# Additive bounds of minimum output entropies for unital channels and an exact qubit formula 

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We investigate minimum output (Rényi) entropy of qubit channels and unital quantum channels. We obtain an exact formula for the minimum output entropy of qubit channels, and bounds for unital quantum channels. Interestingly, our bounds depend only on the operator norm of the matrix representation of the channels on the space of trace-less Hermitian operators. Moreover, since these bounds respect tensor products, we get bounds for the capacity of unital quantum channels, which is saturated by the Werner-Holevo channel. Furthermore, we construct an orthonormal basis, besides the Gell-Mann basis, for the space of trace-less Hermitian operators by using discrete Weyl operators. We apply our bounds to discrete Weyl covariant channels with this basis, and find new examples in which the minimum output Rényi 2-entropy is additive.

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## I. INTRODUCTION

## A. Preliminary

## 1. Maps on Hermitian matrices, norms and entropies

Consider the real vector space of $n \times n$ Hermitian matrices denoted by $H_{n}$. Let $H_{n,+}$ be the positive cone and $H_{n, t}$ the affine space of matrices with trace $t$ in $H_{n}$. Denote also by $H_{n,+, 1}=H_{n,+} \cap H_{n, 1}$ the set of density matrices (quantum states). We will use sometimes the capitals $K$ and $L$ to represent other spaces for notational convenience.

For a linear map $\Phi: H_{n} \rightarrow H_{k}$, we define the $p \rightarrow q$ norm by

$$
\begin{equation*}
\|\Phi\|_{p \rightarrow q} \equiv \max _{0 \neq \rho \in H_{n}} \frac{\|\Phi(\rho)\|_{q}}{\|\rho\|_{p}} . \tag{1}
\end{equation*}
$$

Note that in AH03] it is shown that for $p=1$

$$
\begin{equation*}
\|\Phi\|_{1 \rightarrow q}=\max _{\rho \in H_{n,+1}}\|\Phi(\rho)\|_{q} \tag{2}
\end{equation*}
$$

(Another norm defined for maps between complex vector spaces of matrices is studied briefly in Section IV.)
A quantum channel is a completely positive and trace-preserving map. For a quantum channel $\Phi$, we define the minimum output Rényi $\alpha$-entropy by

$$
\begin{equation*}
S_{\min , \alpha}(\Phi)=\min _{\rho \in H_{n,+, 1}} S_{\alpha}(\Phi(\rho)) \tag{3}
\end{equation*}
$$

Here, $S_{\alpha}(\cdot)$ is the Rényi $\alpha$-entropy:

$$
\begin{equation*}
S_{\alpha}(\sigma)=\frac{1}{1-\alpha} \log \left(\operatorname{Tr} \sigma^{\alpha}\right) \tag{4}
\end{equation*}
$$

which is defined for $\sigma \in H_{k,+, 1}$ and $0 \leq \alpha \leq \infty$; this is well-defined for $\alpha=1, \infty$ by taking limit. For $\alpha=1$, it is the von Neumann entropy:

$$
\begin{equation*}
S(\sigma)=-\operatorname{Tr}[\sigma \log \sigma] \tag{5}
\end{equation*}
$$

[^0]From now on, we write $S(\cdot)=S_{1}(\cdot)$ and $S_{\min }(\cdot)=S_{\min , 1}(\cdot)$. Importantly, $S_{\alpha}(\cdot)$ is non-increasing in $\alpha$.
For a channel $\Phi$, we can see easily that

$$
\begin{equation*}
S_{\min , \alpha}(\Phi)=\frac{\alpha}{1-\alpha} \log \left(\|\Phi\|_{1 \rightarrow \alpha}\right) \tag{6}
\end{equation*}
$$

In general, it is difficult to calculate $S_{\min , \alpha}(\cdot)$ or $\|\Phi\|_{1 \rightarrow \alpha}$, but we get an exact formula for qubit channels in Section III and bounds for unital channels in Section III. Interestingly, these bounds are saturated for depolarizing channel and Werner-Holevo channel, see Remark 11

## 2. Additivity and multiplicativity of channels

Two channels $\Phi$ and $\Omega$ are said to be additive if

$$
\begin{equation*}
S_{\min , \alpha}(\Phi \otimes \Omega)=S_{\min , \alpha}(\Phi)+S_{\min , \alpha}(\Omega) \tag{7}
\end{equation*}
$$

This is equivalent to the multiplicativity:

$$
\begin{equation*}
\|\Phi \otimes \Omega\|_{1 \rightarrow \alpha}=\|\Phi\|_{1 \rightarrow \alpha}\|\Omega\|_{1 \rightarrow \alpha} \tag{8}
\end{equation*}
$$

The above equivalence can be seen from (6). The additivity (or multiplicativity) of channels were conjectured for $p=1$ in KR01 and for $1 \leq p \leq \infty$ in AHW00. They were proved not to hold in general for $1<p$ in HW08 and $p=1$ in Has09. See also [FKM10, BH10, FK10. For $p=1$ it was also shown to be locally additive in GF12. Proofs in terms of asymptotic geometric analysis are found in ASW10, ASW10, Fuk14. Those additivity/non additivity properties are important for communication theory, see Hol06.

For $p=2$, no example is found for additivity violation. Not many additive example are found yet either. Entanglement breaking channels Kin03b, unital qubit channels Kin02 and depolarizing channels Kin03a are proved to be additive as well as some examples in DFH06. We add up other additive examples in Section IIID.

## 3. Complementary channels

This subsection contains facts we use in Section III C 3 and Section IIID. Complementary channels were investigated in relation to additivity questions in Hol05b, KMNR07. The idea of complementary channels is to swap the output and the environment spaces in the framework of Stinespring dilation theorem to create another channel. For pure input states, a channel $\Phi$ and its complementary $\Phi^{C}$ share the same non-zero eigenvalues of output states. Given a channel in the Kraus form:

$$
\Phi(\rho)=\sum_{i} A_{i} \rho A_{i}^{*}
$$

we can define its complementary channel uniquely up to isomorphism:

$$
\left(\Phi^{C}(\rho)\right)_{i, j}=\operatorname{Tr}\left[A_{i} \rho A_{j}^{*}\right]
$$

Moreover, since $(\Phi \otimes \Omega)^{C}=\Phi^{C} \otimes \Omega^{C}$, we have for $1 \leq p \leq \infty$

$$
\begin{equation*}
\|\Phi \otimes \Omega\|_{1 \rightarrow p}=\left\|\Phi^{C} \otimes \Omega^{C}\right\|_{1 \rightarrow p} \tag{9}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
S_{\min , \alpha}(\Phi \otimes \Omega)=S_{\min , \alpha}\left(\Phi^{C} \otimes \Omega^{C}\right) \tag{10}
\end{equation*}
$$

This means that channels are additive if and only if so are their complementary channels:

$$
\begin{equation*}
\|\Phi \otimes \Omega\|_{1 \rightarrow p}=\|\Phi\|_{1 \rightarrow p}\|\Omega\|_{1 \rightarrow p} \Longleftrightarrow\left\|\Phi^{C} \otimes \Omega^{C}\right\|_{1 \rightarrow p}=\left\|\Phi^{C}\right\|_{1 \rightarrow p}\left\|\Omega^{C}\right\|_{1 \rightarrow p} \tag{11}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
S_{\min , \alpha}(\Phi \otimes \Omega)=S_{\min , \alpha}(\Phi)+S_{\min , \alpha}(\Omega) \Longleftrightarrow S_{\min , \alpha}\left(\Phi^{C} \otimes \Omega^{C}\right)=S_{\min , \alpha}\left(\Phi^{C}\right)+S_{\min , \alpha}\left(\Omega^{C}\right) \tag{12}
\end{equation*}
$$

For more details, see Hol05b, KMNR07. This concept of complementarity is applied in Section III C 3 and Section IIID.

## B. Our results

## 1. Qubit inputs

We first consider a quantum channel $\Phi: H_{2} \rightarrow H_{n}$ with a qubit input. In this case, we found a closed formula in Theorem 2 for $\|\Phi\|_{1 \rightarrow 2}$, which also provides a closed formula for $S_{\min , 2}(\Phi)=-\log \|\Phi\|_{1 \rightarrow 2}^{2}$. If in addition also the output space is 2-dimensional, then the formula for $\|\Phi\|_{1 \rightarrow 2}$ can be used to derive a closed formula for $S_{\min , \alpha}$ for any $0 \leq \alpha \leq \infty$ (see corollary 4). We arrive at these formulas using the Bloch representation of a qubit. Since any pure qubit can be identified with a point on the 3-dimensional Bloch sphere, the optimization involved in the calculation of $\|\Phi\|_{1 \rightarrow 2}$ is relatively a simple one. However, if the input dimension is higher than 2 , the optimization is no longer over a three dimensional sphere, and therefore becomes more cumbersome.

