# Infinity-Norm Permutation Covering Codes from Cyclic Groups 

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#### Abstract

We study covering codes of permutations with the $\ell_{\infty}$-metric. We provide a general code construction, which uses smaller building-block codes. We study cyclic transitive groups as building blocks, determining their exact covering radius, and showing linear-time algorithms for finding a covering codeword. We also bound the covering radius of relabeled cyclic transitive groups under conjugation.


## Index Terms

covering codes, $\ell_{\infty}$-metric, relabeling, cyclic group

## I. Introduction

CODING over permutations appears in the literature as early as the works [3], [21]. In a typical setting, the symmetric group of permutations, $S_{n}$, is endowed with a distance function, $d: S_{n} \times S_{n} \rightarrow \mathbb{N}_{0}$, to create a metric. An error correcting code is then defined as a set $C \subseteq S_{n}$, the elements of which are called codewords, such that $d(f, g) \geqslant d_{\text {min }}$, for all $f, g \in C$, $f \neq g$. The largest such $d_{\min }$ is called the minimum distance of the code. It is also well known that $C$ induces a packing of the space, $S_{n}$, by disjoint balls of radius $\left\lfloor\left(d_{\min }-1\right) / 2\right\rfloor$, the packing radius, centered at the codewords.

In this work, we are interested in the dual problem of covering. Instead of packing balls, we are interested in the smallest radius of balls, centered at the codewords, such that their union covers the entire space. This radius is called the covering radius of the code. Equivalently, we are looking for the smallest $r_{\min } \in \mathbb{N}_{0}$ such that every $f \in S_{n}$ has a codeword $g \in C$ with $d(f, g) \leqslant r_{\text {min }}$.

Covering codes over permutations have only recently been studied in depth, starting with the work of [2], and following with [13], [19], all of which only use the Hamming distance over permutations. In [2], the exact size of covering codes over $S_{n}$ and covering radius $n-1$ is found, and bounds are given on the size of covering codes with smaller covering radius. In [13], the authors present a randomized construction for a code and use a certain frequency parameter to bound the covering radius of the code. A survey of error-correcting codes and covering codes over permutations is given in [19].

Motivated by applications to information storage in non-volatile memories, the rank-modulation scheme was recently suggested [10], in which information is stored in the form of permutations. The relevant permutation metrics for this scheme are mainly the $\ell_{\infty}$-metric and Kendall's- $\tau$ metric. Thus, we have works studying error-correcting codes [1], [6], [11], [17], [18], [22], [23], [27], [29], Gray codes and snake-in-the-box codes [8], [9], [25], [26], [28], and related combinatorial questions [15], [16], [20].

Covering codes over permutations with the $\ell_{\infty}$-metric have only been studied in [5], [24]. In [24], various connections between different metrics over permutations were found, thus enabling code construction in the $\ell_{\infty}$-metric based on codes in other metrics. Additionally, bounds on code parameters were given, which were later improved in [5], together with an explicit direct code construction.

The main contribution of this paper is a generalization of the code construction from [5]. This generalization requires smaller building-block covering codes. We study one such building-block code in detail - a cyclic transitive group of $S_{n}$. We derive the exact covering radius of this group, as well as bound its covering radius after relabeling (conjugation). We also provide linear-time covering-codeword algorithm for the codes.

The paper is organized as follows. In Section $\Pi$ we introduce formal definitions and notations used throughout the paper. Section III is devoted to the derivation of the covering radius of the naturally labeled cyclic transitive group. In Section IV we describe the generalized code construction, as well as linear-time algorithms associated with it. We then turn in Section $\nabla$ to studying relabeling of the building-block code and finding bounds on its covering radius. We conclude in Section VI by discussing the results and suggesting open problems.

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## II. Notations and Definitions

For $m, m^{\prime} \in \mathbb{N}$, we denote $\left[m, m^{\prime}\right] \triangleq\left\{m, m+1, \ldots, m^{\prime}\right\}$, as well as $[m] \triangleq[1, m]$. For ease of notation, we write $m \bmod ^{+} n$ to denote the unique $r \in[n]$ such that $n$ divides $m-r$. We then define the cyclic interval

$$
\left[m, m^{\prime}\right] \bmod ^{+} n \triangleq\left\{m \bmod ^{+} n,(m+1) \bmod ^{+} n, \ldots, m^{\prime} \bmod ^{+} n\right\}
$$

The symmetric group of permutations is denoted by $S_{n}$. As will be evident later, it is important for us to fix the permuted elements. Thus, a permutation $f \in S_{n}$ is a bijection between $[n]$ and itself. We shall use either a one-line notation for permutations, where $f=\left[f_{1}, f_{2}, \ldots, f_{n}\right]$ denotes a permutation mapping $i \mapsto f_{i}$ for all $i \in[n]$, or a cycle notation $f=$
 denoted by $f g$, where $(f g)(i)=f(g(i))$ for all $i \in[n]$. The identity permutation is denoted by Id.

The metric of interest in this work is the $\ell_{\infty}$-metric, sometimes also called the Chebyshev metric. The distance function in this metric, denoted $d_{\infty}: S_{n} \times S_{n} \rightarrow \mathbb{N}_{0}$, is defined for all $f, g \in S_{n}$ by

$$
d_{\infty}(f, g) \triangleq \max _{i \in[n]}|f(i)-g(i)|
$$

Since this will be the only distance function of interest, we shall drop the $\infty$ subscript and use only $d$. We note that for all $f, g \in S_{n}$, we have $d(f, g) \leqslant n-1$. It is well known (e.g., see [4]) that $d$ is right invariant (but not left invariant), i.e., for all $f, g, h \in S_{n}$,

$$
d(f h, g h)=d(f, g)
$$

A code $C$ is simply a subset $C \subseteq S_{n}$. Sometimes $C$ will also be a subgroup of $S_{n}$, in which case we may refer to $C$ as a group code. For such a code $C \subseteq S_{n}$, and $f \in S_{n}$, we define the distance between $f$ and $C$ by

$$
d(f, C) \triangleq \min _{g \in C} d(f, g)
$$

The main object of study in this work is now defined.
Definition 1. An $(n, M, r)$ covering code is a subset $C \subseteq S_{n}$, such that $|C|=M$ and $d(f, C) \leqslant r$ for all $f \in S_{n}$, and $r$ is the minimal integer with this property.

Given an $(n, M, r)$ covering code $C$, we call $r(C) \triangleq r$ the covering radius of $C$. In an asymptotic setting it will be useful to define the rate of the code, and its normalized covering radius by

$$
R(C) \triangleq \frac{\log _{2} M}{n}, \quad \rho(C) \triangleq \frac{r}{n-1}
$$

The main focus throughout this paper involves cyclic groups. Since the distance function crucially depends on the permuted elements, we need to define a "natural" description of these group. Additionally, to avoid degenerate cases, we shall only examine transitive cyclic groups. We therefore give the following definition.
Definition 2. For all $n \in \mathbb{N}$, the (natural, transitive) cyclic group, denoted $G_{n} \leqslant S_{n}$, is the group generated by the permutation $(1,2, \ldots, n)$, i.e.,

$$
\begin{equation*}
G_{n} \triangleq\langle(1,2, \ldots, n)\rangle=\left\{(1,2, \ldots, n)^{k}: k \in \mathbb{Z}\right\} \tag{1}
\end{equation*}
$$

It will additionally be helpful to have a notation for permutations that are close enough to the code. If $f, g \in S_{n}$ and $d(f, g) \leqslant \tilde{r}$, we say $f$ is $\tilde{r}$-covered by $g$, and otherwise, we say $f$ is $\tilde{r}$-exposed by $g$. If $C \subseteq S_{n}$ is a code, and $f \in S_{n}$ is $\tilde{r}$-covered by at least one $g \in C$, i.e., $d(f, C) \leqslant \tilde{r}$, we say $f$ is $(\tilde{r}, C)$-covered. Otherwise, $f$ is $\tilde{r}$-exposed by every $g \in C$, and we say $f$ is $(\tilde{r}, C)$-exposed. In the latter case, for every $g \in C$, there exists $i \in[n]$ such that $|f(i)-g(i)|>\tilde{r}$, and we say that the mapping $i \mapsto f(i)$ is $\tilde{r}$-exposed by $g$.

## III. The Covering Radius of the Cyclic Group

In this section we determine the covering radius of the natural transitive cyclic group. This will later be used as a component in a more general construction for covering codes. We first present two bounds on the covering radius, that nearly agree. We then close the small gap to obtain the exact covering radius.

