# Pretty good measures in quantum information theory 

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#### Abstract

Quantum generalizations of Rényi's entropies are a useful tool to describe a variety of operational tasks in quantum information processing. Two families of such generalizations turn out to be particularly useful: the Petz quantum Rényi divergence $\bar{D}_{\alpha}$ and the minimal quantum Rényi divergence $\widetilde{D}_{\alpha}$. In this paper, we prove a reverse Araki-Lieb-Thirring inequality that implies a new relation between these two families of divergences, namely that $\alpha \bar{D}_{\alpha}(\varrho \| \sigma) \leq \widetilde{D}_{\alpha}(\varrho \| \sigma)$ for $\alpha \in[0,1]$ and where $\varrho$ and $\sigma$ are density operators. This bound suggests defining a "pretty good fidelity", whose relation to the usual fidelity implies the known relations between the optimal and pretty good measurement as well as the optimal and pretty good singlet fraction. We also find a new necessary and sufficient condition for optimality of the pretty good measurement and singlet fraction.


## 1 Introduction

As with their classical counterparts, quantum generalizations of Rényi entropies and divergences are powerful tools in information theory. Two families of quantum Rényi divergences have proven particularly useful, finding application to achievability, strong converses, and refined asymptotic analysis of a variety of coding and hypothesis testing problems (for a recent overview, see [1]): the Petz quantum Rényi divergence [2] and the minimal quantum Rényi divergence [3, 4] (also known as sandwiched quantum Rényi divergence). A natural and important issue is the relation between these two families. In this work we prove a novel two-sided bound that relates the two families and discuss its implications.

For two non-negative operators $\varrho \neq 0$ and $\sigma$ and $\alpha \in(0,1) \cup(1, \infty)$, the Petz quantum Rényi divergence is defined as

$$
\bar{D}_{\alpha}(\varrho \| \sigma):= \begin{cases}\frac{1}{\alpha-1} \log \frac{1}{\operatorname{tr} \varrho} \bar{Q}_{\alpha}(\varrho \| \sigma) & \text { if } \sigma \gg \varrho \vee \alpha<1  \tag{1}\\ \infty & \text { otherwise }\end{cases}
$$

where $\bar{Q}_{\alpha}(\varrho \| \sigma):=\operatorname{tr} \varrho^{\alpha} \sigma^{1-\alpha}$ and we use the common convention that $-\log 0=\infty$. Moreover, negative matrix powers should be considered as generalized inverses. The notation $\sigma \gg \varrho$ denotes that the kernel of $\sigma$ is a subset of the kernel of $\varrho$. The minimal quantum Rényi divergence on the other hand is defined by

$$
\widetilde{D}_{\alpha}(\varrho \| \sigma):= \begin{cases}\frac{1}{\alpha-1} \log \frac{1}{\operatorname{tr} \varrho} \widetilde{Q}_{\alpha}(\varrho \| \sigma) & \text { if } \sigma \gg \varrho \vee \alpha<1  \tag{2}\\ \infty & \text { otherwise }\end{cases}
$$

where $\widetilde{Q}_{\alpha}(\varrho \| \sigma):=\operatorname{tr}\left(\sigma^{\frac{1-\alpha}{2 \alpha}} \varrho \sigma^{\frac{1-\alpha}{2 \alpha}}\right)^{\alpha}$. Moreover, we define $D_{0}, D_{1}$ and $D_{\infty}$ as limits of $D_{\alpha}$ for $\alpha \rightarrow 0, \alpha \rightarrow 1$ and $\alpha \rightarrow \infty$, respectively. Throughout this paper we use the convention that statements without either bar or tilde symbols are true for both cases.

The Araki-Lieb-Thirring (ALT) inequality [5, 6] implies that the Petz divergence is larger than or equal to the minimal divergence, i.e., $\bar{D}_{\alpha}(\varrho \| \sigma) \geq \widetilde{D}_{\alpha}(\varrho \| \sigma)$. But what remains unanswered is
how much bigger than the minimal divergence the Petz divergence can be. We settle this question for $\alpha \leq 1$ by showing that $\bar{D}_{\alpha}(\varrho \| \sigma) \leq \frac{1}{\alpha} \widetilde{D}_{\alpha}(\varrho \| \sigma)$ if $\varrho$ and $\sigma$ are normalized. This result follows from a new reversed ALT inequality. (We refer to Theorem 2.1 and Corollary 2.3 for precise statements.)

This result has several applications. In Section 3.1, we define the "pretty good fidelity" as $F_{\mathrm{pg}}(\varrho, \sigma):=\operatorname{tr} \sqrt{\varrho} \sqrt{\sigma}$. The result above then implies that the pretty good fidelity is indeed pretty good in that $F_{\mathrm{pg}} \leq F \leq \sqrt{F_{\mathrm{pg}}}$, where $F$ denotes the usual fidelity defined by $F(\varrho, \sigma):=$ $\operatorname{tr}(\sqrt{\varrho} \sigma \sqrt{\varrho})^{1 / 2}$. Analogous bounds are also known between the pretty good guessing probability and the optimal guessing probability [7] as well as between the pretty good and the optimal achievable singlet fraction [8]. ${ }^{1}$ We show that both of these relations follow by the inequality relating the pretty good fidelity and the fidelity. We thus present a unified picture of the relationship between pretty good quantities and their optimal versions. Additionally, we show that equality conditions for the ALT inequality lead to a new necessary and sufficient condition on the optimality of both pretty good measurement and singlet fraction.

In this paper we consider finite-dimensional Hilbert spaces only, though most of our results can be extended to separable Hilbert spaces. We label Hilbert spaces with capital letters $A, B$, etc. and denote their dimension by $|A|,|B|$, etc.. The set of density operators on $A$, i.e., non-negative operators $\varrho_{A}$ with $\operatorname{tr} \varrho_{A}=1$, is denoted $\mathscr{D}(A)$. We shall also make use of the convention $\frac{1}{0}=\infty$. The Schatten p-norm of any linear operator $L$ is given by

$$
\begin{equation*}
\|L\|_{p}:=\left(\operatorname{tr}|L|^{p}\right)^{\frac{1}{p}} \quad \text { for } \quad p \geq 1, \tag{3}
\end{equation*}
$$

where $|L|:=\sqrt{L^{*} L}$. We may extend this definition to all $p>0$, but note that $\|L\|_{p}$ is not a norm for $p \in(0,1)$ since it does not satisfy the triangle inequality. In the limit $p \rightarrow \infty$ we recover the operator norm and for $p=1$ we obtain the trace norm. Schatten norms are functions of the singular values and thus unitarily invariant. Moreover, they satisfy $\|L\|_{p}=\left\|L^{*}\right\|_{p}$ and $\|L\|_{2 p}^{2}=\left\|L L^{*}\right\|_{p}=\left\|L^{*} L\right\|_{p}$.

## 2 Results

### 2.1 Reverse ALT inequality

The ALT inequality states that for any non-negative operators $A$ and $B, q \geq 0$ and $r \in[0,1]$,

$$
\begin{equation*}
\operatorname{tr}\left(B^{\frac{r}{2}} A^{r} B^{\frac{r}{2}}\right)^{q} \leq \operatorname{tr}\left(B^{\frac{1}{2}} A B^{\frac{1}{2}}\right)^{r q}, \tag{4}
\end{equation*}
$$

and the inequality holds in the opposite direction for $r \geq 1[5,6]$. Our main result is a reversed version of the ALT inequality.

Theorem 2.1 (Reverse ALT inequality). Let $A$ and $B$ be non-negative operators and $q>0$. Then, for $r \in(0,1]$ and $a, b \in(0, \infty]$ such that $\frac{1}{2 r q}=\frac{1}{2 q}+\frac{1}{a}+\frac{1}{b}$, we have

$$
\begin{equation*}
\operatorname{tr}\left(B^{\frac{1}{2}} A B^{\frac{1}{2}}\right)^{r q} \leq\left(\operatorname{tr}\left(B^{\frac{r}{2}} A^{r} B^{\frac{r}{2}}\right)^{q}\right)^{r}\left\|A^{\frac{1-r}{2}}\right\|_{a}^{2 r q}\left\|B^{\frac{1-r}{2}}\right\|_{b}^{2 r q} \tag{5}
\end{equation*}
$$

Meanwhile, for $r \in[1, \infty)$ and $a, b \in(0, \infty]$ such that $\frac{1}{2 q}=\frac{1}{2 r q}+\frac{1}{a}+\frac{1}{b}$, we have

$$
\begin{equation*}
\operatorname{tr}\left(B^{\frac{1}{2}} A B^{\frac{1}{2}}\right)^{r q} \geq\left(\operatorname{tr}\left(B^{\frac{r}{2}} A^{r} B^{\frac{r}{2}}\right)^{q}\right)^{r}\left\|A^{\frac{r-1}{2}}\right\|_{a}^{-2 r q}\left\|B^{\frac{r-1}{2}}\right\|_{b}^{-2 r q} \tag{6}
\end{equation*}
$$

[^0]Proof. For $r=1$ the statement is trivial. Let $r \in(0,1)$ and $q>0$. Recall the generalized Hölder inequality for matrices (see e.g., [9, Exercise IV.2.7] for a proof): For $s, s_{1}, \ldots, s_{n}$ positive real numbers and $\left\{A_{k}\right\}_{k=1}^{n}$ a collection of square matrices, it holds that

$$
\begin{equation*}
\left\|\prod_{k=1}^{n} A_{k}\right\|_{s} \leq \prod_{k=1}^{n}\left\|A_{k}\right\|_{s_{k}} \quad \text { for } \quad \sum_{k=1}^{n} \frac{1}{s_{k}}=\frac{1}{s} \tag{7}
\end{equation*}
$$

Furthermore, we can rewrite the trace-terms in (5) as Schatten (quasi-)norms

$$
\begin{equation*}
\operatorname{tr}\left(B^{\frac{1}{2}} A B^{\frac{1}{2}}\right)^{r q}=\left\|B^{\frac{1}{2}} A^{\frac{1}{2}}\right\|_{2 r q}^{2 r q} \quad \text { and } \quad \operatorname{tr}\left(B^{\frac{r}{2}} A^{r} B^{\frac{r}{2}}\right)^{q}=\left\|B^{\frac{r}{2}} A^{\frac{r}{2}}\right\|_{2 q}^{2 q} \tag{8}
\end{equation*}
$$

