# Success probability of the Babai estimators for box-constrained integer linear models 

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#### Abstract

In many applications including communications, one may encounter a linear model where the parameter vector $\hat{x}$ is an integer vector in a box. To estimate $\hat{x}$, a typical method is to solve a box-constrained integer least squares (BILS) problem. However, due to its high complexity, the box-constrained Babai integer point $x^{\mathrm{BB}}$ is commonly used as a suboptimal solution. In this paper, we first derive formulas for the success probability $P^{\mathrm{BB}}$ of $x^{\mathrm{BB}}$ and the success probability $P^{0 \mathrm{~B}}$ of the ordinary Babai integer point $x^{\text {OB }}$ when $\hat{x}$ is uniformly distributed over the constraint box. Some properties of $P^{\mathrm{BB}}$ and $P^{\mathrm{OB}}$ and the relationship between them are studied. Then, we investigate the effects of some column permutation strategies on $P^{\mathrm{BB}}$. In addition to V-BLAST and SQRD, we also consider the permutation strategy involved in the LLL lattice reduction, to be referred to as LLL-P. On the one hand, we show that when the noise is relatively small, LLL-P always increases $P^{\mathrm{BB}}$ and argue why both V-BLAST and SQRD often increase $P^{\mathrm{BB}}$; and on the other hand, we show that when the noise is relatively large, LLL-P always decreases $P^{\mathrm{BB}}$ and argue why both V BLAST and SQRD often decrease $P^{B B}$. We also derive a column permutation invariant bound on $P^{\mathrm{BB}}$, which is an upper bound and a lower bound under these two opposite conditions, respectively. Numerical results demonstrate our findings. Finally, we consider a conjecture concerning $x^{0 B}$ proposed by Ma et al. We first construct an example to show that the conjecture does not hold in general, and then show that it does hold under some conditions.


Index Terms-Box-constrained integer least squares estimation, Babai integer point, success probability, column permutations, LLL-P, SQRD, V-BLAST.

## I. Introduction

S UPPOSE that we have the following box-constrained linear model:

$$
\begin{align*}
& \boldsymbol{y}=\boldsymbol{A} \hat{\boldsymbol{x}}+\boldsymbol{v}, \quad \boldsymbol{v} \sim \mathcal{N}\left(\mathbf{0}, \sigma^{2} \boldsymbol{I}\right)  \tag{1a}\\
& \hat{\boldsymbol{x}} \in \mathcal{B} \equiv\left\{\boldsymbol{x} \in \mathbb{Z}^{n}: \boldsymbol{\ell} \leq \boldsymbol{x} \leq \boldsymbol{u}, \boldsymbol{\ell}, \boldsymbol{u} \in \mathbb{Z}^{n}\right\} \tag{1b}
\end{align*}
$$

where $\boldsymbol{y} \in \mathbb{R}^{m}$ is an observation vector, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is a deterministic model matrix with full column rank,

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$\hat{\boldsymbol{x}}$ is an unknown integer parameter vector in the box $\mathcal{B}$, $\boldsymbol{v} \in \mathbb{R}^{m}$ is a noise vector following the Gaussian distribution $\mathcal{N}\left(\mathbf{0}, \sigma^{2} \boldsymbol{I}\right)$ with $\sigma$ being known. This model arises in various applications including wireless communications, see e.g., [1], [2]. In this paper, we assume that $\hat{\boldsymbol{x}}$ is random and uniformly distributed over the box $\mathcal{B}$. This assumption is often made for MIMO applications, see, e.g., [3].

A common method to estimate/detect $\hat{\boldsymbol{x}}$ in (1) is to solve the following box-constrained integer least squares (BILS) problem:

$$
\begin{equation*}
\min _{\boldsymbol{x} \in \mathcal{B}}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2} \tag{2}
\end{equation*}
$$

whose solution is the maximum likelihood estimator/detector of $\hat{\boldsymbol{x}}$. Here we would like to make a comment on terminology. In communications, it is proper to use "detect" and "detector" for the constrained case. However, later in this paper we will use "estimate" and "estimator" as an extension of the terminology commonly used in the unconstrained case. A typical approach to solving (2) is discrete search, which usually consists of two stages: reduction and search. In the first stage, orthogonal transformations are used to transform $\boldsymbol{A}$ to an upper triangular matrix $\boldsymbol{R}$. To make the search process more efficient, a column permutation strategy is often used in reduction. Two well-known strategies are V-BLAST [4], [1] and SQRD [5], [6]. The commonly used search methods are the so-called sphere decoding methods [1], [7] and [6], which are the extensions of the Schnorr-Euchner search method [8], a variation of the Fincke-Pohst search method [9], for ordinary integer least squares problems to be mentioned below. There are also some variants of SchnorrEuchner search methods, see, e.g., [10].

If the true parameter vector $\hat{\boldsymbol{x}} \in \mathbb{Z}^{n}$ in the linear model (1a) is not subject to any constraint, then we say (1a) is an ordinary linear model. In this case, to estimate $\hat{\boldsymbol{x}}$, one solves an ordinary integer least squares (OILS) problem (also referred to as the closest vector problem):

$$
\begin{equation*}
\min _{\boldsymbol{x} \in \mathbb{Z}^{n}}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2} \tag{3}
\end{equation*}
$$

and whose solution is referred to as the OILS estimator of $\hat{\boldsymbol{x}}$. Algorithms and theory for OILS problems are surveyed in [11] and [12].
The most widely used reduction strategy in solving (3) is the LLL reduction [13], which consists of two types of operations called size reduction and column permutation. But it is difficult to use it to solve a BILS problem
because after size reductions the box constraint becomes too complicated to handle in the search process. However, one can use its permutation strategy, to be referred to as LLL-P (we referred it to as LLL-permute in [14]). The LLL-P, SQRD and V-BLAST strategies use only the information of $\boldsymbol{A}$ to do the column permutations. Some column permutation strategies which use not only the information of $\boldsymbol{A}$, but also the information of $\boldsymbol{y}$ and the box constraint have also been proposed [15], [6] and [16].

For a fixed constraint box $\mathcal{B}$ in (1b), where all the entries of $\boldsymbol{\ell}$ are equal and all the entries of $\boldsymbol{u}$ are equal, it was shown in [3] that when the signal-to-noise ratio (SNR) is fixed the expected complexity of solving (2) by the FinckePohst search method behaves as an exponential function of the dimension $n$ when $n$ is large enough, although it is dominated by polynomial terms for high SNR and small $n$ [17] [3]. So for some real-time applications, an approximate solution, which can be produced quickly, is computed instead. For the OILS problem, the Babai integer point $\boldsymbol{x}^{\mathrm{OB}}$, to be referred to as the ordinary Babai estimator, which can be obtained by the Babai nearest plane algorithm [18], is an often used approximate solution. Taking the box constraint into account, one can easily modify the Babai nearest plane algorithm to get an approximate solution $\boldsymbol{x}^{\mathrm{BB}}$ to the BILS problem (2), to be referred to as the box-constrained Babai estimator. This estimator is the first point found by the search methods proposed in [7], [1] and [6], and it has been used as a suboptimal solution, see, e.g., [19]. In communications, algorithms for finding the Babai estimators are often referred to as successive interference cancellation detectors. There have been algorithms which find other suboptimal solutions to the BILS problems in communications, see, e.g., [20][29] etc. In this paper we will focus only on the Babai estimators.

In order to see how good an estimator is, one needs to find the probability of the estimator being equal to the true integer parameter vector, which is referred to as success probability [30]. The probability of wrong estimation is referred to as error probability, see, e.g., [26].

For the estimation of $\hat{\boldsymbol{x}}$ in the ordinary linear model (1a), where $\hat{\boldsymbol{x}}$ is supposed to be deterministic, the formula of the success probability $P^{0 B}$ of the ordinary Babai estimator $\boldsymbol{x}^{\mathrm{OB}}$ was first given in [31], which considers a variant form of the ILS problem (3). A simple derivation for an equivalent formula of $P^{\text {ob }}$ was given in [14]. It was shown in [14] that $P^{\mathrm{OB}}$ increases after applying the LLL reduction algorithm or only the LLL-P column permutation strategy, but $P^{\mathrm{OB}}$ may strictly decrease after applying the SQRD and V-BLAST permutation strategies.

The main goal of this paper is to extend the main results we obtained in [14] for the ordinary case to the boxconstrained case. We will present a formula for the success probability $P^{\mathrm{BB}}$ of the box-constrained Babai estimator $\boldsymbol{x}^{\mathrm{BB}}$ and a formula for the success probability $P^{\mathrm{OB}}$ of the
ordinary Babai estimator $\boldsymbol{x}^{\mathrm{OB}}$ when $\hat{\boldsymbol{x}}$ in (1) follows a uniform distribution over the box $\mathcal{B}$. Some properties of $P^{\mathrm{BB}}$ and $P^{\mathrm{OB}}$ and the relationship between them will also be given.

Then we will investigate the effect of the LLL-P column permutation strategy on $P^{\mathrm{BB}}$. We will show that $P^{B B}$ increases under a condition. Surprisingly, we will also show that $P^{\text {BB }}$ decreases after LLL-P is applied under an opposite condition. Roughly speaking, these two opposite conditions are that the noise standard deviation $\sigma$ in (1a) are relatively small and large, respectively. This is different from the ordinary case, where $P^{\mathrm{OB}}$ always increases after the LLL-P strategy is applied. Although our theoretical results for LLL-P cannot be extended to SQRD and VBLAST, our numerical tests indicate that under the two conditions, often (not always) $P^{\mathrm{BB}}$ increases and decreases, respectively, after applying SQRD or V-BLAST. Explanations will be given for these phenomena. These suggest that before we applying LLL-P, SQRD or V-BLAST we should check the conditions. Moreover, we will give a bound on $P^{\mathrm{BB}}$, which is column permutation invariant. It is interesting that the bound is an upper bound under the small noise condition we just mentioned and becomes a lower bound under the opposite condition.

In [32], the authors made a conjecture, based on which a stopping criterion for the search process was proposed to reduce the computational cost of solving the BILS problem. The conjecture is related to the success probability $P^{\mathrm{OB}}$ of the ordinary Babai estimator $\boldsymbol{x}^{\mathrm{OB}}$. We will first show that the conjecture does not always hold and then show it holds under a condition.

The rest of the paper is organized as follows. In Section II, we introduce the QR reduction and the LLL-P, SQRD and V-BLAST column recording strategies. In Section III, we present the formulas for $P^{B B}$ and $P^{\text {oB }}$, study the properties of $P^{\mathrm{BB}}$ and $P^{\mathrm{OB}}$ and the relationship between them. In Section IV, we investigate the effects of the LLLP, SQRD and V-BLAST column permutation strategies and derive a bound on $P^{\mathrm{BB}}$. In Section V, we investigate the conjecture made in [32] and obtain some negative and positive results. Finally, we summarize this paper in Section VI.

Notation. For matrices, we use bold upper-case letters and for vectors we use bold lower-case letters. For $\boldsymbol{x} \in$ $\mathbb{R}^{n}$, we use $\lfloor\boldsymbol{x}\rceil$ to denote its nearest integer vector, i.e., each entry of $\boldsymbol{x}$ is rounded to its nearest integer (if there is a tie, the one with smaller magnitude is chosen). For a vector $\boldsymbol{x}, \boldsymbol{x}_{i: j}$ denotes the subvector of $\boldsymbol{x}$ formed by entries $i, i+1, \ldots, j$. For a matrix $\boldsymbol{A}, \boldsymbol{A}_{i: j, i: j}$ denotes the submatrix of $\boldsymbol{A}$ formed by rows and columns $i, i+1, \ldots, j$.

## II. QR Factorization and Column Reordering

Assume that the model matrix $\boldsymbol{A}$ in the linear model (1a) has the QR factorization

$$
A=\left[\boldsymbol{Q}_{1}, \boldsymbol{Q}_{2}\right]\left[\begin{array}{c}
\boldsymbol{R}  \tag{4}\\
\mathbf{0}
\end{array}\right]
$$

where $\left[\boldsymbol{Q}_{1}, \underset{m-n}{\boldsymbol{Q}_{2}}\right] \in \mathbb{R}^{m \times m}$ is orthogonal and $\boldsymbol{R} \in \mathbb{R}^{n \times n}$ is upper triangular. Without loss of generality, we assume that the diagonal entries of $\boldsymbol{R}$ are positive throughout the paper. Define $\tilde{\boldsymbol{y}}=\boldsymbol{Q}_{1}^{T} \boldsymbol{y}$ and $\tilde{\boldsymbol{v}}=\boldsymbol{Q}_{1}^{T} \boldsymbol{v}$. Then, the linear model (1) is reduced to

$$
\begin{align*}
& \tilde{\boldsymbol{y}}=\boldsymbol{R} \hat{\boldsymbol{x}}+\tilde{\boldsymbol{v}}, \quad \tilde{\boldsymbol{v}} \sim \mathcal{N}\left(\mathbf{0}, \sigma^{2} \boldsymbol{I}\right)  \tag{5a}\\
& \hat{\boldsymbol{x}} \in \mathcal{B} \equiv\left\{\boldsymbol{x} \in \mathbb{Z}^{n}: \boldsymbol{\ell} \leq \boldsymbol{x} \leq \boldsymbol{u}, \boldsymbol{\ell}, \boldsymbol{u} \in \mathbb{Z}^{n}\right\} \tag{5b}
\end{align*}
$$

and the BILS problem (2) is reduced to

$$
\begin{equation*}
\min _{\boldsymbol{x} \in \mathcal{B}}\|\tilde{\boldsymbol{y}}-\boldsymbol{R} \boldsymbol{x}\|_{2}^{2} \tag{6}
\end{equation*}
$$

To solve the reduced problem (6), sphere decoding search algorithms are usually used to find the optimal solution. For search efficiency, one typically adopts a column permutation strategy, such as V-BLAST, SQRD or LLL-P, in the reduction process to obtain a better $\boldsymbol{R}$. For simplicity, we assume that the column permutations are performed on $\boldsymbol{R}$ in (4) no matter which strategy is used, i.e.,

$$
\begin{equation*}
\overline{\boldsymbol{Q}}^{T} \boldsymbol{R P}=\overline{\boldsymbol{R}} \tag{7}
\end{equation*}
$$

where $\overline{\boldsymbol{Q}} \in \mathbb{R}^{n \times n}$ is orthogonal, $\boldsymbol{P} \in \mathbb{Z}^{n \times n}$ is a permutation matrix, and $\overline{\boldsymbol{R}} \in \mathbb{R}^{n \times n}$ is an upper triangular matrix satisfying the properties of the corresponding column permutation strategies. Notice that combining (4) and (7) result in the following QR factorization of the column reordered $\boldsymbol{A}$ :

$$
A P=\tilde{Q}\left[\begin{array}{c}
\overline{\boldsymbol{R}} \\
0
\end{array}\right], \quad \tilde{Q} \equiv Q\left[\begin{array}{cc}
\bar{Q} & 0 \\
0 & I_{m-n}
\end{array}\right]
$$

The V-BLAST strategy determines the columns of $\overline{\boldsymbol{R}}$ from the last to the first. Suppose columns $n, n-1, \ldots, k+$ 1 of $\overline{\boldsymbol{R}}$ have been determined, this strategy chooses a column from $k$ remaining columns of $\boldsymbol{R}$ as the $k$-th column such that $\bar{r}_{k k}$ is maximum over all of the $k$ choices. For more details, including efficient algorithms, see [1], [4], [33]-[35] etc. One may refer to [36] for the performance analysis of V-BLAST.

In contrast to V-BLAST, the SQRD strategy determines the columns of $\overline{\boldsymbol{R}}$ from the first to the last by using the modified Gram-Schmidt algorithm or the Householder QR algorithm. Suppose columns $1,2, \ldots, k-1$ of $\overline{\boldsymbol{R}}$ have been determined. In the $k$-th step of the algorithm, the $k$-th column of $\overline{\boldsymbol{R}}$ we seek is chosen from the remaining $n-k+1$ columns of $\boldsymbol{R}$ such that $\bar{r}_{k k}$ is smallest. For more details, see [5] and [6] etc.

The LLL-P strategy [14] does the column permutations of the LLL reduction algorithm and produces $\overline{\boldsymbol{R}}$ satisfying the Lovász condition:

$$
\begin{equation*}
\delta \bar{r}_{k-1, k-1}^{2} \leq \bar{r}_{k-1, k}^{2}+\bar{r}_{k k}^{2}, \quad k=2,3, \ldots, n \tag{8}
\end{equation*}
$$

where $\delta$ is a parameter satisfying $1 / 4<\delta \leq 1$. Suppose that $\delta r_{k-1, k-1}^{2}>r_{k-1, k}^{2}+r_{k, k}^{2}$ for some $k$. Then we interchange columns $k-1$ and $k$ of $\boldsymbol{R}$. After the permutation, the upper triangular structure of $\boldsymbol{R}$ is no longer maintained. But we can bring $\boldsymbol{R}$ back to an upper triangular matrix by using the Gram-Schmidt orthogonalization technique (see [13]) or by a Givens rotation:

$$
\begin{equation*}
\overline{\boldsymbol{R}}=\boldsymbol{G}_{k-1, k}^{T} \boldsymbol{R} \boldsymbol{P}_{k-1, k} \tag{9}
\end{equation*}
$$

where $\boldsymbol{G}_{k-1, k}$ is an orthogonal matrix and $\boldsymbol{P}_{k-1, k}$ is a permutation matrix, and $\overline{\boldsymbol{R}}$ satisfies

$$
\begin{align*}
\bar{r}_{k-1, k-1}^{2} & =r_{k-1, k}^{2}+r_{k, k}^{2} \\
\bar{r}_{k-1, k}^{2}+\bar{r}_{k, k}^{2} & =r_{k-1, k-1}^{2}  \tag{10}\\
\bar{r}_{k-1, k-1} \bar{r}_{k k} & =r_{k-1, k-1} r_{k k}
\end{align*}
$$

Note that the above operation guarantees that the inequality in (8) holds. For simplicity, later when we refer to a column permutation, we mean the whole process of a column permutation and triangularization. For readers' convenience, we describe the LLL-P strategy in Algorithm 1, which can also be called the LLL-P reduction.

```
Algorithm 1 LLL-P
    set \(\boldsymbol{P}=\boldsymbol{I}_{n}, k=2\);
    while \(k \leq n\) do
        if \(\delta r_{k-1, k-1}^{2}>r_{k-1, k}^{2}+r_{k k}^{2}\) then
            perform a column permutation: \(\boldsymbol{R}=\)
            \(\boldsymbol{G}_{k-1, k}^{T} \boldsymbol{R} \boldsymbol{P}_{k-1, k}\);
            update \(\boldsymbol{P}: \boldsymbol{P}=\boldsymbol{P} \boldsymbol{P}_{k-1, k}\);
            \(k=k-1\), when \(k>2\);
        else
            \(k=k+1 ;\)
        end if
    end while
```

Here we give a remark about the LLL-P algorithm. Note that the LLL-P algorithm is the same as the original LLL algorithm, except that any operations related to size reductions are not performed. When the Lovász condition (8) for two consecutive columns $k-1$ and $k$ of $\boldsymbol{R}$ is not satisfied, the algorithm interchanges the two columns and performs triangularization. We have just shown that the two updated columns satisfy the Lovász condition. The algorithm terminates when the Lovász condition for any two consecutive columns is satisfied. The proof for the convergence of the original LLL algorithm, which does not use the size reduction condition, can be applied here to show the convergence of the LLL-P algorithm. We would like to point out that as the size reduction condition $\left(\left|r_{i j}\right| \leq r_{i i} / 2\right)$ in the LLL reduction is not satisfied any
more, some properties of the LLL reduction are lost in the LLL-P reduction.

