

Sampling Constrained Asynchronous Communication: How to Sleep Efficiently

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Abstract—The minimum energy, and, more generally, the minimum cost, to transmit one bit of information has been recently derived for bursty communication when information is available infrequently at random times at the transmitter. Furthermore, it has been shown that even if the receiver is constrained to sample only a fraction $\rho \in (0, 1]$ of the channel outputs, there is no capacity penalty. That is, for any strictly positive sampling rate ρ , the asynchronous capacity per unit cost is the same as under full sampling, *i.e.*, when $\rho = 1$. Moreover, there is no penalty in terms of decoding delay.

The above results are asymptotic in nature, considering the limit as the number B of bits to be transmitted tends to infinity, while the sampling rate ρ remains fixed. A natural question is then whether the sampling rate $\rho(B)$ can drop to zero without introducing a capacity (or delay) penalty compared to full sampling. We answer this question affirmatively. The main result of this paper is an essentially tight characterization of the minimum sampling rate. We show that any sampling rate that grows at least as fast as $\omega(1/B)$ is achievable, while any sampling rate smaller than $o(1/B)$ yields unreliable communication. The key ingredient in our improved achievability result is a new, multi-phase adaptive sampling scheme for locating transient changes, which we believe may be of independent interest for certain change-point detection problems.

Index Terms—Asynchronous communication; bursty communication; capacity per unit cost; energy; change detection; hypothesis testing; sequential analysis; sparse communication; sampling; synchronization; transient change

I. INTRODUCTION

IN many emerging technologies, communication is sparse and asynchronous, but it is essential that

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when data is available, it is delivered to the destination as timely and reliably as possible.

In [3] the authors characterized capacity per unit cost as a function of the level of asynchronism for the following model. There are B bits of information that are made available to the transmitter at some random time ν , and need to be communicated to the receiver. The B bits are encoded into a codeword of length n , and transmitted over a memoryless channel using a sequence of symbols that have costs associated with them. The rate R per unit cost is B divided by the cost of the transmitted sequence. Asynchronism is captured here by the fact that the random time ν is not known *a priori* to the receiver. However, both transmitter and receiver know that ν is distributed uniformly over a time horizon $\{1, 2, \dots, A\}$. At all times before and after the actual transmission, the receiver observes pure noise.

The goal of the receiver is to reliably decode the information bits by sequentially observing the outputs of the channel. A main result in [3] is a single-letter characterization of the asynchronous capacity per unit cost $C(\beta)$, where $\beta = (\log A)/B$ denotes the *timing uncertainty per information bit*. While this result holds for arbitrary discrete memoryless channels and arbitrary input costs, the underlying model assumes that the receiver is always in the listening mode: every channel output is observed until the decoding instant.

In [8] it is shown that even if the receiver is constrained to observe at most a fraction $\rho \in (0, 1]$ of the channel outputs the asynchronous capacity per unit cost $C(\beta, \rho)$ is *not* impacted by a sparse output sampling, that is

$$C(\beta, \rho) = C(\beta)$$

for any asynchronism level $\beta > 0$ and sampling frequency $\rho \in (0, 1]$. Moreover, the decoding delay is minimal: the elapsed time between when information is available sent and when it is decoded

is asymptotically the same as under full sampling. This result uses the possibility for the receiver to sample adaptively: the next sample can be chosen as a function of past observed samples. In fact, under non-adaptive sampling, it is still possible to achieve the full sampling asynchronous capacity per unit cost, but the decoding delay gets multiplied by a factor $1/\rho$. Therefore, adaptive sampling strategies are of particular interest in the very sparse regime.

The results of [8] provide an achievability scheme when the sampling frequency ρ is a strictly positive constant. This suggests the question whether $\rho = \rho(B)$ can tend to zero as B tends to infinity while still incurring no capacity or delay penalty. The main result of this paper resolves this question. We introduce a novel, multi-phase adaptive sampling algorithm for message detection, and use it to prove an essentially tight asymptotic characterization of the minimum sampling rate needed in order to communicate as efficiently as under full sampling. Informally, we exhibit a communication scheme utilizing this multi-phase sampling method at the receiver that asymptotically achieves vanishing probability of error and possesses the following properties:

1. The scheme achieves the capacity per unit cost under full sampling, that is, there is no rate penalty even though the sampling rate tends to zero;
2. The receiver detects the codeword with minimal delay;
3. The receiver detects changes with minimal sampling rate, in the sense that any scheme that achieves the same order of delay but operates at a lower sampling rate will completely miss the codeword transmission period, regardless of false-alarm probability. The sampling rate converges to 0 in the limit of large B , and our main result characterizes the best possible rate of convergence.

In other words, our communication scheme achieves essentially the minimal sampling rate possible, and incurs no delay or capacity penalty relative to full sampling. A formal statement of the main result is given in Section II.

Related works

The above sparse communication model was first introduced in [2], [10]. These works characterize the *synchronization threshold*, *i.e.*, the largest level

of asynchronism under which it is still possible to communicate reliably. In [9], [10] capacity is defined as the message length divided by the mean elapsed time between when information is available and when it is decoded. For this definition, capacity upper and lower bounds are established and shown to be tight for certain channels. In [9] it is also shown that so called training-based schemes, where synchronization and information transmission are performed separately, need not be optimal in particular in the high rate regime. In [3] capacity is defined with respect to codeword length and is characterized as a function of the level of asynchronism. For the same setup Polyanskiy in [5] investigated the finite length regime and showed that in certain cases dispersion is unaffected by asynchronism even when $\beta > 0$.

In [11], [12] the authors investigated the slotted version of the problem (*i.e.*, the decoder is revealed $\nu \bmod n$) and established error exponent tradeoffs between decoding error, false-alarm, and miss-detection.

In [3], [6] the above bursty communication setup is investigated in a random access configuration and tradeoffs between communication rate and number of users are derived as a function of the level of asynchronism. Finally, in [7] a diamond network is considered and the authors provided bounds on the minimum energy needed to convey one bit across the network.

Paper organization

This paper is organized as follows. In Section II, we recall the asynchronous communication model and related prior results. Then, we state our main result, Theorem 3, which is a stronger version of the results in [8]. Section III states auxiliary results, Theorems 4 and 5, characterizing the performance of our multi-phase sampling algorithm. In Section IV we first prove Theorems 4 and 5, then prove Theorem 3. The achievability part of Theorem 3 uses the multi-phase sampling algorithm for message detection at the receiver, and the converse is essentially an immediate consequence of the converse of Theorem 5.

II. MAIN RESULT: THE SAMPLING RATE REQUIRED IN ASYNCHRONOUS COMMUNICATION

Our main result, Theorem 3 below, is a strengthening of the results of [8]. We recall the model and

results (Theorems 1 and 2) of that paper below to keep the paper self-contained.

Communication is discrete-time and carried over a discrete memoryless channel characterized by its finite input and output alphabets

$$\mathcal{X} \quad \text{and} \quad \mathcal{Y},$$

respectively, and transition probability matrix

$$Q(y|x),$$

for all $y \in \mathcal{Y}$ and $x \in \mathcal{X}$. Without loss of generality, we assume that for all $y \in \mathcal{Y}$ there is some $x \in \mathcal{X}$ for which $Q(y|x) > 0$.

Given $B \geq 1$ information bits to be transmitted, a codebook \mathcal{C} consists of

$$M = 2^B$$

codewords of length $n \geq 1$ composed of symbols from \mathcal{X} .

A randomly and uniformly chosen message m is available at the transmitter at a random time ν , independent of m , and uniformly distributed over $\{1, \dots, A_B\}$, where the integer

$$A = 2^{\beta B}$$

characterizes the *asynchronism level* between the transmitter and the receiver, and where the constant

$$\beta \geq 0$$

denotes the *timing uncertainty per information bit*. While ν is unknown to the receiver, A is known by both the transmitter and the receiver.

We consider one-shot communication, *i.e.*, only one message arrives over the period $\{1, 2, \dots, A\}$. If $A = 1$, the channel is said to be synchronous.

Given ν and m , the transmitter chooses a time $\sigma(\nu, m)$ to start sending codeword $c^n(m) \in \mathcal{C}$ assigned to message m . Transmission cannot start before the message arrives or after the end of the uncertainty window, hence $\sigma(\nu, m)$ must satisfy

$$\nu \leq \sigma(\nu, m) \leq A \quad \text{almost surely.}$$

In the rest of the paper, we suppress the arguments ν and m of σ when these arguments are clear from context.

Before and after the codeword transmission, *i.e.*, before time σ and after time $\sigma + n - 1$, the

receiver observes “pure noise.” Specifically, conditioned on ν and on the message to be conveyed m , the receiver observes independent channel outputs

$$Y_1, Y_2, \dots, Y_{A+n-1}$$

distributed as follows. For

$$1 \leq t \leq \sigma - 1$$

or

$$\sigma + n \leq t \leq A + n - 1,$$

the Y_t 's are “pure noise” symbols, *i.e.*,

$$Y_t \sim Q(\cdot|\star)$$

where \star represents the “idle” symbol. For $\sigma \leq t \leq \sigma + n - 1$

$$Y_t \sim Q(\cdot|c_{t-\sigma+1}(m))$$

where $c_i(m)$ denotes the i th symbol of the codeword $c^n(m)$.