Inequality (5) then follows by an application of the generalized Hölder inequality with $n=3$. Choosing $s=2 r q$, and $s_{1}=b, s_{2}=2 q$, and $s_{3}=a$ for some $a, b \in(0, \infty]$ with $\frac{1}{2 r q}=\frac{1}{2 q}+\frac{1}{a}+\frac{1}{b}$, we find

$$
\begin{equation*}
\operatorname{tr}\left(B^{\frac{1}{2}} A B^{\frac{1}{2}}\right)^{r q}=\left\|B^{\frac{1-r}{2}} B^{\frac{r}{2}} A^{\frac{r}{2}} A^{\frac{1-r}{2}}\right\|_{2 r q}^{2 r q} \leq\left\|B^{\frac{1-r}{2}}\right\|_{b}^{2 r q}\left\|B^{\frac{r}{2}} A^{\frac{r}{2}}\right\|_{2 q}^{2 r q}\left\|A^{\frac{1-r}{2}}\right\|_{a}^{2 r q} \tag{9}
\end{equation*}
$$

Inequality (6) now follows from (5) by substituting $A \rightarrow A^{r}, B \rightarrow B^{r}, r \rightarrow \frac{1}{r}$, and $q \rightarrow q r$.
Remark 2.2. Another reverse ALT inequality was given in [10], where it was shown that for $r \in(0,1)$ and $q>0$ we have

$$
\begin{equation*}
\operatorname{tr}\left(B^{\frac{1}{2}} A B^{\frac{1}{2}}\right)^{r q} \leq\left(\operatorname{tr}\left(B^{\frac{r}{2}} A^{r} B^{\frac{r}{2}}\right)^{q}\right)^{r}\left(\operatorname{tr} A^{r q}\|B\|_{\infty}^{r q}\right)^{1-r} \tag{10}
\end{equation*}
$$

while for $r>1$ the inequality holds in the opposite direction. We recover these inequalities as a corollary of Theorem 2.1 by setting $b=\infty$ and $a=\frac{2 r q}{1-r}$ in (5), and $b=\infty$ and $a=\frac{2 r q}{r-1}$ in (6). We note that there also exists a reverse ALT inequality in terms of matrix means (see e.g. [11]) that however is different to Theorem 2.1.

### 2.2 Relation between the Petz and the minimal divergence

It is known that the minimal quantum Rényi divergence provides a lower bound for all other quantum Rényi divergences satisfying a small number of axiomatic properties (see e.g., [1, §4.2.2] for a precise statement). Hence, in particular, we have $\widetilde{D}_{\alpha}(\varrho \| \sigma) \leq \bar{D}_{\alpha}(\varrho \| \sigma)$ for all $\alpha \in[0, \infty]$. ${ }^{2}$ Theorem 2.1 leads to reversed relations between these two divergences. In the case where $\alpha \in$ $[0,1]$, we find a particularly useful relation of a simple form.

Corollary 2.3. Let $\varrho \neq 0$ and $\sigma$ be two non-negative operators and $\alpha \in[0,1]$. Then

$$
\begin{equation*}
\alpha \bar{D}_{\alpha}(\varrho \| \sigma)+(1-\alpha)(\log \operatorname{tr} \varrho-\log \operatorname{tr} \sigma) \leq \widetilde{D}_{\alpha}(\varrho \| \sigma) \leq \bar{D}_{\alpha}(\varrho \| \sigma) \tag{11}
\end{equation*}
$$

Proof. The second inequality is a direct consequence of the ALT inequality. It thus remains to show the first inequality. We note that it suffices to consider the case $\alpha \in(0,1)$, as $\alpha \in\{0,1\}$ then follows by continuity. By definition, we can reformulate the first inequality of (11) as

$$
\begin{equation*}
\widetilde{Q}_{\alpha}(\varrho \| \sigma) \leq \bar{Q}_{\alpha}(\varrho \| \sigma)^{\alpha}(\operatorname{tr} \varrho)^{\alpha(1-\alpha)}(\operatorname{tr} \sigma)^{(1-\alpha)^{2}} \tag{12}
\end{equation*}
$$

This follows from Theorem 2.1 with $q=1, r=\alpha, A=\varrho, B=\sigma^{\frac{1-\alpha}{\alpha}}, a=\frac{2}{1-\alpha}$, and $b=\frac{2 \alpha}{(1-\alpha)^{2}}$.
There is a well known equality condition for the ALT inequality, which leads to an equality condition for the second inequality of (11).

[^1]Lemma 2.4. For $\alpha \in(0,1)$, we have $\widetilde{D}_{\alpha}(\varrho \| \sigma)=\bar{D}_{\alpha}(\varrho \| \sigma)$ if and only if $\varrho$ and $\sigma$ commute.
Proof. To see this, note that for $r \in(1, \infty)$ and $r q \geq 1$, we have equality in the ALT inequality (4) if and only if $A$ and $B$ commute. Equality for commuting states is obvious; for the other direction, note that we can rewrite (4) using the substitution $r q=q^{\prime}$ as

$$
\begin{equation*}
\left\|\left(B^{\frac{r}{2}} A^{r} B^{\frac{r}{2}}\right)^{\frac{1}{r}}\right\|_{q^{\prime}} \geq\left\|\left(B^{\frac{1}{2}} A B^{\frac{1}{2}}\right)\right\|_{q^{\prime}} . \tag{13}
\end{equation*}
$$

Equality in the inequality (13) for some $r \in(1, \infty)$ (and noting that we have also equality for $r=1$ ) implies that the function $r \rightarrow\left\|\left(B^{\frac{r}{2}} A^{r} B^{\frac{r}{2}}\right)^{\frac{1}{r}}\right\|_{q^{\prime}}$ is not strictly increasing. Therefore, by [12, Theorem 2.1], it follows ${ }^{3}$ that $[A, B]=0$. Let $\varrho, \sigma$ be non negative. Setting $r=1 / \alpha, q=\alpha$ and $A=\varrho^{\alpha}, B=\sigma^{1-\alpha}$ in (4), we conclude that for $\alpha \in(0,1)$ we have that $\widetilde{D}_{\alpha}(\varrho \| \sigma)=\bar{D}_{\alpha}(\varrho \| \sigma)$ if and only if $[\varrho, \sigma]=0$.

For density operators $\varrho$ and $\sigma$ the first inequality of Corollary 2.3 simplifies to

$$
\begin{equation*}
\alpha \bar{D}_{\alpha}(\varrho \| \sigma) \leq \widetilde{D}_{\alpha}(\varrho \| \sigma) \quad \text { for } \quad \alpha \in[0,1] . \tag{14}
\end{equation*}
$$

This bound is simpler than an alternative bound given in [13], which is based on the earlier reversed ALT inequality in (10) and states that $\alpha \bar{D}_{\alpha}(\varrho \| \sigma)-\log \operatorname{tr} \varrho^{\alpha}+(\alpha-1) \log \|\sigma\|_{\infty} \leq \widetilde{D}_{\alpha}(\varrho \| \sigma)$ for density operators $\varrho$ and $\sigma$.

### 2.3 Relations between quantum conditional Rényi entropies

Divergences can be used to define conditional entropies. For any density operator $\varrho_{A B}$ on $A \otimes B$ we define the quantum conditional Rényi entropy of $A$ given $B$ as

$$
\begin{equation*}
H_{\alpha}^{\downarrow}(A \mid B)_{\varrho}:=-D_{\alpha}\left(\varrho_{A B} \| \mathbb{1}_{A} \otimes \varrho_{B}\right) \quad \text { and } \quad H_{\alpha}^{\uparrow}(A \mid B)_{\varrho}:=\sup _{\sigma_{B} \in \mathscr{O}(B)}-D_{\alpha}\left(\varrho_{A B} \| \mathbb{1}_{A} \otimes \sigma_{B}\right) . \tag{15}
\end{equation*}
$$

Note that the special cases $\alpha \in\{0,1, \infty\}$ are defined by taking the limits inside the supremum. ${ }^{4}$ We call the set of all conditional entropies with $\alpha \in(0,1)$ "max-like" and those with $\alpha \in(1, \infty)$ "min-like", owing to the fact that under small changes to the state the entropies in either class are approximately equal [14, 15]. Moreover, min- and max-like entropies are related by some interesting duality relations, which are summarized in the following lemma.

Lemma 2.5 (Duality relations [3,15-19]). Let $\varrho_{A B C}$ be a pure state on $A \otimes B \otimes C$. Then

$$
\begin{array}{lll}
\bar{H}_{\alpha}^{\downarrow}(A \mid B)_{\varrho}+\bar{H}_{\beta}^{\downarrow}(A \mid C)_{\varrho}=0 & \text { when } & \alpha+\beta=2 \text { for } \alpha, \beta \in[0,2] \quad \text { and } \\
\widetilde{H}_{\alpha}^{\uparrow}(A \mid B)_{\varrho}+\widetilde{H}_{\beta}^{\uparrow}(A \mid C)_{\varrho}=0 & \text { when } & \frac{1}{\alpha}+\frac{1}{\beta}=2 \text { for } \alpha, \beta \in\left[\frac{1}{2}, \infty\right] \quad \text { and } \\
\bar{H}_{\alpha}^{\uparrow}(A \mid B)_{\varrho}+\widetilde{H}_{\beta}^{\downarrow}(A \mid C)_{\varrho}=0 & \text { when } & \alpha \beta=1 \text { for } \alpha, \beta \in[0, \infty], \tag{18}
\end{array}
$$

where we use the convention that $\frac{1}{\infty}=0$ and $\infty \cdot 0=1$.

