# Robust Nonnegative Sparse Recovery and the Nullspace Property of 0/1 Measurements 

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#### Abstract

We investigate recovery of nonnegative vectors from non-adaptive compressive measurements in the presence of noise of unknown power. In the absence of noise, existing results in the literature identify properties of the measurement that assure uniqueness in the non-negative orthant. By linking such uniqueness results to nullspace properties, we deduce uniform and robust compressed sensing guarantees for nonnegative least squares. No $\ell_{1}$-regularization is required. As an important proof of principle, we establish that $m \times n$ random i.i.d. $0 / 1$-valued Bernoulli matrices obey the required conditions with overwhelming probability provided that $m=\mathcal{O}(s \log (n / s))$. We achieve this by establishing the robust nullspace property for random 0/1-matrices-a novel result in its own right. Our analysis is motivated by applications in wireless network activity detection.


## I. Introduction

Recovery of lower complexity objects by observations far below the Nyquist rate has applications in physics, applied math, and many engineering disciplines. Moreover, it is one of the key tools for facing challenges in data processing (like big data and the Internet of Things), wireless communications (the 5th generation of the mobile cellular network) and large scale network control. Compressed Sensing (CS), with its original goal of recovering sparse or compressible vectors, has, in particular, stimulated the research community to investigate further in this direction. The aim is to identify compressibility and low-dimensional structures which allow the recovery from low-rate samples with efficient algorithms. In many applications, the objects of interest exhibit further structural constraints which should be exploited in reconstruction algorithms. Take, for instance, the following setting which appears naturally in communication protocols: The components of sparse information carrying vectors are taken from a finite alphabet, or the data vectors are lying in specific subspaces. Similarly, in network traffic estimation and anomaly detection from end-to-end measurements, the parameters are restricted to particular low-dimensional domains. Finally, the signals occurring in imaging problems are typically constrained to non-negative intensities.

Our work is partially inspired by the task of identifying sparse network activation patterns in a large-scale asynchronous wireless network: Suppose that, in order to indicate its presence, each active device node transmits an individual sequence into a noisy wireless channel. All such sequences are multiplied with individual, but unknown,

[^0]channel amplitudes ${ }^{1}$ and finally superimpose at the receiver. The receiver's task then is to detect all active devices and the corresponding channel amplitudes from this global superposition (note that each device is uniquely characterized by the sequence it transmits). This problem can be re-cast as the task of estimating non-negative sparse vectors from noisy linear observations.

Such non-negative and sparse structures also arise naturally in certain empirical inference problems, like network tomography [1], [2], statistical tracking (see e.g. [3]) and compressed imaging of intensity patterns [4]. The underlying mathematical problem has received considerable attention in its own right [5], [6], [7], [8], [9], [10]. It has been shown that measurement matrices $\mathbf{A} \in \mathbb{R}^{m \times n}$ coming from outwardly s-neighborly polytopes [11] and matrices $\mathbf{A}$ whose row span intersects the positive orthant ${ }^{2}$ [12] maintain an intrinsic uniqueness property for non-negative, $s$-sparse vectors. These carry over to the under-determined setting ( $m<n$ ). Such uniqueness properties in turn allow for entirely avoiding CS algorithms in the reconstruction step. From an algorithmic point of view, this is highly beneficial. However, all the statements mentioned above focus on idealized scenarios, where no noise is present in the sampling procedure.

Motivated by device detection, we shall overcome this idealization and devise non-negative recovery protocols that are robust towards any form of additive noise. Our results have the added benefit that no a-priori bound on the noise step is required in the algorithmic reconstruction.

## A. Main Results

Mathematically, we are interested in recovering sparse, entry-wise nonnegative vectors $\mathbf{x} \geq \mathbf{0}$ in $\mathbb{R}^{n}$ from $m \ll n$ noisy linear measurements of the form $y_{i}=\mathbf{a}_{i}^{T} \mathbf{x}+e_{i}$. Here, the vectors $\mathbf{a}_{i} \in \mathbb{R}^{n}$ model the different linear measurement operations and $e_{i}$ is additive noise of arbitrary size and nature. By encompassing all $\mathbf{a}_{i}$ 's as rows of a sampling matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and defining $\mathbf{y}=\left(y_{1}, \ldots, y_{m}\right)^{T}$, as well as $\mathbf{e}=\left(e_{1}, \ldots, e_{m}\right)^{T}$, such a sampling procedure can succinctly be written as

$$
\begin{equation*}
\mathbf{y}=\mathbf{A x}+\mathbf{e} \tag{1}
\end{equation*}
$$

Several conditions on $\mathbf{A}$ are known to be sufficient to ensure that a sparse vector $\mathbf{x}$ can be robustly estimated from measurements $\mathbf{y}$. Here, we focus on uniform reconstruction guarantees. These assure recovery of all $s$-sparse vectors simultaneously. While several sufficient criteria for uniform recovery exist, the nullspace property (NSP) is both necessary and sufficient. In order to properly define a robust version of the NSP, see e.g. [13, Def. 4.21], we need to introduce some notation: Fix $\mathbf{x} \in \mathbb{R}^{n}$ and let $S \subset[n]=\{1, \ldots, n\}$ be a set. We denote the restriction of $\mathbf{x}$ to $S$ by $\mathbf{x}_{S}$ (i.e. $\left(\mathbf{x}_{S}\right)_{i}=\mathbf{x}_{i}$ for $i \in S$ and $\left(\mathbf{x}_{S}\right)_{i}=0$ else). Let $\bar{S}$ be the complement of $S$ in $[n]$, such that $\mathbf{x}=\mathbf{x}_{S}+\mathbf{x}_{\bar{S}}$.

Definition 1 ( $\ell_{2}$-robust nullspace property). A $m \times n$ matrix A satisfies the $\ell_{2}$-robust null space property of order $s$ with parameters $\rho \in(0,1)$ and $\tau>0$, if:

$$
\left\|\mathbf{v}_{S}\right\|_{\ell_{2}} \leq \frac{\rho}{\sqrt{s}}\left\|\mathbf{v}_{\bar{S}}\right\|_{\ell_{1}}+\tau\|\mathbf{A} \mathbf{v}\|_{\ell_{2}} \quad \forall \mathbf{v} \in \mathbb{R}^{n}
$$

[^1]holds for all $S \subset[n]$ with $|S| \leq s$.

This property implies that no $s$-sparse vectors lie in the kernel (or nullspace) of $\mathbf{A}$. Importantly, validity of the NSP also implies

$$
\begin{equation*}
\|\mathbf{x}-\mathbf{z}\|_{\ell_{2}} \leq \frac{C}{\sqrt{s}}\left(\|\mathbf{z}\|_{\ell_{1}}-\|\mathbf{x}\|_{\ell_{1}}\right)+D \tau\|\mathbf{A}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}} \tag{2}
\end{equation*}
$$

for any $s$-sparse $\mathbf{x} \in \mathbb{R}^{n}$ and every $\mathbf{z} \in \mathbb{R}^{n}$ [13, Theorem 4.25]. The constants $C, D$ only depend on the NSP parameter $\rho$ and we refer to Formula (12) below for explicit dependencies. In turn, this relation implies that every $s$-sparse vector $\mathbf{x}$ can be reconstructed from noisy measurements of the form (1) via basis pursuit denoising (BPDN):

$$
\begin{equation*}
\mathbf{x}_{\eta}^{\sharp}=\arg \min \|\mathbf{z}\|_{\ell_{1}} \quad \text { s.t. } \quad\|\mathbf{A} \mathbf{z}-\mathbf{y}\|_{\ell_{2}} \leq \eta . \tag{3}
\end{equation*}
$$

Here, $\eta$ must be an a-priori known upper bound on the noise strength in (1): $\eta \geq\|\mathbf{e}\|_{\ell_{2}}$. Our first main technical contribution is a substantial strengthening of Formula (2) that is valid for non-negative $s$-sparse vectors ( $\mathbf{x} \geq \mathbf{0}$ ):

Theorem 2. Suppose that A obeys the NSP of order $s \leq n$ and moreover admits a strictly-positive linear combination of its rows: $\exists \mathbf{t} \in \mathbb{R}^{m}$ such that $\mathbf{w}=\mathbf{A}^{T} \mathbf{t}>\mathbf{0}$. Then, the following bound holds for any s-sparse $\mathbf{x} \geq \mathbf{0}$ and any $\mathbf{z} \geq \mathbf{0}$ :

$$
\begin{equation*}
\left.\|\mathbf{x}-\mathbf{z}\|_{\ell_{2}} \leq D^{\prime}\left(\|\mathbf{t}\|_{\ell_{2}}+\tau\right)\right)\|\mathbf{A}(\mathbf{z}-\mathbf{x})\|_{\ell_{2}} \tag{4}
\end{equation*}
$$

The constant $D^{\prime}$ only depends on the quality of NSP and the conditioning of the strictly positive vector $\mathbf{w}$.

This statement is a simplified version of Theorem 4 below and we refer to this statement for a more explicit presentation. The crucial difference between (4) and (2) is the fact that no ( $\|\mathbf{z}\|_{\ell_{1}}-\|\mathbf{x}\|_{\ell_{1}}$ )-term occurs in the former. This term is responsible for the $\ell_{1}$-regularization in BPDN. Theorem 2 highlights that this is not necessary in the non-negative case. Instead, a simple nonnegative least squares regression suffices:

$$
\begin{equation*}
\mathbf{x}^{\sharp}=\underset{\mathbf{z} \geq \mathbf{0}}{\arg \min }\|\mathbf{A z}-\mathbf{y}\|_{\ell_{2}} . \tag{5}
\end{equation*}
$$

Under the pre-requisites of Theorem 2, the solution of this optimization problem stably reconstructs any non-negative $s$-sparse vector from noisy measurements (1). We refer to Sec. III-B for a derivation of this claim. Here, we content ourselves with pointing out that this recovery guarantee is (up to multiplicative constants) as strong as existing ones for different reconstruction algorithms. These include the LASSO and Dantzig selectors, as well as basis pursuit denoising (BPDN) (see [13] and references therein). However, on the contrary to them, algorithms for solving (5) require neither an explicit a-priori bound $\eta \geq\|\mathbf{e}\|_{\ell_{2}}$ on the noise, nor an $\|\cdot\|_{\ell_{1}}$ regression term. This simplicity is caused by the non-negativity constraint $\mathbf{z} \geq \mathbf{0}$ and the geometric restrictions it imposes. Also, these assertions stably remain true if we consider approximately sparse target vectors instead of perfectly sparse ones (see Theorem 4 below).

In order to underline the applicability of Theorem 2, we consider nonnegative $0 / 1$-Bernoulli sampling matrices and prove that they meet the requirements of said statement with high probability (w.h.p). This in turn implies:

Theorem 3. Let A be a sampling matrix whose entries are independently chosen from a $0 / 1$-Bernoulli distribution with parameter $p \in[0,1]$, i.e. $\operatorname{Pr}[1]=p$ and $\operatorname{Pr}[0]=1-p$. Fix $s \leq n$ and set

$$
\begin{equation*}
m \geq C \alpha(p) s\left(\log \left(\frac{\mathrm{e} n}{s}\right)+\beta(p)\right) \tag{6}
\end{equation*}
$$

where $\alpha(p), \beta(p)$ are constants depending only on $p$. Then, with probability at least $1-(n+1) \mathrm{e}^{-C^{\prime} p^{2}(1-p)^{2} m}, \mathbf{A}$ allows for stably reconstructing any non-negative s-sparse vector $\mathbf{x}$ from $\mathbf{y}=\mathbf{A}+\mathbf{e}$ via (5). The solution $\mathbf{x}^{\sharp}$ of (5) is guaranteed to obey

$$
\left\|\mathbf{x}^{\sharp}-\mathbf{x}\right\|_{\ell_{2}} \leq \frac{E^{\prime}}{\sqrt{p(1-p)}^{3}} \frac{\|\mathbf{e}\|_{\ell_{2}}}{\sqrt{m}}
$$

where $E^{\prime}$ is constant.

We emphasize two important aspects of this result:

1) 0/1-Bernoulli matrices obey the NSP with overwhelming probability. This novel statement alone assures robust sparse recovery via BPDN (3). Moreover, the required sampling rate is proportional to $s \log (n / s)$ which is optimal.
2) For non-negative vectors we overcome traditional $\ell_{1}$-regularization. We demonstrate this numerically in Figure 1.

Up to our knowledge, this is the first rigorous proof that 0/1-matrices tend to obey a strong version of the nullspace property. The main difference to most existing NSP and RIP results is the fact that the individual random entries of $\mathbf{A}$ are not centered, $\left(\mathbb{E}\left[\mathbf{A}_{k, j}\right]=p \neq 0\right)$. Thus, the covariance matrix of $\mathbf{A}$ admits a condition number of $\kappa\left(\mathbb{E}\left[\mathbf{A}^{T} \mathbf{A}\right]\right)=1+\frac{p n}{1-p}$, which underlines the ensemble's anisotropy. Traditional proof techniques, like establishing an RIP, are either not applicable in such a setting, or yield sub-optimal results [14], [15]. This is not true for Mendelson's small ball method [16], [17] (see also [18]), which we employ in our proof of Theorem 3. We refer to [19] for an excellent survey about the applicability of Mendelson's small ball method in compressed sensing. In the conceptually similar problem of reconstructing low rank matrices from rank-one projective measurements (which arises e.g. from the PhaseLift approach for phase retrieval [20]), applying this technique allowed for establishing strong null space properties, despite a similar degree of anisotropy.

Finally, we point out that the constant $\alpha(p)$ in Theorem 3 diverges for $p \rightarrow 0,1$. This is to be expected, because the inverse problem becomes ill-posed in this regime of sparse (or co-sparse) measurements. Despite our efforts, we do not expect $\alpha(p)$ to be tight in this interesting parameter regime and leave a more detailed analysis of this additional parameter dependence for future work, see Remark 10 below.

