# New MDS Self-Dual Codes from Generalized Reed-Solomon Codes 

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#### Abstract

Both MDS and Euclidean self-dual codes have theoretical and practical importance and the study of MDS self-dual codes has attracted lots of attention in recent years. In particular, determining existence of $q$-ary MDS self-dual codes for various lengths has been investigated extensively. The problem is completely solved for the case where $q$ is even. The current paper focuses on the case where $q$ is odd. We construct a few classes of new MDS self-dual codes through generalized Reed-Solomon codes. More precisely, we show that for any given even length $n$ we have a $q$-ary MDS code as long as $q \equiv 1 \bmod 4$ and $q$ is sufficiently large (say $q \geq 4^{n} \times n^{2}$ ). Furthermore, we prove that there exists a $q$-ary MDS self-dual code of length $n$ if $q=r^{2}$ and $n$ satisfies one of the three conditions: (i) $n \leq r$ and $n$ is even; (ii) $q$ is odd and $n-1$ is an odd divisor of $q-1$; (iii) $r \equiv 3 \bmod 4$ and $n=2 t r$ for any $t \leq(r-1) / 2$.


## Index Terms

Self-dual codes, MDS codes, Generalized Reed-Solomon codes.

## I. Introduction

MDS codes and Euclidean self-dual codes belong to two different categories of block codes. Both classes are of practical and theoretical importance. In recent years, study of MDS self-dual codes (we only consider Euclidean inner product in the following context) has attracted a lot of attention [1]-[3], [9]-[14]. First of all, MDS codes achieve optimal parameters that allow correction of maximal number of errors for a given code rate. Study of various properties of MDS codes, such as classification [15], [20] of MDS codes, non-Reed-Solomon MDS codes [21], balanced MDS codes [6], lowest density MDS codes [4], [17] and existence of MDS codes [7], has been the center of the area. In addition, MDS codes are closely connected to combinatorial design and finite geometry [18, Chapters 11 and 14]. Furthermore, the generalized Reed-Solomon codes are a class of MDS codes and have found wide applications in practice. On the other hand, due to their nice structures, self-dual codes have been attracting attention from both coding theorists, cryptographers and mathematicians. Self-dual codes have found various applications in cryptography (in particular secret sharing) [5], [8], [19] and combinatorics [18]. Thus, it is natural to consider the intersection of these two classes, namely, MDS self-dual codes.

As the parameters of an MDS self-dual code is completely determined by its length, one of the central problems in this topic is to determine existence of MDS self-dual codes for various lengths. The problem is completely solved for the case where $q$ is even [10]. The current paper focuses on the case where $q$ is odd. Our idea is to construct generalized Reed-Solomon code that are self-dual. Thus, the result is of theoretical interest and practical relevance.

[^0]TABLE I
KNOWN RESULTS ON EXISTENCE OF $q$-ARY MDS SELF-DUAL CODES OF EVEN LENGTH $n$

| $q$ | $2 \mid q$ | $2 \nmid q$ | $q=r^{t}, t$ even | $q=r^{t}, r \equiv 3 \bmod 4, t$ odd | $q=r^{t}, r \equiv 1 \bmod 4, t$ odd |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $n$ | $n \leq q$ | $n=q+1$ | $(n-1) \mid(r-1)$ | $n=p^{m}+1$, odd $m$ and prime $p$ <br> and $p \equiv 3 \bmod 4$ | $n=p^{m}+1, m$ odd and prime $p$ <br> and $p \equiv 1 \bmod 4$ |
| Reference | $[10]$ | $[10]$ | $\boxed{11]}$ | $[11]$ | $[11]$ |

## A. Known results

One of the existing constructions of MDS self-dual codes in literature is through constacyclic codes [1], [10], [14] because the generator polynomial of the dual code of a constacyclic code can be determined by the generator polynomial of the code. Some other approaches include orthogonal designs [3], [9] and generalized Reed-Solomon codes [1]. We summarize some known results in the Table I.

Besides the results in Table I, only some sparse lengths $n$ of MDS self-dual codes have been found (see [1]-[3], [9], [12], [14]).