Throughout this section, let $G_{n} \leqslant S_{n}$ denote the natural transitive cyclic group of (1). Since for $n=1,2$, we have $G_{n}=S_{n}$, we trivially have $r\left(G_{1}\right)=r\left(G_{2}\right)=0$. Thus, in what follows we focus on $n \geqslant 3$.

If $f \in S_{n}$ is some permutation, $H \leqslant S_{n}$ a subgroup, and $\tilde{r} \in \mathbb{N}$, we define

$$
A_{i \mapsto f(i)}^{H} \triangleq\left\{h^{-1}(1): i \mapsto f(i) \text { is } \tilde{r} \text {-exposed by } h \in H\right\}
$$

Since we will be mainly interested in the case of $H=G_{n}$, we define

$$
A_{i \mapsto f(i)} \triangleq A_{i \mapsto f(i)}^{G_{n}}
$$

We also define the two sets

$$
B \triangleq[n-\tilde{r}-1] \quad T \triangleq[\tilde{r}+2, n],
$$

for the bottom and top parts of the range $[n]$. In these definitions, to keep the notation simple, the dependence on $n$ and $\tilde{r}$ is implicit. Some simple observations are formalized in the next two lemmas.
Lemma 3. Let $f \in S_{n}$ be any permutation, and $\tilde{r} \in \mathbb{N}$. If $H \leqslant S_{n}$ is a transitive group, $|H|=n$, then $f$ is $(\tilde{r}, H)$-exposed if and only if

$$
\begin{equation*}
\bigcup_{i \in[n]} A_{i \mapsto f(i)}^{H}=[n] . \tag{2}
\end{equation*}
$$

Proof: If (2] holds, since $|H|=n$, it follows that every $h \in H \tilde{r}$-exposes $f$, hence $f$ is $(\tilde{r}, H)$-exposed. In the other direction, if $f$ is $\left(\tilde{r}, G_{n}\right)$-exposed, then every $g \in G_{n} \tilde{r}$-exposes some mapping $i \mapsto f(i)$. Since $\bigcup_{g \in G_{n}}\left\{g^{-1}(1)\right\}=[n]$, the claim follows.
Lemma 4. Let $\tilde{r}, n \in \mathbb{N}, \tilde{r} \geqslant \frac{n}{2}-1$, and $H \leqslant S_{n}$ a transitive subgroup, $|H|=n$. Then for all $i, j \in[n]$,

$$
\left|A_{i \mapsto j}^{H}\right|= \begin{cases}n-\tilde{r}-j & j \in B=[n-\tilde{r}-1], \\ j-\tilde{r}-1 & j \in T=[\tilde{r}+2, n], \\ 0 & \text { otherwise. }\end{cases}
$$

In particular, for $H=G_{n}$, for all $j_{B} \in B, j_{T} \in T$, and $i_{B}, i_{T} \in[n]$,

$$
\begin{aligned}
& A_{i_{B} \rightarrow j_{B}}=\left[i_{B}+1, i_{B}+n-\tilde{r}-j_{B}\right] \bmod ^{+} n, \\
& A_{i_{T} \rightarrow j_{T}}=\left[i_{T}-j_{T}+\tilde{r}+2, i_{T}\right] \bmod ^{+} n .
\end{aligned}
$$

Proof: Consider the first claim. If $i \mapsto j, j \in B$, is $\tilde{r}$-exposed by some $h \in H$, then $h(i) \in[j+\tilde{r}+1, n]$. Thus, since $H$ is transitive and $|H|=n$, there are exactly $n-\tilde{r}-j$ such $h \in H$, proving the claim regarding the size of $A_{i \mapsto j}^{H}$.

Additionally, when considering $H=G_{n} \triangleq\langle(1,2, \ldots, n)\rangle$, we know $h^{-1}(1)=\left(i_{B}-h\left(i_{B}\right)+1\right) \bmod ^{+} n$. Combining this with the range of $h\left(i_{B}\right)$ we get

$$
A_{i_{B} \rightarrow j_{B}}=\left[i_{B}+1, i_{B}+n-\tilde{r}-j_{B}\right] \bmod ^{+} n
$$

The rest of the claims, involving $T, i_{T}$, and $j_{T}$, are proven symmetrically.
We can now prove an upper bound on the covering radius of $G_{n}$.
Lemma 5. For all $n \in \mathbb{N}, n \geqslant 3$,

$$
r\left(G_{n}\right) \leqslant n-\left\lceil\frac{\sqrt{4 n+1}-1}{2}\right\rceil
$$

Proof: Let $f \in S_{n}$ be any permutation, and consider any $\tilde{r} \in \mathbb{N}$ in the range $\frac{n}{2}-1 \leqslant \tilde{r} \leqslant n-1$. Using Lemma 4

$$
\begin{equation*}
\left|\bigcup_{i \in n} A_{i \mapsto f(i)}\right| \leqslant 2 \sum_{i=1}^{n-\tilde{r}-1} i=(n-\tilde{r}-1)(n-\tilde{r}) . \tag{3}
\end{equation*}
$$

By Lemma 3 if

$$
\begin{equation*}
(n-\tilde{r}-1)(n-\tilde{r})<n, \tag{4}
\end{equation*}
$$

then $f$ is $\left(\tilde{r}, G_{n}\right)$-covered. The smallest value of $\tilde{r}$ that satisfies (4) is

$$
\tilde{r}=n-\left\lceil\frac{\sqrt{4 n+1}-1}{2}\right\rceil \text {, }
$$

and since for any $\tilde{r}$ that satisfies (4) we have $r\left(G_{n}\right) \leqslant \tilde{r}$, we obtain the desired bound.
We now move on to a lower bound on the covering radius of $G_{n}$.
Lemma 6. For all $n \in \mathbb{N}, n \geqslant 3$,

$$
r\left(G_{n}\right) \geqslant n-\left\lfloor\frac{\sqrt{4 n+1}+1}{2}\right\rfloor .
$$

Proof: By simple inspection, $r\left(G_{3}\right)=1$, agreeing with the claim. We therefore focus on the remaining case of $n \geqslant 4$. For convenience we define

$$
a \triangleq\left\lfloor\frac{\sqrt{4 n+1}+1}{2}\right\rfloor \quad \tilde{r} \triangleq n-a-1 .
$$

The proof strategy is the following: we shall define a permutation $f_{0} \in S_{n}$ and show that $f_{0}$ is $\left(\tilde{r}, G_{n}\right)$-exposed. It would then follow that $r\left(G_{n}\right) \geqslant \tilde{r}+1=n-a$, which would complete the proof.

We construct a permutation $f_{0} \in S_{n}$ as follows:

$$
\begin{align*}
f_{0}(i) & \triangleq \begin{cases}n-a+k & i=\binom{k+1}{2}, k \in[a], \\
a-\ell+1 & i=2\binom{a+1}{2}-1-\binom{\ell+1}{2}, \ell \in[a], \\
\text { arbitrary } & \text { otherwise, }\end{cases}  \tag{5}\\
& = \begin{cases}j_{T} & i=\left(\begin{array}{c}
a-\left(n-j_{T}\right)+1
\end{array}\right), j_{T} \in T, \\
j_{B} & i=2\binom{a+1}{2}-1-\binom{a-\left(j_{B}-1\right)+1}{2}, j_{B} \in B, \\
\text { arbitrary } & \text { otherwise, }\end{cases}
\end{align*}
$$

for all $i \in[n]$, and where arbitrary entries are set in a way that completes $f_{0}$ to a permutation.
We first contend that $f_{0}$ is well defined. We note that since $n \geqslant 4$ we have $B \cap T=\varnothing$, so the values in the range of $f_{0}$ are distinct. As for the domain, the first two cases of (5) are disjoint, since otherwise we would have $k, \ell \in[a]$ such that

$$
\binom{k+1}{2}+\binom{\ell+1}{2}=2\binom{a+1}{2}-1
$$

This obviously does not hold for $k=\ell=a$, as well as $k, \ell \in[a-1]$. The only remaining case is when $\{k, \ell\}=\{a, a-1\}$. However, it is easy to verify that

$$
\binom{a+1}{2}+\binom{a}{2}=2\binom{a+1}{2}-1
$$

only when $a=1$, which is never the case when $n \geqslant 4$. Hence, $f_{0}$ is indeed a well defined permutation.
We now proceed with showing that $f_{0}$ is $\left(\tilde{r}, G_{n}\right)$-exposed. By examining the first case of (5) and using Lemma 4 we obtain for all $j_{T} \in T$,

$$
\begin{aligned}
\bigcup_{j_{T} \in T} A_{f_{0}^{-1}\left(j_{T}\right) \mapsto j_{T}} & =\bigcup_{k \in[a]}\left[\binom{k+1}{2}-k+1,\binom{k+1}{2}\right] \bmod ^{+} n \\
& =\left[\binom{a+1}{2}\right] \bmod ^{+} n
\end{aligned}
$$