Decoding involves three components:

- a sampling strategy,
- a stopping (decoding) time defined on the sampled process,
- a decoding function defined on the stopped sampled process.

A sampling strategy consists of “sampling times” which are defined as an ordered collection of random time indices

$$\mathcal{S} = \{(S_1, \dots, S_\ell) \subseteq \{1, \dots, A+n-1\} : S_i < S_j, i < j\}$$

where S_j is interpreted as the j th sampling time. The sampling strategy is either non-adaptive or adaptive. It is non-adaptive when the sampling times in \mathcal{S} are independent of Y_1^{A+n-1} . The strategy is adaptive when the sampling times are functions of past observations. This means that S_1 is an arbitrary value in $\{1, \dots, A+n-1\}$, possibly random but independent of Y_1^{A+n-1} , and for $j \geq 2$

$$S_j = g_j(\{Y_{S_i}\}_{i < j})$$

for some (possibly randomized) function

$$g_j : \mathcal{Y}^{j-1} \rightarrow \{S_{j-1} + 1, \dots, A+n-1\}.$$

Given a sampling strategy, the receiver decodes by means of a sequential test (τ, ϕ_τ) where τ

denotes a stopping (decision) time with respect to the sampled output process¹

$$Y_{S_1}, Y_{S_2}, \dots$$

and where ϕ_τ denotes a decoding function based on the stopped sampled output process. Let

$$\mathcal{S}^t \stackrel{\text{def}}{=} \{S_i \in \mathcal{S} : S_i \leq t\}. \quad (1)$$

denote the set of sampling times taken up to time t and let

$$\mathcal{O}^t \stackrel{\text{def}}{=} \{Y_{S_i} : S_i \in \mathcal{S}^t\} \quad (2)$$

denote the corresponding set of channel outputs. The decoding function ϕ_τ is a map

$$\begin{aligned} \phi_\tau : \mathcal{Y}^{|\mathcal{O}^\tau|} &\rightarrow \{1, 2, \dots, M\} \\ \mathcal{O}^\tau &\mapsto \phi_\tau(\mathcal{O}^\tau). \end{aligned}$$

A code $(\mathcal{C}, (\mathcal{S}, \tau, \phi_\tau))$ is defined as a codebook and a decoder composed of a sampling strategy, a decision time, and a decoding function. Throughout the paper, whenever clear from context, we often refer to a code using the codebook symbol \mathcal{C} only, leaving out an explicit reference to the decoder.

Note that a pair (\mathcal{S}, τ) allows only to do message detection but does not provide a message estimate. Such a restricted decoder will later (Section III) be referred simply as a “detector.”

Definition 1 (Error probability). The maximum (over messages) decoding error probability of a code \mathcal{C} is defined as

$$\max_m \mathbb{P}_m(\mathcal{E}_m | \mathcal{C}), \quad (3)$$

where

$$\mathbb{P}_m(\mathcal{E}_m | \mathcal{C}) \stackrel{\text{def}}{=} \frac{1}{A} \sum_{t=1}^A \mathbb{P}_{m,t}(\mathcal{E}_m | \mathcal{C}),$$

where the subscripts “ m, t ” denote conditioning on the event that message m arrives at time $\nu = t$, and where \mathcal{E}_m denotes the error event that the decoded message does not correspond to m , *i.e.*,

$$\mathcal{E}_m \stackrel{\text{def}}{=} \{\phi_\tau(\mathcal{O}^\tau) \neq m\}. \quad (4)$$

¹Recall that a (deterministic or randomized) stopping time τ with respect to a sequence of random variables Y_1, Y_2, \dots is a positive, integer-valued, random variable such that the event $\{\tau = t\}$, conditioned on the realization of Y_1, Y_2, \dots, Y_t , is independent of the realization of Y_{t+1}, Y_{t+2}, \dots for all $t \geq 1$.

Definition 2 (Cost of a code). The (maximum) cost of a code \mathcal{C} with respect to a cost function $k : \mathcal{X} \rightarrow [0, \infty]$ is defined as

$$K(\mathcal{C}) \stackrel{\text{def}}{=} \max_m \sum_{i=1}^n k(c_i(m)).$$

Definition 3 (Sampling frequency of a code). Given $\varepsilon > 0$, the sampling frequency of a code \mathcal{C} , denoted by $\rho(\mathcal{C}, \varepsilon)$, is the relative number of channel outputs that are observed until a message is declared. Specifically, it is defined as the minimum $r \geq 0$ such that

$$\min_m \mathbb{P}_m(|\mathcal{S}_\tau|/\tau \leq r) \geq 1 - \varepsilon.$$

Definition 4 (Delay of a code). Given $\varepsilon > 0$, the (maximum) delay of a code \mathcal{C} , denoted by $d(\mathcal{C}, \varepsilon)$, is defined as the minimum integer l such that

$$\min_m \mathbb{P}_m(\tau - \nu \leq l - 1) \geq 1 - \varepsilon.$$

We now define capacity per unit cost under the constraint that the receiver has access to a limited number of channel outputs:

Definition 5 (Asynchronous capacity per unit cost under sampling constraint). Given $\beta \geq 0$ and a non-increasing sequence of numbers $\{\rho_B\}$, with $0 \leq \rho_B \leq 1$, rate per unit cost \mathbf{R} is said to be achievable if there exists a sequence of codes $\{\mathcal{C}_B\}$ and a sequence of positive numbers ε_B with $\varepsilon_B \xrightarrow{B \rightarrow \infty} 0$ such that for all B large enough

- 1) \mathcal{C}_B operates at timing uncertainty per information bit β ;
- 2) the maximum error probability $\mathbb{P}(\mathcal{E} | \mathcal{C}_B)$ is at most ε_B ;
- 3) the rate per unit cost

$$\frac{B}{K(\mathcal{C}_B)}$$

is at least $\mathbf{R} - \varepsilon_B$;

- 4) the sampling frequency satisfies

$$\rho(\mathcal{C}_B, \varepsilon_B) \leq \rho_B;$$

- 5) the delay satisfies²

$$\frac{1}{B} \log(d(\mathcal{C}_B, \varepsilon_B)) \leq \varepsilon_B.$$

²Throughout the paper logarithms are always intended to be to the base 2.

Given β and $\{\rho_B\}$, the asynchronous capacity per unit cost, denoted by $\mathbf{C}(\beta, \{\rho_B\})$, is the supremum of achievable rates per unit cost.

Two comments are in order. First note that samples occurring after time τ play no role in our performance metrics since error probability, delay, and sampling rate are all functions of \mathcal{O}^τ (defined in (2)). Hence, without loss of generality, for the rest of the paper we assume that the last sample is taken at time τ , *i.e.*, that the sampled process is truncated at time τ . The truncated sampled process is thus given by the collection of sampling times \mathcal{S}^τ (defined in (1)). In particular, we have (almost surely)

$$\mathcal{S}^1 \subseteq \mathcal{S}^2 \subseteq \dots \subseteq \mathcal{S}^\tau = \mathcal{S}^{\tau+1} = \dots = \mathcal{S}^{A_B+n-1}. \quad (5)$$

The second comment concerns the delay constraint 4). The delay constraint is meant to capture the fact that the receiver is able to locate ν_B with high accuracy. More precisely, with high probability, τ_B should be at most sub-exponentially larger than ν_B . This already represents a decent level of accuracy, given that ν_B itself is uniform over an exponentially large interval. However, allowing a sub-exponential delay still seems like a very loose constraint. As Theorem 3 claims, however, we can achieve much greater accuracy. Specifically, if a sampling rate is achievable, it can be achieved with delay linear in B , and if a sampling rate cannot be achieved with linear delay, it cannot be achieved even if we allow a sub-exponential delay.

Notational conventions: We shall use d_B and ρ_B instead of $d(\mathcal{C}_B, \varepsilon_B)$ and $\rho(\mathcal{C}_B, \varepsilon_B)$, respectively, leaving out any explicit reference to \mathcal{C}_B and the sequence of non-negative numbers $\{\varepsilon_B\}$, which we assume satisfies $\varepsilon_B \rightarrow 0$. Under full sampling, *i.e.*, when $\rho_B = 1$ for all B , capacity is simply denoted by $\mathbf{C}(\beta)$, and when the sampling rate is constant, *i.e.*, when $\rho_B = \rho \leq 1$ for all B , capacity is denoted by $\mathbf{C}(\beta, \rho)$.

The main, previously known, results regarding capacity for this asynchronous communication model are the following. First, capacity per unit cost under full sampling is given by the following theorem:

Theorem 1 (Full sampling, Theorem 1 [1]). *For any $\beta \geq 0$*

$$\mathbf{C}(\beta) = \max_X \min \left\{ \frac{I(X; Y)}{\mathbb{E}[k(X)]}, \frac{I(X; Y) + D(Y||Y_\star)}{\mathbb{E}[k(X)](1 + \beta)} \right\} \quad (6)$$

where \max_X denotes maximization with respect to the channel input distribution P_X , where $(X, Y) \sim P_X(\cdot)Q(\cdot|\cdot)$, where Y_\star denotes the random output of the channel when the idle symbol \star is transmitted (*i.e.*, $Y_\star \sim Q(\cdot|\star)$), where $I(X; Y)$ denotes the mutual information between X and Y , and where $D(Y||Y_\star)$ denotes the divergence between the distributions of Y and Y_\star . ■

Theorem 1 characterizes capacity per unit cost under full output sampling, and over codes whose delay grow sub-exponentially with B . As it turns out, the full sampling capacity per unit cost can also be achieved with linear delay and sparse output sampling.