## B. Our results

We show the following result in this paper.
Theorem 1.1 (Main Theorem): Let $q$ be an odd prime power and let $n$ be an even positive integer. Then there exists a $q$-ary MDS self-dual code of length $n$ if $q$ and $n$ satisfy one of the following conditions
(i) $q \equiv 1 \bmod 4$ and $q \geq 4^{n} \times n^{2}$ (see Theorem 3.2(ii));
(ii) $q=r^{2}$ and $n \leq r$ (see Theorem 3.4(i));
(iii) $q=r^{2}$ and $n-1$ is a divisor of $q-1$ (see Theorem 3.4(ii));
(iv) $q=r^{2}, r \equiv 3 \bmod 4$ and $n=2 t r$ for any $t \leq(r-1) / 2$ (see Theorem 3.5).

Remark 1.2: Part (i) of Theorem 1.1 says that for any given even length $n$ we have a $q$-ary MDS code as long as $q \equiv 1 \bmod 4$ and $q$ is sufficiently large (say $q \geq 4^{n} \times n^{2}$ ), while Part (iii) of Theorem 1.1] extends the result of [11] where a stricter condition $(n-1) \mid(r-1)$ is required. In addition, we also use our approach to get MDS self-dual codes in [10]. Note that the approach in [10] is quite different as the main tool for constructing MDS codes in [10] is orthogonal design, while our construction is through generalized Reed-Solomon codes.

## Our techniques

Our idea of constructing MDS self-dual codes is through generalized Reed-Solomon (GRS or generalized RS for short) codes. In this paper, we present two methods to construct generalized RS codes that are self-dual. The first one is to directly find elements $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$ such that $\prod_{1 \leq j \leq n, j \neq i}\left(\alpha_{i}-\alpha_{j}\right)$ is a square element in $\mathbb{F}_{q}$. The second method is to find a sufficient condition under which the homogenous equation system $A \mathbf{x}^{T}=\mathbf{0}$ with $A$ over $\mathbb{F}_{q}$ has a nonzero solution over $\mathbb{F}_{r}$, where $q=r^{2}$ (see (II.4)).

## Organization of the paper

In Section 2, we first study generalized Reed-Solomon codes and their duals, and analyze solutions of a system of homogenous equations. In Section 3, we show that these conditions are satisfied in some cases and consequently we obtain several classes of MDS self-dual codes.

## II. Preliminaries

Let $\mathbb{F}_{q}$ be the finite field of $q$ elements and let $\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right\}$ be $n$ distinct elements of $\mathbb{F}_{q}$. Choose $n$ nonzero elements $v_{1}, v_{2}, \ldots, v_{n}$ of $\mathbb{F}_{q}\left(v_{i}\right.$ may not be distinct). Put $\mathbf{v}=\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ and $\mathbf{a}=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right)$. Then the generalized Reed-Solomon code associated with $\mathbf{a}$ and $\mathbf{v}$ is defined below.

$$
\begin{equation*}
\mathcal{G} R S_{k}(\mathbf{a}, \mathbf{v}):=\left\{\left(v_{1} f\left(\alpha_{1}\right), v_{2} f\left(\alpha_{2}\right), \ldots, v_{n} f\left(\alpha_{n}\right)\right): f(x) \in \mathbb{F}_{q}[x], \operatorname{deg}(f(x)) \leq k-1\right\} \tag{II.1}
\end{equation*}
$$

It is well known that the code $\mathcal{G} R S_{k}(\mathbf{a}, \mathbf{v})$ is a $q$-ary $[n, k, n-k+1]$-MDS code [18, Theorem 9.1.4] and the corresponding dual code is also a GRS code.

Furthermore we consider the extended code of the generalized Reed-Solomon code $\mathcal{G} R S_{k}(\mathbf{a}, \mathbf{v})$ given by

$$
\begin{equation*}
\mathcal{G} R S_{k}(\mathbf{a}, \mathbf{v}, \infty):=\left\{\left(v_{1} f\left(\alpha_{1}\right), v_{2} f\left(\alpha_{2}\right), \ldots, v_{n} f\left(\alpha_{n}\right), f_{k-1}\right): f(x) \in \mathbb{F}_{q}[x], \operatorname{deg}(f(x)) \leq k-1\right\} \tag{II.2}
\end{equation*}
$$