Symmetrically, let $\ell^{\prime} \in[a]$ be the smallest integer such that

$$
2\binom{a+1}{2}-1-\binom{\ell^{\prime}+1}{2} \leqslant n
$$

Then by Lemma 4

$$
\begin{aligned}
& \bigcup_{j_{B} \in B} A_{f_{0}^{-1}\left(j_{B}\right) \mapsto j_{B}} \\
& \quad=\bigcup_{\ell \in\left[\ell^{\prime}, a\right]}\left[2\binom{a+1}{2}-\binom{\ell+1}{2}, 2\binom{a+1}{2}-\binom{\ell+1}{2}+\ell-1\right] \bmod ^{+} n \\
& \quad=\left[\binom{a+1}{2}, 2\binom{a+1}{2}-1-\binom{\ell^{\prime}}{2}\right] \bmod ^{+} n .
\end{aligned}
$$

We now note that

$$
\begin{aligned}
2\binom{a+1}{2}-1 & =\left\lfloor\frac{\sqrt{4 n+1}+1}{2}\right\rfloor\left(\left\lfloor\frac{\sqrt{4 n+1}+1}{2}\right\rfloor+1\right)-1 \\
& >\frac{\sqrt{4 n+1}-1}{2} \cdot \frac{\sqrt{4 n+1}+1}{2}-1 \\
& =n-1
\end{aligned}
$$

and since the expression on the left-hand side is an integer, we get

$$
2\binom{a+1}{2}-1 \geqslant n
$$

Additionally, the choice of $\ell^{\prime}$ ensures that also

$$
2\binom{a+1}{2}-1-\binom{\ell^{\prime}}{2} \geqslant n
$$

It then follows that

$$
\bigcup_{i \in[n]} A_{i \mapsto f_{0}(i)}=[n],
$$

and by Lemma $3, f_{0}$ is $\left(\tilde{r}, G_{n}\right)$-exposed.
Example 7. For $n=7$, from (5) we get

$$
f_{0}=[5, ?, 6, ?, 1,7, ?]
$$

where? represents entries that can be mapped arbitrarily so as to complete a permutation from $S_{7}$. Denote $g=(1,2,3,4,5,6,7)$, so that $G_{7}=\langle g\rangle$. Table $\square$ shows the entries of $f_{0}$ which were mapped to $B \cup T$, and the permutations $g^{k} \in G_{7}$ by which they are 3 -exposed. It also details the relevant $A_{i \mapsto f_{0}(i)}$ sets. We conclude that $r\left(G_{7}\right) \geqslant 4$, since $f_{0}$ is $\left(3, G_{7}\right)$-exposed. From Lemma[5we have $r\left(G_{7}\right) \leqslant 4$. Thus $r\left(G_{7}\right)=4$.

TABLE I
The entries of $f_{0}$ That are explicit in the proof of Lemma 5 THE PERMUTATIONS in $G_{7}$ By Which they are 3-EXposed, and the relevant $A_{i \mapsto f_{0}(i)}$ SETS.

| $f_{0}$ | 3-exposed by | $A_{i \mapsto f_{0}(i)}$ |
| :---: | :---: | :---: |
| $1 \mapsto 5$ | $g^{0}$ | $A_{1 \mapsto 5}=[1]=\{1\}$ |
| $3 \mapsto 6$ | $g^{5}, g^{6}$ | $A_{3 \mapsto 6}=[2,3]=\{2,3\}$ |
| $6 \mapsto 7$ | $g^{2}, g^{3}, g^{4}$ | $A_{6 \mapsto 7}=[4,6]=\{4,5,6\}$ |
| $5 \mapsto 1$ | $g^{0}, g^{1}, g^{2}$ | $A_{5 \mapsto 1}=[6,8] \bmod ^{+} 7=\{6,7,1\}$ |

The upper bound of Lemma 5 and the upper bound of Lemma6 do not match exactly. The gap between the two is eliminated in the following theorem, by improving the upper bound, thus giving the exact covering radius of $G_{n}$.
Theorem 8. For all $n \in \mathbb{N}$,

$$
r\left(G_{n}\right)=n-\left\lfloor\frac{\sqrt{4 n+1}+1}{2}\right\rfloor
$$

Proof: For $n=1,2$ we already know that $r\left(G_{n}\right)=0$, agreeing with the claimed expression. Therefore we consider $n \geqslant 3$. By Lemma 5 and Lemma 6 we have

$$
n-\left\lfloor\frac{\sqrt{4 n+1}+1}{2}\right\rfloor \leqslant r\left(G_{n}\right) \leqslant n-\left\lceil\frac{\sqrt{4 n+1}-1}{2}\right\rceil
$$

Using straightforward analysis, one can see that the lower and upper bounds agree, except when $n=t(t+1), t \in \mathbb{N}$, where there is a gap of 1 between the bounds. To prove the claim we shall strengthen the upper bound to match the lower bound.

For the remainder of the proof we focus on the case of $n=t(t+1), t \in \mathbb{N}$. In this case, there is no need for the floor or ceiling operations, and we would like to prove that

$$
r\left(G_{n}\right)=n-\frac{\sqrt{4 n+1}+1}{2}=t^{2}-1
$$

Denote $\tilde{r} \triangleq t^{2}-1$, and assume to the contrary that there exists $f \in S_{n}$ that is $\left(\tilde{r}, G_{n}\right)$-exposed. Then,

$$
n \stackrel{(\mathrm{a})}{=}\left|\bigcup_{j \in[n]} A_{f^{-1}(j) \mapsto j}\right| \leqslant \sum_{j \in[n]}\left|A_{f^{-1}(j) \mapsto j}\right| \stackrel{(\mathrm{b})}{=}(n-\tilde{r}-1)(n-\tilde{r})=t(t+1)=n
$$

where (a) follows from Lemma 3, and (b) is taken from (3). It follows that the sets $A_{f^{-1}(j) \mapsto j}, j \in[n]$, are all disjoint, and they form a partition of $[n]$.

Define a $B$-set to be any set of the form $A_{f-1\left(j_{B}\right) \mapsto j_{B}}$, with $j_{B} \in B$, and a $T$-set to be any set $A_{f-1\left(j_{T}\right) \mapsto j_{T}}$, with $j_{T} \in T$. Since $\tilde{r} \geqslant \frac{n}{2}-1$, we have $B \cap T=\varnothing$, and thus no $B$-set is also a $T$-set. As noted above, the $B$-sets and $T$-sets partition [n], and therefore there exists some $T$-set immediately to the left (cyclically) of a $B$-set. More precisely, there exist $j_{B} \in B$ and $j_{T} \in T$ such that

$$
\begin{aligned}
A_{f^{-1}\left(j_{T}\right) \mapsto j_{T}} & =\left[k, k+\ell_{T}\right] \bmod ^{+} n \\
A_{f^{-1}\left(j_{B}\right) \mapsto j_{B}} & =\left[k+\ell_{T}+1, k+\ell_{T}+\ell_{B}\right] \bmod ^{+} n
\end{aligned}
$$

for some $k, \ell_{B}, \ell_{T} \in[n]$. But by Lemma 4

$$
\begin{aligned}
A_{f^{-1}\left(j_{T}\right) \rightarrow j_{T}} & =\left[f^{-1}\left(j_{T}\right)-j_{T}+\tilde{r}+2, f^{-1}\left(j_{T}\right)\right] \bmod ^{+} n, \\
A_{f^{-1}\left(j_{B}\right) \rightarrow j_{B}} & =\left[f^{-1}\left(j_{B}\right)+1, f^{-1}\left(j_{B}\right)+n-\tilde{r}-j_{B}\right] \bmod ^{+} n,
\end{aligned}
$$

implying $f^{-1}\left(j_{B}\right)=f^{-1}\left(j_{T}\right)$, and therefore $j_{B}=j_{T}$, but then $B \cap T \neq \varnothing$, a contradiction.

