)$$

$$\mathbf{y}_{2,2} = \begin{bmatrix} \mathbf{H}'_{2,2} & \mathbf{0}_{4 \times 6} & \mathbf{0}_{4 \times 6} & \mathbf{0}_{4 \times 6} \\ \mathbf{0}_{4 \times 6} & \mathbf{H}'_{2,2} & \mathbf{0}_{4 \times 6} & \mathbf{0}_{4 \times 6} \\ \mathbf{0}_{4 \times 6} & \mathbf{0}_{4 \times 6} & \mathbf{H}'_{2,2} & \mathbf{0}_{4 \times 6} \\ \mathbf{0}_{4 \times 6} & \mathbf{0}_{4 \times 6} & \mathbf{0}_{4 \times 6} & \mathbf{H}'_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[2]} \\ \mathbf{s}_2^{[2]} \\ \mathbf{s}_3^{[2]} \\ \mathbf{s}_4^{[2]} \end{bmatrix} \quad (82)$$

respectively, where $\mathbf{H}'_{2,2} = \text{diag}(\mathbf{H}_{2,2,1}, \mathbf{H}_{2,2,2}) \in \mathbb{C}^{4 \times 6}$.

3) *Interference cancellation and achievable LDoF*: From (51), after cancelling all interference vectors in $\mathbf{y}_{1,0}$, user 1 has

$$\left[\frac{\mathbf{y}_{1,0} - (\mathbf{I}_4 \otimes \Phi) \mathbf{y}_{1,2}}{\mathbf{y}_{1,1}} \right] = \begin{bmatrix} \mathbf{H}_{1,1} & \mathbf{0}_{1 \times 3} \\ \mathbf{0}_{1 \times 3} & \mathbf{H}_{1,1} \\ \mathbf{H}_{1,2} & \mathbf{0}_{1 \times 3} \\ \mathbf{0}_{1 \times 3} & \mathbf{H}_{1,2} \\ \mathbf{H}_{1,3} & \mathbf{0}_{1 \times 3} \\ \mathbf{0}_{1 \times 3} & \mathbf{H}_{1,3} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[1]} \\ \mathbf{s}_2^{[1]} \end{bmatrix} \quad (83)$$

Sorting the rows in (83), we have

$$\begin{bmatrix} \mathbf{H}_1 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{H}_1 \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[1]} \\ \mathbf{s}_2^{[1]} \end{bmatrix} \quad (84)$$

Obviously, user 1 can obtain $\mathbf{s}_1^{[1]}$ and $\mathbf{s}_2^{[1]}$ from (84) almost surely.

Similarly, from (51), after cancelling all interference vectors in $\mathbf{y}_{2,0}$, user 2 has

$$\left[\frac{\mathbf{y}_{2,0} - \mathbf{1}_{2 \times 1} \otimes \mathbf{y}_{2,1}}{\mathbf{y}_{2,2}} \right] = \begin{bmatrix} \Phi \otimes \mathbf{H}_{2,1} & \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} \\ \mathbf{0}_{2 \times 6} & \Phi \otimes \mathbf{H}_{2,1} & \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} \\ \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} & \Phi \otimes \mathbf{H}_{2,1} & \mathbf{0}_{2 \times 6} \\ \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} & \mathbf{0}_{2 \times 6} & \Phi \otimes \mathbf{H}_{2,1} \\ \mathbf{H}'_{2,2} & \mathbf{0}_{4 \times 6} & \mathbf{0}_{4 \times 6} & \mathbf{0}_{4 \times 6} \\ \mathbf{0}_{4 \times 6} & \mathbf{H}'_{2,2} & \mathbf{0}_{4 \times 6} & \mathbf{0}_{4 \times 6} \\ \mathbf{0}_{4 \times 6} & \mathbf{0}_{4 \times 6} & \mathbf{H}'_{2,2} & \mathbf{0}_{4 \times 6} \\ \mathbf{0}_{4 \times 6} & \mathbf{0}_{4 \times 6} & \mathbf{0}_{4 \times 6} & \mathbf{H}'_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{s}_1^{[2]} \\ \mathbf{s}_2^{[2]} \\ \mathbf{s}_3^{[2]} \\ \mathbf{s}_4^{[2]} \end{bmatrix} \quad (85)$$

Then, (85) can be decomposed into four segments as in the following.

$$\begin{bmatrix} \Phi \otimes \mathbf{H}_{2,1} \\ \mathbf{H}'_{2,2} \end{bmatrix} \mathbf{s}_i^{[2]} = \begin{bmatrix} \phi_{11} \mathbf{h}_{2,1} & \phi_{12} \mathbf{h}_{2,1} \\ \phi_{11} \mathbf{h}_{2,2} & \phi_{12} \mathbf{h}_{2,2} \\ \mathbf{h}_{2,3} & \mathbf{0}_{1 \times 3} \\ \mathbf{h}_{2,1} & \mathbf{0}_{1 \times 3} \\ \mathbf{0}_{1 \times 3} & \mathbf{h}_{2,3} \\ \mathbf{0}_{1 \times 3} & \mathbf{h}_{2,2} \end{bmatrix} \mathbf{s}_i^{[2]}, \quad i = 1, 2, 3, 4 \quad (86)$$

It can be easily shown that user 2 can obtain $\mathbf{s}_i^{[2]}$ for all i from (86) almost surely.

As a result, the transmitter delivers 6 information symbols to user 1 and 24 information symbols to user 2 during 14 time slots. Consequently, the achievable sum LDof is given by $\frac{15}{7}$.

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