Organization of the Paper: In Section II we explain our motivating application in more detail and rephrase activity detection as a nonnegative sparse recovery problem. Then, we provide an overview on prior work and known results regarding this topic. In Section III we show that recovery guarantees in the presence of noise are governed by the robust nullspace property (see here [13]) under nonnegative constraints. Finally, in Section IV we analyze binary measurement matrices having i.i.d. random $0 / 1$-valued entries. We prove that such matrices admit the NSP with overwhelming probability and moreover meet the additional requirement of Theorem 2.


Fig. 1: Phase transition for NNLS in (5) for i.i.d. 0/1-Bernoulli measurement matrices in the noiseless case. More details are given in Section V.

## II. System Model and Problem Statement

## A. Activity Detection in Wireless Networks

Let $\mathbf{A}=\left(\mathbf{s}_{1}|\cdots| \mathbf{s}_{n}\right) \in \mathbb{R}^{m \times n}$ be a matrix with $n$ real columns $\mathbf{s}_{j} \in \mathbb{R}^{m}$. In our network application [21], the columns $\mathbf{s}_{j}$ are the individual sequences of length $m$ transmitted by the active devices. These sequences are transmitted simultaneously and each of them is multiplied by an individual amplitude that depends on transmit power and other channel conditions. In practice such a scenario can be achieved by using the channel reciprocity principle in time-division multiplexing. This assures that the devices have knowledge about the complex channel coefficients and may perform a pre-multiplication to correct for the phase. All these modulated sequences are superimposed at a single receiver, because the wireless medium is shared by all devices. We model such a situation by an unknown non-negative vector $\mathbf{0} \leq \mathbf{x} \in \mathbb{R}^{n}$, where $x_{i}>0$ indicates that a device with sequence $i$ is active with amplitude $x_{i}$ ( $x_{i}=0$ implies that a device is inactive). We point out that, due to path loss in the channel, the individual received amplitudes $x_{i}$ of each active device are unknown to the receiver as well. Here, we focus on networks that contain a large number $n$ of registered devices, but, at any time, only a small unknown fraction, say $s \ll n$, of these devices are active.

Communicating activity patterns, that is $\operatorname{supp}(\mathbf{x})=\left\{i: x_{i} \neq 0\right\}$, and the corresponding list of received amplitudes/powers ( $\mathbf{x} \geq \mathbf{0}$ itself) in a traditional way would require $\mathcal{O}(n)$ resources. Here, we aim for a reduction of the signaling time $m$ by exploiting the facts that (i) $\mathbf{x} \geq \mathbf{0}$ is non-negative and (ii) the vector $\mathbf{x}$ is $s$-sparse, i.e. $\|\mathbf{x}\|_{\ell_{0}} \leq s$. Hence, we focus on the regime $s \leq m \ll n$. Obviously, in such a scenario the resulting system of linear
equations cannot be directly inverted. A reasonable approach towards recovery is to consider the program:

$$
\arg \min \|\mathbf{z}\|_{\ell_{0}} \text { s.t. } \mathbf{A z}=\mathbf{y} \& \mathbf{z} \geq \mathbf{0}
$$

Combinatorial problems of this type are infamous for being NP-hard in general. A common approach to circumvent this obstacle is to consider convex relaxations. A prominent relaxation is to replace $\|\cdot\|_{\ell_{0}}$ with the $\ell_{1}$-norm. The resulting algorithm can then be re-cast as an efficiently solvable linear program. However, such approaches become more challenging when robustness towards additive noise is required. In particular, if the type and the strength of the noise is itself unknown. In our application, noisy contributions inevitably arise due to quantization, thermal noise and other interferences. If the noisy measurements are of the form (1) (i.e. $\mathbf{y}=\mathbf{A x}+\mathbf{e}$, where the vector $\mathbf{e}$ is an additive distortion) a well-known modification is to consider the BPDN (3) but with an additional nonnegativity constraint:

$$
\begin{equation*}
\arg \min \|\mathbf{z}\|_{\ell_{1}} \text { s.t. }\|\mathbf{A} \mathbf{z}-\mathbf{y}\|_{\ell_{2}} \leq \eta \& \mathbf{z} \geq \mathbf{0} \tag{7}
\end{equation*}
$$

While this problem is algorithmically a bit more complicated than (3), it is still convex and computationally tractable (in principle). In practice, further modifications are necessary to solve such problems sufficiently fast and efficiently, see [22], [21]. However, having access to an a-priori bound $\eta$ on $\|\mathbf{e}\|_{\ell_{2}}$ is essential for (i) posing this problem and (ii) solving it using certain algorithms that involve stopping criteria, or other conditions that depend on the noise level. Suppose, for instance, that $\mathbf{e}$ is i.i.d. normal distributed. Then $\|\mathbf{e}\|_{\ell_{2}}^{2}$ admits a $\chi^{2}$-distribution of order $m$ and feasibility is assured w.h.p., when taking $\eta$ in terms of second moments. However, much less is known for different noise distributions. This in particular includes situations where second moment information about the noise is challenging to acquire.

One option to tackle problems of this kind is to establish a quotient property for the measurement matrix $\mathbf{A}$ [13]. However, this property is geared towards Gaussian measurements and is challenging to establish for different random models of $\mathbf{A}$. In this paper we show that another condition—namely that $\mathbf{A}$ admit a strictly positive linear combination of rows-allows for drawing similar conclusions.

## B. Prior Work on Recovery of Nonnegative Sparse Vectors

One of the first works on non-negative compressed sensing is due to Donoho et al.n [4] on the "nearly black object". It furthers the understanding of the "maximum entropy inversion" method to recover sparse (nearly-black) images in radio astronomy. Donoho and Tanner investigated this subject more directly in Ref. [11]. The central question is: what properties of $\mathbf{A}$ intrinsically ensure that only one solution is feasible for any $s$-sparse $\mathbf{x} \geq \mathbf{0}$ :

$$
\begin{equation*}
\{\mathbf{z} \mid \mathbf{A} \mathbf{z}=\mathbf{A} \mathbf{x} \& \mathbf{z} \geq \mathbf{0}\}=\{\mathbf{x}\} \tag{8}
\end{equation*}
$$

At the center of their work is the notion of outwardly s-neighborly polytopes. Assume w.l.o.g. that all columns $\mathbf{s}_{j}$ of $\mathbf{A}$ are non-zero and define their convex hull

$$
P_{\mathbf{A}}:=\operatorname{conv}\left(\mathbf{s}_{1}, \ldots, \mathbf{s}_{n}\right)
$$

This polytope is called s-neighborly if every set of $s$ vertices spans a face of $P_{\mathbf{A}}$. If this is the case, the polytope $P_{\mathbf{A}}^{0}:=\operatorname{conv}\left(P_{\mathbf{A}} \cup\{\mathbf{0}\}\right)$ is called outwardly s-neighborly. They then move on to prove that the solution to

$$
\arg \min \|\mathbf{x}\|_{\ell_{0}} \quad \text { s.t. } \quad \mathbf{A x}=\mathbf{y}
$$

is unique if and only if $P_{\mathbf{A}}^{0}$ is outwardly $s$-neighborly [11].
Another approach to the same question was introduced in Ref. [12]. They consider full rank $m \times n$-matrices whose row space intersects the positive orthant:

$$
\begin{equation*}
\mathcal{M}^{+}=\left\{\mathbf{A}: \exists \mathbf{t} \in \mathbb{R}^{m} \mathbf{A}^{*} \mathbf{t}>\mathbf{0}\right\} \tag{9}
\end{equation*}
$$

Note that both structures are related in the sense that $\mathbf{A} \in \mathcal{M}^{+}$, if and only if $\mathbf{0} \notin P_{\mathbf{A}}$ [23]. Also, a strictly positive row assures $\mathbf{A} \in \mathcal{M}^{+}$. An extreme case thereof occurs if $\mathbf{A}$ contains the "all-ones" vector $\mathbf{1}_{n}$ in $\mathbb{R}^{n}$. The corresponding measurement yields the $\ell_{1}$-norm $\|\mathbf{x}\|_{\ell_{1}}=\left\langle\mathbf{1}_{n}, \mathbf{x}\right\rangle$ and therefore all admissible vectors in (7) for $\eta=0$ have the same cost. The uniqueness property in such a setting has already been obtained by Fuchs [5] for Vandermonde measurement matrices and for particular real Fourier measurements using convex duality. In these special cases, $m$ distinct columns are linear independent ("full spark") and therefore Eq. (8) holds, provided that $\mathbf{x}$ is sufficiently sparse: $\left\|\mathbf{x}^{(0)}\right\|_{\ell_{0}} \leq \frac{m-1}{2}$.

In Ref. [12], Bruckstein et al. investigated the recovery of nonnegative vectors by (7) and modifications of OMP using a coherence-based approach. They obtained numerical evidence for unique recovery in the regime $s=\mathcal{O}(\sqrt{n})$. Later, Wang and coauthors [23] have analyzed non-negativity priors for vector and matrix recovery using an RIP-based analysis. Concretely, they translated the well-known RIP-result of random i.i.d. $\pm 1$-Bernoulli matrices (see for example [24]) to 0/1-measurements in the following way. Perform measurements using an $(m+1) \times n$ matrix $\mathbf{A}^{1}=\left(\mathbf{1}_{n}^{T} \mid \mathbf{A}^{T}\right)^{T}$ which consists of an all-ones row $\mathbf{1}_{n}$ appended by a random i.i.d. 0/1-valued $m \times n$ matrix A. By construction, the first noiseless measurement on a nonnegative vector $\mathbf{x}$ returns its $\ell_{1}$-norm $\|\mathbf{x}\|_{\ell_{1}}=\left\langle\mathbf{1}_{n}, \mathbf{x}\right\rangle$. Rescaling and subtracting this value from the $m$ remaining measurements then results in $\pm 1$-measurements. This insight allows for an indirect nullspace characterization of $\mathbf{A}$ in terms of the restricted isometry property (RIP) of i.i.d. $\pm 1$-Bernoulli random matrices $\tilde{\mathbf{A}}$. Recall that a matrix obeys the RIP of order $s$, if it acts almost isometrically on $s$-sparse vectors: There exists $\delta_{s} \in[0,1)$ such that $\left|\|\tilde{\mathbf{A}} \mathbf{x}\|_{\ell_{2}}^{2}-\|\mathbf{x}\|_{\ell_{2}}^{2}\right| \leq \delta_{s}\|\mathbf{x}\|_{\ell_{2}}^{2}$ for all $s$-sparse $\mathbf{x}$. Candès showed in [25] that validity of a $2 s$-RIP implies that a $\left(\ell_{1}, \ell_{1}\right)$-nullspace property is valid for each $\mathbf{v} \in \mathbb{R}^{n}$ that is contained in the nullspace $\mathcal{N}(\tilde{\mathbf{A}})$ of $\tilde{\mathbf{A}}$ :

$$
\begin{equation*}
\left\|\mathbf{v}_{S}\right\|_{\ell_{1}} \leq \frac{\sqrt{2} \delta_{2 s}}{1-\delta_{2 s}}\left\|\mathbf{v}_{\bar{S}}\right\|_{\ell_{1}} \tag{10}
\end{equation*}
$$

for all $\mathbf{v} \in \mathcal{N}(\tilde{\mathbf{A}})$ and support sets $S$ of size $|S| \leq s$. Combining this with $\mathcal{N}\left(\mathbf{A}^{1}\right) \subset \mathcal{N}(\tilde{\mathbf{A}})$ then allows for proving unique recovery in regime $s=\mathcal{O}(n)$ with overwhelming probability.

However, so far, all these results manifestly focus on noiseless measurements. Thus, the robustness of these approaches towards noise corruption needs to be examined. Foucart, for instance, considered the $\ell_{1}$-squared nonnegative regularization [10]:

$$
\begin{equation*}
\min _{\mathbf{z} \geq \mathbf{0}}\|\mathbf{z}\|_{\ell_{1}}^{2}+\lambda^{2}\|\mathbf{A} \mathbf{z}-\mathbf{y}\|_{\ell_{2}}^{2} \tag{11}
\end{equation*}
$$

which can be re-cast as nonnegative least-squares problem. He then showed that for stochastic matrices ${ }^{3}$ the solution of (11) converges to the solution of (7) for $\lambda \rightarrow \infty$.

Here, we aim at establishing even stronger recovery guarantees that, among other things, require neither an a-priori noise bound $\eta$, nor a regularization parameter $\lambda$. We have already mentioned that the quotient property would assure such bounds for Gaussian matrices in the optimal regime. But $m \times n$ Gaussian matrices fail to be in $\mathcal{M}^{+}$with probability approaching one as long as $\lim _{n \rightarrow} m / n<\frac{1}{2}$ [23]. On the algorithmic side, there exists variations of certain regression methods where the regularization parameter can be chosen independent of the noise power-see Ref. [26] for more details on this topic. For the LASSO selector, in particular, such modifications are known as the "scaled LASSO" and "square root LASSO" [27], [28].

Non-negativity as a further structural constraint has also been investigated in the statistics community. But these works focus on the averaged case with respect to (sub-)Gaussian additive noise, whereby we consider instantaneous guarantees. Slawski and Hein [9], as well as Meinshausen [8] have recently investigated this averaged setting.

Finally, we note that the measurement setup above using a separate "all ones" row can also casted as a linearly constrained NNLS, i.e., minimizing $\|\mathbf{A x}-\mathbf{y}\|_{\ell_{2}}$ subject to $\mathbf{x} \geq \mathbf{0}$ and $\left\langle\mathbf{1}_{n}, \mathbf{x}\right\rangle=$ const., see for example [22] for a Bayesian recovery approach.

