

# A generalized quantum Slepian-Wolf

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## Abstract

In this work we consider a quantum generalization of the task considered by Slepian and Wolf [1] regarding distributed source compression. In our task Alice, Bob, Charlie and Reference share a joint pure state. Alice and Bob wish to send a part of their respective systems to Charlie without collaborating with each other. We give achievability bounds for this task in the one-shot setting and provide the asymptotic and i.i.d. analysis in the case when there is no side information with Charlie.

Our result implies the result of Abeyesinghe, Devetak, Hayden and Winter in [2] who studied a special case of this problem. As another special case wherein Bob holds trivial registers, we recover the result of Devetak and Yard [3] regarding quantum state redistribution.

## 1 Introduction

In information theory, one of the most fundamental problems is the task of source-compression. The answer to this problem was given by Shannon in his celebrated work [4]. Slepian and Wolf, in their work [1], studied this task in the distributed network setting, which consists of three parties Alice ( $X_1, X_2 \dots X_n$ ), Bob ( $Y_1, Y_2 \dots Y_n$ ) and Charlie, where  $(X_1, Y_1), (X_2, Y_2), \dots (X_n, Y_n)$  are pairs of independent and identically distributed correlated random variables. The goal here is that Alice needs to communicate  $(X_1, X_2, \dots X_n)$  to Charlie and similarly, Bob needs to communicate  $(Y_1, Y_2, \dots Y_n)$  to Charlie. Furthermore, Alice and Bob do not collaborate. From Shannon's result, one can easily see that the amount of total communication needed to accomplish this task is  $nH(X) + nH(Y)$ . However, the surprising feature of the result of Slepian and Wolf is that the amount of total communication only needs to be  $nH(XY)$ . Furthermore, their result implies that there is a trade-off on the amount of communication between (Alice, Charlie) and (Bob, Charlie).

The quantum version of this problem was studied by Abeyesinghe, Devetak, Hayden and Winter in [2]. In this setting, there are four parties, Alice (M), Bob (N), Charlie and Reference (R), where Reference serves as a purifying system for Alice and Bob. The goal is that Alice needs to communicate the register  $M$  to Charlie and Bob needs to communicate the register  $N$  to Charlie, such that the final quantum state between Reference and Charlie is close to the original pure state between Reference, Alice and Bob. The work [2] studied above task in the asymptotic and i.i.d setting. The authors introduced a protocol termed *Fully Quantum Slepian-Wolf* and combined it with Schumacher's compression [5] (using the notion of *time-sharing*) to obtain a rate pair.

The emerging framework of one-shot information theory is providing a new perspective on data compression and channel coding and is relevant in the practical scenarios. It has also led to new insights into the conceptual details of information theoretic protocols. This is largely because the notational complications arising due to many copies of the state are no longer present (although we note that the asymptotic and i.i.d setting also has its own conveniences). One-shot information theory has also found applications in both classical communication complexity [6, 7] and quantum communication complexity [8]. Many quantum tasks have been formulated

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in their one-shot setting, such as quantum state merging ([9, 10], originally introduced in [11]) and quantum state redistribution ([12, 13, 14], originally introduced in [3, 15]). A one-shot version of the distributed source compression with multiple senders was studied in the work [16], where the authors considered the *entanglement consumption* (in place of the quantum communication cost) of the protocol.

Given the importance of one-shot information theory, in this work we consider the one-shot version of the problem studied in [2]. To capture a more general scenario, along with the registers  $M, N$  we also allow Alice, Bob and Charlie to have additional registers  $A, B, C$  respectively. Thus, our setting is as follows, depicted in Figure 1.

**Task 1:** Alice (AM), Bob (BN), Charlie (C) and Reference (R) share a joint pure quantum state. The goal is that Alice needs to communicate the register  $M$  to Charlie and Bob needs to communicate the register  $N$  to Charlie, such that the final quantum state between Reference (R), Alice (A), Bob (B) and Charlie (CMN) is close to the original pure state between the parties. We allow pre-shared entanglement between (Alice, Charlie) and (Bob, Charlie) respectively.

This task is a natural generalization of the aforementioned task considered in [2] and also extends the well studied problem of quantum state redistribution [3, 15]. A special case when  $A$  is trivial was considered by [17] in which they studied the trade-off between the amount of entanglement consumed between Alice and Charlie and the communication between Bob and Charlie. The task is also natural for the well studied simultaneous message passing model [18, 19, 20] in quantum communication complexity. Furthermore, in the setting of quantum communication complexity with three parties, all the parties receive an input and hence can have some side information about the messages from other parties. This is partially captured by above task, a caveat being that the restriction on shared entanglement may not be necessary in general in quantum communication complexity. A special case of the above task, the quantum state redistribution, has found important recent applications in quantum communication complexity [8].

In our results, we shall also consider a *time-reversed* version of the above task, as stated below.

**Task 2:** Alice (A), Bob (B), Charlie (CMN) and Reference (R) share a joint pure quantum state. The goal is that Charlie needs to communicate the register  $M$  to Alice and the register  $N$  to Bob, such that the final quantum state between Reference (R), Alice (M), Bob (N) and Charlie (C) is close to the original pure state between the parties. We allow pre-shared entanglement between (Alice, Charlie) and (Bob, Charlie) respectively.

The motivation to study this task comes from the fact that near-optimal one-shot bounds on the entanglement assisted quantum communication cost of quantum state merging have been obtained by constructing protocols for its time reversed version, quantum state splitting [10, 13].

**Our Results:** Our one-shot result is stated as Theorem 2. We emphasize upon two main ingredients:

- First is that the achievable rate region appears in terms of the max-relative entropy and the hypothesis testing relative entropy.
- Second is that the achievable rate region is a union of a family of achievable rate regions, each characterized by a quantum state that is close to original state  $\Psi$  and satisfies some max-relative entropy constraints.

Using this, we are able to obtain the following achievable rate region for Task 1 in the asymptotic and i.i.d

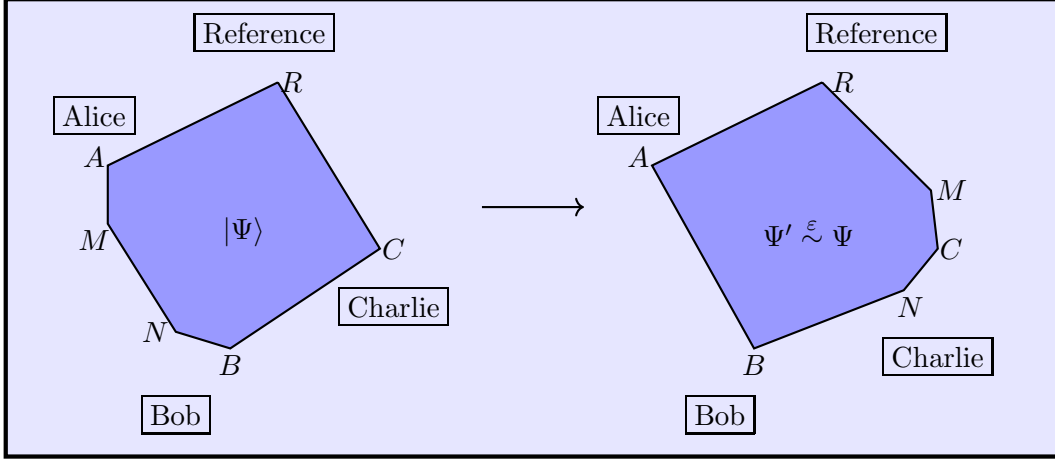


Figure 1: Alice (AM), Bob (BN), Charlie (C) and Reference (R) share a joint pure quantum state  $\Psi$ . The goal is that Alice needs to communicate the register  $M$  to Charlie and Bob needs to communicate the register  $N$  to Charlie, such that the final quantum state  $\Psi'$  between Reference (R), Alice (A), Bob (B) and Charlie (CMN) is close to the original pure state between the parties. Shared entanglement is allowed only between (Alice, Charlie) and (Bob, Charlie) respectively.

setting, when the register  $C$  is trivial:

$$\begin{aligned}
 R_{A \rightarrow C} &\geq \frac{1}{2} (I(RAB : M) - I(A : M)), \\
 R_{B \rightarrow C} &\geq \frac{1}{2} (I(RAB : N) - I(B : N)), \\
 R_{A \rightarrow C} + R_{B \rightarrow C} &\geq \frac{1}{2} \left( I(RAB : M : N) - I(A : M) - I(B : N) \right),
 \end{aligned}$$

where  $R_{A \rightarrow C}$  is the rate of quantum communication from Alice to Charlie,  $R_{B \rightarrow C}$  is the rate of quantum communication from Bob to Charlie and all the information theoretic quantities calculated above are with respect to the state  $\Psi_{RABMN}$  shared between Alice, Bob and Reference. Note that we have used a tripartite version of the mutual information, formally defined in Section 2.

An immediate consequence of the above result is the rate pair obtained for the task considered in [2], with the registers  $A, B$  being trivial. Moreover, if the registers  $B, N$  are trivial in the original task, then the task reduces to that of quantum state redistribution. In this case, the result of Theorem 2 also reproduces the bound given in [3, 15] for quantum state redistribution in the asymptotic and i.i.d. setting.

**Converse bounds:** There are two challenges for obtaining the one-shot converse rate region for our tasks. First is that a matching converse bound for the task of quantum state redistribution, which is a special case of our tasks, is not known in the one-shot setting and is a major open question in quantum information theory. Second, a matching converse for the task considered in [2] is not known even in the asymptotic and i.i.d. setting (as discussed in [2, Section 10]). We are not able to solve any of these challenges, but are able to show a matching one-shot converse for the achievable rate region of Task 2, in the special case where registers  $A, B$  are trivial (more details appear in Section 5). One might be tempted to suggest that this should imply a matching converse for Task 1 in the special case where registers  $A, B$  are trivial (in analogy with quantum state merging and quantum state splitting). Unfortunately this is not the case, since a general protocol for Task 1 might start with Alice and Bob distilling out a pure state on their registers, and then proceeding with a potentially easier communication task. This problem was already recognized in [2, Section 10], which led to a gap between their achievable rate region and their converse. We point out that this problem does not arise in Task 2, as Alice and Bob are not allowed to share entanglement before the protocol starts.

**Techniques:** Along with the inherent challenges of one-shot information theory, an additional challenge for extending the result of [2] is the absence of the notion of time sharing in the one-shot case. The idea of time-sharing is as follows: given two rates  $R = (R_1, R_2)$  and  $R' = (R'_1, R'_2)$  at which Alice and Bob can communicate to Charlie, one can construct a protocol which achieves the rate  $\alpha R + (1 - \alpha)R'$  by using the first protocol for the first  $\alpha n$  copies and using the second protocol for the last  $(1 - \alpha)n$  copies (see [21, Page 534]).

It is clear that this technique cannot extend to the one-shot setting which considers just one copy of the input state. We overcome the obstacle of time sharing in the one-shot case by using the technique of *convex-split* [13] along with *position-based decoding* [22]. The convex-split technique allows one party to prepare a convex combination of states on the registers of other party, if the first party holds a purification of the registers of the second party. The concept of position-based decoding is essentially that of hypothesis testing on a global state.

The technical contribution of this work resides in two aspects. First is that we prove a new version of convex-split lemma [13, Page 3], which we refer to as *tripartite* convex-split lemma, which requires Charlie to prepare a convex combination of quantum states shared between three parties Reference, Alice and Bob. We prove the sufficient conditions which allow Charlie to prepare such convex combination with small error. The second technical contribution is in our asymptotic and i.i.d. analysis of the one-shot bounds. It can be seen that the time-sharing technique, along with the quantum state redistribution protocol of [3, 15], obtains the asymptotic and i.i.d. achievability result mentioned above<sup>1</sup>. Since our one-shot result has no time-sharing involved, we provide an explicit analysis of our bound when there are many independent copies of the state  $\Psi$  shared between the parties, in the case where register  $C$  is absent. For this, we exploit several properties of the quantum information spectrum relative entropy (introduced in [23, 24]; the classical information spectrum approach originated in [25]) to show the existence of a quantum state that is close to the original state  $\Psi$  and satisfies several max-entropy constraints on the reduced systems (given explicitly in the statement of Theorem 4). A special case of this analysis has also appeared in the context of quantum channel coding for the quantum broadcast channel in [22], suggesting a wide applicability of the techniques developed in the proof of Theorem 4.

## Organization

We provide our notations and useful facts in Section 2. We discuss our achievability protocol in Section 3 and the asymptotic and i.i.d. bounds in Section 4. We discuss a converse result in Section 5. We prove the tripartite version of convex-split lemma in Appendix A and give details of the asymptotic and i.i.d. analysis in Appendix B.

## 2 Quantum information theory

Consider a finite dimensional Hilbert space  $\mathcal{H}$  endowed with an inner product  $\langle \cdot, \cdot \rangle$  (in this paper, we only consider finite dimensional Hilbert-spaces). The  $\ell_1$  norm of an operator  $X$  on  $\mathcal{H}$  is  $\|X\|_1 \stackrel{\text{def}}{=} \text{Tr} \sqrt{X^\dagger X}$  and  $\ell_2$  norm is  $\|X\|_2 \stackrel{\text{def}}{=} \sqrt{\text{Tr} X X^\dagger}$ . A quantum state (or a density matrix or a state) is a positive semi-definite matrix on  $\mathcal{H}$  with trace equal to 1. It is called *pure* if and only if its rank is 1. A sub-normalized state is a positive semi-definite matrix on  $\mathcal{H}$  with trace less than or equal to 1. Let  $|\psi\rangle$  be a unit vector on  $\mathcal{H}$ , that is  $\langle \psi, \psi \rangle = 1$ . With some abuse of notation, we use  $\psi$  to represent the state and also the density matrix  $|\psi\rangle\langle\psi|$ , associated with  $|\psi\rangle$ . Given a quantum state  $\rho$  on  $\mathcal{H}$ , *support of  $\rho$* , called  $\text{supp}(\rho)$  is the subspace of  $\mathcal{H}$  spanned by all eigenvectors of  $\rho$  with non-zero eigenvalues.

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<sup>1</sup>The extremal points of the achievable rate region are  $(R_{A \rightarrow C}, R_{B \rightarrow C}) = (\frac{1}{2}I(RB : M|NC), \frac{1}{2}I(RAM : N|C))$  and  $(R_{A \rightarrow C}, R_{B \rightarrow C}) = (\frac{1}{2}I(RBN : M|C), \frac{1}{2}I(RA : N|MC))$ . The first can be achieved by Bob sending  $N$  to Charlie using quantum state redistribution, followed by Alice sending  $M$  to Charlie, again using quantum state redistribution. Second can be achieved in analogous fashion. Any rate pair can then be achieved by time sharing between these two protocols.

A *quantum register*  $A$  is associated with some Hilbert space  $\mathcal{H}_A$ . Define  $|A| \stackrel{\text{def}}{=} \dim(\mathcal{H}_A)$ . Let  $\mathcal{L}(A)$  represent the set of all linear operators on  $\mathcal{H}_A$ . Let  $\mathcal{P}(A)$  represent the set of all positive semidefinite operators on  $\mathcal{H}_A$ . We denote by  $\mathcal{D}(A)$ , the set of quantum states on the Hilbert space  $\mathcal{H}_A$ . State  $\rho$  with subscript  $A$  indicates  $\rho_A \in \mathcal{D}(A)$ . If two registers  $A, B$  are associated with the same Hilbert space, we shall represent the relation by  $A \equiv B$ . Composition of two registers  $A$  and  $B$ , denoted  $AB$ , is associated with Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$ . For two quantum states  $\rho \in \mathcal{D}(A)$  and  $\sigma \in \mathcal{D}(B)$ ,  $\rho \otimes \sigma \in \mathcal{D}(AB)$  represents the tensor product (Kronecker product) of  $\rho$  and  $\sigma$ . The identity operator on  $\mathcal{H}_A$  (and associated register  $A$ ) is denoted  $I_A$ . For any operator  $O$  on  $\mathcal{H}_A$ , we denote by  $\{O\}_+$  the subspace spanned by non-negative eigenvalues of  $O$  and by  $\{O\}_-$  the subspace spanned by negative eigenvalues of  $O$ . For a positive semidefinite operator  $M \in \mathcal{P}(A)$ , the largest and smallest non-zero eigenvalues of  $M$  are denoted by  $\lambda_{\max}(M)$  and  $\lambda_{\min}(M)$ , respectively.

Let  $\rho_{AB} \in \mathcal{D}(AB)$ . We define

$$\rho_B \stackrel{\text{def}}{=} \text{Tr}_A \rho_{AB} \stackrel{\text{def}}{=} \sum_i (\langle i| \otimes I_B) \rho_{AB} (|i\rangle \otimes I_B),$$

where  $\{|i\rangle\}_i$  is an orthonormal basis for the Hilbert space  $\mathcal{H}_A$ . The state  $\rho_B \in \mathcal{D}(B)$  is referred to as the marginal state of  $\rho_{AB}$ . Unless otherwise stated, a register missing from the subscript of a state will represent the partial trace over that register. Given a  $\rho_A \in \mathcal{D}(A)$ , a *purification* of  $\rho_A$  is a pure state  $\rho_{AB} \in \mathcal{D}(AB)$  such that  $\text{Tr}_B \rho_{AB} = \rho_A$ . A purification of a quantum state is not unique.

A quantum map  $\mathcal{E} : \mathcal{L}(A) \rightarrow \mathcal{L}(B)$  is a completely positive and trace preserving (CPTP) linear map (mapping states in  $\mathcal{D}(A)$  to states in  $\mathcal{D}(B)$ ). A *unitary* operator  $U_A : \mathcal{H}_A \rightarrow \mathcal{H}_A$  is such that  $U_A^\dagger U_A = U_A U_A^\dagger = I_A$ . An *isometry*  $V : \mathcal{H}_A \rightarrow \mathcal{H}_B$  is such that  $V^\dagger V = I_A$  and  $V V^\dagger = \Pi_B$ , for a projection  $\Pi_B$  on  $\mathcal{H}_B$ . The set of all unitary operations on register  $A$  is denoted by  $\mathcal{U}(A)$ . For registers  $A$  and  $B$  with  $|A| = |B|$ , the operation that swaps these registers is  $\text{SWAP}_{A,B} \stackrel{\text{def}}{=} \sum_{i,j} |i,j\rangle \langle j,i|$ , for an arbitrary basis  $\{|i\rangle\}_{i=1}^{|A|}, \{|j\rangle\}_{j=1}^{|B|}$  on  $\mathcal{H}_A, \mathcal{H}_B$  respectively.

We shall consider the following information theoretic quantities. Let  $\varepsilon \in (0, 1)$ .