For a quantum channel $\Phi: H_{2} \rightarrow H_{n}$, the closed formula for $\|\Phi\|_{1 \rightarrow 2}$, can also be used to derive an upper bound on the Holevo capacity of $\Phi$. The Holevo capacity is defined by

$$
\begin{equation*}
\chi(\Phi):=\max _{\left\{\left(\rho_{i}, p_{i}\right)\right\}_{i}}\left[S\left(\sum_{i} p_{i} \Phi\left(\rho_{i}\right)\right)-\sum_{i} p_{i} S\left(\Phi\left(\rho_{i}\right)\right)\right] \tag{13}
\end{equation*}
$$

Therefore, for a quantum channel $\Phi$ with output dimension $n$

$$
\chi(\Phi) \leq \log (n)-S_{\min }(\Phi)
$$

where it is known from the results in Kin02 and Hol05a that equality holds for unital qubit channels. From theorem 6 we have $S_{\min }(\Phi) \geq g\left(\|\Phi\|_{1 \rightarrow 2}^{2}\right)$ where the function $g$ is given in Eq. 25). This lower bound is optimal in the sense that if another function $f$ satisfying $S_{\min }(\Phi) \geq f\left(\|\Phi\|_{1 \rightarrow 2}^{2}\right)$, then $g\left(\|\Phi\|_{1 \rightarrow 2}^{2}\right) \geq f\left(\|\Phi\|_{1 \rightarrow 2}^{2}\right)$. In particular, $g\left(\|\Phi\|_{1 \rightarrow 2}^{2}\right) \geq-\log \left(\|\Phi\|_{1 \rightarrow 2}^{2}\right)$; see Fig. 1. Therefore, for a qubit-input channel $\Phi: H_{2} \rightarrow H_{n}$ we obtain the following new upper bound for the Holevo capacity:

$$
\begin{equation*}
\chi(\Phi) \leq \log (n)-g\left(\|\Phi\|_{1 \rightarrow 2}^{2}\right) \tag{14}
\end{equation*}
$$

where the closed expression for $\|\Phi\|_{1 \rightarrow 2}$ is given in Theorem 2, and the function $g$ is defined in Eq. 25). This upper bounds becomes an equality for unital qubit channels. Eq. (14) holds for all channels $\Phi: H_{m} \rightarrow H_{n}$, but the closed expression for $\|\Phi\|_{1 \rightarrow 2}$ (see Theorem 22) holds only for $m=2$.

## 2. Multiplicative bounds and operational meanings

A trace-preserving linear map $\Phi: H_{n} \rightarrow H_{k}$ is called unital if

$$
\begin{equation*}
\Phi\left(I_{n} / n\right)=I_{k} / k \tag{15}
\end{equation*}
$$

In Section III A, we derive lower bounds for the minimum output Rényi 2-entropy for unital quantum channels; we also derive bounds for $p \rightarrow q$ norms for general unital trace-preserving linear maps and apply it to unital quantum channels (i.e. completely positive maps). Interestingly, these bounds respect tensor products, which can be used to bound output Rényi 2-entropy of tensor products of many unital channels, which is stated in Theorem 10. Importantly, these bounds are calculated from a function $\gamma(\cdot)$, given in (40). As you can see in $(33), \gamma(\cdot)$ depends on the operator norm of matrix representation of maps. Let us make some historical notes. In Mon13 an idea of multiplicative bounds is used to bound minimum output Rényi $\infty$-entropy of tensor products of channels. Other multiplicative bounds, for example the operator norm of partially transposed Choi matrices, were found in [FN14] to bound minimum output Rényi 2-entropy of tensor products of channels.

For the rest of this section, we deduce some operational meanings of such multiplicative bounds, which correspond to Theorem 10 in our paper. The capacity $C(\cdot)$ is the maximum ratio in bits per channel use where information can be sent reliably with arbitrary small probability of errors. The formula for the capacity was given in Hol98, SW97:

$$
\begin{equation*}
C(\Phi)=\lim _{N \rightarrow \infty} \frac{1}{N} \chi\left(\Phi^{\otimes N}\right) \tag{16}
\end{equation*}
$$

where $\chi(\cdot)$ is defined in 13 . This immediately gives the following bound

$$
\begin{equation*}
C(\Phi) \leq \log k-\lim _{N \rightarrow \infty} \frac{1}{N} S_{\min }\left(\Phi^{\otimes N}\right) \tag{17}
\end{equation*}
$$

for a unital channel $\Phi: H_{n} \rightarrow H_{k}$. On the other hand, by using the monotonicity of Rényi $\alpha$-entropy and Theorem 10 we have

$$
\begin{equation*}
S_{\min }\left(\Phi^{\otimes N}\right) \geq S_{\min , 2}\left(\Phi^{\otimes N}\right)=-\log \left(\left\|\Phi^{\otimes N}\right\|_{1 \rightarrow 2}^{2}\right) \geq N \log (\gamma(\Phi)) \tag{18}
\end{equation*}
$$

We have proved:
Theorem 1 (A bound for capacity of unital channels). Take a unital quantum channel $\Phi$. Then,

1. The regularized minimum output entropy has the following bound:

$$
\lim _{N \rightarrow \infty} \frac{1}{N} S_{\min }\left(\Phi^{\otimes N}\right) \geq-\log \gamma(\Phi)
$$

2. The capacity has the following upper bound.

$$
C(\Phi) \leq \log k+\log \gamma(\Phi)
$$

where $\gamma(\cdot)$ is defined in 40.
Interestingly, these bounds turn out to be saturated by the Werner-Holevo channel. See Corollary 14

## 3. Examples and applications

After obtaining our general theory in Section III A, for the rest of Section III we work on examples to show how to use it. In Section IIIB, we introduce some orthonormal basis in $H_{n, 0}$ made of discrete Weyl operators. Although, the Gell-Mann basis is quite famous, our new basis gets along with discrete Weyl covariant channels, which are introduced in Section IIIC Previously, discrete Weyl covariant channels are investigated in DFH06 to give a bound which is equivalent to Theorem 17, and found additive examples for $S_{\min , 2}(\cdot)$. In fact, our Theorem 10 can be seen as generalization of Theorem 2 in DFH06. Also, the complementary channels of discrete Weyl covariant channels are studied in Section III C 3 when they are also unital. Based on this study, we give new additive examples for $S_{\min , 2}(\cdot)$ in Section IIID.

In Section IV, we extend Lemma 8 of Section III A to the $p \rightarrow q$ norms when maps are defined on $M_{n}(\mathbb{C})$. This is the usual definition of $p \rightarrow q$ norms, although our definition in 11 is compatible with the concept of channels.

## II. MINIMUM OUTPUT ENTROPY FORMULA WITH QUBIT INPUTS

Let $\Phi: H_{2,+, 1} \rightarrow H_{n,+, 1}$ be a quantum channel, where $H_{n,+, 1}$ is the set of density matrices acting on $\mathbb{C}^{n}$. The minimum entropy output of $\Phi$ is defined by

$$
S_{\min }(\Phi) \equiv \min _{\rho \in H_{2,+, 1}} S(\Phi(\rho))
$$

where $S(\rho)=-\operatorname{Tr}(\rho \log \rho)$. Our goal here is to find a closed formula for this quantity.
For this purpose, we will calculate first the square of the maximum 2-norm of $\Phi$ :

$$
\left(\|\Phi\|_{1 \rightarrow 2}\right)^{2}=\max _{\rho \in H_{n,+, 1}}\|\Phi(\rho)\|_{2}^{2}=\max _{\rho \in H_{n,+, 1}} \operatorname{Tr}\left(\Phi(\rho)^{2}\right)
$$

It is well know that the optimal $\rho$ for both of the equations above is a pure state. In the qubit case, any pure state has the Bloch representation

$$
\rho=\frac{1}{2}(I+\vec{r} \cdot \vec{\lambda})
$$

where $\vec{r}$ is a unit vector in $\mathbb{R}^{3}$, and $\vec{\lambda}=\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$ is a vector consisting of the three $2 \times 2$ traceless Hermitian matrices forming a basis for the three dimensional real vector space of $2 \times 2$ traceless Hermitian matrices. With this representation of $\rho$,

$$
\left(\|\Phi\|_{1 \rightarrow 2}\right)^{2}=\max _{\vec{r} \in \mathbb{R}^{3},\|\vec{r}\|=1} \frac{1}{4} \operatorname{Tr}\left(\Phi(I)^{2}\right)+\frac{1}{2} \sum_{j=1}^{3} r_{j} \operatorname{Tr}\left(\Phi(I) \Phi\left(\lambda_{j}\right)\right)+\frac{1}{4} \sum_{j=1}^{3} \sum_{k=1}^{3} r_{j} r_{k} \operatorname{Tr}\left(\Phi\left(\lambda_{j}\right) \Phi\left(\lambda_{k}\right)\right)
$$