## III. Nullspace Property with Nonnegative constraints

Throughout our work we endow $\mathbb{R}^{n}$ with the partial ordering induced by the nonnegative orthant, i.e. $\mathbf{x} \leq \mathbf{z}$ if and only if $x_{i} \leq z_{i}$ for all $1 \leq i \leq n$. Here, $x_{i}=\left\langle\mathbf{e}_{i}, \mathbf{x}\right\rangle$ are the components of $\mathbf{x}$ with respect to the standard basis $\left\{\mathbf{e}_{i}\right\}_{i=1}^{n}$ of $\mathbb{R}^{n}$. Similarly, we write $\mathbf{x}<\mathbf{z}$ if strict inequality holds in each component. We also write $\mathbf{x} \geq \mathbf{0}$ to indicate that $\mathbf{x}$ is (entry-wise) nonnegative. For $1 \leq p \leq \infty$, we denote the vector $\ell_{p}$-norms by $\|\cdot\|_{\ell_{p}}$ and $\|\cdot\|$ is the usual operator/matrix norm. The $\ell_{1}$-error of the best $s$-term approximation of a vector $\mathbf{x}$ will be denoted by $\sigma_{s}(\mathbf{x})_{\ell_{1}}$.

## A. The robust nullspace property

The implications of a NSP are by now well-established and can be found, for instance, in [13, Sec. 4.3]. Suppose that a matrix $\mathbf{A}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ obeys the $\ell_{2}$-robust nullspace property of order $s(s$-NSP $)$ from Definition 1 . Theorem 4.25 in [13] then states that

$$
\begin{equation*}
\|\mathbf{x}-\mathbf{z}\|_{\ell_{2}} \leq \frac{C}{\sqrt{s}}\left(\|\mathbf{z}\|_{\ell_{1}}-\|\mathbf{x}\|_{\ell_{1}}+2 \sigma_{s}(\mathbf{x})_{\ell_{1}}\right)+D \tau\|\mathbf{A}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}} \tag{12}
\end{equation*}
$$

is true for any $\mathbf{x}, \mathbf{z} \in \mathbb{R}^{n}$. Here, $C=\frac{(1+\rho)^{2}}{1-\rho}$ and $D=\frac{3+\rho}{1-\rho}$ depend only on the NSP parameter $\rho$. Replacing $\mathbf{z}$ with the BPDN minimizer $\mathbf{x}_{\eta}^{\sharp}$ from (3) for the sampling model $\mathbf{y}=\mathbf{A x}+\mathbf{e}$ then implies

$$
\begin{align*}
\left\|\mathbf{x}-\mathbf{x}_{\eta}^{\sharp}\right\|_{\ell_{2}} & \leq \frac{2 C}{\sqrt{s}} \sigma_{s}(\mathbf{x})_{\ell_{1}}+D \tau\left\|\mathbf{y}-\mathbf{e}-\mathbf{A} \mathbf{x}_{\eta}^{\sharp}\right\|_{\ell_{2}} \leq \frac{2 C}{\sqrt{s}} \sigma_{s}(\mathbf{x})_{\ell_{1}}+D \tau\left(\left\|\mathbf{y}-\mathbf{A} \mathbf{x}_{\eta}^{\sharp}\right\|_{\ell_{2}}+\|\mathbf{e}\|_{\ell_{2}}\right) \\
& \leq \frac{2 C}{\sqrt{s}} \sigma_{s}(\mathbf{x})_{\ell_{1}}+2 D \tau \eta \tag{13}
\end{align*}
$$

provided that $\|\mathbf{e}\|_{\ell_{2}} \leq \eta$ is true. This estimate follows from exploiting $\left\|\mathbf{x}_{\eta}^{\sharp}\right\|_{\ell_{1}} \leq\|\mathbf{x}\|_{\ell_{1}}$ and and $\left\|\mathbf{y}-\mathbf{A} \mathbf{x}_{\eta}^{\sharp}\right\|_{\ell_{2}} \leq \eta$. Evidently, it is only true for $\eta \geq\|\mathbf{e}\|_{\ell_{2}}$ which in turn requires some knowledge about the noise corruption.

[^2]
## B. Nonnegative Constraints

Here we will prove a variation of Formula (12) which holds for nonnegative vectors and matrices in $\mathcal{M}^{+} \subset \mathbb{R}^{m \times n}$. For such matrices we define a condition number by

$$
\begin{equation*}
\kappa(\mathbf{A})=\min \left\{\|\mathbf{W}\|\left\|\mathbf{W}^{-1}\right\| \mid \exists \mathbf{t} \in \mathbb{R}^{m} \text { with } \mathbf{W}=\operatorname{diag}\left(\mathbf{A}^{T} \mathbf{t}\right)>0\right\} \tag{14}
\end{equation*}
$$

Note that for diagonal matrices $\mathbf{W}$ with non-negative entries $\kappa(\mathbf{W})=\|\mathbf{W}\|\left\|\mathbf{W}^{-1}\right\|$.

Theorem 4. Suppose that $\mathbf{A} \in \mathcal{M}^{+}$obeys the s-NSP with parameters $\rho$ and $\tau$, and let $\kappa=\kappa(\mathbf{A})$ be its condition number achieved for $\mathbf{t} \in \mathbb{R}^{m}$. If $\kappa \rho<1$, then

$$
\|\mathbf{x}-\mathbf{z}\|_{\ell_{2}} \leq \frac{2 C^{\prime}}{\sqrt{s}} \sigma_{s}(\mathbf{x})_{\ell_{1}}+D^{\prime}\left(\|\mathbf{t}\|_{\ell_{2}}+\tau\right)\|\mathbf{A}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}}
$$

is true for all nonnegative vectors $\mathbf{x}, \mathbf{z} \in \mathbb{R}^{n}$. The constants amount to $C^{\prime}=\frac{\kappa(1+\kappa \rho)^{2}}{1-\kappa \rho}$ and $D^{\prime}=\frac{3+\kappa \rho}{1-\kappa \rho} \max \left\{\kappa,\left\|\mathbf{W}^{-1}\right\|\right\}$.
Comparing this to (12) reveals that the $\ell_{1}$-term $\left(\|\mathbf{z}\|_{\ell_{1}}-\|\mathbf{x}\|_{\ell_{1}}\right)$ is not present anymore. Inserting $\mathbf{y}=\mathbf{A x}+\mathbf{e}$ and applying the triangle inequality results in

$$
\begin{equation*}
\|\mathbf{x}-\mathbf{z}\|_{\ell_{2}} \leq \frac{2 C}{\sqrt{s}} \sigma_{s}(\mathbf{x})_{\ell_{1}}+D\left(\|\mathbf{t}\|_{\ell_{2}}+\tau\right)\left(\|\mathbf{A} \mathbf{z}-\mathbf{y}\|_{\ell_{2}}+\|\mathbf{e}\|_{\ell_{2}}\right) \quad \forall \mathbf{x}, \mathbf{z} \geq \mathbf{0} \tag{15}
\end{equation*}
$$

This observation already highlights that CS-oriented algorithms, which typically minimize the $\ell_{1}$-norm, are not required anymore in the non-negative case. Instead, in order to get good estimates it makes sense to minimize the r.h.s. of the bound over the "free" parameter $\mathbf{z} \geq \mathbf{0}$. Doing so, results in the non-negative least squares fit (5). The sought for vector $\mathbf{x}$ is itself a feasible point of this optimization problem and consequently $\left\|\mathbf{A} \mathbf{x}^{\sharp}-\mathbf{y}\right\|_{\ell_{2}} \leq\|\mathbf{A x}-\mathbf{y}\|_{\ell_{2}}=\|\mathbf{e}\|_{\ell_{2}}$. Inserting this into (15) then implies

$$
\left\|\mathbf{x}-\mathbf{x}^{\sharp}\right\|_{\ell_{2}} \leq \frac{2 C}{\sqrt{s}} \sigma_{s}(\mathbf{x})_{\ell_{1}}+2 D\left(\|\mathbf{t}\|_{\ell_{2}}+\tau\right)\|\mathbf{e}\|_{\ell_{2}}
$$

which is comparable to (13). However, rather than depending on an a-priori noise bound $\eta$, the reconstruction error scales proportionally to $\|\mathbf{e}\|_{\ell_{2}}$ itself.

We will require two auxiliary statements in order to prove Theorem 4:

Lemma 5. Suppose that $\mathbf{A}$ obeys the $s-N S P$ with parameters $\rho$ and $\tau$, and set $\mathbf{W}=\operatorname{diag}(\mathbf{w})$, where $\mathbf{w}>\mathbf{0}$ is strictly positive. Then, $\mathbf{A} \mathbf{W}^{-1}$ also obeys the s-NSP with parameters $\tilde{\rho}=\kappa(\mathbf{W}) \rho$ and $\tilde{\tau}=\|\mathbf{W}\| \tau$.

Proof. The fact that $\mathbf{W}$ is diagonal assures $\mathbf{W}^{-1} \mathbf{v}_{S}=\left(\mathbf{W}^{-1} \mathbf{v}\right)_{S}$ (same for $\left.\bar{S}\right)$. Also, A obeys the $s$-NSP by assumption. Consequently

$$
\begin{aligned}
\left\|\mathbf{v}_{S}\right\|_{\ell_{2}} & =\left\|\mathbf{W} \mathbf{W}^{-1} \mathbf{v}_{S}\right\|_{\ell_{2}} \leq\|\mathbf{W}\|\left\|\left(\mathbf{W}^{-1} \mathbf{v}\right)_{S}\right\|_{\ell_{2}} \leq\|\mathbf{W}\|\left(\frac{\rho}{\sqrt{s}}\left\|\left(\mathbf{W}^{-1} \mathbf{v}\right)_{\bar{S}}\right\|_{\ell_{2}}+\tau\left\|\mathbf{A} \mathbf{W}^{-1} \mathbf{v}\right\|_{\ell_{2}}\right) \\
& \leq \frac{\|\mathbf{W}\|\left\|\mathbf{W}^{-1}\right\| \rho}{\sqrt{s}}\left\|\mathbf{v}_{\bar{S}}\right\|_{\ell_{1}}+\|\mathbf{W}\| \tau\left\|\mathbf{A} \mathbf{W}^{-1} \mathbf{v}\right\|_{\ell_{2}}=\frac{\tilde{\rho}}{\sqrt{s}}\left\|\mathbf{v}_{\bar{S}}\right\|_{\ell_{1}}+\tilde{\tau}\left\|\mathbf{A} \mathbf{W}^{-1} \mathbf{v}\right\|_{\ell_{2}}
\end{aligned}
$$

is true for every set $S$ with $|S| \leq s$.

Lemma 6. Fix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and suppose that $\mathbf{w}=\mathbf{A}^{T} \mathbf{t}$ is strictly positive for some $\mathbf{t} \in \mathbb{R}^{m}$. Also, set $\mathbf{W}=\operatorname{diag}(\mathbf{w})$.
Then, the following relation holds for any pair of non-negative vectors $\mathbf{x}, \mathbf{z} \geq \mathbf{0}$ in $\mathbb{R}^{n}$ :

$$
\|\mathbf{W} \mathbf{z}\|_{\ell_{1}}-\|\mathbf{W} \mathbf{x}\|_{\ell_{1}} \leq\|\mathbf{t}\|_{\ell_{2}}\|\mathbf{A}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}}
$$

Proof. Note that, by construction, $\mathbf{W}$ is symmetric and preserves entry-wise non-negativity. These features together with positivity of $z$ imply

$$
\|\mathbf{W} \mathbf{z}\|_{\ell_{1}}=\left\langle\mathbf{1}_{n}, \mathbf{W} \mathbf{z}\right\rangle=\left\langle\mathbf{W} \mathbf{1}_{n}, \mathbf{z}\right\rangle=\left\langle\operatorname{diag}\left(\mathbf{A}^{T} \mathbf{t}\right) \mathbf{1}_{n}, \mathbf{z}\right\rangle=\left\langle\mathbf{A}^{T} \mathbf{t}, \mathbf{z}\right\rangle=\langle\mathbf{t}, \mathbf{A} \mathbf{z}\rangle .
$$

An analogous reformulation is true for $\|\mathbf{W} \mathbf{x}\|_{\ell_{1}}$ and combining these two reveals

$$
\|\mathbf{W} \mathbf{z}\|_{\ell_{1}}-\|\mathbf{W} \mathbf{x}\|_{\ell_{1}}=\langle\mathbf{t}, \mathbf{A}(\mathbf{z}-\mathbf{x})\rangle \leq\|\mathbf{t}\|_{\ell_{2}}\|\mathbf{A}(\mathbf{z}-\mathbf{x})\|_{\ell_{2}}
$$

due to Cauchy-Schwarz.
Proof of Theorem 4. The assumption $\mathbf{A} \in \mathcal{M}^{+}$assures that there exists $\mathbf{t} \in \mathbb{R}^{m}$ such that $\mathbf{w}=\mathbf{A}^{T} \mathbf{t}>\mathbf{0}$ and we define $\mathbf{W}:=\operatorname{diag}(\mathbf{w})$. By assumption, $\mathbf{W}$ is invertible and admits a condition number $\kappa=\|\mathbf{W}\|\left\|\mathbf{W}^{-1}\right\|$. Thus, we may write

$$
\|\mathbf{x}-\mathbf{z}\|_{\ell_{2}}=\left\|\mathbf{W}^{-1} \mathbf{W}(\mathbf{x}-\mathbf{z})\right\|_{\ell_{2}} \leq\left\|\mathbf{W}^{-1}\right\|\|\mathbf{W}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}}
$$

for any pair $\mathbf{x}, \mathbf{z}>\mathbf{0}$. Since A obeys the $s$-NSP, Lemma 5 assures that $\mathbf{A W}^{-1}$ also admits a $s$-NSP, albeit with parameters $\tilde{\rho}=\kappa \rho$ and $\tilde{\tau}=\|\mathbf{W}\| \tau$. Thus, from (12) we conclude the following for vectors $\mathbf{W} \mathbf{x}$ and $\mathbf{W} \mathbf{z}$ :

$$
\begin{aligned}
\|\mathbf{W}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}} & \leq \frac{1}{\sqrt{s}} \frac{(1+\kappa \rho)^{2}}{1-\kappa \rho}\left(\|\mathbf{W} \mathbf{z}\|_{\ell_{1}}-\|\mathbf{W} \mathbf{x}\|_{\ell_{1}}+2 \sigma_{s}(\mathbf{W} \mathbf{x})_{\ell_{1}}\right)+\frac{3+\kappa \rho}{1-\kappa \rho}\|\mathbf{W}\| \tau\|\mathbf{A}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}} \\
& \leq \frac{2(1+\kappa \rho)^{2}}{1-\kappa \rho}\|\mathbf{W}\| \frac{\sigma_{s}(\mathbf{x})_{\ell_{1}}}{\sqrt{s}}+\left(\frac{(1+\kappa \rho)^{2}}{1-\kappa \rho} \frac{\|\mathbf{t}\|_{\ell_{2}}}{\sqrt{s}}+\frac{3+\kappa \rho}{1-\kappa \rho}\|\mathbf{W}\| \tau\right)\|\mathbf{A}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}} \\
& \leq \frac{2(1+\kappa \rho)^{2}}{1-\kappa \rho}\|\mathbf{W}\| \frac{\sigma_{s}(\mathbf{x})_{\ell_{1}}}{\sqrt{s}}+\frac{3+\kappa \rho}{1-\kappa \rho}\left(\|\mathbf{t}\|_{\ell_{2}}+\|\mathbf{W}\| \tau\right)\|\mathbf{A}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}}
\end{aligned}
$$