1. **Fidelity** ([26], see also [27]) For  $\rho_A, \sigma_A \in \mathcal{D}(A)$ ,

$$F(\rho_A, \sigma_A) \stackrel{\text{def}}{=} \|\sqrt{\rho_A} \sqrt{\sigma_A}\|_1.$$

For classical probability distributions  $P = \{p_i\}$ ,  $Q = \{q_i\}$ ,

$$F(P, Q) \stackrel{\text{def}}{=} \sum_i \sqrt{p_i \cdot q_i}.$$

2. **Purified distance** ([28]) For  $\rho_A, \sigma_A \in \mathcal{D}(A)$ ,

$$P(\rho_A, \sigma_A) = \sqrt{1 - F^2(\rho_A, \sigma_A)}.$$

3.  **$\varepsilon$ -ball** For  $\rho_A \in \mathcal{D}(A)$ ,

$$\mathcal{B}^\varepsilon(\rho_A) \stackrel{\text{def}}{=} \{\rho'_A \in \mathcal{D}(A) \mid P(\rho_A, \rho'_A) \leq \varepsilon\}.$$

4. **Von-Neumann entropy** ([29]) For  $\rho_A \in \mathcal{D}(A)$ ,

$$S(\rho_A) \stackrel{\text{def}}{=} -\text{Tr}(\rho_A \log \rho_A).$$

5. **Relative entropy** ([30]) For  $\rho_A \in \mathcal{D}(A)$ ,  $\sigma_A \in \mathcal{P}(A)$  such that  $\text{supp}(\rho_A) \subset \text{supp}(\sigma_A)$ ,

$$D(\rho_A \parallel \sigma_A) \stackrel{\text{def}}{=} \text{Tr}(\rho_A \log \rho_A) - \text{Tr}(\rho_A \log \sigma_A).$$

6. **Max-relative entropy** ([31]) For  $\rho_A, \sigma_A \in \mathcal{P}(A)$  such that  $\text{supp}(\rho_A) \subset \text{supp}(\sigma_A)$ ,

$$D_{\max}(\rho_A \| \sigma_A) \stackrel{\text{def}}{=} \inf\{\lambda \in \mathbb{R} : \rho_A \preceq 2^\lambda \sigma_A\}.$$

7. **Smooth max-relative entropy** ([31], see also [32]) For  $\rho_A \in \mathcal{D}(A), \sigma_A \in \mathcal{P}(A)$  such that  $\text{supp}(\rho_A) \subset \text{supp}(\sigma_A)$ ,

$$D_{\max}^\varepsilon(\rho_A \| \sigma_A) \stackrel{\text{def}}{=} \sup_{\rho'_A \in \mathcal{B}^\varepsilon(\rho_A)} D_{\max}(\rho'_A \| \sigma_A).$$

8. **Hypothesis testing relative entropy** ([33], see also [23]) For  $\rho_A \in \mathcal{D}(A), \sigma_A \in \mathcal{P}(A)$ ,

$$D_H^\varepsilon(\rho_A \| \sigma_A) \stackrel{\text{def}}{=} \sup_{0 \preceq \Pi \preceq I, \text{Tr}(\Pi \rho_A) \geq 1 - \varepsilon} \log \left( \frac{1}{\text{Tr}(\Pi \sigma_A)} \right).$$

9. **Information spectrum relative entropy** ([23, 24]) For  $\rho_A \in \mathcal{D}(A), \sigma_A \in \mathcal{P}(A)$  such that  $\text{supp}(\rho_A) \subset \text{supp}(\sigma_A)$ ,

$$D_s^\varepsilon(\rho_A \| \sigma_A) \stackrel{\text{def}}{=} \sup\{R : \text{Tr}(\rho_A \{\rho_A - 2^R \sigma_A\}_+) \geq 1 - \varepsilon\}.$$

10. **Information spectrum relative entropy [Alternate definition]** For  $\rho_A \in \mathcal{D}(A), \sigma_A \in \mathcal{P}(A)$  such that  $\text{supp}(\rho_A) \subset \text{supp}(\sigma_A)$ ,

$$\tilde{D}_s^\varepsilon(\rho_A \| \sigma_A) \stackrel{\text{def}}{=} \inf\{R : \text{Tr}(\rho_A \{\rho_A - 2^R \sigma_A\}_-) \geq 1 - \varepsilon\}.$$

11. **Mutual information** For  $\rho_{AB} \in \mathcal{D}(AB)$ ,

$$I(A : B)_\rho \stackrel{\text{def}}{=} S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = D(\rho_{AB} \| \rho_A \otimes \rho_B).$$

12. **tripartite mutual information** For  $\rho_{ABC} \in \mathcal{D}(ABC)$ ,

$$\begin{aligned} I(A : B : C)_\rho &\stackrel{\text{def}}{=} S(\rho_A) + S(\rho_B) + S(\rho_C) - S(\rho_{ABC}) \\ &= D(\rho_{ABC} \| \rho_A \otimes \rho_B \otimes \rho_C). \end{aligned}$$

We will use the following facts.

**Fact 1** (Triangle inequality for purified distance, [34, 28]). *For quantum states  $\rho_A, \sigma_A, \tau_A \in \mathcal{D}(A)$ ,*

$$P(\rho_A, \sigma_A) \leq P(\rho_A, \tau_A) + P(\tau_A, \sigma_A).$$

**Fact 2** (Monotonicity under quantum operations, [35],[36]). *For quantum states  $\rho, \sigma \in \mathcal{D}(A)$ , and quantum operation  $\mathcal{E}(\cdot) : \mathcal{L}(A) \rightarrow \mathcal{L}(B)$ , it holds that*

$$\begin{aligned} D_{\max}(\rho \| \sigma) &\geq D_{\max}(\mathcal{E}(\rho) \| \mathcal{E}(\sigma)), \\ F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &\geq F(\rho, \sigma), \\ D_H^\varepsilon(\rho \| \sigma) &\geq D_H^\varepsilon(\mathcal{E}(\rho) \| \mathcal{E}(\sigma)). \end{aligned}$$

*In particular, for bipartite states  $\rho_{AB}, \sigma_{AB} \in \mathcal{D}(AB)$ , it holds that*

$$\begin{aligned} D_{\max}(\rho_{AB} \| \sigma_{AB}) &\geq D_{\max}(\rho_A \| \sigma_A), \\ F(\rho_{AB}, \sigma_{AB}) &\geq F(\rho_A, \sigma_A), \\ D_H^\varepsilon(\rho_{AB} \| \sigma_{AB}) &\geq D_H^\varepsilon(\rho_A \| \sigma_A). \end{aligned}$$

**Fact 3** (Uhlmann's Theorem, [27]). Let  $\rho_A, \sigma_A \in \mathcal{D}(A)$ . Let  $\rho_{AB} \in \mathcal{D}(AB)$  be a purification of  $\rho_A$  and  $|\sigma\rangle_{AC} \in \mathcal{D}(AC)$  be a purification of  $\sigma_A$ . There exists an isometry  $V : C \rightarrow B$  such that,

$$F(|\theta\rangle\langle\theta|_{AB}, |\rho\rangle\langle\rho|_{AB}) = F(\rho_A, \sigma_A),$$

where  $|\theta\rangle_{AB} = (\mathbf{I}_A \otimes V)|\sigma\rangle_{AC}$ .

Following fact implies the Pinsker's inequality.

**Fact 4** (Lemma 5, [37]). For quantum states  $\rho_A, \sigma_A \in \mathcal{D}(A)$ ,

$$F(\rho, \sigma) \geq 2^{-\frac{1}{2}D(\rho\|\sigma)}.$$

**Fact 5** (Lemma B.7, [10]). For a quantum state  $\rho_{AB} \in \mathcal{D}(AB)$ , it holds that  $D_{\max}(\rho_{AB} \parallel \rho_A \otimes \frac{\mathbf{I}_B}{|B|}) \leq 2 \log |B|$ .

**Fact 6** (Gentle measurement lemma, [38, 39]). Let  $\rho$  be a quantum state and  $0 \preceq A \preceq \mathbf{I}$  be an operator. Then

$$F(\rho, \frac{A\rho A}{\text{Tr}(A^2\rho)}) \geq \sqrt{\text{Tr}(A^2\rho)}.$$

**Lemma 1.** Consider a pure quantum state  $|\rho\rangle_{ORA} = \sum_i \sqrt{p_i} |i\rangle_O |\rho^i\rangle_{RA}$  and an isometry  $\mathcal{A} = \sum_i P_i \otimes |i\rangle_{O'}$ , such that  $0 < P_i < \mathbf{I}_A$ ,  $\sum_i P_i^2 = \mathbf{I}_A$ . Define the state  $|\rho'\rangle_{ORAO'} \stackrel{\text{def}}{=} \sum_i \sqrt{p_i} |i\rangle_O |\rho^i\rangle_{RA} |i\rangle_{O'}$  and let  $q_i \stackrel{\text{def}}{=} \text{Tr}(P_i^2 \rho_A^i)$ . Then it holds that

$$F(\rho'_{ORAO'}, \mathcal{A}\rho_{ORA}\mathcal{A}^\dagger) \geq \sum_i p_i q_i.$$

*Proof.* Consider the state

$$\mathcal{A}|\rho\rangle_{ORA} = \sum_{i,j} \sqrt{p_i} |i\rangle_O (\mathbf{I}_R \otimes P_j) |\rho^i\rangle_{RA} |j\rangle_{O'}.$$

We compute

$$\begin{aligned} F(\rho'_{ORAO'}, \mathcal{A}\rho_{ORA}\mathcal{A}^\dagger) &= |(\sum_i \sqrt{p_i} \langle i|_O \langle \rho^i|_{RA} \langle i|_{O'}) \\ &\quad (\sum_{i,j} \sqrt{p_i} |i\rangle_O (\mathbf{I}_R \otimes P_j) |\rho^i\rangle_{RA} |j\rangle_{O'})| \\ &= |\sum_i p_i \langle \rho^i|_{RA} (\mathbf{I}_R \otimes P_i) |\rho^i\rangle_{RA}| \\ &= \sum_i p_i \text{Tr}(P_i \rho_A^i) \geq \sum_i p_i \text{Tr}(P_i^2 \rho_A^i), \end{aligned}$$

where the last inequality follows from the fact that  $P_i^2 \preceq P_i$ , which is implied by  $P_i \preceq \mathbf{I}_A$ . This completes the proof by the definition of  $q_i$ .  $\square$

**Fact 7** ([14]). Let  $\varepsilon, \delta \in (0, 1)$  such that  $2\varepsilon + \delta < 1$ . Let  $\rho, \sigma$  be quantum states such that  $P(\rho, \sigma) \leq \varepsilon$ . Let  $0 \preceq \Pi \preceq \mathbf{I}$  be an operator such that  $\text{Tr}(\Pi\rho) \geq 1 - \delta^2$ . Then  $\text{Tr}(\Pi\sigma) \geq 1 - (2\varepsilon + \delta)^2$ . If  $\delta = 0$ , then  $\text{Tr}(\Pi\sigma) \geq 1 - \varepsilon^2$ .

**Fact 8** (Hayashi-Nagaoka inequality, [23]). Let  $0 \preceq S \preceq \mathbf{I}, T$  be positive semi-definite operators. Then

$$\mathbf{I} - (S + T)^{-\frac{1}{2}} S (S + T)^{-\frac{1}{2}} \preceq 2(\mathbf{I} - S) + 4T.$$

### 3 Achievable rate region for distributed quantum source compression with side information

We define our tasks formally below.

**Task 1:** There are four parties Alice, Bob, Charlie and Reference. Furthermore, Alice ( $AM$ ), Bob ( $BN$ ), Reference ( $R$ ) and Charlie ( $C$ ) share the joint pure state  $|\Psi\rangle_{RAMBNC}$ . Alice and Bob wish to communicate their registers  $M$  and  $N$  to Charlie such that the final state shared between Alice ( $A$ ), Bob ( $B$ ), Reference ( $R$ ) and Charlie ( $CMN$ ) is  $\Phi_{RABCMN}$  with the property that  $P(\Phi, \Psi) \leq \varepsilon$ , where  $\varepsilon \in (0, 1)$  is an error parameter. To accomplish this task, Alice and Charlie are also allowed pre-shared entanglement. Similarly, Bob and Charlie are allowed the same. See Figure 1.

To accomplish Task 1, we will first consider the *time-reversed* version defined as follows.

**Task 2:** There are four parties Alice, Bob, Charlie and Reference. Furthermore, Alice ( $A$ ), Bob ( $B$ ), Reference ( $R$ ) and Charlie ( $CMN$ ) share the joint pure state  $|\Psi\rangle_{RAMBNC}$ . Charlie wishes to communicate her register  $M$  to Alice and  $N$  to Bob such that the final state shared between Alice ( $AM$ ), Bob ( $BN$ ), Reference ( $R$ ) and Charlie ( $C$ ) is  $\Phi_{RAMBNC}$  with the property that  $P(\Phi, \Psi) \leq \varepsilon$ , where  $\varepsilon \in (0, 1)$  is an error parameter. To accomplish this task, Alice and Charlie are also allowed pre-shared entanglement. Similarly, Bob and Charlie are allowed the same.

#### Main result: Achievable rate region for Task 2

**Theorem 1.** Fix  $\varepsilon_1, \varepsilon_2, \delta \in (0, 1)$  such that  $\varepsilon_1 + 5\varepsilon_2 + 2\sqrt{\delta} < 1$ . Let Alice ( $A$ ), Bob ( $B$ ), Reference ( $R$ ) and Charlie ( $CMN$ ) share the pure state  $|\Psi\rangle_{RAMBNC}$ . There exists an entanglement assisted quantum protocol with pre-shared entanglement of the form  $|\theta_1\rangle \otimes |\theta_2\rangle$  (where  $|\theta_1\rangle$  is shared between Alice, Charlie in some registers  $E_{AC}$  and  $|\theta_2\rangle$  is shared between Bob, Charlie in some registers  $E_{BC}$ ), such that at the end of the protocol following properties hold.

- The global shared state is  $|\Phi\rangle_{RAMBNCE'_{AC}E'_{BC}}$  with  $R$  belonging to Reference, ( $AM$ ) belonging to Alice, ( $BN$ ) belonging to Bob,  $C$  belonging to Charlie,  $E'_{AC}$  belonging to (Alice, Charlie) and  $E'_{BC}$  belonging to (Bob, Charlie).
- There exist states  $|\theta'_1\rangle_{E'_{AC}}$  and  $|\theta'_2\rangle_{E'_{BC}}$  such that  $P(|\Phi\rangle\langle\Phi|, |\Psi\rangle\langle\Psi| \otimes |\theta'_1\rangle\langle\theta'_1| \otimes |\theta'_2\rangle\langle\theta'_2|) \leq \varepsilon_1 + 5\varepsilon_2 + 2\sqrt{\delta}$ .

The number of qubits that Charlie sends to Alice and Bob are  $R_{C \rightarrow A}$  and  $R_{C \rightarrow B}$  be respectively, where the pair  $(R_{C \rightarrow A}, R_{C \rightarrow B})$  lie in the union of the following achievable rate region: for every  $\Psi'_{RABCMN} \in \mathcal{B}^{\varepsilon_1}(\Psi_{RABCMN})$  such that  $\Psi'_{RAB} \preceq 2^\delta \Psi_{RAB}$  and states  $\sigma_M, \omega_N$ :

$$\begin{aligned}
R_{C \rightarrow A} &\geq \frac{1}{2} \left( D_{\max}(\Psi'_{RABM} \| \Psi_{RAB} \otimes \sigma_M) - D_H^{\varepsilon_2^2}(\Psi_{AM} \| \Psi_A \otimes \sigma_M) + \log \frac{1}{\varepsilon_2^2 \delta} \right), \\
R_{C \rightarrow B} &\geq \frac{1}{2} \left( D_{\max}(\Psi'_{RABN} \| \Psi_{RAB} \otimes \omega_N) - D_H^{\varepsilon_2^2}(\Psi_{BN} \| \Psi_B \otimes \omega_N) + \log \frac{1}{\varepsilon_2^2 \delta} \right), \\
R_{C \rightarrow A} + R_{C \rightarrow B} &\geq \frac{1}{2} \left( D_{\max}(\Psi'_{RABMN} \| \Psi_{RAB} \otimes \sigma_M \otimes \omega_N) - D_H^{\varepsilon_2^2}(\Psi_{AM} \| \Psi_A \otimes \sigma_M) \right. \\
&\quad \left. - D_H^{\varepsilon_2^2}(\Psi_{BN} \| \Psi_B \otimes \omega_N) + \log \frac{1}{\varepsilon_2^2 \delta} \right).
\end{aligned}$$

*Proof.* We divide our proof into the following steps.



**1. Quantum states and registers involved in the proof:** Fix  $\Psi'_{RAB} \in \mathcal{B}^{\varepsilon_1}(\Psi_{RAB})$ , states  $\sigma_M, \omega_N$  and the pair  $(R_{C \rightarrow A}, R_{C \rightarrow B})$  as mentioned in the lemma. Let  $R_A \stackrel{\text{def}}{=} 2 \cdot R_{C \rightarrow A}$  and  $R_B \stackrel{\text{def}}{=} 2 \cdot R_{C \rightarrow B}$ . Let  $r_A, r_B$  be such that

$$r_A \leq D_H^{\varepsilon_2^2}(\Psi_{AM} \| \Psi_A \otimes \sigma_M) + 2 \log \varepsilon_2,$$

$$r_B \leq D_H^{\varepsilon_2^2}(\Psi_{BN} \| \Psi_B \otimes \omega_N) + 2 \log \varepsilon_2.$$

Let  $\Pi_{AM}^A$  and  $\Pi_{BN}^B$  be projectors achieving the optimum in the definitions of  $D_H^{\varepsilon_2^2}(\Psi_{AM} \| \Psi_A \otimes \sigma_M)$  and  $D_H^{\varepsilon_2^2}(\Psi_{BN} \| \Psi_B \otimes \omega_N)$  respectively.

Introduce registers  $M_1, M_2, \dots, M_{2R_A+r_A}$  such that for all  $i$ ,  $M_i \equiv M$  and  $N_1, N_2, \dots, N_{2R_B+r_B}$  such that for all  $i$ ,  $N_i \equiv N$ . For brevity, we define  $\sigma^{(-j)} \stackrel{\text{def}}{=} \sigma_{M_1} \otimes \dots \otimes \sigma_{M_{j-1}} \otimes \sigma_{M_{j+1}} \otimes \dots \otimes \sigma_{M_{2R_A+r_A}}$  and  $\omega^{(-k)} \stackrel{\text{def}}{=} \omega_{N_1} \otimes \dots \otimes \omega_{N_{k-1}} \otimes \omega_{N_{k+1}} \otimes \dots \otimes \omega_{N_{2R_B+r_B}}$ . Consider the states,

$$\mu_{RABM_1 \dots M_{2R_A+r_A} N_1 \dots N_{2R_B+r_B}} \stackrel{\text{def}}{=} \frac{1}{2^{R_A+r_A+R_B+r_B}} \times \sum_{j=1, k=1}^{2R_A+r_A, 2R_B+r_B} \Psi_{RABM_j N_k} \otimes \sigma^{(-j)} \otimes \omega^{(-k)}$$

$$\xi_{RABM_1 \dots M_{2R_A+r_A} N_1 \dots N_{2R_B+r_B}} \stackrel{\text{def}}{=} \Psi_{RAB} \otimes \sigma_{M_1} \dots \otimes \sigma_{M_{2R_A+r_A}} \otimes \omega_{N_1} \dots \otimes \omega_{N_{2R_B+r_B}}.$$

Note that  $\Psi_{RAB} = \mu_{RAB}$ . Let

$$|\theta\rangle = |\sigma\rangle_{M'_1 M_1} \otimes \dots \otimes |\sigma\rangle_{M'_{2R_A+r_A} M_{2R_A+r_A}} \otimes |\omega\rangle_{N'_1 N_1} \otimes \dots \otimes |\omega\rangle_{N'_{2R_B+r_B} N_{2R_B+r_B}}$$

be a purification of  $\sigma_{M_1} \dots \otimes \sigma_{M_{2R_A+r_A}} \otimes \omega_{N_1} \dots \otimes \omega_{N_{2R_B+r_B}}$ . Let

$$|\xi\rangle \stackrel{\text{def}}{=} |\Psi\rangle_{RABCMN} \otimes |\theta\rangle_{M'_1 \dots M'_{2R_A+r_A} N'_1 \dots N'_{2R_B+r_B} M_1 \dots M_{2R_A+r_A} N_1 \dots N_{2R_B+r_B}}$$

be a purification of  $\xi_{RABM_1 \dots M_{2R_A+r_A} N_1 \dots N_{2R_B+r_B}}$ .