where $f_{k-1}$ stands for the coefficient of $x^{k-1}$ in $f(x)$. The following result can be easily derived from [18].
Lemma 2.1: The code $\mathcal{G} R S_{k}(\mathbf{a}, \mathbf{v}, \infty)$ defined in (II.2) is a $q$-ary $[n+1, k, n+2-k]$-MDS code.
For any distinct elements $\alpha_{1}, \ldots, \alpha_{n}$ of $\mathbb{F}_{q}$, put $\mathbf{a}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and denote by $A_{\mathbf{a}}$ the matrix

$$
\left(\begin{array}{cccc}
1 & 1 & \cdots & 1  \tag{II.3}\\
\alpha_{1} & \alpha_{2} & \cdots & \alpha_{n} \\
\alpha_{1}^{2} & \alpha_{2}^{2} & \cdots & \alpha_{n}^{2} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{1}^{n-2} & \alpha_{2}^{n-2} & \cdots & \alpha_{n}^{n-2}
\end{array}\right)
$$

Lemma 2.2: The solution space of the equation system $A_{\mathbf{a}} \mathbf{x}^{T}=\mathbf{0}$ has dimension 1 and $\left\{\mathbf{u}=\left(u_{1}, \ldots, u_{n}\right)\right\}$ is a basis of this solution space, where $u_{i}=\prod_{1 \leq j \leq n, j \neq i}\left(\alpha_{i}-\alpha_{j}\right)^{-1}$. Furthermore, for any two polynomials $f(x), g(x) \in \mathbb{F}_{q}[x]$ with $\operatorname{deg}(f) \leq k-1$ and $\operatorname{deg}(g) \leq n-k-1$, one has $\sum_{i=1}^{n} f\left(\alpha_{i}\right)\left(u_{i} g\left(\alpha_{i}\right)\right)=0$.

Proof: It is easy to see that the rank of $A_{\mathrm{a}}$ is $n-1$. Thus, the solution space has dimension 1. Furthermore, it is straightforward to verify that $\mathbf{u}$ is a nonzero solution.

Since $\mathbf{u}$ is a solution of $A_{a} \mathbf{x}=\mathbf{0}$, it is easy to see that the Euclidean inner product of $\left(\alpha_{1}^{i}, \ldots, \alpha_{n}^{i}\right)$ and $\left(u_{1} \alpha_{1}^{j}, \ldots, u_{n} \alpha_{n}^{j}\right)$ is zero for all $0 \leq i \leq k-1$ and $0 \leq j \leq n-k-1$. This implies that $\sum_{i=1}^{n} f\left(\alpha_{i}\right)\left(u_{i} g\left(\alpha_{i}\right)\right)=0$ for any two polynomials $f(x), g(x) \in \mathbb{F}_{q}[x]$ with $\operatorname{deg}(f) \leq k-1$ and $\operatorname{deg}(g) \leq n-k-1$.

Lemma 2.3: Let 1 be all-one word of length $n$. Then one has the following results.
(i) The dual code of $\mathcal{G} R S_{k}(\mathbf{a}, \mathbf{1})$ is $\mathcal{G} R S_{n-k}(\mathbf{a}, \mathbf{u})$, where $\mathbf{u}=\left(u_{1}, u_{2}, \ldots, u_{n}\right)$ with $u_{i}=\prod_{1 \leq j \leq n, j \neq i}\left(\alpha_{i}-\alpha_{j}\right)^{-1}$.
(ii) If $1 \leq k \leq q-1$, then the dual code of $\mathcal{G} R S_{k}(\mathbf{a}, \mathbf{1}, \infty)$ is $\mathcal{G} R S_{q-k+1}(\mathbf{a}, \mathbf{1}, \infty)$.

Proof: By the second statement of Lemma 2.2, we know that $\mathcal{G} R S_{n-k}(\mathbf{a}, \mathbf{u})$ is orthogonal to $\mathcal{G} R S_{k}(\mathbf{a}, \mathbf{1})$. Thus, part (i) follows from the fact that $\operatorname{dim}\left(\mathcal{G} R S_{k}(\mathbf{a}, \mathbf{1})\right)+\operatorname{dim}\left(\mathcal{G} R S_{n-k}(\mathbf{a}, \mathbf{u})\right)=k+n-k=n$.