With the QR factorization (7), we define

$$
\begin{array}{ll}
\overline{\boldsymbol{y}}=\overline{\boldsymbol{Q}}^{T} \tilde{\boldsymbol{y}}, & \hat{\boldsymbol{z}}=\boldsymbol{P}^{T} \hat{\boldsymbol{x}}, \quad \overline{\boldsymbol{v}}=\overline{\boldsymbol{Q}}^{T} \tilde{\boldsymbol{v}}  \tag{11}\\
\boldsymbol{z}=\boldsymbol{P}^{T} \boldsymbol{x}, & \overline{\boldsymbol{\ell}}=\boldsymbol{P}^{T} \boldsymbol{\ell}, \quad \overline{\boldsymbol{u}}=\boldsymbol{P}^{T} \boldsymbol{u}
\end{array}
$$

Then the linear model (5) is transformed to

$$
\begin{align*}
& \overline{\boldsymbol{y}}=\overline{\boldsymbol{R}} \hat{\boldsymbol{z}}+\overline{\boldsymbol{v}}, \quad \overline{\boldsymbol{v}} \sim \mathcal{N}\left(\mathbf{0}, \sigma^{2} \boldsymbol{I}\right)  \tag{12a}\\
& \hat{\boldsymbol{z}} \in \overline{\mathcal{B}}=\left\{\boldsymbol{z} \in \mathbb{Z}^{n}: \overline{\boldsymbol{\ell}} \leq \boldsymbol{z} \leq \overline{\boldsymbol{u}}, \overline{\boldsymbol{\ell}}, \overline{\boldsymbol{u}} \in \mathbb{Z}^{n}\right\} \tag{12b}
\end{align*}
$$

and the BILS problem (6) is transformed to

$$
\begin{equation*}
\min _{\boldsymbol{z} \in \overline{\mathcal{B}}}\|\overline{\boldsymbol{y}}-\overline{\boldsymbol{R}} \boldsymbol{z}\|_{2}^{2} \tag{13}
\end{equation*}
$$

whose solution is the BILS estimator of $\hat{\boldsymbol{z}}$.

## III. Success Probabilities of the Babai ESTIMATORS

We consider the reduced box-constrained linear model (5). The same analysis can be applied to the transformed reduced linear model (12).

The box-constrained Babai estimator $\boldsymbol{x}^{\mathrm{BB}}$ of $\hat{\boldsymbol{x}}$ in (5), a suboptimal solution to (6), can be computed as follows:

$$
\begin{align*}
& c_{i}^{\mathrm{BB}}=\left(\tilde{y}_{i}-\sum_{j=i+1}^{n} r_{i j} x_{j}^{\mathrm{BB}}\right) / r_{i i}, \\
& x_{i}^{\mathrm{BB}}= \begin{cases}\ell_{i}, & \text { if }\left\lfloor c_{i}^{\mathrm{BB}}\right\rfloor \leq \ell_{i} \\
\left\lfloor c_{i}^{\mathrm{BB}}\right\rceil, & \text { if } \ell_{i}<\left\lfloor c_{i}^{\mathrm{BB}}\right\rfloor<u_{i} \\
u_{i}, & \text { if }\left\lfloor c_{i}^{\mathrm{BB}}\right\rfloor \geq u_{i}\end{cases} \tag{14}
\end{align*}
$$

for $i=n, n-1, \ldots, 1$, where $\sum_{n+1}^{n} \cdot=0$. If we do not take the box constraint into account, we get the ordinary Babai estimator $\boldsymbol{x}^{\text {ов }}$ :

$$
\begin{equation*}
c_{i}^{\mathrm{OB}}=\left(\tilde{y}_{i}-\sum_{j=i+1}^{n} r_{i j} x_{j}^{\mathrm{OB}}\right) / r_{i i}, \quad x_{i}^{\mathrm{OB}}=\left\lfloor c_{i}^{\mathrm{OB}}\right\rceil \tag{15}
\end{equation*}
$$

for $i=n, n-1, \ldots, 1$.
In the following, we give formulas for the success probabilities of $\boldsymbol{x}^{\mathrm{BB}}$ and $\boldsymbol{x}^{\mathrm{OB}}$.

Theorem 1: Suppose that in (1) $\hat{\boldsymbol{x}}$ is uniformly distributed over the constraint box $\mathcal{B}$, and $\hat{\boldsymbol{x}}$ and $\boldsymbol{v}$ are independent. Suppose that (1) is transformed to (5) through the QR factorization (4). Then the success probabilities of the box-constrained Babai estimator $\boldsymbol{x}^{\mathrm{BB}}$ and the ordinary Babai estimator $\boldsymbol{x}^{\mathrm{OB}}$, which are respectively defined in (14) and (15), are

$$
\begin{align*}
P^{\mathrm{BB}} & \equiv \operatorname{Pr}\left(\boldsymbol{x}^{\mathrm{BB}}=\hat{\boldsymbol{x}}\right) \\
& =\prod_{i=1}^{n}\left[\frac{1}{u_{i}-\ell_{i}+1}+\frac{u_{i}-\ell_{i}}{u_{i}-\ell_{i}+1} \operatorname{erf}\left(\frac{r_{i i}}{2 \sqrt{2} \sigma}\right)\right],  \tag{16}\\
P^{\mathrm{OB}} & \equiv \operatorname{Pr}\left(\boldsymbol{x}^{\mathrm{OB}}=\hat{\boldsymbol{x}}\right)=\prod_{i=1}^{n} \operatorname{erf}\left(\frac{r_{i i}}{2 \sqrt{2} \sigma}\right), \tag{17}
\end{align*}
$$

where the error function is

$$
\operatorname{erf}(\zeta)=\frac{2}{\sqrt{\pi}} \int_{0}^{\zeta} \exp \left(-t^{2}\right) d t
$$

Proof. To simplify notation, we denote

$$
\begin{equation*}
\phi_{\sigma}(\zeta)=\operatorname{erf}\left(\frac{\zeta}{2 \sqrt{2} \sigma}\right) \tag{18}
\end{equation*}
$$

which will be used in this proof and other places.
Since the random vectors $\hat{\boldsymbol{x}}$ and $\boldsymbol{v}$ in (1) are independent, $\hat{\boldsymbol{x}}$ and $\tilde{\boldsymbol{v}}$ in (5) are also independent. From (5a),

$$
\tilde{y}_{i}=r_{i i} \hat{x}_{i}+\sum_{j=i+1}^{n} r_{i j} \hat{x}_{j}+\tilde{v}_{i}, \quad i=n, n-1, \ldots, 1
$$

Then from (14), we obtain
$c_{i}^{\mathrm{BB}}=\hat{x}_{i}+\sum_{j=i+1}^{n} \frac{r_{i j}}{r_{i i}}\left(\hat{x}_{j}-x_{j}^{\mathrm{BB}}\right)+\frac{\tilde{v}_{i}}{r_{i i}}, \quad i=n, n-1, \ldots, 1$.
Therefore, if $x_{i+1}^{\mathrm{BB}}=\hat{x}_{i+1}, \cdots, x_{n}^{\mathrm{BB}}=\hat{x}_{n}$ and $\hat{x}_{i}$ is fixed, we have $c_{i}^{\text {BB }} \sim \mathcal{N}\left(\hat{x}_{i}, \sigma^{2} / r_{i i}^{2}\right)$. Thus,

$$
\begin{equation*}
\frac{\left(c_{i}^{\mathrm{BB}}-\hat{x}_{i}\right) r_{i i}}{\sqrt{2} \sigma} \sim \mathcal{N}\left(0, \frac{1}{2}\right) \tag{20}
\end{equation*}
$$

To simplify notation, we denote events

$$
E_{i}=\left(x_{i}^{\mathrm{BB}}=\hat{x}_{i}, \ldots, x_{n}^{\mathrm{BB}}=\hat{x}_{n}\right), \quad i=1, \ldots, n .
$$

Then, applying the chain rule of conditional probabilities yields

$$
\begin{equation*}
P^{\mathrm{BB}}=\operatorname{Pr}\left(E_{1}\right)=\prod_{i=1}^{n} \operatorname{Pr}\left(x_{i}^{\mathrm{BB}}=\hat{x}_{i} \mid E_{i+1}\right) \tag{21}
\end{equation*}
$$

where $E_{n+1}$ is the sample space $\Omega$ leading to $\operatorname{Pr}\left(x_{n}^{\mathrm{BB}}=\right.$ $\left.\hat{x}_{n} \mid E_{n+1}\right)=\operatorname{Pr}\left(x_{n}^{\mathrm{BB}}=\hat{x}_{n}\right)$.

Since events $\hat{x}_{i}=\ell_{i}, \ell_{i}<\hat{x}_{i}<u_{i}$ and $\hat{x}_{i}=u_{i}$ are independent, by (14), we have

$$
\begin{align*}
& \operatorname{Pr}\left(x_{i}^{\mathrm{BB}}=\hat{x}_{i} \mid E_{i+1}\right) \\
= & \operatorname{Pr}\left(\left(\hat{x}_{i}=\ell_{i}, c_{i}^{\mathrm{BB}} \leq \ell_{i}+1 / 2\right) \mid E_{i+1}\right) \\
+ & \operatorname{Pr}\left(\left(\ell_{i}<\hat{x}_{i}<u_{i}, \hat{x}_{i}-1 / 2 \leq c_{i}^{\mathrm{BB}}<\hat{x}_{i}+1 / 2\right) \mid E_{i+1}\right) \\
+ & \operatorname{Pr}\left(\left(\hat{x}_{i}=u_{i}, c_{i}^{\mathrm{BB}} \geq u_{i}-1 / 2\right) \mid E_{i+1}\right) . \tag{22}
\end{align*}
$$

In the following we will use this simple result: if $\bar{E}_{1}, \bar{E}_{2}$ and $\bar{E}_{3}$ are three events, and $\bar{E}_{2}$ and $\bar{E}_{3}$ are independent, then

$$
\begin{equation*}
\operatorname{Pr}\left(\left(\bar{E}_{1}, \bar{E}_{2}\right) \mid \bar{E}_{3}\right)=\operatorname{Pr}\left(\bar{E}_{1}\right) \operatorname{Pr}\left(\bar{E}_{2} \mid\left(\bar{E}_{1}, \bar{E}_{3}\right)\right) \tag{23}
\end{equation*}
$$

This can easily be proved. In fact,

$$
\begin{aligned}
\operatorname{Pr}\left(\left(\bar{E}_{1}, \bar{E}_{2}\right) \mid \bar{E}_{3}\right) & =\frac{\operatorname{Pr}\left(\bar{E}_{1}, \bar{E}_{2}, \bar{E}_{3}\right)}{\operatorname{Pr}\left(\bar{E}_{3}\right)} \\
& =\operatorname{Pr}\left(\bar{E}_{1}\right) \frac{\operatorname{Pr}\left(\bar{E}_{1}, \bar{E}_{2}, \bar{E}_{3}\right)}{\operatorname{Pr}\left(\bar{E}_{1}, \bar{E}_{3}\right)} \\
& =\operatorname{Pr}\left(\bar{E}_{1}\right) \operatorname{Pr}\left(\bar{E}_{2} \mid\left(\bar{E}_{1}, \bar{E}_{3}\right)\right)
\end{aligned}
$$

where the second equality follows from the fact that $\bar{E}_{1}$ and $\bar{E}_{3}$ are independent.

Thus, by (22) and (23), we obtain

$$
\begin{align*}
& \operatorname{Pr}\left(x_{i}^{\mathrm{BB}}=\hat{x}_{i} \mid E_{i+1}\right) \\
= & \operatorname{Pr}\left(\hat{x}_{i}=\ell_{i}\right) \operatorname{Pr}\left(c_{i}^{\mathrm{BB}} \leq \ell_{i}+1 / 2 \mid\left(\hat{x}_{i}=\ell_{i}, E_{i+1}\right)\right) \\
+ & \operatorname{Pr}\left(\ell_{i}<\hat{x}_{i}<u_{i}\right) \\
\times & \operatorname{Pr}\left(\hat{x}_{i}-1 / 2 \leq c_{i}^{\mathrm{BB}}<\hat{x}_{i}+1 / 2 \mid\left(\ell_{i}<\hat{x}_{i}<u_{i}, E_{i+1}\right)\right) \\
+ & \operatorname{Pr}\left(\hat{x}_{i}=u_{i}\right) \operatorname{Pr}\left(c_{i}^{\mathrm{BB}} \geq u_{i}-1 / 2 \mid\left(\hat{x}_{i}=u_{i}, E_{i+1}\right)\right) . \tag{24}
\end{align*}
$$

Since $\hat{\boldsymbol{x}}$ is uniformly distributed over the box $\mathcal{B}$, for the first factors of the three terms on the right-hand side of (24), we have

$$
\begin{aligned}
\operatorname{Pr}\left(\hat{x}_{i}=\ell_{i}\right) & =\frac{1}{u_{i}-\ell_{i}+1} \\
\operatorname{Pr}\left(\ell_{i}<\hat{x}_{i}<u_{i}\right) & =\frac{u_{i}-\ell_{i}-1}{u_{i}-\ell_{i}+1} \\
\operatorname{Pr}\left(\hat{x}_{i}=u_{i}\right) & =\frac{1}{u_{i}-\ell_{i}+1} .
\end{aligned}
$$

By (18) and (20), for the second factors of these three terms, we have

$$
\left.\begin{array}{rl} 
& \operatorname{Pr}\left(c_{i}^{\mathrm{BB}} \leq \ell_{i}+1 / 2 \mid\left(\hat{x}_{i}=\ell_{i}, E_{i+1}\right)\right) \\
= & \operatorname{Pr}\left(\left.\frac{\left(c_{i}^{\mathrm{BB}}-\hat{x}_{i}\right) r_{i i}}{\sqrt{2} \sigma} \leq \frac{r_{i i}}{2 \sqrt{2} \sigma} \right\rvert\,\left(\hat{x}_{i}=\ell_{i}, E_{i+1}\right)\right) \\
= & \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{r_{i i}}{2 \sqrt{2} \sigma}} \exp \left(-t^{2}\right) d t=\frac{1}{2}\left[1+\phi_{\sigma}\left(r_{i i}\right)\right], \\
\operatorname{Pr}\left(\hat{x}_{i}-1 / 2 \leq c_{i}^{\mathrm{BB}}<\hat{x}_{i}+1 / 2 \mid\left(\ell_{i}<\hat{x}_{i}<u_{i}, E_{i+1}\right)\right) \\
= & \operatorname{Pr}\left(\left.\left|\frac{\left(c_{i}^{\mathrm{BB}}-\hat{x}_{i}\right) r_{i i}}{\sqrt{2} \sigma}\right| \leq \frac{r_{i i}}{2 \sqrt{2} \sigma} \right\rvert\,\left(\ell_{i}<\hat{x}_{i}<u_{i}, E_{i+1}\right)\right) \\
= & \frac{1}{\sqrt{\pi}} \int_{-\frac{r_{i i}}{2 \sqrt{2} \sigma}}^{\frac{r_{i i}}{2 \sqrt{2}}} \exp \left(-t^{2}\right) d t=\phi_{\sigma}\left(r_{i i}\right), \\
= & \operatorname{Pr}\left(c_{i}^{\mathrm{BB}} \geq u_{i}-1 / 2 \mid\left(\hat{x}_{i}=u_{i}, E_{i+1}\right)\right) \\
= & \left.\left.\frac{1}{\sqrt{\pi}} \int_{\left.-\frac{r_{i i}}{2 \sqrt{\mathrm{BB}}}-\hat{x}_{i}\right) r_{i i}}^{\sqrt{2} \sigma} \exp \left(-t^{2}\right) d t=\frac{r_{i i}}{2 \sqrt{2} \sigma} \right\rvert\,\left(\hat{x}_{i}=u_{i}, E_{i+1}\right)\right) \\
2
\end{array} 1^{\infty}+\phi_{\sigma}\left(r_{i i}\right)\right] . .
$$

Combining the equalities above with (24) yields

$$
\begin{aligned}
& \operatorname{Pr}\left(x_{i}^{\mathrm{BB}}=\hat{x}_{i} \mid E_{i+1}\right) \\
= & \frac{1}{2\left(u_{i}-\ell_{i}+1\right)}\left[1+\phi_{\sigma}\left(r_{i i}\right)\right]+\frac{u_{i}-\ell_{i}-1}{u_{i}-\ell_{i}+1} \phi_{\sigma}\left(r_{i i}\right) \\
& +\frac{1}{2\left(u_{i}-\ell_{i}+1\right)}\left[1+\phi_{\sigma}\left(r_{i i}\right)\right] \\
= & \frac{1}{u_{i}-\ell_{i}+1}+\frac{u_{i}-\ell_{i}}{u_{i}-\ell_{i}+1} \phi_{\sigma}\left(r_{i i}\right)
\end{aligned}
$$

which, with (18) and (21), yields (16).
Now we consider the success probability of the ordinary Babai estimator $\boldsymbol{x}^{\mathrm{OB}}$. Everything in the first three
paragraphs of this proof still holds if we replace each superscript BB by OB . But we need to make more significant changes to the last two paragraphs. We change (22) and (24) as follows:

$$
\begin{aligned}
& \operatorname{Pr}\left(x_{i}^{\mathrm{OB}}=\hat{x}_{i} \mid E_{i+1}\right) \\
= & \operatorname{Pr}\left(\left(\ell_{i} \leq \hat{x}_{i} \leq u_{i}, \hat{x}_{i}-1 / 2 \leq c_{i}^{\mathrm{OB}}<\hat{x}_{i}+1 / 2\right) \mid E_{i+1}\right) \\
= & \operatorname{Pr}\left(\ell_{i} \leq \hat{x}_{i} \leq u_{i}\right) \\
\times & \operatorname{Pr}\left(\hat{x}_{i}-1 / 2 \leq c_{i}^{\mathrm{OB}}<\hat{x}_{i}+1 / 2 \mid\left(\ell_{i} \leq \hat{x}_{i} \leq u_{i}, E_{i+1}\right)\right)
\end{aligned}
$$

Here

$$
\begin{aligned}
& \operatorname{Pr}\left(\ell_{i} \leq \hat{x}_{i} \leq u_{i}\right)=1 \\
& \operatorname{Pr}\left(\hat{x}_{i}-1 / 2 \leq c_{i}^{\mathrm{OB}}<\hat{x}_{i}+1 / 2 \mid\left(\ell_{i} \leq \hat{x}_{i} \leq u_{i}, E_{i+1}\right)\right) \\
& \quad=\phi_{\sigma}\left(r_{i i}\right)
\end{aligned}
$$

Thus

$$
\operatorname{Pr}\left(x_{i}^{\mathrm{OB}}=\hat{x}_{i} \mid E_{i+1}\right)=\phi_{\sigma}\left(r_{i i}\right)
$$

Then (17) follows from (18) and (21) with each superscript $B B$ replaced by OB.