## IV. Codes Constructed from the Cyclic Group

Using $G_{n}$ as a covering code, now that its covering radius has been determined, has severe limitations. Most notably, there is just one code of each length, and no flexibility in code parameters. We overcome this by providing a more general code construction which uses $G_{n}$ as an internal building block. This construction is a generalization of the covering-code construction of [5]. It enables us to construct a covering code $C_{n} \subseteq S_{n}$, using existing covering codes $C_{m} \subseteq S_{m}, m \leqslant n$.

## A. Code Construction and Parameters

Before describing the construction we first define permutation projections.
Definition 9. Let $I=\left\{i_{1}, i_{2}, \ldots, i_{m}\right\} \subseteq[n]$ be a subset of indices, $i_{1}<i_{2}<\cdots<i_{m}$. For a permutation $f \in S_{n}$ we define $\left.f\right|_{I}$ to be the permutation in $S_{m}$ that preserves the relative order of the sequence $f\left(i_{1}\right), f\left(i_{2}\right), \ldots, f\left(i_{m}\right)$, i.e., $g=\left.f\right|_{I}$ if for all $j, j^{\prime} \in[m]$, we have $g(j)<g\left(j^{\prime}\right)$ if and only if $f\left(i_{j}\right)<f\left(i_{j^{\prime}}\right)$. We also define

$$
\left.f\right|^{I} \triangleq\left(\left.f^{-1}\right|_{I}\right)^{-1}
$$

Intuitively, from the definition above, to compute $\left.f\right|_{I}$ we take its one-line notation, keep only the coordinates of $f$ from $I$, and then rename them to the elements of $[m]$ while keeping the relative order. In contrast, to compute $\left.f\right|^{I}$, we keep only the one-line notation values of $f$ that are from $I$, and rename those to $[m]$ while keeping the relative order.
Example 10. Let $n=6, f=[6,1,3,5,2,4] \in S_{6}$, and $I=\{3,5,6\}$. Then

$$
\left.f\right|_{I}=[2,1,3]
$$

since we keep entries 3 , 5 , and 6 of $f$, giving us $[3,2,4]$, which we then rename to $[2,1,3]$. Similarly, we have

$$
\left.f\right|^{I}=[3,1,2],
$$

since we keep the values 3,5 , and 6 of $f$, giving us $[6,3,5]$, which we then rename to $[3,1,2]$.
To simplify notation, it will become convenient to define a projection using the empty set. Thus, for $I=\varnothing$ and $f \in S_{n}$ we define $\left.f\right|_{I}=\left.f\right|^{I} \triangleq[]$, where [] denotes the unique permutation over zero elements.

We now present the code construction.
Construction A. Let $m, n \in \mathbb{N}, m \leqslant n$. We define the indices sets

$$
I_{i} \triangleq[i m+1,(i+1) m] \cap[n]
$$

for all $i \in\left[0,\left\lfloor\frac{n}{m}\right\rfloor\right]$. We construct the code $C_{n} \subseteq S_{n}$ defined by

$$
C_{n} \triangleq\left\{f \in S_{n}:\left.f\right|^{I_{i}} \in C_{\left|I_{i}\right|}, i \in\left[0,\left\lfloor\frac{n}{m}\right\rfloor\right]\right\}
$$

where $C_{\left|I_{i}\right|} \subseteq S_{\left|I_{i}\right|}$ are covering codes, called the building-block codes.
We note that in the above construction, all the indices sets are of size $m$, except for the last one which is of size $n$ mod $m$. Thus, when $m \mid n$ the last indices set is empty, and $C_{0} \triangleq\{[]\}=\{\mathrm{Id}\} \subseteq S_{0}$ is degenerate, containing only the unique empty (identity) permutation. We define $r\left(C_{0}\right) \triangleq 0$. We also mention that a more general construction is possible, in which the indices sets form an arbitrary partition of $[n]$.

The code construction of [5] is a special case of Construction A] in which $C_{m} \triangleq\{\operatorname{Id}\} \subseteq S_{m}$, and $C_{n \bmod m} \triangleq\{\operatorname{Id}\} \subseteq$ $S_{n \bmod m}$.
Lemma 11. The code $C_{n}$ from Construction is an $(n, M, r)$ code, where

$$
M=\frac{n!}{(m!)^{\lfloor n / m\rfloor}(n \bmod m)!}\left|C_{m}\right|^{\lfloor n / m\rfloor}\left|C_{n \bmod m}\right|
$$

and

$$
r=\max \left\{r\left(C_{m}\right), r\left(C_{n \bmod m}\right)\right\}
$$

Proof: The cardinality of the code, $M$, is easily obtainable by noting that we first need to partition the $n$ coordinates into $\left\lfloor\frac{n}{m}\right\rfloor$ sets of size $m$, and one set of size $n \bmod m$. There are

$$
\binom{n}{m, m, \ldots, m, n \bmod m}=\frac{n!}{(m!)^{\lfloor n / m\rfloor}(n \bmod m)!}
$$

ways of doing so. We then assign values to each set from the corresponding set $I_{i}$. The number of ways to do so is exactly $\left|C_{m}\right|^{\lfloor n / m\rfloor}\left|C_{n \bmod m}\right|$.

The covering radius is also straightforward. Given a permutation $f \in S_{n}$, assume the values of $I_{i}$ are found in positions given by $J_{i} \subseteq[n]$. By the properties of the code $C_{\left|I_{i}\right|}$, there exists a codeword $g \in C_{n}$, such that the restrictions of $f$ and $g$ to positions $J_{i}$ are at most $r\left(C_{\left|I_{i}\right|}\right)$ distance apart. Since we can make this hold for all $i \in\left[0,\left\lfloor\frac{n}{m}\right\rfloor\right]$ simultaneously, we have

$$
r \leqslant \max \left\{r\left(C_{m}\right), r\left(C_{n \bmod m}\right)\right\}
$$

This is met with equality, since we can easily find a permutation $f \in S_{n}$ within this distance from $C_{n}$ : take $f^{\prime} \in S_{m}$ such that $d\left(f^{\prime}, C_{m}\right)=r\left(C_{m}\right)$. Construct $f \in S_{n}$ such that $\left.f\right|^{I_{0}}=f^{\prime}$ and then $d\left(f, C_{n}\right) \geqslant r\left(C_{m}\right)$. If necessary, repeat analogously for $C_{n \bmod m}$ to obtain a permutation $f \in S_{n}$ such that $d\left(f, C_{n}\right) \geqslant r\left(C_{n \bmod m}\right)$.

Next, we take a closer look at this code construction using $G_{n}$ as the building block code.
Corollary 12. Let $m, n \in \mathbb{N}, m \leqslant n$. Then the code $C_{n}$ from Construction A, with building-block codes $C_{m}=G_{m}$ and $C_{n \bmod m}=G_{n \bmod m}$, is an $(n, M, r)$ code, where

$$
M=\left\{\begin{array}{lll}
\frac{n!}{((m-1)!)^{\frac{\pi}{m}}} n! & n \equiv 0 & (\bmod m) \\
\frac{n!}{((m-1)!)^{\left\lfloor\frac{n}{m}\right\rfloor}((n \bmod m)-1)!} & n \neq 0 & (\bmod m)
\end{array}\right.
$$

and

$$
r=m-\left\lfloor\frac{\sqrt{4 m+1}+1}{2}\right\rfloor
$$

Here we use the convention that $G_{0}=\{[]\}$.
Proof: The proof follows from substituting the parameters of the cyclic group into Lemma 11 and noting that $r\left(G_{m}\right)$ is monotone non-decreasing in $m$.
Lemma 13. Let $n, m \in \mathbb{N}, m \leqslant n$. Then the code $C_{n}$ of Construction with $C_{m}=G_{m}$ and $C_{n \bmod m}=G_{n} \bmod m$, has the following rate,

$$
\begin{equation*}
R=-\rho\left\lfloor\frac{1}{\rho}\right\rfloor \log _{2} \rho-\left(1-\rho\left\lfloor\frac{1}{\rho}\right\rfloor\right) \log _{2}\left(1-\rho\left\lfloor\frac{1}{\rho}\right\rfloor\right)+o(1) \tag{6}
\end{equation*}
$$

where $\rho \triangleq \rho\left(C_{n}\right)$ is the normalized covering radius of $C_{n}, R \triangleq R\left(C_{n}\right)$ is the rate of $C_{n}$, and o(1) denotes a function that tends to 0 as $n$ tends to infinity.