Consider the following purification of  $\mu_{RABM_1 \dots M_{2R_A+r_A} N_1 \dots N_{2R_B+r_B}}$ ,

$$\frac{1}{\sqrt{2^{R_A+r_A} \cdot 2^{R_B+r_B}}} \sum_{j=1, k=1}^{2R_A+r_A, 2R_B+r_B} |j, k\rangle_{JK} \otimes |\Psi\rangle_{RABCM_j N_k} \otimes |\sigma^{(-j)}\rangle \otimes |0\rangle_{M'_j} \otimes |\omega^{(-k)}\rangle \otimes |0\rangle_{N'_k},$$

where

$$|\sigma^{(-j)}\rangle \stackrel{\text{def}}{=} |\sigma\rangle_{M'_1 M_1} \otimes \dots \otimes |\sigma\rangle_{M'_{j-1} M_{j-1}} \otimes |\sigma\rangle_{M'_{j+1} M_{j+1}} \otimes \dots \otimes |\sigma\rangle_{M'_{2R_A+r_A} M_{2R_A+r_A}}$$

and

$$|\omega^{(-k)}\rangle \stackrel{\text{def}}{=} |\omega\rangle_{N'_1 N_1} \otimes \dots \otimes |\omega\rangle_{N'_{k-1} N_{k-1}} \otimes |\omega\rangle_{N'_{k+1} N_{k+1}} \otimes \dots \otimes |\omega\rangle_{N'_{2R_B+r_B} N_{2R_B+r_B}}$$

and  $\forall j \in [2R_A+r_A] : |\sigma\rangle_{M'_j M_j}$  is a purification of  $\sigma_{M_j}$  and  $\forall k \in [2R_B+r_B] : |\omega\rangle_{N'_k N_k}$  is a purification of  $\omega_{N_k}$ .

We decompose the register  $J$  into registers  $J_1, J_2$  satisfying  $|J_1| = 2^{R_A}, |J_2| = 2^{r_A}$ . Similarly, we decompose the register  $K$  into registers  $K_1, K_2$  satisfying  $|K_1| = 2^{R_B}, |K_2| = 2^{r_B}$ . Using this, we obtain the following state as a purification of  $\mu_{RABM_1 \dots M_{2R_A+r_A} N_1 \dots N_{2R_B+r_B}}$  on registers  $RABCM_1 \dots M_{2R_A+r_A} N_1 \dots N_{2R_B+r_B}$  as a purification of  $\mu_{RABM_1 \dots M_{2R_A+r_A} N_1 \dots N_{2R_B+r_B}}$  on registers  $RABCM_1 \dots M_{2R_A+r_A} N_1 \dots N_{2R_B+r_B}$ :

$$|\mu\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2^{R_A+r_A} \cdot 2^{R_B+r_B}}} \sum_{j_1, j_2, k_1, k_2} |j_1, j_2, k_1, k_2\rangle_{J_1 J_2 K_1 K_2} \otimes |\Psi\rangle_{RABCM_j N_k} \otimes |\sigma^{(-j)}\rangle \otimes |0\rangle_{M'_j} \otimes |\omega^{(-k)}\rangle \otimes |0\rangle_{N'_k},$$

where,  $j_1 \in [1 : 2^{R_A}]$ ,  $j_2 \in [1 : 2^{r_A}]$ ,  $k_1 \in [1 : 2^{R_B}]$ ,  $k_2 \in [1 : 2^{r_B}]$ ,  $j \stackrel{\text{def}}{=} (j_1 - 1)2^{r_A} + j_2$ ,  $k \stackrel{\text{def}}{=} (k_1 - 1)2^{r_B} + k_2$ . Henceforth, we shall take the convention that  $j = (j_1 - 1)2^{r_A} + j_2$ ,  $k = (k_1 - 1)2^{r_B} + k_2$ , whenever it is clear from the context.

Using the tripartite convex-split lemma (Lemma 2) and choice of  $R_A + r_A, R_B + r_B$ , we have

$$F^2(\xi_{RABM_1 \dots M_{2^{R_A+r_A}} N_1 \dots N_{2^{R_B+r_B}}}, \mu_{RABM_1 \dots M_{2^{R_A+r_A}} N_1 \dots N_{2^{R_B+r_B}}}) \geq 1 - (\varepsilon_1 + 2\sqrt{\delta})^2.$$

Let  $|\xi'\rangle$  be a purification of  $\xi_{RABM_1 \dots M_{2^{R_A+r_A}} N_1 \dots N_{2^{R_B+r_B}}}$  (guaranteed by Uhlmann's theorem, Fact 3) such that,

$$\begin{aligned} F^2(|\xi'\rangle\langle\xi'|, |\mu\rangle\langle\mu|) &= F^2(\xi_{RABM_1 \dots M_{2^{R_A+r_A}} N_1 \dots N_{2^{R_B+r_B}}}, \mu_{RABM_1 \dots M_{2^{R_A+r_A}} N_1 \dots N_{2^{R_B+r_B}}}) \\ &\geq 1 - (\varepsilon_1 + 2\sqrt{\delta})^2. \end{aligned} \quad (1)$$

Let  $V' : CMNM'_1 \dots M'_{2^{R_A+r_A}} N'_1 \dots N'_{2^{R_B+r_B}} \rightarrow J_1 J_2 K_1 K_2 CM'_1 \dots M'_{2^{R_A+r_A}} N'_1 \dots N'_{2^{R_B+r_B}}$  be an isometry (guaranteed by Uhlmann's theorem, Fact 3) such that,

$$V'|\xi\rangle = |\xi'\rangle.$$

## 2. The protocol: Consider the following protocol $\mathcal{P}$ :

1. Alice, Bob, Charlie and Reference start by sharing the state  $|\xi\rangle$  between themselves where Alice holds registers  $AM_1 \dots M_{2^{R_A+r_A}}$ , Bob holds the registers  $BN_1 \dots N_{2^{R_B+r_B}}$ , Charlie holds the registers  $CMNM'_1 \dots M'_{2^{R_A+r_A}} N'_1 \dots N'_{2^{R_B+r_B}}$  and Reference holds the register  $R$ . Note that  $|\Psi\rangle_{RABCMN}$  is provided as input to the protocol and  $|\theta\rangle$  is the additional shared entanglement of the form  $|\theta\rangle = |\theta_1\rangle \otimes |\theta_2\rangle$ , with  $|\theta_1\rangle$  shared between (Alice, Charlie) in registers  $E_{AC} \stackrel{\text{def}}{=} M_1 M'_1 \dots M_{2^{R_A+r_A}} M'_{2^{R_A+r_A}}$  and  $|\theta_2\rangle$  shared between (Bob, Charlie) in registers  $E_{BC} \stackrel{\text{def}}{=} N_1 N'_1 \dots N_{2^{R_B+r_B}} N'_{2^{R_B+r_B}}$ .
2. Charlie applies the isometry  $V'$  to obtain state the  $|\xi'\rangle$ .
  - At this stage of the protocol, the global state  $|\xi'\rangle$  is close to the state  $|\mu\rangle$ , where Alice holds the registers  $AM_1 \dots M_{2^{R_A+r_A}}$ , Bob holds the registers  $BN_1 N_2 \dots N_{2^{R_B+r_B}}$ , Charlie holds the registers  $CJ_1 J_2 K_1 K_2 M'_1 \dots M'_{2^{R_A+r_A}} N'_1 \dots N'_{2^{R_B+r_B}}$  and Reference holds the register  $R$ .
3. Charlie measures the registers  $J_1, K_1$  and obtains the measurement outcomes  $(j_1, k_1) \in [1 : 2^{R_A}] \times [1 : 2^{R_B}]$ . He sends  $j_1$  to Alice and  $k_1$  to Bob using  $\frac{R_A}{2}$  and  $\frac{R_B}{2}$  qubits of respective quantum communication. Charlie, Alice and Bob employ coherent superdense coding ([40, 41]) using fresh entanglement to achieve this. Let the final register obtained with Alice be  $J'_1$  and with Bob be  $K'_1$ .
  - Note that the additional entanglement for coherent superdense coding is still shared between (Alice, Charlie) and (Bob, Charlie) respectively. Furthermore, the coherent superdense coding scheme does not output any registers other than  $J'_1$  and  $K'_1$ .
  - If global state in step 2 above were  $|\mu\rangle$ , the global state at this step would be

$$\begin{aligned} |\mu^{(2)}\rangle &\stackrel{\text{def}}{=} \frac{1}{\sqrt{2^{R_A+r_A} \cdot 2^{R_B+r_B}}} \sum_{j_1, j_2, k_1, k_2} |j_1, j_2, k_1, k_2\rangle_{J_1 J_2 K_1 K_2} \otimes |j_1, k_1\rangle_{J'_1, K'_1} \\ &\quad \otimes |\Psi\rangle_{RABCM_j N_k} \otimes |\sigma^{(-j)}\rangle \otimes |0\rangle_{M'_j} \otimes |\omega^{(-k)}\rangle \otimes |0\rangle_{N'_k}. \end{aligned}$$

- For a fixed pair  $(j_1, k_1)$ , we define the state

$$|\mu_{(j_1, k_1)}^{(2)}\rangle = \frac{1}{\sqrt{2^{r_A+r_B}}} \sum_{j_2, k_2} |j_2, k_2\rangle_{J_2 K_2} \otimes |\Psi\rangle_{RABCM_j N_k} \otimes |\sigma^{(-j)}\rangle \otimes |0\rangle_{M'_j} \otimes |\omega^{(-k)}\rangle \otimes |0\rangle_{N'_k}.$$

4. This is the hypothesis testing step. Conditioned on  $(j_1, k_1)$ , Alice and Bob consider the following operations. Let

$$\Pi_{j_2}^A \stackrel{\text{def}}{=} \Pi_{AM_j}^A \otimes I_{M_1} \otimes \dots \otimes I_{M_{j-1}} \otimes I_{M_{j+1}} \dots \otimes I_{M_{2^{r_A}+r_A}},$$

with  $j = (j_1 - 1)2^{r_A} + j_2$  and  $\Pi^A \stackrel{\text{def}}{=} \sum_{j_2} \Pi_{j_2}^A$ . Let

$$\Pi_{k_2}^B \stackrel{\text{def}}{=} \Pi_{BN_k}^B \otimes I_{M_1} \otimes \dots \otimes I_{M_{k-1}} \otimes I_{M_{k+1}} \dots \otimes I_{M_{2^{r_B}+r_B}},$$

with  $k = (k_1 - 1)2^{r_B} + k_2$  and  $\Pi^B \stackrel{\text{def}}{=} \sum_{k_2} \Pi_{k_2}^B$ . Let  $(\Pi^A)^0$ , which is the operator  $\Pi^A$  raised to the power 0, represent the support of  $\Pi^A$ . Similarly let  $(\Pi^B)^0$  represent the support of  $\Pi^B$ .

Alice applies the isometry  $\sum_{j_1} |j_1\rangle\langle j_1|_{J'_1} \otimes \mathcal{A}_{j_1}$ , where

$$\mathcal{A}_{j_1} \stackrel{\text{def}}{=} \sum_{j_2} \sqrt{(\Pi^A)^{-\frac{1}{2}} \Pi_{j_2}^A (\Pi^A)^{-\frac{1}{2}}} \otimes |j_2\rangle_{J'_2} + \sqrt{I - (\Pi^A)^0} \otimes |0\rangle_{J'_2}.$$

Bob applies the isometry  $\sum_{k_1} |k_1\rangle\langle k_1|_{K'_1} \otimes \mathcal{B}_{k_1}$ , where

$$\mathcal{B}_{k_1} \stackrel{\text{def}}{=} \sum_{k_2} \sqrt{(\Pi^B)^{-\frac{1}{2}} \Pi_{k_2}^B (\Pi^B)^{-\frac{1}{2}}} \otimes |k_2\rangle_{K'_2} + \sqrt{I - (\Pi^B)^0} \otimes |0\rangle_{K'_2}.$$

Above operations are coherent versions of the position-based decoding operation [22] and the outcome  $|0\rangle$  corresponds to Alice or Bob not being able to decode any location. Define

$$|\mu_{(j_1, k_1)}^{(3)}\rangle := \mathcal{A}_{j_1} \otimes \mathcal{B}_{k_1} |\mu_{(j_1, k_1)}^{(2)}\rangle.$$

- If the global state on Step 2 above were  $|\mu\rangle$ , the resulting global state at this step would be

$$|\mu^{(3)}\rangle = \frac{1}{\sqrt{2^{r_A+r_B}}} \sum_{j_1, k_1} |j_1, k_1\rangle_{J_1 K_1} \otimes |j_1, k_1\rangle_{J'_1 K'_1} \otimes |\mu_{(j_1, k_1)}^{(3)}\rangle.$$

- Define the state

$$|\mu^{(4)}\rangle = \frac{1}{\sqrt{2^{r_A+r_B}}} \sum_{j_1, k_1} |j_1, k_1\rangle_{J_1 K_1} \otimes |j_1, k_1\rangle_{J'_1 K'_1} \otimes |\mu_{(j_1, k_1)}^{(4)}\rangle,$$

where

$$\begin{aligned} |\mu_{(j_1, k_1)}^{(4)}\rangle &\stackrel{\text{def}}{=} \frac{1}{\sqrt{2^{r_A+r_B}}} \sum_{j_2, k_2} |j_2, k_2\rangle_{J_2 K_2} \otimes |j_2, k_2\rangle_{J'_2 K'_2} \otimes |\Psi\rangle_{RABCM_j N_k} \otimes \\ &|\sigma^{(-j)}\rangle \otimes |0\rangle_{M'_j} \otimes |\omega^{(-k)}\rangle \otimes |0\rangle_{N'_k}. \end{aligned}$$

- We shall show in Claim 1 that  $|\mu^{(3)}\rangle$  is close to  $|\mu^{(4)}\rangle$ .

5. Alice and Bob introduce the registers  $M, N$  in the states  $|0\rangle_M, |0\rangle_N$  respectively. Alice applies the operation

$$\sum_{j_1, j_2} |j_1, j_2\rangle\langle j_1, j_2|_{J'_1 J'_2} \otimes \text{SWAP}_{M, M_j},$$

where  $j = (j_1 - 1) \cdot 2^{r_A} + j_2$ . Similarly, Bob applies the operation

$$\sum_{k_1, k_2} |k_1, k_2\rangle\langle k_1, k_2|_{K'_1 K'_2} \otimes \text{SWAP}_{N, N_k},$$

where  $k = (k_1 - 1) \cdot 2^{r_B} + k_2$ .

- If the global state after step 4 were  $|\mu^{(4)}\rangle$ , the resulting global state at this step would be

$$\begin{aligned}
|\mu^{(5)}\rangle &\stackrel{\text{def}}{=} \frac{1}{\sqrt{2^{R_A+r_A} \cdot 2^{R_B+r_B}}} \sum_{j_1, j_2, k_1, k_2} |j_1, j_2, k_1, k_2\rangle_{J_1 J_2 K_1 K_2} \otimes |j_1, k_1, j_2, k_2\rangle_{J'_1 K'_1 J'_2 K'_2} \\
&\quad \otimes |\Psi\rangle_{RAMBNC} \otimes |\sigma^{(-j)}\rangle \otimes |0, 0\rangle_{M'_j M_j} \otimes |\omega^{(-k)}\rangle \otimes |0, 0\rangle_{N'_k N_k} \\
&= |\Psi\rangle_{RAMBNC} \otimes \left( \frac{1}{\sqrt{2^{R_A+r_A}}} \sum_{j_1, j_2} |j_1, j_2\rangle_{J_1 J_2} |j_1, j_2\rangle_{J'_1 J'_2} \otimes |\sigma^{(-j)}\rangle \otimes |0, 0\rangle_{M'_j M_j} \right) \\
&\quad \otimes \left( \frac{1}{\sqrt{2^{R_B+r_B}}} \sum_{k_1, k_2} |k_1, k_2\rangle_{K_1 K_2} |k_1, k_2\rangle_{K'_1 K'_2} \otimes |\omega^{(-k)}\rangle \otimes |0, 0\rangle_{N'_k N_k} \right),
\end{aligned}$$

which is of the form  $|\Psi\rangle_{RAMBNC} \otimes |\theta'_1\rangle_{E'_{AC}} \otimes |\theta'_2\rangle_{E'_{BC}}$ .

Let  $E'_{AC}$  represent the registers  $J_1 J_2 J'_1 J'_2 M_1 M'_1 \dots M_{2^{R_A+r_A}} M'_{2^{R_A+r_A}}$  and  $E'_{BC}$  represent the registers  $K_1 K_2 K'_1 K'_2 N_1 N'_1 \dots N_{2^{R_B+r_B}} N'_{2^{R_B+r_B}}$ . Let  $|\Phi\rangle\langle\Phi|_{RAMBNCE'_{AC}E'_{BC}}$  be the output of the protocol  $\mathcal{P}$ .

**3. Analysis of the protocol:** We shall show that

$$P(\mu^{(5)}, \Phi_{RAMBNCE'_{AC}E'_{BC}}) \leq \varepsilon_1 + 2\sqrt{\delta} + 5\varepsilon_2.$$

For this, we shall use the following relations:  $P(|\xi'\rangle\langle\xi'|, |\mu\rangle\langle\mu|) \leq \varepsilon_1 + 2\sqrt{\delta}$  (Equation 1) and  $P(\mu^{(3)}, \mu^{(4)}) \leq 5\varepsilon_2$  (to be shown in Claim 1 below).

Let  $|\mu^{(6)}\rangle$  be the global quantum state obtained after the application of Step 5 of the protocol  $\mathcal{P}$  on state  $|\mu^{(3)}\rangle$ . Thus,  $|\mu^{(6)}\rangle$  is also the output obtained after the application of Steps 3 – 5 on the state  $|\mu\rangle$ . Using the monotonicity of fidelity under quantum operation (Fact 2), we have

$$P(\mu^{(6)}, \mu^{(5)}) \leq P(\mu^{(3)}, \mu^{(4)}) \leq 5\varepsilon_2.$$

Moreover, by another application of monotonicity of fidelity under quantum operation (Fact 2) and the observation that  $|\mu^{(6)}\rangle$  is the output of the action of Steps 3 – 5 on the state  $|\mu\rangle$ , we obtain

$$P(\mu^{(6)}, \Phi_{RAMBNCE'_{AC}E'_{BC}}) \leq P(|\mu\rangle\langle\mu|, |\xi'\rangle\langle\xi'|) \leq \varepsilon_1 + 2\sqrt{\delta}.$$

Thus, by the triangle inequality for purified distance (Fact 1) we conclude that

$$P(\mu^{(5)}, \Phi_{RAMBNCE'_{AC}E'_{BC}}) \leq P(|\xi'\rangle\langle\xi'|, |\mu\rangle\langle\mu|) + P(\mu^{(3)}, \mu^{(4)}) \leq \varepsilon_1 + 2\sqrt{\delta} + 5\varepsilon_2.$$

This is equivalent to the statement

$$P(|\Psi\rangle\langle\Psi|_{RAMBNC} \otimes |\theta'_1\rangle\langle\theta'_1|_{E'_{AC}} \otimes |\theta'_2\rangle\langle\theta'_2|_{E'_{BC}}, \Phi_{RAMBNCE'_{AC}E'_{BC}}) \leq \varepsilon_1 + 2\sqrt{\delta} + 5\varepsilon_2.$$

The number of qubits communicated by Charlie to Alice and Charlie to Bob in  $\mathcal{P}$  is  $\frac{R_A}{2}$  and  $\frac{R_B}{2}$  respectively.