From the proof of (17), we observe that the formula holds no matter what distribution of $\hat{\boldsymbol{x}}$ is over the box $\mathcal{B}$. Furthermore, the formula is identical to the one for the success probability of the ordinary Babai estimator $\boldsymbol{x}^{\text {Ов }}$ when $\hat{\boldsymbol{x}}$ in (1) is deterministic and is not subject to any box constraint; for more details, see [14].

The following result shows the relationship between $P^{\text {BB }}$ and $P^{\mathrm{OB}}$.

Corollary 1: Under the same assumption as in Theorem 1,

$$
\begin{equation*}
\lim _{\leq n, u_{i}-\ell_{i} \rightarrow \infty} P^{P^{\mathrm{OB}}}<P^{\mathrm{BB}}, \tag{25}
\end{equation*}
$$

Proof. Note that

$$
\phi_{\sigma}\left(r_{i i}\right)=\operatorname{erf}\left(r_{i i} /(2 \sqrt{2} \sigma)\right)<1
$$

Thus

$$
\begin{aligned}
\phi_{\sigma}\left(r_{i i}\right) & =\frac{1}{u_{i}-\ell_{i}+1} \phi_{\sigma}\left(r_{i i}\right)+\frac{u_{i}-\ell_{i}}{u_{i}-\ell_{i}+1} \phi_{\sigma}\left(r_{i i}\right) \\
& <\frac{1}{u_{i}-\ell_{i}+1}+\frac{u_{i}-\ell_{i}}{u_{i}-\ell_{i}+1} \phi_{\sigma}\left(r_{i i}\right) .
\end{aligned}
$$

Then, by Theorem 1, we can conclude that (25) holds, and we can also see (26) holds.

Corollary 2: Under the same assumption as in Theorem $1, P^{\mathrm{BB}}$ and $P^{\mathrm{oB}}$ increase when $\sigma$ decreases and

$$
\lim _{\sigma \rightarrow 0} P^{\mathrm{BB}}=\lim _{\sigma \rightarrow 0} P^{\mathrm{OB}}=1
$$

Proof. For a given $\zeta$, when $\sigma$ decreases $\operatorname{erf}\left(r_{i i} /(2 \sqrt{2} \sigma)\right)$ increases and $\lim _{\sigma \rightarrow 0} \operatorname{erf}\left(r_{i i} /(2 \sqrt{2} \sigma)\right)=1$. Then from Theorem 1 , we immediately see that the corollary holds.

## IV. Effects of LLL-P, SQRD and V-BLAST on $P^{\text {bB }}$

Suppose that we perform the QR factorization (7) by using a column permutation strategy, such as LLL-P, SQRD or V-BLAST, then we have the reduced boxconstrained linear model (12). For (12) we can define its corresponding Babai point $\boldsymbol{z}^{\mathrm{BB}}$, and use it as an estimator of $\hat{\boldsymbol{z}}$, which is equal to $\boldsymbol{P}^{T} \hat{\boldsymbol{x}}$, or equivalently we use $\boldsymbol{P} \boldsymbol{z}^{\mathrm{BB}}$ as an estimator of $\hat{\boldsymbol{x}}$.

In this section, we will investigate how LLL-P, SQRD and V-BLAST column permutation strategies affect the success probability $P^{B B}$ of the box-constrained Babai estimator.

## A. Effect of LLL-P on $P^{B B}$

The LLL-P strategy involves a sequence of permutations of two consecutive columns of $\boldsymbol{R}$. To investigate how LLL-P affects $P^{\mathrm{BB}}$, we first look at one column permutation. Suppose that $\delta r_{k-1, k-1}^{2}>r_{k-1, k}^{2}+r_{k k}^{2}$ for some $k$ for the $\boldsymbol{R}$ in (5). After the permutation of columns $k-1$ and $k, \boldsymbol{R}$ becomes $\overline{\boldsymbol{R}}=\boldsymbol{G}_{k-1, k}^{T} \boldsymbol{R} \boldsymbol{P}_{k-1, k}$ (see (9)). Then with the transformations given in (11), where $\overline{\boldsymbol{Q}}=\boldsymbol{G}_{k-1, k}$ and $\boldsymbol{P}=\boldsymbol{P}_{k-1, k}$, (5) is transformed to (12). We will compare $\operatorname{Pr}\left(\boldsymbol{x}^{\mathrm{BB}}=\hat{\boldsymbol{x}}\right)$ and $\operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{BB}}=\hat{\boldsymbol{z}}\right)$.

To prove our main results, we need the following two lemmas.

Lemma 1: Given $\alpha>0$, define

$$
\begin{equation*}
f(\zeta, \alpha)=\left(1-2 \zeta^{2}\right)(1+\alpha \operatorname{erf}(\zeta))-\frac{2 \alpha}{\sqrt{\pi}} \zeta \exp \left(-\zeta^{2}\right) \tag{27}
\end{equation*}
$$

for $\zeta \geq 0$. Then, $f(\zeta, \alpha)$ is a strictly decreasing function of $\zeta$ and has a unique zero $r(\alpha)$, i.e.,

$$
\begin{equation*}
f(r(\alpha), \alpha)=0 \tag{28}
\end{equation*}
$$

When $\zeta>r(\alpha), f(\zeta, \alpha)<0$ and when $\zeta<r(\alpha)$, $f(\zeta, \alpha)>0$. Furthermore, $0<r(\alpha)<1 / \sqrt{2}, r(\alpha)$ is a strictly decreasing function of $\alpha$, and $\lim _{\alpha \rightarrow \infty} r(\alpha)=0$.

Proof. By some simple calculations, $\stackrel{\alpha \rightarrow \infty}{\text { we obtain }}$

$$
\frac{\partial f(\zeta, \alpha)}{\partial \zeta}=-4 \zeta(1+\alpha \operatorname{erf}(\zeta))
$$

Thus, for any $\zeta \geq 0$ and $\alpha>0, \partial f(\zeta, \alpha) / \partial \zeta \leq 0$, where the equality holds if and only $\zeta=0$. Therefore, $f(\zeta, \alpha)$ is a strictly decreasing function of $\zeta$.

Note that $f(0, \alpha)=1>0$ and $f(1 / \sqrt{2}, \alpha)<0$ for $\alpha>$ 0 , by the implicit function theorem, there exists a unique $r(\alpha)$, which is continuously differentiable with respect to $\alpha$, such that (28) holds and $0<r(\alpha)<1 / \sqrt{2}$. Since $f(\zeta, \alpha)$ is strictly decreasing with respect to $\zeta$, when $\zeta>$ $r(\alpha), f(\zeta, \alpha)<0$ and when $\zeta<r(\alpha), f(\zeta, \alpha)>0$.

In the following, we show that $r(\alpha)$ is a strictly decreasing function of $\alpha$. From (28), we have
$\left(1-2 r^{2}(\alpha)\right)(1+\alpha \operatorname{erf}(r(\alpha)))=\frac{2 \alpha}{\sqrt{\pi}} r(\alpha) \exp \left(-r^{2}(\alpha)\right)$.

Taking the derivative for both sides of (29) with respect to $\alpha$ yields

$$
\begin{aligned}
& -2 r(\alpha) r^{\prime}(\alpha)(1+\alpha \operatorname{erf}(r(\alpha)))+ \\
& \left(1-2 r^{2}(\alpha)\right)\left(\operatorname{erf}(r(\alpha))+\frac{2 \alpha}{\sqrt{\pi}} r^{\prime}(\alpha) \exp \left(-r^{2}(\alpha)\right)\right) \\
= & \frac{2}{\sqrt{\pi}} r(\alpha) \exp \left(-r^{2}(\alpha)\right) \\
& +\frac{2 \alpha}{\sqrt{\pi}} r^{\prime}(\alpha) \exp \left(-r^{2}(\alpha)\right)\left(1-2 r^{2}(\alpha)\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& 2 r(\alpha) r^{\prime}(\alpha)(1+\alpha \operatorname{erf}(r(\alpha))) \\
= & \left(1-2 r^{2}(\alpha)\right) \operatorname{erf}(r(\alpha))-\frac{2}{\sqrt{\pi}} r(\alpha) \exp \left(-r^{2}(\alpha)\right) \\
= & -\frac{1}{\alpha}\left(1-2 r^{2}(\alpha)\right)
\end{aligned}
$$

where the latter equality follows from (29). Hence

$$
r^{\prime}(\alpha)=-\frac{\left(1-2 r^{2}(\alpha)\right)}{2 \alpha r(\alpha)(1+\alpha \operatorname{erf}(r(\alpha)))}<0
$$

Finally, we show that $\lim _{\alpha \rightarrow \infty} r(\alpha)=0$. Since $r(\alpha)$ is continuously differentiable with respect to $\alpha$ and $r(\alpha)>0$ for $\alpha>0, \lim _{\alpha \rightarrow \infty} r(\alpha)$ exists. Let $\eta=\lim _{\alpha \rightarrow \infty} r(\alpha)$, by the fact that $r(\alpha)$ is strictly decreasing with $\alpha$, we obtain that $0 \leq \eta \leq 1 / \sqrt{2}$.

From (29), we have

$$
\begin{aligned}
& \left(1-2 r^{2}(\alpha)\right) \operatorname{erf}(r(\alpha))-\frac{2}{\sqrt{\pi}} r(\alpha) \exp \left(-r^{2}(\alpha)\right) \\
= & \frac{1-2(r(\alpha))^{2}}{\alpha} .
\end{aligned}
$$

Then we take limits on both sides of the above equation as $\alpha \rightarrow \infty$, resulting in

$$
\left(1-2 \eta^{2}\right) \operatorname{erf}(\eta)-\frac{2}{\sqrt{\pi}} \eta \exp \left(-\eta^{2}\right)=0
$$

Since $0 \leq \eta \leq 1 / \sqrt{2}$, one can conclude from the above equation that $\lim _{\alpha \rightarrow+\infty} r(\alpha)=\eta=0$.

Remark 1: Given $\alpha$, we can easily solve (28) by a numerical method, e.g., the Newton method, to find $r(\alpha)$.

Lemma 2: Given $\alpha, \beta>0$, define

$$
\begin{equation*}
g(\zeta, \alpha, \beta)=(1+\alpha \operatorname{erf}(\zeta))(1+\alpha \operatorname{erf}(\beta / \zeta)), \quad \zeta>0 \tag{30}
\end{equation*}
$$

Then, when

$$
\begin{equation*}
\min \{\sqrt{\beta}, \beta / r(\alpha)\} \leq \zeta<\max \{\sqrt{\beta}, \beta / r(\alpha)\} \tag{31}
\end{equation*}
$$

where $r(\alpha)$ is defined in Lemma $1, g(\zeta, \alpha, \beta)$ is a strictly decreasing function of $\zeta$.

Proof. By the definition of $g$, we can easily obtain

$$
\begin{aligned}
\frac{\partial g(\zeta, \alpha, \beta)}{\partial \zeta}= & \frac{2 \alpha}{\sqrt{\pi} \zeta}(1+\alpha \operatorname{erf}(\zeta))(1+\alpha \operatorname{erf}(\beta / \zeta)) \\
& \times[h(\zeta, \alpha)-h(\beta / \zeta, \alpha)]
\end{aligned}
$$

where

$$
\begin{equation*}
h(\zeta, \alpha)=\frac{\zeta \exp \left(-\zeta^{2}\right)}{1+\alpha \operatorname{erf}(\zeta)} \tag{32}
\end{equation*}
$$

It is easy to see that in order to show the result, we need only to show $h(\zeta, \alpha)<h(\beta / \zeta, \alpha)$ under the condition (31) with $\zeta \neq \beta / \zeta$.

By some simple calculations and (27), we have

$$
\begin{equation*}
\frac{\partial h(\zeta, \alpha)}{\partial \zeta}=\frac{\exp \left(-\zeta^{2}\right)}{(1+\alpha \operatorname{erf}(\zeta))^{2}} \times f(\zeta, \alpha) \tag{33}
\end{equation*}
$$

Now we assume that $\zeta$ satisfies (31) with $\zeta \neq \beta / \zeta$. If $\sqrt{\beta}<\beta / r(\alpha)$, by (31), we have $\zeta>\beta / \zeta>r(\alpha)$, and then from Lemma 1, in this case, $f(\zeta, \alpha)<0$, thus $\partial h(\zeta, \alpha) / \partial \zeta<0$, i.e., $h(\zeta, \alpha)$ is a strictly deceasing function of $\zeta$, thus $h(\zeta, \alpha)<h(\beta / \zeta, \alpha)$. If $\sqrt{\beta}>\beta / r(\alpha)$, by (31), we obtain $\zeta<\beta / \zeta<r(\alpha)$, and then from Lemma 1, $f(\zeta, \alpha)>0$, thus $\partial h(\zeta, \alpha) / \partial \zeta>0$, i.e., $h(\zeta, \alpha)$ is a strictly increasing function of $\zeta$, thus again $h(\zeta, \alpha)<h(\beta / \zeta, \alpha)$.

With the above lemmas, we can show how the success probability of the box-constrained Babai estimator changes after two consecutive columns are swapped when the LLLP strategy is applied. Specifically, we have the following theorem.

Theorem 2: Suppose that in (1) the box $\mathcal{B}$ is a cube with edge length of $d, \hat{x}$ is uniformly distributed over $\mathcal{B}$, and $\hat{\boldsymbol{x}}$ and $\boldsymbol{v}$ are independent. Suppose that (1) is transformed to (5) through the QR factorization (4) and $\delta r_{k-1, k-1}^{2}>$ $r_{k-1, k}^{2}+r_{k k}^{2}$. After the permutation of columns $k-1$ and $k$ of $\boldsymbol{R}$ and triangularization (see (9)), (5) is transformed to (12).

1) If $r_{k k} \geq 2 \sqrt{2} \sigma r(d)$, where $r(\cdot)$ is defined in Lemma 1, then after the permutation, the success probability of the box-constrained Babai estimator increases, i.e.,

$$
\begin{equation*}
\operatorname{Pr}\left(\boldsymbol{x}^{\mathrm{BB}}=\hat{\boldsymbol{x}}\right) \leq \operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{BB}}=\hat{\boldsymbol{z}}\right) . \tag{34}
\end{equation*}
$$

2) If $r_{k-1, k-1} \leq 2 \sqrt{2} \sigma r(d)$, then after the permutation, the success probability of the box-constrained Babai estimator decreases, i.e.,

$$
\begin{equation*}
\operatorname{Pr}\left(\boldsymbol{x}^{\mathrm{BB}}=\hat{\boldsymbol{x}}\right) \geq \operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{BB}}=\hat{\boldsymbol{z}}\right) \tag{35}
\end{equation*}
$$

Furthermore, the equality in each of (34) and (35) holds if and only if $r_{k-1, k}=0$.

Proof. When $r_{k-1, k}=0$, by Theorem 1, we see the equalities in (34) and (35) hold. In the following we assume $r_{k-1, k} \neq 0$ and show the strict inequalities in (34) and (35) hold.

Define

$$
\begin{equation*}
\beta \equiv \frac{r_{k-1, k-1}}{2 \sqrt{2} \sigma} \frac{r_{k k}}{2 \sqrt{2} \sigma}=\frac{\bar{r}_{k-1, k-1}}{2 \sqrt{2} \sigma} \frac{\bar{r}_{k k}}{2 \sqrt{2} \sigma} \tag{36}
\end{equation*}
$$

where for the second equality, see (10). Using $\delta r_{k-1, k-1}^{2}>r_{k-1, k}^{2}+r_{k k}^{2}$ and the equalities in (10), we can easily verify that

$$
\begin{align*}
& \sqrt{\beta} \leq \max \left\{\frac{\bar{r}_{k-1, k-1}}{2 \sqrt{2} \sigma}, \frac{\bar{r}_{k k}}{2 \sqrt{2} \sigma}\right\} \\
&<\max \left\{\frac{r_{k-1, k-1}}{2 \sqrt{2} \sigma}, \frac{r_{k k}}{2 \sqrt{2} \sigma}\right\}=\frac{r_{k-1, k-1}}{2 \sqrt{2} \sigma} \\
&=\frac{\beta}{r_{k k} /(2 \sqrt{2} \sigma)},  \tag{37}\\
& \frac{\beta}{r_{k-1, k-1} /(2 \sqrt{2} \sigma)}=\frac{r_{k k}}{2 \sqrt{2} \sigma}=\min \left\{\frac{r_{k-1, k-1}}{2 \sqrt{2} \sigma}, \frac{r_{k k}}{2 \sqrt{2} \sigma}\right\} \\
&<\min \left\{\frac{\bar{r}_{k-1, k-1}}{2 \sqrt{2} \sigma}, \frac{\bar{r}_{k k}}{2 \sqrt{2} \sigma}\right\} \leq \sqrt{\beta} \tag{38}
\end{align*}
$$

Now we prove part 1. Note that after the permutation, $r_{k-1, k-1}$ and $r_{k k}$ change, but other diagonal entries of $\boldsymbol{R}$ do not change. Then by Theorem 1, we can easily observe that (34) is equivalent to

$$
\begin{align*}
& {\left[\frac{1}{d+1}+\frac{d}{d+1} \operatorname{erf}\left(\frac{r_{k-1, k-1}}{2 \sqrt{2} \sigma}\right)\right] } \\
& \times\left[\frac{1}{d+1}+\frac{d}{d+1} \operatorname{erf}\left(\frac{r_{k k}}{2 \sqrt{2} \sigma}\right)\right] \\
& \leq\left[\frac{1}{d+1}+\frac{d}{d+1} \operatorname{erf}\left(\frac{\bar{r}_{k-1, k-1}}{2 \sqrt{2} \sigma}\right)\right] \\
& \times\left[\frac{1}{d+1}+\frac{d}{d+1} \operatorname{erf}\left(\frac{\bar{r}_{k k}}{2 \sqrt{2} \sigma}\right)\right] \tag{39}
\end{align*}
$$

By (30), we can see that (39) is equivalent to

$$
\begin{align*}
& g\left(\max \left\{\frac{r_{k-1, k-1}}{2 \sqrt{2} \sigma}, \frac{r_{k k}}{2 \sqrt{2} \sigma}\right\}, d, \beta\right) \\
\leq & g\left(\max \left\{\frac{\bar{r}_{k-1, k-1}}{2 \sqrt{2} \sigma}, \frac{\bar{r}_{k k}}{2 \sqrt{2} \sigma}\right\}, d, \beta\right) . \tag{40}
\end{align*}
$$

If $r_{k k} \geq 2 \sqrt{2} \sigma r(d)$, then the right-hand side of the last equality in (37) satisfies

$$
\begin{equation*}
\frac{\beta}{r_{k k} /(2 \sqrt{2} \sigma)} \leq \frac{\beta}{r(d)} \tag{41}
\end{equation*}
$$

Then by combining (37) and (41) and applying Lemma 2 we can conclude that the strict inequality in (40) holds.