Note that the matrix $A_{j k} \equiv \operatorname{Tr}\left(\Phi\left(\lambda_{j}\right) \Phi\left(\lambda_{k}\right)\right)$ is a real symmetric matrix. Therefore, for the purpose of our calculation, we will choose the basis $\lambda_{1}, \lambda_{2}, \lambda_{3}$ to be the basis that diagonalizes the symmetric matrix $A_{j k}$; that is, w.l.o.g we assume that $\operatorname{Tr}\left(\Phi\left(\lambda_{j}\right) \Phi\left(\lambda_{k}\right)\right)=a_{j} \delta_{j k}$. Further, we denote by $b_{j} \equiv \operatorname{Tr}\left(\Phi(I) \Phi\left(\lambda_{j}\right)\right)$. With these notations $\left(\|\Phi\|_{1 \rightarrow 2}\right)^{2}$ is given by

$$
\begin{equation*}
\left(\|\Phi\|_{1 \rightarrow 2}\right)^{2}=\frac{1}{4} \operatorname{Tr}\left(\Phi(I)^{2}\right)+\frac{1}{4} \max _{\vec{r} \in \mathbb{R}^{3},\|\vec{r}\|=1} \sum_{j=1}^{3}\left(2 b_{j} r_{j}+a_{j} r_{j}^{2}\right) \tag{19}
\end{equation*}
$$

In order to maximize the function $f(\vec{r}) \equiv \sum_{j=1}^{3}\left(2 b_{j} r_{j}+a_{j} r_{j}^{2}\right)$ on the unit sphere we define the constrain function $g(\vec{r})=\|\vec{r}\|^{2}-1$ and use the Lagrange multipliers technique. Denoting by $\alpha \in \mathbb{R}$ the Lagrange multiplier, the condition $\nabla f=\alpha \nabla g$ gives the relation

$$
r_{j}=\frac{b_{j}}{\alpha-a_{j}}
$$

where the coefficient $\alpha$ is determined from the constraint equation

$$
\begin{equation*}
\sum_{j=1}^{3} \frac{b_{j}^{2}}{\left(\alpha-a_{j}\right)^{2}}=1 \tag{20}
\end{equation*}
$$

We therefore obtain a closed formula for $\|\Phi\|_{1 \rightarrow 2}$, which we summarize in the following theorem.
Theorem 2 (An exact formula for qubit inputs). Let $\Phi: H_{2,+, 1} \rightarrow H_{n,+, 1}$ be a quantum channel, then

$$
\begin{equation*}
\|\Phi\|_{1 \rightarrow 2}=\frac{1}{2} \sqrt{\operatorname{Tr}\left(\Phi(I)^{2}\right)+\sum_{j=1}^{3} \frac{2 \alpha-a_{j}}{\left(\alpha-a_{j}\right)^{2}} b_{j}^{2}} \tag{21}
\end{equation*}
$$

with $a_{j}$ the eigenvalues of $A_{j k} \equiv \operatorname{Tr}\left(\Phi\left(\lambda_{j}\right) \Phi\left(\lambda_{k}\right)\right), b_{j} \equiv \operatorname{Tr}\left(\Phi(I) \Phi\left(\lambda_{j}\right)\right)$, and $\alpha$ determined by Eq. 20.
Next, we state basic facts about qubit states:
Proposition 3. For qubit states, we have the following properties.

1. Take $\rho \in H_{2}$, then $\rho$ is a pure state if and only if $\operatorname{Tr} \rho=1$ and $\operatorname{Tr} \rho^{2}=1$.
2. For $\rho \in H_{2,+, 1}$, we have

$$
S_{\alpha}(\rho)=h_{2, \alpha} \circ f(\rho)
$$

Here, $h_{2, \alpha}(x)=\frac{1}{1-\alpha}\left(x^{\alpha}+(1-x)^{\alpha}\right)$ is the binary $\alpha$-Rényi entropy, and

$$
\begin{equation*}
f(x) \equiv \frac{1+\sqrt{2 x^{2}-1}}{2} \tag{22}
\end{equation*}
$$

The first statement is equivalent to the idea of the Bloch sphere, and the second statement leads to:
Corollary 4 (An exact formula for qubit channels). For a qubit channel $\Phi: H_{2,+, 1} \rightarrow H_{2,+, 1}$ the minimum $\alpha$-Rényi entropy output is given by

$$
\begin{equation*}
S_{\min , \alpha}(\Phi)=h_{2, \alpha} \circ f\left(\|\Phi\|_{1 \rightarrow 2}\right) \tag{23}
\end{equation*}
$$

where and $\|\Phi\|_{1 \rightarrow 2}$ is given by (21) and $f(\cdot)$ is defined in 22.
While the formula above holds only for qubit channels, we can still use theorem 2 to derive a tight lower bound for $S_{\min }(\Phi)$ with $\Phi: H_{2,+, 1} \rightarrow H_{n,+, 1}$ and $n>3$. The lower bound follows from the following lemma.
Lemma 5. Let $\left\{p_{j}\right\}_{j=1}^{n}$ be a probability distribution. For a given fixed value of the "index of coincidence", $\sum_{j=1}^{n} p_{j}^{2} \equiv$ $c$, the lowest possible value of the Rényi entropy is achieved by the probability distribution:

$$
\begin{equation*}
(\underbrace{\frac{1+\Delta}{1+k}, \ldots, \frac{1+\Delta}{1+k}}_{k}, \frac{1-k \Delta}{1+k}) \tag{24}
\end{equation*}
$$



FIG. 1: The functions $g(c)$ and $-\log _{2}(c)$ are drawn in purple and blue, respectively. Note that $g(c) \geq-\log _{2}(c)$ with equality exactly at $c=1,1 / 2,1 / 3,1 / 4, \ldots$ Furthermore, $g(c)+\log _{2}(c) \rightarrow 0$ in the limit $c \rightarrow 0$.
and the value is given by

$$
g_{\alpha}(c) \equiv \begin{cases}\frac{1}{1-\alpha} \log \left(k\left(\frac{1+\Delta}{1+k}\right)^{\alpha}+\left(\frac{1-k \Delta}{1+k}\right)^{\alpha}\right) & \text { if } 1<\alpha<2  \tag{25}\\ \log (k+1)-\frac{1}{1+k}[k(1+\Delta) \log (1+\Delta)+(1-k \Delta) \log (1-k \Delta)] & \text { if } \alpha=1\end{cases}
$$

where $k \equiv\left\lfloor\frac{1}{c}\right\rfloor$ and $\Delta \equiv \sqrt{c-(1-c) k^{-1}}$. We write $g(\cdot)=g_{1}(\cdot)$. Moreover, $g(c)$ is a continuous and monotonically non-increasing function of c; see Fig. 1.

Note that if $c \geq 1 / 2$ then $g(c)=h_{2}\left(\frac{1+\sqrt{2 c-1}}{2}\right)$, where $h_{2}$ is the binary Shannon entropy. The proof of the lemma above follows directly from the results in HT01, BS03.
Theorem 6. Let $\Phi: H_{m,+, 1} \rightarrow H_{n,+, 1}$ be a quantum channel, then

$$
\begin{equation*}
S_{\min , \alpha}(\Phi) \geq g_{\alpha}\left(\|\Phi\|_{1 \rightarrow 2}^{2}\right) \tag{26}
\end{equation*}
$$

where the function $g_{\alpha}$ is defined in Eq. 25), and equality holds for qubit output channels (i.e. if $n=2$ ).
Remarks: (1) This lower bound is optimal in the sense that for any other function $f$ satisfying $S_{\min , \alpha}(\Phi) \geq$ $f\left(\|\Phi\|_{1 \rightarrow 2}^{2}\right), g_{\alpha}\left(\|\Phi\|_{1 \rightarrow 2}^{2}\right) \geq f\left(\|\Phi\|_{1 \rightarrow 2}^{2}\right)$. In particular, $g\left(\|\Phi\|_{1 \rightarrow 2}^{2}\right) \geq-\log \left(\|\Phi\|_{1 \rightarrow 2}^{2}\right)$ as shown in Fig. 1. (2) For $m=2$ and $n>2$, the closed expression given in Eq. 21) can be used in Eq. 26) to obtain a tight lower bound on $S_{\min }(\Phi)$.