Proof: From Corollary 12

$$
\rho=\frac{r\left(C_{n}\right)}{n-1}=\frac{m-\left\lfloor\frac{\sqrt{4 m+1}+1}{2}\right\rfloor}{n-1}=\frac{m}{n}-o(1)
$$

Therefore, $m=n \rho+o(n)$. Notice that $n \bmod m=n-m\left\lfloor\frac{n}{m}\right\rfloor$, hence, by rewriting $\left|C_{n}\right|$ from Corollary 12 we get

$$
\begin{aligned}
\left|C_{n}\right|=2^{R n}= & \frac{n!}{(m!)^{\left\lfloor\frac{n}{m}\right\rfloor}(n \bmod m)!} m\left\lfloor\frac{n}{m}\right\rfloor(n \bmod m) \\
= & \frac{n!}{((n \rho+o(n))!)^{\left\lfloor\frac{n}{n \rho+o(n)}\right\rfloor}\left(n-(n \rho+o(n))\left\lfloor\frac{n}{n \rho+o(n)}\right\rfloor\right)!} \\
& \cdot(n \rho+o(n))^{\left\lfloor\frac{n}{n \rho+o(n)}\right\rfloor\left(n-(n \rho+o(n))\left\lfloor\frac{n}{n \rho+o(n)}\right\rfloor\right) .}
\end{aligned}
$$

It is now a matter of using Stirling's approximation (e.g., [7]),

$$
n!=\left(\frac{n}{e}\right)^{n} 2^{o(n)}
$$

and standard analysis techniques, to arrive at the desired form.
We observe that (6) is the same as the rate obtained by the construction of [5], which uses only $C_{m}=\{\operatorname{Id}\}$. However, the rate is a rather crude measure. Upon closer inspection, we shall now show the code parameters of Corollary 12 are superior to those of [5].

To avoid clutter, let us consider the case of $n=t m$, where $t, m \in \mathbb{N}$. We use Construction A with $C_{m}=G_{m}$ to obtain a code we denote as $C_{n}^{\text {cyc }}$. This code has cardinality given by Corollary 12

$$
M_{n}^{\mathrm{cyc}}=\frac{(m t)!}{((m-1)!)^{t}}
$$

Its covering radius is

$$
r \triangleq r\left(C_{n}^{\mathrm{cyc}}\right)=m-\left\lfloor\frac{\sqrt{4 m+1}+1}{2}\right\rfloor
$$

For a fair comparison with the code of [5], we construct one with the same length $n$, and same covering radius $r$. Such a code is a special case of Construction A using the building-block codes $C_{r+1}=\{\operatorname{Id}\}$ and $C_{n \bmod (r+1)}=\{\operatorname{Id}\}$. We call the resulting code $C_{n}^{\text {Id }}$, and its cardinality (see also [5]) is given by

$$
M_{n}^{\mathrm{Id}}=\frac{(m t)!}{((r+1)!)^{\left\lfloor\frac{n}{r+1}\right\rfloor}(n \bmod (r+1))!}
$$

For the comparison, we first observe that

$$
\begin{equation*}
r \leqslant m-\sqrt{m}+1 \tag{7}
\end{equation*}
$$

We also recall Stirling's approximation in more detail,

$$
\begin{equation*}
\sqrt{2 \pi n}\left(\frac{n}{e}\right)^{n} \leqslant n!\leqslant \sqrt{2 \pi n} \cdot e^{\frac{1}{12 n}}\left(\frac{n}{e}\right)^{n} \tag{8}
\end{equation*}
$$

We now have

$$
M_{n}^{\mathrm{cyc}}=\frac{(t m)!\cdot m^{t}}{(m!)^{t}} \stackrel{\text { (a) }}{\leqslant} \frac{\sqrt{2 \pi t m}\left(\frac{t m}{e}\right)^{t m} e^{\frac{1}{12 t m}} \cdot m^{t}}{\left(\frac{m}{e}\right)^{t m}(2 \pi m)^{\frac{t}{2}}} \stackrel{(\mathrm{~b})}{\leqslant} 2 \sqrt{t} \cdot m^{t} t^{t m}
$$

where (a) is obtained by using (8), and (b) is by rearrangement and noting that $e^{\frac{1}{12 t m}} \leqslant 2$.
To bound $M^{\text {Id }}$ we write

$$
n=t m=q(r+1)+s
$$

where $q, s \in \mathbb{Z}, s \in[0, r]$. We then have

$$
\begin{aligned}
M_{n}^{\mathrm{Id}} & =\frac{(t m)!}{((r+1)!)^{q} \cdot s!} \\
& \stackrel{(a)}{\geqslant} \frac{\sqrt{2 \pi t m}\left(\frac{t m}{e}\right)^{t m}}{(2 \pi(r+1))^{\frac{q}{2}} e^{\frac{q}{12(r+1)}\left(\frac{r+1}{e}\right)^{(r+1) q}} \cdot(2 \pi s)^{\frac{1}{2}} e^{\frac{q}{12(r+1)}+\frac{1}{12 s}}\left(\frac{s}{e}\right)^{s}} \\
& \stackrel{(b)}{\geqslant} \frac{t^{t m}}{\left(\frac{r+1}{m}\right)^{(r+1) q}\left(\frac{s}{m}\right)^{s} 2^{2 t+1}(2 \pi t m)^{t}} \\
& \stackrel{(c)}{\geqslant} \frac{t^{t m}}{\left(\frac{r+1}{m}\right)^{t m} 2^{2 t+1}(2 \pi t m)^{t}} \\
& \stackrel{(\mathrm{~d})}{\geqslant} \frac{t^{t m}}{\left(1-\frac{1}{\sqrt{m}}+\frac{2}{m}\right)^{t m} 2^{2 t+1}(2 \pi t m)^{t}} \\
& \stackrel{t^{t m}}{\geqslant} \frac{\left(e^{-\sqrt{m}+2}\right)^{t} 2^{2 t+1}(2 \pi t m)^{t}}{(1)}
\end{aligned}
$$

where (a) is due to (8), (b) is by rearrangement and noting that $q \leqslant 2 t$, (c) is due to $s \leqslant m$, (d) is due to (7), and (e) is due to $1+x \leqslant e^{x}$. It now follows that

$$
\frac{M^{c y c}}{M^{\mathrm{Id}}} \leqslant 2^{2 t+2} \sqrt{t}(2 \pi t)^{t}\left(m^{2} e^{-\sqrt{m}+2}\right)^{t}
$$

Thus, for any fixed $t \in \mathbb{N}$, and $m$ tending to infinity, the codes $C_{n}^{\text {cyc }}$ are sub-exponentially better than $C_{n}^{\text {Id }}$ of [5] in terms of size.

As a final note, we mention the fact that we may improve the parameters of Corollary 12 by picking $C_{m}=G_{m}$, but $C_{n \bmod m}=\{\operatorname{Id}\}$, whenever $(n \bmod m)-1 \leqslant r\left(G_{m}\right)$, as this would decrease the resulting code size while maintaining its covering radius.

## B. Covering-Codeword Algorithm

A common task associated with covering codes is, given a covering code $C \subseteq S_{n}$ and a permutation $f \in S_{n}$, to find a codeword $g \in C$ such that $d(f, g) \leqslant r(C)$, i.e., find a codeword covering $f$. The code $G_{n}$ is small, and a trivial algorithm measuring the distance between the given $f$ and each of the $n$ codewords of $G_{n}$ (returning an $r\left(G_{n}\right)$-covering codeword) runs in $O\left(n^{2}\right)$ time. However, this might be improved upon, and we now describe a more efficient algorithm.

```
Algorithm 1 Finding a covering codeword \(g \in G_{n}\)
    Input: any permutation \(f \in S_{n}\)
    Output: a codeword \(g \in G_{n}\) with \(d(f, g) \leqslant r\left(G_{n}\right)\)
    Initialization: \(V\) is an array of size \(n, V[i] \leftarrow 0, \forall i \in[n], a \leftarrow\left\lfloor\frac{\sqrt{4 n+1}-1}{2}\right\rfloor\)
    for \(i=1\) to \(n\) do
        if \(f(i) \leqslant a\) then
            for \(j=i+1\) to \(i+a-(f(i)-1)\) do
            \(V\left[j \bmod ^{+} n\right] \leftarrow 1\)
            end for
        else if \(f(i) \geqslant n-a+1\) then
            for \(j=i-(a-(n-f(i)))+1\) to \(i\) do
                \(V\left[j \bmod ^{+} n\right] \leftarrow 1\)
            end for
        end if
    end for
    for \(i=1\) to \(n\) do
        if \(V[i]=0\) then
            return \([n-i+2, \ldots, n, 1, \ldots, n-i+1] \in G_{n}\)
        end if
    end for
```