**4. Hypothesis testing succeeds with high probability:** We show the following claim.

**Claim 1.** *It holds that  $F^2(\mu^{(3)}, \mu^{(4)}) \geq 1 - 24\varepsilon_2^2$ .*

*Proof.* We shall prove that for every  $(j_1, k_1)$ , it holds that  $F^2(\mu^{(3)}_{(j_1, k_1)}, \mu^{(4)}_{(j_1, k_1)}) \geq 1 - 12\varepsilon_2^2$ , from which the claim is immediate. Appealing to symmetry, it is sufficient to consider the case  $(j_1, k_1) = (1, 1)$ , for which  $j = j_2$  and  $k = k_2$ . Let  $p_{\tilde{j}_2, \tilde{k}_2 | j_2, k_2}$  be defined as follows:

$$p_{\tilde{j}_2, \tilde{k}_2 | j_2, k_2} \stackrel{\text{def}}{=} \text{Tr} \left( (\Pi^A)^{-\frac{1}{2}} \Pi_{\tilde{j}_2}^A (\Pi^A)^{-\frac{1}{2}} \otimes (\Pi^B)^{-\frac{1}{2}} \Pi_{\tilde{k}_2}^B (\Pi^B)^{-\frac{1}{2}} |\Psi\rangle\langle\Psi|_{ABM_{j_2} N_{k_2}} \otimes \sigma^{-j_2} \otimes \omega^{-k_2} \right),$$

if  $\tilde{j}_2, \tilde{k}_2 \neq 0$ ,

$$p_{\tilde{j}_2, 0 | j_2, k_2} \stackrel{\text{def}}{=} \text{Tr} \left( (\Pi^A)^{-\frac{1}{2}} \Pi_{\tilde{j}_2}^A (\Pi^A)^{-\frac{1}{2}} \otimes (\text{I} - (\Pi^B)^0) |\Psi\rangle\langle\Psi|_{ABM_{j_2} N_{k_2}} \otimes \sigma^{-j_2} \otimes \omega^{-k_2} \right),$$

if  $\tilde{j}_2 \neq 0$ ,

$$p_{0, \tilde{k}_2 | j_2, k_2} \stackrel{\text{def}}{=} \text{Tr} \left( (\text{I} - (\Pi^A)^0) \otimes (\Pi^B)^{-\frac{1}{2}} \Pi_{\tilde{k}_2}^B (\Pi^B)^{-\frac{1}{2}} |\Psi\rangle\langle\Psi|_{ABM_{j_2} N_{k_2}} \otimes \sigma^{-j_2} \otimes \omega^{-k_2} \right),$$

if  $\tilde{k}_2 \neq 0$  and

$$p_{0, 0 | j_2, k_2} \stackrel{\text{def}}{=} \text{Tr} \left( (\text{I} - (\Pi^A)^0) \otimes (\text{I} - (\Pi^B)^0) |\Psi\rangle\langle\Psi|_{ABM_{j_2} N_{k_2}} \otimes \sigma^{-j_2} \otimes \omega^{-k_2} \right).$$

From Lemma 1, it holds that

$$F(\mathcal{A}_1 \otimes \mathcal{B}_1(\mu_{(1,1)}^{(2)}, \mu_{(1,1)}^{(4)})) \geq \frac{1}{2^{r_A+r_B}} \sum_{j_2, k_2} p_{j_2, k_2 | j_2, k_2}.$$

Since

$$\frac{1}{2^{r_A+r_B}} \sum_{j_2, k_2, \tilde{j}_2, \tilde{k}_2} p_{\tilde{j}_2, \tilde{k}_2 | j_2, k_2} = 1,$$

we have that

$$\begin{aligned} \frac{1}{2^{r_A+r_B}} \sum_{j_2, k_2} p_{j_2, k_2 | j_2, k_2} &= 1 - \frac{1}{2^{r_A+r_B}} \sum_{j_2, k_2} \sum_{\tilde{j}_2 \neq j_2, \tilde{k}_2 \neq k_2} p_{\tilde{j}_2, \tilde{k}_2 | j_2, k_2} \\ &= 1 - \sum_{\tilde{j}_2 \neq 1, \tilde{k}_2 \neq 1} p_{\tilde{j}_2, \tilde{k}_2 | 1, 1}, \end{aligned}$$

where the last line follows by symmetry under interchange of registers  $M_{j_2}, N_{k_2}$ . Now, consider

$$\begin{aligned} \sum_{\tilde{j}_2 \neq 1, \tilde{k}_2 \neq 1} p_{\tilde{j}_2, \tilde{k}_2 | 1, 1} &= \text{Tr} \left( \left( \text{I}_A \otimes \text{I}_B - (\Pi^A)^{-\frac{1}{2}} \Pi_1^A (\Pi^A)^{-\frac{1}{2}} \otimes (\Pi^B)^{-\frac{1}{2}} \Pi_1^B (\Pi^B)^{-\frac{1}{2}} \right) |\Psi\rangle\langle\Psi|_{ABM_1 N_1} \otimes \sigma^{-1} \otimes \omega^{-1} \right) \\ &\stackrel{a}{\leq} \text{Tr} \left( \left( \text{I}_A - (\Pi^A)^{-\frac{1}{2}} \Pi_1^A (\Pi^A)^{-\frac{1}{2}} \right) \otimes \text{I}_B \cdot |\Psi\rangle\langle\Psi|_{ABM_1 N_1} \otimes \sigma^{-1} \otimes \omega^{-1} \right) \\ &\quad + \text{Tr} \left( \text{I}_A \otimes \left( \text{I}_B - (\Pi^B)^{-\frac{1}{2}} \Pi_1^B (\Pi^B)^{-\frac{1}{2}} \right) \cdot |\Psi\rangle\langle\Psi|_{ABM_1 N_1} \otimes \sigma^{-1} \otimes \omega^{-1} \right) \\ &\stackrel{b}{\leq} \text{Tr} \left( \left( 2(\text{I}_A - \Pi_1^A) + 4 \sum_{\tilde{j}_2 \neq 1} \Pi_{\tilde{j}_2}^A \right) |\Psi\rangle\langle\Psi|_{AM_1} \otimes \sigma^{-1} \right) \\ &\quad + \text{Tr} \left( \left( 2(\text{I}_B - \Pi_1^B) + 4 \sum_{\tilde{k}_2 \neq 1} \Pi_{\tilde{k}_2}^B \right) |\Psi\rangle\langle\Psi|_{BN_1} \otimes \omega^{-1} \right) \\ &\stackrel{c}{\leq} 4(\varepsilon_2^2) + 4 \cdot 2^{r_A - D_H^{\varepsilon_2^2}(\Psi_{AM} \| \Psi_{A \otimes \sigma_M})} + 4 \cdot 2^{r_B - D_H^{\varepsilon_2^2}(\Psi_{BN} \| \Psi_{B \otimes \omega_N})} \leq 12\varepsilon_2^2, \end{aligned}$$

where in (a) we use the operator inequality

$$(\text{I} - P \otimes Q) \leq \text{I} \otimes (\text{I} - Q) + (\text{I} - P) \otimes \text{I},$$

for positive semidefinite operators  $P, Q \preceq I$ ; (b) follows from the Hayashi-Nagaoka inequality (Fact 8), (c) follows from the definition of  $\Pi^A, \Pi^B$  and the choice of  $r_A, r_B$ . This implies that  $\frac{1}{2^{r_A+r_B}} \sum_{j_2, k_2} p_{j_2, k_2 | j_2, k_2} \geq 1 - 12\varepsilon_2^2$ .

Thus,

$$F^2(\mu_{(1,1)}^{(3)}, \mu_{(1,1)}^{(4)}) = F^2(\mathcal{A}_1 \otimes \mathcal{B}_1(\mu_{(1,1)}^{(2)}), \mu_{(1,1)}^{(4)}) \geq (1 - 12\varepsilon_2^2)^2 \geq 1 - 24\varepsilon_2^2,$$

from which the claim concludes.  $\square$

This completes the proof of the theorem.  $\square$

## Achievable rate region for Task 1

**Theorem 2.** Fix  $\varepsilon_1, \varepsilon_2, \delta \in (0, 1)$  such that  $\varepsilon_1 + 5\varepsilon_2 + 2\sqrt{\delta} < 1$ . Let Alice (AM), Bob (BN), Reference (R) and Charlie (C) share the pure state  $|\Psi\rangle_{RAMBNC}$ . There exists an entanglement assisted quantum protocol, with entanglement shared only between (Alice, Charlie) and (Bob, Charlie), such that at the end of the protocol, Alice (A), Bob (B), Reference (R) and Charlie (CMN) share the state  $\Phi'_{RABCMN}$  with the property that  $P(\Phi', |\Psi\rangle\langle\Psi|) \leq \varepsilon_1 + 5\varepsilon_2 + 2\sqrt{\delta}$ . The number of qubits that Alice sends to Charlie is  $R_{A \rightarrow C}$  and that Bob sends to Charlie is  $R_{B \rightarrow C}$ , where the pair  $(R_{A \rightarrow C}, R_{B \rightarrow C})$  lie in the union of the following achievable rate region: for every  $\Psi'_{RABCMN} \in \mathcal{B}^{\varepsilon_1}(\Psi_{RABCMN})$  such that  $\Psi'_{RAB} \preceq 2^\delta \Psi_{RAB}$  and states  $\sigma_M, \omega_N$ :

$$\begin{aligned} R_{A \rightarrow C} &\geq \frac{1}{2} \left( D_{\max}(\Psi'_{RABM} \| \Psi_{RAB} \otimes \sigma_M) - D_H^{\varepsilon_2^2}(\Psi_{AM} \| \Psi_A \otimes \sigma_M) + \log \frac{1}{\varepsilon_2^2 \delta} \right), \\ R_{B \rightarrow C} &\geq \frac{1}{2} \left( D_{\max}(\Psi'_{RABN} \| \Psi_{RAB} \otimes \omega_N) - D_H^{\varepsilon_2^2}(\Psi_{BN} \| \Psi_B \otimes \omega_N) + \log \frac{1}{\varepsilon_2^2 \delta} \right), \\ R_{A \rightarrow C} + R_{B \rightarrow C} &\geq \frac{1}{2} \left( D_{\max}(\Psi'_{RABMN} \| \Psi_{RAB} \otimes \sigma_M \otimes \omega_N) - D_H^{\varepsilon_2^2}(\Psi_{AM} \| \Psi_A \otimes \sigma_M) \right. \\ &\quad \left. - D_H^{\varepsilon_2^2}(\Psi_{BN} \| \Psi_B \otimes \omega_N) + \log \frac{1}{\varepsilon_2^2 \delta} \right). \end{aligned}$$

*Proof.* Consider the protocol  $\mathcal{P}$  as obtained in Theorem 1 for the Task 2, with the starting state  $|\Psi\rangle\langle\Psi|_{RAMBNC} \otimes |\theta_1\rangle\langle\theta_1|_{E_{AC}} \otimes |\theta_2\rangle\langle\theta_2|_{E_{BC}}$  (where  $\theta_1$  and  $\theta_2$  serve as pre-shared entanglement) and the final state  $|\Phi\rangle\langle\Phi|_{RABCMNE'_{AC}E'_{BC}}$ . Furthermore, as promised by Theorem 1, there exist states  $|\theta'_1\rangle_{E'_{AC}}$  and  $|\theta'_2\rangle_{E'_{BC}}$  such that

$$P(|\Phi\rangle\langle\Phi|, |\Psi\rangle\langle\Psi| \otimes |\theta'_1\rangle\langle\theta'_1| \otimes |\theta'_2\rangle\langle\theta'_2|) \leq \varepsilon_1 + 5\varepsilon_2 + 2\sqrt{\delta}.$$

Since the protocol can be viewed as a unitary by Charlie, followed by quantum communication from Charlie to Alice and Bob and then subsequent unitaries by Alice and Bob, this protocol can be reversed to obtain a protocol  $\mathcal{P}'$ . We take  $\mathcal{P}'$  as the desired protocol for above task and let  $\Phi'_{RABCMNE_{AC}E_{BC}}$  be the state obtained by running  $\mathcal{P}'$  on  $|\Psi\rangle\langle\Psi| \otimes |\theta'_1\rangle\langle\theta'_1| \otimes |\theta'_2\rangle\langle\theta'_2|$  (where  $|\Psi\rangle$  serves as input to the protocol,  $|\theta'_1\rangle$  serves as the shared entanglement between (Alice, Charlie) and  $|\theta'_2\rangle$  serves as the shared entanglement between (Bob, Charlie)). From the relation  $\mathcal{P}'(|\Phi\rangle\langle\Phi|) = |\Psi\rangle\langle\Psi| \otimes |\theta'_1\rangle\langle\theta'_1| \otimes |\theta'_2\rangle\langle\theta'_2|$  and the monotonicity of fidelity under quantum operations (Fact 2), we conclude

$$\begin{aligned} P(|\Psi\rangle\langle\Psi| \otimes |\theta'_1\rangle\langle\theta'_1| \otimes |\theta'_2\rangle\langle\theta'_2|, \Phi') &= P(\mathcal{P}'(|\Phi\rangle\langle\Phi|), \mathcal{P}'(|\Psi\rangle\langle\Psi| \otimes |\theta'_1\rangle\langle\theta'_1| \otimes |\theta'_2\rangle\langle\theta'_2|)) \\ &\leq P(|\Phi\rangle\langle\Phi|, |\Psi\rangle\langle\Psi| \otimes |\theta'_1\rangle\langle\theta'_1| \otimes |\theta'_2\rangle\langle\theta'_2|) \\ &\leq \varepsilon_1 + 5\varepsilon_2 + 2\sqrt{\delta}. \end{aligned}$$

This completes the proof.  $\square$

## 4 Achievable rate region in the asymptotic and i.i.d. setting

In this section, we re-derive the result of [2], but without the use of time-sharing. Consider the asymptotic and i.i.d. version of Task 1 (Section 3) in the special case where register  $C$  is trivial, i.e., the joint state between Alice, Bob and Reference is the  $n$ -fold tensor product of the state  $|\Psi\rangle_{RAMBN}$ . Using Theorem 2 (with  $\sigma_M \rightarrow \Psi_M, \omega_N \rightarrow \Psi_N$ ), Theorem 4 (in Appendix B) for the pure state  $|\Psi\rangle_{RAMBN}$  and Fact 10, we conclude that the rate pair  $(R_{A \rightarrow C}, R_{B \rightarrow C})$  is asymptotic and i.i.d. achievable for Task 1, if it satisfies the following constraints:

$$\begin{aligned} R_{A \rightarrow C} &\geq \frac{1}{2} (I(RAB : M)_\Psi - I(A : M)_\Psi), \\ R_{B \rightarrow C} &\geq \frac{1}{2} (I(RAB : N)_\Psi - I(B : N)_\Psi), \\ R_{A \rightarrow C} + R_{B \rightarrow C} &\geq \frac{1}{2} \left( I(RAB : M : N)_\Psi - I(A : M)_\Psi - I(B : N)_\Psi \right). \end{aligned}$$

**Quantum version of the achievable rate region obtained by Slepian and Wolf:** An immediate corollary of above achievable rate region is the following. Consider the task in [2], which is a quantum version of the Slepian-Wolf protocol [1]. Alice ( $M^n$ ), Bob ( $N^n$ ), Reference ( $R^n$ ) share the joint pure state  $|\Psi\rangle_{RMN}^{\otimes n}$ . Alice and Bob wish to communicate their registers  $M^n$  and  $N^n$  to Charlie such that the final state shared between Reference ( $R^n$ ) and Charlie ( $M^n N^n$ ) is  $\Phi_{R^n M^n N^n}$  such that  $\lim_{n \rightarrow \infty} P(\Psi_{RMN}^{\otimes n}, \Phi_{R^n M^n N^n}) = 0$ . To accomplish this task, there exists an entanglement assisted protocol (with the entanglement shared between (Alice, Charlie) and (Bob, Charlie)) if the amount of communication from Alice to Charlie ( $R_{A \rightarrow C}$ ) and Bob to Charlie ( $R_{B \rightarrow C}$ ) satisfy the following constraints

$$\begin{aligned} R_{A \rightarrow C} &\geq \frac{1}{2} I(R : M)_\Psi, \\ R_{B \rightarrow C} &\geq \frac{1}{2} I(R : N)_\Psi, \\ R_{A \rightarrow C} + R_{B \rightarrow C} &\geq \frac{1}{2} I(R : M : N)_\Psi. \end{aligned}$$

## 5 Converse bound

In this section, we establish a converse for Task 2 in the absence of registers  $A, B$ . This matches with the achievable rate region in Theorem 1. We show the following theorem.

**Theorem 3.** *Fix  $\varepsilon \in (0, 1)$ . Let Reference ( $R$ ) and Charlie ( $CMN$ ) share the pure state  $|\Psi\rangle_{RCMN}$ . Let  $\mathcal{P}$  be an entanglement-assisted quantum protocol with the following properties.*

- *The pre-shared entanglement is of the form  $|\theta_1\rangle_{E_A E_{C_A}} \otimes |\theta_2\rangle_{E_B E_{C_B}}$ , where  $|\theta_1\rangle_{E_A E_{C_A}}$  is shared between Alice ( $E_A$ ), Charlie ( $E_{C_A}$ ) and  $|\theta_2\rangle_{E_B E_{C_B}}$  is shared between Bob ( $E_B$ ), Charlie ( $E_{C_B}$ ).*
- *The quantum communication is from Charlie to Alice with  $R_{C \rightarrow A}$  qubits and from Charlie to Bob with  $R_{C \rightarrow B}$  qubits.*
- *At the end of the protocol, the joint quantum state  $\Phi_{RMNC}$  shared between Alice ( $M$ ), Bob ( $N$ ), Reference ( $R$ ) and Charlie ( $C$ ) satisfies  $\Phi_{RMNC} \in \mathcal{B}^\varepsilon(\Psi_{RMNC})$ .*

*Then there exists a quantum state  $\Psi'_{RMNC} \in \mathcal{B}^\varepsilon(\Psi_{RMNC})$  with  $\Psi'_R = \Psi_R$  and quantum states  $\sigma_M, \omega_N$  such*

that

$$\begin{aligned}
R_{C \rightarrow A} &\geq \frac{1}{2} D_{\max}(\Psi'_{RM} \| \Psi_R \otimes \sigma_M), \\
R_{C \rightarrow B} &\geq \frac{1}{2} D_{\max}(\Psi'_{RN} \| \Psi_R \otimes \omega_N), \\
R_{C \rightarrow A} + R_{C \rightarrow B} &\geq \frac{1}{2} D_{\max}(\Psi'_{RMN} \| \Psi_R \otimes \sigma_M \otimes \omega_N).
\end{aligned}$$

*Proof.* A protocol  $\mathcal{P}$  has the following steps, where the registers  $Q_A, Q_B$  serve as the message registers from Charlie to Alice and Bob, respectively.

- Charlie applies an encoding map  $\mathcal{E} : CMNE_{C_A}E_{C_B} \rightarrow CQ_AQ_B$  and communicates  $Q_A$  and  $Q_B$  to Alice and Bob, respectively.
- Alice applies a decoding map  $\mathcal{D}_A : E_AQ_A \rightarrow M$  and Bob applies a decoding map  $\mathcal{D}_B : E_BQ_B \rightarrow N$ .
- The final quantum state is obtained in the registers  $RCMN$ .