## III. MULTIPLICATIVE BOUNDS FOR UNITAL CHANNELS

In this section, we leave qubit channels for higher dimensional cases. In Section III A we get bounds for the minimum output Rényi $\alpha$-entropy with $0 \leq \alpha \leq 2$. Considering the fact that getting exact values is difficult, it is interesting to get some bounds, which are tight for some examples in Remark 11. Also, our bonds are tight for some class of maps which include depolarizing channel and Werner-Holevo channel. Interestingly, Theorem 10 our main theorem in this section gives bounds for tensor products of unital channels. It's operational meanings are explained in Section IB 2. Historically, a bound for the maximum 2-norm of covariant channels are obtained in KMNR07, which was generalized in DFH06 to tensor products of discrete Weyl covariant channels. Below, we extend these ideas to unital channels. Towards the end of Section III A, we define a sufficient condition for unital channels to have additive properties. In particular, with this observation, out bound for the capacity turns out to be tight for Werner-Holevo channel.

A new orthonormal basis made of discrete Weyl operators is given in Section IIIB, and applications of our bound to discrete Weyl covariant channels and their complementary channels are made in Section III C 2 and Section III C 3, The definition of discrete Weyl covariant channels are given in Section IIIC 1] Finally in Section [IIID we give additive examples based on the additivity test developed in Section III A.

## A. General formula

Take an orthonormal basis in $H_{n, 0}: \vec{M}=\left(M_{1}, \ldots, M_{n^{2}-1}\right)$ so that $\left(M_{0}=I_{n} / \sqrt{n}, M_{1}, \ldots, M_{n^{2}-1}\right)$ forms an orthonormal basis of $H_{n}$ (an inner product on $H_{n}$ is defined by $\langle A, B\rangle=\operatorname{Tr}(A B)$ ). Take another space $K_{m}$ and then for any $\rho \in H_{n} \otimes K_{m}=L_{n m}$ we write

$$
\begin{equation*}
\rho=\sum_{i=0}^{n^{2}-1} M_{i} \otimes \rho_{i}=\frac{I_{n}}{\sqrt{n}} \otimes \rho_{0}+\sum_{i=1}^{n^{2}-1} M_{i} \otimes \rho_{i} . \tag{27}
\end{equation*}
$$

Here, $\rho_{i}=\operatorname{Tr}_{H}\left[\left(M_{i} \otimes I_{m}\right) \rho\right] \in K_{m}$, and in particular

$$
\begin{equation*}
\rho_{0}=\operatorname{Tr}_{H}\left[\left(\frac{I_{n}}{\sqrt{n}} \otimes I_{m}\right) \rho\right]=\frac{\rho_{K}}{\sqrt{n}} \tag{28}
\end{equation*}
$$

with $\rho_{K} \equiv \operatorname{Tr}_{H}[\rho] \in K_{m}$. (Note that $\rho \in L_{n m,+, 1}$ implies $\rho_{K} \in K_{m,+, 1}$.) Moreover,

$$
\begin{equation*}
\operatorname{Tr}\left[\rho^{2}\right]=\sum_{i=0}^{n^{2}-1} \operatorname{Tr}\left[\rho_{i}^{2}\right] \tag{29}
\end{equation*}
$$

Take a trace-preserving linear map $\Phi: H_{n} \rightarrow H_{k}$ and fix an orthonormal basis in $H_{k, 0}$ to be $N_{1}, \ldots, N_{k^{2}-1}$. Since $\Phi$ is a linear map, it is written by a $\left(k^{2}\right) \times\left(n^{2}\right)$ real matrix denoted by $\tilde{B}_{\Phi}$. In case $\Phi$ is unital,

$$
\tilde{B}_{\Phi}=\left(\begin{array}{cc}
\sqrt{\frac{n}{k}} & 0  \tag{30}\\
0 & B_{\Phi}
\end{array}\right)
$$

where $B_{\Phi}: H_{n, 0} \rightarrow H_{k, 0}$. To obtain the matrix $B_{\Phi}$, we calculate

$$
\begin{equation*}
\left(B_{\Phi}\right)_{i, j}=\operatorname{Tr}\left[N_{i} \Phi\left(M_{j}\right)\right] \in \mathbb{R} . \tag{31}
\end{equation*}
$$

Then, we introduce a positive (symmetric) matrix:

$$
\begin{equation*}
A_{\Phi}=B_{\Phi}^{T} B_{\Phi}: H_{n, 0} \rightarrow H_{n, 0} \tag{32}
\end{equation*}
$$

which does not depend on choice of the above basis $N_{1}, \ldots, N_{k^{2}-1}$ in $H_{k, 0}$. Indeed, we get the matrix $A_{\Phi}$ directly by

$$
\begin{equation*}
\left(A_{\Phi}\right)_{i, j}=\operatorname{Tr}\left[\Phi\left(M_{i}\right) \Phi\left(M_{j}\right)\right] . \tag{33}
\end{equation*}
$$

To see this formula,

$$
\begin{equation*}
\left(A_{\Phi}\right)_{i, j}=\sum_{l=1}^{k^{2}-1}\left(B_{\Phi}\right)_{l, i}\left(B_{\Phi}\right)_{l, j}=\sum_{l=1}^{k^{2}-1} \operatorname{Tr}\left[N_{l} \Phi\left(M_{i}\right)\right] \cdot \operatorname{Tr}\left[N_{l} \Phi\left(M_{j}\right)\right]=\operatorname{Tr}\left[\Phi\left(M_{i}\right) \Phi\left(M_{j}\right)\right] . \tag{34}
\end{equation*}
$$

Note that $N_{1}, \ldots, N_{k^{2}-1}$ are also orthonormal in the complex matrix space, and the inner product is Euclidean. In the analyses below, $\left\|A_{\Phi}\right\|_{\infty}$ plays a key role but this quantity does not depend on choice of basis ( $M_{0}, \ldots, M_{n^{2}-1}$ ) in $H_{n, 0}$ either.
Remark 7. $A_{\Phi}$ can be diagonalized if we choose a proper basis in the domain of $\Phi$. Such examples are studied in Section IIIC 2 and Section III C3.

First, we have an important lemma as generalization of Theorem 1 in DFH06:
Lemma 8. For a trace-preserving unital linear map $\Phi: H_{n} \rightarrow H_{k}$ and for a Hermitian matrix $\rho \in H_{n} \otimes K_{m}$,

$$
\operatorname{Tr}\left[(\Phi \otimes \operatorname{id}(\rho))^{2}\right] \leq\left(\frac{1}{k}-\frac{\left\|A_{\Phi}\right\|_{\infty}}{n}\right) \operatorname{Tr}\left[\rho_{K}^{2}\right]+\left\|A_{\Phi}\right\|_{\infty} \cdot \operatorname{Tr}\left[\rho^{2}\right],
$$

where id is the identity map on $K_{m}$.

Proof. By using the decomposition (27), we calculate

$$
\begin{align*}
\operatorname{Tr}\left[(\Phi \otimes \operatorname{id}(\rho))^{2}\right] & =\operatorname{Tr}\left[\left(\Phi\left(\frac{1}{n} I_{n}\right) \otimes \rho_{K}+\sum_{i=1}^{n^{2}-1} \Phi\left(M_{i}\right) \otimes \rho_{i}\right)^{2}\right]  \tag{35}\\
& =\frac{1}{k} \operatorname{Tr}\left[\rho_{K}^{2}\right]+\sum_{i, j=1}^{n^{2}-1} \operatorname{Tr}\left[\Phi\left(M_{i}\right) \Phi\left(M_{j}\right)\right] \cdot \operatorname{Tr}\left[\rho_{i} \rho_{j}\right] \tag{36}
\end{align*}
$$

Here, for the second inequality, we used the fact that $\Phi$ is unital and trace-preserving.
Since $A_{\Phi} \equiv B_{\Phi}^{T} B_{\Phi}$ is symmetric, by choosing $\left(M_{i}\right)_{i=1}^{n^{2}-1}$ properly, we can assume that $A_{\Phi}=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{n^{2}-1}\right)$ with these entries non-increasing (remember (33)). Then,

$$
\begin{align*}
(36) & =\frac{1}{k} \cdot \operatorname{Tr}\left[\rho_{K}^{2}\right]+\sum_{i=1}^{n^{2}-1} \lambda_{i} \cdot \operatorname{Tr}\left[\rho_{i}^{2}\right] \leq \frac{1}{k} \operatorname{Tr}\left[\rho_{K}^{2}\right]+\lambda_{1} \sum_{i=1}^{n^{2}-1} \operatorname{Tr}\left[\rho_{i}^{2}\right]  \tag{37}\\
& =\left(\frac{1}{k}-\frac{\lambda_{1}}{n}\right) \operatorname{Tr}\left[\rho_{K}^{2}\right]+\lambda_{1} \sum_{i=0}^{n^{2}-1} \operatorname{Tr}\left[\rho_{i}^{2}\right]=\left(\frac{1}{k}-\frac{\lambda_{1}}{n}\right) \operatorname{Tr}\left[\rho_{K}^{2}\right]+\lambda_{1} \operatorname{Tr}\left[\rho^{2}\right] \tag{38}
\end{align*}
$$

Here we used (28) and 29). To finish the proof, notice that $\lambda_{1}=\left\|A_{\Phi}\right\|_{\infty}$.
Remark 9. Set $m=1$ in Lemma 8, then the conditions $\operatorname{Tr} \rho=1$ and $\operatorname{Tr} \rho^{2}=1$ make (38) into

$$
\begin{equation*}
(38)=\frac{1}{k}+\left(1-\frac{1}{n}\right)\left\|A_{\Phi}\right\|_{\infty} \tag{39}
\end{equation*}
$$

Here, again, $A_{\Phi}$ is given in (33). Note that we did not use the positivity condition of $\rho$. This fact shares spirits with Section II.