For part (ii), we denote

$$
\mathbf{c}_{i}=\left\{\begin{array}{ll}
\left(\alpha_{1}^{i}, \alpha_{2}^{i}, \ldots, \alpha_{q}^{i}, 0\right) & \text { if } i \neq k-1, \\
\left(\alpha_{1}^{i}, \alpha_{2}^{i}, \ldots, \alpha_{q}^{i}, 1\right) & \text { if } i=k-1 ;
\end{array} \quad \overline{\mathbf{c}}_{i}= \begin{cases}\left(\alpha_{1}^{i}, \alpha_{2}^{i}, \ldots, \alpha_{q}^{i}, 0\right) & \text { if } i \neq q-k \\
\left(\alpha_{1}^{i}, \alpha_{2}^{i}, \ldots, \alpha_{q}^{i}, 1\right) & \text { if } i=q-k\end{cases}\right.
$$

Consider the dot product of $\mathbf{c}_{\ell}$ and $\overline{\mathbf{c}}_{m}$ with $0 \leq \ell \leq k-1$ and $0 \leq m \leq q-k$. If $\ell=m=0$, then both $\mathbf{c}_{\ell}$ and $\mathbf{c}_{m}$ are $(1,0)$, where 1 is the all-one word of length $q$. Thus, the dot product $\left\langle\mathbf{c}_{\ell}, \overline{\mathbf{c}}_{m}\right\rangle$ is 0 . If $\ell=k-1$ and $m=q-k$, then $\mathbf{c}_{\ell}=\left(\alpha_{1}^{\ell}, \alpha_{2}^{\ell}, \ldots, \alpha_{q}^{\ell}, 1\right)$ and $\overline{\mathbf{c}}_{m}=\left(\alpha_{1}^{m}, \alpha_{2}^{m}, \ldots, \alpha_{q}^{m}, 1\right)$. Thus, $\left\langle\mathbf{c}_{\ell}, \overline{\mathbf{c}}_{m}\right\rangle=1+\sum_{i=1}^{q} \alpha_{i}^{q-1}=0$. Now assume that $0<\ell+m<q-1$. Without loss of generality, let $\ell>0$. Then $\mathbf{c}_{\ell}=\left(\alpha_{1}^{\ell}, \alpha_{2}^{\ell}, \ldots, \alpha_{q}^{\ell}, 0\right)$. Thus, $\left\langle\mathbf{c}_{\ell}, \overline{\mathbf{c}}_{m}\right\rangle=\sum_{i=1}^{q} \alpha_{i}^{\ell+m}=0$ since $1 \leq \ell+m \leq q-2$. This completes the proof of Part (ii).
The following corollary follows immediately from Lemma 2.3 .
Corollary 2.4: Let $n$ be an even number.
(i) Let $\lambda \in \mathbb{F}_{q}^{*}$. If $w_{i}=\lambda \prod_{1 \leq j \leq n, j \neq i}\left(\alpha_{i}-\alpha_{j}\right)^{-1}$ is equal to $v_{i}^{2}$ for some $v_{i} \in \mathbb{F}_{q}$ for all $i=1,2, \ldots, n$, then the code $\mathcal{G} R S_{n / 2}(\mathbf{a}, \mathbf{v})$ is MDS self-dual.
(ii) If $q$ is odd, then the code $\mathcal{G} R S_{(q+1) / 2}(\mathbf{a}, \mathbf{1}, \infty)$ is self-dual.

Proof: To prove Part (i), let $f(x), g(x) \in \mathbb{F}_{q}[x]$ with $\operatorname{deg}(f) \leq \frac{n}{2}-1$ and $\operatorname{deg}(g) \leq \frac{n}{2}-1$. By the second statement of Lemma [2.2, we have $\sum_{i=1}^{n} f\left(\alpha_{i}\right)\left(u_{i} g\left(\alpha_{i}\right)\right)=0$, where $u_{i}=\prod_{1 \leq j \leq n, j \neq i}\left(\alpha_{i}-\alpha_{j}\right)^{-1}$ for $i=1,2, \ldots, n$. Hence,

$$
0=\lambda \sum_{i=1}^{n} f\left(\alpha_{i}\right)\left(u_{i} g\left(\alpha_{i}\right)\right)=\sum_{i=1}^{n} f\left(\alpha_{i}\right)\left(\lambda u_{i} g\left(\alpha_{i}\right)\right)=\sum_{i=1}^{n}\left(v_{i} f\left(\alpha_{i}\right)\right)\left(v_{i} g\left(\alpha_{i}\right)\right) .
$$