Here, we invoked Lemma 6, as well as the relation $\sigma_{s}(\mathbf{W} \mathbf{x})_{\ell_{1}} \leq\|\mathbf{W}\| \sigma_{s}(\mathbf{x})_{\ell_{1}}$. So, in summary we obtain

$$
\begin{aligned}
\|\mathbf{x}-\mathbf{z}\|_{\ell_{2}} & \leq\left\|\mathbf{W}^{-1}\right\|\|\mathbf{W}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}} \\
& \leq \frac{2 \kappa(1+\kappa \rho)^{2}}{1-\kappa \rho} \frac{\sigma_{s}(\mathbf{x})_{\ell_{1}}}{\sqrt{s}}+\frac{3+\kappa \rho}{1-\kappa \rho}\left(\left\|\mathbf{W}^{-1}\right\|\|\mathbf{t}\|_{\ell_{2}}+\kappa \tau\right)\|\mathbf{A}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}} \\
& \leq \frac{2 C^{\prime}}{\sqrt{s}} \sigma(\mathbf{x})_{\ell_{1}}+D^{\prime}\left(\|\mathbf{t}\|_{\ell_{2}}+\tau\right)\|\mathbf{A}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}}
\end{aligned}
$$

with $C^{\prime}=\frac{\kappa(1+\kappa \rho)^{2}}{1-\kappa \rho}$ and $D^{\prime}=\frac{3+\kappa \rho}{1-\kappa \rho} \max \left\{\kappa,\left\|\mathbf{W}^{-1}\right\|\right\}$.

## IV. Robust NSP for 0/1-BERNOULLI matrices

In this section, we prove our second main result, Theorem 3. Said statements summarizes two results, namely (i) 0/1-Bernoulli matrices $\mathbf{A}$ with $m=C s \log (n / s)$ rows obey the robust null space property of order $s$ w.h.p. and (ii) the row space of $\mathbf{A}^{T}$ allows for constructing a strictly positive vector $\mathbf{w}=\mathbf{A}^{T} \mathbf{t}>\mathbf{0}$ (that is sufficiently well-conditioned). We will first state the main ideas and prove both statements in subsequent subsections.

## A. Sampling model and overview of main proof ideas

Let us start by formally defining the concept of a $0 / 1$-Bernoulli matrix.
Definition 7. We call $\mathbf{A} \in \mathbb{R}^{m \times n}$ a 0/l-Bernoulli matrix with parameter $p \in[0,1]$, if every matrix element $A_{i, j}$ of $\mathbf{A}$ is an independent realization of a Bernoulli random variable $b$ with parameter $p$, i.e.

$$
\operatorname{Pr}[b=1]=p \quad \text { and } \quad \operatorname{Pr}[b=0]=1-p .
$$

Recall that $\mathbb{E}[b]=p$ and $\operatorname{Var}(b)=\mathbb{E}\left[(b-\mathbb{E}[b])^{2}\right]=p(1-p)$. By construction, the $m$ rows $\mathbf{a}_{1}, \ldots, \mathbf{a}_{m}$ of such a 0/1-Bernoulli matrix are independent and obey

$$
\mathbb{E}\left[\mathbf{a}_{k}\right]=\sum_{j=1}^{n} \mathbb{E}\left[A_{k, j}\right] \mathbf{e}_{j}=p \sum_{j=1}^{n} \mathbf{e}_{j}=p \mathbf{1}
$$

This expected behavior of the individual rows will be crucial for addressing the second point in Theorem 3: Setting

$$
\mathbf{w}:=\frac{1}{p m} \sum_{k=1}^{m} \mathbf{a}_{k}=\mathbf{A}^{T}\left(\frac{1}{p m} \mathbf{1}_{m}\right)
$$

results in a random vector $\mathbf{w} \in \mathbb{R}^{n}$ that obeys $\mathbb{E}[\mathbf{w}]=\mathbf{1}>\mathbf{0}$. Applying a large deviation bound will in turn imply that a realization of $\mathbf{w}$ will w.h.p. not deviate too much from its expectation. This in turn ensures strict positivity. We will prove this in Subsection IV-C.

However, when turning our focus to establishing null space properties for $\mathbf{A}$, working with $0 / 1$-Bernoulli entries renders such a task more challenging. The simple reason for such a complication is that the individual random entries of $\mathbf{A}$ are not centered, i.e. $\mathbb{E}\left[A_{k, j}\right]=p \neq 0$. Combining this with independence of the individual entries yields

$$
\mathbb{E}\left[\mathbf{a}_{k} \mathbf{a}_{k}^{T}\right]=p^{2} \mathbf{1}_{n} \mathbf{1}_{n}^{T}+p(1-p) \mathbb{I}
$$

This matrix admits a condition number of $\kappa\left(\mathbb{E}\left[\mathbf{a}_{k} \mathbf{a}_{k}^{T}\right]\right)=1+\frac{p n}{1-p}$ which underlines the ensemble's anisotropy. Traditional proof techniques, e.g. establishing an RIP, are either not applicable, or yield sub-optimal results [14], [15]. This is not true for Mendelson's small ball method [16], [17] (see also [18])—a strong general purpose tool whose applicability only requires row-wise independence. It was shown in Ref. [19] that this technique allows for establishing the NSP for a variety of compressed sensing scenarios. Our derivation is inspired by the techniques presented in []loc. cit.. Moreover, a similar approach is applicable to the conceptually-related problem of low rank matrix reconstruction [29].

## B. Null Space Properties for 0/1-Bernoulli matrices

Recall that Definiton 1 states that a $m \times n$ matrix $\mathbf{A}$ obeys the robust null space property with parameters $\rho \in(0,1)$ and $\tau>0$, if

$$
\begin{equation*}
\left\|\mathbf{v}_{S}\right\|_{\ell_{2}} \leq \frac{\rho}{\sqrt{s}}\left\|\mathbf{v}_{\bar{S}}\right\|_{\ell_{1}}+\tau\|\mathbf{A} \mathbf{v}\|_{\ell_{2}} \tag{16}
\end{equation*}
$$

is true for all vectors $\mathbf{v} \in \mathbb{R}^{n}$ and support sets $S \in[n]$ with support size $|S| \leq s$. Demanding such generality in the choice of the support set is in fact not necessary, see e.g. [13, Remark 4.2]. For a fixed vector $\mathbf{v}$, the above
condition holds for any index set $S$, if it holds for an index set $S_{\text {max }}$ containing the $s$ largest (in modulus) entries of $\mathbf{v}$. Introducing the notation $\mathbf{v}_{s}:=\mathbf{v}_{S_{\max }}$ and $\mathbf{v}_{c}:=\mathbf{v}_{\bar{S}_{\text {max }}}$, the robust null space property (16) holds, provided that every vector $\mathbf{v} \in \mathbb{R}^{n}$ obeys

$$
\begin{equation*}
\left\|\mathbf{v}_{s}\right\|_{\ell_{2}} \leq \frac{\rho}{\sqrt{s}}\left\|\mathbf{v}_{c}\right\|_{\ell_{1}}+\tau\|\mathbf{A} \mathbf{v}\|_{\ell_{2}} \tag{17}
\end{equation*}
$$

Note that this requirement is invariant under re-scaling and we may w.l.o.g. assume $\|\mathbf{v}\|_{\ell_{2}}=1$. Moreover, for fixed parameters $s$ and $\rho$, any vector $\mathbf{v}$ obeying $\left\|\mathbf{v}_{s}\right\|_{\ell_{2}} \leq \frac{\rho}{\sqrt{s}}\left\|\mathbf{v}_{c}\right\|_{\ell_{1}}$ is guaranteed to fulfill (17) by default. Consequently, when aiming to establish null space properties, it suffices to establish condition (17) for the set of unit-norm vectors that do not obey this criterion:

$$
T_{\rho, s}:=\left\{\mathbf{v} \in \mathbb{R}^{n}:\|\mathbf{v}\|_{\ell_{2}}=1,\left\|\mathbf{v}_{s}\right\|_{\ell_{2}}>\frac{\rho}{\sqrt{s}}\left\|\mathbf{v}_{c}\right\|_{\ell_{1}}\right\}
$$

As a result, a matrix $\mathbf{A}$ obeys the NSP (16), if

$$
\begin{equation*}
\inf \left\{\|\mathbf{A} \mathbf{v}\|_{\ell_{2}}: \mathbf{v} \in T_{\rho, s}\right\}>\frac{1}{\tau} \tag{18}
\end{equation*}
$$

holds, where $\tau>0$ is the second parameter appearing in (16). For random $m \times n$ matrices $\mathbf{A}$ with independent and identically distributed rows $\mathbf{a}_{1}, \ldots, \mathbf{a}_{m} \in \mathbb{R}^{n}$ - which is the case here - Mendelson's small ball method [16], [17], [18] provides a general purpose tool for establishing such lower bounds with high probability:

Theorem 8 (Koltchinskii, Mendelson; Tropp's version [18]). Fix $E \subset \mathbb{R}^{n}$ and let $\mathbf{a}_{1}, \ldots, \mathbf{a}_{m}$ be independent copies of a random vector $\mathbf{a} \in \mathbb{R}^{n}$. Set $\mathbf{h}=\frac{1}{\sqrt{m}} \sum_{k=1}^{m} \epsilon_{k} \mathbf{a}_{k}$, where $\epsilon_{1}, \ldots, \epsilon_{m}$ is a Rademacher sequence. For $\xi>0$ define

$$
Q_{\xi}(E, \mathbf{a})=\inf _{\mathbf{u} \in E} \operatorname{Pr}[|\langle\mathbf{a}, \mathbf{u}\rangle| \geq \xi], \quad \text { as well as } \quad W_{m}(E, \mathbf{a})=\mathbb{E}\left[\sup _{\mathbf{u} \in E}\langle\mathbf{h}, \mathbf{u}\rangle\right] .
$$

Then, for any $\xi>0$ and $t \geq 0$, the following is true with probability at least $1-\mathrm{e}^{-2 t^{2}}$ :

$$
\begin{equation*}
\inf _{\mathbf{v} \in E}\left(\sum_{k=1}^{m}\left|\left\langle\mathbf{a}_{k}, \mathbf{v}\right\rangle\right|^{2}\right)^{1 / 2} \geq \xi \sqrt{m} Q_{2 \xi}(E, \mathbf{a})-\xi t-2 W_{m}(E, \mathbf{a}) \tag{19}
\end{equation*}
$$

In our concrete application, the random vector $\mathbf{a}=\sum_{i=1}^{n} b_{i} \mathbf{e}_{i} \in \mathbb{R}^{n}$ has i.i.d. 0/1-Bernoulli entries $b_{i}$ with parameter $p$ and $E=T_{\rho, r}$. We bound the marginal tail function $Q_{2 \xi}\left(T_{\rho, s}, \mathbf{a}\right)$ from below using a Paley-Zygmund inequality. Detailed in the appendix this calculation yields

$$
\begin{equation*}
\operatorname{Pr}[|\langle\mathbf{a}, \mathbf{z}\rangle| \geq \theta \sqrt{p(1-p)}] \geq \frac{4}{13} p(1-p)\left(1-\theta^{2}\right)^{2} \quad \forall \mathbf{z} \in S^{n-1} \text { and } \theta \in[0,1] \tag{20}
\end{equation*}
$$

for any $\theta \in[0,1]$ and any $\mathbf{z} \in \mathbb{R}^{n}$ obeying $\|\mathbf{z}\|_{\ell_{2}}=1$. This, in particular includes any $\mathbf{z} \in T_{\rho, r}$ and consequently

$$
Q_{2 \xi_{0}}\left(T_{\rho, s}, \mathbf{a}\right) \geq \frac{4 p(1-p)(3 / 4)^{2}}{13}>\frac{p(1-p)}{6} \quad \text { for } \quad \xi_{0}=\frac{1}{4} \sqrt{p(1-p)}
$$

In order to bound the mean empirical width $W_{m}\left(T_{\rho, r}, \mathbf{a}\right)$, we follow the approach outlined in Ref. [19]. Note that $T_{\rho, s}$ contains the set of all $s$-sparse vectors with unit length:

$$
\begin{equation*}
\Sigma_{s}^{2}=\left\{\mathbf{v} \in \mathbb{R}^{n}:\|\mathbf{v}\|_{\ell_{0}} \leq s,\|\mathbf{v}\|_{\ell_{2}}=1\right\} \tag{21}
\end{equation*}
$$