[^2]
### 2.3.1 Relations between max-like entropies

As a direct consequence of Corollary 2.3, we find the following relation between conditional max-like entropies.
Corollary 2.6. For $\alpha \in[0,1]$ and $\varrho_{A B} \in \mathscr{D}(A \otimes B)$, we have that

$$
\begin{align*}
& \bar{H}_{\alpha}^{\downarrow}(A \mid B)_{\varrho} \leq \widetilde{H}_{\alpha}^{\downarrow}(A \mid B)_{\varrho} \leq \alpha \bar{H}_{\alpha}^{\downarrow}(A \mid B)_{\varrho}+(1-\alpha) \log |A| \quad \text { and }  \tag{19}\\
& \bar{H}_{\alpha}^{\uparrow}(A \mid B)_{\varrho} \leq \widetilde{H}_{\alpha}^{\uparrow}(A \mid B)_{\varrho} \leq \alpha \bar{H}_{\alpha}^{\dagger}(A \mid B)_{\varrho}+(1-\alpha) \log |A| . \tag{20}
\end{align*}
$$

We can further improve the upper bounds in (19) and (20) by removing the second term if $\varrho_{A B}$ has a special structure consisting of a quantum and a classical part that is handled coherently.
Proposition 2.7. Let $|\varrho\rangle_{X X^{\prime} B B^{\prime}}=\sum_{x} \sqrt{P_{x}}|x\rangle_{X}|x\rangle_{X^{\prime}}\left|\xi_{x}\right\rangle_{B B^{\prime}}$ be a pure state on $X \otimes X^{\prime} \otimes B \otimes B^{\prime}$, where $X^{\prime} \simeq X, p_{x} \in[0,1]$ with $\sum_{x} p_{x}=1$, and the pure states $\left|\xi_{x}\right\rangle_{B B^{\prime}}$ are arbitrary. Then

$$
\begin{array}{lll}
\widetilde{H}_{\alpha}^{\downarrow}\left(X \mid X^{\prime} B\right)_{\varrho} \leq \alpha \bar{H}_{\alpha}^{\downarrow}\left(X \mid X^{\prime} B\right)_{\varrho} & \text { for } \alpha \in[0,1] & \text { and } \\
\widetilde{H}_{\alpha}^{\uparrow}\left(X \mid X^{\prime} B\right)_{\varrho} \leq \alpha \bar{H}_{\alpha}^{\uparrow}\left(X \mid X^{\prime} B\right)_{\varrho} & \text { for } \alpha \in\left[\frac{1}{2}, 1\right] . & \tag{22}
\end{array}
$$

States $\varrho_{X X^{\prime} B}$ are sometimes called "classically coherent" as the classical information is treated coherently, i.e. fully quantum-mechanically.

Proof of Proposition 2.7. It is known that $\widetilde{D}_{1}=\bar{D}_{1}$ (see for example [1]), and hence the claim is trivial in the case $\alpha=1$. Using (15) as well as (1) and (2) , one can see that it suffices to show that

$$
\begin{array}{ll}
\widetilde{Q}_{\alpha}\left(\varrho_{X X^{\prime} B} \| \mathbb{1}_{X} \otimes \varrho_{X^{\prime} B}\right) \leq \bar{Q}_{\alpha}\left(\varrho_{X X^{\prime} B} \| \mathbb{1}_{X} \otimes \varrho_{X^{\prime} B}\right)^{\alpha} & \text { for } \alpha \in(0,1) \quad \text { and } \\
\widetilde{Q}_{\alpha}\left(\varrho_{X X^{\prime} B} \| \mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}\right) \leq \bar{Q}_{\alpha}\left(\varrho_{X X^{\prime} B} \| \mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}\right)^{\alpha} & \text { for } \alpha \in\left[\frac{1}{2}, 1\right), \tag{24}
\end{array}
$$

for all density operators $\sigma_{X^{\prime} B}$ (the case $\alpha=0$ then follows by continuity).
The marginal state $\varrho_{X^{\prime} B}$ appearing in (23) is a classical quantum (cq) state by assumption. Importantly, by the monotonicity of the Rényi divergence, we need only prove (24) for cq states $\sigma_{X^{\prime} B}$ in order to show (22). Indeed, by Lemma A. 1 of Appendix A, the supremum arising in equation (22) can be taken only over cq states.

Now define the unitary $U_{X X^{\prime}}:=\sum_{x^{\prime}, x}\left|x-x^{\prime}\right\rangle\left\langle\left. x\right|_{X} \otimes \mid x^{\prime}\right\rangle\left\langle\left. x^{\prime}\right|_{X^{\prime}}\right.$, where arithmetic inside the ket is taken modulo $|X|$, and observe that $U_{X X^{\prime}} \otimes \mathbb{1}_{B}$ leaves the state $\mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}$ invariant (here we use the assumption that $\sigma_{X^{\prime} B}$ is a cq state). Hence, by unitary invariance of $Q_{\alpha}$, we find

$$
\begin{align*}
Q_{\alpha}\left(\varrho_{X X^{\prime} B} \| \mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}\right) & =Q_{\alpha}\left(\left(U_{X X^{\prime}} \otimes \mathbb{1}_{B}\right) \varrho_{X X^{\prime} B}\left(U_{X X^{\prime}}^{*} \otimes \mathbb{1}_{B}\right) \| \mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}\right)  \tag{25}\\
& =Q_{\alpha}\left(|0\rangle\left\langle\left. 0\right|_{X} \otimes \sum_{x, x^{\prime}} \sqrt{p_{x} p_{x^{\prime}}} \mid x\right\rangle\left\langle\left. x^{\prime}\right|_{X^{\prime}} \otimes \operatorname{tr}_{B^{\prime}} \mid \xi_{x}\right\rangle\left\langle\left.\xi_{x^{\prime}}\right|_{B B^{\prime}} \| \mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}\right)\right.  \tag{26}\\
& =Q_{\alpha}\left(\sum_{x, x^{\prime}} \sqrt{p_{x} p_{x^{\prime}}}|x\rangle\left\langle\left. x^{\prime}\right|_{X^{\prime}} \otimes \operatorname{tr}_{B^{\prime}} \mid \xi_{x}\right\rangle\left\langle\left.\xi_{x^{\prime}}\right|_{B B^{\prime}} \| \sigma_{X^{\prime} B}\right),\right. \tag{27}
\end{align*}
$$

where we used the multiplicity of the trace under tensor products in the last equality. The claim now follows by a direct application of Corollary 2.3 (or more precisely of (12) applied to density operators):

$$
\begin{align*}
\widetilde{Q}_{\alpha}\left(\varrho_{X X^{\prime} B} \| \mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}\right) & =\widetilde{Q}_{\alpha}\left(\sum_{x, x^{\prime}} \sqrt{p_{x} p_{x^{\prime}}}|x\rangle\left\langle\left. x^{\prime}\right|_{X^{\prime}} \otimes \operatorname{tr}_{B^{\prime}} \mid \xi_{x}\right\rangle\left\langle\left.\xi_{x^{\prime}}\right|_{B B^{\prime}} \| \sigma_{X^{\prime} B}\right)\right.  \tag{28}\\
& \leq \bar{Q}_{\alpha}\left(\sum_{x, x^{\prime}} \sqrt{p_{x} p_{x^{\prime}}}|x\rangle\left\langle\left. x^{\prime}\right|_{X^{\prime}} \otimes \operatorname{tr}_{B^{\prime}} \mid \xi_{x}\right\rangle\left\langle\left.\xi_{x^{\prime}}\right|_{B B^{\prime}} \| \sigma_{X^{\prime} B}\right)^{\alpha}\right.  \tag{29}\\
& =\bar{Q}_{\alpha}\left(\varrho_{X X^{\prime} B} \| \mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}\right)^{\alpha} . \tag{30}
\end{align*}
$$

This shows inequality (24) for cq states $\sigma_{X^{\prime} B}$, and hence (22). Moreover, we recover inequality (23) by setting $\sigma_{X_{B}^{\prime} B}=\varrho_{X^{\prime} B}$.

### 2.3.2 Relations between min-like entropies

We can use duality relations for conditional entropies (see Lemma 2.5) and Corollary 2.6 to derive new bounds for conditional min-like entropies.

Lemma 2.8. For $\alpha \in[1,2]$ and $\varrho_{A B} \in \mathscr{D}(A \otimes B)$, we have that ${ }^{5}$

$$
\begin{align*}
& \tilde{H}_{\alpha}^{\downarrow}(A \mid B)_{\varrho} \leq \alpha \widetilde{H}_{\frac{1}{2-\alpha}}^{\uparrow-}(A \mid B)_{\varrho}+(\alpha-1) \log |A| \quad \text { and }  \tag{31}\\
& \bar{H}_{\alpha}^{\downarrow}(A \mid B)_{\varrho} \leq \frac{1}{2-\alpha}\left(\bar{H}_{\frac{1}{2-\alpha}}^{\uparrow}(A \mid B)_{\varrho}+(\alpha-1) \log |A|\right) . \tag{32}
\end{align*}
$$

Proof. Let $\tau_{A B C}$ be a purification of $\varrho_{A B}$ on $A \otimes B \otimes C$, i.e., $\tau_{A B C}$ is a pure state with $\operatorname{tr}_{C} \tau_{A B C}=\varrho_{A B}$. Then, we find

$$
\begin{equation*}
\widetilde{H}_{\alpha}^{\downarrow}(A \mid B)_{\tau}=-\bar{H}_{\frac{1}{\alpha}}^{\uparrow}(A \mid C)_{\tau} \leq-\alpha \tilde{H}_{\frac{1}{\alpha}}^{\uparrow}(A \mid C)_{\tau}+(\alpha-1) \log |A|=\alpha \tilde{H}_{\frac{1}{2-\alpha}}^{\uparrow}(A \mid B)_{\tau}+(\alpha-1) \log |A| \tag{33}
\end{equation*}
$$

where we used Corollary 2.6 for the inequality and duality relations in the first and third equality. Similarly, we find

$$
\begin{align*}
\bar{H}_{\alpha}^{\downarrow}(A \mid B)_{\tau} & =-\bar{H}_{2-\alpha}^{\downarrow}(A \mid C)_{\tau}  \tag{34}\\
& \leq \frac{1}{2-\alpha}\left(-\tilde{H}_{2-\alpha}^{\downarrow}(A \mid C)_{\tau}+(\alpha-1) \log |A|\right)  \tag{35}\\
& =\frac{1}{2-\alpha}\left(\bar{H}_{\frac{1}{2-\alpha}}^{\uparrow}(A \mid B)_{\tau}+(\alpha-1) \log |A|\right) \tag{36}
\end{align*}
$$

where we again used Corollary 2.6 for the inequality and duality relations in the first and third equality.