Define³

$$n_B^*(\beta, \mathbf{R}) \stackrel{\text{def}}{=} \frac{B}{\mathbf{R} \max\{\mathbb{E}[k(X)] : X \in \mathcal{P}(\mathbf{R})\}} = \Theta(B) \quad (7)$$

where $\mathcal{P}(\mathbf{R})$ is defined as the set

$$\left\{ X : \min \left\{ \frac{I(X; Y)}{\mathbb{E}[k(X)]}, \frac{I(X; Y) + D(Y||Y_\star)}{\mathbb{E}[k(X)](1 + \beta)} \right\} \geq \mathbf{R} \right\}. \quad (8)$$

The quantity $n_B^*(\beta, \mathbf{R})$ quantifies the minimum detection delay as a function of the asynchronism level and rate per unit cost, under full sampling:

Theorem 2 (Minimum delay, constant sampling rate, Theorem 3 [8]). *Fix $\beta \geq 0$, $\mathbf{R} \in (0, \mathbf{C}(\beta))$, and $\rho \in (0, 1]$. For any codes $\{\mathcal{C}_B\}$ that achieve rate per unit cost \mathbf{R} at timing uncertainty β , and operating at constant sampling rate $0 < \rho_B = \rho$, we have*

$$\liminf_{B \rightarrow \infty} \frac{d_B}{n_B^*(\beta, \mathbf{R})} \geq 1.$$

Furthermore, there exist codes $\{\mathcal{C}_B\}$ that achieve rate \mathbf{R} with (a) timing uncertainty β , (b) sampling rate $\rho_B = \rho$, and (c) delay

$$\limsup_{B \rightarrow \infty} \frac{d_B}{n_B^*(\beta, \mathbf{R})} \leq 1.$$

Theorem 2 says that the minimum delay achieved by rate $\mathbf{R} \in (0, \mathbf{C}(\beta))$ codes is $n_B^*(\beta, \mathbf{R})$ for any constant sampling rate $\rho \in (0, 1]$. This naturally

³Throughout the paper we use the standard “big-O” Landau notation to characterize growth rates (see, *e.g.*, [4, Chapter 3]). These growth rates, *e.g.*, $\Theta(B)$ or $o(B)$, are intended in the limit $B \rightarrow \infty$, unless stated otherwise.

suggests the question ‘‘What is the minimum sampling rate of codes that achieve rate \mathbf{R} and minimum delay $n_B^*(\beta, \mathbf{R})$?’’ Our main result is the following theorem, which states that the minimum sampling rate essentially decreases as $1/B$:

Theorem 3 (Minimum delay, minimum sampling rate). *Consider a sequence of codes $\{\mathcal{C}_B\}$ that operate under timing uncertainty per information bit $\beta > 0$. If*

$$\rho_B d_B = o(1), \quad (9)$$

the receiver does not even sample a single component of the sent codeword with probability tending to one. Hence, the average error probability tends to one whenever $\mathbf{R} > 0$, $d_B = O(B)$, and $\rho_B = o(1/B)$.

Moreover, for any $\mathbf{R} \in (0, \mathcal{C}(\beta)]$ and any sequence of sampling rates satisfying $\rho_B = \omega(1/B)$, there exist codes $\{\mathcal{C}_B\}$ that achieve rate \mathbf{R} at (a) timing uncertainty β , (b) sampling rate ρ_B , and (c) delay

$$\limsup_{B \rightarrow \infty} \frac{d_B}{n_B^*(\beta, \mathbf{R})} \leq 1.$$

If $\mathbf{R} > 0$, the minimum delay $n_B^*(\beta, \mathbf{R})$ is $O(B)$ by Theorem 2 and (7), so Theorem 3 gives an essentially tight characterization of the minimum sampling rate; a necessary condition for achieving the minimum delay is that ρ_B be at least $\Omega(1/B)$, and any $\rho_B = \omega(1/B)$ is sufficient.

That sampling rates of order $o(1/d_B)$ are not achievable is certainly intuitively plausible and even essentially trivial to prove when restricted to non-adaptive sampling. To see this note that by the definition of delay, with high probability decoding happens no later than instant $\nu + d_B$. Therefore, without essential loss of generality, we may assume that information is being transmitted only within period $\{\nu, \nu + 1, \dots, \nu + d_B\}$. Hence, if sampling is non-adaptive and its rate is of order $o(1/d_B)$ then with high probability (over ν) information transmission will occur during one unsampled period of duration d_B . This in turn implies a high error probability. The main contribution in the converse argument is that it also handles adaptive sampling.

Achievability rests on a new multi-phase procedure to efficiently detect the sent message. This detector, whose performance is the focus of Section III, is a much more fine grained procedure than the one used to establish Theorem 2. To establish

achievability of Theorem 2, a two-mode detector is considered, consisting of a baseline mode operating at low sampling rate, and a high rate mode. The detector starts in the baseline mode and, if past observed samples suggest the presence of a change in distribution, the detector changes to the high rate mode which acts as a confirmation phase. At the end of the confirmation phase the detector either decides to stop, or decides to reverse to the baseline mode in case the change is unconfirmed.

The detector proposed in this paper (see Section III for the setup and Section IV-C for the description of the procedure) has multiple confirmation phases, each operating at a higher sampling rate than the previous phase. Whenever a confirmation phase is passed, the detector switches to the next confirmation phase. As soon as a change is unconfirmed, the procedure is aborted and the detector returns to the low rate baseline mode. The detector only stops if the change is confirmed by all confirmation phases. Having multiple confirmation phases instead of just one, as for Theorem 2, is key to reducing the rate from a constant to essentially $1/B$, as it allows us to aggressively reject false-alarms without impacting the ability to detect the message.

III. SAMPLING CONSTRAINED TRANSIENT CHANGE-DETECTION

This section focuses on one key aspect of asynchronous communication, namely, that we need to quickly detect the presence of a message with a sampling constrained detector. As there is only one possible message, the problem amounts to a pure (transient) change-point detection problem. Related results are stated in Theorems 4 and 5. These results and their proofs are the key ingredients for proving Theorem 3.

A. Model

The transient change-detection setup we consider in this section is essentially a simpler version of the asynchronous communication problem stated in Section II. Specifically, rather than having a codebook of 2^B messages, we consider a binary hypothesis testing version of the problem. There is a single codeword, so no information is being conveyed, and our goal is simply to detect when the codeword was transmitted.

Proceeding more formally, let P_0 and P_1 be distributions defined over some finite alphabet \mathcal{Y} and with finite divergence

$$D(P_1||P_0) \stackrel{\text{def}}{=} \sum_y P_1(y) \log[P_1(y)/P_0(y)].$$

There is no parameter B in our problem, but in analogy with Section II, let n denote the length of the transient change. Let ν be uniformly distributed over

$$\{1, 2, \dots, A = 2^{\alpha n}\}.$$

where the integer A denotes the *uncertainty level* and where α the corresponding *uncertainty exponent*, respectively.

Given P_0 and P_1 , process $\{Y_t\}$ is defined similarly as in Section II. Conditioned on the value of ν , the Y_t 's are i.i.d. according to P_0 for

$$1 \leq t < \nu$$

or

$$\nu_n + n \leq t \leq A + n - 1$$

and i.i.d. according to P_1 for $\nu \leq t \leq \nu + n - 1$. Process $\{Y_t\}$ is thus i.i.d. P_0 except for a brief period of duration n where it is i.i.d. P_1 .

Sampling strategies are defined as in Section II, but since we now only have a single message, we formally define the relevant performance metrics below.

Definition 6 (False-alarm probability). For a given detector (\mathcal{S}, τ) the probability of false-alarm is defined as

$$\mathbb{P}(\tau < \nu) = \mathbb{P}_0(\tau < \nu)$$

where \mathbb{P}_0 denotes the joint distribution over τ and ν when the observations are drawn from the P_0 -product distribution. In other words, the false-alarm probability is the probability that the detector stops before the transient change has started.

Definition 7 (Detection delay). For a given detector (\mathcal{S}, τ) and $\varepsilon > 0$, the delay, denoted by $d((\mathcal{S}, \tau), \varepsilon)$, is defined as the minimum $l \geq 0$ such that

$$\mathbb{P}(\tau - \nu \leq l - 1) \geq 1 - \varepsilon.$$

Remark: The reader might wonder why we chose the above definition of delay, as opposed to, for example, measuring delay by $\mathbb{E}[\max(0, \tau - \nu)]$. The above definition corresponds to capturing the ‘‘typical’’ delay, without incurring a large penalty in

the tail event where τ is much larger than ν , say because we missed the transient change completely. We are able to characterize optimal performance tightly with the above definition, but expected delay would also be of interest, and an analysis of the optimal performance under this metric is an open problem for future research.

Definition 8 (Sampling rate). For a given detector (\mathcal{S}, τ) and $\varepsilon > 0$, the sampling rate, denoted by $\rho((\mathcal{S}, \tau), \varepsilon)$, is defined as the minimum $r \geq 0$ such that

$$\mathbb{P}(|\mathcal{S}^\tau|/\tau \leq r) \geq 1 - \varepsilon.$$

Achievable sampling rates are defined analogously to Section II, but we include a formal definition for completeness.