Several existing results, such as [30, Lemma 3] (see also [31, Lemma 4.5] and [19, Lemma 3.2] for a generalization to normalization in any $\ell_{q}$-norm), state that a converse relation is also true:

$$
\begin{equation*}
T_{\rho, s} \subset \sqrt{1+(1+1 / \rho)^{2}} \operatorname{conv}\left(\Sigma_{s}^{2}\right) \subseteq \frac{3}{\rho} \operatorname{conv}\left(\Sigma_{s}^{2}\right) . \tag{22}
\end{equation*}
$$

Here, conv $\left(\Sigma_{s}^{2}\right)$ denotes the convex hull of $\Sigma_{s}^{2}$. This in turn implies

$$
W_{m}\left(T_{\rho, s}, \mathbf{a}\right)=\mathbb{E}\left[\sup _{\mathbf{u} \in T_{\rho, r}}\langle\mathbf{h}, \mathbf{u}\rangle\right] \leq \frac{3}{\rho} \mathbb{E}\left[\sup _{\mathbf{u} \in \operatorname{conv}\left(\Sigma_{s}^{2}\right)}\langle\mathbf{h}, \mathbf{u}\rangle\right]=\frac{3}{\rho} W_{m}\left(\Sigma_{s}^{2}, \mathbf{a}\right),
$$

where the last equation is due to the fact that the supremum of the linear function $\langle\mathbf{u}, \mathbf{h}\rangle$ over the convex set $\operatorname{conv}\left(\Sigma_{s}^{2}\right)$ is attained at its extremal set $\Sigma_{s}^{2}$. The quantity $W_{m}\left(\Sigma_{s}^{2}, \mathbf{a}\right)$ corresponds to the supremum of the stochastic process $X_{\mathbf{u}}=\langle\mathbf{u}, \mathbf{h}\rangle$ indexed by $\mathbf{u} \in \Sigma_{s}^{2}$. This stochastic process is centered $\left(\mathbb{E}\left[X_{\mathbf{u}}\right]=0\right)$ and inherits subgaussian marginals from the fact that the individual entries of a are subgaussian random variables. Dudley's inequality, see e.g. [13, Sec. 8.6], allows for bounding the supremum of such centered, subgaussian stochastic processes. A computation detailed in the appendix yields

$$
\begin{equation*}
W_{m}\left(\Sigma_{s}^{2}, \mathbf{a}\right) \leq 20 \theta(p) \sqrt{s\left(\log \left(\frac{\mathrm{e} n}{s}\right)+\frac{p^{2}}{\theta^{2}(p)}\right)}, \quad \text { where } \quad \theta(p)=\sqrt{\frac{2 p-1}{2 \log \left(\frac{p}{1-p}\right)}} \tag{23}
\end{equation*}
$$

is the subgaussian parameter associated with the centered Bernoulli random variable $\tilde{b}$ with parameter $p$ [32]: $\operatorname{Pr}[\tilde{b}=1-p]=p$ and $\operatorname{Pr}[\tilde{b}=-p]=1-p$.

Fixing $t_{0}=\frac{p(1-p)}{12} \sqrt{m}$ and inserting these bounds into Formula (19) reveals

$$
\begin{aligned}
\inf _{\mathbf{v} \in T_{\rho, s}}\|\mathbf{A} \mathbf{v}\|_{\ell_{2}} & \geq \xi_{0} \sqrt{m} Q_{2 \xi_{0}}\left(T_{\rho, s}, \mathbf{a}\right)-\xi_{0} t_{0}-2 W_{m}\left(T_{\rho, s}, \mathbf{a}\right) \\
& \geq \frac{1}{48} \sqrt{p(1-p)} \sqrt{3} \sqrt{m}-120 \frac{\theta(p)}{\rho} \sqrt{s\left(\log \left(\frac{\mathrm{e} n}{s}\right)+\frac{p^{2}}{\theta^{2}(p)}\right)}
\end{aligned}
$$

with probability at least $1-\mathrm{e}^{-\frac{1}{72} p^{2}(1-p)^{2} m}$. In order to assure strict positivity of this bound, we set

$$
m \geq C_{1} \frac{2 \theta^{2}(p)}{p^{3}(1-p)^{3} \rho^{2}} s\left(\log \left(\frac{\mathrm{e} n}{s}\right)+\frac{p^{2}}{\theta^{2}(p)}\right)
$$

where $C_{1}>0$ is a sufficiently large constant. Then the inequality above assures that there is another constant $C_{2}>0$ (whose size only depends on $C_{1}$ ) such that

$$
\inf _{\mathbf{v} \in T_{\rho, s}} \geq \frac{1}{C_{2}} \sqrt{p(1-p)}{ }^{3} \sqrt{m}
$$

Comparing this bound to Eq. (18) allows us to set $\tau=\frac{C_{2}}{\sqrt{p(1-p)} \sqrt{m}}$. This is the main result of this section:
Theorem 9. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a $0 / 1$-Bernoulli matrix with parameter $p \in[0,1]$. Fix $s \leq n$ and $\rho \in[0,1]$ and set

$$
\begin{equation*}
m=\frac{C_{1}}{\rho^{2}} \alpha(p) s\left(\log \left(\frac{\mathrm{e} n}{s}\right)+\beta(p)\right) \tag{24}
\end{equation*}
$$

with $\alpha(p)=\frac{2 p-1}{p^{3}(1-p)^{3} \log \left(\frac{p}{1-p}\right)}$ and $\beta(p)=\frac{2 p^{2} \log \left(\frac{p}{1-p}\right)}{2 p-1}$. Then, with probability of failure bounded by $\mathrm{e}^{-\frac{p^{2}(1-p)^{2}}{72} m}$, A obeys the robust NSP of order $s$ with parameters $\rho$ and $\tau=\frac{C_{2}}{\sqrt{p(1-p)} \sqrt{m}}$. Here, $C_{1}, C_{2}>0$ denote absolute constants that only depend on each other.

This is a more detailed version of the first claim presented in Theorem 3. We see that the sampling rate, the size of the NSP-parameter $\tau$ and the probability bound all depend on the Bernoulli parameter $p \in[0,1]$.

Remark 10. While the sampling rate (24) is optimal in terms of sparsity s and problem dimension $n$, this is not the case for its dependence on the parameter p. In fact, the first version of this work (e.g. see [33]) achieved a strictly better constant $\tilde{\alpha}(p)=\frac{1}{p^{2}(1-p)^{2}}$ at the cost of a sub-optimal sampling rate of order $s \log (n)$. However, we do not know if this result accurately describes the correct behavior for the practically relevant case of sparse (co-sparse) measurements $p \rightarrow 0(p \rightarrow 1)$. We intend to address this question in future work.

Finally, we point out that when opting for a standard Bernoulli process, i.e. $p=\frac{1}{2}$, the assertions of Theorem 9 considerably simplify:

Corollary 11. Fix $s \leq n, \rho \in[0,1]$ and let A be a standard $(m \times n) 0 / 1$-Bernoulli matrix (i.e. $p=\frac{1}{2}$ ) with

$$
m \geq \frac{C_{1}}{128 \rho^{2}} s \log (n)
$$

Then with probability at least $1-\mathrm{e}^{-\frac{m}{1152}}$ this matrix obeys the NSP of order $s$ with parameters $\rho$ and $\tau=\frac{C_{2}}{8 \sqrt{m}}$. Here, $C_{1}$ and $C_{2}$ are the constants from Theorem 9.

## C. 0/1-Bernoulli matrices lie in $\mathcal{M}_{+}$

We now move on to showing that $0 / 1$-Bernoulli matrices are very likely to admit the second requirement of Theorem 4. Namely, that there exists a vector $\mathbf{w}=\mathbf{A}^{T} \mathbf{t}$ that is strictly positive (this is equivalent to demanding $\mathbf{A} \in \mathcal{M}_{+}$). Concretely, we show that setting $\mathbf{t}=\frac{1}{p m} \mathbf{1}_{m}$ w.h.p. results in a strictly positive vector $\mathbf{w} \in \mathbb{R}^{n}$ whose conditioning obeys

$$
\begin{equation*}
\kappa(\mathbf{w})=\frac{\max _{k}\left|\left\langle\mathbf{e}_{k}, \mathbf{w}\right\rangle\right|}{\min _{k}\left|\left\langle\mathbf{e}_{k}, \mathbf{w}\right\rangle\right|} \leq 3 \tag{25}
\end{equation*}
$$

To do so, we note that $\mathbf{w}=\frac{1}{p m} \sum_{k=1}^{m} \mathbf{a}_{k}$ has expectation $\mathbb{E}[\mathbf{w}]=\mathbf{1}_{n}$, which is-up to re-scaling-the unique non-negative vector admitting $\kappa\left(\mathbf{1}_{n}\right)=1$. After having realized this, it suffices to use a concentration inequality to prove that w.h.p. $\mathbf{w}$ does not deviate too much from its expectation. We do this by invoking a large deviation bound.

Theorem 12. Suppose that $\mathbf{A}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ is a $0 / 1$-Bernoulli matrix with parameter $p \in[0,1]$ and set

$$
\begin{equation*}
\mathbf{w}=\mathbf{A}^{T} \mathbf{t} \in \mathbb{R}^{n} \quad \text { with } \quad \mathbf{t}=\frac{1}{p m} \mathbf{1}_{m} \in \mathbb{R}^{m} \tag{26}
\end{equation*}
$$

Then with probability at least $1-n \mathrm{e}^{-\frac{3}{8} p(1-p) m}$

$$
\begin{equation*}
\max _{i}\left|\left\langle\mathbf{e}_{i}, \mathbf{w}\right\rangle\right| \leq \frac{3}{2} \quad \text { and } \quad \min _{i}\left|\left\langle\mathbf{e}_{i}, \mathbf{w}\right\rangle\right| \geq \frac{1}{2} \tag{27}
\end{equation*}
$$

This in turn implies (25).

Proof. Instead of showing the claim directly, we prove the stronger statement:

$$
\begin{equation*}
\left|\left\langle\mathbf{e}_{i}, \mathbf{w}\right\rangle-1\right| \leq \frac{1}{2} \quad 1 \leq i \leq n \tag{28}
\end{equation*}
$$

is true with probability of failure bounded by $n \mathrm{e}^{-\frac{3}{8} p(1-p) m}$. If such a bound is true for all $i$, it is also valid for maximal and minimal vector components and we obtain

$$
\max _{i}\left|\left\langle\mathbf{e}_{i}, \mathbf{w}\right\rangle\right| \leq \max _{k}\left|\left\langle\mathbf{e}_{i}, \mathbf{w}\right\rangle-1\right|+1 \leq \frac{3}{2} \quad \text { and } \quad \min _{k}\left|\left\langle\mathbf{e}_{i}, \mathbf{w}\right\rangle\right| \geq 1-\max _{i}\left|\left\langle\mathbf{e}_{i}, \mathbf{w}\right\rangle-1\right| \geq \frac{1}{2}
$$

as claimed. In order to prove (28), we fix $1 \leq i \leq n$ and focus on

$$
\left|\left\langle\mathbf{e}_{i}, \mathbf{w}\right\rangle-1\right|=\left|\frac{1}{p m} \sum_{k=1}^{m}\left\langle\mathbf{e}_{i}, \mathbf{a}_{k}\right\rangle-1\right|=\frac{1}{p m}\left|\sum_{k=1}^{m}\left(b_{k, i}-\mathbb{E}\left[b_{k, i}\right]\right)\right| .
$$

Here, we have used $\left\langle\mathbf{e}_{i}, \mathbf{a}_{k}\right\rangle=\left\langle\mathbf{e}_{k}, \mathbf{A} \mathbf{e}_{i}\right\rangle=b_{k, i}$, which is an indepenent instance of a $0 / 1$-Bernoulli random variable with parameter $p$. Thus we are faced with bounding the deviation of a sum of $m$ centered, independent random variables $c_{k}:=b_{k, i}-\mathbb{E}\left[b_{k, i}\right]$ from its mean. Each such variable obeys

$$
\left|c_{k}\right| \leq \max \{p, 1-p\} \leq 1 \quad \text { and } \quad \mathbb{E}\left[c_{k}^{2}\right]=\operatorname{Var}\left(b_{k, i}\right)=p(1-p)
$$

Applying a Bernstein inequality [13, Theorem 7.30] reveals

$$
\operatorname{Pr}\left[\left|\left\langle\mathbf{e}_{i}, \mathbf{w}\right\rangle-1\right| \geq \frac{1}{2}\right] \leq \operatorname{Pr}\left[\left|\left\langle\mathbf{e}_{i}, \mathbf{w}\right\rangle-1\right| \geq \frac{1-p}{2}\right]=\operatorname{Pr}\left[\left|\sum_{k=1}^{m} c_{k}\right| \geq \frac{m p(1-p)}{2}\right] \leq \exp \left(-\frac{3}{8} p(1-p) m\right)
$$

Combining this statement with a union bound assures that $\left|\left\langle\mathbf{e}_{i}, \mathbf{w}\right\rangle-1\right|<\frac{1}{2}$ is simultaneously true for all $1 \leq i \leq n$ with probability at least $1-n \mathrm{e}^{-\frac{3}{8} p(1-p) m}$.