This implies that $\mathcal{G} R S_{n / 2}^{\perp}(\mathbf{a}, \mathbf{v})=\mathcal{G} R S_{n / 2}(\mathbf{a}, \mathbf{v})$.
Part (ii) is a direct result of Lemma [2.3(ii).
For the rest of this section, we provide another sufficient condition under which a GRS code is self-dual. For this purpose, we assume that $q=r^{2}$.

Let us consider a system of equations over $\mathbb{F}_{r^{2}}$ given by

$$
\begin{equation*}
A \mathbf{x}^{T}=\mathbf{0} \tag{II.4}
\end{equation*}
$$

where $A$ is an $(n-1) \times n$ matrix of rank $n-1$ over $\mathbb{F}_{r^{2}}$. One knows that (II.4) must have at least one nonzero solution over $\mathbb{F}_{r^{2}}$. However, for our application, we are curious about the question whether (II.4) has a nonzero solution over $\mathbb{F}_{r}$. In this section, we give some sufficient and necessary conditions under which (II.4) has a nonzero solution over $\mathbb{F}_{r}$.

Lemma 2.5: The equation (II.4) has a nonzero solution in $\mathbb{F}_{r}^{n}$ if and only if $\mathbf{c}^{r}$ is a solution of (II.4) whenever $\mathbf{c}$ is a solution of (II.4), where $\mathbf{c}^{r}$ is obtained from $\mathbf{c}$ by raising every coordinate to its $r$ th power.

Proof: If (II.4) has a nonzero solution $\mathbf{b}$ in $\mathbb{F}_{r}^{n}$, then the solution space of (II.4) is $\mathbb{F}_{r^{2}} \cdot \mathbf{b}=\left\{\alpha \mathbf{b}: \alpha \in \mathbb{F}_{r^{2}}\right\}$ since the solution space has dimension 1 over $\mathbb{F}_{r^{2}}$. Thus, for every solution $\lambda \mathbf{b}$, we have $(\lambda \mathbf{b})^{r}=\lambda^{r} \mathbf{b} \in \mathbb{F}_{r^{2}} \cdot \mathbf{b}$.

Conversely, assume that $\mathbf{c}^{r}$ is a solution of (II.4) for a nonzero solution $\mathbf{c}$ of (III.4). Choose a basis $\{1, \alpha\}$ of $\mathbb{F}_{r^{2}}$ over $\mathbb{F}_{r}$. Consider the two elements $\mathbf{w}_{1}:=\mathbf{c}+\mathbf{c}^{r}$ and $\mathbf{w}_{2}:=\alpha \mathbf{c}+\alpha^{r} \mathbf{c}^{r}$. It is clear that both $\mathbf{w}_{1}$ and $\mathbf{w}_{2}$ are solutions of (II.4) in $\mathbb{F}_{r}^{n}$. On the other hand, we have

$$
\binom{\mathbf{c}}{\mathbf{c}^{r}}=\left(\begin{array}{cc}
1 & 1 \\
\alpha & \alpha^{r}
\end{array}\right)^{-1}\binom{\mathbf{w}_{1}}{\mathbf{w}_{2}} .
$$

This implies that one of $\mathbf{w}_{1}$ and $\mathbf{w}_{2}$ must be nonzero, otherwise $\mathbf{c}$ is equal to zero. This completes the proof.
The condition given in Lemma 2.5 can be converted to a condition on the coefficient matrix of the equation (II.4) as shown below.

Lemma 2.6: Let $A$ be the coefficient matrix of the equation (II.4). Then the equation (II.4) has a nonzero solution in $\mathbb{F}_{r}^{n}$ if and only if $A^{(r)}$ and $A$ are row equivalent, where $A^{(r)}$ is obtained from $A$ by raising every entry to its $r$ th power.