The proof for part 2 is similar. The inequality (35) is equivalent to

$$
\begin{align*}
& g\left(\min \left\{\frac{r_{k-1, k-1}}{2 \sqrt{2} \sigma}, \frac{r_{k k}}{2 \sqrt{2} \sigma}\right\}, d, \beta\right) \\
\geq & g\left(\min \left\{\frac{\bar{r}_{k-1, k-1}}{2 \sqrt{2} \sigma}, \frac{\bar{r}_{k k}}{2 \sqrt{2} \sigma}\right\}, d, \beta\right) \tag{42}
\end{align*}
$$

If $r_{k-1, k-1} \leq 2 \sqrt{2} \sigma r(d)$, then the left-hand side of the first equality in (38) satisfies

$$
\begin{equation*}
\frac{\beta}{r(d)} \leq \frac{\beta}{r_{k-1, k-1} /(2 \sqrt{2} \sigma)} \tag{43}
\end{equation*}
$$

Then by combining (38) and (43) and applying Lemma 2 we can conclude that the strict inequality in (42) holds.

We make a few remarks about Theorem 2.
Remark 2: In the theorem, $\mathcal{B}$ is assumed to be a cube, not a more general box. This restriction simplified the theoretical analysis. Furthermore, in practical applications, such as communications, indeed $\mathcal{B}$ is often a cube.

Remark 3: After the permutation, the larger one of $r_{k-1, k-1}$ and $r_{k k}$ becomes smaller (see (37)) and the smaller one becomes larger (see (38)), so the gap between $r_{k-1, k-1}$ and $r_{k k}$ becomes smaller. This makes $P^{\mathrm{BB}}$ increase under the condition $r_{k k} \geq 2 \sqrt{2} \sigma r(d)$ or decrease under the condition $r_{k-1, k-1} \leq 2 \sqrt{2} \sigma r(d)$. It is natural to ask for fixed $r_{k-1, k-1}$ and $r_{k k}$, when will $P^{\mathrm{BB}}$ increase most or decrease most after the permutation under the corresponding conditions? From the proof we observe that $P^{\text {вв }}$ will become maximal when the first inequality in (37) becomes an equality or minimal when the last inequality in (38) becomes an equality under the corresponding conditions. Either of the two equalities holds if and only if $\bar{r}_{k-1, k-1}=\bar{r}_{k k}$, which is equivalent to $r_{k-1, k}^{2}+r_{k k}^{2}=r_{k-1, k-1} r_{k k}$ by (10).

Remark 4: The case where $r_{k k}<2 \sqrt{2} \sigma r(d)<$ $r_{k-1, k-1}$ is not covered by the theorem. For this case, $P^{\mathrm{BB}}$ may increase or decrease after the permutation, for more details, see the simulations in Sec. IV-D.

Based on Theorem 2, we can establish the following general result for the LLL-P strategy.

Theorem 3: Suppose that in (1) the box $\mathcal{B}$ is a cube with edge length of $d, \hat{\boldsymbol{x}}$ is uniformly distributed over $\mathcal{B}$, and $\hat{\boldsymbol{x}}$ and $\boldsymbol{v}$ are independent. Suppose that (1) is first transformed to (5) through the QR factorization (4) and then to (12) through the QR factorization (7) where the LLL-P strategy is used for column permutations.

1) If the diagonal entries of $\boldsymbol{R}$ in (5) satisfies

$$
\begin{equation*}
\min _{1 \leq i \leq n} r_{i i} \geq 2 \sqrt{2} \sigma r(d), \tag{44}
\end{equation*}
$$

where $r(\cdot)$ is defined in Lemma 1, then

$$
\begin{equation*}
\operatorname{Pr}\left(\boldsymbol{x}^{\mathrm{BB}}=\hat{\boldsymbol{x}}\right) \leq \operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{BB}}=\hat{\boldsymbol{z}}\right) \tag{45}
\end{equation*}
$$

2) If the diagonal entries of $\boldsymbol{R}$ in (5) satisfies

$$
\begin{equation*}
\max _{1 \leq i \leq n} r_{i i} \leq 2 \sqrt{2} \sigma r(d) \tag{46}
\end{equation*}
$$

then

$$
\begin{equation*}
\operatorname{Pr}\left(\boldsymbol{x}^{\mathrm{BB}}=\hat{\boldsymbol{x}}\right) \geq \operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{BB}}=\hat{\boldsymbol{z}}\right) . \tag{47}
\end{equation*}
$$

And the equalities in (45) and (47) hold if and only if no column permutation occurs in the process or whenever two consecutive columns, say $k-1$ and $k$, are permuted, $r_{k-1, k}=0$.

Proof. It is easy to show that after each column permutation, the smaller one of the two diagonal entries of $\boldsymbol{R}$ involved in the permutation either keeps unchanged (the
involved super-diagonal entry is 0 in this case) or strictly increases, while the larger one either keeps unchanged or strictly decreases (see (37) and (38)). Thus, after each column permutation, the minimum of the diagonal entries of $\boldsymbol{R}$ either keeps unchanged or strictly increases and the maximum either keeps unchanged or strictly decreases, so the diagonal entries of any upper triangular $\overline{\boldsymbol{R}}$ produced after a column permutation satisfies $\min _{1 \leq i \leq n} r_{i i} \leq \bar{r}_{k k} \leq$ $\max _{1<i \leq n} r_{i i}$ for all $k=1, \ldots, n$. Then the conclusion follows $1 \leq i \leq n$ from Theorem 2.

We make some remarks about Theorem 3.
Remark 5: The quantity $r(d)$ is involved in the conditions. To get some idea about how large it is, we compute it for a few different $d=2^{k}-1$. For $k=1,2,3,4,5$, the corresponding values of $r$ are $0.5939,0.4926,0.4042$, $0.3286,0.2653$. They are decreasing with $k$ as proved in Lemma 1. As $d \rightarrow \infty, r(d) \rightarrow 0$. Thus, when $d$ is large enough, the condition (44) will be satisfied. By Corollary 1 , taking the limit as $d \rightarrow \infty$ on both sides of (45), we obtain the following result proved in [14]:

$$
\operatorname{Pr}\left(\boldsymbol{x}^{\mathrm{OB}}=\hat{\boldsymbol{x}}\right) \leq \operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{OB}}=\hat{\boldsymbol{z}}\right),
$$

i.e., LLL-P always increases the success probability of the ordinary Babai estimator.

Remark 6: The two conditions (44) and (46) also involve the noise standard deviation $\sigma$. When $\sigma$ is small, (44) is likely to hold, so applying LLL-P is likely to increase $P^{\mathrm{BB}}$, and when $\sigma$ is large, (46) is likely to hold, so applying LLL-P is likely to decrease $P^{\text {BB }}$. It is quite surprising that when $\sigma$ is large enough applying LLL-P will decrease $P^{B B}$. Thus, before applying LLL-P, one needs to check the conditions (44) and (46). If (44) holds, one has confidence to apply LLL-P. If (46) holds, one should not apply it. If both do not hold, i.e., $\min _{1 \leq i \leq n} r_{i i}<2 \sqrt{2} \sigma r(d)<\max _{1 \leq i \leq n} r_{i i}$, applying LLL-P may increase or decrease $P^{8 B}$.

## B. Effects of SQRD and V-BLAST on $P^{B B}$

SQRD and V-BLAST have been used to find better ordinary and box-constrained Babai estimators in the literature. It has been demonstrated in [14] that unlike LLLP, both SQRD and V-BLAST may decrease the success probability $P^{0 \mathrm{~B}}$ of the ordinary Babai estimator when the parameter vector $\hat{\boldsymbol{x}}$ is deterministic and not subject to any constraint.

We would like to know how SQRD and V-BLAST affect $P^{\mathrm{Bb}}$. Unlike LLL-P, both SQRD and V-BLAST usually involve two non-consecutive columns permutations, resulting in the changes of all diagonal entries between and including the two columns. This makes it very difficult to analyze under what condition $\boldsymbol{P}^{\mathrm{BB}}$ increases or decreases. We will use numerical test results to show the effects of SQRD and V-BLAST on $P^{\text {BB }}$ with explanations.
In Theorem 2 we showed that if the condition (44) holds, then applying LLL-P will increase $\boldsymbol{P}^{\mathrm{BB}}$ and if
(46) holds, then applying LLL-P will decrease $P^{\mathrm{BB}}$. The following example shows that both SQRD and V-BLAST may decrease $P^{8 B}$ even if (44) holds, and they may increase $P^{\text {BB }}$ even if (46) holds.

Example 1: Let $d=1$ and consider two matrices:
$\boldsymbol{R}^{(1)}=\left[\begin{array}{ccc}3.5 & 3 & 0 \\ 0 & 1 & -1.5 \\ 0 & 0 & 1\end{array}\right], \quad \boldsymbol{R}^{(2)}=\left[\begin{array}{ccc}1 & -1.5 & 1.5 \\ 0 & 0.8 & -1 \\ 0 & 0 & 0.42\end{array}\right]$.
Applying SQRD, V-BLAST and LLL-P to $\boldsymbol{R}^{(1)}$ and $\boldsymbol{R}^{(2)}$, we obtain

$$
\begin{aligned}
\boldsymbol{R}_{S}^{(1)} & =\left[\begin{array}{ccc}
1.8028 & -0.8321 & 0 \\
0 & 3.0509 & 3.4417 \\
0 & 0 & 0.6364
\end{array}\right] \\
\boldsymbol{R}_{V}^{(1)}=\boldsymbol{R}_{L}^{(1)} & =\left[\begin{array}{ccc}
3.1623 & 3.3204 & -0.4743 \\
0 & 1.1068 & 1.4230 \\
0 & 0 & 1
\end{array}\right], \\
\boldsymbol{R}_{V}^{(2)} & =\left[\begin{array}{ccc}
1.7 & -1.7941 & -0.8824 \\
0 & 0.4556 & -0.1823 \\
0 & 0 & 0.4338
\end{array}\right] \\
\boldsymbol{R}_{S}^{(2)} & =\boldsymbol{R}_{L}^{(2)}=\boldsymbol{R}^{(2)}
\end{aligned}
$$

If $\sigma=0.2$, then it is easy to verify that for both $\boldsymbol{R}^{(1)}$ and $\boldsymbol{R}^{(2)}$, (44) holds (note that $2 \sqrt{2} r(d)=2 \sqrt{2} r(1)=$ 1.6798). Simple calculations by using (16) give

$$
P^{\mathrm{BB}}\left(\boldsymbol{R}^{(1)}\right)=0.9876, \quad P^{\mathrm{BB}}\left(\boldsymbol{R}^{(2)}\right)=0.8286,
$$

and
$P^{\mathrm{BB}}\left(\boldsymbol{R}_{S}^{(1)}\right)=0.9442, \quad P^{\mathrm{BB}}\left(\boldsymbol{R}_{V}^{(1)}\right)=P^{\mathrm{BB}}\left(\boldsymbol{R}_{L}^{(1)}\right)=0.9910$,
$P^{\mathrm{BB}}\left(\boldsymbol{R}_{V}^{(2)}\right)=0.7513, \quad P^{\mathrm{BB}}\left(\boldsymbol{R}_{S}^{(2)}\right)=P^{\mathrm{BB}}\left(\boldsymbol{R}_{L}^{(2)}\right)=0.8286$.
Thus, SQRD decreases $P^{\text {BB }}$, while V-BLAST and LLL-P increase $P^{\mathrm{BB}}$ for $\boldsymbol{R}^{(1)}$, and V-BLAST decreases $P^{\mathrm{BB}}$, while SQRD and LLL-P keep $P^{\mathrm{BB}}$ unchanged for $\boldsymbol{R}^{(2)}$.

If $\sigma=2.2$, then it is easy to verify that for both $\boldsymbol{R}^{(1)}$ and $\boldsymbol{R}^{(2)}$, (46) holds. Simple calculations by using (16) give

$$
P^{\mathrm{BB}}\left(\boldsymbol{R}^{(1)}\right)=0.2738, \quad P^{\mathrm{BB}}\left(\boldsymbol{R}^{(2)}\right)=0.1816 .
$$

Then
$P^{\mathrm{BB}}\left(\boldsymbol{R}_{S}^{(1)}\right)=0.2777, \quad P^{\mathrm{BB}}\left(\boldsymbol{R}_{V}^{(1)}\right)=P^{\mathrm{BB}}\left(\boldsymbol{R}_{L}^{(1)}\right)=0.2700$, $P^{\mathrm{BB}}\left(\boldsymbol{R}_{V}^{(2)}\right)=0.1898, \quad P^{\mathrm{BB}}\left(\boldsymbol{R}_{S}^{(2)}\right)=P^{\mathrm{BB}}\left(\boldsymbol{R}_{L}^{(2)}\right)=0.1816$.

Thus, SQRD increases $P^{\text {BB }}$, while V-BLAST and LLL-P decrease $P^{\mathrm{BB}}$ for $\boldsymbol{R}^{(1)}$, and V-BLAST increases $P^{\mathrm{BB}}$, while SQRD and LLL-P keep $P^{\text {Bb }}$ unchanged for $\boldsymbol{R}^{(2)}$.

Although Example 1 indicates that under the condition (44), unlike LLL-P, both SQRD and V-BLAST may decrease $P^{\mathrm{BB}}$, often they increase $P^{\mathrm{BB}}$. This is the reason why SQRD and V-BLAST (especially the latter) have often been used to increase the accuracy of the Babai estimator in practice. Example 1 also indicates that under the condition (46), unlike LLL-P, both SQRD and V-BLAST
may increase $P^{\mathrm{BB}}$, but often they decrease $P^{\mathrm{BB}}$. This is the opposite of what we commonly believe. Later we will give numerical test results to show both phenomena. In the following we give some explanations.

It is easy to show that like LLL-P, V-BLAST increases $\min _{1 \leq i \leq} r_{i i}$ (not strictly) after each permutation and like LLL-P, SQRD decreases $\max _{1 \leq i \leq n} r_{i i}$ (not strictly) after each permutation. The relation between V-BLAST and SQRD can be found in [34] and [28]. Thus if the condition (44) holds before applying V-BLAST, it will also hold after applying it; and if the condition (46) holds before applying SQRD, it will also hold after applying it. Often applying V-BLAST decreases $\max _{1 \leq i \leq n} r_{i i}$ and applying SQRD increases $\min _{1 \leq i \leq} r_{i i}$ (both may not be true sometimes, see Example 1). Thus often the gaps between the large diagonal entries and the small ones of $\boldsymbol{R}$ decrease after applying SQRD or V-BLAST. From the proof of Theorem 2 we see reducing the gaps will likely increase $P^{\mathrm{BB}}$ under (44) and decrease $P^{\mathrm{BB}}$ under (46). Thus it is likely both SQRD and V-BLAST will increase $P^{B B}$ under (44) and decrease it under (46). We will give further explanations in the next subsection.

## C. A bound on $P^{B B}$

In this subsection we give a bound on $P^{B B}$, which is an upper bound under (44) and becomes a lower bound under (46). This bound can help us to understand what a column permutation strategy should try to achieve.

Theorem 4: Suppose that the assumptions in Theorem 1 hold. Let the box $\mathcal{B}$ in (1b) be a cube with edge length of $d$ and denote $\gamma=(\operatorname{det}(\boldsymbol{R}))^{1 / n}$.

1) If the condition (44) holds, then

$$
\begin{equation*}
\operatorname{Pr}\left(\boldsymbol{x}^{\mathrm{BB}}=\hat{\boldsymbol{x}}\right) \leq\left[\frac{1}{d+1}+\frac{d}{d+1} \operatorname{erf}\left(\frac{\gamma}{2 \sqrt{2} \sigma}\right)\right]^{n} \tag{48}
\end{equation*}
$$

2) If the condition (46) holds, then

$$
\begin{equation*}
\operatorname{Pr}\left(\boldsymbol{x}^{\mathrm{BB}}=\hat{\boldsymbol{x}}\right) \geq\left[\frac{1}{d+1}+\frac{d}{d+1} \operatorname{erf}\left(\frac{\gamma}{2 \sqrt{2} \sigma}\right)\right]^{n} \tag{49}
\end{equation*}
$$

The equality in either (48) or (49) holds if and only if $r_{i i}=\gamma$ for $i=1, \ldots, n$.

Proof. We prove only part 1. Part 2 can be proved similarly. Note that $\gamma^{n}=\prod_{i=1}^{n} r_{i i}$. Obviously, if $r_{i i}=\gamma$ for $i=1, \ldots, n$, then by (16) the equality in (48) holds. In the following we assume that there exist $j$ and $k$ such that $r_{j j} \neq r_{k k}$, we only need to show that the strict inequality (48) holds.

Denote $F(\zeta)=\ln \left(1+d \operatorname{erf}\left(\frac{\exp (\zeta)}{2 \sqrt{2} \sigma}\right)\right), \eta_{i}=\ln \left(r_{i i}\right)$ for $i=1,2, \ldots, n$ and $\eta=\frac{1}{n} \sum_{i=1}^{n} \eta_{i}$, then by (16), (48) is equivalent to

$$
\frac{1}{n} \sum_{i=1}^{n} F\left(\eta_{i}\right)<F(\eta)
$$

Since $\min _{1 \leq i \leq} r_{i i} \geq 2 \sqrt{2} \sigma r(d)$ and $r_{j j} \neq r_{k k}$, it suffices to show that $F(\zeta)$ is a strict concave function on $(\ln (2 \sqrt{2} \sigma r(d)),+\infty)$. Therefore, we only need to show that $F^{\prime \prime}(\zeta)<0$ when $\zeta>\ln (2 \sqrt{2} \sigma r(d))$.