Corollary 2.9. Let $\alpha \in[1,2]$ and $\varrho_{X B}$ be a cq state on $X \otimes B$, i.e., $\varrho_{X B}=\sum_{x} p_{x}|x\rangle\left\langle\left. x\right|_{X} \otimes\left(\varrho_{x}\right)_{B}\right.$ where $\left(\varrho_{x}\right)_{B}$ are density operators and $p_{x} \in[0,1]$, such that $\sum_{x} p_{x}=1$. Then

$$
\begin{align*}
& \tilde{H}_{\alpha}^{\downarrow}(X \mid B)_{\varrho} \leq \alpha \tilde{H}_{\frac{1}{2-\alpha}}^{\uparrow}(X \mid B)_{\varrho} \quad \text { and }  \tag{37}\\
& \bar{H}_{\alpha}^{\downarrow}(X \mid B)_{\varrho} \leq \frac{1}{2-\alpha} \bar{H}_{\frac{1}{2-\alpha}}^{\uparrow}(X \mid B)_{\varrho} \tag{38}
\end{align*}
$$

Proof. The proof proceeds analogously to the proof of Lemma 2.8, but we can make use of the improved bounds given in Proposition 2.7: Let $|\tau\rangle_{X X^{\prime} B B^{\prime}}=\sum_{x} \sqrt{p_{x}}|x\rangle_{X}|x\rangle_{X^{\prime}}\left|\xi_{x}\right\rangle_{B B^{\prime}}$ where $\left|\xi_{x}\right\rangle_{B B^{\prime}}$ purifies $\left(\varrho_{x}\right)_{B}$. The system $X^{\prime} \otimes B^{\prime}$ corresponds to the system $C$ in the proof of Lemma 2.8 and the state on $X \otimes X^{\prime} \otimes B^{\prime}$, i.e., $\tau_{X X^{\prime} B^{\prime}}$, is a classical-coherent state as required for Proposition 2.7 (note that the role of $B$ and $B^{\prime}$ are interchanged here and in the statement of Proposition 2.7).

We note that the special case $\alpha=2$ of the inequalities (31) and (37) was already shown in [8].

### 2.3.3 Equality condition for max-like entropies

In this section, we give a necessary and sufficient condition on a density operator $\varrho_{A B}$, such that the entropies $\bar{H}_{\alpha}^{\uparrow}(A \mid B)_{\varrho}$ and $\widetilde{H}_{\alpha}^{\uparrow}(A \mid B)_{\varrho}$ are equal for $\alpha \in\left[\frac{1}{2}, 1\right)$. To derive the necessary condition, let $\alpha \in(0,1)$. In the proof of Lemma 1 of [16], it is shown that the optimizer $\sigma_{B}^{\star}$ of $\bar{H}_{\alpha}^{\uparrow}(A \mid B)_{\varrho}=\sup _{\sigma_{B} \in \mathscr{D}(B)}-\bar{D}_{\alpha}\left(\varrho_{A B} \| \mathbb{1}_{A} \otimes \sigma_{B}\right)$ is given by

$$
\begin{equation*}
\sigma_{B}^{\star}=\frac{\left(\operatorname{tr}_{A} \varrho_{A B}^{\alpha}\right)^{\frac{1}{\alpha}}}{\operatorname{tr}\left(\operatorname{tr}_{A} \varrho_{A B}^{\alpha}\right)^{\frac{1}{\alpha}}} \tag{39}
\end{equation*}
$$

${ }^{5}$ We use again the convention that $\frac{1}{0}=\infty$.

By the ALT inequality [5, 6], we then find that

$$
\begin{equation*}
\bar{H}_{\alpha}^{\uparrow}(A \mid B)_{\varrho}=-\bar{D}_{\alpha}\left(\varrho_{A B} \| \mathbb{1}_{A} \otimes \sigma_{B}^{\star}\right) \leq \sup _{\sigma_{B} \in \mathscr{D}(B)}-\widetilde{D}_{\alpha}\left(\varrho_{A B} \| \mathbb{1}_{A} \otimes \sigma_{B}\right) \tag{40}
\end{equation*}
$$

According to Lemma 2.4, a necessary condition for equality in (40) is that $\left[\varrho_{A B}, \mathbb{1}_{A} \otimes \sigma_{B}^{\star}\right]=0$. Assume now that $\alpha \in\left[\frac{1}{2}, 1\right.$ ). To show that this condition is also sufficient for equality in (40), it suffices to show that the function $\sigma_{B} \mapsto-\widetilde{D}_{\alpha}\left(\varrho_{A B} \| \mathbb{1}_{A} \otimes \sigma_{B}\right)$ or equivalently $\sigma_{B} \mapsto \widetilde{Q}_{\alpha}\left(\varrho_{A B} \| \mathbb{1}_{A} \otimes\right.$ $\sigma_{B}$ ) attains its global maximum at $\sigma_{B}=\sigma_{B}^{\star}$ if $\left[\varrho_{A B}, \mathbb{1}_{A} \otimes \sigma_{B}^{\star}\right]=0$. The proof of this fact is based on standard derivative techniques, albeit for matrices, and is given in Appendix B. The results are summarized in the following Lemma.

Lemma 2.10 (Equality condition for entropies). Let $\alpha \in\left[\frac{1}{2}, 1\right), \varrho_{A B}$ be a density operator and $\hat{\sigma}_{B}^{\star}:=\operatorname{tr}_{A} \varrho_{A B}^{\alpha}$. Then, the following are equivalent

1. $\bar{H}_{\alpha}^{\uparrow}(A \mid B)_{\varrho}=\tilde{H}_{\alpha}^{\uparrow}(A \mid B)_{\varrho}$
2. $\left[\varrho_{A B}, \mathbb{1}_{A} \otimes \hat{\sigma}_{B}^{\star}\right]=0$.

## 3 Pretty good fidelity and the quality of pretty good measures

Our main results yield a unified framework relating pretty good measures often used in quantum information to their optimal counterparts.

### 3.1 Pretty good fidelity

Let $\varrho$ and $\sigma$ be two density operators throughout this subsection. We define the pretty good fidelity of $\varrho$ and $\sigma$ by

$$
\begin{equation*}
F_{\mathrm{pg}}(\varrho, \sigma):=\bar{Q}_{\frac{1}{2}}(\varrho, \sigma)=\operatorname{tr} \sqrt{\varrho} \sqrt{\sigma} . \tag{41}
\end{equation*}
$$

This quantity was called the "quantum affinity" in [20] and is nothing but the fidelity of the "pretty good purification" introduced in [21]: Letting $|\Omega\rangle_{A A^{\prime}}=\sum_{k}|k\rangle_{A}|k\rangle_{A^{\prime}}$, the canonical purification with respect to $|\Omega\rangle_{A A^{\prime}}$ of $\varrho$ is $\left|\Psi_{\varrho}\right\rangle_{A A^{\prime}}=\left(\sqrt{\varrho}_{A} \otimes \mathbb{1}_{A^{\prime}}\right)|\Omega\rangle_{A A^{\prime}}$, and thus

$$
\begin{equation*}
F_{\mathrm{pg}}(\varrho, \sigma)=\left\langle\Psi_{\varrho} \mid \Psi_{\sigma}\right\rangle_{A A^{\prime}} \tag{42}
\end{equation*}
$$

Recall that the usual fidelity is given by

$$
\begin{equation*}
F(\varrho, \sigma):=\widetilde{Q}_{\frac{1}{2}}(\varrho, \sigma)=\|\sqrt{\varrho} \sqrt{\sigma}\|_{1}=\max _{V_{A^{\prime}}}\left\langle\Psi_{\varrho}\right|\left(\mathbb{1}_{A} \otimes V_{A^{\prime}}\right)\left|\Psi_{\sigma}\right\rangle_{A A^{\prime}} \tag{43}
\end{equation*}
$$

where the maximum is taken over all unitary operators $V_{A^{\prime}}$ and the final equality follows from Uhlmann's theorem [22]. Therefore, it is clear that $F_{\mathrm{pg}}(\varrho, \sigma) \leq F(\varrho, \sigma)$. This can also be seen from the ALT inequality directly (cf. Corollary 2.3 for $\alpha=\frac{1}{2}$ ), and therefore, by Lemma 2.4, we have that $F_{\mathrm{pg}}(\varrho, \sigma)=F(\varrho, \sigma)$ if and only if $[\varrho, \sigma]=0$. The reverse ALT inequality implies a bound in the opposite direction; a similar approach using the Hölder inequality is given in [23]. By choosing $\alpha=1 / 2$, it follows from Corollary 2.3 that the fidelity is also upper bounded by the square root of the pretty good fidelity, i.e.,

$$
\begin{equation*}
F_{\mathrm{pg}}(\varrho, \sigma) \leq F(\varrho, \sigma) \leq \sqrt{F_{\mathrm{pg}}(\varrho, \sigma)} \tag{44}
\end{equation*}
$$

Hence the pretty good fidelity is indeed pretty good.
Recall that the trace distance between two density operators $\varrho$ and $\sigma$ is defined by $\delta(\varrho, \sigma):=$ $\frac{1}{2}\|\varrho-\sigma\|_{1}$. An important property of the fidelity is its relation to the trace distance [24]:

$$
\begin{equation*}
1-F(\varrho, \sigma) \leq \delta(\varrho, \sigma) \leq \sqrt{1-F(\varrho, \sigma)^{2}} \tag{45}
\end{equation*}
$$

Indeed the pretty good fidelity satisfies the same relation,

$$
\begin{equation*}
1-F_{\mathrm{pg}}(\varrho, \sigma) \leq \delta(\varrho, \sigma) \leq \sqrt{1-F_{\mathrm{pg}}(\varrho, \sigma)^{2}} . \tag{46}
\end{equation*}
$$

The upper bound follows immediately by combining the upper bound in (45) with the lower bound in (44). The lower bound was first shown in [25] (see also [23]).

### 3.2 Relation to bounds for the pretty good measurement and singlet fraction

In this section we show that together with entropy duality, the relation between fidelity and pretty good fidelity in (44) implies the known optimality bounds of the pretty good measurement and the pretty good singlet fraction. Let us first consider the optimal and pretty good singlet fraction. Define $R(A \mid B)_{\varrho}$ to be the largest achievable overlap with the maximally entangled state one can obtain from $\varrho_{A B}$ by applying a quantum channel on $B$. Formally,

$$
\begin{equation*}
R(A \mid B)_{\varrho}:=\max _{\mathscr{E}_{B \rightarrow A^{\prime}}} F\left(|\Phi\rangle\left\langle\left.\Phi\right|_{A A^{\prime}},\left(\mathbb{1}_{A} \otimes \mathscr{E}_{B \rightarrow A^{\prime}}\right) \varrho_{A B}\right)^{2},\right. \tag{47}
\end{equation*}
$$

where $|\Phi\rangle_{A A^{\prime}}=\frac{1}{\sqrt{|A|}} \sum_{k}|k\rangle_{A}|k\rangle_{A^{\prime}}$ and the maximization is over all completely positive, tracepreserving maps $\mathscr{E}_{B \rightarrow A^{\prime}}$. In [18] it was shown that

$$
\begin{equation*}
\widetilde{H}_{\infty}^{\uparrow}(A \mid B)_{\varrho}=-\log |A| R(A \mid B)_{\varrho} . \tag{48}
\end{equation*}
$$

A "pretty good" map $\mathscr{E}_{\text {pg }}$ was considered in [26], and it was shown that

$$
\begin{equation*}
\tilde{H}_{2}^{\downarrow}(A \mid B)_{\varrho}=-\log |A| R_{\mathrm{pg}}(A \mid B)_{\varrho}, \tag{49}
\end{equation*}
$$

where $R_{\mathrm{pg}}(A \mid B)_{\varrho}$ is the overlap obtained by using $\mathscr{E}_{\mathrm{pg}}$. Clearly $R_{\mathrm{pg}}(A \mid B)_{\varrho} \leq R(A \mid B)_{\varrho}$, but the case $\alpha=2$ in (31), which comes from (44) via entropy duality, implies that we also have

$$
\begin{equation*}
R_{\mathrm{pg}}(A \mid B)_{\varrho} \leq R(A \mid B)_{\varrho} \leq \sqrt{R_{\mathrm{pg}}(A \mid B)_{\varrho}} . \tag{50}
\end{equation*}
$$

This was also shown in [8]. Note that in the special case where $\varrho_{A B}$ has the form of a Choi state, i.e., $\operatorname{tr}_{B} \varrho_{A B}=\frac{1}{|A|} \mathbb{1}_{A}$, this statement also follows from [7].