Proof: It is easy to see that $\mathbf{c}^{r}$ is a solution of $A^{(r)} \mathbf{x}^{T}=\mathbf{0}$ whenever $\mathbf{c}$ is a solution of (II.4) and vice versa. By Lemma 2.1) this implies that the equation (II.4) has a nonzero solution in $\mathbb{F}_{r}^{n}$ if and only if the equation $A^{(r)} \mathbf{x}^{T}=\mathbf{0}$ and the equation (II.4) have the same solution space, i.e., $A^{(r)}$ and $A$ are row equivalent.

Example 2.7: Let $m$ be a divisor of $r^{2}-1$ and let $n=m+1$. Let $\alpha_{2}, \ldots, \alpha_{n}$ be all the $m$ th roots of unity. We claim that the system $A_{\mathbf{a}} \mathbf{x}=\mathbf{0}$ has a nonzero solution in $\mathbb{F}_{r}^{n}$, where $\mathbf{a}=\left(\alpha_{1}=0, \alpha_{2}, \ldots, \alpha_{n}\right)$. To prove this, it is sufficient to show that the rows of $A_{\mathrm{a}}^{(r)}$ are a permutation of the rows of $A_{\mathrm{a}}$. The first row of the two matrices are identical. Hence, it is sufficient to show that the last $n-2=m-1$ rows of $A_{\mathrm{a}}^{(r)}$ are a permutation of those of $A_{\mathrm{a}}$. To see this, we notice that the powers in the last $m-1$ rows of $A_{\mathbf{a}}^{(r)}$ consist of $\{1 \cdot r, 2 \cdot r, \ldots,(m-1) \cdot r\}$, while the powers in the last $m-1$ rows of $A_{\mathbf{a}}$ consist of $\{1,2 \ldots, m-1\}$. Thus, the desired result follows from the fact that the set $\{1 \cdot r(\bmod m), 2 \cdot r(\bmod m), \ldots,(m-1) \cdot r$ $(\bmod m)\}$ and the set $\{1,2 \ldots, m-1\}$ are identical.

Example 2.8: Let $\alpha_{1}, \ldots, \alpha_{n}$ be all the $n$ distinct elements of $\mathbb{F}_{r}$. Then $A_{\mathbf{a}}$ is a matrix over $\mathbb{F}_{r}$ and it is clear that system $A_{\mathrm{a}} \mathrm{x}=\mathbf{0}$ has a nonzero solution in $\mathbb{F}_{r}^{n}$. On the other hand, if we apply Lemma [2.6 we can also see that $A_{\mathrm{a}} \mathbf{x}=\mathbf{0}$ has a nonzero solution in $\mathbb{F}_{r}^{n}$ since $A_{\mathbf{a}}$ and $A_{\mathbf{a}}^{(r)}$ are equal and hence row equivalent.

Lemma 2.9: Let $n$ be an even number and let $q=r^{2}$. Let $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$ be $n$ distinct elements of $\mathbb{F}_{q}$. If $A_{\mathbf{a}} \mathbf{x}=\mathbf{0}$ has a nonzero solution $\mathbf{w}=\left(w_{1}, \ldots, w_{n}\right)$ in $\mathbb{F}_{r}$, then the code $\mathcal{G} R S_{n / 2}(\mathbf{a}, \mathbf{v})$ is an MDS self-dual code over $\mathbb{F}_{q}$, where $w_{i}=v_{i}^{2}$ for all $1 \leq i \leq n$.

Proof: Since $w_{i}$ belongs to $\mathbb{F}_{r}$, there exists an element $v_{i} \in \mathbb{F}_{q}$ such that $w_{i}=v_{i}^{2}$. As the dimension of the solution space of $A_{\mathbf{a}} \mathbf{x}=\mathbf{0}$ is 1 , by Lemma [2.2, we must have $w_{i}=\lambda \prod_{1 \leq j \leq n, j \neq i}\left(\alpha_{i}-\alpha_{j}\right)^{-1} \neq 0$ for some $\lambda \in \mathbb{F}_{r^{2}}^{*}$. The desired result follows from Corollary 2.4 (i).