Definition 9 (Achievable sampling rate). Fix $\alpha \geq 0$, and fix a sequence of non-increasing values $\{\rho_n\}$ with $0 \leq \rho_n \leq 1$. Sampling rates $\{\rho_n\}$ are said to be achievable at uncertainty exponent α if there exists a sequence of detectors $\{(\mathcal{S}_n, \tau_n)\}$ such that for all n large enough

- 1) (\mathcal{S}_n, τ_n) operates under uncertainty level $A_n = 2^{\alpha n}$,
- 2) the false-alarm probability $\mathbb{P}(\tau_n < \nu_n)$ is at most ε_n ,
- 3) the sampling rate satisfies $\rho((\mathcal{S}_n, \tau_n), \varepsilon_n) \leq \rho_n$,
- 4) the delay satisfies

$$\frac{1}{n} \log(d((\mathcal{S}_n, \tau_n), \varepsilon_n)) \leq \varepsilon_n$$

for some sequence of non-negative numbers $\{\varepsilon_n\}$ such that $\varepsilon_n \xrightarrow{n \rightarrow \infty} 0$.

Notational conventions: We shall use d_n and ρ_n instead of $d((\mathcal{S}_n, \tau_n), \varepsilon_n)$ and $\rho((\mathcal{S}_n, \tau_n), \varepsilon_n)$, respectively, leaving out any explicit reference to the detectors and the sequence of non-negative numbers $\{\varepsilon_n\}$, which we assume satisfies $\varepsilon_n \rightarrow 0$.

B. Results

Define

$$n^*(\alpha) \stackrel{\text{def}}{=} \frac{n\alpha}{D(P_1||P_0)} = \Theta(n). \quad (10)$$

Theorem 4 (Detection, full sampling). *Under full sampling* ($\rho_n = 1$):

- 1) *the supremum of the set of achievable uncertainty exponents is $D(P_1||P_0)$;*

2) any detector that achieves uncertainty exponent $\alpha \in (0, D(P_1||P_0))$ has a delay that satisfies

$$\liminf_{n \rightarrow \infty} \frac{d_n}{n^*(\alpha)} \geq 1;$$

3) any uncertainty exponent $\alpha \in (0, D(P_1||P_0))$ is achievable with delay satisfying

$$\limsup_{n \rightarrow \infty} \frac{d_n}{n^*(\alpha)} \leq 1.$$

Hence, the shortest detectable⁴ change is of size

$$n_{\min}(A_n) = \frac{\log A_n}{D(P_1||P_0)}(1 \pm o(1)) \quad (11)$$

by Claim 1) of Theorem 4, assuming $A_n \gg 1$. In this regime, change duration and minimum detection delay are essentially the same by Claims 2)-3) and (10), *i.e.*,

$$n^*(\alpha = (\log A_n)/n_{\min}(A_n)) = n_{\min}(A_n)(1 \pm o(1))$$

whereas in general minimum detection delay could be smaller than change duration.

The next theorem says that the minimum sampling rate needed to achieve the same detection delay as under full sampling decreases essentially as $1/n$. Moreover, any detector that tries to operate below this sampling limit will have a huge delay.

Theorem 5 (Detection, sparse sampling). *Fix $\alpha \in (0, D(P_1||P_0))$. Any sampling rate*

$$\rho_n = \omega(1/n)$$

is achievable with delay satisfying

$$\limsup_{n \rightarrow \infty} \frac{d_n}{n^*(\alpha)} \leq 1.$$

Conversely, if

$$\rho_n = o(1/n)$$

the detector samples only from distribution P_0 (i.e., it completely misses the change) with probability tending to one. This implies that the delay is $\Theta(A_n = 2^{\alpha n})$ whenever the probability of false-alarm tends to zero.

⁴By detectable we mean with vanishing false-alarm probability and subexponential delay.

IV. PROOFS

Typicality convention

A length $q \geq 1$ sequence v^q over \mathcal{V}^q is said to be typical with respect to some distribution P over \mathcal{V} if⁵

$$\|\hat{P}_{v^q} - P\| \leq q^{-1/3}$$

where \hat{P}_{v^q} denotes the empirical distribution (or type) of v^q .

Typical sets have large probability. Quantitatively, a simple consequence of Chebyshev's inequality is that

$$P^q(\|\hat{P}_{V^q} - P\| \leq q^{-1/3}) = 1 - O(q^{-1/3}) \quad (q \rightarrow \infty) \quad (12)$$

where P^q denotes the q -fold product distribution of P . Also, for any distribution \tilde{P} over \mathcal{V} we have

$$P^q(\|\hat{P}_{V^q} - \tilde{P}\| \leq q^{-1/3}) \leq 2^{-q(D(\tilde{P}||P) - o(1))}. \quad (13)$$

About rounding

Throughout computations, we ignore issues related to the rounding of non-integer quantities, as they play no role asymptotically.

A. Proof of Theorem 4

The proof of Theorem 4 is essentially a Corollary of [2, Theorem]. We sketch the main arguments.

1) : To establish achievability of $D(P_1||P_0)$ one uses the same sequential typicality detection procedure as in the achievability of [2, Theorem]. For the converse argument, we use similar arguments as for the converse of [2, Theorem]. For this latter setting, achieving α means that we can drive the probability of the event $\{\tau_n \neq \nu_n + n - 1\}$ to zero. Although this performance metric differs from ours—vanishing probability of false-alarm and subexponential delay—a closer look at the converse argument of [2, Theorem] reveals that if $\alpha > D(P_1||P_0)$ there are exponentially many sequences of length n that are “typical” with respect to the posterior distribution. This, in turn, implies that either the probability of false-alarm is bounded away from zero, or the delay is exponential.

⁵ $\|\cdot\|$ refers to the L_1 -norm.

2) : Consider stopping times $\{\tau_n\}$ that achieve delay $\{d_n\}$, and vanishing false-alarm probability (recall the notational conventions for d_n at the end of Section III-A). We define the “effective process” $\{\tilde{Y}_i\}$ as the process whose change has duration $\min\{d_n, n\}$ (instead of n).

Effective output process: The effective process $\{\tilde{Y}_i\}$ is defined as follows. Random variable \tilde{Y}_i is equal to Y_i for any index i such that

$$1 \leq i \leq \nu_n + \min\{d_n, n\} - 1$$

and

$$\{\tilde{Y}_i : \nu_n + \min\{d_n, n\} \leq i \leq A_n + n - 1\}$$

is an i.i.d. P_0 process independent of $\{Y_i\}$. Hence, the effective process differs from the true process over the period $\{1, 2, \dots, \tau_n\}$ only when $\{\tau_n \geq \nu_n + d_n\}$ with $d_n < n$.

Genie aided statistician: A genie aided statistician observes the entire effective process (of duration $A_n + n - 1$) and is informed that the change occurred over one of

$$r_n \stackrel{\text{def}}{=} \left\lfloor \frac{A_n + n - 1 - (\nu_n \bmod d_n)}{d_n} \right\rfloor \quad (14)$$

consecutive (disjoint) blocks of duration d_n . The genie aided statistician produces a time interval of size d_n which corresponds to an estimate of the change in distribution and is declared to be correct only if this interval corresponds to the change in distribution.

Observe that since τ_n achieves false-alarm probability ε_n and delay d_n on the true process $\{Y_i\}$, the genie aided statistician achieves error probability at most $2\varepsilon_n$. The extra ε_n comes from the fact τ_n stops after time $\nu_n + d_n - 1$ (on $\{Y_i\}$) with probability at most ε_n . Therefore, with probability at most ε_n the genie aided statistician observes a process that may differ from the true process.

By using the same arguments as for the converse of [2, Theorem], but on the process $\{\tilde{Y}_i\}$ parsed into consecutive slots of size d_n , we can conclude that if

$$\liminf_{n \rightarrow \infty} \frac{d_n}{n^*(\alpha)} < 1$$

then the error probability of the genie aided decoder tends to one.

3) : To establish achievability apply the same sequential typicality test as in the achievability part of [2, Theorem]. ■

B. Proof of Theorem 5: Converse

As alluded to earlier (see discussion after Theorem 3), it is essentially trivial to prove that sampling rates of order $o(1/n)$ are not achievable when we restrict to non-adaptive sampling, that is when all sampling times are independent of $\{Y_t\}$. The main contribution of the converse, and the reason why it is somewhat convoluted, is that it handles adaptive sampling as well.

Consider a sequence of detectors $\{(\mathcal{S}_n, \tau_n)\}$ that achieves, for some false-alarm probability $\varepsilon_n \rightarrow 0$, sampling rate $\{\rho_n\}$ and communication delay d_n (recall the notational conventions for d_n and ρ_n at the end of Section III-A).

We show first that if

$$\rho_n = o(1/n) \quad (15)$$

then any detector, irrespective of delay, will take only P_0 -generated samples with probability asymptotically tending to one. This, in turn, will imply that the delay is exponential, since by assumption the false-alarm probability vanishes.

In the sequel, we use $\mathbb{P}(\cdot)$ to denote the (unconditional) joint distribution of the output process Y_1, Y_2, \dots and ν , and we use $\mathbb{P}_0(\cdot)$ to denote the distribution of the output process $Y_1, Y_2, \dots, Y_{A_n+n-1}$ when no change occurs, that is a P_0 -product distribution.