Now let $\varrho_{X B}=\sum_{x} p_{x}|x\rangle\left\langle\left. x\right|_{X} \otimes\left(\varrho_{x}\right)_{B}\right.$ be a cq state, and consider an observer with access to the system $B$ who would like to guess the variable $X$. Denote by $p_{\text {guess }}(X \mid B)$ the optimal guessing probability which can be achieved by performing a POVM on the system $B$. It was shown in [18] that

$$
\begin{equation*}
\tilde{H}_{\infty}^{\uparrow}(X \mid B)_{\varrho}=-\log p_{\text {guess }}(X \mid B) . \tag{51}
\end{equation*}
$$

On the other hand, it is also known that [27]

$$
\begin{equation*}
\widetilde{H}_{2}^{\downarrow}(X \mid B)_{\varrho}=-\log p_{\text {guess }}^{\mathrm{pg}}(X \mid B), \tag{52}
\end{equation*}
$$

where $p_{\text {guess }}^{\mathrm{pg}}(X \mid B)$ denotes the guessing probability of the pretty good measurement introduced in [28,29]. Clearly $p_{\text {guess }}^{\mathrm{pg}}(X \mid B) \leq p_{\text {guess }}(X \mid B)$, but the case $\alpha=2$ in (37), which again comes from (44) via entropy duality, also implies that

$$
\begin{equation*}
p_{\text {guess }}^{\mathrm{pg}}(X \mid B) \leq p_{\text {guess }}(X \mid B) \leq \sqrt{p_{\text {guess }}^{\mathrm{pg}}(X \mid B)} . \tag{53}
\end{equation*}
$$

This was originally shown in [7].

### 3.3 Optimality conditions for pretty good measures

Our framework also yields a novel optimality condition for the pretty good measures. Supposing $\tau_{A B C}$ is a purification of $\varrho_{A B}$, the duality relations for Rényi entropies (cf. Lemma 2.5) imply

$$
\begin{equation*}
\tilde{H}_{2}^{\downarrow}(A \mid B)_{\tau}=\tilde{H}_{\infty}^{\uparrow}(A \mid B)_{\tau} \quad \Longleftrightarrow \quad \bar{H}_{1 / 2}^{\uparrow}(A \mid C)_{\tau}=\tilde{H}_{1 / 2}^{\uparrow}(A \mid C)_{\tau} \tag{54}
\end{equation*}
$$

Applying the equality condition for max-like conditional entropies, using Lemma 2.10, we find that the pretty good singlet fraction and pretty good measurement are optimal if and only if $\left[\tau_{A C}, \mathbb{1}_{A} \otimes \hat{\sigma}_{C}^{\star}\right]=0$, where $\hat{\sigma}_{C}^{\star}:=\operatorname{tr}_{A} \sqrt{\tau_{A C}}$. Alternately, this specific equality condition $(\alpha=1 / 2)$ can be established via weak duality of semidefinite programs, as described in Appendix C.

As a simple example of optimality of the pretty good singlet fraction, consider the case of a pure bipartite $\varrho_{A B}$. Then every purification $\tau_{A B C}=\varrho_{A B} \otimes \xi_{C}$ for some pure $\xi_{C}$. Thus, $\tau_{A C}=$ $\varrho_{A} \otimes \xi_{C}$, and it follows immediately that the optimality condition is satisfied. Optimality also holds for arbitrary mixtures of pure states, i.e., for states of the form $\varrho_{A B Y}=\sum_{y} q_{y}\left|\psi_{y}\right\rangle\left\langle\left.\psi_{y}\right|_{A B} \otimes \mid y\right\rangle\left\langle\left. y\right|_{Y}\right.$ with some arbitrary distribution $q_{y}$, provided both $B$ and $Y$ are used in the entanglement recovery operation. Here any purification takes the form $|\tau\rangle_{A B Y Y^{\prime}}=\sum_{y} \sqrt{q_{y}}\left|\psi_{y}\right\rangle_{A B}|y\rangle_{Y}|y\rangle_{Y^{\prime}}$. Hence, we have that $\tau_{A Y^{\prime}}=\sum_{y} q_{y} \operatorname{tr}_{B}\left|\psi_{y}\right\rangle\left\langle\left.\psi_{y}\right|_{A B} \otimes \mid y\right\rangle\left\langle\left. y\right|_{Y^{\prime}}\right.$, a state in which $Y^{\prime}$ is classical, for which it is easy to see that the optimality condition holds.

The optimality condition for the pretty good measurement can be simplified using the classical coherent nature of the state $\tau_{A C}$, which results in a condition formulated in terms of the Gram matrix. Suppose $\varrho_{X B}=\sum_{x} p_{x}|x\rangle\left\langle\left. x\right|_{X} \otimes\left(\varrho_{x}\right)_{B}\right.$ describes the ensemble of mixed states $\left(\varrho_{x}\right)_{B}$, for which a natural purification is given by

$$
\begin{equation*}
|\tau\rangle_{X X^{\prime} B B^{\prime}}=\sum_{x} \sqrt{p_{x}}|x\rangle_{X}|x\rangle_{X^{\prime}}\left|\xi_{x}\right\rangle_{B B^{\prime}} \tag{55}
\end{equation*}
$$

where $\left|\xi_{x}\right\rangle_{B B^{\prime}}$ denotes a purification of $\left(\varrho_{x}\right)_{B}$. Then we define the (generalized) Gram matrix $G$

$$
\begin{equation*}
G_{X^{\prime} B^{\prime}}:=\sum_{x, x^{\prime}} \sqrt{p_{x} p_{x^{\prime}}}|x\rangle\left\langle\left. x^{\prime}\right|_{X^{\prime}} \otimes \operatorname{tr}_{B} \mid \xi_{x}\right\rangle\left\langle\left.\xi_{x^{\prime}}\right|_{B B^{\prime}}\right. \tag{56}
\end{equation*}
$$

This definition reverts to the usual Gram matrix when the states $\left(\varrho_{x}\right)_{B}$ are pure and system $B^{\prime}$ is trivial. Observe that we are in the setting of Proposition 2.7; using the unitary $U_{X X^{\prime}}$ introduced in its proof, we find that $\left(U_{X X^{\prime}} \otimes \mathbb{1}_{B^{\prime}}\right) \tau_{X X^{\prime} B^{\prime}}\left(U_{X X^{\prime}}^{*} \otimes \mathbb{1}_{B^{\prime}}\right)=|0\rangle\left\langle\left. 0\right|_{X} \otimes G_{X^{\prime} B^{\prime}}\right.$. Hence, $\sqrt{\tau_{X X^{\prime} B^{\prime}}}=$ $\left(U_{X X^{\prime}}^{*} \otimes \mathbb{1}_{B^{\prime}}\right)\left(|0\rangle\left\langle\left. 0\right|_{X} \otimes \sqrt{G_{X^{\prime} B^{\prime}}}\right)\left(U_{X X^{\prime}} \otimes \mathbb{1}_{B^{\prime}}\right)\right.$ and a further calculation shows that $\operatorname{tr}_{X} \sqrt{\tau_{X X^{\prime} B^{\prime}}}=$ $\hat{\sigma}_{X^{\prime} B^{\prime}}^{\star}$, with

$$
\begin{equation*}
\hat{\sigma}_{X^{\prime} B^{\prime}}^{\star}:=\sum_{x}|x\rangle\left\langle\left. x\right|_{X^{\prime}} \otimes\langle x| \sqrt{G_{X^{\prime} B^{\prime}}} \mid x\right\rangle_{X^{\prime}} \tag{57}
\end{equation*}
$$

Note that $[M, N]=0$ is equivalent to $\left[U M U^{*}, U N U^{*}\right]=0$ for any square matrices $M, N$ and any unitary $U$. Therefore, we find that the equality condition $\left[\tau_{X X^{\prime} B^{\prime}}, \mathbb{1}_{X} \otimes \hat{\sigma}_{X^{\prime} B^{\prime}}^{\star}\right]=0$ is equivalent to $\left[|0\rangle\left\langle\left. 0\right|_{X} \otimes G_{X^{\prime} B^{\prime}}, \mathbb{1}_{X} \otimes \hat{\sigma}_{X^{\prime} B^{\prime}}^{\star}\right]=\left[G_{X^{\prime} B^{\prime}}, \hat{\sigma}_{X^{\prime} B^{\prime}}^{\star}\right]=0\right.$. Thus we have shown the following result:

Lemma 3.1 (Optimality condition for the pretty good measurement). The pretty good measurement is optimal for distinguishing states in the ensemble $\left\{p_{x}, \varrho_{x}\right\}$ if and only if $\left[G_{X^{\prime} B^{\prime}}, \hat{\sigma}_{X^{\prime} B^{\prime}}^{\star}\right]=0$.

In the case of distinguishing pure states, we recover Theorem 2 of [30] (which was first shown in [31]). To see this, observe that $B^{\prime}$ is now trivial and $G_{X^{\prime}}$ is the usual Gram matrix. Moreover, $\hat{\sigma}_{X^{\prime}}^{\star}$ is now the diagonal of the square root of $G_{X^{\prime}}$, and the commutation condition of Lemma 3.1 becomes $\left[G_{X^{\prime}}, \hat{\sigma}_{X^{\prime}}^{\star}\right]=0$, which is equivalent to the condition in equation (11) of [30] (in the case of the pretty good measurement). Reformulating what it means for the Gram matrix $G_{X^{\prime}}$ to commute with the diagonal matrix $\hat{\sigma}_{X^{\prime}}^{\star}$ then leads to Theorem 3 of [30].