## III. MDS SELF-DUAL CODES

## A. MDS self-dual codes over $\mathbb{F}_{q}$ for sufficiently large $q$

Let us start with a lemma.
Lemma 3.1: For any given $n$, if $q \geq 4^{n} \times n^{2}$, then there exists a subset $S=\left\{\alpha_{1}, \alpha_{2}, \ldots, \ldots, \alpha_{n}\right\}$ of $\mathbb{F}_{q}$ such that $\alpha_{j}-\alpha_{i}$ are nonzero square elements for all $1 \leq i<j \leq n$.

Proof: If $q$ is even, it is clearly true as every element of $\mathbb{F}_{q}$ is a square.
Now assume that $q$ is odd. We prove it by induction on $n$. For $n=2$, we can let $S=\{0,1\}$. Suppose that there exists a subset $T=\left\{\alpha_{1}, \alpha_{2}, \ldots, \ldots, \alpha_{n-1}\right\}$ of $\mathbb{F}_{q}$ of size $n-1$ such that $\alpha_{j}-\alpha_{i}$ are nonzero square elements for all $1 \leq i<j \leq n-1$.

Let $\alpha$ be a primitive element of $\mathbb{F}_{q}$ and let $\chi$ be the multiplicative quadratic character defined by $\chi\left(\alpha^{i}\right)=\alpha^{i(q-1) / 2}$ and $\chi(0)=0$. It is clear that $i$ is even if and only if $\chi\left(\alpha^{i}\right)=1$. Let $N$ denote the number of elements $\beta$ of $\mathbb{F}_{q}$ such that $\chi\left(\beta-\alpha_{i}\right)=1$ for all $i=1,2, \ldots, n-1$. Then by [16, Exercise 5.64], one has

$$
\begin{equation*}
\left|N-\frac{q}{2^{n-1}}\right| \leq\left(\frac{n-3}{2}+\frac{1}{2^{n-1}}\right) \sqrt{q}+\frac{n-1}{2} . \tag{III.1}
\end{equation*}
$$

Thus, by (III.1) and our condition on $n$ and $q$, we have

$$
N \geq \frac{q}{2^{n-1}}-\left(\frac{n-3}{2}+\frac{1}{2^{n-1}}\right) \sqrt{q}-\frac{n-1}{2}>0
$$

This implies that there exists an element $\alpha_{n}$ such that $\alpha_{n}-\alpha_{i}$ are nonzero square elements of $\mathbb{F}_{q}$ for all $i=1,2, \ldots, n-1$. The proof is completed.

Theorem 3.2: Let $n$ be an even integer. If $n$ and $q$ satisfy one of the following three conditions, then there exists a $q$-ary $[n, n / 2, n / 2+1]$-MDS self-dual code.
(i) $q$ is even and $n \leq q$;
(ii) $q \equiv 1 \bmod 4$, and $q \geq 4^{n} \times n^{2}$;
(iii) $q$ is odd, $n=q+1$.

Proof: If $q$ is even, then every element of $\mathbb{F}_{q}$ is a square. Thus, Case (i) follows from Corollary 2.4(i).
By Lemma 3.1, there exists a subset $S=\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right\}$ such that $\alpha_{j}-\alpha_{i}$ are square elements for all $1 \leq i<j \leq n$. As $q \equiv 1 \bmod 4,-1$ is a square since $-1=\alpha^{(q-1) / 2}$, where $\alpha$ is a primitive element of $\mathbb{F}_{q}$. Thus, $\beta-\gamma$ is a nonzero square for any two distinct elements $\beta, \gamma \in S$. Therefore, $\prod_{1 \leq j \leq n, j \neq i}\left(\alpha_{i}-\alpha_{j}\right)^{-1}$ are nonzero square elements of $\mathbb{F}_{q}$ for all $i=1,2, \ldots, n$. Case (ii) follows from Corollary 2.4(i) as well.

Case (iii) is the result of Corollary 2.4(ii).
Remark 3.3: (i) The results of Parts (i) and (iii) of Theorem 3.2 were given in [10]. Here a different proof is given.
(ii) The result of Part (ii) of Theorem 3.2 implies that MDS self-dual code with length $n$ always exists when alphabet size $q$ is exponential in $n$.
B. MDS self-dual codes over $\mathbb{F}_{q}$ with $q=r^{2}$

Theorem 3.4: Let $n$ be an even integer. If $n$ and $q=r^{2}$ satisfy one of the following two conditions, then there exists a $q$-ary $[n, n / 2, n / 2+1]$-MDS self-dual code.
(i) $n \leq r$;
(ii) $q$ is odd and $n-1$ is a divisor of $q-1$.