To simplify notation, denote $\xi=\exp (\zeta) /(2 \sqrt{2} \sigma)$. By some simple calculations, we obtain

$$
F^{\prime}(\zeta)=\frac{d \xi \exp \left(-\xi^{2}\right)}{1+d \operatorname{erf}(\xi)}=d h(\xi, d)
$$

where $h(\cdot, \cdot)$ is defined in (32). Then

$$
F^{\prime \prime}(\zeta)=d \xi(\partial h(\xi, d) / \partial \xi)
$$

By the proof of Lemma 2, $\partial h(\xi, d) / \partial \xi<0$ when $\xi>$ $r(d)$. Thus, we can conclude that $F^{\prime \prime}(\zeta)<0$ when $\zeta>$ $\ln (2 \sqrt{2} \sigma r(d))$, completing the proof.

Now we make some remarks about Theorem 4.
Remark 7: The quantity $\gamma$ is invariant with respect to column permutations, i.e., for $\boldsymbol{R}$ and $\overline{\boldsymbol{R}}$ in (7), we have the same $\gamma$ no matter what the permutation matrix $\boldsymbol{P}$ is. Thus the bounds in (48) and (49), which are actually the same quantity, are invariant with respect to column permutations. Although the condition (44) is variant with respect to column permutations, if it holds before applying LLL-P or V-BLAST, it will hold afterwards, since the minimum of the diagonal entries of $\overline{\boldsymbol{R}}$ will not be smaller than that of $\boldsymbol{R}$ after applying LLL-P or V-BLAST. Similarly, the condition (46) is also variant with respect to column permutations. But if it holds before applying LLL-P or SQRD, it will hold afterwards, since the maximum of the diagonal entries of $\overline{\boldsymbol{R}}$ will not be larger than that of $\boldsymbol{R}$ after applying LLL-P or SQRD.

Remark 8: The equalities in (48) and (49) are reached if all the diagonal entries of $\boldsymbol{R}$ are identical. This suggests that if the gaps between the larger entries and small entries become smaller after permutations, it is likely that $P^{\mathrm{BB}}$ increases under the condition (44) or decreases under the condition (46). As we know, the gap between the largest one and the smallest one decreases after applying LLL-P. Numerical tests indicate usually this is also true for both V-BLAST and SQRD. Thus both V-BLAST and SQRD will likely bring $P^{\mathrm{BB}}$ closer to the bound under the two opposite conditions, respectively.

Remark 9: When $d \rightarrow \infty$, by Lemma 1, $r(d) \rightarrow 0$, thus the condition in part 1 of Theorem 4 becomes $\max _{1 \leq i \leq n} r_{i i} \geq$ 0 , which always holds. Taking the limit as $d \rightarrow \infty$ on both sides of (48) and using Corollary 1, we obtain

$$
\begin{equation*}
\operatorname{Pr}\left(\boldsymbol{x}^{\mathrm{OB}}=\hat{\boldsymbol{x}}\right) \leq(\operatorname{erf}(\gamma /(2 \sqrt{2} \sigma)))^{n} \tag{50}
\end{equation*}
$$

The above result was obtained in [37] and a simple proof was provided in [14].

## D. Numerical tests

We have shown that if (44) holds, then LLL-P increases $P^{\mathrm{BB}}$ and (48) is an upper bound on $P^{\mathrm{BB}}$; and if (46) holds,
then the LLL-P decreases $P^{\mathrm{BB}}$ and (49) is a lower bound on $P^{B B}$. Example 1 in Sec. IV-B indicates that this conclusion does not always hold for SQRD and V-BLAST. To further understand the effects of LLL-P, SQRD and V-BLAST on $P^{\mathrm{BB}}$ and to see how close they bring their corresponding $P^{\mathrm{BB}}$ to the bounds given by (48) and (49), we performed some numerical tests by Matlab. For comparisons, we also performed tests for $P^{\text {ов }}$.

First we performed tests for the following two cases:

- Case 1. $\boldsymbol{A}$ is an $n \times n$ matrix whose entries are chosen independently and randomly according to a zero mean Gaussian distribution with variance $1 / 2$.
- Case 2. $\boldsymbol{A}=\boldsymbol{U} \boldsymbol{D} \boldsymbol{V}^{T}, \boldsymbol{U}, \boldsymbol{V}$ are random orthogonal matrices obtained by the QR factorization of matrices whose entries are chosen independently and andomly according to the standard Gaussian distribution and $\boldsymbol{D}$ is an $n \times n$ diagonal matrix with $d_{i i}=10^{3(n / 2-i) /(n-1)}$. The condition number of $\boldsymbol{A}$ is 1000 .
In the tests for each case, we first chose $n=4$ and $\mathcal{B}=[0,1]^{4}$ and took different noise standard deviation $\sigma$ to test different situations according to the conditions (44) and (46) imposed in Theorems 3 and 4 . The edge length $d$ of $\mathcal{B}$ is 1 . So in (44) and (46), $2 \sqrt{2} r(d)=2 \sqrt{2} r(1)=$ 1.6798. Details about choosing $\sigma$ will be given later.

We use $P^{\mathrm{BB}}, P_{L}^{\mathrm{BB}}, P_{S}^{\mathrm{BB}}$ and $P_{V}^{\mathrm{BB}}$ respectively denote the success probability of the box-constrained Babai estimator corresponding to QR factorization (i.e., no permutations are involved), LLL-P, SQRD and V-BLAST, and use $\mu^{\text {BB }}$ to denote the right-hand side of (48) or (49), which is an upper bound if (44) holds and a lower bound if (46) holds. Similarly, $P^{\mathrm{OB}}, P_{L}^{\mathrm{OB}}, P_{S}^{\mathrm{OB}}$ and $P_{V}^{\mathrm{OB}}$ respectively denote the success probability of the ordinary Babai estimator corresponding to QR factorization, LLL-P, SQRD and VBLAST. We use $\mu^{\mathrm{OB}}$ to denote the right-hand side of (50), which is an upper bound on $P^{\mathrm{OB}}, P_{L}^{\mathrm{OB}}, P_{S}^{\mathrm{OB}}$ and $P_{V}^{\mathrm{OB}}$. For each case, we performed 10 runs (notice that for each run we have different $\boldsymbol{A}, \hat{\boldsymbol{x}}$ and $\boldsymbol{v}$ due to randomness) and the results are displayed in Tables I-VI.

In Tables I and II, $\sigma=\sigma_{1} \equiv \min _{1 \leq i \leq n} r_{i i} / 1.8$. It is easy to verify that the condition (44) holds. This means that $P^{\mathrm{BB}} \leq$ $P_{L}^{\mathrm{BB}}$ by Theorem 3 and $P^{\mathrm{BB}}, P_{L}^{\mathrm{BB}}, P_{V}^{\mathrm{BB}} \leq \mu^{\mathrm{BB}}$ by Theorem 4 and Remark 7. The numerical results given in Tables I and II are consistent with the theoretical results. The numerical results also indicate that SQRD and V-BLAST usually increase (not strictly) $P^{\mathrm{BB}}$, although there is one exceptional case for SQRD in Table II. We observe that the permutation strategies increase $P^{\mathrm{BB}}$ more significantly for Case 2 than for Case 1. The reason is that $\boldsymbol{A}$ is more ill-conditioned for Case 2, resulting in larger gaps between the diagonal entries of $\boldsymbol{R}$, which can usually be reduced more effectively by the permutation strategies. We also observe that $P_{S}^{\mathrm{BB}} \leq \mu^{\mathrm{BB}}$ in both tables. Although in theory the inequality may not hold as we cannot guarantee the condition (44) holds after applying SQRD, usually SQRD
can make $\min _{1 \leq i \leq n} r_{i i}$ larger. Thus if (44) holds before applying SQRD, it is likely that the condition still holds after applying it. Thus it is likely that $P_{S}^{\mathrm{BB}} \leq \mu^{\mathrm{BB}}$ holds.

Tables III and IV are opposite to Tables I and II. In both tables, $\sigma=\sigma_{2} \equiv \max _{1 \leq i \leq n} r_{i i} / 1.6$, then the condition (46) holds. This means that $P^{\mathrm{BB}} \geq P_{L}^{\mathrm{BB}}$ by Theorem 3 and $P^{\mathrm{BB}}, P_{L}^{\mathrm{BB}}, P_{S}^{\mathrm{BB}} \geq \mu^{\mathrm{BB}}$ by Theorem 4 and Remark 7. The numerical results given in the two tables are consistent with the theoretical results. The results in the two tables also indicate that both SQRD and V-BLAST decrease (not strictly) $P^{\mathrm{BB}}$, although Example 1 shows that neither is always true under the condition (46). We also observe that $P_{V}^{\mathrm{BB}} \geq \mu^{\mathrm{BB}}$ in both tables. Although in theory the inequality may not hold as we cannot guarantee the condition (46) holds after applying V-BLAST, usually V-BLAST can make $\max _{1 \leq i \leq n} r_{i i}$ smaller. Thus if (46) holds before applying V-BLAST, it is likely the condition still holds after applying it. Thus it is likely $P_{V}^{\mathrm{BB}} \geq \mu^{\mathrm{BB}}$ holds.

In Tables V and VI,

$$
\sigma=\sigma_{3} \equiv\left(0.3 \max _{1 \leq i \leq n} r_{i i}+0.7 \min _{1 \leq i \leq n} r_{i i}\right) / 1.68
$$

In this case,

$$
\min _{1 \leq i \leq n} r_{i i} \leq \frac{1.68}{1.6798} 2 \sqrt{2} \sigma r(d) \leq \max _{1 \leq i \leq n} r_{i i}
$$

indicating that (46) does not hold and it is very likely that (44) does not hold either. In theory we do not have results that cover this situation. The numerical results in the two tables indicate all of the three permutation strategies can either increase or decrease $P^{\mathrm{BB}}$ strictly and $\mu^{\mathrm{BB}}$ can be larger or smaller than $P^{\mathrm{BB}}, P_{L}^{\mathrm{BB}}, P_{S}^{\mathrm{BB}}$ and $P_{V}^{\mathrm{BB}}$. The reason we chose 0.3 and 0.7 rather than a more natural choice of 0.5 and 0.5 in defining $\sigma$ here is that we may not be able to observe both increasing and decreasing phenomena due to limited runs.

Now we make comments on the success probability of ordinary Babai points. From Tables I-VI, we observe that LLL-P always increases (not strictly) $P^{\mathrm{OB}}$, and SQRD and V-BLAST almost always increases $P^{\mathrm{oB}}$ (there is one exceptional case for SQRD in Table II and two exceptional cases for V-BLAST in Table VI). Thus the ordinary case is different from the box-constrained case. We also observe $P^{\mathrm{OB}} \leq P^{\mathrm{BB}}$ for the same permutation strategies. Sometimes the difference between the two is large (see Tables IV and VI).

Each of Tables I-VI displays the results for only 10 runs due to space limitation. To make up for this shortcoming, we give Tables VII and VIII, which display some statistics for 1000 runs on the data generated exactly the same way as the data for the 10 runs. Specifically these two tables display the number of runs, in which $P^{\mathrm{BB}}\left(P^{\mathrm{OB}}\right)$ strictly increases, keeps unchanged and strictly decreases after each of the three permutation strategies is applied for Case 1 and Case 2, respectively. In the two tables, $\sigma_{1}, \sigma_{2}$ and $\sigma_{3}$ are defined in the same as those used in Tables I-VI.

From Tables VII and VIII, we can see that often these permutation strategies increase or decrease $P^{\mathrm{BB}}$ for the same data. The numerical results given in all the tables suggest that if the condition (44) holds, we should have confidence to use any of these permutation strategies; and if the condition (46) holds we should not use any of them.

Tables VII and VIII do not show which permutation strategy increases $P^{\mathrm{BB}}$ most for small $\sigma$. The information on this given in Tables I-VI are limited. In the following we give more test results to investigate this.
As the main application of this research is in digit communications, we used the MIMO model in the new tests. For a fixed dimension, a fixed type of QAM and a fixed $E_{b} / N_{0}$, we randomly generated 200 complex channel matrices whose entries independently and identically follow the standard complex normal distribution, and for each generated channel matrix, we randomly generated 500 pairs of complex signal vector (whose entries are uniformly distributed according to the QAM constellation) and complex noise vector (whose entries are independently and identically normally distributed), resulting in 10000 instances of a complex linear model. Each complex instance was then transformed to an instance of the real linear model (1).

Unlike the previous tests, we compare the experimental error probabilities of the box-constrained Babai estimators (i.e., the ratio of the number of runs that the Babai point is not equal to the true parameter vector $\hat{\boldsymbol{x}}$ to 10000) corresponding to QR, LLL-P, SQRD and V-BLAST, and the theoretical bound on the error probability of a Babai estimator (i.e., the difference between 1 and the bound on its success probability (see (48))).
Figures 1 and 2 respectively display the experimental error probability corresponding to the QR factorization, and the three permutation strategies, and the average theoretical bound over the 10000 runs versus $E_{b} / N_{0}=5: 5: 30$ for the $4 \times 4 \mathrm{MIMO}$ system with $16-\mathrm{QAM}$ and $64-$ QAM. Similarly, Figures 3 and 4 respectively show the corresponding results for the $8 \times 8$ MIMO system with 16-QAM and 64-QAM. And Figures 5 and 6 show the corresponding results for the $16 \times 16$ MIMO system with 16-QAM and 64-QAM, respectively.
From Figures 1-6, we can see that on average all of the three column permutation strategies decrease the error probability of the Babai point and the error bound is a lower bound (this is because (44) usually holds, which ensures (48)). These Figures also show that the effect of VBLAST is much more significant than that of LLL-P and SQRD, which have more or less the same performance. This phenomenon is similar to that for $P^{\mathrm{OB}}$, as shown in [14].

## V. On the Conjecture proposed in [32]

In [32], a conjecture was made on the ordinary Babai estimator, based on which a stopping criterion was then

TABLE I
SUCCESS PROBABILITIES AND BOUNDS FOR CASE $1, \sigma=\min _{1 \leq i \leq n} r_{i i} / 1.8$

| $\sigma$ | $P^{\mathrm{BB}}$ | $P_{L}^{\mathrm{BB}}$ | $P_{S}^{\mathrm{BB}}$ | $P_{V}^{\mathrm{BB}}$ | $\mu^{\mathrm{BB}}$ | $P^{\mathrm{OB}}$ | $P_{L}^{\mathrm{OB}}$ | $P_{S}^{\mathrm{OB}}$ | $P_{V}^{\mathrm{OB}}$ | $\mu^{\mathrm{OB}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0738 | 0.8159 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.6319 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.1537 | 0.7632 | 0.8423 | 0.8423 | 0.8423 | 0.9083 | 0.5503 | 0.6988 | 0.6988 | 0.6988 | 0.8231 |
| 0.1575 | 0.7938 | 0.9491 | 0.9491 | 0.9491 | 0.9698 | 0.5977 | 0.8998 | 0.8998 | 0.8998 | 0.9403 |
| 0.2170 | 0.7235 | 0.8577 | 0.8577 | 0.8577 | 0.8670 | 0.4893 | 0.7300 | 0.7300 | 0.7300 | 0.7477 |
| 0.1285 | 0.8133 | 0.8534 | 0.8534 | 0.8521 | 0.9882 | 0.6278 | 0.7070 | 0.7070 | 0.7049 | 0.9766 |
| 0.1676 | 0.6809 | 0.7529 | 0.7529 | 0.7529 | 0.8896 | 0.4255 | 0.5375 | 0.5375 | 0.5375 | 0.7885 |
| 0.3665 | 0.7039 | 0.7273 | 0.7273 | 0.7273 | 0.8004 | 0.4629 | 0.5093 | 0.5093 | 0.5093 | 0.6324 |
| 0.1968 | 0.6892 | 0.7320 | 0.7320 | 0.7385 | 0.8073 | 0.4420 | 0.5103 | 0.5103 | 0.5270 | 0.6439 |
| 0.3322 | 0.7087 | 0.7317 | 0.7317 | 0.7317 | 0.7665 | 0.4718 | 0.5156 | 0.5156 | 0.5156 | 0.5765 |
| 0.5221 | 0.4754 | 0.4754 | 0.4754 | 0.4754 | 0.4758 | 0.1899 | 0.1899 | 0.1899 | 0.1899 | 0.1910 |

TABLE II
SUCCESS PROBABILITIES AND BOUNDS FOR CASE $2, \sigma=\min _{1 \leq i \leq n} r_{i i} / 1.8$

| $\sigma$ | $P^{\mathrm{BB}}$ | $P_{L}^{\mathrm{BB}}$ | $P_{S}^{\mathrm{BB}}$ | $P_{V}^{\mathrm{BB}}$ | $\mu^{\mathrm{BB}}$ | $P^{\mathrm{BB}}$ | $P_{L}^{\mathrm{OB}}$ | $P_{S}^{\mathrm{OB}}$ | $P_{V}^{\mathrm{OB}}$ | $\mu^{\mathrm{OB}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0101 | 0.8155 | 0.9452 | 0.9354 | 0.9452 | 1.0000 | 0.6312 | 0.8905 | 0.8708 | 0.8905 | 1.0000 |
| 0.0130 | 0.7983 | 0.9839 | 0.9839 | 0.9839 | 1.0000 | 0.6045 | 0.9679 | 0.9679 | 0.9679 | 1.0000 |
| 0.0173 | 0.8159 | 0.9793 | 0.9793 | 0.9793 | 1.0000 | 0.6319 | 0.9586 | 0.9586 | 0.9586 | 1.0000 |
| 0.0066 | 0.8159 | 0.9913 | 0.9913 | 0.9967 | 1.0000 | 0.6319 | 0.9826 | 0.9826 | 0.9933 | 1.0000 |
| 0.0177 | 0.8106 | 0.9998 | 0.9998 | 0.9998 | 1.0000 | 0.6236 | 0.9997 | 0.9997 | 0.9997 | 1.0000 |
| 0.0060 | 0.8159 | 0.9841 | 0.9841 | 0.9998 | 1.0000 | 0.6319 | 0.9681 | 0.9681 | 0.9996 | 1.0000 |
| 0.0168 | 0.7833 | 0.8098 | 0.7625 | 0.8159 | 1.0000 | 0.5813 | 0.6224 | 0.5250 | 0.6319 | 1.0000 |
| 0.0150 | 0.8159 | 0.9999 | 0.9999 | 0.9999 | 1.0000 | 0.6319 | 0.9998 | 0.9998 | 0.9998 | 1.0000 |
| 0.0231 | 0.8159 | 0.9999 | 0.9999 | 0.9999 | 1.0000 | 0.6319 | 0.9999 | 0.9999 | 0.9999 | 1.0000 |
| 0.0211 | 0.7912 | 0.9696 | 0.9696 | 0.9892 | 1.0000 | 0.5935 | 0.9393 | 0.9393 | 0.9784 | 1.0000 |