Lemma 14. Let $n \in \mathbb{N}$ and $f \in S_{n}$. Algorithm 1 returns a codeword $g \in G_{n}$ such that $d(f, g) \leqslant r\left(G_{n}\right)$.
Proof: Let $\tilde{r} \triangleq r\left(G_{n}\right)$, which means $a=n-\tilde{r}-1$. The inner loops on $j$ assign 1 to the entries of $V$ corresponding to the elements of $A_{i \mapsto f(i)}$ (see proof of Lemma 6). Hence, at the end of the first for loop on $i$,

$$
V[i]=0 \Longleftrightarrow i \notin \bigcup_{i \in[n]} A_{i \mapsto f(i)} .
$$

The second for loop on $i$ finds $i \in[n]$ such that $V[i]=0$. From Theorem 8 such $i$ must exist. We conclude that the codeword $g \in G_{n}$, such that $g(i)=1, \tilde{r}$-covers $f$, and we return it.

Algorithm 1 is more efficient than the trivial brute-force algorithm. We note that $a=O(\sqrt{n})$, and therefore, each of the inner loops is entered $O(\sqrt{n})$ times, performing $O(\sqrt{n})$ iterations each time. Thus, in total, the algorithm runs in $O(n)$ time.

Having this algorithm for the building-block code $G_{n}$, we may extend it in a natural way to the code studied in Corollary 12 to also run in $O(n)$ time. We omit the tedious details.

## V. Relabeling the Cyclic Group

Following the definition of the natural transitive cyclic group,

$$
G_{n} \triangleq\langle(1,2, \ldots, n)\rangle \subseteq S_{n}
$$

as given in Definition 2, it is tempting to ask what happens when we take a non-natural transitive cyclic group. Thus, we are interested in the groups of the form

$$
G_{n}^{h} \triangleq h G_{n} h^{-1} \triangleq\left\langle h(1,2, \ldots, n) h^{-1}\right\rangle=\langle(h(1), h(2), \ldots, h(n))\rangle \subseteq S_{n}
$$

for some $h \in S_{n}$. A similar, more general question, was asked in [23], where an error-correcting code $C \subseteq S_{n}$ was relabeled by conjugation,

$$
C^{h} \triangleq h C h^{-1} \triangleq\left\{h g h^{-1} \quad: g \in C\right\}
$$

$h \in S_{n}$, and its minimum distance was studied as a function of $C$ and $h$. It was shown there that the minimum distance could drastically change due to relabeling, moving from the minimum possible 1 , to the maximum possible $n-1$, for some codes. Additionally, every error-correcting code could be relabeled so that its minimum distance is reduced to either 1 or 2 . In this section we study the covering radius of relabelings of $G_{n}$.

Definition 15. Let $C \subseteq S_{n}$ be a covering code. We denote by $\mathcal{L}_{\text {min }}(C)$ (respectively, $\mathcal{L}_{\max }(C)$ ) the minimal (respectively, maximal) achievable covering radius among all relabelings of $C$, i.e.,

$$
\begin{aligned}
& \mathcal{L}_{\text {min }}(C) \triangleq \min _{h \in S_{n}} r\left(C^{h}\right), \\
& \mathcal{L}_{\max }(C) \triangleq \max _{h \in S_{n}} r\left(C^{h}\right) .
\end{aligned}
$$

We first consider $\mathcal{L}_{\text {max }}\left(G_{n}\right)$. Again, the cases of $n=1,2$ are degenerate, and we therefore only consider $n \geqslant 3$.
Theorem 16. For all $n \in \mathbb{N}, n \geqslant 3$,

$$
\mathcal{L}_{\max }\left(G_{n}\right)=n-\left\lceil\frac{\sqrt{4 n+1}-1}{2}\right\rceil .
$$

Proof: Let $h \in S_{n}$ be any permutation. We begin by noting that since $G_{n}$ is a transitive group, so is $G_{n}^{h}$. Thus, Lemma 3 and Lemma 4 apply. Now Lemma 5 also holds for $G_{n}^{h}$ since it only relies on the two above-mentioned lemmas. Thus,

$$
\mathcal{L}_{\max }\left(G_{n}\right) \leqslant n-\left\lceil\frac{\sqrt{4 n+1}-1}{2}\right\rceil .
$$

Additionally, whenever $n \neq t(t+1), t \in \mathbb{N}$, we have by Theorem 8

$$
\mathcal{L}_{\max }\left(G_{n}\right) \geqslant r\left(G_{n}\right)=n-\left\lfloor\frac{\sqrt{4 n+1}+1}{2}\right\rfloor=n-\left\lceil\frac{\sqrt{4 n+1}-1}{2}\right\rceil .
$$

Let us define

$$
a \triangleq\left\lceil\frac{\sqrt{4 n+1}-1}{2}\right\rceil, \quad \tilde{r} \triangleq n-a-1 .
$$

To complete this proof, we must show that for values of $n$ such that $n=t(t+1), t \in \mathbb{N}, t \geqslant 2$, there exists $h \in S_{n}$ such that $r\left(G_{n}^{h}\right)=n-a$. Notice that in this case, $\frac{\sqrt{4 n+1}-1}{2}$ is an integer, which yields $n=a(a+1)$.

We contend that the permutation $h \triangleq(1,2) \in S_{n}$ will suffice, proving it by constructing a permutation $f_{0} \in S_{n}$ such that $f_{0}$ is $\left(\tilde{r}, G_{n}^{h}\right)$-exposed, giving us

$$
r\left(G_{n}^{h}\right) \geqslant d\left(f_{0}, G_{n}^{h}\right) \geqslant \tilde{r}+1=n-a .
$$

We construct a permutation $f_{0} \in S_{n}$ as follows:

$$
f_{0}(i) \triangleq \begin{cases}1 & i=1  \tag{9}\\ n & i=2 \\ n-a+1 & i=3 \\ a-k & i=\binom{k+1}{2}+a+2, k \in[0, a-2] \\ n-a+1+\ell & i=n-a+2-\binom{\ell+1}{2}, \ell \in[a-2] \\ \text { arbitrary } & \text { otherwise },\end{cases}
$$

for all $i \in[n]$, and where arbitrary entries are set in a way that completes $f_{0}$ to a permutation.
We first note that $f_{0}$ is well defined. The domain intervals in the definition are disjoint since $a \geqslant 2, n=a(a+1)=2\binom{a+1}{2}$, and

$$
\binom{a-1}{2}+a+2<2\binom{a+1}{2}-a+2-\binom{a-1}{2} .
$$

As for the range intervals, the fourth and fifth cases in 96 are $[2, a]$ and $[n-a+2, n-1]$ respectively, and are clearly disjoint, and disjoint from the first three cases. These two sets will be of further interest, so we define

$$
\begin{aligned}
& \tilde{B} \triangleq B \backslash\{1\}=[2, a], \\
& \tilde{T} \triangleq T \backslash\{n-a+1, n\}=[n-a+2, n-1] .
\end{aligned}
$$

Thus, $\tilde{B} \cap \tilde{T}=\varnothing$.