Let the quantum state on the registers  $RQ_AE_AQ_BE_B$  after Alice and Bob receive Charlie's message be  $\Omega_{RQ_AE_AQ_BE_B}$ . Observe that  $\Omega_{RE_AE_B} = \Psi_R \otimes (\theta_1)_{E_A} \otimes (\theta_2)_{E_B}$  and the final state  $\Phi_{RMNC}$  is equal to  $\mathcal{D}_A \otimes \mathcal{D}_B(\Omega_{RQ_AE_AQ_BE_B})$ . We have the following relations using Fact 5,

$$\begin{aligned}
\log |Q_A| &\geq \frac{1}{2} D_{\max} \left( \Omega_{RQ_AE_A} \left\| \Omega_{RE_A} \otimes \frac{I_{Q_A}}{|Q_A|} \right\| \right) = \frac{1}{2} D_{\max} \left( \Omega_{RQ_AE_A} \left\| \Psi_R \otimes (\theta_1)_{E_A} \otimes \frac{I_{Q_A}}{|Q_A|} \right\| \right), \\
\log |Q_B| &\geq \frac{1}{2} D_{\max} \left( \Omega_{RQ_BE_B} \left\| \Omega_{RE_B} \otimes \frac{I_{Q_B}}{|Q_B|} \right\| \right) = \frac{1}{2} D_{\max} \left( \Omega_{RQ_BE_B} \left\| \Psi_R \otimes (\theta_2)_{E_B} \otimes \frac{I_{Q_B}}{|Q_B|} \right\| \right), \\
\log(|Q_A| \cdot |Q_B|) &\geq \frac{1}{2} D_{\max} \left( \Omega_{RQ_AE_AQ_BE_B} \left\| \Omega_{RE_AE_B} \otimes \frac{I_{Q_A}}{|Q_A|} \otimes \frac{I_{Q_B}}{|Q_B|} \right\| \right) \\
&= \frac{1}{2} D_{\max} \left( \Omega_{RQ_AE_AQ_BE_B} \left\| \Psi_R \otimes (\theta_1)_{E_A} \otimes (\theta_2)_{E_B} \otimes \frac{I_{Q_A}}{|Q_A|} \otimes \frac{I_{Q_B}}{|Q_B|} \right\| \right). \tag{2}
\end{aligned}$$

Define  $\Psi'_{RMNC} \stackrel{\text{def}}{=} \Phi_{RMNC} = \mathcal{D}_A \otimes \mathcal{D}_B(\Omega_{RQ_AE_AQ_BE_B})$ . It holds that  $\Psi'_R = \Phi_R = \Omega_R = \Psi_R$ . Further, define  $\sigma_M \stackrel{\text{def}}{=} \mathcal{D}_A \left( (\theta_1)_{E_A} \otimes \frac{I_{Q_A}}{|Q_A|} \right)$  and  $\omega_N \stackrel{\text{def}}{=} \mathcal{D}_B \left( (\theta_2)_{E_B} \otimes \frac{I_{Q_B}}{|Q_B|} \right)$ . Applying the monotonicity of max-relative entropy under quantum operations (Fact 2) in Equation 2, we obtain

$$\begin{aligned}
\log |Q_A| &\geq \frac{1}{2} D_{\max} \left( \mathcal{D}_A(\Omega_{RQ_AE_A}) \left\| \Psi_R \otimes \mathcal{D}_A \left( (\theta_1)_{E_A} \otimes \frac{I_{Q_A}}{|Q_A|} \right) \right\| \right) \\
&= \frac{1}{2} D_{\max}(\Psi'_{RM} \| \Psi_R \otimes \sigma_M), \\
\log |Q_B| &\geq \frac{1}{2} D_{\max} \left( \mathcal{D}_B(\Omega_{RQ_BE_B}) \left\| \Psi_R \otimes \mathcal{D}_B \left( (\theta_2)_{E_B} \otimes \frac{I_{Q_B}}{|Q_B|} \right) \right\| \right) \\
&= \frac{1}{2} D_{\max}(\Psi'_{RN} \| \Psi_R \otimes \omega_N), \\
\log(|Q_A| \cdot |Q_B|) &\geq \frac{1}{2} D_{\max} \left( \mathcal{D}_A \otimes \mathcal{D}_B(\Omega_{RQ_AE_AQ_BE_B}) \left\| \Psi_R \otimes \mathcal{D}_A \left( (\theta_1)_{E_A} \otimes \frac{I_{Q_A}}{|Q_A|} \right) \otimes \mathcal{D}_B \left( (\theta_2)_{E_B} \otimes \frac{I_{Q_B}}{|Q_B|} \right) \right\| \right) \\
&= \frac{1}{2} D_{\max}(\Psi'_{RMN} \| \Psi_R \otimes \sigma_M \otimes \omega_N).
\end{aligned}$$

Since  $R_{C \rightarrow A} = \log |Q_A|$ ,  $R_{C \rightarrow B} = \log |Q_B|$ , the theorem concludes.  $\square$



## Conclusion

In this work, we have studied two distributed quantum source compression tasks characterized by two senders-one receiver and one sender-two receivers, respectively. These cases generalize the distributed source compression tasks studied in [2] and [17] and the task of quantum state redistribution [3, 15]. We have obtained one-shot achievable rate regions for these tasks and shown matching converse in a special case.

An important question for our one-shot achievability results is to connect them to the relative entropy in the asymptotic and i.i.d. setting. We are able to achieve this for a special case of our Task (which is still general enough to include the task considered in [2]). But it is not clear to us how to achieve the same for our most general setting. In fact, a similar problem arises in attempting to extend our work to more complicated network scenarios. We are able to obtain one-shot achievable rate regions in complicated network scenarios that closely resemble the achievable rate region in Theorem 1. However, we are unable to show that such rate regions converge appropriately in the asymptotic and i.i.d. setting. We leave this as an important question to be pursued for future work.

Finally, we leave unexplored the study of a variant of our tasks where all the parties are allowed to pre-share tri-partite entanglement. Such a scenario might allow us to prove near optimal converse bounds for Task 1. This would be in striking contrast with the classical analogue studied by Slepian and Wolf [1], where the optimal rate region can be achieved without any shared randomness between the senders.

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## References

- [1] D. Slepian and J. Wolf, “Noiseless coding of correlated information sources,” *IEEE Transactions on Information Theory*, vol. 19, pp. 471–480, Jul 1973.
- [2] A. Abeyesinghe, I. Devetak, P. Hayden, and A. Winter, “The mother of all protocols: restructuring quantum information’s family tree,” *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 465, no. 2108, pp. 2537–2563, 2009.
- [3] I. Devetak and J. Yard, “Exact cost of redistributing multipartite quantum states,” *Phys. Rev. Lett.*, vol. 100, no. 230501, 2008.
- [4] C. E. Shannon, “A mathematical theory of communication,” *The Bell System Technical Journal*, vol. 27, pp. 379–423, July 1948.
- [5] B. Schumacher, “Quantum coding,” *Phys. Rev. A.*, vol. 51, pp. 2738–2747, 1995.
- [6] P. Harsha, R. Jain, D. McAllester, and J. Radhakrishnan, “The communication complexity of correlation,” *IEEE Transactions on Information Theory*, vol. 56, pp. 438–449, 2010.
- [7] M. Braverman and A. Rao, “Information equals amortized communication,” in *Proceedings of the 52nd Symposium on Foundations of Computer Science*, FOCS ’11, (Washington, DC, USA), pp. 748–757, IEEE Computer Society, 2011.
- [8] D. Touchette, “Quantum information complexity,” in *Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing*, STOC ’15, (New York, NY, USA), pp. 317–326, ACM, 2015.

- [9] M. Berta, “Single-shot quantum state merging.” Master’s thesis, ETH Zurich, <http://arxiv.org/abs/0912.4495>, 2009.
- [10] M. Berta, M. Christandl, and R. Renner, “The Quantum Reverse Shannon Theorem based on one-shot information theory,” *Commun. Math. Phys.*, vol. 306, no. 3, pp. 579–615, 2011.
- [11] M. Horodecki, J. Oppenheim, and A. Winter, “Quantum state merging and negative information,” *Communications in Mathematical Physics*, vol. 269, no. 1, pp. 107–136, 2007.
- [12] M. Berta, M. Christandl, and D. Touchette, “Smooth entropy bounds on one-shot quantum state redistribution,” *IEEE Transactions on Information Theory*, vol. 62, pp. 1425–1439, March 2016.
- [13] A. Anshu, V. K. Devabathini, and R. Jain, “Quantum communication using coherent rejection sampling,” *Phys. Rev. Lett.*, vol. 119, p. 120506, Sep 2017.
- [14] A. Anshu, R. Jain, and N. A. Warsi, “A one-shot achievability result for quantum state redistribution,” *IEEE Transactions on Information Theory*, vol. PP, no. 99, pp. 1–1, 2017.
- [15] J. T. Yard and I. Devetak, “Optimal quantum source coding with quantum side information at the encoder and decoder,” *IEEE Transactions on Information Theory*, vol. 55, pp. 5339–5351, 2009.
- [16] N. Dutil and P. Hayden, “One-shot multiparty state merging.” <https://arxiv.org/abs/1011.1974>, 2010.
- [17] M. H. Hsieh and S. Watanabe, “Fully quantum source compression with a quantum helper,” in *2015 IEEE Information Theory Workshop - Fall (ITW)*, pp. 307–311, Oct 2015.
- [18] D. Gavinsky, O. Regev, and R. de Wolf, “Simultaneous communication protocols with quantum and classical messages,” *Chicago Journal of Theoretical Computer Science*, vol. 2008, December 2008.
- [19] R. Jain and H. Klauck, “New results in the simultaneous message passing model via information theoretic techniques,” in *2009 24th Annual IEEE Conference on Computational Complexity*, pp. 369–378, July 2009.
- [20] D. Gavinsky, “Quantum versus classical simultaneity in communication complexity.” <https://arxiv.org/abs/1705.07211>, 2017.
- [21] T. M. Cover and J. A. Thomas, *Elements of information theory*. Wiley Series in Telecommunications, New York, NY, USA: John Wiley & Sons, 1991.
- [22] A. Anshu, R. Jain, and N. Warsi, “One shot entanglement assisted classical and quantum communication over noisy quantum channels: A hypothesis testing and convex split approach.” <https://arxiv.org/abs/1702.01940>, 2017.
- [23] M. Hayashi and H. Nagaoka, “General formulas for capacity of classical-quantum channels,” *IEEE Transactions on Information Theory*, vol. 49, pp. 1753–1768, July 2003.
- [24] H. Nagaoka and M. Hayashi, “An information-spectrum approach to classical and quantum hypothesis testing for simple hypotheses,” *IEEE Transactions on Information Theory*, vol. 53, pp. 534–549, Feb 2007.
- [25] T. S. Han and S. Verdú, “Approximation theory of output statistics,” *IEEE Transactions on Information Theory*, vol. 39, pp. 752–772, May 1993.
- [26] R. Jozsa, “Fidelity for mixed quantum states,” *Journal of Modern Optics*, vol. 41, no. 12, pp. 2315–2323, 1994.

- [27] A. Uhlmann, “The ”transition probability” in the state space of a  $\ast$ -algebra,” *Rep. Math. Phys.*, vol. 9, pp. 273–279, 1976.
- [28] A. Gilchrist, N. K. Langford, and M. A. Nielsen, “Distance measures to compare real and ideal quantum processes,” *Phys. Rev. A*, vol. 71, p. 062310, Jun 2005.
- [29] J. V. Neumann, *Mathematische Grundlagen der Quantenmechanik*. Berlin, Germany: Springer, 1932.
- [30] H. Umegaki, “Conditional expectation in an operator algebra, i,” *Tohoku Math. J. (2)*, vol. 6, no. 2-3, pp. 177–181, 1954.
- [31] N. Datta, “Min- and max- relative entropies and a new entanglement monotone,” *IEEE Transactions on Information Theory*, vol. 55, pp. 2816–2826, 2009.
- [32] R. Jain, J. Radhakrishnan, and P. Sen, “A property of quantum relative entropy with an application to privacy in quantum communication,” *J. ACM*, vol. 56, pp. 33:1–33:32, Sept. 2009.
- [33] F. Buscemi and N. Datta, “The quantum capacity of channels with arbitrarily correlated noise,” *IEEE Transactions on Information Theory*, vol. 56, pp. 1447–1460, 2010.
- [34] M. Tomamichel, “A framework for non-asymptotic quantum information theory.” PhD Thesis, ETH Zurich, <http://arXiv.org/abs/1203.2142>, 2012.
- [35] H. Barnum, C. M. Cave, C. A. Fuch, R. Jozsa, and B. Schumacher, “Noncommuting mixed states cannot be broadcast,” *Phys. Rev. Lett.*, vol. 76, no. 15, pp. 2818–2821, 1996.
- [36] G. Lindblad, “Completely positive maps and entropy inequalities,” *Commun. Math. Phys.*, vol. 40, pp. 147–151, 1975.
- [37] R. Jain, J. Radhakrishnan, and S. P, “A lower bound for the bounded round quantum communication complexity of set disjointness,” in *44th Annual IEEE Symposium on Foundations of Computer Science, 2003. Proceedings.*, pp. 220–229, Oct 2003.
- [38] A. Winter, “Coding theorem and strong converse for quantum channels.,” *IEEE Transactions on Information Theory*, vol. 45, no. 7, pp. 2481–2485, 1999.
- [39] T. Ogawa and H. Nagaoka, “A new proof of the channel coding theorem via hypothesis testing in quantum information theory,” in *Information Theory, 2002. Proceedings. 2002 IEEE International Symposium on*, pp. 73–, 2002.
- [40] A. Harrow, “Coherent communication of classical messages,” *Phys. Rev. Lett.*, vol. 92, p. 097902, Mar 2004.
- [41] C. H. Bennett and S. J. Wiesner, “Communication via one- and two-particle operators on einstein-podolsky-rosen states,” *Phys. Rev. Lett.*, vol. 69, no. 20, pp. 2881–2884, 1992.
- [42] M. Tomamichel and M. Hayashi, “A hierarchy of information quantities for finite block length analysis of quantum tasks,” *IEEE Transactions on Information Theory*, vol. 59, pp. 7693–7710, Nov 2013.
- [43] K. Li, “Second-order asymptotics for quantum hypothesis testing,” *Ann. Statist.*, vol. 42, pp. 171–189, 02 2014.

## A A variant of the convex-split lemma

In this section, we prove a tripartite variant of the convex split lemma, used in the proof of Theorem 1. We shall use the following fact.

**Fact 9** ([13]). *Let  $\mu_1, \mu_2, \dots, \mu_n, \theta$  be quantum states and  $\{p_1, p_2, \dots, p_n\}$  be a probability distribution. Let  $\mu = \sum_i p_i \mu_i$  be the average quantum state. Then*

$$D(\mu \parallel \theta) = \sum_i p_i (D(\mu_i \parallel \theta) - D(\mu_i \parallel \mu)).$$

Using this fact, we prove the following statement.

**Lemma 2** (tripartite convex-split lemma). *Let  $\varepsilon, \delta \in (0, 1)$  such that  $\varepsilon + 2\sqrt{\delta} < 1$ . Let  $\rho_{RAB} \in \mathcal{D}(RAB), \sigma_A \in \mathcal{D}(A), \omega_B \in \mathcal{D}(B)$  be quantum states and  $\rho'_{RAB}$  be a quantum state satisfying  $\rho'_{RAB} \in \mathcal{B}^\varepsilon(\rho_{RAB})$ . For some  $R_1, R_2 \geq 1$ , consider the following state*

$$\begin{aligned} \tau_{RA_1 \dots A_{2^{R_A}} B_1 \dots B_{2^{R_B}}} &\stackrel{\text{def}}{=} \frac{1}{2^{R_A + R_B}} \sum_{i=1}^{2^{R_A}} \sum_{j=1}^{2^{R_B}} \rho_{RA_i B_j} \\ &\otimes \sigma_{A_1} \otimes \dots \otimes \sigma_{A_{i-1}} \otimes \sigma_{A_{i+1}} \otimes \dots \otimes \sigma_{A_{2^{R_A}}} \\ &\otimes \omega_{B_1} \otimes \dots \otimes \omega_{B_{j-1}} \otimes \omega_{B_{j+1}} \dots \otimes \omega_{B_{2^{R_B}}} \end{aligned}$$

on the registers  $A_1, A_2, \dots, A_{2^{R_A}}, B_1, B_2, \dots, B_{2^{R_B}}$ , where  $\forall i \in [2^{R_A}], j \in [2^{R_B}] : \rho_{RA_i B_j} = \rho_{RAB}, \rho_{RA_i} = \rho_{RA}$  and  $\rho_{RB_j} = \rho_{RB}$ . If

$$\begin{aligned} R_A &\geq D_{\max}(\rho'_{RA} \parallel \rho_R \otimes \sigma_A) + \log \frac{1}{\delta}, \\ R_B &\geq D_{\max}(\rho'_{RB} \parallel \rho_R \otimes \omega_B) + \log \frac{1}{\delta}, \\ R_A + R_B &\geq D_{\max}(\rho'_{RAB} \parallel \rho_R \otimes \sigma_A \otimes \omega_B) + \log \frac{1}{\delta}, \\ \rho'_R &\preceq 2^\delta \rho_R \end{aligned}$$

then

$$\begin{aligned} P(\tau_{RA_1 A_2 \dots A_{2^{R_A}} B_1 B_2 \dots B_{2^{R_B}}}, \rho_R \otimes \sigma_{A_1} \otimes \sigma_{A_2} \otimes \dots \otimes \sigma_{A_{2^{R_A}}} \\ \otimes \omega_{B_1} \otimes \omega_{B_2} \dots \otimes \omega_{B_{2^{R_B}}}) \leq \varepsilon + 2\sqrt{\delta}. \end{aligned}$$

The proof closely follows the original proof of the convex split lemma from [13].