TABLE III
SUCCESS PROBABILITIES AND BOUNDS FOR CASE $1, \sigma=\max \left(r_{i i}\right) / 1.6$

| $\sigma$ | $P^{\mathrm{BB}}$ | $P_{L}^{\mathrm{BB}}$ | $P_{S}^{\mathrm{BB}}$ | $P_{V}^{\mathrm{BB}}$ | $\mu^{\mathrm{BB}}$ | $P^{\mathrm{OB}}$ | $P_{L}^{\mathrm{OB}}$ | $P_{S}^{\mathrm{OB}}$ | $P_{V}^{\mathrm{OB}}$ | $\mu^{\mathrm{OB}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1.1726 | 0.1557 | 0.1310 | 0.1310 | 0.1380 | 0.1121 | 0.0005 | 0.0006 | 0.0006 | 0.0006 | 0.0006 |
| 0.6432 | 0.2756 | 0.2756 | 0.2756 | 0.2756 | 0.2731 | 0.0387 | 0.0387 | 0.0387 | 0.0387 | 0.0395 |
| 0.5962 | 0.2915 | 0.2912 | 0.2912 | 0.2909 | 0.2900 | 0.0472 | 0.0473 | 0.0473 | 0.0475 | 0.0478 |
| 1.2435 | 0.1875 | 0.1632 | 0.1673 | 0.1632 | 0.1571 | 0.0040 | 0.0044 | 0.0044 | 0.0044 | 0.0045 |
| 0.8332 | 0.1873 | 0.1769 | 0.1769 | 0.1769 | 0.1750 | 0.0070 | 0.0074 | 0.0074 | 0.0074 | 0.0074 |
| 0.4875 | 0.2709 | 0.2709 | 0.2709 | 0.2709 | 0.2667 | 0.0356 | 0.0356 | 0.0356 | 0.0356 | 0.0366 |
| 0.9684 | 0.2769 | 0.2709 | 0.2709 | 0.2709 | 0.2688 | 0.0358 | 0.0369 | 0.0369 | 0.0369 | 0.0375 |
| 0.9971 | 0.1846 | 0.1665 | 0.1665 | 0.1665 | 0.1588 | 0.0043 | 0.0046 | 0.0046 | 0.0046 | 0.0047 |
| 1.2791 | 0.1501 | 0.1308 | 0.1308 | 0.1308 | 0.1294 | 0.0015 | 0.0016 | 0.0016 | 0.0016 | 0.0016 |
| 0.6327 | 0.2641 | 0.2564 | 0.2564 | 0.2564 | 0.2556 | 0.0301 | 0.0316 | 0.0316 | 0.0316 | 0.0318 |

TABLE IV
SUCCESS PROBABILITIES AND BOUNDS FOR CASE $2, \sigma=\max \left(r_{i i}\right) / 1.6$

| $\sigma$ | $P^{\mathrm{BB}}$ | $P_{L}^{\mathrm{BB}}$ | $P_{S}^{\mathrm{BB}}$ | $P_{V}^{\mathrm{BB}}$ | $\mu^{\mathrm{BB}}$ | $P^{\mathrm{BB}}$ | $P_{L}^{\mathrm{OB}}$ | $P_{S}^{\mathrm{OB}}$ | $P_{V}^{\mathrm{OB}}$ | $\mu^{\mathrm{OB}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3.9438 | 0.1064 | 0.0947 | 0.0987 | 0.0947 | 0.0709 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.4510 | 0.1173 | 0.1173 | 0.1173 | 0.1173 | 0.0764 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.5790 | 0.1788 | 0.1640 | 0.1640 | 0.1640 | 0.1363 | 0.0019 | 0.0019 | 0.0019 | 0.0019 | 0.0021 |
| 5.3809 | 0.1011 | 0.0701 | 0.0701 | 0.0701 | 0.0686 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.2574 | 0.1140 | 0.1023 | 0.0954 | 0.0954 | 0.0777 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 3.7623 | 0.1099 | 0.0801 | 0.0801 | 0.0757 | 0.0713 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 3.9225 | 0.1063 | 0.0834 | 0.0834 | 0.0834 | 0.0709 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.3198 | 0.1153 | 0.1153 | 0.1153 | 0.1153 | 0.0900 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| 1.2416 | 0.1394 | 0.1108 | 0.1108 | 0.1108 | 0.0920 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| 0.8411 | 0.1719 | 0.1532 | 0.1532 | 0.1532 | 0.1090 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0005 |

proposed for the sphere decoding search process for solving the BILS problem (2). In this section, we first introduce this conjecture, then give an example to show that this conjecture may not hold in general, and finally we show that the conjecture holds under some conditions.
parameter vector $\hat{\boldsymbol{x}}$ in the box-constrained linear model (1). The method proposed in [32] first ignores the box constraint (1b). Instead of using the column permutations in (7), it performs the LLL reduction:

$$
\begin{equation*}
\overline{\boldsymbol{Q}}^{T} \boldsymbol{R} \boldsymbol{Z}=\overline{\boldsymbol{R}} \tag{51}
\end{equation*}
$$

TABLE V
SUCCESS PROBABILITIES AND BOUNDS FOR CASE $1, \sigma=\left(0.3 \max \left(r_{i i}\right)+0.7 \min _{1 \leq i \leq n} r_{i i}\right) / 1.68$

| $\sigma$ | $P^{\mathrm{BB}}$ | $P_{L}^{\mathrm{BB}}$ | $P_{S}^{\mathrm{BB}}$ | $P_{V}^{\mathrm{BB}}$ | $\mu^{\mathrm{BB}}$ | $P^{\mathrm{OB}}$ | $P_{L}^{\mathrm{OB}}$ | $P_{S}^{\mathrm{OB}}$ | $P_{V}^{\mathrm{OB}}$ | $\mu^{\mathrm{OB}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.2848 | 0.4208 | 0.4336 | 0.4336 | 0.3184 | 0.2846 | 0.0154 | 0.0165 | 0.0165 | 0.0252 | 0.0451 |
| 0.6313 | 0.4720 | 0.4829 | 0.4829 | 0.4829 | 0.4863 | 0.1630 | 0.1932 | 0.1932 | 0.1932 | 0.2017 |
| 0.4328 | 0.4540 | 0.4599 | 0.4599 | 0.4599 | 0.4623 | 0.1517 | 0.1673 | 0.1673 | 0.1673 | 0.1776 |
| 0.6105 | 0.5054 | 0.5061 | 0.5061 | 0.5061 | 0.5092 | 0.2123 | 0.2161 | 0.2161 | 0.2161 | 0.2259 |
| 0.3306 | 0.4268 | 0.3807 | 0.3807 | 0.3805 | 0.3484 | 0.0321 | 0.0539 | 0.0539 | 0.0539 | 0.0829 |
| 0.2600 | 0.5055 | 0.5103 | 0.5103 | 0.5103 | 0.5252 | 0.1544 | 0.1868 | 0.1868 | 0.1868 | 0.2437 |
| 0.4743 | 0.4235 | 0.4283 | 0.4283 | 0.4283 | 0.4259 | 0.0631 | 0.1225 | 0.1225 | 0.1225 | 0.1437 |
| 0.5878 | 0.4104 | 0.4161 | 0.4161 | 0.4161 | 0.4170 | 0.1159 | 0.1304 | 0.1304 | 0.1304 | 0.1359 |
| 0.3977 | 0.4429 | 0.4431 | 0.4431 | 0.4431 | 0.4477 | 0.1477 | 0.1479 | 0.1479 | 0.1479 | 0.1636 |
| 0.6273 | 0.4684 | 0.4684 | 0.4684 | 0.4684 | 0.4696 | 0.1792 | 0.1792 | 0.1792 | 0.1792 | 0.1848 |

TABLE VI
SUCCESS PROBABILITIES AND BOUNDS FOR CASE $2, \sigma=\left(0.3 \max \left(r_{i i}\right)+0.7 \min _{1 \leq i \leq n} r_{i i}\right) / 1.68$

| $\sigma$ | $P^{\mathrm{BB}}$ | $P_{L}^{\mathrm{BB}}$ | $P_{S}^{\mathrm{BB}}$ | $P_{V}^{\mathrm{BB}}$ | $\mu^{\mathrm{BB}}$ | $P^{\mathrm{OB}}$ | $P_{L}^{\mathrm{OB}}$ | $P_{S}^{\mathrm{OB}}$ | $P_{V}^{\mathrm{OB}}$ | $\mu^{\mathrm{OB}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1.0377 | 0.1608 | 0.1324 | 0.1324 | 0.1625 | 0.0987 | 0.0001 | 0.0002 | 0.0002 | 0.0002 | 0.0002 |
| 0.3648 | 0.2774 | 0.2774 | 0.2774 | 0.2405 | 0.1987 | 0.0034 | 0.0034 | 0.0034 | 0.0025 | 0.0126 |
| 0.7603 | 0.1681 | 0.1758 | 0.1758 | 0.1758 | 0.1150 | 0.0003 | 0.0005 | 0.0005 | 0.0005 | 0.0007 |
| 0.8769 | 0.1835 | 0.2062 | 0.1713 | 0.2062 | 0.1067 | 0.0002 | 0.0003 | 0.0004 | 0.0003 | 0.0004 |
| 0.4708 | 0.2794 | 0.2352 | 0.2352 | 0.2352 | 0.1590 | 0.0010 | 0.0030 | 0.0030 | 0.0030 | 0.0048 |
| 1.1983 | 0.1572 | 0.1319 | 0.1319 | 0.1319 | 0.0932 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| 1.0001 | 0.1758 | 0.1596 | 0.1596 | 0.1464 | 0.1003 | 0.0001 | 0.0002 | 0.0002 | 0.0001 | 0.0002 |
| 0.8523 | 0.1671 | 0.1733 | 0.1733 | 0.1715 | 0.1082 | 0.0002 | 0.0003 | 0.0003 | 0.0003 | 0.0005 |
| 0.2128 | 0.3478 | 0.3478 | 0.3728 | 0.3478 | 0.3539 | 0.0599 | 0.0599 | 0.0711 | 0.0599 | 0.0866 |
| 0.3956 | 0.2188 | 0.2117 | 0.2117 | 0.1973 | 0.1844 | 0.0047 | 0.0047 | 0.0047 | 0.0034 | 0.0093 |

TABLE VII
Number of runs out of 1000 in which $P^{\text {bB }}$ and $P^{\text {Ob }}$ Changes for Case 1

|  |  | $P^{\mathrm{BB}}$ |  |  | $P^{\text {OB }}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\sigma$ | Sesult | LLLategy | SQRD | V-BLAST | LLL-P | SQRD | V-BLAST |
|  | Strict increase | No change | 933 | 928 | 951 | 933 | 922 |
|  | Strict decrease | 07 | 47 | 42 | 67 | 47 | 42 |
|  | Strict increase | 0 | 25 | 7 | 0 | 31 | 5 |
| $\sigma_{2}$ | No change | 58 | 40 | 6 | 942 | 947 | 950 |
|  | Strict decrease | 942 | 935 | 97 | 58 | 40 | 37 |
| $\sigma_{3}$ | Strict increase | 781 | 797 | 740 | 0 | 13 | 13 |
|  | No change | 58 | 40 | 37 | 58 | 945 | 952 |
|  | Strict decrease | 161 | 163 | 223 | 0 | 40 | 37 |
|  |  |  |  |  |  | 11 |  |

TABLE VIII
Number of runs out of 1000 in which $P^{\text {BB }}$ and $P^{\text {OB }}$ CHANGES For Case 2

|  |  | $P^{\mathrm{BB}}$ |  |  | $P^{\text {OB }}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\sigma$ | Result | LLL-P | SQRD | V-BLAST | LLL-P | SQRD | V-BLAST |
|  | Strict increase | No change | 858 | 803 | 938 | 858 | 800 |
|  | Strict decrease | 0 | 76 | 56 | 142 | 76 | 538 |
|  | Strict increase | 0 | 121 | 6 | 0 | 124 | 6 |
| $\sigma_{2}$ | No change | 94 | 46 | 69 | 906 | 944 | 831 |
|  | Strict decrease | 906 | 931 | 883 | 94 | 46 | 48 |
| $\sigma_{3}$ | Strict increase | 134 | 189 | 97 | 906 | 10 | 121 |
|  | No change | 94 | 46 | 48 | 94 | 46 | 840 |
|  | Strict decrease | 772 | 765 | 855 | 0 | 11 | 48 |



Fig. 1. Error probability of Babai point and bound versus $E_{b} / N_{0}=$ $5: 5: 30$ for the $4 \times 4$ MIMO system with $16-$ QAM


Fig. 2. Error probability of Babai point and bound versus $E_{b} / N_{0}=$ $5: 5: 30$ for the $4 \times 4$ MIMO system with 64 -QAM


Fig. 3. Error probability of Babai point and bound versus $E_{b} / N_{0}=$ $5: 5: 30$ for the $8 \times 8$ MIMO system with 16 -QAM


Fig. 4. Error probability of Babai point and bound versus $E_{b} / N_{0}=$ $5: 5: 30$ for the $8 \times 8$ MIMO system with 64-QAM


Fig. 5. Error probability of Babai point and bound versus $E_{b} / N_{0}=$ $5: 5: 30$ for the $16 \times 16$ MIMO system with 16 -QAM


Fig. 6. Error probability of Babai point and bound versus $E_{b} / N_{0}=$ 5:5:30 for the $16 \times 16$ MIMO system with 64-QAM
and $\operatorname{det}(\boldsymbol{Z})= \pm 1$ ) and the upper triangular $\overline{\boldsymbol{R}}$ is LLL reduced, i.e., it satisfies the Lovász condition (8) and the size-reduce condition:
$\left|\bar{r}_{i k}\right| \leq \frac{1}{2} \bar{r}_{i i}, \quad k=i+1, i+2, \ldots, n, \quad i=1,2, \ldots, n-1$.
Then, with $\overline{\boldsymbol{y}}=\overline{\boldsymbol{Q}}^{T} \tilde{\boldsymbol{y}}, \overline{\boldsymbol{v}}=\overline{\boldsymbol{Q}}^{T} \tilde{\boldsymbol{v}}$ and $\hat{\boldsymbol{z}}=\boldsymbol{Z}^{-1} \hat{\boldsymbol{x}}$, the ordinary linear model (5a) becomes

$$
\begin{equation*}
\overline{\boldsymbol{y}}=\overline{\boldsymbol{R}} \hat{\boldsymbol{z}}+\overline{\boldsymbol{v}}, \quad \overline{\boldsymbol{v}} \sim \mathcal{N}\left(\mathbf{0}, \sigma^{2} \boldsymbol{I}\right) \tag{52}
\end{equation*}
$$

For the reduced model, one can find its ordinary Babai estimator $\boldsymbol{z}^{\text {ОВ }}$ (c.f. (15)):

$$
\begin{equation*}
c_{i}^{\mathrm{OB}}=\left(\bar{y}_{i}-\sum_{j=i+1}^{n} \bar{r}_{i j} z_{j}^{\mathrm{OB}}\right) / \bar{r}_{i i}, \quad z_{i}^{\mathrm{OB}}=\left\lfloor c_{i}^{\mathrm{OB}}\right\rceil \tag{53}
\end{equation*}
$$

for $i=n, n-1, \ldots, 1$, where $\sum_{j=n+1}^{n}=0$. Define $\overline{\boldsymbol{x}}=\boldsymbol{Z} \boldsymbol{z}^{\mathrm{OB}}$. In [32], $\overline{\boldsymbol{x}}$ is used to estimate the true parameter vector $\hat{\boldsymbol{x}}$. If $\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}}$, then a vector error (VE) is said to have occurred. Note that $\overline{\boldsymbol{x}}$ may be outside the constraint box $\mathcal{B}$ in (1b). If $\overline{\boldsymbol{x}} \in \mathcal{B}$, then $\overline{\boldsymbol{x}}$ is called a valid vector, otherwise, i.e., $\overline{\boldsymbol{x}} \notin \mathcal{B}, \overline{\boldsymbol{x}}$ is called an invalid vector. The conjecture proposed in [32] is: a VE is most likely to occur if $\overline{\boldsymbol{x}}$ is invalid; conversely, if $\overline{\boldsymbol{x}}$ is valid, there is little chance that the vector is in error.

From the definition of VE, if $\overline{\boldsymbol{x}}$ is invalid, then VE must occur. So in the following, we will only consider the second part of the conjecture, i.e., $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B}) \approx 0$.

## A. The conjecture does not always hold

In this subsection, we first show that $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B})$ can be very close to 1 , then give a specific example to show $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B}) \geq 0.9275$ and finally perform some Matlab simulations to illustrate this example.

Theorem 5: For any given $\epsilon>0$, any fixed dimension $n \geq 2$, any box $\mathcal{B}$ and any standard deviation $\sigma$ of the noise vector $\boldsymbol{v}$, there always exists a box-constrained linear model in the form of (5), where $\hat{\boldsymbol{x}}$ is uniformly distributed over the box $\mathcal{B}$, such that

$$
\begin{equation*}
\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B}) \geq 1-\frac{1}{u_{1}-\ell_{1}+1}-\epsilon \tag{54}
\end{equation*}
$$

Proof. Note that

$$
\begin{align*}
\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B}) & =\frac{\operatorname{Pr}(\overline{\boldsymbol{x}} \in \mathcal{B})-\operatorname{Pr}(\overline{\boldsymbol{x}}=\hat{\boldsymbol{x}}, \overline{\boldsymbol{x}} \in \mathcal{B})}{\operatorname{Pr}(\overline{\boldsymbol{x}} \in \mathcal{B})} \\
& =1-\frac{\operatorname{Pr}(\overline{\boldsymbol{x}}=\hat{\boldsymbol{x}})}{\operatorname{Pr}(\overline{\boldsymbol{x}} \in \mathcal{B})} \tag{55}
\end{align*}
$$

where the second equality is due to the fact that $\hat{\boldsymbol{x}} \in \mathcal{B}$. Thus, to prove the theorem, it suffices to show that for any given $\epsilon>0$ there exists a box-constrained linear model such that

$$
\begin{equation*}
\frac{\operatorname{Pr}(\overline{\boldsymbol{x}}=\hat{\boldsymbol{x}})}{\operatorname{Pr}(\overline{\boldsymbol{x}} \in \mathcal{B})} \leq \frac{1}{u_{1}-\ell_{1}+1}+\epsilon \tag{56}
\end{equation*}
$$

For any fixed $\sigma$ and $\mathcal{B}$, to construct the linear model, we need only to construct a matrix $\boldsymbol{R} \in \mathbb{R}^{m \times n}$. Define

$$
\boldsymbol{R}=\left[\begin{array}{cc}
r_{11} & 0.5 r_{11} \boldsymbol{e}^{T} \\
\mathbf{0} & r_{22} \boldsymbol{I}_{n-1, n-1}
\end{array}\right], \quad 0<r_{11} \leq r_{22}
$$

where $\boldsymbol{e}=[1, \ldots, 1]^{T} \in \mathbb{R}^{n-1}$. We will show how to choose $r_{11}$ such that (56) holds.