With $g \triangleq(1,2, \ldots, n) \in S_{n}$, and $G_{n} \triangleq\langle g\rangle$, we write the elements of $G_{n}^{h}$ explicitly,

$$
\begin{aligned}
& h_{0} \triangleq h g^{0} h^{-1} \\
& h_{1} \triangleq h g^{1} h^{-1} \\
&=[1,2, \ldots, n], \\
& h_{2} \triangleq h g^{2} h^{-1} \\
&=[4,1,4,5, \ldots, n, 2], \\
& h_{i} \triangleq h g^{i} h^{-1} \\
& h_{n-2} \triangleq h g^{n-2} h^{-1} \\
& h_{n-1} \triangleq h g^{n-1} h^{-1} \\
& h_{n}=[n, n, n, n, 2,1], \\
&n, 1,3,4, \ldots, n-1] .
\end{aligned}
$$

To prove that $f_{0}$ is $\left(\tilde{r}, G_{n}^{h}\right)$-exposed we shall use Lemma 3 ,
The mapping $1 \mapsto f_{0}(1)=1$ is $\tilde{r}$-exposed by $\left\{h_{n-a-1}, h_{n-a}, \ldots, h_{n-2}\right\}$, hence,

$$
A_{1 \mapsto 1}^{\mathrm{G}_{n}^{h}}=[4, a+3] .
$$

The mapping $2 \mapsto f_{0}(2)=n$ is $\tilde{r}$-exposed by $\left\{h_{0}, h_{1}, \ldots, h_{a-1}\right\}$, hence

$$
A_{2 \mapsto n}^{G_{n}^{h}}=[n-a+3, n+2] \bmod ^{+} n=\{n-a+3, n-a+4, \ldots, n, 1,2\} .
$$

The mapping $3 \mapsto f_{0}(3)=n-a+1$ is $\tilde{r}$-exposed solely by $h_{n-1}$, thus

$$
A_{2 \mapsto n-a+1}^{\mathrm{G}_{n}^{h}}=\{3\}
$$

Now consider a mapping $i_{B} \mapsto f_{0}\left(i_{B}\right)=j_{B}$, with $j_{B} \in \tilde{B}$, and we get

$$
A_{i_{B} \mapsto j_{B}}^{G_{n}^{h}}=\left[i_{B}+2, i_{B}+2+a-j_{B}\right],
$$

and in total,

$$
\bigcup_{j_{B} \in \tilde{B}} A_{f_{0}^{-1}\left(j_{B}\right) \mapsto j_{B}}^{G_{n}^{h}}=\left[a+4,\binom{a+1}{2}+3\right]=\left[a+4, \frac{n}{2}+3\right]
$$

Similarly, for $i_{T} \mapsto f_{0}\left(i_{T}\right)=j_{T}$ such that $j_{T} \in \tilde{T}$ we get

$$
A_{i_{T} \mapsto j_{T}}^{\mathrm{G}_{n}^{h}}=\left[i_{T}+n-j_{T}-a+2, i_{T}+1\right]
$$

and in total,

$$
\bigcup_{j_{T} \in \tilde{T}} A_{f_{0}^{-1}\left(j_{T}\right) \mapsto j_{T}}^{G_{n}^{h}}=\left[n-\binom{a+1}{2}+4, n-a+2\right]=\left[\frac{n}{2}+4, n-a+2\right] .
$$

In conclusion, taking the union of all the above we obtain

$$
\bigcup_{j \in[n]} A_{f_{0}^{-1}(j) \mapsto j}^{G_{n}^{h}}=[n],
$$

and by Lemma 3 we have that $f_{0}$ is $\left(\tilde{r}, G_{n}^{h}\right)$-exposed.
We now move on to studying $\mathcal{L}_{\text {min }}$. Unlike $\mathcal{L}_{\text {max }}$, we provide only a weak lower bound on $\mathcal{L}_{\text {min }}$, which depends only on the size of the code. We recall the definition of a ball of radius $r$ and centered at $g \in S_{n}$,

$$
\mathcal{B}_{n, r}(g) \triangleq\left\{f \in S_{n}: d(f, g) \leqslant r\right\}
$$

Since the $\ell_{\infty}$-metric is right invariant, the size of a ball does not depend on the choice of center, and thus we denote its size as $\left|\mathcal{B}_{n, r}\right|$.
Lemma 17. Let $C \subseteq S_{n}$ be a code. If $\tilde{r} \in \mathbb{N}$ is such that

$$
\begin{equation*}
|C| \cdot\left|\mathcal{B}_{n, \tilde{r}-1}\right|<\left|S_{n}\right| \tag{10}
\end{equation*}
$$

then

$$
\mathcal{L}_{\min }(C) \geqslant \tilde{r}
$$

Proof: The claim is quite trivial. Inequality (10) simply states that $|C|$ balls of radius $\tilde{r}-1$ cannot cover $S_{n}$, hence $r(C) \geqslant \tilde{r}$. For all $h \in S_{n}$ we have $|C|=\left|C^{h}\right|$, hence $r\left(C^{h}\right) \geqslant \tilde{r}$.

Specializing Lemma 17 to $|C|=n$, gives us the following corollary, which applies to $G_{n}$ as well.

Corollary 18. For all large enough $n \in \mathbb{N}, C \subseteq S_{n},|C|=n$,

$$
\mathcal{L}_{\min }(C) \geqslant n-\lceil\sqrt{2 n \ln n+2 n}\rceil .
$$

Proof: The following upper bound on the size of a ball is given in [14],

$$
\left|\mathcal{B}_{r, n}\right| \leqslant \begin{cases}((2 r+1)!)^{\frac{n-2 r}{2 r+1}} \prod_{i=r+1}^{2 r}(i!)^{\frac{2}{i}} & 0 \leqslant r \leqslant \frac{n-1}{2}, \\ (n!)^{\frac{2 r+2-n}{n}} \prod_{i=r+1}^{n-1}(i!)^{\frac{2}{\tau}} & \frac{n-1}{2} \leqslant r \leqslant n-1,\end{cases}
$$

and whose proof is an immediate application of Bregman's upper bound on the permanent. We contend that only the second case of this bound is of relevance to us, as we will prove shortly. Thus, if we find $\tilde{r} \geqslant \frac{n+1}{2}$ such that

$$
\begin{equation*}
\frac{|C| \cdot\left|\mathcal{B}_{n, \tilde{r}-1}\right|}{\left|S_{n}\right|}=\frac{\left|\mathcal{B}_{n, \tilde{r}-1}\right|}{(n-1)!} \leqslant \frac{1}{(n-1)!}(n!)^{\frac{2 \tilde{F}-n}{n}} \prod_{i=\tilde{r}}^{n-1}(i!)^{\frac{2}{i}}<1, \tag{11}
\end{equation*}
$$

then by Lemma 17 we will have $\mathcal{L}_{\text {min }}(C) \geqslant \tilde{r}$.
Let us therefore define the auxiliary function,

$$
F(n, \tilde{r}) \triangleq \frac{1}{(n-1)!}(n!)^{\frac{2 \tilde{r}-n}{n}} \prod_{i=\tilde{r}}^{n-1}(i!)^{\frac{2}{i}} .
$$

As a first step we show that for all $n \geqslant 11$,

$$
F\left(n,\left\lceil\frac{n+1}{2}\right\rceil\right)<1
$$

Due to parity, we consider the cases of even $n$ and odd $n$ separately. We shall prove the former, and omit the proof for odd $n$ since it is similar. For the case of even $n$, we prove the claim for $n=12$, and then show the function is monotonically decreasing in $n$.

For $n=12$ we have,

$$
F(12,7) \approx 0.9644<1 .
$$

Next, we consider

$$
\begin{aligned}
\frac{F\left(n, \frac{n+2}{2}\right)}{F\left(n+2, \frac{n+4}{2}\right)} & =\frac{n \cdot(n!)^{\frac{2-n}{n}} \cdot \prod_{i=\frac{n+2}{2}}^{n-1}(i!)^{\frac{2}{\tau}}}{(n+2) \cdot((n+2)!)^{-\frac{n}{n+2}} \cdot \prod_{i=\frac{n+4}{2}}^{n+1}(i!)^{\frac{2}{i}}} \\
& =\frac{n(n+1) \cdot\left(\left(\frac{n+2}{2}\right)!\right)^{\frac{4}{n+2}}}{((n+2)!)^{\frac{2}{n+2}} \cdot((n+1)!)^{\frac{2}{n+1}}} \\
& \geqslant \frac{e^{2}}{4} \cdot \frac{n}{(n+1) \cdot(\pi(n+2))^{\frac{1}{(n+1)(n+2)}} \cdot e^{\frac{1}{6}\left(\frac{1}{(n+1)^{2}}+\frac{1}{(n+2)^{2}}\right)}},
\end{aligned}
$$

where for the inequality we used (8) and trivial bounding techniques. We now note that $\exp \left(\frac{1}{6}\left(\frac{1}{(n+1)^{2}}+\frac{1}{(n+2)^{2}}\right)\right)$ and $(\pi(n+2))^{\frac{1}{(n+1)(n+2)}}$ are monotonically decreasing in $n$, and $\frac{n}{n+1}$ is monotonically increasing. Hence,

$$
\frac{F\left(n, \frac{n+2}{2}\right)}{F\left(n+2, \frac{n+4}{2}\right)} \geqslant \frac{F(12,7)}{F(14,8)} \approx 1.649>1,
$$

and so $F\left(n,\left\lceil\frac{n+1}{2}\right\rceil\right)$ is monotonically decreasing in $n$ for even $n$. A similar proof holds for odd $n$.