## 4 Conclusions

We have given a novel reverse ALT inequality (see Theorem 2.1) that answers the question of how much bigger the Petz quantum Rényi divergence can be compared to the minimal quantum Rényi divergence for $\alpha \leq 1$. More precisely, together with the standard ALT inequality it implies that $\alpha \bar{D}_{\alpha}(\varrho \| \sigma) \leq \widetilde{D}_{\alpha}(\varrho \| \sigma) \leq \bar{D}_{\alpha}(\varrho \| \sigma)$ for $\alpha \leq 1$ and any density operators $\varrho$ and $\sigma$. This bound leads to an elegant unified framework of pretty good constructions in quantum information theory, and the ALT equality condition leads to a simple necessary and sufficient condition for their optimality. Previously it was observed that the min entropy $\widetilde{H}_{\infty}^{\uparrow}$ characterizes optimal measurement and singlet fraction, while $\widetilde{H}_{2}^{\downarrow}$ is the "pretty good min entropy" since it characterizes pretty good measurement and singlet fraction. On the other hand, we can think of $\bar{H}_{1 / 2}^{\uparrow}$ as the "pretty good max entropy" since it is based on the pretty good fidelity instead of the (usual) fidelity itself as in the max entropy $\widetilde{H}_{1 / 2}^{\uparrow}$. Entropy duality then beautifully links the two, as the (pretty good) max entropy is dual to the (pretty good) min entropy, and the known optimality bounds can be seen to stem from the lower bound on the pretty good fidelity in (44). Indeed, that such a unified picture might be possible was the original inspriation to look for a reverse ALT inequality of the form given in Theorem 2.1. It is also interesting to note that both the pretty good min and max entropies appear in achievability proofs of information processing tasks, the former in randomness extraction against quantum adversaries [32] and the latter in the data compression with quantum side information [33].

For future work, it would be interesting to elaborate more on the novel reverse ALT inequality (see Theorem 2.1). It is know that the ALT inequality implies the Golden-Thompson (GT) inequality $[34,35]$ via the Lie-Trotter product formula. Reverse versions of the GT inequality are well-studied [36]. It would be thus interesting to see if Theorem 2.1 can be related to the reverse GT inequality. Recent progress on proving multivariate trace inequalities [37] (see also [38]) suggests the possibility of an $n$-matrix extension of the reversed ALT inequality.

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## Appendix A Optimal marginals for classically coherent states

This appendix details the argument that cq states are optimal in the conditional entropy expressions for classically coherent states. First we recall the data processing inequality (DPI), which states that for all completely positive, trace-preserving maps $\mathscr{E}$ and for all non-negative operators $\varrho$ and $\sigma$, we have

$$
\begin{equation*}
D(\varrho \| \sigma) \geq D(\mathscr{E}(\varrho) \| \mathscr{E}(\sigma)) . \tag{58}
\end{equation*}
$$

It was shown that $\bar{D}_{\alpha}$ satisfies the DPI for $\alpha \in(0,1) \cup(1,2]$ in [2], while [39] (see also [17]) shows that $\widetilde{D}_{\alpha}$ satisfies the DPI for $\alpha \in\left[\frac{1}{2}, \infty\right]$. Following the approach taken in [40, Lemma A.1] to establish a similar result for the smooth min entropy, we can show

Lemma A.1. Let $|\varrho\rangle_{X X^{\prime} B B^{\prime}}=\sum_{x} \sqrt{P_{x}}|x\rangle_{X}|x\rangle_{X^{\prime}}\left|\xi_{x}\right\rangle_{B B^{\prime}}$ be a pure state on $X \otimes X^{\prime} \otimes B \otimes B^{\prime}$, where $p_{x} \in[0,1]$ with $\sum_{x} p_{x}=1$, and $X^{\prime} \simeq X$. Then, for any density operator $\sigma_{X^{\prime} B}$, we have that

$$
\begin{equation*}
Q_{\alpha}\left(\varrho_{X X^{\prime} B} \| \mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}\right) \leq Q_{\alpha}\left(\varrho_{X X^{\prime} B} \| \mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}^{c l}\right) \quad \text { for } \alpha \in\left[\frac{1}{2}, 1\right), \tag{59}
\end{equation*}
$$

where $\sigma_{X^{\prime} B}^{c l}:=\sum_{x}|x\rangle\left\langle\left. x\right|_{X^{\prime}} \otimes\langle x| \sigma_{X^{\prime} B} \mid x\right\rangle_{X^{\prime}}$.

Proof. Let $P_{X X^{\prime}}=\sum_{x}|x\rangle\left\langle\left. x\right|_{X} \otimes \mid x\right\rangle\left\langle\left. x\right|_{X^{\prime}}\right.$ and define the quantum channel $\mathscr{E}$ from $X \otimes X^{\prime}$ to itself by $\mathscr{E}(\cdot):=P_{X X^{\prime}}(\cdot) P_{X X^{\prime}}+\left(\mathbb{1}_{X X^{\prime}}-P_{X X^{\prime}}\right)(\cdot)\left(\mathbb{1}_{X X^{\prime}}-P_{X X^{\prime}}\right)$. Since $P_{X X^{\prime}}|\Psi\rangle_{X X^{\prime} B B^{\prime}}=|\Psi\rangle_{X X^{\prime} B B^{\prime}}, \mathscr{E}_{X X^{\prime}} \otimes \mathscr{I}_{B}$ leaves the density operator $\varrho_{X X^{\prime} B}$ invariant. By the DPI we then have, for $\alpha \in\left[\frac{1}{2}, 1\right)$,

$$
\begin{align*}
Q_{\alpha}\left(\varrho_{X X^{\prime} B} \| \mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}\right) & \leq Q_{\alpha}\left(\varrho_{X X^{\prime} B} \| \varepsilon_{X X^{\prime}} \otimes \mathscr{J}_{B}\left(\mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}\right)\right)  \tag{60}\\
& =Q_{\alpha}\left(\varrho_{X X^{\prime} B} \|\left(P_{X X^{\prime}} \otimes \mathbb{1}_{B}\right)\left(\mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}\right)\left(P_{X X^{\prime}} \otimes \mathbb{1}_{B}\right)\right) . \tag{61}
\end{align*}
$$

In the second line we use the fact that $Q_{\alpha}$ is indifferent to parts of its second argument which are not contained in the support of its first argument. Observe that $\left(P_{X X^{\prime}} \otimes \mathbb{1}_{B}\right)\left(\mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}\right)\left(P_{X X^{\prime}} \otimes \mathbb{1}_{B}\right)=$ $\sum_{x}|x\rangle\left\langle\left. x\right|_{X} \otimes \mid x\right\rangle\left\langle\left. x\right|_{X^{\prime}} \sigma_{X^{\prime} B} \mid x\right\rangle\left\langle\left. x\right|_{X^{\prime}} \leq \mathbb{1}_{X} \otimes \sigma_{X^{\prime} B}^{\mathrm{cl}}\right.$. Inequality (59) now follows directly from the dominance property of $D_{\alpha}$ (see e.g., [1]), which states (in terms of $Q_{\alpha}$ ) that $Q_{\alpha}(\varrho \| \sigma) \leq Q_{\alpha}\left(\varrho \| \sigma^{\prime}\right)$ for any non-negative operators $\varrho, \sigma, \sigma^{\prime}$ with $\sigma \leq \sigma^{\prime}$.

## Appendix B Sufficient condition for equality of max-like entropies

In this Appendix, we show that, for $\alpha \in\left[\frac{1}{2}, 1\right)$, the function $f_{\alpha}: \mathscr{D}(B) \ni \sigma_{B} \mapsto \widetilde{Q}_{\alpha}\left(\varrho_{A B} \| \mathbb{1}_{A} \otimes\right.$ $\sigma_{B}$ ) attains its global maximum at $\sigma_{B}=\sigma_{B}^{\star}$ if $\left[\varrho_{A B}, \mathbb{1}_{A} \otimes \sigma_{B}^{\star}\right]=0$. We use the notation of Section 2.3.3. The following lemma is similar to Lemma 5.1 of [41].

Lemma B.1. Let $I \subset \mathbb{R}$ be open and $t_{0} \in I$. Let $A(t)$ be a matrix whose entries are smooth functions of $t \in I$ and $A(t)>0$ for all $t \in I$. Further, let $B$ be a matrix such that $\left[B, A\left(t_{0}\right)\right]=0$. Then,

$$
\begin{equation*}
\left.\frac{d}{d t}\right|_{t=t_{0}} \operatorname{tr} B A(t)^{r}=r \operatorname{tr} B A\left(t_{0}\right)^{r-1} A^{\prime}\left(t_{0}\right) \quad \text { for } \quad r \in \mathbb{R} \tag{62}
\end{equation*}
$$

where $A^{\prime}\left(t_{0}\right):=\left.\frac{d}{d t}\right|_{t=t_{0}} A(t)$.
Proof. Note that it is straightforward to adapt Theorem 3.5 of [41] to the complex case. Therefore, by setting $\alpha=0$ in the equation (26) of [41], we find that

$$
\begin{equation*}
\left.\frac{d}{d t}\right|_{t=t_{0}} \operatorname{tr} B A(t)^{r}=r \operatorname{tr} B A^{\prime}\left(t_{0}\right) A\left(t_{0}\right)^{r-1}+r \operatorname{tr} B H_{0, r} A\left(t_{0}\right)^{r-1}, \tag{63}
\end{equation*}
$$

where $H_{0, r}$ is defined in equation (27) of [41]. Since $\left[A\left(t_{0}\right), B\right]=0$, a short calculation shows that $\operatorname{tr} B H_{0, r} A\left(t_{0}\right)^{r-1}=0$.