*Proof.* For brevity, we set

$$\begin{aligned} k_1 &\stackrel{\text{def}}{=} D_{\max}(\rho'_{RAB} \parallel \rho_R \otimes \sigma_A \otimes \omega_B), \\ k_2 &\stackrel{\text{def}}{=} D_{\max}(\rho'_{RA} \parallel \rho_R \otimes \sigma_A), \\ k_3 &\stackrel{\text{def}}{=} D_{\max}(\rho'_{RB} \parallel \rho_R \otimes \omega_B). \end{aligned}$$

We shall work with the state

$$\begin{aligned} \tau'_{RA_1 \dots A_{2^{R_A}} B_1 \dots B_{2^{R_B}}} &\stackrel{\text{def}}{=} \frac{1}{2^{R_A} \cdot 2^{R_B}} \sum_{i=1}^{2^{R_A}} \sum_{j=1}^{2^{R_B}} \rho'_{RA_i B_j} \\ &\otimes \sigma_{A_1} \otimes \dots \otimes \sigma_{A_{i-1}} \otimes \sigma_{A_{i+1}} \otimes \dots \otimes \sigma_{A_{2^{R_A}}} \\ &\otimes \omega_{B_1} \otimes \dots \otimes \omega_{B_{j-1}} \otimes \omega_{B_{j+1}} \dots \otimes \omega_{B_{2^{R_B}}}. \end{aligned}$$

Define,

$$\rho^{-(i,j)} \stackrel{\text{def}}{=} \sigma_{A_1} \otimes \dots \otimes \sigma_{A_{i-1}} \otimes \sigma_{A_{i+1}} \otimes \dots \otimes \sigma_{A_{2R_A}} \\ \otimes \omega_{B_1} \dots \otimes \omega_{B_{j-1}} \otimes \omega_{B_{j+1}} \dots \otimes \omega_{B_{2R_B}},$$

$$\rho \stackrel{\text{def}}{=} \sigma_{A_1} \otimes \sigma_{A_2} \otimes \dots \otimes \sigma_{A_{2R_A}} \otimes \omega_{B_1} \otimes \omega_{B_2} \dots \omega_{B_{2R_B}}.$$

Then

$$\tau'_{RA_1 A_2 \dots A_{2R_A} B_1 B_2 \dots B_{2R_B}} = \frac{1}{2^{R_A} \cdot 2^{R_B}} \sum_{i,j} \rho'_{RA_i B_j} \otimes \rho^{-(i,j)}.$$

Now, we use Fact 9 to express

$$\begin{aligned} & D\left(\tau'_{RA_1 \dots A_{2R_A} B_1 \dots B_{2R_B}} \parallel \rho_R \otimes \rho\right) \\ &= \frac{1}{2^{R_A} \cdot 2^{R_B}} \sum_{i,j} D\left(\rho'_{RA_i B_j} \otimes \rho^{-(i,j)} \parallel \rho_R \otimes \rho\right) \\ &= \frac{1}{2^{R_A} \cdot 2^{R_B}} \sum_{i,j} D\left(\rho'_{RA_i B_j} \otimes \right. \\ &\quad \left. \rho^{-(i,j)} \parallel \tau'_{RA_1 A_2 \dots A_{2R_A} B_1 B_2 \dots B_{2R_B}}\right). \end{aligned} \tag{3}$$

Note that,

$$\begin{aligned} & D\left(\rho'_{RA_i B_j} \otimes \rho^{-(i,j)} \parallel \rho_R \otimes \rho\right) \\ &= D\left(\rho'_{RA_i B_j} \parallel \rho_R \otimes \sigma_{A_i} \otimes \omega_{B_j}\right), \\ & D\left(\rho'_{RA_i B_j} \otimes \rho^{-(i,j)} \parallel \tau'_{RA_1 A_2 \dots A_{2R_A} B_1 B_2 \dots B_{2R_B}}\right) \\ &\geq D\left(\rho'_{RA_i B_j} \parallel \tau'_{RA_i B_j}\right), \end{aligned}$$

as relative entropy decreases under partial trace. Moreover,

$$\begin{aligned} \tau'_{RA_i B_j} &= \frac{1}{2^{R_A} \cdot 2^{R_B}} \rho'_{RA_i B_j} \\ &+ \frac{1}{2^{R_A}} \left(1 - \frac{1}{2^{R_B}}\right) \rho'_{RA_i} \otimes \omega_{B_j} \\ &+ \frac{1}{2^{R_A}} \left(1 - \frac{1}{2^{R_B}}\right) \sigma_{A_i} \otimes \rho'_{RB_j} \\ &+ \left(1 - \frac{1}{2^{R_A}} - \frac{1}{2^{R_B}} + \frac{1}{2^{R_A} \cdot 2^{R_B}}\right) \rho'_R \otimes \sigma_{A_i} \otimes \omega_{B_j}. \end{aligned}$$

By assumption,

$$\begin{aligned} \rho'_{RA_i B_j} &\preceq 2^{k_1} \rho_R \otimes \sigma_{A_i} \otimes \omega_{B_j}, \quad \rho'_{RA_i} \preceq 2^{k_2} \rho_R \otimes \sigma_{A_i}, \\ \rho'_{RB_j} &\preceq 2^{k_3} \rho_R \otimes \omega_{B_j}, \quad \rho'_R \preceq 2^\delta \rho_R. \end{aligned}$$

Hence

$$\tau'_{RA_i B_j} \preceq \left(2^\delta + \frac{2^{k_2}}{2^{R_A}} + \frac{2^{k_3}}{2^{R_B}} + \frac{2^{k_1}}{2^{R_A} \cdot 2^{R_B}}\right) \rho_R \otimes \sigma_{A_i} \otimes \omega_{B_j}.$$

Since  $\log(A) \preceq \log(B)$  if  $A \preceq B$ , for positive semi-definite matrices  $A$  and  $B$ , we have

$$\begin{aligned}
& D(\rho'_{RA_i B_j} \parallel \tau'_{RA_i B_j}) \\
&= \text{Tr}(\rho'_{RA_i B_j} \log \rho'_{RA_i B_j}) - \text{Tr}(\rho'_{RA_i B_j} \log \tau'_{RA_i B_j}) \\
&\geq \text{Tr}(\rho'_{RA_i B_j} \log \rho'_{RA_i B_j}) \\
&\quad - \text{Tr}(\rho'_{RA_i B_j} \log(\rho_R \otimes \sigma_{A_i} \otimes \omega_{B_j})) \\
&\quad - \log \left( 2^\delta + \frac{2^{k_2}}{2^{R_A}} + \frac{2^{k_3}}{2^{R_B}} + \frac{2^{k_1}}{2^{R_A} \cdot 2^{R_B}} \right) \\
&= D(\rho'_{RA_i B_j} \parallel \rho_R \otimes \sigma_{A_i} \otimes \omega_{B_j}) \\
&\quad - \log \left( 2^\delta + \frac{2^{k_2}}{2^{R_A}} + \frac{2^{k_3}}{2^{R_B}} + \frac{2^{k_1}}{2^{R_A} \cdot 2^{R_B}} \right).
\end{aligned}$$

Using in Equation 3, we find that

$$\begin{aligned}
& D(\tau'_{RA_1 A_2 \dots A_{2^{R_A}} B_1 B_2 \dots B_{2^{R_B}}} \parallel \rho_R \otimes \rho) \\
&\leq \frac{1}{2^{R_A} \cdot 2^{R_B}} \sum_{i,j} D(\rho'_{RA_i B_j} \parallel \rho_R \otimes \sigma_{A_i} \otimes \omega_{B_j}) \\
&\quad - \frac{1}{2^{R_A} \cdot 2^{R_B}} \sum_{i,j} D(\rho'_{RA_i B_j} \parallel \rho_R \otimes \sigma_{A_i} \otimes \omega_{B_j}) \\
&\quad + \log \left( 2^\delta + \frac{2^{k_2}}{2^{R_A}} + \frac{2^{k_3}}{2^{R_B}} + \frac{2^{k_1}}{2^{R_A} \cdot 2^{R_B}} \right) \\
&\leq \log(1 + \delta + 3\delta).
\end{aligned}$$

Above, the last inequality follows by the lower bound on  $2^{R_A}, 2^{R_B}$  and the fact that  $\delta < 1$ . Thus, by Fact 4 (which is the improved version of Pinsker's inequality), we obtain

$$P(\tau'_{RA_1 A_2 \dots A_{2^{R_A}} B_1 B_2 \dots B_{2^{R_B}}}, \rho_R \otimes \rho) \leq \sqrt{4\delta}.$$

Since  $P(\tau'_{A_1 A_2 \dots A_{2^{R_A}} B_1 B_2 \dots B_{2^{R_B}}}, \tau_{A_1 A_2 \dots A_{2^{R_A}} B_1 B_2 \dots B_{2^{R_B}}}) \leq P(\rho'_{AB}, \rho_{AB}) \leq \varepsilon$ , triangle inequality for Purified distance (Fact 1) shows that

$$P(\tau_{RA_1 A_2 \dots A_{2^{R_A}} B_1 B_2 \dots B_{2^{R_B}}}, \rho_R \otimes \rho) \leq \varepsilon + 2\sqrt{\delta}.$$

This proves the lemma.  $\square$

## B Asymptotic and i.i.d. analysis

An important property of the smooth information theoretic quantities is that in the asymptotic and i.i.d. setting, they converge to the relative entropy based quantities. In this section, we show this property for the one-shot bounds we obtain in Theorem 2, in the case where register  $C$  is absent.

### Facts used in the proof

Following fact ensures that the smooth max-relative entropy and hypothesis testing relative entropy converge to suitable quantities in the asymptotic and i.i.d. setting.

**Fact 10** ([42, 43]). Let  $\varepsilon \in (0, 1)$  and  $n$  be an integer. Let  $\rho^{\otimes n}, \sigma^{\otimes n}$  be quantum states. Define  $V(\rho\|\sigma) = \text{Tr}(\rho(\log \rho - \log \sigma)^2) - (\text{D}(\rho\|\sigma))^2$  and  $\Phi(x) = \int_{-\infty}^x \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$ . It holds that

$$\text{D}_{\max}^{\varepsilon}(\rho^{\otimes n}\|\sigma^{\otimes n}) = n\text{D}(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon) + O(\log n),$$

and

$$\text{D}_{\text{H}}^{\varepsilon}(\rho^{\otimes n}\|\sigma^{\otimes n}) = n\text{D}(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon) + O(\log n).$$

Following fact can be viewed as a triangle inequality for smooth max-relative entropy.

**Fact 11.** For  $\rho_A \in \mathcal{D}(A), \sigma_A, \tau_A \in \mathcal{P}(A)$ , it holds that

$$\text{D}_{\max}^{\varepsilon}(\rho_A\|\tau_A) \leq \text{D}_{\max}^{\varepsilon}(\rho_A\|\sigma_A) + \text{D}_{\max}(\sigma_A\|\tau_A).$$

*Proof.* Let  $k \stackrel{\text{def}}{=} \text{D}_{\max}(\sigma_A\|\tau_A)$ , which implies that  $\sigma_A \preceq 2^k \tau_A$ . Let  $\rho'_A \in \mathcal{B}^{\varepsilon}(\rho_A)$  be the state achieving the infimum in  $R \stackrel{\text{def}}{=} \text{D}_{\max}^{\varepsilon}(\rho_A\|\sigma_A)$ . Then  $\rho'_A \preceq 2^R \sigma_A \preceq 2^{R+k} \tau_A$ . This implies that  $\text{D}_{\max}(\rho'_A\|\tau_A) \leq R + k$ , which concludes the fact using the inequality  $\text{D}_{\max}^{\varepsilon}(\rho_A\|\tau_A) \leq \text{D}_{\max}(\rho'_A\|\tau_A)$ .  $\square$

Following is an important fact that relates the information spectrum relative entropy to the max-relative entropy.

**Fact 12** (Lemma 12 and Proposition 13,[42]). Let  $\varepsilon, \delta \in (0, 1)$  such that  $\varepsilon^2 + \delta < 1$ . For quantum state  $\rho_A \in \mathcal{D}(A)$  and  $\sigma \in \mathcal{P}(A)$ , it holds that

$$\begin{aligned} \text{D}_s^{1-\varepsilon^2-\delta}(\rho_A\|\sigma_A) - 2\log \frac{1}{\delta} - 2 &\leq \text{D}_{\max}^{\varepsilon}(\rho_A\|\sigma_A) \leq \\ \text{D}_s^{1-\varepsilon^2+\delta}(\rho_A\|\sigma_A) + \log v(\sigma) + 2\log \frac{1}{\varepsilon} + \log \frac{1}{\delta}, \end{aligned}$$

where  $v(\sigma_A)$  is the number of distinct eigenvalues of  $\sigma_A$ . It also holds that

$$\tilde{\text{D}}_s^{\varepsilon^2+\delta}(\rho_A\|\sigma_A) - 2\log \frac{1}{\delta} - 2 \leq \text{D}_{\max}^{\varepsilon}(\rho_A\|\sigma_A).$$

*Proof.* The first part is essentially that given in [42] (Proposition 13 and Lemma 12). For the second part, we note that the proof in [42] (Proposition 12, Equation 23) directly proceeds for this case as well: setting  $R \stackrel{\text{def}}{=} \text{D}_{\max}^{\varepsilon}(\rho_A\|\sigma_A)$ , it is shown that for any  $\delta' > 0$ , it holds that

$$\text{Tr}(\rho_A\{\rho_A - 2^{R+\delta'}\sigma_A\}_-) \geq (\sqrt{1-\varepsilon^2} - 2^{-\delta'/2})^2.$$

Setting  $\delta' = \log \frac{1}{\delta}$ , the inequality follows.  $\square$

Following Fact immediately follows from the above Fact.

**Fact 13.** Let  $\varepsilon, \varepsilon_1 \in (0, 1)$ . For quantum states  $\rho_A, \rho'_A \in \mathcal{D}(A)$  and  $\sigma_A, \sigma'_A \in \mathcal{P}(A)$ , the following properties hold.

1. If  $\rho'_A \in \mathcal{B}^{\varepsilon}(\rho_A)$ , then

$$\begin{aligned} \text{D}_s^{1-\varepsilon^2-3\varepsilon_1^2}(\rho'_A\|\sigma_A) &\leq \text{D}_s^{1-\varepsilon_1^2}(\rho_A\|\sigma_A) + \log v(\sigma) \\ &\quad + 8\log \frac{1}{\varepsilon_1} \end{aligned}$$

and

$$\begin{aligned} \tilde{\text{D}}_s^{\varepsilon^2+3\varepsilon_1^2}(\rho'_A\|\sigma_A) &\leq \text{D}_s^{1-\varepsilon_1^2}(\rho_A\|\sigma_A) + \log v(\sigma) \\ &\quad + 8\log \frac{1}{\varepsilon_1}. \end{aligned}$$

2.

$$\begin{aligned}\tilde{D}_s^{4\varepsilon_1^2}(\rho_A\|\sigma'_A) &\leq D_s^{1-\varepsilon_1^2}(\rho_A\|\sigma_A) + D_{\max}(\sigma_A\|\sigma'_A) \\ &\quad + \log v(\sigma) + 4\log \frac{1}{\varepsilon_1}.\end{aligned}$$

*Proof.* The items are proved as follows.

1. Consider the following series of inequalities, which follow by application of Fact 12 and the observation that  $D_{\max}^\delta(\rho_A\|\sigma_A) \geq D_{\max}^{\delta+\varepsilon}(\rho'_A\|\sigma_A)$  for any  $\delta > 0$ .

$$\begin{aligned}D_s^{1-\varepsilon_1^2}(\rho_A\|\sigma_A) &\geq \\ D_{\max}^{\sqrt{2\varepsilon_1}}(\rho_A\|\sigma_A) - \log v(\sigma) - 4\log \frac{1}{\varepsilon_1} \\ &\geq D_{\max}^{\sqrt{2\varepsilon_1}+\varepsilon}(\rho'_A\|\sigma_A) - \log v(\sigma) - 4\log \frac{1}{\varepsilon_1} \\ &\geq D_s^{1-3\varepsilon_1^2-\varepsilon^2}(\rho'_A\|\sigma_A) - \log v(\sigma) - 8\log \frac{1}{\varepsilon_1}.\end{aligned}$$

Second expression of the item follows similarly.

2. Let  $k \stackrel{\text{def}}{=} D_{\max}(\sigma_A\|\sigma'_A)$ . Consider the following series of inequalities, which follow by the application of Fact 12 and Fact 11.

$$\begin{aligned}D_s^{1-\varepsilon_1^2}(\rho_A\|\sigma_A) &\geq D_{\max}^{\sqrt{2\varepsilon_1}}(\rho_A\|\sigma_A) - \log v(\sigma) - 4\log \frac{1}{\varepsilon_1} \\ &\geq D_{\max}^{\sqrt{2\varepsilon_1}}(\rho_A\|\sigma'_A) - k - \log v(\sigma) - 4\log \frac{1}{\varepsilon_1} \\ &\geq \tilde{D}_s^{4\varepsilon_1^2}(\rho_A\|\sigma'_A) - k - \log v(\sigma) - 4\log \frac{1}{\varepsilon_1}.\end{aligned}$$

□

Following fact relates the two definitions of the information spectrum relative entropy used in our proofs.

**Fact 14.** Fix an  $\varepsilon \in (0, 1)$ . For quantum state  $\rho_A \in \mathcal{D}(A)$  and operator  $\sigma_A \in \mathcal{P}(A)$ , we have that

$$D_s^\varepsilon(\rho_A\|\sigma_A) \leq \tilde{D}_s^{1-\varepsilon}(\rho_A\|\sigma_A).$$

*Proof.* Let  $R$  achieve the infimum in the definition of  $\tilde{D}_s^{1-\varepsilon}(\rho_A\|\sigma_A)$ . Thus, we have that  $\text{Tr}(\rho_A\{\rho_A - 2^R\sigma_A\}_-) \geq \varepsilon$ . Using the relation  $\{\rho_A - 2^R\sigma_A\}_- + \{\rho_A - 2^R\sigma_A\}_+ = I_A$ , we obtain  $\text{Tr}(\rho_A\{\rho_A - 2^R\sigma_A\}_+) < 1 - \varepsilon$ . From the fact that  $\text{Tr}(\rho_A\{\rho_A - 2^R\sigma_A\}_+)$  is monotonically decreasing in  $R$  (as shown in [24], Equation 17), we conclude the proof. □

Following two facts are about some special properties of information spectrum relative entropy.

**Fact 15.** Let  $\varepsilon \in (0, 1)$ . Let  $\rho_A, \sigma_A \in \mathcal{D}(A)$  be quantum states such that  $\rho_A \in \text{supp}(\Pi_A)$  for some projector  $\Pi_A$  that commutes with  $\sigma_A$ . Then it holds that,

$$D_s^\varepsilon(\rho_A\|\sigma_A) = D_s^\varepsilon(\rho_A\|\Pi_A\sigma_A\Pi_A).$$



*Proof.* For any  $k > 0$ , it holds that

$$\rho_A - 2^k \sigma_A = \Pi_A \rho_A \Pi_A - 2^k \Pi_A \sigma_A \Pi_A - 2^k (\mathbb{I}_A - \Pi_A) \sigma_A (\mathbb{I}_A - \Pi_A).$$

Thus,

$$\begin{aligned} \{\rho_A - 2^k \sigma_A\}_+ &= \{\Pi_A \rho_A \Pi_A - 2^k \Pi_A \sigma_A \Pi_A\}_+ \\ &= \{\rho_A - 2^k \Pi_A \sigma_A \Pi_A\}_+. \end{aligned}$$

This completes the proof by using the definition of information spectrum relative entropy.  $\square$

**Fact 16.** Let  $\varepsilon \in (0, 1)$ , Let  $\rho_A \in \mathcal{D}(A)$  be a pure quantum state and  $\mu_A$  be the maximally mixed state on register  $A$ . Then

$$D_s^\varepsilon(\rho_A \| \mu_A) = D_{\max}(\rho_A \| \mu_A).$$

*Proof.* By definition, we have  $D_s^\varepsilon(\rho_A \| \mu_A) = \sup R : \text{Tr}(\rho_A \{\rho_A - 2^R \mu_A\}_+) \geq 1 - \varepsilon$ . Now the projector  $\{\rho_A - 2^R \mu_A\}_+$  contains the support of  $\rho_A$  if  $R < \log \dim(A)$  and otherwise is a null projector. The constraint  $\text{Tr}(\rho_A \{\rho_A - 2^R \mu_A\}_+) \geq 1 - \varepsilon > 0$  requires that  $\{\rho_A - 2^R \mu_A\}_+$  be a non-null projector. This implies that  $D_s^\varepsilon(\rho_A \| \mu_A) = \log \dim(A)$ , which is equal to the value of  $D_{\max}(\rho_A \| \mu_A)$ . This completes the proof.  $\square$

Following fact relates a projection on one system of a bipartite pure state to a projection on the other system.

**Fact 17.** Let  $\rho_{AB} \in \mathcal{D}(AB)$  be a pure quantum state. Let  $\Pi_A$  be a projector on the support of  $\rho_A$  that commutes with  $\rho_A$ . Then there exists a projector  $\Pi_B$  acting on register  $B$  (which we refer to as a dual to the projector  $\Pi_A$ ) such that  $\Pi_A \rho_{AB} \Pi_A = \Pi_B \rho_{AB} \Pi_B$ ,  $\Pi_B$  commutes with  $\rho_B$  and belongs to the support of  $\rho_B$ .

*Proof.* Let  $d$  be the dimension of the subspace corresponding to  $\Pi_A$ . Since  $\Pi_A$  and  $\rho_A$  commute, there exists a basis  $\{|e_i\rangle_A\}_{i=1}^{|A|}$  on register  $A$  and a basis  $\{|f_i\rangle_B\}_{i=1}^{|B|}$  on register  $B$  such that

$$\Pi_A = \sum_{i=1}^d |e_i\rangle\langle e_i|_A, \quad |\rho\rangle_{AB} = \sum_i \lambda_i |e_i\rangle_A |f_i\rangle_B.$$

Define  $\Pi_B \stackrel{\text{def}}{=} \sum_{i=1}^d |f_i\rangle\langle f_i|_B$ . It can be verified that  $\Pi_B$  satisfies the properties mentioned in the statement.  $\square$

We shall also use the well known Chernoff bounds.