By definition of achievable sampling rates $\{\rho_n\}$ we have

$$1 - o(1) \leq \mathbb{P}(|\mathcal{S}^{\tau_n}| \leq \tau_n \rho_n). \quad (16)$$

The following lemma, proved thereafter, says if (15) holds then with probability tending to one the detector samples only P_0 -distributed samples with probability tending to one:

Lemma 1. For any $\alpha > 0$, if $\rho_n = o(1/n)$ then

$$\begin{aligned} \mathbb{P}(\{\nu_n, \nu_n + 1, \dots, \nu_n + n - 1\} \cap \mathcal{S}^{\tau_n} = \emptyset) \\ \geq 1 - o(1). \end{aligned} \quad (17)$$

This, as we now show, implies that the delay is exponential.

On the one hand, since the probability of false-alarm vanishes, we have

$$\begin{aligned} o(1) &\geq \mathbb{P}(\tau_n < \nu_n) \\ &\geq \mathbb{P}(\tau_n < A_n/2 | \nu_n \geq A_n/2) / 2 \\ &= \mathbb{P}_0(\tau_n < A_n/2) / 2. \end{aligned}$$

This implies

$$\mathbb{P}_0(\tau_n < A_n/2) \leq o(1),$$

and, therefore,

$$\begin{aligned} \mathbb{P}(\tau_n \geq A_n/2) &\geq \mathbb{P}(\tau_n \geq A_n/2 | \nu_n > A_n/2)/2 \\ &= \mathbb{P}_0(\tau_n \geq A_n/2)/2 \\ &= 1/2 - o(1). \end{aligned} \quad (18)$$

Now, define events

$$\begin{aligned} \mathcal{A}_1 &\stackrel{\text{def}}{=} \{\tau_n \geq A_n/2\}, \\ \mathcal{A}_2 &\stackrel{\text{def}}{=} \{|\mathcal{S}^{\tau_n}| \leq \tau_n \rho_n\}, \\ \mathcal{A}_3 &\stackrel{\text{def}}{=} \{\{\nu_n, \nu_n + 1, \dots, \nu_n + n - 1\} \cap \mathcal{S}^{\tau_n} = \emptyset\}, \end{aligned}$$

and let $\mathcal{A} \stackrel{\text{def}}{=} \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$.

From (16), (17), and (18), we get

$$\mathbb{P}(\mathcal{A}) = 1/2 - o(1). \quad (19)$$

We now argue that when event \mathcal{A} happens, the detector misses the change which might have occurred, say, before time $A_n/4$, thereby implying a delay $\Theta(A_n)$ since $\tau_n \geq A_n/2$ on \mathcal{A} .

When event \mathcal{A} happens, the detector takes $o(A_n/n)$ samples (this follows from event \mathcal{A}_2 since by assumption $\rho_n = o(1/n)$). Therefore, within $\{1, 2, \dots, A_n/4\}$ there are at least $A_n/4 - o(A_n)$ time intervals of length n that are unsampled. Each of these corresponds to a possible change. Therefore, conditioned on event \mathcal{A} , with probability at least $1/4 - o(1)$ the change happens before time $A_n/4$, whereas $\tau_n \geq A_n/2$. Hence the delay is $\Theta(A_n)$, since the probability of \mathcal{A} is asymptotically bounded away from zero by (19). ■

Proof of Lemma 1: We have

$$\begin{aligned} &\mathbb{P}(\{\nu_n, \nu_n + 1, \dots, \nu_n + n - 1\} \cap \mathcal{S}^{\tau_n} = \emptyset) \\ &= \mathbb{P}(\{\{\nu_n, \nu_n + 1, \dots, \nu_n + n - 1\} \cap \mathcal{S}^{\nu_n+n-1} = \emptyset\}) \\ &\geq \mathbb{P}(\{\{\nu_n, \nu_n + 1, \dots, \nu_n + n - 1\} \cap \mathcal{S}^{\nu_n+n-1} = \emptyset\} \\ &\quad \cap \{|\mathcal{S}^{\nu_n+n-1}| \leq k\}) \\ &= \sum_{s:|s| \leq k} \sum_{j \in \mathcal{J}_s} \mathbb{P}(\mathcal{S}^{\nu_n+n-1} = s, \nu_n = j) \\ &= \sum_{s:|s| \leq k} \sum_{j \in \mathcal{J}_s} \mathbb{P}_0(\mathcal{S}^{\nu_n+n-1} = s) \mathbb{P}(\nu_n = j) \\ &\geq \frac{A_n - k \cdot n}{A_n} \sum_{s:|s| \leq k} \mathbb{P}_0(\mathcal{S}^{\nu_n+n-1} = s) \\ &= \frac{A_n - k \cdot n}{A_n} \mathbb{P}_0(|\mathcal{S}^{\nu_n+n-1}| \leq k) \end{aligned} \quad (20)$$

for any $k \in \{1, 2, \dots, A_n\}$, where we defined the set of indices

$$\mathcal{J}_s \stackrel{\text{def}}{=} \{j : \{j, j + 1, \dots, j + n - 1\} \cap s = \emptyset\}.$$

The first equality in (20) holds by the definition of \mathcal{S}^t (see (1)) and by (5). The third equality holds because event $\{\mathcal{S}^{\nu_n+n-1} = s\}$ involves random variables whose indices are not in \mathcal{J}_s . Hence samples in s are all distributed according to the nominal distribution \mathbb{P}_0 (P_0 -product distribution). The last inequality holds by the property

$$|\mathcal{S}^{a+b}| \leq |\mathcal{S}^a| + b \quad (21)$$

which follows from the definition of \mathcal{S}^t .

Since $\tau_n \leq A_n + n - 1$ from (16) we get

$$\begin{aligned} 1 - o(1) &\leq \mathbb{P}(|\mathcal{S}^{\tau_n}| \leq (A_n + n - 1)\rho_n) \\ &\leq \mathbb{P}(|\mathcal{S}^{\nu_n-1}| \leq (A_n + n - 1)\rho_n) \end{aligned} \quad (22)$$

where the second inequality holds by (5).

Now,

$$\begin{aligned} &\mathbb{P}(|\mathcal{S}^{\nu_n-1}| \leq (A_n + n - 1)\rho_n) \\ &= \sum_{t=1}^{A_n} \mathbb{P}(|\mathcal{S}^{t-1}| \leq (A_n + n - 1)\rho_n, \nu_n = t) \\ &= \sum_{t=1}^{A_n} \mathbb{P}_0(|\mathcal{S}^{t-1}| \leq (A_n + n - 1)\rho_n) \mathbb{P}(\nu_n = t) \\ &\leq \sum_{t=n}^{A_n+n-1} \mathbb{P}_0(|\mathcal{S}^{t-1}| \leq (A_n + n - 1)\rho_n) \mathbb{P}(\nu_n = t) \\ &\quad + \sum_{t=1}^{n-1} \mathbb{P}(\nu_n = t) \\ &\leq \mathbb{P}_0(|\mathcal{S}^{\nu_n+n-1}| \leq (A_n + n - 1)\rho_n) \\ &\quad + n/A_n \\ &\leq \mathbb{P}_0(|\mathcal{S}^{\nu_n+n-1}| \leq (A_n + n - 1)\rho_n) \\ &\quad + o(1) \end{aligned} \quad (23)$$

where the last equality holds since $A_n = 2^{\alpha n}$.

From (23) and (22) we have

$$1 - o(1) \leq \mathbb{P}_0(|\mathcal{S}^{\nu_n+n-1}| \leq (A_n + n - 1)\rho_n). \quad (24)$$

Letting

$$k \stackrel{\text{def}}{=} k_n \stackrel{\text{def}}{=} (A_n + n - 1)\rho_n, \quad (25)$$

and assuming that $\rho_n = o(1/n)$ we get

$$k_n \cdot n = o(A_n)$$

and hence from (20) and (24)

$$\begin{aligned} \mathbb{P}(\{\nu_n, \nu_n + 1, \dots, \nu_n + n - 1\} \cap \mathcal{S}^{\tau_n} = \emptyset) \\ \geq 1 - o(1) \end{aligned}$$

which concludes the proof. \blacksquare

C. Proof of Theorem 5: Achievability

We describe a detection procedure that asymptotically achieves minimum delay $n^*(\alpha)$ and any sampling rate that is $\omega(1/n)$ whenever $\alpha \in (0, D(P_0||P_1))$.