Lemma B.2. Set $I=(-\delta, \delta) \subset \mathbb{R}$ for some $\delta>0$ and let $A(t)$ be a matrix whose entries are smooth functions of $t \in I$ and $A(t)>0$ for all $t \in I$. For $B$ a density operator such that $[B, A(0)]=0$,

$$
\begin{equation*}
\left.\frac{d}{d t}\right|_{t=0} \widetilde{Q}_{\alpha}(B \| A(t))=(1-\alpha) \operatorname{Retr} B^{\alpha} A(0)^{-\alpha} A^{\prime}(0) \quad \text { for } \quad \alpha \in(0,1), \tag{64}
\end{equation*}
$$

where $A^{\prime}\left(t_{0}\right):=\left.\frac{d}{d t}\right|_{t=t_{0}} A(t)$ for $t_{0} \in I$.
Proof. To simplify the notation, let us define $\beta:=\frac{1-\alpha}{2 \alpha}$. We set $B_{\varepsilon}:=B+\varepsilon \mathbb{1}>0$ for some $\varepsilon>0$. Using Lemma B. 1 (with $A=A(t)^{\beta} B_{\varepsilon} A(t)^{\beta}$ and $B=\mathbb{1}$ ), we find

$$
\begin{align*}
\left.\frac{d}{d t}\right|_{t=t_{0}} & \operatorname{tr}\left(A(t)^{\beta} B_{\varepsilon} A(t)^{\beta}\right)^{\alpha} \\
& =\left.\alpha \operatorname{tr}\left(A\left(t_{0}\right)^{\beta} B_{\varepsilon} A\left(t_{0}\right)^{\beta}\right)^{\alpha-1} \frac{d}{d t}\right|_{t=t_{0}}\left(A(t)^{\beta} B_{\varepsilon} A(t)^{\beta}\right)  \tag{65}\\
& =\alpha \operatorname{tr}\left(A\left(t_{0}\right)^{\beta} B_{\varepsilon} A\left(t_{0}\right)^{\beta}\right)^{\alpha-1}\left(\left.\frac{d}{d t}\right|_{t=t_{0}} A(t)^{\beta} B_{\varepsilon} A\left(t_{0}\right)^{\beta}+\left.A\left(t_{0}\right)^{\beta} B_{\varepsilon} \frac{d}{d t}\right|_{t=t_{0}} A(t)^{\beta}\right) . \tag{66}
\end{align*}
$$

This can be simplified by noting that for any Hermitian matrix $H$ and any matrix $C$,

$$
\begin{equation*}
\operatorname{tr} H\left(C+C^{*}\right)=\operatorname{tr} H C+\operatorname{tr} H C^{*}=\operatorname{tr} H C+\operatorname{tr} H^{*} C^{*}=\operatorname{tr} H C+(\operatorname{tr} H C)^{*}=2 \operatorname{Re} \operatorname{tr} H C \tag{67}
\end{equation*}
$$

Using this we obtain

$$
\begin{align*}
\left.\frac{d}{d t}\right|_{t=t_{0}} \widetilde{Q}_{\alpha}\left(B_{\varepsilon} \| A(t)\right) & =\left.2 \alpha \operatorname{Retr}\left(A\left(t_{0}\right)^{\beta} B_{\varepsilon} A\left(t_{0}\right)^{\beta}\right)^{\alpha-1} \frac{d}{d t}\right|_{t=t_{0}} A(t)^{\beta} B_{\varepsilon} A\left(t_{0}\right)^{\beta}  \tag{68}\\
& =\left.2 \alpha \operatorname{Retr} A\left(t_{0}\right)^{-\beta}\left(A\left(t_{0}\right)^{\beta} B_{\varepsilon} A\left(t_{0}\right)^{\beta}\right)^{\alpha} \frac{d}{d t}\right|_{t=t_{0}} A(t)^{\beta} \tag{69}
\end{align*}
$$

Taking the limit $\varepsilon \rightarrow 0$ yields

$$
\begin{equation*}
\left.\lim _{\varepsilon \rightarrow 0} \frac{d}{d t}\right|_{t=t_{0}} \widetilde{Q}_{\alpha}\left(B_{\varepsilon} \| A(t)\right)=\left.2 \alpha \operatorname{Re} \operatorname{tr} A\left(t_{0}\right)^{-\beta}\left(A\left(t_{0}\right)^{\beta} B A\left(t_{0}\right)^{\beta}\right)^{\alpha} \frac{d}{d t}\right|_{t=t_{0}} A(t)^{\beta} . \tag{70}
\end{equation*}
$$

At $t_{0}=0$ the righthand side can be simplified by again making use of Lemma B. 1 as well as $[A(0), B]=0$ :

$$
\begin{align*}
\left.\lim _{\varepsilon \rightarrow 0} \frac{d}{d t}\right|_{t=0} \widetilde{Q}_{\alpha}\left(B_{\varepsilon} \| A(t)\right) & =\left.2 \alpha \operatorname{Retr} B^{\alpha} A(0)^{\beta(2 \alpha-1)} \frac{d}{d t}\right|_{t=0} A(t)^{\beta}  \tag{71}\\
& =(1-\alpha) \operatorname{Retr} B^{\alpha} A(0)^{-\alpha} A^{\prime}(0) \tag{72}
\end{align*}
$$

It remains to be shown that the limit can be interchanged with the derivative. This follows if we ensure that $\left.\frac{d}{d t}\right|_{t=t_{0}} \widetilde{Q}_{\alpha}\left(B_{\varepsilon} \| A(t)\right)$ converges uniformly in $t_{0} \in[-\delta / 2, \delta / 2]$ for $\varepsilon \rightarrow 0$. To show uniform convergence, it suffices to show

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \sup _{t_{0} \in[-\delta / 2, \delta / 2]}\left\|\left.A\left(t_{0}\right)^{-\beta}\left[\left(A\left(t_{0}\right)^{\beta} B_{\varepsilon} A\left(t_{0}\right)^{\beta}\right)^{\alpha}-\left(A\left(t_{0}\right)^{\beta} B A\left(t_{0}\right)^{\beta}\right)^{\alpha}\right] \frac{d}{d t}\right|_{t=t_{0}} A(t)^{\beta}\right\|_{1}=0 \tag{73}
\end{equation*}
$$

where we used that $|\operatorname{tr}(M)| \leqslant\|M\|_{1}$ for any square matrix $M$ (see, e.g., [9, Exercise IV 2.12]). By the generalized Hölder inequality for matrices (see (7)), we find that it is enough to show that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \sup _{t_{0} \in[-\delta / 2, \delta / 2]}\left\|A\left(t_{0}\right)^{-\beta}\right\|_{\infty}\left\|\left(A\left(t_{0}\right)^{\beta} B_{\varepsilon} A\left(t_{0}\right)^{\beta}\right)^{\alpha}-\left(A\left(t_{0}\right)^{\beta} B A\left(t_{0}\right)^{\beta}\right)^{\alpha}\right\|_{1}\left\|\left.\frac{d}{d t}\right|_{t=t_{0}} A(t)^{\beta}\right\|_{\infty}=0 \tag{74}
\end{equation*}
$$

Note that the infinity-norm terms are bounded on the compact interval $t_{0} \in[-\delta / 2, \delta / 2]$, as $A(t)^{\beta}$ is continuously differentiable for $A(t)>0$. Thus, we need only show that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \sup _{t_{0} \in[-\delta / 2, \delta / 2]}\left\|\left(A\left(t_{0}\right)^{\beta} B_{\varepsilon} A\left(t_{0}\right)^{\beta}\right)^{\alpha}-\left(A\left(t_{0}\right)^{\beta} B A\left(t_{0}\right)^{\beta}\right)^{\alpha}\right\|_{1}=0 \tag{75}
\end{equation*}
$$

Since $t \rightarrow t^{\alpha}$ is operator monotone for $\alpha \in[0,1]$ (Löwner's theorem [42]), the matrix inside the trace norm is positive, and hence (75) is equivalent to

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \sup _{t_{0} \in[-\delta / 2, \delta / 2]}\left\|\left(A\left(t_{0}\right)^{\beta} B_{\varepsilon} A\left(t_{0}\right)^{\beta}\right)^{\alpha}\right\|_{1}-\left\|\left(A\left(t_{0}\right)^{\beta} B A\left(t_{0}\right)^{\beta}\right)^{\alpha}\right\|_{1}=0 \tag{76}
\end{equation*}
$$

Note that $\varepsilon \mapsto\left\|\left(A\left(t_{0}\right)^{\beta} B_{\varepsilon} A\left(t_{0}\right)^{\beta}\right)^{\alpha}\right\|_{1}$ is monotonically decreasing (again by Löwner's theorem). Then, by Dini's theorem, it converges uniformly to $\left\|\left(A\left(t_{0}\right)^{\beta} B A\left(t_{0}\right)^{\beta}\right)^{\alpha}\right\|_{1}$, which proves (76), and hence the desired uniformity of the convergence.

We are now ready to calculate the derivative of the function $f_{\alpha}$ at $\sigma_{B}=\sigma_{B}^{\star}$.

Lemma B.3. Let $\alpha \in\left[\frac{1}{2}, 1\right)$ and $\varrho_{A B} \in \mathscr{D}(A \otimes B)$ be such that $\left[\varrho_{A B}, \mathbb{1}_{A} \otimes \sigma_{B}^{\star}\right]=0$. Then the function $f_{\alpha}: \mathscr{D}(B) \ni \sigma_{B} \mapsto \widetilde{Q}_{\alpha}\left(\varrho_{A B} \| \mathbb{1}_{A} \otimes \sigma_{B}\right)$ attains its global maximum at $\sigma_{B}^{\star}$ as defined in (39).
Proof. First consider the case $\varrho_{A B}>0$ for simplicity; we return to the rank-deficient case be$\widetilde{\sim}_{\alpha}$ low. Since $(\varrho, \sigma) \mapsto \widetilde{Q}_{\alpha}(\varrho \| \sigma)$ is jointly concave [17, 39], the function $f_{\alpha}: \mathscr{D}(B) \ni \sigma_{B} \mapsto$ $\widetilde{Q}_{\alpha}\left(\varrho_{A B} \| \mathbb{1}_{A} \otimes \sigma_{B}\right)$ is concave. As $\mathscr{D}(B)$ is a convex set, it suffices to show that $f_{\alpha}$ has an extreme point at $\sigma_{B}^{\star}$ (which is then also a global maximum). Observe that $\sigma_{B}^{\star}>0$ by definition, and therefore all states $\sigma_{B}(t)$ along arbitrary paths of states through $\sigma_{B}(0)=\sigma_{B}^{\star}$ have full rank for all $t$ sufficiently close to zero. Thus, we may use Lemma B. 2 to compute the derivative along any such path and find

$$
\begin{align*}
\left.\frac{d}{d t}\right|_{t=0} \widetilde{Q}_{\alpha}\left(\varrho_{A B}| | \mathbb{1}_{A} \otimes \sigma_{B}(t)\right) & =(1-\alpha) \operatorname{Retr} \varrho_{A B}^{\alpha}\left(\mathbb{1}_{A} \otimes \sigma_{B}^{\star}\right)^{-\alpha}\left(\left.\mathbb{1}_{A} \otimes \frac{d}{d t}\right|_{t=0} \sigma_{B}(t)\right)  \tag{77}\\
& =(1-\alpha) \operatorname{Retr}\left(\left.\operatorname{tr}_{A}\left(\varrho_{A B}^{\alpha}\right)\left(\sigma_{B}^{\star}\right)^{-\alpha} \frac{d}{d t}\right|_{t=0} \sigma_{B}(t)\right)=0 \tag{78}
\end{align*}
$$