Note that $\boldsymbol{R}$ is already LLL reduced, thus, $\overline{\boldsymbol{x}}=\boldsymbol{z}^{\mathrm{OB}}=$ $\boldsymbol{x}^{\mathrm{OB}}$ and $\hat{\boldsymbol{x}}=\hat{\boldsymbol{z}}$. Then, by (17) and (18),

$$
\begin{equation*}
\operatorname{Pr}(\overline{\boldsymbol{x}}=\hat{\boldsymbol{x}})=\phi_{\sigma}\left(r_{11}\right)\left(\phi_{\sigma}\left(r_{22}\right)\right)^{n-1} \tag{57}
\end{equation*}
$$

Obviously, with event $E_{2} \equiv\left(x_{2}^{\mathrm{OB}}=\hat{x}_{2}, \ldots, x_{n}^{\mathrm{OB}}=\hat{x}_{n}\right)$,

$$
\begin{align*}
\operatorname{Pr}(\overline{\boldsymbol{x}} \in \mathcal{B}) & =\operatorname{Pr}\left(\boldsymbol{x}^{\mathrm{OB}} \in \mathcal{B}\right) \geq \operatorname{Pr}\left(x_{1}^{\mathrm{OB}} \in\left[\ell_{1}, u_{1}\right], E_{2}\right) \\
& =\operatorname{Pr}\left(x_{1}^{\mathrm{OB}} \in\left[\ell_{1}, u_{1}\right] \mid E_{2}\right) \cdot \operatorname{Pr}\left(E_{2}\right) \\
& =\operatorname{Pr}\left(x_{1}^{\mathrm{OB}} \in\left[\ell_{1}, u_{1}\right] \mid E_{2}\right)\left(\phi_{\sigma}\left(r_{22}\right)\right)^{n-1} \tag{58}
\end{align*}
$$

where the last equality follows from (17) and (18). Therefore, by (57) and (58), to show (56) it suffices to show that there exists an $r_{11}>0$ such that

$$
\begin{equation*}
\frac{\phi_{\sigma}\left(r_{11}\right)}{\operatorname{Pr}\left(x_{1}^{\mathrm{OB}} \in\left[\ell_{1}, u_{1}\right] \mid E_{2}\right)} \leq \frac{1}{u_{1}-\ell_{1}+1}+\epsilon \tag{59}
\end{equation*}
$$

In the following we derive an expression for $\operatorname{Pr}\left(x_{1}^{\mathrm{OB}} \in\right.$ $\left[\ell_{1}, u_{1}\right] \mid E_{2}$ ) and then use it to show that (59) holds for some $r_{11}>0$. From the proof of Theorem 1, we see that if $x_{i}^{\mathrm{OB}}=\hat{x}_{i}$ for $i=n, n-1, \ldots, 2$ and $\hat{x}_{1}$ is fixed, then (cf. (20))

$$
\begin{equation*}
\frac{\left(c_{1}^{\mathrm{OB}}-\hat{x}_{1}\right) r_{11}}{\sqrt{2} \sigma} \sim \mathcal{N}\left(0, \frac{1}{2}\right) \tag{60}
\end{equation*}
$$

Since $x_{1}^{\mathrm{OB}}=\left\lfloor c_{1}^{\mathrm{OB}}\right\rceil$ and $\hat{\boldsymbol{x}}$ is uniformly distributed over the box $\mathcal{B}$,

$$
\begin{aligned}
& \operatorname{Pr}\left(x_{1}^{\mathrm{OB}} \in\left[\ell_{1}, u_{1}\right] \mid E_{2}\right) \\
= & \sum_{i=0}^{u_{1}-\ell_{1}} \operatorname{Pr}\left(\hat{x}_{1}=\ell_{1}+i, x_{1}^{\mathrm{OB}} \in\left[\ell_{1}, u_{1}\right] \mid E_{2}\right) \\
= & \sum_{i=0}^{u_{1}-\ell_{1}} \operatorname{Pr}\left(\hat{x}_{1}=\ell_{1}+i\right) \\
= & \frac{\operatorname{Pr}\left(x_{1}^{\mathrm{OB}} \in\left[\ell_{1}, u_{1}\right] \mid\left(\hat{x}_{1}=\ell_{1}+i, E_{2}\right)\right)}{u_{1}-\ell_{1}+1} \times \\
& \sum_{i=0}^{u_{1}-\ell_{1}} \operatorname{Pr}\left(c_{1}^{\mathrm{OB}} \in\left[\ell_{1}-1 / 2, u_{1}+1 / 2\right] \mid\left(\hat{x}_{1}=\ell_{1}+i, E_{2}\right)\right) \\
= & \frac{1}{u_{1}-\ell_{1}+1} \sum_{i=0}^{u_{1}-\ell_{1}} \operatorname{Pr}\left(\frac{\left(c_{1}^{\mathrm{OB}}-\hat{x}_{1}\right) r_{11}}{\sqrt{2} \sigma}\right. \\
= & \frac{1}{2\left(u_{1}-\ell_{1}+1\right)} \sum_{i=0}^{2 \sqrt{2} \sigma}\left[\phi_{\sigma}\left(\left(2 u_{1}-2 \ell_{1}-2 i+1\right) r_{11}\right)\right. \\
& \left.+\phi_{\sigma}\left((2 i+1) r_{11}\right)\right]
\end{aligned}
$$

where the second equality follows from (23), and the last equality is due to (18) and (60).

It is easy to verify by L'Hôpital's rule that

$$
\lim _{r_{11} \rightarrow 0} \frac{\phi_{\sigma}\left(r_{11}\right)}{\alpha}=\frac{1}{u_{1}-\ell_{1}+1}
$$

where

$$
\begin{aligned}
& \alpha=\frac{1}{2\left(u_{1}-\ell_{1}+1\right)} \sum_{i=0}^{u_{1}-\ell_{1}}\left[\phi_{\sigma}\left(\left(2 u_{1}-2 \ell_{1}-2 i+1\right) r_{11}\right)\right. \\
& \left.\quad+\phi_{\sigma}\left((2 i+1) r_{11}\right)\right]
\end{aligned}
$$

Therefore, for any $\epsilon>0$, there exists an $r_{11}>0$ such that (59) holds, completing the proof.

As $u_{1}-\ell_{1}+1$ is at least 2 , Theorem 5 shows that $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B})$ can be at least $1 / 2-\epsilon$. Note that $u_{1}-\ell_{1}+1$ can be arbitrarily large and we can choose $\ell_{1}$, $u_{1}$ and $r_{11}$ such that $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B})$ can be arbitrarily close to 1 . In the following, we give a specific example to show that $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B}) \geq 0.9275$ and give some simulation results.

Example 2: For any fixed $n$ and $\sigma$, let $\epsilon=0.01$ and $\mathcal{B}=[0,15]^{n}$, and define

$$
\boldsymbol{R}=\left[\begin{array}{cc}
0.04 \sigma & 0.02 \sigma \boldsymbol{e}^{T}  \tag{61}\\
\mathbf{0} & 10 \sigma \boldsymbol{I}_{n-1, n-1}
\end{array}\right]
$$

It is easy to verify that this matrix $\boldsymbol{R}$ satisfies the conditions given in the proof of Theorem 5. Then by (54), we have $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B}) \geq 0.9275$.

We use MATLAB to do some simulations to illustrate the probability. In the simulations, for any fixed $n$ and $\sigma$, we generated an $n \times n$ matrix $\boldsymbol{R}$ by using (61). After fixing $\boldsymbol{R}$, we gave 10000 runs to generate 10000 pairs of $\hat{\boldsymbol{x}}$ and $\tilde{\boldsymbol{v}}$ according to their distributions, producing $10000 \tilde{\boldsymbol{y}}$ 's according to (5a). For each $\tilde{\boldsymbol{y}}$, we found the Babai point $\boldsymbol{x}^{\text {OB }}$ by using (15). For each pair of $\boldsymbol{R}$ and $\sigma$, we computed the theoretical probability $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}})$ denoted by $P_{\text {err }}$ by using (57) (notice that $P_{\text {err }}=1-P^{\text {ов }}$ since $\overline{\boldsymbol{x}}=\boldsymbol{x}^{\text {OB }}$ here) and the corresponding experimental probability $P_{e x}$ (i.e., the ratio of the number of runs in which $\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}}$ to 10000). We also computed the experimental probability $P_{b}$ of $\overline{\boldsymbol{x}} \in$ $\mathcal{B}$ (i.e., the ratio of the number of runs in which $\overline{\boldsymbol{x}} \in \mathcal{B}$ to 10000) and the experimental probability $P_{e}$ corresponding to $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B})$, i.e., $P_{e}=1-\left(1-P_{e x}\right) / P_{b}$ (cf. (55)).

Tables IX and X respectively display those probabilities versus $n=5: 5: 40$ with $\sigma=0.1$ and versus $\sigma=0.1$ : $0.1: 0.8$ with $n=20$. From these two tables, we can see that the values of $P_{e}$ are larger than 0.9275 except the case that $n=40$ in Table IX, in which $P_{e}$ is smaller than 0.9275 , but it is close to the latter. Thus the test results are consistent with the theoretical result. We also observe that $P_{t h}$ is very small and $P_{e x}$ is a good approximation to $P_{t h}$. In Tables IX the values of $P_{e r r}$ are actually different, but very close because $\phi_{\sigma}\left(r_{22}\right)$ is very close to 1 (c.f. (57)) and in Tables X the values of $P_{\text {err }}$ are exactly equal
because in (57) $\phi_{\sigma}\left(r_{11}\right)$ and $\phi_{\sigma}\left(r_{22}\right)$ are independent of $\sigma$. This experiment confirms that even if $\overline{\boldsymbol{x}}$ is valid, there may be a large chance that it is in error.

TABLE IX
Probabilities versus $n=5: 5: 40$ with $\sigma=0.1$

| $\boldsymbol{n}$ | $P_{e r r}$ | $P_{e x}$ | $P_{b}$ | $P_{e}$ |
| :---: | :---: | :---: | :---: | :---: |
| 5 | 0.9840 | 0.9840 | 0.2484 | 0.9356 |
| 10 | 0.9840 | 0.9850 | 0.2485 | 0.9396 |
| 15 | 0.9840 | 0.9836 | 0.2469 | 0.9336 |
| 20 | 0.9840 | 0.9837 | 0.2499 | 0.9348 |
| 25 | 0.9840 | 0.9827 | 0.2564 | 0.9325 |
| 30 | 0.9840 | 0.9828 | 0.2500 | 0.9312 |
| 35 | 0.9840 | 0.9828 | 0.2473 | 0.9304 |
| 40 | 0.9840 | 0.9818 | 0.2434 | 0.9252 |

TABLE X
Probabilities versus $\sigma=0.1: 0.1: 0.8$ With $n=20$

| $\sigma$ | $P_{e r r}$ | $P_{e x}$ | $P_{b}$ | $P_{e}$ |
| :---: | :---: | :---: | :---: | :---: |
| 0.1 | 0.9840 | 0.9841 | 0.2503 | 0.9365 |
| 0.2 | 0.9840 | 0.9832 | 0.2522 | 0.9334 |
| 0.3 | 0.9840 | 0.9844 | 0.2434 | 0.9359 |
| 0.4 | 0.9840 | 0.9844 | 0.2399 | 0.9350 |
| 0.5 | 0.9840 | 0.9825 | 0.2435 | 0.9281 |
| 0.6 | 0.9840 | 0.9827 | 0.2475 | 0.9301 |
| 0.7 | 0.9840 | 0.9843 | 0.2541 | 0.9382 |
| 0.8 | 0.9840 | 0.9838 | 0.2517 | 0.9356 |

## B. The conjecture holds under some conditions

In this subsection, we will show that the conjecture holds under some conditions.

Recall that $\overline{\boldsymbol{x}}=\boldsymbol{Z} \boldsymbol{z}^{\mathrm{OB}}$ and $\hat{\boldsymbol{x}}=\boldsymbol{Z} \hat{\boldsymbol{z}}$, thus $\operatorname{Pr}(\overline{\boldsymbol{x}}=\hat{\boldsymbol{x}})=$ $\operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{OB}}=\hat{\boldsymbol{z}}\right)$. Then, by (55), we have

$$
\begin{equation*}
\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B}) \leq 1-\operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{OB}}=\hat{\boldsymbol{z}}\right) \tag{62}
\end{equation*}
$$

So, if $\operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{OB}}=\hat{\boldsymbol{z}}\right) \approx 1$, then the conjecture holds. From Corollary 2 we see that when $\sigma$ is small enough, we have $\operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{OB}}=\hat{\boldsymbol{z}}\right) \approx 1$. But the upper bound given in (62) is not sharp because it was derived from (55) by using the inequality $\operatorname{Pr}(\overline{\boldsymbol{x}} \in \mathcal{B}) \leq 1$. We will give a sharper upper bound on $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B})$ based on a sharper upper bound on $\operatorname{Pr}(\overline{\boldsymbol{x}} \in \mathcal{B})$.
Since $\overline{\boldsymbol{x}}=\boldsymbol{Z} \boldsymbol{z}^{\mathrm{OB}}, \overline{\boldsymbol{x}} \in \mathcal{B}$ if and only if $\boldsymbol{z}^{\mathrm{OB}} \in \mathcal{E} \equiv$ $\left\{\boldsymbol{Z}^{-1} \boldsymbol{s} \mid \forall \boldsymbol{s} \in \mathcal{B}\right\}$. Thus $\operatorname{Pr}(\overline{\boldsymbol{x}} \in \mathcal{B})=\operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{OB}} \in \mathcal{E}\right)$. But the set $\mathcal{E}$ is a parallelotope and it is difficult to analyze $\operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{OB}} \in \mathcal{E}\right)$. Thus in the following we will give a box $\mathcal{F}$ which contains $\mathcal{E}$, then we analyze $\operatorname{Pr}\left(\boldsymbol{z}^{\text {OB }} \in \mathcal{F}\right)$. Let $\boldsymbol{U}=\left(u_{i j}\right)=\boldsymbol{Z}^{-1}$ and define for $i, j=1,2, \ldots, n$,

$$
\mu_{i j}=\left\{\begin{array}{ll}
\ell_{j}, & \text { if } u_{i j} \geq 0 \\
u_{j}, & \text { if } u_{i j}<0
\end{array}, \quad \nu_{i j}= \begin{cases}u_{j}, & \text { if } u_{i j} \geq 0 \\
\ell_{j}, & \text { if } u_{i j}<0\end{cases}\right.
$$

Then define $\bar{\ell} \in \mathbb{Z}^{n}$ and $\overline{\boldsymbol{u}} \in \mathbb{Z}^{n}$ as follows:

$$
\begin{equation*}
\bar{\ell}_{i}=\sum_{j=1}^{n} u_{i j} \mu_{i j}, \quad \bar{u}_{i}=\sum_{j=1}^{n} u_{i j} \nu_{i j}, \quad i=1,2, \ldots, n \tag{63}
\end{equation*}
$$

It is easy to observe that

$$
\begin{equation*}
\mathcal{E} \subseteq \mathcal{F} \equiv\left\{\boldsymbol{z} \in \mathbb{Z}^{n}: \bar{\ell} \leq \boldsymbol{z} \leq \overline{\boldsymbol{u}}\right\} \tag{64}
\end{equation*}
$$

Actually it is easy to observe that $\mathcal{F}$ is the smallest box including $\mathcal{E}$.

With the above preparation, we now give the following result.

Theorem 6: Suppose that the assumptions in Theorem 1 hold and the linear model (5a) is transformed to the linear model (52) through the LLL reduction (51). Then the estimator $\overline{\boldsymbol{x}}$ defined as $\overline{\boldsymbol{x}}=\boldsymbol{Z} \boldsymbol{z}^{\mathrm{OB}}$ satisfies

$$
\begin{align*}
& \operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B}) \\
\leq & 1-\prod_{i=1}^{n} \frac{\operatorname{erf}\left(\bar{r}_{i i} /(2 \sqrt{2} \sigma)\right)}{\operatorname{erf}\left(\left(\bar{u}_{i}-\bar{\ell}_{i}+1\right) \bar{r}_{i i} /(2 \sqrt{2} \sigma)\right)} \tag{65}
\end{align*}
$$

where $\bar{\ell}$ and $\bar{u}$ are defined in (63).
Proof. Since $\operatorname{Pr}(\overline{\boldsymbol{x}} \in \mathcal{B})=\operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{OB}} \in \mathcal{E}\right)$, it follows from (64) that

$$
\begin{equation*}
\operatorname{Pr}(\overline{\boldsymbol{x}} \in \mathcal{B}) \leq \operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{OB}} \in \mathcal{F}\right) \tag{66}
\end{equation*}
$$

In the following, we will show

$$
\begin{equation*}
\operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{OB}} \in \mathcal{F}\right) \leq \prod_{i=1}^{n} \phi_{\sigma}\left(\left(\bar{u}_{i}-\bar{\ell}_{i}+1\right) \bar{r}_{i i}\right) \tag{67}
\end{equation*}
$$

where $\phi_{\sigma}(\cdot)$ is defined in (18). Then combining (66), (67) and the fact that $\operatorname{Pr}(\overline{\boldsymbol{x}}=\hat{\boldsymbol{x}})=\operatorname{Pr}\left(\boldsymbol{z}^{\mathrm{OB}}=\hat{\boldsymbol{z}}\right)=$ $\prod_{i=1}^{n} \phi_{\sigma}\left(\bar{r}_{i i}\right)$ (see (17) and (18)), we can conclude that (65) holds from (55).

To show (67), instead of analyzing the probability of $z^{\text {OB }}$ on its left-hand side, we will analyze an equivalent probability of $\overline{\boldsymbol{v}}$ as we know its distribution.