Then, we define

$$
\gamma(\Phi) \equiv \begin{cases}\frac{1}{k}+\left(1-\frac{1}{n}\right)\left\|A_{\Phi}\right\|_{\infty} & \text { if } n \geq k \cdot\left\|A_{\Phi}\right\|_{\infty}  \tag{40}\\ \left\|A_{\Phi}\right\|_{\infty} & \text { if } n<k \cdot\left\|A_{\Phi}\right\|_{\infty}\end{cases}
$$

With these notations we are ready to present the main theorem of this section:
Theorem 10 (Bounds for unital channels). We have the following bounds $0 \leq \alpha \leq 2$ and $N \in \mathbb{N}$.

1. Take two linear maps $\Phi$ and $\Omega$ defined between spaces of Hermitian matrices, where $\Phi$ is unital and tracepreserving and $\Omega$ is completely positive. Then,

$$
\|\Phi \otimes \Omega\|_{1 \rightarrow 2} \leq \sqrt{\gamma(\Phi)} \cdot\|\Omega\|_{1 \rightarrow 2} \quad \text { and } \quad S_{\min , \alpha}(\Phi \otimes \Omega) \geq-\log (\gamma(\Phi))+S_{\min , 2}(\Omega)
$$

2. For a sequence of unital channels $\left(\Phi_{i}\right)_{i=1}^{N}$,

$$
\left\|\bigotimes_{i=1}^{N} \Phi_{i}\right\|_{1 \rightarrow 2} \leq \prod_{i=1}^{N} \sqrt{\gamma\left(\Phi_{i}\right)} \quad \text { and } \quad S_{\min , \alpha}\left(\bigotimes_{i=1}^{N} \Phi_{i}\right) \geq-\sum_{i=1}^{N} \log \left(\gamma\left(\Phi_{i}\right)\right)
$$

Proof. First, we prove the statements for the norms. Suppose that $H_{n}$ and $K_{m}$ are domains of $\Phi$ and $\Omega$, respectively. For $\rho \in H_{n} \otimes K_{m}=L_{n m}$ Lemma 8 implies that

$$
\begin{equation*}
\operatorname{Tr}\left[(\Phi \otimes \Omega(\rho))^{2}\right] \leq\left(\frac{1}{k}-\frac{\left\|A_{\Phi}\right\|_{\infty}}{n}\right) \operatorname{Tr}\left[\Omega\left(\rho_{K}\right)^{2}\right]+\left\|A_{\Phi}\right\|_{\infty} \cdot \operatorname{Tr}\left[(\operatorname{id} \otimes \Omega(\rho))^{2}\right] \tag{41}
\end{equation*}
$$

On the other hand, for $\rho \in L_{n m,+, 1}$

$$
\operatorname{Tr}\left[(\mathrm{id} \otimes \Omega(\rho))^{2}\right] \leq\|\Omega\|_{1 \rightarrow 2}^{2}
$$

which was proved in AHW00. This proves the first statement for because $\rho_{K} \in H_{n,+, 1}$. Note that when $k \cdot\left\|A_{\Phi}\right\|_{\infty}>n$ we ignore the first term in $\sqrt[41]{ }$ to get the bound. The second statement is shown for by inductive applications of the first statement.

Next, the above result can translate into the case of Rényi entropy with $\alpha=2$. To complete the proof remember the monotonicity: $S_{\alpha}(\cdot) \geq S_{2}(\cdot)$ for $0 \leq \alpha \leq 2$.

Remark 11. The above bounds in Lemma 8 and Theorem 10 are saturated by the following class of linear maps on $H_{n}$ :

$$
\begin{equation*}
\Psi(\rho)=t \rho^{\star}+(1-t) \frac{\operatorname{Tr}[\rho]}{n} I_{n} \tag{42}
\end{equation*}
$$

with $t \in \mathbb{R}$ such that $|t| \leq 1$, and $\star \in\{1, T\}$ where $T$ is transpose. Special cases of these maps are called depolarizing channel or Werner-Holevo channels [WH02]. Indeed, those bounds are saturated when the inequality in (37) has no gap, while the map $\Psi$ just rescales vectors by multiplying $t$ in $H_{n, 0}$ up to rotations, so that $A_{\Psi}=|t| I$. As a consequence, we have

$$
\|\Psi \otimes \Omega\|_{1 \rightarrow 2}=\|\Psi\|_{1 \rightarrow 2} \cdot\|\Omega\|_{1 \rightarrow 2}
$$

for $\Psi$ in (42) and $\Omega$ a completely positive map. See the proof of Theorem 13 .
Moreover, we can generalize the above additivity statement, which is the generalization of Theorem 2 in [DFH06]. To this end, we give the following definition.

Definition 12. We define a condition called $\mathcal{C}_{\text {add }}$ in such a way that a unital trace-preserving linear map $\Phi: H_{n} \rightarrow H_{k}$ satisfies $\mathcal{C}_{\text {add }}$ if the following two equivalent conditions are satisfied.

1. $\gamma(\Phi)=\|\Phi\|_{1 \rightarrow 2}^{2}$.
2. There exists a state which is supported within $H_{n, 0}$ by the eigenspaces of the largest eigenvalue of $A_{\Phi}$ and $n \geq k\left\|A_{\Phi}\right\|_{\infty}$.
The above equivalence is clear if we look into conditions when the inequality in (37) is saturated.
Then, as a corollary we have
Theorem 13 (Additivity test). Take two linear maps $\Phi$ and $\Omega$ defined between spaces of Hermitian matrices, where $\Phi$ is unital, trace-preserving and satisfying $\mathcal{C}_{a d d}$ (Definition 12), and $\Omega$ is completely positive. Then,

$$
\|\Phi \otimes \Omega\|_{1 \rightarrow 2}=\|\Phi\|_{1 \rightarrow 2} \cdot\|\Omega\|_{1 \rightarrow 2} \quad \text { and } \quad S_{\min , 2}(\Phi \otimes \Omega)=S_{\min , 2}(\Phi)+S_{\min , 2}(\Omega)
$$

In particular, for a sequence of unital quantum channels $\left(\Phi_{i}\right)_{i=1}^{N}$ satisfying $\mathcal{C}_{a d d}$,

$$
\left\|\bigotimes_{i=1}^{N} \Phi_{i}\right\|_{1 \rightarrow 2}=\prod_{i=1}^{N}\left\|\Phi_{i}\right\|_{1 \rightarrow 2} \quad \text { and } \quad S_{\min , \alpha}\left(\bigotimes_{i=1}^{N} \Phi_{i}\right)=\sum_{i=1}^{N} S_{\min , 2}\left(\Phi_{i}\right)
$$

Proof. We have

$$
\begin{equation*}
\|\Phi\|_{1 \rightarrow 2} \cdot\|\Omega\|_{1 \rightarrow 2} \leq\|\Phi \otimes \Omega\|_{1 \rightarrow 2} \leq\|\Phi\|_{1 \rightarrow 2} \cdot\|\Omega\|_{1 \rightarrow 2} \tag{43}
\end{equation*}
$$

The first bound is trivial and the second comes from $\mathcal{C}_{\text {add }}$.
The following corollary justify our bound on the capacity in Theorem 1 .
Corollary 14. Suppose $\Psi: H_{n} \rightarrow H_{n}$ is the Werner-Holevo channel:

$$
\begin{equation*}
\Psi(\rho)=\frac{\operatorname{Tr}[\rho] I_{n}-\rho^{T}}{n-1} \tag{44}
\end{equation*}
$$

Then, the bound in Theorem 1 is saturated:

$$
\begin{equation*}
C(\Psi)=\log n+\log \gamma(\Psi) \tag{45}
\end{equation*}
$$

Proof. By the contra-variant property of the channel we have Hol05a

$$
\begin{equation*}
C(\Psi)=\log n-\lim _{N \rightarrow \infty} \frac{1}{N} S_{\min }\left(\Psi^{\otimes N}\right) \tag{46}
\end{equation*}
$$

On the other hand,

$$
\begin{equation*}
S_{\min }\left(\Psi^{\otimes N}\right) \leq N S_{\min }(\Psi)=N S_{\min , 2}(\Psi)=-N \log (\gamma(\Psi)) \tag{47}
\end{equation*}
$$

Here, the first equality comes from WE05, and the second holds because $\Psi$ satisfies $\mathcal{C}_{\text {add }}$. Hence,

$$
\begin{equation*}
\log n+\log (\gamma(\Psi)) \leq C(\Psi) \tag{48}
\end{equation*}
$$

Then, Theorem 1 completes the proof by sandwich theorem.