Fix $\alpha \in (0, D(P_1||P_0))$ and pick $\varepsilon > 0$ small enough so that

$$n^*(\alpha)(1 + 2\varepsilon) \leq n. \quad (26)$$

Suppose we want to achieve some sampling rate $\rho_n = f(n)/n$ where $f(n) = \omega(1)$ is some arbitrary increasing function (upper bounded by n without loss of generality). For concreteness, it might be helpful for the reader to take $f(n) = \log \log \log \log(n)$. Define

$$\bar{\Delta}(n) \stackrel{\text{def}}{=} n/f(n)^{1/3}$$

$$s\text{-instants} \stackrel{\text{def}}{=} \{t = j\bar{\Delta}(n), j \in \mathbb{N}^*\},$$

and recursively define

$$\Delta_0(n) \stackrel{\text{def}}{=} f(n)^{1/3}$$

$$\Delta_i(n) \stackrel{\text{def}}{=} \min\{2^{c\Delta_{i-1}(n)}, n^*(\alpha)(1 + \varepsilon)\}$$

for $i \in 1, 2, \dots, \ell$ where ℓ denotes the smallest integer such that $\Delta_\ell(n) = n^*(\alpha)(1 + \varepsilon)$. The constant c in the definition of $\Delta_i(n)$ can be any fixed value such that

$$0 < c < D(P_1||P_0).$$

The detector starts sampling in phases at the first s -instant (*i.e.*, at time $t = \bar{\Delta}(n)$) as follows:

- 1 **Preamble detection (phase zero):** Take $\Delta_0(n)$ consecutive samples and check if they are typical with respect to P_1 . If the test is negative, meaning that $\Delta_0(n)$ samples are not typical, skip samples until the next s -instant and repeat the procedure *i.e.*, sample and test $\Delta_0(n)$ observations. If the test is positive, proceed to confirmation phases.
- 2 **Preamble confirmations (variable duration, $\ell - 1$ phases at most):** Take another $\Delta_1(n)$ consecutive samples and check if they are

typical with respect to P_1 . If the test is negative, skip samples until the next s -instant and repeat Phase zero (that is, test $\Delta_0(n)$ samples). If the test is positive, perform a second confirmation phase with $\Delta_1(n)$ replaced with $\Delta_2(n)$, and so forth. Note that each confirmation phase is performed on a new set of samples. If $\ell - 1$ consecutive confirmation phases (with respect to the same s -instant) are positive, the receiver moves to the full block sampling phase.

- 3 **Full block sampling (ℓ -th phase):** Take another

$$\Delta_\ell(n) = n^*(\alpha)(1 + \varepsilon)$$

samples and check if they are typical with respect to P_1 . If they are typical, stop. Otherwise, skip samples until the next s -instant and repeat Phase zero. If by time $A_n + n - 1$ no sequence is found to be typical, stop.

Note that with our $f(n) = \log \log \log \log(n)$ example, we have two preamble confirmation phases followed by the last full block sampling phase.

For the probability of false-alarm we have

$$\begin{aligned} \mathbb{P}(\tau_n < \nu_n) &\leq 2^{\alpha n} \cdot 2^{-n^*(\alpha)(1+\varepsilon)(D(P_1||P_0)-o(1))} \\ &= 2^{-n\alpha\Theta(\varepsilon)} \\ &= o(1) \end{aligned} \quad (27)$$

because whenever the detector stops, the previous

$$n^*(\alpha)(1 + \varepsilon)$$

samples are necessarily typical with respect to P_1 . Therefore, the inequality (27) follows from (13) and a union bound over time indices. The equality in (27) follows directly from the definition of $n^*(\alpha)$ (see (10)).

Next, we analyze the delay of the proposed scheme. We show that

$$\mathbb{P}(\tau_n \leq \nu_n + (1 + 2\varepsilon)n^*(\alpha)) = 1 - o(1). \quad (28)$$

To see this, note that by the definition of $\bar{\Delta}(n)$ and because each $\Delta_i(n)$ is exponentially larger than the previous $\Delta_{i-1}(n)$,

$$\bar{\Delta}(n) + \sum_{i=0}^{\ell} \Delta_i(n) \leq (1 + 2\varepsilon)n^*(\alpha)$$

for n large enough. Applying (12) and taking a union bound, we see that when the samples are distributed according to P_1 , the series of $\ell + 1$

hypothesis tests will all be positive with probability $1 - o(1)$. Specifically,

$$\mathbb{P}(\text{any test fails}) \leq \sum_{i=0}^{\ell} O(\Delta_i(n))^{-\frac{1}{3}} = o(1). \quad (29)$$

Since ε can be made arbitrarily small, from (27) and (28) we deduce that the detector achieves minimum delay (see Theorem 4, Claim 2) .

Finally, to show that the above detection procedure achieves sampling rate

$$\rho_n = f(n)/n$$

we need to establish that

$$\mathbb{P}(|\mathcal{S}^{\tau_n}|/\tau_n \geq \rho_n) \xrightarrow{n \rightarrow \infty} 0. \quad (30)$$

To prove this, we first compute the sampling rate of the detector when run over an i.i.d.- P_0 sequence, that is, a sequence with no transient change. As should be intuitively clear, this will essentially give us the desired result, since in the true model, the duration of the transient change, n , is negligible with respect to A_n anyway.

To get a handle on the sampling rate of the detector over an i.i.d.- P_0 sequence, we start by computing the expected number of samples N taken by the detector at any given s -instant, when the detector is started at that specific s -instant and the observations are all i.i.d. P_0 . Clearly, this expectation does not depend on the s -instant.⁶ We have

$$\mathbb{E}_0 N \leq \Delta_0(n) + \sum_{i=0}^{\ell-1} p_i \cdot \Delta_{i+1}(n) \quad (31)$$

where p_i denotes the probability that the i -th confirmation phase is positive given that the detector actually reaches the i -th confirmation phase, and \mathbb{E}_0 denotes expectation with respect to an i.i.d.- P_0 sequence. Since each phase uses new, and therefore, independent, observations, from (13) we conclude that

$$p_i \leq 2^{-\Delta_i(n)(D(P_1||P_0)-o(1))}.$$

Using the definition of $\Delta_i(n)$, and recalling that $0 < c < D(P_1||P_0)$, this implies that the sum in the second term of (31) is negligible, and

$$\mathbb{E}_0 N_s = \Delta_0(n)(1 + o(1)). \quad (32)$$

⁶Boundary effects due to the fact that A_n need not be a multiple of $\bar{\Delta}_n$ play no role asymptotically and thus are ignored.

Therefore, the expected number of samples taken by the detector up to any given time t can be upper bounded as

$$\begin{aligned} \mathbb{E}_0 |\mathcal{S}^t| &\leq \frac{t}{\bar{\Delta}(n)} \Delta_0(n)(1 + o(1)) \\ &= t \frac{f(n)^{2/3}}{n} (1 + o(1)). \end{aligned} \quad (33)$$

This, as we now show, implies that the detector has the desired sampling rate. We have

$$\begin{aligned} \mathbb{P}(|\mathcal{S}^{\tau_n}|/\tau_n \geq \rho_n) &\leq \mathbb{P}(|\mathcal{S}^{\tau_n}|/\tau_n \geq \rho_n, \nu_n \leq \tau_n \leq \nu_n + (1 + 2\varepsilon)n^*(\alpha)) \\ &\quad + 1 - \mathbb{P}(\nu_n \leq \tau_n \leq \nu_n + (1 + 2\varepsilon)n^*(\alpha)) \\ &\leq \mathbb{P}(|\mathcal{S}^{\tau_n}|/\tau_n \geq \rho_n, \nu_n \leq \tau_n \leq \nu_n + n) \\ &\quad + 1 - \mathbb{P}(\nu_n \leq \tau_n \leq \nu_n + (1 + 2\varepsilon)n^*(\alpha)) \end{aligned} \quad (34)$$

where the second inequality holds for ε small enough by the definition of $n^*(\alpha)$.

The fact that

$$1 - \mathbb{P}(\nu_n \leq \tau_n < \nu_n + (1 + 2\varepsilon)n^*(\alpha)) = o(1) \quad (35)$$

follows from (27) and (28). For the first term on the right-hand side of the second inequality in (34), we have

$$\begin{aligned} \mathbb{P}(|\mathcal{S}^{\tau_n}|/\tau_n \geq \rho_n, \nu_n \leq \tau_n \leq \nu_n + n) &\leq \mathbb{P}(|\mathcal{S}^{\nu_n+n}| \geq \nu_n \rho_n) \\ &\leq \mathbb{P}(|\mathcal{S}^{\nu_n-1}| \geq \nu_n \rho_n - n - 1). \end{aligned} \quad (36)$$

Since \mathcal{S}_{ν_n-1} represents sampling times before the transient change, the underlying process is i.i.d. P_0 , so we can use our previous bound on the sampling rate to analyze \mathcal{S}_{ν_n-1} . Conditioned on reasonably large values of ν_n , in particular, all ν_n satisfying

$$\nu_n \geq \sqrt{A_n} = 2^{\alpha n} \quad (37)$$

we have

$$\begin{aligned} \mathbb{P}(|\mathcal{S}^{\nu_n-1}| \geq \nu_n \rho_n - n - 1 | \nu_n) &\leq \frac{\mathbb{E}_0 |\mathcal{S}^{\nu_n}|}{\nu_n \rho_n - n - 1} \\ &\leq \frac{f(n)^{2/3}(1 + o(1))}{n(\rho_n - (n+1)/\nu_n)} \\ &\leq \frac{f(n)^{2/3}(1 + o(1))}{n\rho_n(1 - o(1))} \\ &= \frac{(1 + o(1))}{f(n)^{1/3}(1 - o(1))} \\ &= o(1) \end{aligned} \quad (38)$$

where the second inequality holds by (33); where the third inequality holds by (37) and because $\rho_n = \omega(1/n)$; and where the last two equalities hold by the definitions of ρ_n and $f(n)$.