**Fact 18** (Chernoff bounds). Let  $\varepsilon \in (0, 1)$ . Let  $X_1, \dots, X_n$  be independent random variables, with each  $X_i \in [0, 1]$  always. Let  $X \stackrel{\text{def}}{=} X_1 + \dots + X_n$  and  $\mu \stackrel{\text{def}}{=} \frac{\mathbb{E}X}{n} = \frac{\mathbb{E}X_1 + \dots + \mathbb{E}X_n}{n}$ . Then

$$\begin{aligned} \Pr(X \geq n(\mu + \varepsilon)) &\leq \exp\left(-n \frac{\varepsilon^2}{3\mu}\right) \\ \Pr(X \leq n(\mu - \varepsilon)) &\leq \exp\left(-n \frac{\varepsilon^2}{2\mu}\right). \end{aligned}$$

## Statement of the main theorem

Now we proceed to the main result of this section, which shows that given a pure state  $\rho$ , one can find a pure state  $\psi$  close to  $\rho$  that satisfies several constraints on the max-relative entropy.

**Theorem 4.** Let  $\delta \in (0, \frac{1}{6000})$ . Let  $\rho_{RMN} \in \mathcal{D}(RMN)$  be a pure quantum state. Fix an integer  $n$  such that:

$$n > 10^5 \cdot \log \frac{2}{\delta} \cdot \max \left\{ \frac{S(\rho_M) \cdot \log(1/\lambda_{\min}(\rho_M))}{\delta^2}, \frac{S(\rho_N) \cdot \log(1/\lambda_{\min}(\rho_N))}{\delta^2}, \frac{S(\rho_R) \cdot \log(1/\lambda_{\min}(\rho_R))}{\delta^2} \right\}.$$

Then there exists a pure quantum state  $\psi_{R^n M^n N^n}$  such that

1.  $P(\psi_{R^n M^n N^n}, \rho_{RMN}^{\otimes n}) \leq 60\sqrt{\delta}$ .
2.  $\psi_{R^n} \preceq (1 + 2000\delta)\rho_R^{\otimes n}$ .
3.  $D_{\max}(\psi_{R^n N^n} \| \rho_R^{\otimes n} \otimes \rho_N^{\otimes n}) \leq D_{\max}^{\sqrt{\delta}}(\rho_{RN}^{\otimes n} \| \rho_R^{\otimes n} \otimes \rho_N^{\otimes n}) + 10 \log \frac{1}{\delta} + 12n\delta + O(\log n)$ .
4.  $D_{\max}(\psi_{R^n M^n} \| \rho_R^{\otimes n} \otimes \rho_M^{\otimes n}) \leq D_{\max}^{\sqrt{\delta}}(\rho_{RM}^{\otimes n} \| \rho_R^{\otimes n} \otimes \rho_M^{\otimes n}) + 10 \log \frac{1}{\delta} + 12n\delta + O(\log n)$ .
5.  $D_{\max}(\psi_{R^n M^n N^n} \| \rho_R^{\otimes n} \otimes \rho_M^{\otimes n} \otimes \rho_N^{\otimes n}) \leq D_{\max}^{\sqrt{\delta}}(\rho_{RMN}^{\otimes n} \| \rho_R^{\otimes n} \otimes \rho_M^{\otimes n} \otimes \rho_N^{\otimes n}) + 10 \log \frac{1}{\delta} + 12n\delta + O(\log n)$ .

## A warm-up lemma

We first prove a simpler version of Theorem 4, wherein we assume that each of the marginals of the quantum state  $\rho_{RMN}$  are maximally mixed. More formally, we show the following lemma (note that the statement below is in fact in one-shot).

**Lemma 3.** Let  $\delta \in (0, \frac{1}{5})$ . Let  $\rho_{RMN} \in \mathcal{D}(RMN)$  be a pure quantum state such that  $\rho_R = \frac{I_R}{|R|}, \rho_M = \frac{I_M}{|M|}, \rho_N = \frac{I_N}{|N|}$ . Then there exists a pure quantum state  $\rho''_{RMN} \in \mathcal{B}^{5\delta}(\rho_{RMN})$  such that

1.  $\rho''_R \preceq \frac{\rho_R}{1-10\delta^2}$ .
2.  $D_{\max}(\rho''_{RN} \| \rho_R \otimes \rho_N) \leq D_{\max}^{\delta}(\rho_{RN} \| \rho_R \otimes \rho_N) + 6 \log \frac{1}{\delta}$ .
3.  $D_{\max}(\rho''_{RM} \| \rho_R \otimes \rho_M) \leq D_{\max}^{\delta}(\rho_{RM} \| \rho_R \otimes \rho_M) + 6 \log \frac{1}{\delta}$ .
4.  $D_{\max}(\rho''_{RMN} \| \rho_R \otimes \rho_M \otimes \rho_N) \leq D_{\max}^{\delta}(\rho_{RMN} \| \rho_R \otimes \rho_M \otimes \rho_N) + 21 \log \frac{1}{\delta}$ .

*Proof.* Below, we use the fact that maximally mixed quantum states have exactly one eigenvalue. From Fact 12 (which relates the information spectrum relative entropy to the max-relative entropy) and Fact 14 (which relates the two definitions of information spectrum relative entropy), we conclude the following relations.

$$\begin{aligned} & D_s^{1-2\delta^2}(\rho_{RMN} \| \rho_R \otimes \rho_M \otimes \rho_N) \\ & \leq \tilde{D}_s^{2\delta^2}(\rho_{RMN} \| \rho_R \otimes \rho_M \otimes \rho_N) \\ & \leq D_{\max}^{\delta}(\rho_{RMN} \| \rho_R \otimes \rho_M \otimes \rho_N) + 5 \log \frac{1}{\delta}, \\ & D_s^{1-2\delta^2}(\rho_{RM} \| \rho_R \otimes \rho_M) \leq \tilde{D}_s^{2\delta^2}(\rho_{RM} \| \rho_R \otimes \rho_M) \\ & \leq D_{\max}^{\delta}(\rho_{RM} \| \rho_R \otimes \rho_M) + 5 \log \frac{1}{\delta}, \\ & D_s^{1-2\delta^2}(\rho_{RN} \| \rho_R \otimes \rho_N) \leq \tilde{D}_s^{2\delta^2}(\rho_{RN} \| \rho_R \otimes \rho_N) \\ & \leq D_{\max}^{\delta}(\rho_{RN} \| \rho_R \otimes \rho_N) + 5 \log \frac{1}{\delta}. \end{aligned} \tag{4}$$

Let  $k$  be the minimum achieved in  $\tilde{D}_s^{2\delta^2}(\rho_{RM} \parallel \rho_R \otimes \rho_M)$ . Let  $\Pi \stackrel{\text{def}}{=} \{\rho_{RM} - 2^k \rho_R \otimes \rho_M\}_-$  and define the state

$$\rho'_{RMN} \stackrel{\text{def}}{=} \frac{\Pi \rho_{RMN} \Pi}{\text{Tr}(\Pi \rho_{RMN})}.$$

It holds that  $\text{Tr}(\Pi \rho_{RM}) \geq 1 - 2\delta^2$ . We prove the following properties of  $\rho'_{RMN}$ . First item shows that  $\rho'$  is close to  $\rho$ . Second and third items show that  $\rho'$  now satisfies the desired max-relative entropy constraints on systems  $R$  and  $R, M$ . Fourth item shows that  $\rho'$  still retains the information spectrum relative entropy properties of  $\rho$ .

**Claim 2.** *It holds that*

1.  $P(\rho'_{RMN}, \rho_{RMN}) \leq \sqrt{2}\delta$ .
2.  $\rho'_R \preceq \frac{\rho_R}{1-2\delta^2}$ .
3.  $D_{\max}(\rho'_{RM} \parallel \rho_R \otimes \rho_M) \leq \tilde{D}_s^{2\delta^2}(\rho_{RM} \parallel \rho_R \otimes \rho_M) + \log \frac{1}{1-2\delta^2}$ .
4.  $\tilde{D}_s^{8\delta^2}(\rho'_{RN} \parallel \rho_R \otimes \rho_N) \leq D_{\max}^\delta(\rho_{RN} \parallel \rho_R \otimes \rho_N) + 13 \log \frac{1}{\delta}$ .
5.  $\tilde{D}_s^{8\delta^2}(\rho'_{RMN} \parallel \rho_R \otimes \rho_M \otimes \rho_N) \leq D_{\max}^\delta(\rho_{RMN} \parallel \rho_R \otimes \rho_M \otimes \rho_N) + 13 \log \frac{1}{\delta}$ .

*Proof.* We prove each item as follows.

1. From the Gentle measurement lemma 6, we have  $F^2(\rho'_{RMN}, \rho_{RMN}) \geq \text{Tr}(\Pi \rho_{RMN}) \geq 1 - 2\delta^2$ . Thus,  $P(\rho'_{RMN}, \rho_{RMN}) \leq \sqrt{2}\delta$ .
2. Since  $\rho_R \otimes \rho_M$  commutes with  $\rho_{RM}$ ,  $\Pi$  commutes with  $\rho_{RM}$  as well. Thus,

$$\rho'_{RM} = \frac{\Pi \rho_{RM} \Pi}{\text{Tr}(\Pi \rho_{RMN})} \preceq \frac{\rho_{RM}}{1 - 2\delta^2}.$$

Thus, we conclude the statement by tracing out register  $M$ .

3. By definition of  $\Pi$ , we have

$$\rho'_{RM} = \frac{\Pi \rho_{RM} \Pi}{\text{Tr}(\Pi \rho_{RMN})} \preceq \frac{2^k \rho_R \otimes \rho_M}{1 - 2\delta^2}.$$

This concludes the statement from the definition of  $k$ .

4. From Equation 4 and Fact 13 (which relates the information spectrum relative entropies of two close-by quantum states), we have

$$\begin{aligned} & \tilde{D}_s^{8\delta^2}(\rho'_{RN} \parallel \rho_R \otimes \rho_N) \\ & \leq D_s^{1-2\delta^2}(\rho_{RN} \parallel \rho_R \otimes \rho_N) + 8 \log \frac{1}{\delta^2} \\ & \leq D_{\max}^\delta(\rho_{RN} \parallel \rho_R \otimes \rho_N) + 13 \log \frac{1}{\delta}. \end{aligned}$$

5. Similar arguments as above imply that

$$\begin{aligned} & \tilde{D}_s^{8\delta^2}(\rho'_{RMN} \parallel \rho_R \otimes \rho_M \otimes \rho_N) \\ & \leq D_{\max}^\delta(\rho_{RMN} \parallel \rho_R \otimes \rho_M \otimes \rho_N) + 13 \log \frac{1}{\delta}. \end{aligned}$$

□

Let  $k'$  be the minimum achieved in  $\tilde{D}_s^{8\delta^2}(\rho'_{RN} \parallel \rho_R \otimes \rho_N)$ . Let  $\Pi' \stackrel{\text{def}}{=} \{\rho'_{RN} - 2^{k'} \rho_R \otimes \rho_N\}_-$  and define the state

$$\rho''_{RMN} \stackrel{\text{def}}{=} \frac{\Pi' \rho'_{RMN} \Pi'}{\text{Tr}(\Pi' \rho'_{RMN})}.$$

It holds that  $\text{Tr}(\Pi' \rho'_{RN}) \geq 1 - 8\delta^2$ . We prove the following properties for  $\rho''_{RMN}$ . First property says that  $\rho''$  is close to  $\rho$ . Second and third properties say that the max-relative entropy constraints on the registers  $R$  and  $R, N$  hold for  $\rho''$ . Fourth property says that the max-relative entropy constraint on  $\rho'_{RM}$  continues to hold for  $\rho''$  as well, even when the projector  $\Pi'$  acted on a different subsystem. This uses Fact 17 in its argument. Fifth property shows that the max-entropy constraint holds on all the registers  $RMN$ .

**Claim 3.** *It holds that*

1.  $P(\rho''_{RMN}, \rho_{RMN}) \leq 5\delta$ .
2.  $\rho''_R \preceq \frac{\rho_R}{1-10\delta^2}$ .
3.  $D_{\max}(\rho''_{RN} \parallel \rho_R \otimes \rho_N) \leq D_{\max}^{\delta}(\rho_{RN} \parallel \rho_R \otimes \rho_N) + 6 \log \frac{1}{\delta}$ .
4.  $D_{\max}(\rho''_{RM} \parallel \rho_R \otimes \rho_M) \leq D_{\max}^{\delta}(\rho_{RM} \parallel \rho_R \otimes \rho_M) + 6 \log \frac{1}{\delta}$ .
5.  $D_{\max}(\rho''_{RMN} \parallel \rho_R \otimes \rho_M \otimes \rho_N) \leq D_{\max}^{\delta}(\rho_{RMN} \parallel \rho_R \otimes \rho_M \otimes \rho_N) + 21 \log \frac{1}{\delta}$ .

*Proof.* We prove each item as follows.

1. This follows from the application of the Gentle measurement lemma 6.
2. This follows similarly along the lines of Item 2 in Claim 2.
3. This follows along the lines similar to Item 3, Claim 2.
4. We note that the projector  $\Pi'$  commutes with  $\rho'_{RN}$  and belongs to its support. Since  $\rho'_{RMN}$  is a pure state, there exists a dual projector  $\tilde{\Pi}$  acting on register  $M$  that belongs to the support of  $\rho'_M$  and commutes with  $\rho'_M$  (Fact 17). Further,  $\tilde{\Pi}$  satisfies  $\text{Tr}(\tilde{\Pi} \rho'_M) = \text{Tr}(\Pi' \rho'_{RN})$  and  $\rho'' = \frac{\tilde{\Pi} \rho'_{RMN} \tilde{\Pi}}{\text{Tr}(\tilde{\Pi} \rho'_{RMN})}$ . Thus, we find

$$\begin{aligned} \rho''_{RM} &\preceq \frac{\tilde{\Pi} \rho'_{RM} \tilde{\Pi}}{1 - 8\delta^2} \preceq \frac{2^k \tilde{\Pi} \rho_R \otimes \rho_M \tilde{\Pi}}{1 - 10\delta^2} \\ &= \frac{2^k \rho_R \otimes \tilde{\Pi} \rho_M \tilde{\Pi}}{1 - 10\delta^2} \preceq \frac{2^k \rho_R \otimes \rho_M}{1 - 10\delta^2}. \end{aligned}$$

Above, the second inequality follows from Item 3, Claim 2. This concludes the item, from the definition of  $k$  and Equation 4.

5. Along the lines similar to Item 5, Claim 2, we have that

$$\begin{aligned} &\tilde{D}_s^{25\delta^2}(\rho''_{RMN} \parallel \rho_R \otimes \rho_M \otimes \rho_N) \\ &\leq D_{\max}^{\delta}(\rho_{RMN} \parallel \rho_R \otimes \rho_M \otimes \rho_N) + 21 \log \frac{1}{\delta}. \end{aligned}$$

But  $\rho''_{RMN}$  is a pure quantum state. Thus from the choice of  $\delta$ , which ensures that  $25\delta^2 < 1$  and Fact 16 (which shows that the smooth max-relative entropy and the information spectrum relative entropy coincide for pure states), we obtain

$$\begin{aligned} &D_{\max}(\rho''_{RMN} \parallel \rho_R \otimes \rho_M \otimes \rho_N) \\ &\leq D_{\max}^{\delta}(\rho_{RMN} \parallel \rho_R \otimes \rho_M \otimes \rho_N) + 21 \log \frac{1}{\delta}. \end{aligned}$$

□

□

This completes the proof of the lemma.

## Proof of the main theorem

Now we proceed to the proof of Theorem 4. Its proof roughly follows the proof of Lemma 3 above. Additional care is required in the arguments due to the fact that the marginals of  $\rho_{RMN}^{\otimes n}$  are not exactly uniform.

*Proof of Theorem 4.* Our proof is divided into three main steps.

**Typical projection onto the subsystems  $R, M, N$ :** For brevity, we set  $\rho_{R^n M^n N^n} \stackrel{\text{def}}{=} \rho_{RMN}^{\otimes n}$ . Let  $\Pi_{R^n}$  be the projector onto the eigenvectors of  $\rho_{R^n}$  with eigenvalues in the range  $[2^{-n(S(\rho_R)+\delta)}, 2^{-n(S(\rho_R)-\delta)}]$ . Similarly, define  $\Pi_{M^n}, \Pi_{N^n}$ . Let  $\mu_{R^n}, \mu_{M^n}, \mu_{N^n}$  be the uniform distributions in the support of  $\Pi_{R^n}, \Pi_{M^n}, \Pi_{N^n}$  respectively. Using Chernoff bounds (Fact 18), we have that

$$\text{Tr}(\Pi_{R^n} \rho_{R^n}) \geq 1 - 2 \cdot \exp\left(-\frac{\delta^2 \cdot n}{S(\rho_R) \cdot \log(1/\lambda_{\min}(\rho_R))}\right) \geq 1 - \delta$$

for the choice of  $n$ . Similarly,  $\text{Tr}(\Pi_{M^n} \rho_{M^n}) \geq 1 - \delta$  and  $\text{Tr}(\Pi_{N^n} \rho_{N^n}) \geq 1 - \delta$ .

Thus, the dimensions of the projectors  $\Pi_{R^n}, \Pi_{M^n}, \Pi_{N^n}$  are in the range  $[(1 - \delta)2^{n(S(\rho_R)-\delta)}, 2^{n(S(\rho_R)+\delta)}]$ ,  $[(1 - \delta)2^{n(S(\rho_M)-\delta)}, 2^{n(S(\rho_M)+\delta)}]$  and  $[(1 - \delta)2^{n(S(\rho_N)-\delta)}, 2^{n(S(\rho_N)+\delta)}]$ , respectively. Following relations are now easy to observe.

$$\begin{aligned} (1 - \delta)2^{-2n\delta} \Pi_{R^n} \rho_{R^n} \Pi_{R^n} &\preceq \mu_{R^n} \\ &\preceq (1 + \delta)2^{2n\delta} \Pi_{R^n} \rho_{R^n} \Pi_{R^n} \preceq (1 + \delta)2^{2n\delta} \rho_{R^n}, \\ (1 - \delta)2^{-2n\delta} \Pi_{M^n} \rho_{M^n} \Pi_{M^n} &\preceq \mu_{M^n} \\ &\preceq (1 + \delta)2^{2n\delta} \Pi_{M^n} \rho_{M^n} \Pi_{M^n} \preceq (1 + \delta)2^{2n\delta} \rho_{M^n}, \\ (1 - \delta)2^{-2n\delta} \Pi_{N^n} \rho_{N^n} \Pi_{N^n} &\preceq \mu_{N^n} \\ &\preceq (1 + \delta)2^{2n\delta} \Pi_{N^n} \rho_{N^n} \Pi_{N^n} \preceq (1 + \delta)2^{2n\delta} \rho_{N^n}. \end{aligned} \tag{5}$$

Now, define the state

$$\rho'_{R^n M^n N^n} \stackrel{\text{def}}{=} \frac{(\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n}) \rho_{R^n M^n N^n} (\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n})}{\text{Tr}(\rho_{R^n M^n N^n} (\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n}))}.$$

We will establish the following claims about  $\rho'_{R^n M^n N^n}$ .