## D. Proof of Theorem 3

Finally, these two results can be combined to yield Theorem 3. It readily follows from taking a union bound over the individual probabilities of failure. Theorem 9 requires a sampling rate of

$$
\begin{equation*}
m=\frac{C_{1}}{\rho^{2}} \alpha(p) s\left(\log \left(\frac{\mathrm{e} n}{s}\right)+\beta(p)\right) \tag{29}
\end{equation*}
$$

to assure that a corresponding 0/1-Bernoulli matrix obeys a strong version of the NSP with probability at least $1-\mathrm{e}^{-\frac{p^{2}(1-p)^{2}}{72} m}$. On the other hand, Theorem 12 asserts that choosing $\mathbf{w}=\mathbf{A}^{T} \frac{1}{p m} \mathbf{1}_{m}$ for 0/1-Bernoulli matrices $\mathbf{A}$ results in a well-conditioned and strictly positive vector $\mathbf{w}$ with probability at least $1-n \mathrm{e}^{-\frac{3}{8} p(1-p) m}$. The probability that either of these assertions fails to hold can be controlled by the union bound over both probabilities of failure:

$$
\mathrm{e}^{-\frac{p^{2}(1-p)^{2}}{72} m}+n \mathrm{e}^{-\frac{3 p(1-p)}{8} m} \leq(n+1) \mathrm{e}^{-\frac{p^{2}(1-p)^{2}}{72} m}
$$

Finally, we focus on 0/1-Bernoulli matrices $\mathbf{A}$ for which both statements are true and whose sampling rate exceeds (29). Theorem 9 then implies that $\mathbf{A}$ obeys the $s$-NSP with a pre-selected parameter $\rho \in[0,1]$ and $\tau=\frac{C_{2}}{\sqrt{p(1-p)} \sqrt{m}}$. Moreover, the choice $\mathbf{t}=\frac{1}{p m} \mathbf{1}_{m}$ in Theorem 12 results in $\|\mathbf{t}\|_{\ell_{2}}=\frac{1}{p \sqrt{m}}$. As a result, Theorem 4 implies the following for any $\mathbf{x}, \mathbf{z} \geq \mathbf{0}$ :

$$
\left.\begin{array}{rl}
\|\mathbf{x}-\mathbf{z}\|_{\ell_{2}} & \leq \frac{2 C^{\prime}}{\sqrt{s}} \sigma_{s}(\mathbf{x})_{\ell_{1}}+D^{\prime}\left(\|\mathbf{t}\|_{\ell_{2}}+\tau\right)\|\mathbf{A}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}} \\
& =\frac{2 C^{\prime}}{\sqrt{s}} \sigma_{s}(\mathbf{x})_{\ell_{1}}+D^{\prime}\left(\frac{1}{p \sqrt{m}}+\frac{C_{2}}{\sqrt{p(1-p)}}{ }^{3} \sqrt{m}\right.
\end{array}\right)\|\mathbf{A}(\mathbf{x}-\mathbf{z})\|_{\ell_{2}} .
$$

Setting $\mathbf{z}=\mathbf{x}^{\sharp} \geq \mathbf{0}$ to be the solution of NNLS (5) in turn assures

$$
\left\|\mathbf{A}\left(\mathbf{x}-\mathbf{x}^{\sharp}\right)\right\|_{\ell_{2}} \leq\|\mathbf{e}\|_{\ell_{2}}+\left\|\mathbf{y}-\mathbf{x}^{\sharp}\right\|_{\ell_{2}} \leq 2\|\mathbf{e}\|_{\ell_{2}}
$$

and the claim follows with $E^{\prime}=2 D^{\prime}\left(1+C_{2}\right)$.

## V. Numerical Experiments

This section is devoted to numerical tests regarding the nonnegative least squares (NNLS) estimation in (5). To benchmark it, we compare this to the results obtained with basis pursuit denoising (BPDN) in (3). The NNLS has been computed using the lsqnonneg function in MATLAB which implements the "active-set" Lawson-Hanson algorithm [34]. For the BPDN the SPGL1 toolbox has been used [35].

In a first test we have evaluated numerically the phase transition of NNLS in the $0 / 1$-Bernoulli setting for the noiseless case. The dimension and sparsity parameters are generated uniformly (in this order) in the ranges $n \in[10 \ldots 500], m \in[10 \ldots n]$ and $s \in[1 \ldots m]$. Thus, the sparsity/density variable is $r=s / m$ and the subsampling ratio is $\delta=m / n$. The $m \times n$ measurement matrix $\mathbf{A}$ is generated using the i.i.d. $0 / 1$-Bernoulli model with $p=1 / 2$. The nonnegative $s$-sparse signal $\mathbf{0} \leq \mathbf{x} \in \mathbb{R}^{n}$ to recover is created as follows: the random support $\operatorname{supp}(\mathbf{x})$ is obtained from taking the first $s$ elements of a random (uniformly-distributed) permutation of the indices $(1 \ldots n)$. On this support each component is the absolute value of an i.i.d. standard (zero mean, unit variance) Gaussian, i.e., $x_{i}=\left|g_{i}\right|$ with $g_{i} \sim N(0,1)$ for all $i \in \operatorname{supp}(\mathbf{x})$. We consider one individual recovery to be successful if $\|\mathbf{x}-\hat{\mathbf{x}}\|_{\ell_{2}} \leq 10^{-3}\|\mathbf{x}\|_{\ell_{2}}$. The resulting phase transition diagram, shown in Figure 1 above, demonstrates that NNLS indeed reliable recovers nonnegative sparse vectors without any $\ell_{1}$-regularization.


Fig. 2: Comparison of NNLS in (5) with BPDN in (3) for i.i.d. 0/1-Bernoulli matrices in the noisy setting.

In the second experiment we consider the noisy case. Apart from its simplicity, the important feature of NNLS is that no a-priori norm assumptions on the noise are necessary. This is not the case for BPDN. Theorem 3 implies that the NNLS estimate $x^{\sharp}$ obeys

$$
\begin{equation*}
\left\|\mathbf{x}-\mathbf{x}^{\sharp}\right\|_{\ell_{2}} \leq \frac{D^{\prime}}{8 \sqrt{m}}\|\mathbf{e}\|_{\ell_{2}} \tag{30}
\end{equation*}
$$

A similar bound is valid for the BPDN (see (13)) estimate $\mathbf{x}_{\eta}$ provided that $\|\mathbf{e}\|_{\ell_{2}} \leq \eta$. Note, that achieving this requires knowledge of $\|\mathbf{e}\|_{\ell_{2}}$. Interestingly, even under this prerequisite (BPDN indeed uses here the instantaneous norm $\eta:=\|\mathbf{e}\|_{\ell_{2}}$ of the noise) the performance of NNLS is considerably better then BPDN in our setting. This is visualized in Figure 2 where each component $e_{j}$ of $\mathbf{e}$ is i.i.d. Gaussian distributed with zero mean and variance $\sigma_{e}^{2}=1 / 100$. There recovery has been identified as "successful" if (30) is fulfilled for $\frac{D^{\prime}}{8}=\sqrt{10}$.

Finally, we show in Figure 3 that NNLS is not as well-suited for uniform recovery with Gaussian measurements. We have considered the noiseless scenario and generated random $m \times n$ i.i.d. Gaussian and 0/1-Bernoulli matrices where $n=100$ and $m=20 \ldots 80$. For each generated matrix we have tested 10000 random $s$-sparse vectors with $s=5$ as explained above. We counted an event as successful, if all 10000 test vectors were recovered within the bound $\left\|\mathbf{x}-\mathbf{x}^{\sharp}\right\|_{\ell_{2}} \leq 10^{-3}\|\mathbf{x}\|_{\ell_{2}}$. We repeated this procedure 200 times and accumulated the results for every $m$.


Fig. 3: Comparison of NNLS in (5) for i.i.d. Gaussian and 0/1-Bernoulli matrices

## VI. Conclusions

In this work we have shown that non-negativity is an important additional property when recovering sparse vectors. This additional structural constraint is relevant in many applications. Here, we provided activity detection in wireless networks using individual sequences as concrete example. There, designing measurement matrices such that convex hull of its columns (the sequences) is sufficiently well-separated from the origin allow for remarkably simple and robust recovery algorithms. Crucially, these are robust to noise and blind in a sense that no regularization and a-priori information on the noise is required. We have demonstrated this feature by strengthening the implications of the
robust nullspace property for the non-negative setting. Furthermore, we have proved that i.i.d. binary measurements fulfill w.h.p. this property and are simultaneously well-conditioned. Therefore, they can be used for recovering nonnegative and sparse vectors in the optimal regime.

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## Appendix

Here we provide derivations of the two bounds (23) and (20) on which we built our argument that $0 / 1$-Bernoulli matrices obey the robust NSP. Since both are rather technical and not essential for understanding the main ideas, we decided to present them in this appendix.

## A. An upper bound on $W_{m}\left(\Sigma_{s}^{2}\right.$, a) for 0/1-Bernoulli matrices

We will follow Ref. [19] and use Dudley's inequality to bound the mean empirical width $W_{m}\left(\Sigma_{s}^{2}, \mathbf{a}\right)$ in Mendelson's small ball method; see also Ref. [18] for a similar approach. Recall that $\mathbf{a}=\sum_{i=1}^{n} b_{i} \mathbf{e}_{i} \in \mathbb{R}^{n}$ is a random vector whose entries are i.i.d. Bernoulli random variables with parameter $p$. We decompose a into $\tilde{\mathbf{a}}+p \mathbf{1}$, where each entry $\tilde{b}_{i}$ of $\tilde{\mathbf{a}}$ is an i.i.d. copy of the centered Bernoulli random variable

$$
\tilde{b}= \begin{cases}1-p & \text { with prob. } p  \tag{31}\\ -p & \text { with prob. } 1-p\end{cases}
$$

Likewise, we introduce $\tilde{\mathbf{h}}=\frac{1}{\sqrt{m}} \sum_{k=1}^{m} \epsilon_{k} \tilde{\mathbf{a}}_{k}$ and note that

$$
\begin{aligned}
W_{m}\left(\Sigma_{s}^{2}, \mathbf{a}\right) & =\mathbb{E}\left[\sup _{\mathbf{u} \in \Sigma_{s}^{2}}\langle\mathbf{u}, \mathbf{h}\rangle\right]=\mathbb{E}\left[\sup _{\mathbf{u} \in \Sigma_{s}^{2}}\left\langle\mathbf{u}, \tilde{\mathbf{h}}+\frac{p}{\sqrt{m}} \sum_{k=1}^{m} \epsilon_{k} \mathbf{1}_{n}\right\rangle\right] \\
& \leq \mathbb{E}\left[\sup _{\mathbf{u} \in \Sigma_{s}^{2}}\langle\mathbf{u}, \tilde{\mathbf{h}}\rangle\right]+p \mathbb{E}\left[\left|\frac{1}{\sqrt{m}} \sum_{k=1}^{m} \epsilon_{k}\right|\right] \sup _{\mathbf{u} \in \Sigma_{s}^{2}}\left\langle\mathbf{u}, \mathbf{1}_{n}\right\rangle \\
& =W_{m}\left(\Sigma_{s}^{2}, \tilde{\mathbf{a}}\right)+p \sqrt{\frac{s}{m}} \mathbb{E}\left[\left|\sum_{k=1}^{m} \epsilon_{k}\right|\right]
\end{aligned}
$$

because $\left\langle\mathbf{u}, \mathbf{1}_{n}\right\rangle \leq\|\mathbf{u}\|_{\ell_{1}} \leq \sqrt{s}\|\mathbf{u}\|_{\ell_{2}}=\sqrt{s}$ for any $\mathbf{u} \in \Sigma_{s}^{2}$ (and this chain of inequalities is tight). The second term in this expression can be bounded via a Khintchine-type inequality. Corollary 8.7 in [13] implies (with $q=1$ and $\mathbf{c}=\mathbf{1}_{m} \in \mathbb{R}^{m}$ )

$$
p \sqrt{\frac{s}{m}} \mathbb{E}\left[\left|\sum_{k=1}^{m} \epsilon_{k}\right|\right] \leq \frac{\sqrt{s} p 2^{3 / 4} \mathrm{e}^{-1 / 2}\left\|\mathbf{1}_{m}\right\|_{\ell_{2}}}{\sqrt{m}} \leq \sqrt{2 s} p
$$

The remaining term $W_{m}\left(\Sigma_{s}^{2}, \tilde{\mathbf{a}}\right)$ corresponds to the supremum of the stochastic process $\tilde{X}_{\mathbf{u}}=\langle\mathbf{u}, \tilde{\mathbf{h}}\rangle$ which is indexed by $\mathbf{u} \in \Sigma_{s}^{2}$. This process is centered $\left(\mathbb{E}\left[X_{\mathbf{u}}\right]=0 \forall \mathbf{u} \in \Sigma_{s}^{2}\right)$ and also subgaussian. A centered random variable $X$ is subgaussian with parameter $\theta$, if its moment generating function obeys

$$
\begin{equation*}
\mathbb{E}\left[\mathrm{e}^{\lambda X}\right] \leq \mathrm{e}^{\frac{1}{2} \theta^{2} \lambda^{2}} \quad \forall \lambda \in \mathbb{R} \tag{32}
\end{equation*}
$$

We refer to [13, Sec. 7.4$]$ and $[36, S e c .5 .2 .3]$ for a thorough introduction to subgaussian random variables. Here, we content ourselves with stating that (32) implies

$$
\operatorname{Pr}[|X| \geq t] \leq 2 \mathrm{e}^{-\frac{t^{2}}{2 \theta^{2}}} \quad \forall t \geq 0
$$

see e.g. [13, Proposition 7.24]. Thus, every random variable obeying (32) has a subgaussian tail behavior governed by $\theta^{2}$. The centered random variable $\tilde{b}$ is a particular instance of a subgaussian random variable. The exact value of its subgaussian parameter has been determined in Ref. [32]:

$$
\theta(p)=\sqrt{\frac{2 p-1}{2 \log \left(\frac{p}{1-p}\right)}} \quad p \in[0,1]
$$

This includes the special cases $\theta(0)=\theta(1)=0$ and $\theta\left(\frac{1}{2}\right)=\frac{1}{2}$. The stochastic process $\tilde{X}_{\mathbf{u}}$ inherits this subgaussian behavior. More precisely, standard results such as [13, Theorem 7.27] imply

$$
\mathbb{E}\left[\mathrm{e}^{\lambda\left(\tilde{X}_{\mathbf{u}}-\tilde{X}_{\mathbf{v}}\right)}\right] \leq \mathrm{e}^{\frac{1}{2} \theta^{2}(p)\|\mathbf{u}-\mathbf{v}\|_{\ell_{2}}^{2} \lambda^{2}} \quad \forall \lambda \in \mathbb{R}, \forall \mathbf{u}, \mathbf{v} \in \Sigma_{s}^{2}
$$