Removing the conditioning on ν_n ,

$$\begin{aligned} & \mathbb{P}(|\mathcal{S}^{\nu_n-1}| \geq \nu_n \rho_n - n - 1) \\ & \leq \mathbb{P}(|\mathcal{S}^{\nu_n-1}| \geq \nu_n \rho_n - n - 1, \nu_n \geq \sqrt{A_n}) \\ & \quad + \mathbb{P}(\nu_n < \sqrt{A_n}) \\ & = o(1) \end{aligned} \tag{39}$$

by (38) and the fact that ν_n is uniformly distributed over $\{1, 2, \dots, A_n\}$. Hence, from (36), the first term on the right-hand side of the second inequality in (34) vanishes.

This yields (30).

D. Discussion

There is obviously a lot of flexibility around the quickest detection procedure described in Section IV-C. Its main feature is the sequence of binary hypothesis tests, which manages to reject the hypothesis that a change occurred with as few samples as possible when the samples are drawn from P_0 , while maintaining a high probability of detecting the transient change.

It may be tempting to simplify the detection procedure by considering, say, only two phases, a preamble phase and the full block phase. Such a scheme, which is similar in spirit to the one proposed in [8], would not work, as it would produce either a much higher level of false-alarm, or a much higher sampling rate. We provide an intuitive justification for this below, thereby highlighting the role of the multiphase procedure.

Consider a two phase procedure, a preamble phase followed by a full block phase. Each time we switch to the second phase, we take $\Theta(n)$ samples. Therefore, if we want to achieve a vanishing sampling rate, then necessarily the probability of switching from the preamble phase to the full block phase under P_0 should be $o(1/n)$. By Sanov's theorem, such a probability can be achieved only if the preamble phase makes its decision to switch to the full block phase based on at least $\omega(\log n)$ samples, taken over time windows of size $\Theta(n)$. This translates into a sampling rate of $\omega((\log n)/n)$ at best, and we know that this is suboptimal, since any sampling rate $\omega(1/n)$ is achievable.

The reason a two-phase scheme does not yield a sampling rate lower than $\omega((\log n)/n)$ is that it is too coarse. To guarantee a vanishing sampling rate, the decision to switch to the full block phase should be based on at least $\log(n)$ samples, which in turn yields a suboptimal sampling rate. The important observation is that the (average) sampling rate of the two-phase procedure essentially corresponds to the sampling rate of the first phase, but the first phase also controls the decision to switch to the full block phase and sample continuously for a long period of order n . In the multiphase procedure, however, we can separate these two functions. The first phase controls the sampling rate, but passing the first phase only leads us to a second phase, a much less costly decision than immediately switching to full block sampling. By allowing multiple phases, we can ensure that when the decision to ultimately switch to full sampling occurs, it only occurs because we have accumulated a significant amount of evidence that we are in the middle of the transient change. In particular, note that many other choices would work for the length and probability thresholds used in each phase of our sampling scheme. The main property we rely on is that the lengths and probability thresholds be chosen so that the sampling rate is dominated by the first phase.

E. Proof of Theorem 3

In this section, we prove Theorem 3. A reader familiar with the proofs presented in [8] will recognize Theorem 3 as a corollary of Theorem 5, but we include a detailed proof below for interested readers unfamiliar with the prior work [8].

1) *Converse of Theorem 3:* By using the same arguments as for Lemma 1, and simply replacing n with d_B , one readily sees that if

$$\rho_B d_B = o(1) \tag{40}$$

then

$$\begin{aligned} & \mathbb{P}(\{\nu_B, \nu_B + 1, \dots, \nu_B + d_B - 1\} \cap \mathcal{S}^{\tau_B} = \emptyset) \\ & \geq (1 - o(1)). \end{aligned} \tag{41}$$

Since the decoder samples no codeword symbol with probability approaching one, the decoding error probability will tend to one whenever the rate is positive (so that $(M - 1)/M$ tends to one).

2) *Achievability of Theorem 3:* Fix $\beta > 0$. We show that any $\mathbf{R} \in (0, \mathbf{C}(\beta)]$ is achievable with codes $\{\mathcal{C}_B\}$ whose delays satisfy $d(\mathcal{C}_B, \varepsilon_B) \leq n_B^*(\beta, \mathbf{R})(1 + o(1))$ whenever the sampling rate ρ_B is such that

$$\rho_B = \frac{f(B)}{B}$$

for some $f(B) = \omega(1)$.

Let $X \sim P$ be some channel input and let Y denote the corresponding output, *i.e.*, $(X, Y) \sim P(\cdot)Q(\cdot|\cdot)$. For the moment we only assume that X is such that $I(X; Y) > 0$. Further, we suppose that the codeword length n is linearly related to B , *i.e.*,

$$\frac{B}{n} = q$$

for some fixed constant $q > 0$. We shall specify this linear dependency later to accommodate the desired rate \mathbf{R} . Further, let

$$\tilde{f}(n) \stackrel{\text{def}}{=} f(q \cdot n)/q$$

and

$$\tilde{\rho}_n \stackrel{\text{def}}{=} \frac{\tilde{f}(n)}{n}.$$

Hence, by definition we have

$$\tilde{\rho}_n = \rho_B.$$

Let a be some arbitrary fixed input symbol such that

$$Q(\cdot|a) \neq Q(\cdot|\star).$$

Below we introduce the quantities $\bar{\Delta}(n)$ and $\Delta_i(n)$, $1 \leq i \leq \ell$, which are defined as in Section IV-C but with P_0 replaced with $Q(\cdot|\star)$, P_1 replaced with $Q(\cdot|a)$, $f(n)$ replaced with $\tilde{f}(n)$, and $n^*(\alpha)$ replaced with n .

Codewords: preamble followed by constant composition information symbols. Each codeword $c^n(m)$ starts with a common preamble that consists of $\bar{\Delta}(n)$ repetitions of symbol a . The remaining

$$n - \bar{\Delta}(n)$$

components

$$c_{\bar{\Delta}(n)+1}^n(m)$$

of $c^n(m)$ of each message m carry information and are generated as follows. For message 1, randomly generate length $n - \bar{\Delta}(n)$ sequences $x^{n-\bar{\Delta}(n)}$ i.i.d.

according to P until when $x^{n-\bar{\Delta}(n)}$ is typical with respect to P . In this case we let

$$c_{\bar{\Delta}(n)+1}^n(1) \stackrel{\text{def}}{=} x^{n-\bar{\Delta}(n)},$$

move to message 2, and repeat the procedure until when a codeword has been assigned to each message.

From (12), for any fixed m no repetition will be required to generate $c_{\bar{\Delta}(n)+1}^n(m)$ with probability tending to one as $n \rightarrow \infty$. Moreover, by construction the codewords are essentially of constant composition, *i.e.*, each symbol appears roughly the same number of times in all codewords, and all codewords have cost

$$n\mathbb{E}[k(X)](1 + o(1))$$

as $n \rightarrow \infty$.

Codeword transmission time. Define the set of start instants

$$s\text{-instants} \stackrel{\text{def}}{=} \{t = j\bar{\Delta}(n), j \in \mathbb{N}^*\}.$$

Codeword transmission start time $\sigma(m, \nu_n)$ corresponds to the first s -instant $\geq \nu_n$ (regardless of m).

Sampling and decoding procedures. The decoder first tries to detect the preamble by using a similar detection procedure as in the achievability of Theorem 5, then applies a standard message decoding isolation map.

Starting at the first s -instant (*i.e.*, at time $t = \bar{\Delta}(n)$), the decoder samples in phases as follows.

- 1 **Preamble test (phase zero):** Take $\Delta_0(n)$ consecutive samples and check if they are typical with respect to $Q(\cdot|a)$. If the test turns negative, the decoder skips samples until the next s -instant when it repeats the procedure. If the test turns positive, the decoder moves to the confirmation phases.
- 2 **Preamble confirmations (variable duration, $\ell - 1$ phases at most):** The decoder takes another $\Delta_1(n)$ consecutive samples and checks if they are typical with respect to $Q(\cdot|a)$. If the test turns negative the decoder skips samples until the next s -instant when it repeats Phase zero (and tests $\Delta_0(n)$ samples). If the test turns positive, the decoder performs a second confirmation phase based on new $\Delta_2(n)$ samples, and so forth. If $\ell - 1$ consecutive confirmation phases (with respect to the same

s -instant) turn positive, the decoder moves to the message sampling phase.

3 Message sampling and isolation (ℓ -th phase):

Take another n samples and check if among these samples there are $n - \bar{\Delta}(n)$ consecutive samples that are jointly typical with the $n - \bar{\Delta}(n)$ information symbols of one of the codewords. If one codeword is typical, stop and declare the corresponding message. If more than one codeword is typical declare one message at random. If no codeword is typical, the decoder stops sampling until the next s -instant and repeats Phase zero. If by time $A_B + n - 1$ no codeword is found to be typical, the decoder declares a random message.

Error probability. Error probability and delay are evaluated in the limit $B \rightarrow \infty$ with $A_B = 2^{\beta B}$ and with

$$q = \frac{B}{n} < \min \left\{ I(X; Y), \frac{I(X; Y) + D(Y||Y_*)}{1 + \beta} \right\}. \quad (42)$$

We first compute the error probability averaged over codebooks and messages. Suppose message m is transmitted and denote by \mathcal{E}_m the error event that the decoder stops and outputs a message $m' \neq m$. Then we have

$$\mathcal{E}_m \subseteq \mathcal{E}_{0,m} \cup_{m' \neq m} (\mathcal{E}_{1,m'} \cup \mathcal{E}_{2,m'}), \quad (43)$$

where events $\mathcal{E}_{0,m}$, $\mathcal{E}_{1,m'}$, and $\mathcal{E}_{2,m'}$ are defined as

- $\mathcal{E}_{0,m}$: at the s -instant corresponding to σ , the preamble test phase or one of the preamble confirmation phases turns negative, or $c_{\bar{\Delta}(n)+1}^n(m)$ is not found to be typical by time $\sigma + n - 1$;
- $\mathcal{E}_{1,m'}$: the decoder stops at a time $t < \sigma$ and declares m' ;
- $\mathcal{E}_{2,m'}$: the decoder stops at a time t between σ and $\sigma + n - 1$ (including σ and $\sigma + n - 1$) and declares m' .