In our proof, we need to use the basic result: given $v \sim \mathcal{N}\left(0, \sigma^{2}\right)$ and $\eta>0$, for any $\zeta \in \mathbb{R}$,

$$
\begin{equation*}
\operatorname{Pr}(v \in[\zeta, \zeta+\eta]) \leq \operatorname{Pr}(v \in[-\eta / 2, \eta / 2])=\phi_{\sigma}(\eta) \tag{68}
\end{equation*}
$$

By (18), the equality in (68) obviously holds. In the following, we show the inequality holds, i.e., equivalently show

$$
\int_{\zeta}^{\zeta+\eta} \exp \left(-\frac{t^{2}}{2 \sigma^{2}}\right) d t \leq \int_{-\eta / 2}^{\eta / 2} \exp \left(-\frac{t^{2}}{2 \sigma^{2}}\right) d t
$$

For any fixed $\eta>0$, the left-hand side of the above inequality is a function of $\zeta$. Furthermore, it can be easily verified that the derivative of this function equal to zero if $\zeta=-\eta / 2$, and it is positive when $\zeta<-\eta / 2$ and negative when $\zeta>-\eta / 2$. Thus, this function achieves the maximal value when $\zeta=-\eta / 2$, so the inequality holds.

From (52) and (53), we observe that $z^{\mathrm{OB}}$ is a function of $\overline{\boldsymbol{v}}$. To emphasize this, we write it as $\boldsymbol{z}^{\mathrm{OB}}(\overline{\boldsymbol{v}})$. When $\overline{\boldsymbol{v}}$ changes, $\boldsymbol{z}^{\mathrm{OB}}(\overline{\boldsymbol{v}})$ may change too. In the following analysis, we assume that $\hat{\boldsymbol{z}}$ is fixed and $\overline{\boldsymbol{v}}$ satisfies the model (52).

For later uses, for $=n, n-1, \ldots, 1$, define

$$
\begin{gather*}
\mathcal{G}_{i}=\left\{\boldsymbol{w}_{i: n} \mid \overline{\boldsymbol{y}}_{i: n}=\overline{\boldsymbol{R}}_{i: n, i: n} \hat{\boldsymbol{z}}_{i: n}+\boldsymbol{w}_{i: n}, \boldsymbol{w}_{i: n} \in \mathbb{R}^{n-i+1}\right. \\
\left.z_{k}^{\mathrm{OB}}\left(\boldsymbol{w}_{k: n}\right) \in\left[\bar{\ell}_{k}, \bar{u}_{k}\right], k=i, i+1, \ldots, n\right\} \tag{69}
\end{gather*}
$$

From (69), it is easy to verify that $\boldsymbol{z}^{\mathrm{OB}}(\overline{\boldsymbol{v}}) \in \mathcal{F}$ if and only if $\overline{\boldsymbol{v}} \in \mathcal{G}_{1}$. Therefore, (67) is equivalent to

$$
\begin{equation*}
\operatorname{Pr}\left(\overline{\boldsymbol{v}}_{1: n} \in \mathcal{G}_{1}\right) \leq \prod_{i=1}^{n} \phi_{\sigma}\left(\left(\bar{u}_{i}-\bar{\ell}_{i}+1\right) \bar{r}_{i i}\right) . \tag{70}
\end{equation*}
$$

We prove (70) by induction. First, we prove the base case:

$$
\operatorname{Pr}\left(\bar{v}_{n} \in \mathcal{G}_{n}\right) \leq \phi_{\sigma}\left(\left(\bar{u}_{n}-\bar{\ell}_{n}+1\right) \bar{r}_{n n}\right) .
$$

By (53) and (52), we have

$$
c_{n}^{\mathrm{OB}}=\frac{\bar{y}_{n}}{\bar{r}_{n n}}=\frac{\bar{r}_{n n} \hat{z}_{n}+\bar{v}_{n}}{\bar{r}_{n n}}=\hat{z}_{n}+\frac{\bar{v}_{n}}{\bar{r}_{n n}} .
$$

Since $z_{n}^{\mathrm{OB}}\left(\bar{v}_{n}\right)=\left\lfloor c_{n}^{\mathrm{OB}}\right\rceil$, by (69),

$$
\begin{aligned}
& \operatorname{Pr}\left(\bar{v}_{n} \in \mathcal{G}_{n}\right) \\
= & \operatorname{Pr}\left(\hat{z}_{n}+\frac{\bar{v}_{n}}{\bar{r}_{n n}} \in\left[\bar{\ell}_{n}-1 / 2, \bar{u}_{n}+1 / 2\right]\right) \\
= & \operatorname{Pr}\left(\bar{v}_{n} \in\left[\left(\bar{\ell}_{n}-\hat{z}_{n}-1 / 2\right) \bar{r}_{n n},\left(\bar{u}_{n}-\hat{z}_{n}+1 / 2\right) \bar{r}_{n n}\right]\right) \\
\leq & \phi_{\sigma}\left(\left(\bar{u}_{n}-\bar{\ell}_{n}+1\right) \bar{r}_{n n}\right),
\end{aligned}
$$

where the inequality follows from (68).
Suppose for some $i>1$, we have

$$
\begin{equation*}
\operatorname{Pr}\left(\overline{\boldsymbol{v}}_{i: n} \in \mathcal{G}_{i}\right) \leq \prod_{k=i}^{n} \phi_{\sigma}\left(\left(\bar{u}_{k}-\bar{\ell}_{k}+1\right) \bar{r}_{k k}\right) \tag{71}
\end{equation*}
$$

Now we want to prove

$$
\begin{equation*}
\operatorname{Pr}\left(\overline{\boldsymbol{v}}_{i-1: n} \in \mathcal{G}_{i-1}\right) \leq \prod_{k=i-1}^{n} \phi_{\sigma}\left(\left(\bar{u}_{k}-\bar{\ell}_{k}+1\right) \bar{r}_{k k}\right) . \tag{72}
\end{equation*}
$$

We partition the set $\mathcal{G}_{i}$ into a sequence of disjoint subsets. To do that, for $i=n, n-1, \ldots, 1$, we first define the discrete set

$$
\mathcal{H}_{i}=\left\{\left.\sum_{j=i}^{n} \frac{r_{i-1, j}}{r_{i-1, i-1}}\left(\hat{z}_{j}-z_{j}^{\mathrm{OB}}\left(\boldsymbol{w}_{j: n}\right)\right) \right\rvert\, \boldsymbol{w}_{i: n} \in \mathcal{G}_{i}\right\} .
$$

Then, for any $t \in \mathcal{H}_{i}$, we define

$$
\begin{aligned}
& \mathcal{G}_{i, t}=\left\{\boldsymbol{w}_{i: n} \mid \boldsymbol{w}_{i: n} \in \mathcal{G}_{i}\right. \text { such that } \\
& \left.\qquad \sum_{j=i}^{n} \frac{r_{i-1, j}}{r_{i-1, i-1}}\left(\hat{z}_{j}-z_{j}^{\mathrm{OB}}\left(\boldsymbol{w}_{j: n}\right)\right)=t\right\} .
\end{aligned}
$$

It is easy to verify that $\cup_{t \in \mathcal{H}_{i}} \mathcal{G}_{i, t}=\mathcal{G}_{i}$ and $\mathcal{G}_{i, t_{1}} \cap \mathcal{G}_{i, t_{2}}=$ $\varnothing$ for $t_{1}, t_{2} \in \mathcal{H}_{i}$ and $t_{1} \neq t_{2}$. Therefore,

$$
\begin{equation*}
\operatorname{Pr}\left(\overline{\boldsymbol{v}}_{i: n} \in \mathcal{G}_{i}\right)=\sum_{t \in \mathcal{H}_{i}} \operatorname{Pr}\left(\overline{\boldsymbol{v}}_{i: n} \in \mathcal{G}_{i, t}\right) \tag{73}
\end{equation*}
$$

and

$$
\begin{align*}
& \operatorname{Pr}\left(\overline{\boldsymbol{v}}_{i-1: n} \in \mathcal{G}_{i-1}\right) \\
= & \operatorname{Pr}\left(\overline{\boldsymbol{v}}_{i: n} \in \mathcal{G}_{i}, z_{i-1}^{\mathrm{OB}}\left(\overline{\boldsymbol{v}}_{i-1: n}\right) \in\left[\bar{\ell}_{i-1}, \bar{u}_{i-1}\right]\right) \\
= & \sum_{t \in \mathcal{H}_{i}} \operatorname{Pr}\left(\overline{\boldsymbol{v}}_{i: n} \in \mathcal{G}_{i, t}, z_{i-1}^{\mathrm{OB}}\left(\overline{\boldsymbol{v}}_{i-1: n}\right) \in\left[\bar{\ell}_{i-1}, \bar{u}_{i-1}\right]\right) \\
= & \sum_{t \in \mathcal{H}_{i}} \operatorname{Pr}\left(\overline{\boldsymbol{v}}_{i: n} \in \mathcal{G}_{i, t}\right) \\
& \times \operatorname{Pr}\left(z_{i-1}^{\mathrm{OB}}\left(\overline{\boldsymbol{v}}_{i-1: n}\right) \in\left[\bar{\ell}_{i-1}, \bar{u}_{i-1}\right] \mid \overline{\boldsymbol{v}}_{i: n} \in \mathcal{G}_{i, t}\right) . \tag{74}
\end{align*}
$$

Now we derive a bound on the second probability of each term on the right-hand side of (74). By (53) and (52), we have

$$
\begin{aligned}
c_{i-1}^{\mathrm{OB}}= & \frac{\bar{r}_{i-1, i-1} \hat{z}_{i-1}}{\bar{r}_{i-1, i-1}}+\frac{\sum_{j=i}^{n} \bar{r}_{i-1, j}\left(\hat{z}_{j}-z_{j}^{\mathrm{OB}}\left(\overline{\boldsymbol{v}}_{j: n}\right)\right)}{\bar{r}_{i-1, i-1}} \\
& +\frac{\bar{v}_{i-1}}{\bar{r}_{i-1, i-1}} \\
= & \hat{z}_{i-1}+t^{\prime}+\frac{\bar{v}_{i-1}}{\bar{r}_{i-1, i-1}}
\end{aligned}
$$

where

$$
t^{\prime}=\sum_{j=i}^{n} \frac{\bar{r}_{i-1, j}}{\bar{r}_{i-1, i-1}}\left(\hat{z}_{j}-z_{j}^{\mathrm{OB}}\left(\overline{\boldsymbol{v}}_{j: n}\right)\right) .
$$

If $\overline{\boldsymbol{v}}_{i: n} \in \mathcal{G}_{i, t}, t^{\prime} \in \mathcal{H}_{i}$. Since $z_{i-1}^{\mathrm{OB}}\left(\overline{\boldsymbol{v}}_{i-1: n}\right)=\left\lfloor c_{i-1}^{\mathrm{OB}}\right\rceil$,

$$
\begin{align*}
& \operatorname{Pr}\left(z_{i-1}^{\mathrm{OB}}\left(\overline{\boldsymbol{v}}_{i-1: n}\right) \in\left[\bar{\ell}_{i-1}, \bar{u}_{i-1}\right] \mid \overline{\boldsymbol{v}}_{i: n} \in \mathcal{G}_{i, t}\right) \\
= & \operatorname{Pr}\left(\hat{z}_{i-1}+t^{\prime}+\frac{\bar{v}_{i-1}}{\bar{r}_{i-1, i-1}} \in\left[\bar{\ell}_{i-1}-1 / 2, \bar{u}_{i-1}+1 / 2\right]\right) \\
= & \operatorname{Pr}\left(\overline { v } _ { i - 1 } \in \left[\left(\bar{\ell}_{i-1}-\hat{z}_{i-1}-t^{\prime}-1 / 2\right) \bar{r}_{i-1, i-1}\right.\right. \\
& \left.\left.\left(\bar{u}_{i-1}-\hat{z}_{i-1}-t^{\prime}+1 / 2\right) \bar{r}_{i-1, i-1}\right]\right) \\
\leq & \phi_{\sigma}\left(\left(\bar{u}_{i-1}-\bar{\ell}_{i-1}+1\right) \bar{r}_{i-1, i-1}\right) \tag{75}
\end{align*}
$$

where the inequality follows from (68). Thus, from (74) it follows that

$$
\begin{aligned}
& \operatorname{Pr}\left(\overline{\boldsymbol{v}}_{i-1: n} \in \mathcal{G}_{i-1}\right) \\
\leq & \sum_{t \in \mathcal{H}_{i}} \operatorname{Pr}\left(\overline{\boldsymbol{v}}_{i: n} \in \mathcal{G}_{i, t}\right) \phi_{\sigma}\left(\left(\bar{u}_{i-1}-\bar{\ell}_{i-1}+1\right) \bar{r}_{i-1, i-1}\right) \\
= & \operatorname{Pr}\left(\overline{\boldsymbol{v}}_{i: n} \in \mathcal{G}_{i}\right) \phi_{\sigma}\left(\left(\bar{u}_{i-1}-\bar{\ell}_{i-1}+1\right) \bar{r}_{i-1, i-1}\right)
\end{aligned}
$$

where the equality is due to (73). Then the inequality (72) follows by using the induction hypothesis (71). Therefore, the inequality (70), or the equivalent inequality (67), holds for any fixed $\hat{\boldsymbol{z}}$.

Since (67) holds for any fixed $\hat{\boldsymbol{z}}$, it is easy to argue that it holds no matter what distribution of $\hat{\boldsymbol{z}}$ is over the box $\mathcal{F}$, so the theorem is proved.

By Theorem 6, if $\prod_{i=1}^{n} \frac{\phi_{\sigma}\left(\bar{r}_{i i}\right)}{\phi_{\sigma}\left(\left(\bar{u}_{i}-\ell_{i}+1\right) \bar{r}_{i i}\right)} \approx 1$, then $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B}) \approx 0$, i.e., the conjecture holds. Again, the condition will be satisfied when the noise standard deviation $\sigma$ is sufficiently small. Simulations in [32] showed that for practical MIMO systems often $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B}) \approx 0$.

Here we make a comment on the upper bound in (65). The derivation of (65) was based on the two inequalities (66) and (67). The inequality (66) was established based on the fact that $\mathcal{E} \subseteq \mathcal{F}$ in (64). If the absolute values of the entries of the unimodular matrix $\boldsymbol{Z}^{-1}$ are big, then it is likely that $\mathcal{F}$ is much bigger than $\mathcal{E}$ although $\mathcal{F}$ is the smallest box containing $\mathcal{E}$, making the inequality (66) loose. Otherwise it will be tight; in particular, when $Z=$ $\boldsymbol{I}$, then $\mathcal{E}=\mathcal{F}$ and the inequality (66) becomes an equality. In establishing the inequality (67) we used the inequality (68) (see (75)), which is simple but may not be tight if $\zeta$ is not close to $-\eta / 2$. Thus the inequality (67) may not be tight. Overall, the upper bound in (65) may not be tight sometimes, but it is always tighter than the upper bound given by (62). The following example shows that the former can be significantly tighter than the latter and can be a sharp bound.

Example 3: We use exactly the same data generated in Example 2 to compute the upper bounds in (62) and (65), which are denoted by $\mu_{e b 1}$ and $\mu_{e b 2}$, respectively. The results for $n=5: 5: 40$ with $\sigma=0.1$ are given in Table XI. To see how tight they are, the values of $P_{e}$ given in Table IX are displayed here again. Recall $P_{e}$ is the experimental probability corresponding to the theoretical probability $\operatorname{Pr}(\overline{\boldsymbol{x}} \neq \hat{\boldsymbol{x}} \mid \overline{\boldsymbol{x}} \in \mathcal{B})$ in (62) and (65).

From Table XI, we can see the upper bound $\mu_{e b 2}$ is obviously tighter than the upper bound $\mu_{e b 1}$ and $\mu_{e b 2}$ is close to $P_{e}$. When $n=10, P_{e}>\mu_{e b 2}$, this is because there are some deviations between the experimental values and the theoretical values. The values of $\mu_{e b 1}$ are actually not exactly the same for different $n$, but they are very close. This is also true for $\mu_{e b 2}$.

TABLE XI
$P_{e}$ AND BOUNDS VERSUS $n=5: 5: 40$ WITH $\sigma=0.1$

| $\boldsymbol{n}$ | $P_{e}$ | $\mu_{e b 1}$ | $\mu_{e b 2}$ |
| :---: | :---: | :---: | :---: |
| 5 | 0.9356 | 0.9840 | 0.9364 |
| 10 | 0.9396 | 0.9840 | 0.9364 |
| 15 | 0.9336 | 0.9840 | 0.9364 |
| 20 | 0.9348 | 0.9840 | 0.9364 |
| 25 | 0.9325 | 0.9840 | 0.9364 |
| 30 | 0.9312 | 0.9840 | 0.9364 |
| 35 | 0.9304 | 0.9840 | 0.9364 |
| 40 | 0.9252 | 0.9840 | 0.9364 |

## VI. Summary and future work

We have presented formulas for the success probability $P^{\mathrm{BB}}$ of the box-constrained Babai estimator and the success probability $P^{\text {OB }}$ of the ordinary Babai estimator for the linear model where the true integer parameter vector $\hat{\boldsymbol{x}}$ is uniformly distributed over the constraint box and the noise vector follows a normal distribution. The properties of $P^{B B}$ and $P^{\circ \mathrm{B}}$ and the relationship between them were given.

The effects of the column permutations on $P^{\mathrm{BB}}$ by the LLL-P, SQRD and V-BLAST column permutation strategies have been investigated. When the noise is relatively small, we showed that LLL-P always increases
$P^{\mathrm{BB}}$ and argued why both SQRD and V-BLAST usually increase $P^{\mathrm{BB}}$; and when the noise is relatively large, LLL-P always decreases $P^{\mathrm{BB}}$ and argued why both SQRD and VBLAST usually decrease $P^{\mathrm{BB}}$. The latter contradicts with what we commonly believed. And it suggests that we should check the conditions given in the paper before we apply these strategies. We also provided a column permutation invariant bound on $P^{\mathrm{BB}}$. This bound helped us to understand the effects of these column permutation strategies on $P^{\mathrm{BB}}$. Our theoretical findings were supported by numerical test results.

We have given an example to show that the conjecture proposed in [24] does not always hold and imposed a condition under which the conjecture holds.

LLL-P has better theory than V-BLAST and SQRD in terms of their effects on $P^{\mathrm{BB}}$. But our numerical experiments indicated often V-BLAST is more effective than LLL-P and SQRD. Developing a more effective column permutation strategy with solid theory will be investigated in the future. These three permutation column permutation strategies use only the information of $\boldsymbol{A}$. The effects of the column permutation strategies which use all available information of the model such as those proposed in [15], [6] and [16] need to be investigated.

Recently the success probability of the BILS estimator has been given in [38]. We intend to study the relationship between it and $P^{\mathrm{BB}}$.

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