This implies that $\tilde{X}_{\mathbf{u}}$ is a centered subgaussian stochastic process with associated (pseudo-) metric $d(\mathbf{u}, \mathbf{v})=$ $\theta(p)\|\mathbf{u}-\mathbf{v}\|_{\ell_{2}}$, see e.g. [13, Definition 8.22]. Dudley's inequality, see c.f. [13, Theorem 8.23], applies to such stochastic processes and yields

$$
W_{m}\left(\Sigma_{s}^{2}, \tilde{\mathbf{a}}\right)=\mathbb{E}\left[\sup _{\mathbf{u} \in \Sigma_{s}^{2}} \tilde{X}_{\mathbf{u}}\right] \leq 4 \sqrt{2} \int_{0}^{\infty} \sqrt{\log \left(\mathcal{N}\left(\Sigma_{s}^{2}, \theta(p)\|\cdot\|_{\ell_{2}}, u\right)\right)} \mathrm{d} u
$$

where $\mathcal{N}\left(\Sigma_{s}^{2}, \theta(p)\|\cdot\|_{\ell_{2}}, u\right)$ denotes the covering number, i.e. the smallest integer $N$ such that there exists a subset $F$ of $\Sigma_{s}^{2}$ with $|F| \leq N$ and $\min _{\mathbf{y} \in F} \theta(p)\|\mathbf{y}-\mathbf{x}\|_{\ell_{2}} \leq u$ for all $\mathbf{x} \in \Sigma_{s}^{2}$. We refer to [13, Appendix C.2] for a concise introduction of covering numbers and their properties. In particular,

$$
\int_{0}^{\infty} \sqrt{\log \left(\mathcal{N}\left(\Sigma_{s}^{2}, \theta(p)\|\cdot\|_{\ell_{2}}, u\right)\right)} \mathrm{d} u=\theta(p) \int_{0}^{1} \sqrt{\log \left(\mathcal{N}\left(\Sigma_{s}^{2},\|\cdot\|_{\ell_{2}}, v\right)\right)} \mathrm{d} v
$$

which follows from $\mathcal{N}\left(\Sigma_{s}^{2}, \theta(p)\|\cdot\|_{\ell_{2}}\right)=\mathcal{N}\left(\Sigma_{s}^{2},\|\cdot\|_{\ell_{2}}, \frac{u}{\theta(p)}\right)$, a change of variables in the integration $\left(v=\frac{u}{\theta(p)}\right)$ and the fact that $\Sigma_{s}^{2}$ is contained in the $\ell_{2}$-unit ball. This last fact implies that $\mathcal{N}\left(\Sigma_{s}^{2},\|\cdot\|_{\ell_{2}}, v\right)=1$ for any $v \geq 1$ and the corresponding integrand vanishes. For $v \in[0,1]$, the covering number of $\Sigma_{s}^{2}$ can be estimated in the following way: There are $\binom{n}{s}$ different ways to choose the support $S$ of an $s$-sparse vector in $\mathbb{R}^{n}$. In turn, normalization of $\Sigma_{s}^{2}$ assures that each such vector is contained in an $s$-dimensional unit ball $B_{S}$. A volumetric argument in turn implies $\mathcal{N}\left(B_{S},\|\cdot\|_{\ell_{2}}, v\right) \leq\left(1+\frac{2}{v}\right)^{s}$, see e.g. [13, Prop. C.3]. Using subadditivity of covering numbers, we conclude

$$
\mathcal{N}\left(\Sigma_{s}^{2},\|\cdot\|_{\ell_{2}}, v\right) \leq\binom{ n}{s} \max _{|S|=s} \mathcal{N}\left(B_{S},\|\cdot\|_{\ell_{2}}, v\right) \leq\binom{ n}{s}\left(1+\frac{2}{v}\right)^{s} \leq\left(\frac{\mathrm{e} n}{s}\right)^{s}\left(1+\frac{2}{v}\right)^{s}
$$

where the last inequality is due to Stirling's formula. Combining these estimates yields

$$
\begin{aligned}
W_{m}\left(\Sigma_{s}^{2}, \tilde{\mathbf{a}}\right) & \leq 4 \sqrt{2} \theta(p) \int_{0}^{1} \sqrt{\log \left(\left(\frac{e n}{s}\right)^{s}\left(1+\frac{2}{v}\right)^{s}\right)} \mathrm{d} v \\
& \leq 4 \sqrt{2} \theta(p)\left(\sqrt{s \log \left(\frac{\mathrm{e} n}{s}\right)} \int_{0}^{1} \mathrm{~d} v+\sqrt{s} \int_{0}^{1} \sqrt{\ln \left(1+\frac{2}{v}\right)} \mathrm{d} v\right) \\
& \leq 4 \sqrt{2} \theta(p)\left(\sqrt{s \log \left(\frac{\mathrm{e} n}{s}\right)}+\sqrt{s \log (3 \mathrm{e})}\right)
\end{aligned}
$$

where the last estimate follows from bounding the second integral, see e.g. [13, Lemma C.9].
Summarizing the results from this paragraph, we conclude

$$
\begin{aligned}
W_{m}\left(\Sigma_{s}^{2}, \mathbf{a}\right) & \leq \sqrt{2 s} \theta(p)\left(4 \sqrt{\log \left(\frac{e n}{s}\right)}+4 \sqrt{\log (3 \mathrm{e})}+\frac{p}{\theta(p)}\right) \leq \sqrt{2 s} \theta(p)\left(10 \sqrt{\log \left(\frac{\mathrm{e} n}{s}\right)}+\frac{p}{\theta(p)}\right) \\
& \leq 20 \sqrt{s} \theta(p) \sqrt{\log \left(\frac{\mathrm{e} n}{s}\right)+\frac{p^{2}}{\theta^{2}(p)}}
\end{aligned}
$$

where the last line follows from $\sqrt{a}+\sqrt{b} \leq \sqrt{2(a+b)}$ for any $a, b \geq 0$.

## B. Bounding $\operatorname{Pr}\left[|\langle\mathbf{a}, \mathbf{z}\rangle| \geq \theta\|\mathbf{z}\|_{\ell_{2}}\right]$ for $0 / 1-$ Bernoulli vectors

In this final section we prove that for any unit vector $\mathbf{z}=\left(z_{1}, \ldots, z_{n}\right)^{T} \in \mathbb{R}^{n}\left(\|\mathbf{z}\|_{\ell_{2}}=1\right)$ and any $\theta \in[0,1 / 2]$, the bound

$$
\begin{equation*}
\operatorname{Pr}[|\langle\mathbf{a}, \mathbf{z}\rangle| \geq \theta \sqrt{p(1-p)}] \geq \frac{4}{13} p(1-p)\left(1-\theta^{2}\right)^{2} \tag{33}
\end{equation*}
$$

holds in the Bernoulli setting. Here, the probability is taken over instances of i.id. Bernoulli vectors $\mathbf{a}=\sum_{i=1}^{n} b_{i} \mathbf{e}_{i} \in$ $\mathbb{R}^{n}$ with parameter $p$. Up to the multiplicative constant $\frac{4}{13}$, this bound is tight. To see this, set $\mathbf{z}_{0}=\frac{1}{\sqrt{2}}\left(\mathbf{e}_{1}-\mathbf{e}_{2}\right)$ and observe

$$
\operatorname{Pr}\left[\left|\left\langle\mathbf{a}, \mathbf{z}_{0}\right\rangle\right| \geq \theta \sqrt{p(1-p)}\right]=\operatorname{Pr}\left[\left|b_{1}-b_{2}\right| \geq \theta \sqrt{2 p(1-p)}\right]=2 p(1-p)
$$

for any $0 \leq \theta \leq \frac{1}{\sqrt{2 p(1-p)}}$. Letting $\theta \rightarrow 0$ then establishes tightness. We note in passing that a direct exploitation of the subgaussian properties of a would lead to considerably weaker results.

The derivation of Formula (33) is going to rely on the Paley-Zygmund inequality and a few standard, but rather tedious, moment calculations for Bernoulli processes. We start by exploiting

$$
\begin{equation*}
\operatorname{Pr}[|\langle\mathbf{a}, \mathbf{z}\rangle| \geq \theta \sqrt{p(1-p)}]=\operatorname{Pr}\left[\langle\mathbf{a}, \mathbf{z}\rangle^{2} \geq \theta^{2} p(1-p)\right] \tag{34}
\end{equation*}
$$

because the latter expression is easier to handle. Introducing the nonnegative random variable $S:=\langle\mathbf{a}, \mathbf{z}\rangle^{2}=$ $\sum_{i, j=1}^{n} b_{i} b_{j} z_{i} z_{j}$,, we see

$$
\begin{equation*}
\mathbb{E}[S]=\sum_{i \neq j} \mathbb{E}\left[b_{i}\right] \mathbb{E}\left[b_{j}\right] z_{i} z_{j}+\sum_{i=1}^{n} \mathbb{E}\left[b_{i}^{2}\right] z_{i}^{2}=p^{2}\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle^{2}+p(1-p)\|\mathbf{z}\|_{\ell_{2}}^{2} \geq p(1-p) \tag{35}
\end{equation*}
$$

This calculation together with (34) implies

$$
\begin{equation*}
\operatorname{Pr}[|\langle\mathbf{a}, \mathbf{z}\rangle| \geq \theta \sqrt{p(1-p)}] \geq \operatorname{Pr}\left[S \geq \theta^{2} \mathbb{E}[S]\right] \tag{36}
\end{equation*}
$$

Since $S \geq 0$ by definition, the Paley-Zygmund inequality implies

$$
\begin{equation*}
\operatorname{Pr}\left[S \geq \theta^{2} \mathbb{E}[S]\right] \geq \frac{\left(1-\theta^{2}\right)^{2} \mathbb{E}[S]}{\operatorname{Var}(S)+\mathbb{E}[S]^{2}} \tag{37}
\end{equation*}
$$

We have already computed $\mathbb{E}[S]$ in (35), but we still have to compute its variance. We defer this calculation to the very end of this section and for now simply state its result:

$$
\begin{equation*}
\operatorname{Var}(S)=2 \mathbb{E}[S]^{2}-2 p^{4}\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle^{4}+4 p^{2}(1-p)(1-2 p)\langle\mathbf{1}, \mathbf{z}\rangle \sum_{i=1}^{n} z_{i}^{3}+p(1-p)(1-6 p(1-p))\|\mathbf{z}\|_{\ell_{4}}^{4} \tag{38}
\end{equation*}
$$

We now move on to bound these contributions individually by a multiple of $\mathbb{E}[S]^{2}$. We omit the second term and for the third term obtain

$$
\begin{aligned}
4 p^{2}(1-p)(1-2 p)\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle \sum_{i=1}^{n} z_{i}^{3} & \leq 4 p^{2}(1-p)^{2}\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle\|\mathbf{z}\|_{\ell_{2}}^{3}=4 p^{2}(1-p)^{2}\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle \leq 4 p^{2}(1-p)^{2} \max \left\{\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle^{2}, 1\right\} \\
& \leq \frac{2}{p}\left(p^{2}\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle^{2}+p(1-p)\right)^{2}=\frac{2}{p} \mathbb{E}[S]^{2}
\end{aligned}
$$

because $\|\mathbf{z}\|_{\ell_{2}}=1$. The fourth term can be bounded via

$$
p(1-p)(1-6 p(1-p))\|\mathbf{z}\|_{\ell_{4}}^{4} \leq p(1-p)\|\mathbf{z}\|_{\ell_{2}}^{4} \leq \frac{1}{p(1-p)} \mathbb{E}[S]^{2}
$$

and combining all these bounds implies

$$
\operatorname{Var}(S) \leq\left(2+\frac{2}{p}+\frac{1}{p(1-p)}\right) \mathbb{E}[S]^{2}=\frac{3-2 p^{2}}{p(1-p)} \mathbb{E}[S]^{2} \leq \frac{3}{p(1-p)} \mathbb{E}[S]^{2}
$$