**Claim 4.** *It holds that*

1.  $F^2(\rho'_{R^n M^n N^n}, \rho_{R^n M^n N^n}) \geq 1 - 64\delta$ .
2.  $\text{Tr}(\rho_{R^n M^n N^n} (\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n})) \geq 1 - 64\delta$ .
3.  $\rho'_{R^n M^n N^n} \in \text{supp}(\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n})$ .
4.  $\rho'_{R^n} \preceq \frac{1}{1-64\delta} \rho_{R^n}, \rho'_{M^n} \preceq \frac{1}{1-64\delta} \rho_{M^n},$   
 $\rho'_{N^n} \preceq \frac{1}{1-64\delta} \rho_{N^n}.$

*Proof.* We prove each item in a sequence below.

1. This is a straightforward application of Gentle measurement lemma (Fact 6) and Fact 7.

2. This follows along similar lines as argued above.
3. This follows since  $(\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n})\rho_{R^n M^n N^n}(\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n}) \preceq \Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n}$ .
4. We proceed as follows for  $\rho'_{R^n}$ .

$$\begin{aligned}
\rho'_{R^n} &= \frac{1}{\text{Tr}(\rho_{R^n M^n N^n}(\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n}))} \\
&\quad \text{Tr}_{M^n N^n}((\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n})\rho_{R^n M^n N^n} \\
&\quad ((\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n}))) \\
&= \frac{1}{\text{Tr}(\rho_{R^n M^n N^n}(\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n}))} \\
&\quad \text{Tr}_{M^n N^n}((\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n})\rho_{R^n M^n N^n} \\
&\quad ((\Pi_{R^n} \otimes \mathbf{I}_{M^n} \otimes \mathbf{I}_{N^n}))) \\
&\preceq \frac{1}{\text{Tr}(\rho_{R^n M^n N^n}(\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n}))} \\
&\quad \text{Tr}_{M^n N^n}((\Pi_{R^n} \otimes \mathbf{I}_{M^n} \otimes \mathbf{I}_{N^n})\rho_{R^n M^n N^n} \\
&\quad ((\Pi_{R^n} \otimes \mathbf{I}_{M^n} \otimes \mathbf{I}_{N^n}))) \\
&= \frac{1}{\text{Tr}(\rho_{R^n M^n N^n}(\Pi_{R^n} \otimes \Pi_{M^n} \otimes \Pi_{N^n}))} \Pi_{R^n} \rho_{R^n} \Pi_{R^n} \\
&\preceq \frac{1}{1 - 64\delta} \rho_{R^n}.
\end{aligned}$$

Last inequality is due to Item 2 above and the fact that  $\Pi_{R^n}$  is a projector onto certain eigenspace of  $\rho_{R^n}$ . Same argument holds for  $\rho_{M^n}$  and  $\rho_{N^n}$ .

□

**Switching to the information spectrum relative entropy:** Using above claim, we now proceed to the second step of our proof. As a corollary from the Claim (Item 1), along with Fact 12, we conclude

$$\begin{aligned}
&D_s^{1-90\delta}(\rho'_{R^n M^n N^n} \| \rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}) - 2 \log \frac{1}{\delta} \\
&\leq D_{\max}^{9\sqrt{\delta}}(\rho'_{R^n M^n N^n} \| \rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}) \\
&\leq D_{\max}^{\sqrt{\delta}}(\rho_{R^n M^n N^n} \| \rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}).
\end{aligned} \tag{6}$$

Fact 15 implies that

$$\begin{aligned}
&D_s^{1-90\delta}(\rho'_{R^n M^n N^n} \| \rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}) = \\
&D_s^{1-90\delta}\left(\rho'_{R^n M^n N^n} \| \Pi_{R^n} \rho_{R^n} \Pi_{R^n} \right. \\
&\quad \left. \otimes \Pi_{M^n} \rho_{M^n} \Pi_{M^n} \otimes \Pi_{N^n} \rho_{N^n} \Pi_{N^n}\right).
\end{aligned}$$

Let  $v \stackrel{\text{def}}{=} v(\Pi_{R^n} \rho_{R^n} \Pi_{R^n} \otimes \Pi_{M^n} \rho_{M^n} \Pi_{M^n} \otimes \Pi_{N^n} \rho_{N^n} \Pi_{N^n})$ , which is the number of distinct eigenvalues of

$\Pi_{R^n} \rho_{R^n} \Pi_{R^n} \otimes \Pi_{M^n} \rho_{M^n} \Pi_{M^n} \otimes \Pi_{N^n} \rho_{N^n} \Pi_{N^n}$ . We apply Fact 13 along with Equation 5 to conclude that

$$\begin{aligned} & D_s^{1-90\delta} \left( \rho'_{R^n M^n N^n} \parallel \Pi_{R^n} \rho_{R^n} \Pi_{R^n} \otimes \right. \\ & \left. \Pi_{M^n} \rho_{M^n} \Pi_{M^n} \otimes \Pi_{N^n} \rho_{N^n} \Pi_{N^n} \right) \\ & \geq \tilde{D}_s^{400\delta} (\rho'_{R^n M^n N^n} \parallel \mu_{R^n} \otimes \mu_{M^n} \otimes \mu_{N^n}) \\ & - \log \frac{2^{6n\delta}}{(1-\delta)^3} - \log v - 5 \log \frac{1}{\delta}. \end{aligned}$$

Combining this with Equation 6, we conclude that

$$\begin{aligned} & \tilde{D}_s^{400\delta} (\rho'_{R^n M^n N^n} \parallel \mu_{R^n} \otimes \mu_{M^n} \otimes \mu_{N^n}) \\ & \leq D_{\max}^{\sqrt{\delta}} (\rho_{R^n M^n N^n} \parallel \rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}) \\ & + 8 \log \frac{1}{\delta} + 6n\delta + \log v. \end{aligned} \tag{7}$$

In the same way, we can argue that

$$\begin{aligned} & \tilde{D}_s^{400\delta} (\rho'_{R^n M^n} \parallel \mu_{R^n} \otimes \mu_{M^n}) \\ & \leq D_{\max}^{\sqrt{\delta}} (\rho_{R^n M^n} \parallel \rho_{R^n} \otimes \rho_{M^n}) \\ & + 8 \log \frac{1}{\delta} + 6n\delta + \log v. \end{aligned} \tag{8}$$

and

$$\begin{aligned} & \tilde{D}_s^{400\delta} (\rho'_{R^n N^n} \parallel \mu_{R^n} \otimes \mu_{N^n}) \\ & \leq D_{\max}^{\sqrt{\delta}} (\rho_{R^n N^n} \parallel \rho_{R^n} \otimes \rho_{N^n}) \\ & + 8 \log \frac{1}{\delta} + 6n\delta + \log v. \end{aligned} \tag{9}$$

**Removing large eigenvalues from a subsystem:** Let  $k$  be the minimum achieved in  $\tilde{D}_s^{400\delta}(\rho'_{R^n M^n} \parallel \mu_{R^n} \otimes \mu_{M^n})$ . For brevity, set  $\Pi' \stackrel{\text{def}}{=} \{\rho'_{R^n M^n} - 2^k \mu_{R^n} \otimes \mu_{M^n}\}_-$  and define the state

$$\rho''_{R^n M^n N^n} \stackrel{\text{def}}{=} \frac{\Pi' \rho'_{R^n M^n N^n} \Pi'}{\text{Tr}(\Pi' \rho'_{R^n M^n N^n})}.$$

It holds that  $\text{Tr}(\Pi' \rho'_{A^n B^n}) \geq 1 - 400\delta$ . We prove the following properties for  $\rho''_{R^n M^n N^n}$ .

**Claim 5.** *It holds that*

1.  $P(\rho''_{R^n M^n N^n}, \rho_{R^n M^n N^n}) \leq 30\sqrt{\delta}$ .
2.  $\rho''_{R^n} \preceq (1 + 1000\delta)\rho_{R^n}$ ,  $\rho''_{M^n} \preceq (1 + 1000\delta)\rho_{M^n}$ ,  $\rho''_{N^n} \preceq (1 + 1000\delta)\rho_{N^n}$ . Furthermore,  $\rho''_{R^n} \in \text{supp}(\Pi_{R^n})$ ,  $\rho''_{M^n} \in \text{supp}(\Pi_{M^n})$  and  $\rho''_{N^n} \in \text{supp}(\Pi_{N^n})$ .
3.  $D_{\max}(\rho''_{R^n M^n} \parallel \rho_{R^n} \otimes \rho_{M^n}) \leq D_{\max}^{\sqrt{\delta}}(\rho_{R^n M^n} \parallel \rho_{R^n} \otimes \rho_{M^n}) + 9 \log \frac{1}{\delta} + 12n\delta + \log v$ .
4.  $\tilde{D}_s^{1300\delta}(\rho''_{R^n M^n N^n} \parallel \mu_{R^n} \otimes \mu_{M^n} \otimes \mu_{N^n}) \leq D_{\max}^{\sqrt{\delta}}(\rho_{R^n M^n N^n} \parallel \rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}) + 8 \log \frac{1}{\delta} + 6n\delta + \log v$ .  
 $\tilde{D}_s^{1300\delta}(\rho''_{R^n N^n} \parallel \mu_{R^n} \otimes \mu_{N^n}) \leq D_{\max}^{\sqrt{\delta}}(\rho_{R^n N^n} \parallel \rho_{R^n} \otimes \rho_{N^n}) + 8 \log \frac{1}{\delta} + 6n\delta + \log v$ .

*Proof.* We prove the items in the respective sequence.

1. From Gentle measurement lemma 6, we have that  $F^2(\rho''_{R^n M^n N^n}, \rho'_{R^n M^n N^n}) \geq \text{Tr}(\Pi' \rho'_{R^n M^n N^n}) \geq 1 - 400\delta$ . Using Claim 4 (Item 1) and triangle inequality for purified distance (Fact 1), we obtain that  $P(\rho''_{R^n M^n N^n}, \rho_{R^n M^n N^n}) \leq 30\sqrt{\delta}$ .
2. Since  $\mu_{R^n} \otimes \mu_{M^n}$  is uniform in the support of  $\rho'_{R^n M^n}$ ,  $\rho'_{R^n M^n}$  commutes with  $\mu_{R^n} \otimes \mu_{M^n}$ . This immediately implies that  $\Pi'$  commutes with  $\rho'_{R^n M^n}$ . Thus, we conclude that

$$\rho''_{R^n M^n} = \frac{\Pi' \rho'_{R^n M^n} \Pi'}{\text{Tr}(\Pi' \rho'_{R^n M^n})} \preceq \frac{\rho'_{R^n M^n}}{1 - 400\delta},$$

where the inequality follows from the relation  $\text{Tr}(\Pi' \rho'_{R^n M^n}) \geq 1 - 400\delta$ .

Invoking Claim 4 (Item 4), we obtain

$$\rho''_{R^n} \preceq \frac{\rho'_{R^n}}{1 - 400\delta} \preceq \frac{\rho_{R^n}}{(1 - 400\delta)(1 - 10\delta)} \preceq \frac{\rho_{R^n}}{1 - 410\delta}.$$

Similarly, we obtain  $\rho''_{M^n} \preceq \frac{\rho_{M^n}}{1 - 410\delta}$ . The inequality  $\rho''_{N^n} \preceq \frac{\rho'_{N^n}}{1 - 400\delta}$  follows from the fact that  $\Pi'$  does not act on register  $N^n$ . First part of the item now follows since  $\frac{1}{1 - 410\delta} < 1 + 1000\delta$  for the choice of  $\delta$ .

For the second part, we use the fact that  $\rho'_{R^n} \in \text{supp}(\Pi_{R^n})$  and the relation  $\rho''_{R^n} \preceq \frac{\rho'_{R^n}}{1 - 400\delta}$  established above. Same argument holds for  $\rho''_{M^n}, \rho''_{N^n}$ .

3. By definition of  $\Pi'$ , we have that

$$\Pi' \rho'_{R^n M^n} \Pi' \preceq 2^k \Pi' \mu_{R^n} \otimes \mu_{M^n} \Pi' \preceq 2^k \mu_{R^n} \otimes \mu_{M^n},$$

where last inequality holds since  $\mu_{R^n} \otimes \mu_{M^n}$  is uniform and  $\Pi'$  is in its support. Thus,

$$\rho''_{R^n M^n} = \frac{\Pi' \rho'_{R^n M^n} \Pi'}{\text{Tr}(\Pi' \rho'_{R^n M^n})} \preceq \frac{2^k}{1 - 400\delta} \cdot \mu_{R^n} \otimes \mu_{M^n}.$$

From Equation 5, this further implies that

$$\rho''_{R^n M^n} \preceq \frac{(1 + \delta)^2 \cdot 2^{2n\delta} \cdot 2^k}{1 - 400\delta} \cdot \rho_{R^n} \otimes \rho_{M^n}.$$

This proves the item after using Equation 8 to upper bound  $k$ .

4. Applying Fact 14, we conclude from Equation 7 that

$$\begin{aligned} & D_s^{1-400\delta}(\rho'_{R^n M^n N^n} \| \mu_{R^n} \otimes \mu_{M^n} \otimes \mu_{N^n}) \\ & \leq D_{\max}^{\sqrt{\delta}}(\rho_{R^n M^n N^n} \| \rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}) \\ & + 8 \log \frac{1}{\delta} + 6n\delta + \log v. \end{aligned}$$

Now we use Fact 13 along with the Item 1 above, which says that  $P(\rho''_{R^n M^n N^n}, \rho_{R^n M^n N^n}) \leq 30\sqrt{\delta}$ , to conclude that

$$\begin{aligned} & \tilde{D}_s^{1300\delta}(\rho''_{R^n M^n N^n} \| \mu_{R^n} \otimes \mu_{M^n} \otimes \mu_{N^n}) \\ & \leq D_{\max}^{\sqrt{\delta}}(\rho_{R^n M^n N^n} \| \rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}) \\ & + 8 \log \frac{1}{\delta} + 6n\delta + \log v. \end{aligned}$$

Second expression in this item follows similarly using Equation 9.



□

**Removing large eigenvalues from another subsystem:** Let  $k'$  be the minimum achieved in  $\tilde{D}_s^{1300\delta}(\rho''_{R^n N^n} \|\mu_{R^n} \otimes \mu_{N^n})$ . For brevity, set  $\Pi'' \stackrel{\text{def}}{=} \{\rho''_{R^n N^n} - 2^{k'} \mu_{R^n} \otimes \mu_{N^n}\}_-$  and define the state

$$\rho'''_{R^n M^n N^n} \stackrel{\text{def}}{=} \frac{\Pi'' \rho''_{R^n M^n N^n} \Pi''}{\text{Tr}(\Pi'' \rho''_{R^n M^n N^n})}.$$

It holds that  $\text{Tr}(\Pi'' \rho''_{A^n B^n}) \geq 1 - 1300\delta$ . We prove the following properties for  $\rho'''_{R^n M^n N^n}$ .

**Claim 6.** *It holds that*

1.  $P(\rho'''_{R^n M^n N^n}, \rho_{R^n M^n N^n}) \leq 60\sqrt{\delta}$ .
2.  $\rho'''_{R^n} \preceq (1 + 2000\delta)\rho_{R^n}$ .
3.  $D_{\max}(\rho'''_{R^n N^n} \|\rho_{R^n} \otimes \rho_{N^n}) \leq D_{\max}^{\sqrt{\delta}}(\rho_{R^n N^n} \|\rho_{R^n} \otimes \rho_{N^n}) + 10 \log \frac{1}{\delta} + 12n\delta + \log v$ .
4.  $D_{\max}(\rho'''_{R^n M^n} \|\rho_{R^n} \otimes \rho_{M^n}) \leq D_{\max}^{\sqrt{\delta}}(\rho_{R^n M^n} \|\rho_{R^n} \otimes \rho_{M^n}) + 10 \log \frac{1}{\delta} + 12n\delta + \log v$ .
5.  $D_{\max}(\rho'''_{R^n M^n N^n} \|\rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}) \leq D_{\max}^{\sqrt{\delta}}(\rho_{R^n M^n N^n} \|\rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}) + 10 \log \frac{1}{\delta} + 12n\delta + \log v$ .

*Proof.* We prove each item as follows.

1. This follows similarly along the lines of Item 1 in Claim 5.
2. This follows similarly along the lines of Item 2 in Claim 5.
3. This follows similarly along the lines of Item 3 in Claim 5 and also uses Item 4 of Claim 5.
4. We note that the projector  $\Pi''$  commutes with  $\rho''_{R^n N^n}$  and belongs to its support. Since  $\rho''_{R^n M^n N^n}$  is a pure state, there exists a dual projector  $\tilde{\Pi}$  acting on register  $M^n$  that belongs to the support of  $\rho''_{M^n}$  and commutes with  $\rho''_{M^n}$  such that  $\text{Tr}(\tilde{\Pi} \rho''_{R^n M^n N^n}) = \text{Tr}(\Pi'' \rho''_{R^n M^n N^n})$  and  $\rho'''_{R^n M^n N^n} = \frac{\tilde{\Pi} \rho''_{R^n M^n N^n} \tilde{\Pi}}{\text{Tr}(\tilde{\Pi} \rho''_{R^n M^n N^n})}$ .

Thus, we find

$$\begin{aligned} \rho'''_{R^n M^n} &\preceq \frac{\tilde{\Pi} \rho''_{R^n M^n} \tilde{\Pi}}{1 - 1300\delta} \preceq \frac{2^k \tilde{\Pi} \mu_{R^n} \otimes \mu_{M^n} \tilde{\Pi}}{1 - 1700\delta} \\ &= \frac{2^k \mu_{R^n} \otimes \tilde{\Pi} \mu_{M^n} \tilde{\Pi}}{1 - 1700\delta} \preceq \frac{2^k \mu_{R^n} \otimes \mu_{M^n}}{1 - 1700\delta}. \end{aligned}$$

Above, the second inequality is immediate from the definition of  $\rho''_{R^n M^n}$  (and also appears in the proof of Item 3, Claim 5). Last inequality follows from the observation that  $\tilde{\Pi}$  is in the support of  $\rho''_{M^n}$  and hence in the support of  $\Pi_{M^n}$ , as implied by Claim 5 (Item 2). This ensures that  $\tilde{\Pi}$  is in the support of  $\mu_{M^n}$ . The item concludes by using Equation 5.

5. From Claim 5 (Item 4), and Fact 13, we conclude that

$$\begin{aligned} &D_s^{1-5000\delta}(\rho'''_{R^n M^n N^n} \|\mu_{R^n} \otimes \mu_{M^n} \otimes \mu_{N^n}) \\ &\leq D_{\max}^{\sqrt{\delta}}(\rho_{R^n M^n N^n} \|\rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}) \\ &\quad + 10 \log \frac{1}{\delta} + 6n\delta + \log v. \end{aligned}$$

But observe that  $\rho'''_{R^n M^n N^n}$  is a pure state. Thus, from the choice of  $\delta$  which ensures that  $5000\delta < 1$  and Fact 16, we have that

$$\begin{aligned} & D_{\max}(\rho'''_{R^n M^n N^n} \| \mu_{R^n} \otimes \mu_{M^n} \otimes \mu_{N^n}) \\ & \leq D_{\max}^{\sqrt{\delta}}(\rho_{R^n M^n N^n} \| \rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}) \\ & \quad + 10 \log \frac{1}{\delta} + 6n\delta + \log v. \end{aligned}$$

The item now follows from Equation 5.

□

Now the value of  $v$ , which is the number of distinct eigenvalues of  $\Pi_{R^n} \rho_{R^n} \Pi_{R^n} \otimes \Pi_{M^n} \rho_{M^n} \Pi_{M^n} \otimes \Pi_{N^n} \rho_{N^n} \Pi_{N^n}$ , is upper bounded by the number of distinct eigenvalues of  $\rho_{R^n} \otimes \rho_{M^n} \otimes \rho_{N^n}$ . This is at most  $n^{2|R|+2|M|+2|N|}$ . This proves the theorem, combining with the Claim 6 and setting  $\psi_{R^n M^n N^n} = \rho'''_{R^n M^n N^n}$ . □