Proof: In case (i), we can choose $n$ distinct elements $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$ of $\mathbb{F}_{r}$. Then the system $A_{\mathbf{a}} \mathbf{x}=\mathbf{0}$ has a nonzero solution in $\mathbb{F}_{r}^{n}$, By Lemma 2.9, there exists a $q$-ary $[n, n / 2, n / 2+1]$-MDS self-dual code.

As $n-1$ is a divisor of $q-1$, by Example 2.7 we can find $n$ distinct elements $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$ of $\mathbb{F}_{q}$ such that the system $A_{\mathrm{a}} \mathbf{x}=\mathbf{0}$ has a nonzero solution in $\mathbb{F}_{r}^{n}$. Thus, Case (ii) follows from Lemma 2.9. This completes the proof.

Theorem 3.5: Let $q=r^{2}$ and $r \equiv 3 \bmod 4$, then there exists a $q$-ary [2tr, $\left.\operatorname{tr}, \operatorname{tr}+1\right]$-MDS self-dual code for any $1 \leq t \leq$ $(r-1) / 2$.

Proof: Label elements of $\mathbb{F}_{r}$ by $\left\{a_{1}, a_{2}, \ldots, a_{r}\right\}$. Assume that $\gamma$ is a primitive element of $\mathbb{F}_{q}$ and let $\beta=\gamma^{(r+1) / 2}$. Put $n=2 t r$ and $\alpha_{\ell r+k}=a_{\ell} \beta+a_{k}$ for all $1 \leq \ell \leq 2 t$ and $1 \leq k \leq r$. By Corollary 2.4(i), it is sufficient to show that $\prod_{1 \leq j \leq n, j \neq i}\left(\alpha_{i}-\alpha_{j}\right)^{-1}$ is a square of $\mathbb{F}_{q}$ for all $1 \leq i \leq n$.

Write $i=\ell_{0} r+k_{0}$ for some $1 \leq \ell_{0} \leq 2 t$ and $1 \leq k_{0} \leq r$. Then

$$
\begin{equation*}
v_{\ell_{0}}:=\prod_{\ell_{0} r+1 \leq j \leq \ell_{0} r+r, j \neq \ell_{0} r+k_{0}}\left(\alpha_{\ell_{0} r+k_{0}}-\alpha_{j}\right)=\prod_{1 \leq j \leq r, j \neq k_{0}}\left(a_{k_{0}}-a_{j}\right) \in \mathbb{F}_{r} \tag{III.2}
\end{equation*}
$$

Thus, $v_{\ell_{0}}$ is a square in $\mathbb{F}_{q}$ since it is an element of $\mathbb{F}_{r}$. Furthermore, for $\ell \neq \ell_{0}$, we have
$v_{\ell}:=\prod_{\ell r+1 \leq j \leq \ell r+r}\left(\alpha_{\ell_{0} r+k_{0}}-\alpha_{j}\right)=\prod_{1 \leq j \leq r}\left(\left(a_{\ell_{0}}-a_{\ell}\right) \beta+a_{k_{0}}-a_{j}\right)=\left(\left(a_{\ell_{0}}-a_{\ell}\right) \beta\right)^{r}-\left(a_{\ell_{0}}-a_{\ell}\right) \beta=\left(a_{\ell_{0}}-a_{\ell}\right) \beta\left(\beta^{r-1}-1\right)$.
This implies that $v_{\ell}$ is a square in $\mathbb{F}_{q}$ as well since $a_{\ell_{0}}-a_{\ell}$ and $\beta^{r-1}-1=-2$ are elements of $\mathbb{F}_{r}$ and $\beta=\gamma^{(r+1) / 2}$ is a square. Our result follows from the fact that $\prod_{1 \leq j \leq n, j \neq i}\left(\alpha_{i}-\alpha_{j}\right)^{-1}=\prod_{\ell=1}^{2 t} v_{\ell}^{-1}$.

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