## B. A basis made of discrete Weyl operators in the real space of Hermitian matrices

To obtain the matrix $A_{\Phi}$ for a unital channel $\Phi$, we need to fix a basis in $H_{n, 0}$, where $H_{n}$ is the domain of $\Phi$. For this purpose, Gell-Mann basis could be the first candidate. However, in this section we develop another orthonormal basis by using the discrete Weyl operators.

Define the discrete Weyl operators by

$$
\begin{equation*}
W_{x, y}=U^{x} V^{y} \tag{49}
\end{equation*}
$$

where $U, V \in \mathcal{U}(n)$ are unitary matrices as follows.

$$
\begin{equation*}
U|m\rangle=|m+1\rangle \quad \text { and } \quad V|m\rangle=\exp \left(\frac{2 \pi \mathrm{i}}{n} m\right)|m\rangle \tag{50}
\end{equation*}
$$

for $m, x, y \in\{0,1, \ldots, n-1\}=\mathbb{Z}_{n}$, which is the group of integers modulo $n$. Note that for $(x, y) \neq\left(x^{\prime}, y^{\prime}\right)$ we have

$$
\begin{equation*}
\operatorname{Tr}\left[W_{x, y}^{*} W_{x^{\prime}, y^{\prime}}\right]=0 \tag{51}
\end{equation*}
$$

In particular, $\operatorname{Tr} W_{x, y}=0$ for $(x, y) \neq(0,0)$.
Before introducing our basis, we set up four subsets of $\mathbb{Z}_{n} \times \mathbb{Z}_{n}$ by the following conditions.
(i) $1 \leq x<y \leq n-1$
(ii) $1 \leq x=y \leq\left\lfloor\frac{n}{2}\right\rfloor$
(iii) $x=0, \quad 1 \leq y \leq\left\lfloor\frac{n}{2}\right\rfloor$
(iv) $y=0, \quad 1 \leq x \leq\left\lfloor\frac{n}{2}\right\rfloor$
where $\lfloor r\rfloor$ is the largest integer such that $r \geq\lfloor r\rfloor$. We define a set $\mathcal{S}_{n}=\left\{(x, y) \in \mathbb{Z}_{n} \times \mathbb{Z}_{n}\right.$ : $(x, y)$ satisfies (i), (ii), (iii) or (iv) $\}$. Now, set $\mathcal{S}_{n}^{F}=\left\{\left(\frac{n}{2}, \frac{n}{2}\right),\left(0, \frac{n}{2}\right),\left(\frac{n}{2}, 0\right)\right\} \subset \mathcal{S}_{n}$ and $\mathcal{S}_{n}^{G H}=S_{n} \backslash \mathcal{S}_{n}^{F}$. Then, we define the following Hermitian matrices to construct an orthonormal basis in $H_{n, 0}$ :

$$
\left.\begin{array}{ll}
F_{x, y}=\frac{1}{\sqrt{n}} W_{x, y} & \text { for }(x, y) \in \mathcal{S}_{n}^{F} \\
G_{x, y}=\frac{1}{\sqrt{2 n}}\left(W_{x, y}+W_{x, y}^{*}\right)  \tag{52}\\
H_{x, y}=\frac{1}{\sqrt{2 n} i}\left(W_{x, y}-W_{x, y}^{*}\right)
\end{array}\right\} \quad \text { for }(x, y) \in \mathcal{S}_{n}^{G H}
$$

Note that $S_{n}^{F}=\emptyset$ when $n$ is odd.
Proposition 15. The matrices defined in 52) form an orthonormal basis in $H_{n, 0}$.

Proof. We start with counting the number of matrices defined in 52 . When $n$ is odd,

$$
\begin{equation*}
\binom{n-1}{2} \cdot 2+\frac{n-1}{2} \cdot 2+\frac{n-1}{2} \cdot 2+\frac{n-1}{2} \cdot 2=n^{2}-1 . \tag{53}
\end{equation*}
$$

When $n$ is even,

$$
\begin{equation*}
\binom{n-1}{2} \cdot 2+\left(\frac{n}{2}-1\right) \cdot 2+\left(\frac{n}{2}-1\right) \cdot 2+\left(\frac{n}{2}-1\right) \cdot 2+3=n^{2}-1 \tag{54}
\end{equation*}
$$

Hence, we show that they are orthogonal. To this end, we note that

$$
\begin{equation*}
\operatorname{Tr}\left[W_{x, y} W_{x^{\prime}, y^{\prime}}\right]=c \operatorname{Tr}\left[W_{x+x^{\prime}, y+y^{\prime}}\right] \tag{55}
\end{equation*}
$$

for some complex number $c$ with modulo 1 .
First, we claim that $G_{x, y}$ and $H_{x, y}$ are orthogonal. Indeed,

$$
\begin{equation*}
\operatorname{Tr}\left[\left(W_{x, y}+W_{x, y}^{*}\right)\left(W_{x, y}-W_{x, y}^{*}\right)\right]=\operatorname{Tr}\left[W_{x, y}^{2}+\left(W_{x, y}^{*}\right)^{2}\right]=\operatorname{Tr}\left[c W_{2 x, 2 y}+\bar{c} W_{-2 x,-2 y}\right]=0 \tag{56}
\end{equation*}
$$

where $c$ is some complex number with modulo 1 . The last equality holds for (i), (iii) and (iv) because $x \neq y$ implies that $2 x$ and $2 y$ can not be 0 at the same time unless $(x, y) \in \mathcal{S}_{n}^{F}$. For (ii), $2 x=2 y \neq 0$ unless $(x, y) \in \mathcal{S}_{n}^{F}$, again.

Next, we prove orthogonality when $(x, y) \neq\left(x^{\prime}, y^{\prime}\right)$. Orthogonality within (iii) and (iv) is clear because we have $x+x^{\prime} \neq 0$ or $y+y^{\prime} \neq 0$ for $(x, y) \neq\left(x^{\prime}, y^{\prime}\right)$. Also, we know orthogonality between the first two cases (i) and (ii)) and the last two cases ((iii) and (iv)). Indeed, for $(x, y)$ from the first group and $\left(x^{\prime}, y^{\prime}\right)$ the second, $x^{\prime}=0$ implies $x+x^{\prime} \neq 0$ and $y^{\prime}=0 y+y^{\prime} \neq 0$.

Hence, we show orthogonality within (i) and (ii) as a whole to finish the proof. Take $(x, y),\left(x^{\prime}, y^{\prime}\right)$ from (i) and (ii). Since $(x, y) \neq\left(x^{\prime}, y^{\prime}\right)$, firstly $x+x^{\prime}=0$ implies $y+y^{\prime} \neq 0$, and secondly $y+y^{\prime}=0$ implies $x+x^{\prime} \neq 0$.

## C. Weyl covariant channels as examples

## 1. Discrete Weyl covariant channels

The discrete Weyl covariant channels are defined by

$$
\begin{equation*}
\Psi(\rho)=\sum_{(x, y) \in \mathbb{Z}_{n} \times \mathbb{Z}_{n}} p_{x, y} W_{x, y} \rho W_{x, y}^{*} \tag{57}
\end{equation*}
$$

where $\left(p_{x, y}\right)_{(x, y)}$ is a probability distribution. Remember $W_{x, y}$ are defined in 49 . The name comes from the property that

$$
\begin{equation*}
\Psi\left(W_{a, b} \rho W_{a, b}^{*}\right)=W_{a, b} \Psi(\rho) W_{a, b}^{*} \tag{58}
\end{equation*}
$$

for all $(a, b) \in \mathbb{Z}_{n} \times \mathbb{Z}_{n}$. This is true because the discrete Weyl operators are commuting up to constants with modulo 1. I.e.,

$$
\begin{equation*}
W_{x, y} W_{a, b}=c_{x, y, a, b} W_{a, b} W_{x, y} \tag{59}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{x, y, a, b}=\exp \left(\frac{2 \pi i}{n}(a y-x b)\right) \tag{60}
\end{equation*}
$$

See [DFH06] for more details about the discrete Weyl covariant channels. Not surprisingly, our matrix $A_{\Psi}$ in (32) is diagonal in our basis defined in Section IIIB, which you can see below.