From Sanov's theorem,

$$\mathbb{P}_m(\mathcal{E}_{0,m}) = \varepsilon_1(B) \quad (44)$$

where $\varepsilon_1(B) = o(1)$. Note that this equality holds pointwise (and not only on average over codebooks) for any specific (non-random) codeword $c^n(m)$ since, by construction, they all satisfy the constant composition property

$$\|\hat{P}_{c_{\bar{\Delta}+1}^n(m)} - P\| \leq (n - \bar{\Delta})^{-1/3} = o(1) \quad (45)$$

as $n \rightarrow \infty$.

Using analogous arguments as in the achievability of [1, Proof of Theorem 1], we obtain the upper bounds

$$\mathbb{P}_m(\mathcal{E}_{1,m'}) \leq 2^{\beta B} \cdot 2^{-n(I(X;Y)+D(Y||Y_*)-o(1))}$$

and

$$\mathbb{P}_m(\mathcal{E}_{2,m'}) \leq 2^{-n(I(X;Y)-o(1))}$$

which are both valid for any fixed $\varepsilon > 0$ provided that B is large enough. Hence from the union bound

$$\mathbb{P}_m(\mathcal{E}_{1,m'} \cup \mathcal{E}_{2,m'}) \leq 2^{-n(I(X;Y)-o(1))} + 2^{\beta B} \cdot 2^{-n(I(X;Y)+D(Y||Y_*)-o(1))}.$$

Taking a second union bound over all possible wrong messages, we get

$$\begin{aligned} \mathbb{P}_m(\cup_{m' \neq m} (\mathcal{E}_{1,m'} \cup \mathcal{E}_{2,m'})) &\leq 2^B \left(2^{-n(I(X;Y)-o(1))} \right. \\ &\quad \left. + 2^{\beta B} \cdot 2^{-n(I(X;Y)+D(Y||Y_*)-o(1))} \right) \\ &\stackrel{\text{def}}{=} \varepsilon_2(B) \end{aligned} \quad (46)$$

where $\varepsilon_2(B) = o(1)$ because of (42).

Combining (43), (44), (46), we get from the union bound

$$\begin{aligned} \mathbb{P}_m(\mathcal{E}_m) &\leq \varepsilon_1(B) + \varepsilon_2(B) \\ &= o(1) \end{aligned} \quad (47)$$

for any m .

Delay. We now show that the delay of our coding scheme is at most $n(1 + o(1))$. Suppose codeword $c^n(m)$ is sent. If

$$\tau_B > \sigma + n$$

then necessarily $c_{\bar{\Delta}+1}^n(m)$ is not typical with the corresponding channel outputs. Hence

$$\begin{aligned} \mathbb{P}_m(\tau_B - \sigma \leq n) &\geq 1 - \mathbb{P}_m(\mathcal{E}_{0,m}) \\ &= 1 - \varepsilon_1(B) \end{aligned} \quad (48)$$

by (44). Since $\sigma \leq \nu_B + \bar{\Delta}(n)$ and $\bar{\Delta}(n) = o(n)$ we get⁷

$$\mathbb{P}_m(\tau_B - \nu_B \leq n(1 + o(1))) \geq 1 - \varepsilon_1(B).$$

Since this inequality holds for any codeword $c^n(m)$ that satisfies (45), the delay is no more than $n(1 + o(1))$. Furthermore, from (47) there exists a specific

⁷Recall that B/n is kept fixed and $B \rightarrow \infty$.

non-random code \mathcal{C} whose error probability, averaged over messages, is less than $\varepsilon_1(n) + \varepsilon_2(n) = o(1)$ whenever condition (42) is satisfied. Removing the half of the codewords with the highest error probability, we end up with a set \mathcal{C}' of 2^{B-1} codewords whose maximum error probability satisfies

$$\max_m \mathbb{P}_m(\mathcal{E}_m) \leq o(1) \quad (49)$$

whenever condition (42) is satisfied.

Since any codeword has cost $n\mathbb{E}[k(X)](1+o(1))$, condition (42) is equivalent to

$$\mathbf{R} < \min \left\{ \frac{I(X; Y)}{\mathbb{E}[k(X)](1+o(1))}, \frac{I(X; Y) + D(Y||Y_*)}{\mathbb{E}[k(X)](1+o(1))(1+\beta)} \right\} \quad (50)$$

where

$$\mathbf{R} \stackrel{\text{def}}{=} \frac{B}{K(\mathcal{C}')}$$

denotes the rate per unit cost of \mathcal{C}' .

Thus, to achieve a given $\mathbf{R} \in (0, \mathbf{C}(\beta))$ it suffices to choose the input distribution and the codeword length as

$$X = \arg \max\{\mathbb{E}[k(X')] : X' \in \mathcal{P}(\mathbf{R})\}$$

and

$$n = n_B^*(\beta, \mathbf{R})$$

(see (7) and (8)). By a previous argument the corresponding delay is no larger than $n_B^*(\beta, \mathbf{R})(1+o(1))$.

Sampling rate. For the sampling rate, a very similar analysis to the achievability proof of Theorem 5 (see from equation (30) onwards with $f(n)$, ρ_n , $n^*(\alpha)$, and A_n replaced with $\tilde{f}(n)$, $\tilde{\rho}_n$, $n^*(\beta, \mathbf{R})$, and A_B , respectively) shows that

$$\mathbb{P}_m(|\mathcal{S}^{\tau_B}|/\tau_B \geq \rho_B) \xrightarrow{B \rightarrow \infty} 0. \quad (51)$$

Note that the arguments that establish (51) rely only on the preamble detection procedure. In particular, they do not use (50) and hold for any codeword length n_B as long as $n_B = \Theta(B)$.

V. CONCLUSION

We have proved an essentially tight characterization of the sampling rate required to have no capacity or delay penalty for the asynchronous communication model of [8]. The key ingredient in our results is a new, multi-phase, adaptive sampling scheme used to detect when the received signal's

distribution switches from the pure noise distribution to the codeword distribution. As noted above, there is a lot of flexibility around the quickest detection procedure described in Section IV-C, but a simple, two level generalization of the sampling algorithm from [8] is insufficient to achieve the optimal sampling rate. Instead, a fine-grained, multi-level scheme is needed.

REFERENCES

- [1] V. Chandar, A. Tchamkerten, and D. Tse. Asynchronous capacity per unit cost. *Information Theory, IEEE Transactions on*, 59(3):1213–1226, march 2013.
- [2] V. Chandar, A. Tchamkerten, and G. Wornell. Optimal sequential frame synchronization. *Information Theory, IEEE Transactions on*, 54(8):3725–3728, 2008.
- [3] Venkat Chandar, Aslan Tchamkerten, and David Tse. Asynchronous capacity per unit cost. *Information Theory, IEEE Transactions on*, 59(3):1213–1226, 2013.
- [4] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. *Introduction to Algorithms, 2nd edition*. MIT Press, McGraw-Hill Book Company, 2000.
- [5] Yury Polyanskiy. Asynchronous communication: Exact synchronization, universality, and dispersion. *Information Theory, IEEE Transactions on*, 59(3):1256–1270, 2013.
- [6] Sara Shahi, Daniela Tuninetti, and Natasha Devroye. On the capacity of strong asynchronous multiple access channels with a large number of users. In *Information Theory (ISIT), 2016 IEEE International Symposium on*, pages 1486–1490. IEEE, 2016.
- [7] I. Shomorony, R. Etkin, F. Parvaresh, and A.S. Avestimehr. Bounds on the minimum energy-per-bit for bursty traffic in diamond networks. In *Information Theory Proceedings (ISIT), 2012 IEEE International Symposium on*, pages 801–805. IEEE, 2012.
- [8] A. Tchamkerten, V. Chandar, and G. Caire. Energy and sampling constrained asynchronous communication. *Information Theory, IEEE Transactions on*, 60(12):7686–7697, Dec 2014.
- [9] A. Tchamkerten, V. Chandar, and G. W. Wornell. Asynchronous communication: Capacity bounds and suboptimality of training. *Information Theory, IEEE Transactions on*, 59(3):1227–1255, march 2013.
- [10] A. Tchamkerten, V. Chandar, and G.W. Wornell. Communication under strong asynchronism. *Information Theory, IEEE Transactions on*, 55(10):4508–4528, 2009.
- [11] Da Wang. Distinguishing codes from noise: fundamental limits and applications to sparse communication. Master's thesis, Massachusetts Institute of Technology, 2010.
- [12] N. Weinberger and N. Merhav. Codeword or noise? exact random coding exponents for joint detection and decoding. *Information Theory, IEEE Transactions on*, 60(9):5077–5094, Sept 2014.