Thus far we showed there exists $\tilde{r} \geqslant \frac{n+1}{2}$ that satisfies (11) (in particular, $\tilde{r}=\lceil(n+1) / 2\rceil$ does). We would now like to find such $\tilde{r}$ as large as possible. We observe the following sequence of inequalities, where we take $n \geqslant 1$, and $\frac{n+1}{2} \leqslant \tilde{r} \leqslant n-1$.

$$
\begin{align*}
F(n, \tilde{r}) & \triangleq \frac{1}{(n-1)!}(n!)^{\frac{2 \tilde{r}-n}{n}} \prod_{i=\tilde{r}}^{n-1}(i!)^{\frac{2}{i}} \\
& \stackrel{(a)}{\leqslant} n \cdot\left(\frac{n}{e}\right)^{2 \tilde{r}-2 n} \cdot \prod_{i=\tilde{r}}^{n-1}(2 \pi i)^{\frac{1}{i}} e^{\frac{1}{6 i}}\left(\frac{i}{e}\right)^{2} \\
& \stackrel{(\mathrm{~b})}{\leqslant} n \cdot n^{2 \tilde{r}-2 n} \cdot(2 \pi \tilde{r})^{\frac{n-\tilde{r}}{\tilde{r}}} e^{\frac{n-\tilde{r}}{6 \tilde{r}}}\left(\frac{(n-1)!}{(\tilde{r}-1)!}\right)^{2} \\
& \stackrel{\text { (c) }}{\leqslant} n \cdot n^{2 \tilde{r}-2 n} \cdot \pi n e^{\frac{1}{6}} \cdot\left(\frac{(n-1)!}{(\tilde{r}-1)!}\right)^{2} \\
& \stackrel{\text { (d) }}{\leqslant} \pi e^{\frac{1}{6}} e^{\frac{1}{6(n-1)}} e^{2 \tilde{r}-2 n} n^{2} \cdot \frac{\left(\frac{n-1}{n}\right)^{2 n} \frac{1}{n-1}}{\left(\frac{\tilde{r}-1}{n}\right)^{2 \tilde{r}} \frac{1}{\tilde{r}-1}} \\
& \stackrel{(e)}{\leqslant} \pi e^{-\frac{109}{60} e^{2 \tilde{r}-2 n} n^{2} \cdot \frac{1}{\left(\frac{\tilde{r}-1}{\tilde{r}}\right)^{2 \tilde{r}}\left(\frac{\tilde{r}}{n}\right)^{2 \tilde{r}}}} \\
& \stackrel{\text { (f) }}{\leqslant} \pi e^{-\frac{109}{60}}\left(\frac{6}{5}\right)^{12} \cdot e^{2 \tilde{r}-2 n} n^{2}\left(\frac{n}{\tilde{r}}\right)^{2 \tilde{r}} \tag{12}
\end{align*}
$$

where (a) follows from (8), (b) follows by noting that $(2 \pi i)^{\frac{1}{i}}$ and $e^{\frac{1}{6 i}}$ are decreasing in $i$ and then replacing $i$ by $\tilde{r}$, (c) follows by noting that $(2 \pi \tilde{r})^{\frac{n-\tilde{r}}{r}}$ and $e^{\frac{n-\tilde{r}}{6 r}}$ are decreasing in $\tilde{r}$ and replacing $\tilde{r}$ by $\frac{n}{2}$, (d) follows again by use of (8), (e) follows by noting that $\exp \left(\frac{1}{6(n-1)}\right)$ is decreasing in $n$ and substituting $n=11$, that $((n-1) / n)^{2 n} \leqslant e^{-2}$, and that $\frac{\tilde{r}-1}{n-1}<1$, and finally, (f) follows by noting that $((\tilde{r}-1) / \tilde{r})^{2 \tilde{r}}$ is increasing in $\tilde{r}$ and replacing $\tilde{r}$ (since $n \geqslant 11$ and $\tilde{r} \geqslant \frac{n}{2}$ ) by $\tilde{r}=6$.

We note that taking $\tilde{r}=n-\sqrt{2 n \ln n+2 n}$, by 12 we get

$$
\lim _{n \rightarrow \infty} F(n, n-\sqrt{2 n \ln n+2 n}) \leqslant \pi e^{-\frac{109}{60}}\left(\frac{6}{5}\right)^{12} \frac{1}{e^{2}}<1
$$

It now follows that for large enough $n$,

$$
F(n, n-\sqrt{2 n \ln n+2 n})<1
$$

and then

$$
\mathcal{L}_{\min }(C) \geqslant n-\lceil\sqrt{2 n \ln n+2 n}\rceil
$$

as claimed.

## VI. DISCUSSION

In this paper we found the exact covering radius of the (natural) transitive cyclic group, $G_{n}$, and used it to construct new covering codes. These codes often exhibit better parameters than known covering-code constructions, while still allowing a linear-time covering-codeword algorithm.

The methods we described may be extended to larger groups, e.g., the dihedral group, though at a cost of a growing gap between the lower and upper bounds on the covering radius. Thus, in the case of the (naturally labeled) dihedral group, $D_{n} \leqslant S_{n}$, defined by,

$$
D_{n} \triangleq\left\langle(1,2, \ldots, n), \prod_{i=1}^{\lfloor n / 2\rfloor}(i, n-i)\right\rangle
$$

we can obtain

$$
n-\left\lfloor\frac{\sqrt{4 n+1}+1}{2}\right\rfloor \geqslant r\left(D_{n}\right) \geqslant \begin{cases}n-\left\lceil\frac{\sqrt{288 n+297}-3}{16}\right. \\ n-\left\lceil\frac{\sqrt{288 n+737}-1}{16}\right. \\ n-\left\lceil\frac{\sqrt{18 n-18}}{4}\right\rceil & n \in[4,9] \\ n \in[10,911] \\ & n \geqslant 912\end{cases}
$$

The tedious proof follows the same logic as that presented in Section III] and the interested reader may find it in [12]. We believe a more elegant treatment is needed.

Another gap exhibited in this work is between $\mathcal{L}_{\min }\left(G_{n}\right)$ and $\mathcal{L}_{\max }\left(G_{n}\right)$. First, we note an interesting contrast with the case of error-correcting codes (as described in [23]). When relabeling error-correcting codes, the minimum distance of any
code, including $G_{n}$, may be reduced to either 1 or 2 . The minimum distance of $G_{n}$ is $\lceil n / 2\rceil$, and the best possible minimum distance after relabeling is $n-\left\lceil\frac{\sqrt{4 n-3}-1}{2}\right\rceil$, which bears a striking resemblance to $r\left(G_{n}\right)$.

In light of Section III and Section V it appears that the covering radius of $G_{n}$ and its conjugate, has much less variance. This is evident from the small gap between $\mathcal{L}_{\min }\left(G_{n}\right)$ and $\mathcal{L}_{\max }\left(G_{n}\right)$, not to mention the fact that $r\left(G_{n}\right)=\mathcal{L}_{\max }\left(G_{n}\right)$ in most cases. We ran brute-force computer search, checking all possible relabelings of $G_{n}, n \in[3,10]$. For this range,

$$
\mathcal{L}_{\min }\left(G_{n}\right)=r\left(G_{n}\right)=\mathcal{L}_{\max }\left(G_{n}\right)
$$

for all $n \in[3,10] \backslash\{6\}$, and

$$
\mathcal{L}_{\min }\left(G_{6}\right)=r\left(G_{6}\right)=\mathcal{L}_{\max }\left(G_{6}\right)-1
$$

where of the 6 ! labeling permutations, 264 give covering radius $3=r\left(G_{n}\right)$, and 456 give covering radius $4=\mathcal{L}_{\max }\left(G_{n}\right)$. The gap between $r\left(G_{6}\right)$ and $\mathcal{L}_{\max }\left(G_{6}\right)$ is a consequence of Theorem 16 It is now tempting to conjecture that for all $n \in \mathbb{N}$, $\mathcal{L}_{\min }\left(G_{n}\right)=r\left(G_{n}\right)$. We leave this conjecture, and the determination of the covering radius of other groups, as open questions for future work.

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    This work was supported in part by the Israel Science Foundation (ISF) grant No. 130/14.