## 2. How our formulas work with discrete Weyl covariant channels

Lemma 16. For the channel $\Psi$ in (57), the matrix $A_{\Psi}$ in (33) is diagonal with respect to the basis in (52), and the diagonal entries are given by $\left|c_{a, b}\right|^{2}$ where

$$
\begin{equation*}
c_{a, b}=\sum_{x, y} p_{x, y} c_{x, y, a, b} \tag{61}
\end{equation*}
$$

Proof.

$$
\begin{equation*}
\Psi\left(W_{a, b}^{*}\right)=\left(\sum_{x, y} p_{x, y} W_{x, y} W_{a, b} W_{x, y}^{*}\right)^{*}=\sum_{x, y} p_{x, y} \bar{c}_{x, y, a, b} W_{a, b}^{*} \tag{62}
\end{equation*}
$$

where $c_{x, y, a, b}$ is defined in 60. Hence,

$$
\begin{equation*}
\Psi\left(F_{a, b}\right)=\frac{1}{\sqrt{n}} c_{a, b} W_{a, b}, \quad \Psi\left(G_{a, b}\right)=\frac{1}{\sqrt{2 n}}\left(c_{a, b} W_{a, b}+\bar{c}_{a, b} W_{a, b}^{*}\right), \quad \Psi\left(H_{a, b}\right)=\frac{1}{\sqrt{2 n} i}\left(c_{a, b} W_{a, b}-\bar{c}_{a, b} W_{a, b}^{*}\right) \tag{63}
\end{equation*}
$$

These are orthogonal because the complex numbers $c_{x, y}$ respect the proof of Proposition 15 . Then, omitting the subscripts, we calculate

$$
\begin{align*}
& \operatorname{Tr}(\Psi(F))^{2}=|c|^{2} \quad \text { and } \\
& \operatorname{Tr}(\Psi(G))^{2}=\frac{1}{2 n} \operatorname{Tr}\left[c^{2} W^{2}+2|c|^{2} I_{n}+(\bar{c})^{2}\left(W^{*}\right)^{2}\right]=|c|^{2}=\operatorname{Tr}(\Psi(H))^{2} \tag{64}
\end{align*}
$$

This completes the proof.
Now we recover Theorem 1 of [DFH06:
Theorem 17. For $\Psi$ in (57) we have

$$
\gamma(\Phi)= \begin{cases}\frac{1}{n}+\left(1-\frac{1}{n}\right) \max _{(a, b) \in \mathcal{S}_{n}}\left|c_{a, b}\right|^{2} & \text { if } \max _{(a, b) \in \mathcal{S}_{n}}\left|c_{a, b}\right| \leq 1  \tag{65}\\ \max _{(a, b) \in \mathcal{S}_{n}}\left|c_{a, b}\right|^{2} & \text { if } \max _{(a, b) \in \mathcal{S}_{n}}\left|c_{a, b}\right|>1\end{cases}
$$

where $c_{a, b}$ is defined in 61) or 63). Here, $\gamma(\cdot)$ is defined in 40).
Proof. By Lemma 16

$$
\begin{equation*}
\left\|A_{\Psi}\right\|_{\infty}=\max _{(a, b) \in \mathcal{S}_{n}}\left|c_{a, b}\right|^{2} \tag{66}
\end{equation*}
$$

Applying this to 40 completes the proof.

## 3. Complementary channels and their bounds

In this section, we study the following subset of discrete Weyl covariant channels, whose complementary channels are unital:

$$
\begin{equation*}
\Psi(\rho)=\frac{1}{k} \sum_{l=1}^{k} W_{l} \rho W_{l}^{*} \tag{67}
\end{equation*}
$$

Here, $W_{l}=W_{x_{l}, y_{l}}$ for a $k$-sequence $\left(\left(x_{l}, y_{l}\right)\right)_{l=1}^{k} \subseteq \mathbb{Z}_{n} \times \mathbb{Z}_{n}$ with $1 \leq k \leq n^{2}$; in the sequence there is no multiplicity. Their complementary channels are written by

$$
\begin{equation*}
\left[\Psi^{C}(\rho)\right]_{l, m}=\frac{1}{k} \operatorname{Tr}\left[W_{l} \rho W_{m}^{*}\right], \quad 1 \leq l, m \leq k \tag{68}
\end{equation*}
$$

Again, our matrix $A_{\Psi^{C}}$ is diagonal in our basis:

Lemma 18. For $\Psi^{C}$ in (68), the matrix $A_{\Psi^{C}}$ in (33) is diagonal with respect to the basis in 52) and the diagonal entries are $\left\{\frac{n}{2 k^{2}} \cdot N(a, b)\right\}_{(a, b) \in \mathcal{S}_{n}}$. Here,

$$
\begin{align*}
N(a, b) \equiv \# & \left\{(l, m) \in \mathbb{Z}_{k} \times \mathbb{Z}_{k}:\left(x_{m}-x_{l}, y_{m}-y_{l}\right)=(a, b)\right\}  \tag{69}\\
& +\#\left\{(l, m) \in \mathbb{Z}_{k} \times \mathbb{Z}_{k}:\left(x_{m}-x_{l}, y_{m}-y_{l}\right)=(-a,-b)\right\}
\end{align*}
$$

Proof. First, note that $W_{x_{m}, y_{m}}^{*} W_{x_{l}, y_{l}}=u_{l, m} W_{x_{l}-x_{m}, y_{l}-y_{m}}$ for some complex number $u_{l, m}$ of modulus 1, which we do not specify. Then,

$$
\begin{align*}
{\left[\Psi^{C}\left(F_{a, b}\right)\right]_{l, m} } & =\frac{\sqrt{n}}{k} \cdot\left[v_{l, m, a, b} \cdot \delta_{a, x_{m}-x_{l}} \delta_{b, y_{m}-y_{l}}\right] \\
{\left[\Psi^{C}\left(G_{a, b}\right)\right]_{l, m} } & =\frac{\sqrt{n}}{\sqrt{2} \cdot k} \cdot\left[v_{l, m, a, b} \cdot \delta_{a, x_{m}-x_{l}} \delta_{b, y_{m}-y_{l}}+\bar{v}_{l, m, a, b} \cdot \delta_{a, x_{l}-x_{m}} \delta_{b, y_{l}-y_{m}}\right]  \tag{70}\\
{\left[\Psi^{C}\left(H_{a, b}\right)\right]_{l, m} } & =\frac{\sqrt{n}}{\sqrt{2} \cdot k} \cdot\left[-\mathrm{i} v_{l, m, a, b} \cdot \delta_{a, x_{m}-x_{l}} \delta_{b, y_{m}-y_{l}}+\mathrm{i} \bar{v}_{l, m, a, b} \cdot \delta_{a, x_{l}-x_{m}} \delta_{b, y_{l}-y_{m}}\right]
\end{align*}
$$

where $v_{l, m a, b}$ is some complex number of modulus 1 , which we do not specify. In particular, the above matrices have zero diagonal entries. Note that $x_{m}-x_{l}=x_{l}-x_{m}$ and $y_{m}-y_{l}=y_{l}-y_{m}$ imply that $\left(x_{l}, y_{l}\right)=\left(x_{m}, y_{m}\right)$ for $(x, y) \in S_{n}^{G H}$. This implies that for fixed $l, m$, at least one of the two terms in each of $\left[\Psi^{C}\left(G_{a, b}\right)\right]_{l, m}$ and $\left[\Psi^{C}\left(H_{a, b}\right)\right]_{l, m}$ must vanish; the first term for both or the second for both.