Therefore $\sigma_{B}^{\star}$ is the optimizer in this case.
For $\varrho_{A B}$ not strictly positive, we can restrict the set of marginal states $\sigma_{B}$ to the support of $\sigma_{B}^{\star}$ and replay the above argument. To see this, first observe that the support of $\sigma_{B}^{\star}$ is the same as that of $\varrho_{B}$. Furthermore, as noted in [3], the DPI for $\widetilde{D}_{\alpha}$ implies that the maximum of $f_{\alpha}$ is always attained at a density matrix $\sigma_{B}^{\star}$ satisfying $\sigma_{B}^{\star} \ll \varrho_{B}$. Therefore, we can restrict the domain of the function $f_{\alpha}$ to the set $\mathscr{P}(B):=\left\{\sigma_{B} \in \mathscr{D}(B): \sigma_{B} \ll \sigma_{B}^{\star}\right\}$. Now observe that $\operatorname{ker}\left(\mathbb{1}_{A} \otimes \sigma_{B}^{\star}\right) \subseteq \operatorname{ker}\left(\varrho_{A B}\right)$. For any $|\psi\rangle_{B}$ we have $\langle\psi| \varrho_{B}|\psi\rangle_{B}=\sum_{k}\left\langle\left. k\right|_{A}\left\langle\left.\psi\right|_{B} \varrho_{A B} \mid k\right\rangle_{A} \mid \psi\right\rangle_{B}$. By positivity of $\varrho_{A B} \geq 0$, each $|\psi\rangle_{B} \in \operatorname{ker}\left(\sigma_{B}^{\star}\right)=\operatorname{ker}\left(\varrho_{B}\right)$ leads to a set of states $|k\rangle_{A} \otimes|\psi\rangle_{B} \in \operatorname{ker}\left(\varrho_{A B}\right)$. This implies that projecting $\varrho_{A B}$ to the support of $\mathbb{1}_{A} \otimes \sigma_{B}^{\star}$ has no effect on $\widetilde{Q}_{\alpha}$. Hence, we can restrict all operators in the problem to this subspace, where again all states in $\mathscr{P}(B)$ sufficiently close to $\sigma_{B}^{\star}$ have full rank.

Appendix C Optimality condition for pretty good measures via semidefinite programming
Here we derive the optimality condition for pretty good measures via weak duality of semidefinite programs. In terms of fidelity and pretty good fidelity, the optimality condition in (54) reads

$$
\begin{equation*}
F_{\mathrm{pg}}\left(\tau_{A C}, \mathbb{1}_{A} \otimes \sigma_{C}^{\star}\right)=\sup _{\sigma \in \mathscr{D}(C)} F\left(\tau_{A C}, \mathbb{1}_{A} \otimes \sigma_{C}\right) \tag{79}
\end{equation*}
$$

where $\sigma_{C}^{\star}$ is as in (39) with $\alpha=1 / 2$. Lemma 2.4 implies that $\left[\tau_{A C}, \mathbb{1}_{A} \otimes \sigma_{C}^{\star}\right.$ ] $=0$ is necessary for (79) to hold. Sufficiency, meanwhile, is the statement that $\sigma_{C}^{\star}$ is the optimizer on the righthand side. We can show this by formulating the optimization as a semidefinite program and finding a matching upper bound using the dual program.

In particular, following [43], the optimal value of the (primal) semidefinite program

$$
\begin{align*}
\gamma=\text { sup } & \operatorname{tr} W_{A C A^{\prime} C^{\prime}} \tau_{A C A^{\prime} C^{\prime}} \\
\text { s.t. } & \operatorname{tr}_{A^{\prime} C^{\prime}} W_{A C A^{\prime} C^{\prime}} \leq \mathbb{1}_{A} \otimes \sigma_{C} \\
& \operatorname{tr} \sigma_{C} \leq 1  \tag{80}\\
& W_{A C A^{\prime} C^{\prime}}, \sigma_{C} \geq 0
\end{align*}
$$

satisfies $\gamma=\sup _{\sigma \in \mathscr{D}(C)} F\left(\tau_{A C}, \mathbb{1}_{A} \otimes \sigma_{C}\right)^{2}$. Here $A^{\prime} \simeq A, C^{\prime} \simeq C$, and we take $\tau_{A C A^{\prime} C^{\prime}}$ to be the canonical purification of $\tau_{A C}$ as in Section 3.1. Using Watrous's general form for semidefinite programs we can easily derive the dual, which turns out to be

$$
\begin{align*}
\beta=\inf & \mu \\
\text { s.t. } & Z_{A C} \otimes \mathbb{1}_{A^{\prime} C^{\prime}} \geq \tau_{A C A^{\prime} C^{\prime}} \\
& \mu \mathbb{1}_{C} \geq \operatorname{tr}_{A} Z_{A C}  \tag{81}\\
& \mu, Z_{A C} \geq 0
\end{align*}
$$

By weak duality $\gamma \leq \beta$, but the following choice of $\mu$ and $Z_{A C}$ gives $\beta=F_{\mathrm{pg}}\left(\tau_{A C}, \mathbb{1}_{A} \otimes \sigma_{C}^{\star}\right)^{2}$ and therefore (79):

$$
\begin{equation*}
\mu^{\star}=\left(\operatorname{tr} \sqrt{\tau_{A C}} \sqrt{\mathbb{1}_{A} \otimes \sigma_{C}^{\star}}\right)^{2} \quad \text { and } \quad Z_{A C}^{\star}=\operatorname{tr}\left(\sqrt{\tau_{A C}} \sqrt{\mathbb{1}_{A} \otimes \sigma_{C}^{\star}}\right) \tau_{A C}^{1 / 2}\left(\mathbb{1}_{A} \otimes \sigma_{C}^{\star}\right)^{-1 / 2} \tag{82}
\end{equation*}
$$

Here the inverse of $\mathbb{1}_{A} \otimes \sigma_{C}^{\star}$ is taken on its support. To see that the first feasibility constraint is satisfied, start with the operator inequality

$$
\begin{equation*}
\mathbb{1}_{A C A^{\prime} C^{\prime}} \operatorname{tr} \sqrt{\tau_{A C}} \sqrt{\mathbb{1}_{A} \otimes \sigma_{C}^{\star}} \geq\left(\tau_{A C}^{1 / 4}\left(\mathbb{1}_{A} \otimes \sigma_{C}^{\star}\right)^{1 / 4} \otimes \mathbb{1}_{A^{\prime} C^{\prime}}\right) \Omega_{A C A^{\prime} C^{\prime}}\left(\tau_{A C}^{1 / 4}\left(\mathbb{1}_{A} \otimes \sigma_{C}^{\star}\right)^{1 / 4} \otimes \mathbb{1}_{A^{\prime} C^{\prime}}\right) \tag{83}
\end{equation*}
$$

which holds because the righthand side is the canonical purification of the positive operator $\sqrt{\tau_{A C}} \sqrt{\mathbb{1}_{A} \otimes \sigma_{C}^{\star}}$ and the trace factor on the left is its normalization. Conjugating both sides by $\tau_{A C}^{1 / 4}\left(\mathbb{1}_{A} \otimes \sigma_{C}^{\star}\right)^{-1 / 4} \otimes \mathbb{1}_{A^{\prime} C^{\prime}}$ preserves the positivity ordering and gives

$$
\begin{equation*}
\operatorname{tr}\left(\sqrt{\tau_{A C}} \sqrt{\mathbb{1}_{A} \otimes \sigma_{C}^{\star}}\right) \tau_{A C}^{1 / 2}\left(\mathbb{1}_{A} \otimes \sigma_{C}^{\star}\right)^{-1 / 2} \otimes \mathbb{1}_{A^{\prime} C^{\prime}} \geq\left(\tau_{A C}^{1 / 2} \otimes \mathbb{1}_{A^{\prime} C^{\prime}}\right) \Omega_{A C A^{\prime} C^{\prime}}\left(\tau_{A C}^{1 / 2} \otimes \mathbb{1}_{A^{\prime} C^{\prime}}\right) \tag{84}
\end{equation*}
$$

where we used that $\operatorname{ker}\left(\mathbb{1}_{A} \otimes \sigma_{C}^{\star}\right) \subseteq \operatorname{ker}\left(\tau_{A C}\right)$ (just as in the proof of Lemma B.3), ensuring that $\tau_{A C}\left(\mathbb{1}_{A} \otimes \sigma_{C}^{\star}\right)^{-1}\left(\mathbb{1}_{A} \otimes \sigma_{C}^{\star}\right)=\tau_{A C}$. Note that inequality (84) shows that $Z_{A C}^{\star} \otimes \mathbb{1}_{A^{\prime} C^{\prime}} \geq \tau_{A C A^{\prime} C^{\prime}}$. Meanwhile, the second constraint is satisfied (with equality in the case where $\sigma_{C}^{\star}$ has full rank) because direct calculation shows that $\operatorname{tr}_{A} \tau_{A C}^{1 / 2}\left(\mathbb{1}_{A} \otimes \sigma_{C}^{\star}\right)^{-1 / 2} \leq \mathbb{1}_{C} \operatorname{tr} \sqrt{\tau_{A C}} \sqrt{\mathbb{1}_{A} \otimes \sigma_{C}^{\star}}$.

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[^0]:    ${ }^{1}$ Note that "singlet" refers to a maximally entangled state (and not necessarily to the maximally entangled two-qubit state) [8].

[^1]:    ${ }^{2}$ Alternatively, this follows directly from the ALT inequality.

[^2]:    ${ }^{3}$ Here we use our assumption that $q^{\prime} \geq 1$, since in this case $\|\cdot\|_{q^{\prime}}$ is a strictly increasing norm.
    ${ }^{4}$ We are following the notation in [1]. Note that $H_{\text {min }}(A \mid B)_{\varrho \mid \varrho}=\widetilde{H}_{\infty}^{\downarrow}(A \mid B)_{\varrho}, H_{\text {min }}(A \mid B)_{\varrho}=\widetilde{H}_{\infty}^{\uparrow}(A \mid B)_{\varrho}$ and $H_{\max }(A \mid B)_{\varrho}=\tilde{H}_{\frac{1}{2}}^{\uparrow}(A \mid B)_{\varrho}$ are also often used notations.