Inserting this upper bound into the Paley-Zygmund estimate (37) yields

$$
\operatorname{Pr}[|\langle\mathbf{a}, \mathbf{z}\rangle| \geq \theta \sqrt{p(1-p)}] \geq \frac{\left(1-\theta^{2}\right)^{2} \mathbb{E}[S]^{2}}{\operatorname{Var}(S)+\mathbb{E}[S]^{2}} \geq \frac{\left(1-\theta^{2}\right)^{2} \mathbb{E}[S]^{2}}{\left(\frac{3}{p(1-p)}+1\right) \mathbb{E}[S]^{2}} \geq \frac{4}{13} p(1-p)\left(1-\theta^{2}\right)^{2}
$$

as claimed in (20) and (33), respectively. In the last line, we have used $p(1-p) \leq \frac{1}{4}$ for any $p \in[0,1]$.
Finally, we provide the derivation of Equation (38). We use our knowledge of $\mathbb{E}[S]=p^{2}\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle^{2}+p(1-p)\|\mathbf{z}\|_{\ell_{2}}^{2}$ together with the elementary formula

$$
\left(b_{i}-p\right)\left(b_{j}-p\right)=\left(b_{i} b_{j}-p^{2}\right)-p b_{i}-p b_{j}+2 p^{2}
$$

to rewrite $S-\mathbb{E}[S]$ as

$$
\begin{aligned}
S-\mathbb{E}[S] & =\sum_{i, j=1}^{n} b_{i} b_{j} z_{i} z_{j}-p^{2} \sum_{i \neq j} z_{i} z_{j}-p \sum_{i=1}^{n} z_{i}^{2}=\sum_{i \neq j}\left(b_{i} b_{j}-p^{2}\right) z_{i} z_{j}+\sum_{i=1}^{n}\left(b_{i}^{2}-p\right) z_{i}^{2} \\
& =\sum_{i \neq j}\left(\left(b_{i}-p\right)\left(b_{j}-p\right)+p b_{i}+p b_{j}-2 p^{2}\right) z_{i} z_{j}+\sum_{i=1}^{n}\left(b_{i}^{2}-p\right) z_{i}^{2} \\
& =\sum_{i \neq j}\left(b_{i}-p\right)\left(b_{j}-p\right) z_{i} z_{j}+\sum_{i=1}^{n}\left(b_{i}^{2}-p\right) z_{i}^{2}+p \sum_{i \neq j} b_{i} z_{i} z_{j}+p \sum_{j \neq i} b_{j} z_{j} z_{i}-2 p^{2} \sum_{i \neq j} z_{i} z_{j} \\
& =\sum_{i \neq j}\left(b_{i}-p\right)\left(b_{j}-p\right) z_{i} z_{j}+\sum_{i=1}^{n}\left(b_{i}^{2}-p\right) z_{i}^{2}+2 p \sum_{i, j=1}^{n} b_{i} z_{i} z_{j}-2 p \sum_{i=1}^{n} b_{i} z_{i}^{2}-2 p^{2} \sum_{i, j=1}^{n} z_{i} z_{j}+2 p^{2} \sum_{i=1}^{n} z_{i}^{2} \\
& =\sum_{i \neq j}\left(b_{i}-p\right)\left(b_{j}-p\right) z_{i} z_{j}+\sum_{i=1}^{n}\left(b_{i}^{2}-p\right) z_{i}^{2}+2 p \sum_{i, j=1}^{n}\left(b_{i}-p\right) z_{i} z_{j}-2 p \sum_{i=1}^{n}\left(b_{i}-p\right) z_{i}^{2} \\
& =2 \sum_{i<j}\left(b_{i}-p\right)\left(b_{j}-p\right) z_{i} z_{j}+2 p\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle \sum_{i=1}^{n}\left(b_{i}-p\right) z_{i}+(1-2 p) \sum_{i=1}^{n}\left(b_{i}-p\right) z_{i}^{2} .
\end{aligned}
$$

Here we have exploited symmetry in the first term and $b_{i}^{2}=b_{i}$ to further simplify that expression. For notational simplicity, it makes sense to re-introduce the centered random variable $\tilde{b}_{i}:=b_{i}-p$ :

$$
S-\mathbb{E}[S]=2 \sum_{i<j} \tilde{b}_{i} \tilde{b}_{j} z_{i} z_{j}+2 p\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle \sum_{i=1}^{n} \tilde{b}_{i} z_{i}+(1-2 p) \sum_{i=1}^{n} \tilde{b}_{i} z_{i}^{2}
$$

Employing the binomial formula $(a+b+c)^{2}=a^{2}+2 a b+2 a c+b^{2}+2 b c+c^{2}$, we obtain

$$
\begin{aligned}
\operatorname{Var}(S) & =\mathbb{E}\left[(S-\mathbb{E}[S])^{2}\right]=4 \sum_{i<j} \sum_{k<l} \mathbb{E}\left[\tilde{b}_{i} \tilde{b}_{j} \tilde{b}_{k} \tilde{b}_{l}\right] z_{i} z_{j} z_{k} z_{l}+8 p\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle \sum_{i<j} \sum_{k=1}^{n} \mathbb{E}\left[\tilde{b}_{i} \tilde{b}_{j} \tilde{b}_{k}\right] z_{i} z_{j} z_{k} \\
& +4(1-2 p) \sum_{i<j} \sum_{k=1}^{n} \mathbb{E}\left[\tilde{b}_{i} \tilde{b}_{j} \tilde{b}_{k}\right] z_{i} z_{j} z_{k}^{2}+4 p^{2}\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle^{2} \sum_{i, j=1}^{n} \mathbb{E}\left[\tilde{b}_{i} \tilde{b}_{j}\right] z_{i} z_{j} \\
& +4 p(1-2 p)\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle \sum_{i, j=1}^{n} \mathbb{E}\left[\tilde{b}_{i} \tilde{b}_{j}\right] z_{i} z_{j}^{2}+(1-2 p)^{2} \sum_{i, j=1}^{n} \mathbb{E}\left[\tilde{b}_{i} \tilde{b}_{j}\right] z_{i}^{2} z_{j}^{2}
\end{aligned}
$$

Centeredness of $\tilde{b}$ together with the summation constraints $(i<j)$ and $(k<l)$ implies that summands in the first term vanish, unless $i=k$ and $j=l$. This in turn implies

$$
\begin{aligned}
4 \sum_{i<j} \sum_{k<l} \mathbb{E}\left[\tilde{b}_{i} \tilde{b}_{j} \tilde{b}_{k} \tilde{b}_{l}\right] z_{i} z_{j} z_{k} z_{l} & =4 \sum_{i<j} \mathbb{E}\left[\tilde{b}_{i}^{2}\right] \mathbb{E}\left[\tilde{b}_{j}^{2}\right] z_{i}^{2} z_{j}^{2}=2 p^{2}(1-p)^{2} \sum_{i \neq j} z_{i}^{2} z_{j}^{2} \\
& =2 p^{2}(1-p)^{2}\left(\sum_{i, j=1}^{n} z_{i}^{2} z_{j}^{2}-\sum_{i=1}^{n} z_{i}^{4}\right)=2 p^{2}(1-p)^{2}\left(\|\mathbf{z}\|_{\ell_{2}}^{4}-\|\mathbf{z}\|_{\ell_{4}}^{4}\right) .
\end{aligned}
$$

Using a similar argument allows us to conclude that the second and third term must identically vanish (because the index constraints $i<j$ prevents $i=j=k$ and, consequently, at least one index must always remain unpaired). We can exploit $\mathbb{E}\left[\tilde{b}_{i} \tilde{b}_{j}\right]=p(1-p) \delta_{i, j}$ in the remaining terms to conclude

$$
\begin{aligned}
\operatorname{Var}(S) & =2 p^{2}(1-p)^{2}\left(\|\mathbf{z}\|_{\ell_{2}}^{4}-\|\mathbf{z}\|_{\ell_{4}}^{4}\right)+4 p^{3}(1-p)\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle^{2}\|\mathbf{z}\|_{\ell_{2}}^{2} \\
& +4 p^{2}(1-p)(1-2 p)\left\langle\mathbf{1}_{n}, \mathbf{z}\right\rangle \sum_{i=1}^{n} z_{i}^{3}+p(1-p)(1-2 p)^{2}\|\mathbf{z}\|_{\ell_{4}}^{4}
\end{aligned}
$$

Slightly rewriting this expression then yields the result presented in (38)

## References

[1] Y. Vardi, "Network Tomography: Estimating Source-Destination Traffic Intensities from Link Data," J. Am. Stat. Assoc., vol. 91, no. 433, pp. 365-377, 1996.
[2] R. Castro, M. Coates, G. Liang, R. Nowak, and B. Yu, "Network Tomography: Recent Developments," Stat. Sci., vol. 19, pp. 499-517, 2004.
[3] J. E. Boyd and J. Meloche, "Evaluation of statistical and multiple-hypothesis tracking for video traffic surveillance," Mach. Vision Appl., vol. 13, no. 5-6, pp. 344-351, 2003.
[4] D. L. Donoho, I. M. Johnstone, J. C. Hoch, and S. A. S, "Maximum Entropy and the Nearly Black Object," J. Roy. Stat. Soc. B Met., vol. 54, no. 1, 1992.
[5] J. J. Fuchs, "Sparsity and uniqueness for some specific under-determined linear systems," ICASSP, IEEE International Conference on Acoustics, Speech and Signal Processing - Proceedings, vol. V, pp. 729-732, 2005.
[6] G. Zhang, S. Jiao, X. Xu, and L. Wang, "Compressed sensing and reconstruction with Bernoulli matrices," in IEEE International Conference on Information and Automation, 2010, pp. 455-460.
[7] M. A. Khajehnejad, A. G. Dimakis, W. Xu, and B. Hassibi, "Sparse recovery of nonnegative signals with minimal expansion," IEEE Trans. on Signal Process., vol. 59, no. 1, pp. 196-208, 2011.
[8] N. Meinshausen, "Sign-constrained least squares estimation for high-dimensional regression," Electron. J. Stat., vol. 7, no. 1, pp. 1607-1631, 2013.
[9] M. Slawski and M. Hein, "Sparse recovery by thresholded non-negative least squares," Electron. J. Stat., vol. 7, 2013.
[10] S. Foucart and D. Koslicki, "Sparse Recovery by Means of Nonnegative Least Squares," IEEE Signal Proc. Let., vol. 21, no. 4, 2014.
[11] D. L. Donoho and J. Tanner, "Sparse nonnegative solution of underdetermined linear equations by linear programming." P. Natl. Acad. Sci. USA, vol. 102, no. 27, pp. 9446-9451, 2005.
[12] A. M. Bruckstein, M. Elad, and M. Zibulevsky, "On the uniqueness of non-negative sparse \& redundant representations," ICASSP, IEEE International Conference on Acoustics, Speech and Signal Processing - Proceedings, no. 796, pp. 5145-5148, 2008.
[13] S. Foucart and H. Rauhut, A Mathematical Introduction to Compressive Sensing, ser. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2013.
[14] M. Rudelson and S. Zhou, "Reconstruction from anisotropic random measurements," IEEE Trans. Inform. Theory, vol. 59, no. 6, pp. 3434-3447, June 2013.
[15] "RIPless compressed sensing from anisotropic measurements," Lin. Alg. Appl., vol. 441, pp. 110 - 123, 2014.
[16] S. Mendelson, "Learning without concentration," J. ACM, vol. 62, no. 3, pp. 21:1-21:25, Jun. 2015.
[17] V. Koltchinskii and S. Mendelson, "Bounding the smallest singular value of a random matrix without concentration," Int. Math. Res. Notices, vol. 2015, no. 23, pp. 12991-13008, 2015.
[18] J. A. Tropp, Sampling Theory, a Renaissance: Compressive Sensing and Other Developments. Cham: Springer International Publishing, 2015, ch. Convex Recovery of a Structured Signal from Independent Random Linear Measurements, pp. 67-101.
[19] S. Dirksen, G. Lecue, and H. Rauhut, "On the gap between restricted isometry properties and sparse recovery conditions," IEEE Trans. Inform. Theory, vol. PP, no. 99, pp. 1-1, 2016.
[20] E. J. Candes, T. Strohmer, and V. Voroninski, "Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming," Commun Pure Appl. Math., vol. 66, no. 8, pp. 1241-1274, 2013.
[21] Y. Chang, P. Jung, C. Zhou, and S. Stanczak, "Block compressed sensing based distributed resource allocation for m2m communications," in IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP). IEEE, 2016, pp. 3791-3795.
[22] J. P. Vila and P. Schniter, "An empirical-bayes approach to recovering linearly constrained non-negative sparse signals," IEEE Trans. Signal Process., vol. 62, no. 18, pp. 4689-4703, 2014.
[23] M. Wang, W. Xu, and A. Tang, "A unique "nonnegative" solution to an underdetermined system: From vectors to matrices," IEEE Trans. Inform. Theory, vol. 59, no. 3, pp. 1007-1016, 2011.
[24] R. Baraniuk, M. Davenport, R. A. DeVore, and M. Wakin, "A Simple Proof of the Restricted Isometry Property for Random Matrices," Constr. Approx., vol. 28, no. 3, pp. 253-263, jan 2008.
[25] E. J. Candes, "The restricted isometry property and its implications for compressed sensing," CR Acad. Sci. I-Math, vol. 346, no. Paris, Serie I, pp. 589-592, may 2008.
[26] C. Giraud, S. Huet, N. Verzelen et al., "High-dimensional regression with unknown variance," Stat. Sci., vol. 27, no. 4, pp. 500-518, 2012.
[27] T. Sun and C.-H. Zhang, "Scaled sparse linear regression," preprint arXiv:1104.4595, 2011.
[28] N. Städler, P. Bühlmann, and S. van de Geer, "Rejoinder: $\ell_{1}$-penalization for mixture regression models," Test, vol. 19, pp. 209-256, 2010.
[29] M. Kabanava, R. Kueng, H. Rauhut, and U. Terstiege, "Stable low-rank matrix recovery via null space properties," Inf. Inference, vol. 5, no. 4, pp. 405-441, 2016.
[30] M. Kabanava and H. Rauhut, "Analysis $\ell_{1}$-recovery with frames and gaussian measurements," Acta Appl. Math., vol. 140, no. 1, pp. 173-195, 2015.
[31] M. Rudelson and R. Vershynin, "On sparse reconstruction from fourier and gaussian measurements," Commun. Pure Appl. Math., vol. 61, no. 8, pp. 1025-1045, 2008.
[32] V. Buldygin and K. Moskvichova, "The sub-gaussian norm of a binary random variable," Theory Probab. Math. Stat., vol. 86, pp. 33-49, 2013.
[33] R. Kueng and J. P., "Robust Nonnegative Sparse Recovery and 0/1-Bernoulli Measurements," in IEEE Inf. Theory Workshop (ITW), 2016.
[34] C. L. Lawson and R. J. Hanson, Solving Least Squares Problems. Prentice-Hall, 1974.
[35] E. van den Berg and M. P. Friedlander, "Probing the pareto frontier for basis pursuit solutions," SIAM J. Sci. Comput., vol. 31, no. 2, pp. 890-912, 2008.
[36] R. Vershynin, "Introduction to the non-asymptotic analysis of random matrices," in Compressed sensing. Cambridge Univ. Press, Cambridge, 2012, pp. 210-268.


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[^1]:    ${ }^{1}$ This can be justified under certain assumptions like pre-multiplications using channel reciprocity in time-division multiplexing.
    ${ }^{2}$ See Eq. (9) below for a precise definition.

[^2]:    ${ }^{3}$ Recall that a matrix is stochastic, if all entries are non-negative and all columns sum up to one.