Let $\Theta=F, G, H$ and $(a, b) \neq\left(a^{\prime}, b^{\prime}\right)$, then two matrices $\Psi^{C}\left(\Theta_{a, b}\right)$ and $\Psi^{C}\left(\Theta_{a^{\prime}, b^{\prime}}\right)$ have non-zero elements at different positions with no overlap, which shows $\operatorname{Tr}\left[\Psi^{C}\left(\Theta_{a, b}\right)\left[\Psi^{C}\left(\Theta_{a^{\prime}, b^{\prime}}\right)\right]^{*}\right]=0$. I.e., any two of $n^{2} \times n^{2}$ matrices in 70 are orthogonal to each other if we choose two different pairs of $(a, b) \in \mathbb{Z}_{n} \times \mathbb{Z}_{n}$. Hence, showing $\operatorname{Tr}\left[\Psi^{C}\left(G_{a, b}\right)\left[\Psi^{C}\left(H_{a, b}\right)\right]^{*}\right]=$ 0 implies that $A_{\Psi^{C}}$ is diagonal. Indeed,

$$
\begin{equation*}
\operatorname{Tr}\left[\Psi^{C}\left(G_{a, b}\right)\left[\Psi^{C}\left(H_{a, b}\right)\right]^{*}\right]=\frac{n}{2 k^{2}} \cdot \sum_{l \neq k}\left[\mathrm{i} \cdot \delta_{a, x_{m}-x_{l}} \delta_{b, y_{m}-y_{l}}-\mathrm{i} \cdot \delta_{a, x_{l}-x_{m}} \delta_{b, y_{l}-y_{m}}\right]=\frac{n}{2 k^{2}} \cdot \sum_{l<k} 0=0 \tag{71}
\end{equation*}
$$

To get diagonal entries of $A_{\Psi^{C}}$,

$$
\begin{align*}
\operatorname{Tr}\left[\Psi^{C}\left(F_{a, b}\right)\left[\Psi^{C}\left(F_{a, b}\right)\right]^{*}\right] & =\frac{n}{k^{2}} \cdot \sum_{1 \leq l, m \leq k}\left[\delta_{a, x_{m}-x_{l}} \delta_{b, x_{m}-y_{l}}\right]  \tag{72}\\
\operatorname{Tr}\left[\Psi^{C}\left(G_{a, b}\right)\left[\Psi^{C}\left(G_{a, b}\right)\right]^{*}\right] & =\frac{n}{2 k^{2}} \cdot \sum_{1 \leq l, m \leq k}\left[\delta_{a, x_{m}-x_{l}} \delta_{b, u_{m}-y_{l}}+\delta_{a, x_{l}-x_{m}} \delta_{b, y_{l}-y_{m}}\right]  \tag{73}\\
& =\operatorname{Tr}\left[\Psi^{C}\left(H_{a, b}\right)\left[\Psi^{C}\left(H_{a, b}\right)\right]^{*}\right] . \tag{74}
\end{align*}
$$

Note that in 72,,$\delta_{a, x_{m}-x_{l}} \delta_{b, u_{m}-y_{l}}=\delta_{a, x_{l}-x_{m}} \delta_{b, y_{l}-y_{m}}$.
Theorem 19. For the complementary channel $\Psi^{C}$ of the channel defined in in (57) we have

$$
\gamma\left(\Psi^{C}\right)= \begin{cases}\frac{1}{k}+\frac{1}{2 k^{2}}(n-1) \max _{(a, b) \in \mathcal{S}_{n}} N(a, b) & \text { if } \max _{(a, b) \in \mathcal{S}_{n}} N(a, b) \leq 2  \tag{75}\\ \frac{1}{2 k^{2}} \max _{(a, b) \in \mathcal{S}_{n}} N(a, b) & \text { if } \max _{(a, b) \in \mathcal{S}_{n}} N(a, b)>2\end{cases}
$$

where $N(a, b)$ is defined in 69). Here, $\gamma(\cdot)$ is defined in 40).

## D. Additive examples by discrete Weyl covariant channels

In this section, we use Theorem 13 to have some examples of discrete Weyl covariant channels which show additivity. To do so, we need to look into $\mathcal{C}_{\text {add }}$ in Definition 12. Since the preceding paper [DFH06] has such examples in terms of $\Psi$ in terms of Lemma 16, we construct examples in terms of Lemma 18. Interestingly, the condition we consider is purely algebraic.

Example 20. For $n=5$, the sequence $(1,2),(2,3),(1,4),(2,4)$ in the definitions 67 and 68 gives additive examples $\Psi$ and $\Psi^{C}$ in the sense of Theorem 13 .

Proof. We prove additivity for $\Phi^{C}$, then additivity is also true for $\Phi$ by 11 and 12 . First, we draw the table of $(x, y)-\left(x^{\prime}, y^{\prime}\right)$ where $(x, y)$ name columns and $\left(x^{\prime}, y^{\prime}\right)$ rows.

|  | $(1,2)$ | $(2,3)$ | $(1,4)$ | $(2,4)$ |
| :--- | :--- | :--- | :--- | :--- |
| $(1,2)$ | $(0,0)$ | $(1,1)$ | $(0,2)$ | $(1,2)$ |
| $(2,3)$ | $(4,4)$ | $(0,0)$ | $(4,1)$ | $(0,1)$ |
| $(1,4)$ | $(0,3)$ | $(1,4)$ | $(0,0)$ | $(1,0)$ |
| $(2,4)$ | $(4,3)$ | $(0,4)$ | $(4,0)$ | $(0,0)$ |

Then, $N(a, b)$ in 69 becomes as follows:

$$
N(a, b)= \begin{cases}2 & \text { if }(a, b)=(1,2),(1,4),(1,1),(0,1),(0,2),(1,0)  \tag{76}\\ 0 & \text { if }(a, b)=(1,3),(2,3),(2,4),(3,4),(2,2),(2,0)\end{cases}
$$

Note that $(a, b)$ 's which give 2 constitute the eigenspaces for the largest eigenvalues of $A_{\Psi}$. Among them, we pick up $(0,1)$ and $(0,2)$ so that $G_{0,1}, H_{0,1}, G_{0,2}, H_{0,2}$ generate $V, V^{2}, V^{3}, V^{4}$ where $V$ is defined in 50). Since $\frac{1}{5}\left[I+V+V^{2}+\right.$ $\left.V^{3}+V^{4}\right]$ is a rank-one projection, the channel $\Psi^{C}$ satisfies $\mathcal{C}_{a d d}$, so that Theorem 13 can be applied.

## IV. NORMS OF SUPER OPERATORS

In this section, we reinterpret our bound on $\|\cdot\|_{p \rightarrow 2}$ of Lemma 8 in the complex settings. Up to now we focused on linear maps $\Phi: H_{n} \rightarrow H_{k}$, but our results are compatible with maps $\Phi: M_{n}(\mathbb{C}) \rightarrow M_{k}(\mathbb{C})$. In fact, an orthonormal basis $\left(M_{i}\right)_{i=0}^{n^{2}-1}$ in $H_{n}$ becomes one in $M_{n}(\mathbb{C})$ if we span them over $\mathbb{C}$. To pursue this problem, we introduce another norm for $\Phi$ :

$$
\begin{equation*}
|\Phi|_{p \rightarrow q}=\max _{0 \neq \rho \in M_{n}(\mathbb{C})} \frac{\|\Phi(\rho)\|_{q}}{\|\rho\|_{p}} \tag{77}
\end{equation*}
$$

In general, $\|\Phi\|_{p \rightarrow q} \leq|\Phi|_{p \rightarrow q}$, but the equality holds when $\Phi$ is completely positive Wat05.
Theorem 21. Take linear maps $\Phi$ and $\Omega$ where $\Phi$ is unital and trace-preserving. Then, for $1 \leq p \leq 2$

$$
|\Phi \otimes \Omega|_{p \rightarrow 2}^{2} \leq \begin{cases}k^{-1} n^{2-\frac{2}{p}}+\left(1-n^{1-\frac{2}{p}}\right)\left\|A_{\Phi}\right\|_{\infty} \cdot|\Omega|_{p \rightarrow 2}^{2} & \text { if } k \cdot\left\|A_{\Phi}\right\|_{\infty} \leq n  \tag{78}\\ \left\|A_{\Phi}\right\|_{\infty} \cdot|\Omega|_{p \rightarrow 2}^{2} & \text { if } k \cdot\left\|A_{\Phi}\right\|_{\infty}>n\end{cases}
$$

Proof. First, Lemma 8 and (41) can naturally be extended to our case. Since $\|\rho\|_{p}=1$ we have

$$
\begin{equation*}
\operatorname{Tr}\left[(\mathrm{id} \otimes \Omega(\rho))^{2}\right] \leq|\mathrm{id} \otimes \Omega|_{p \rightarrow 2}^{2}=|\Omega|_{p \rightarrow 2}^{2} \tag{79}
\end{equation*}
$$

The equality was proved in Wat05.
Next, if $\left\|A_{\Phi}\right\|_{\infty} \geq \frac{n}{k}$ then the first term in 41 is non-positive, which we ignore to get the bound $\left\|A_{\Phi}\right\|_{\infty} \cdot|\Omega|_{p \rightarrow 2}^{2}$. Otherwise we bound the first term by

$$
\begin{equation*}
\operatorname{Tr}\left[\Omega\left(\rho_{k}\right)^{2}\right] \leq n^{2-\frac{2}{p}}|\Omega|_{p \rightarrow 2}^{2} \tag{80}
\end{equation*}
$$

because $\left\|\rho_{K}\right\|_{p} \leq n^{1-\frac{1}{p}}$. This completes the proof.
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